Model-Based Reinforcement Learning with Multinomial Logistic Function Approximation

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More Recent Results in RL

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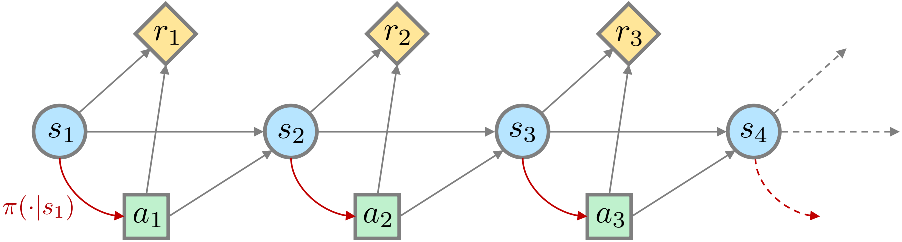
More Recent Results in RL

RL with function approximation has made significant advances in empirical studies. However,

- Theoretical understanding of these methods is still limited
- Most existing theoretical works in RL with function approximation consider linear function approximation
- Trying to close the gap between theory and empirical findings

Markov Decision Processes (MDPs)



A finite-horizon Markov Decision Processes (MDPs), $\mathcal{M} = (\mathcal{S}, \mathcal{A}, H, P, r)$

- $\mathcal{S}$: State space
- $\mathcal{A}$: Set of actions
- H: Length of horizon
- $P = \{\mathbb{P}(\cdot \mid s, a) \mid (s, a) \in \mathcal{S} \times \mathcal{A}\}$: Collection of transition probability
- r: Reward function

Value Functions & Performance Measure

Value function of a policy π

$$Q_h^\pi(s,a) := \mathbb{E}_\pi\left[\sum_{h'=h}^{H} r\left(s_{h'}, \pi(s_{h'}, h')\right) \mid s_h = s, a_h = a\right]$$

$$V_h^\pi(s) = Q^\pi(s, \pi_h(s))$$

Optimal value function & policy

$$Q_h^*(s,a) = \sup_\pi Q_h^\pi(s,a)$$

$$\pi_h^*(s) := \operatorname*{argmax}_{a\in\mathcal{A}} Q_h^*(s,a)$$

Performance measure

$$\textbf{Regret}_\pi(K) := \sum_{k=1}^{K}(V_1^* - V_1^{\pi_k})(s_{k,1})$$

- K: total number of episodes
- $T = KH$: total number of timesteps

Existing Works: Tabular Methods

A large number of works both on model-based and model-free methods

- Model-based: (Jaksch et al., 2010; Osband and Roy, 2014; Azar et al., 2017; Dann et al., 2017; Agrawal and Jia, 2017; Ouyang et al., 2017)
- Model-free: (Jin et al., 2018; Osband et al., 2019; Russo, 2019; Zhang et al., 2020, 2021)

Model-based and model-free methods can achieve $\widetilde{\mathcal{O}}(H\sqrt{SAT})$ regret.

- *optimal up to logarithmic factors* (Jin et al., 2018; Zhang et al., 2020).
- $S = |\mathcal{S}|$: the total number of states
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But these methods do not perform well with large $\mathcal{S}$ & $\mathcal{A}$.

- No generalization across states (or actions)

Existing Works: Linear Function Approximation

Low-rank linear MDPs (Model-free): $\mathbb{P}(s' \mid s, a) = \langle \phi(s, a), \boldsymbol{\mu}^*(s') \rangle$

- Optimism: `LSVI-UCB` $\widetilde{\mathcal{O}}(d^{3/2}H^{3/2}\sqrt{T})$ (Jin et al., 2020)
- Randomization: `OPT-RLSVI` $\widetilde{\mathcal{O}}(d^2H^2\sqrt{T})$ (Zanette et al., 2020)

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Bilinear transition model (Model-based): $\mathbb{P}(s' \mid s, a) = \phi(s, a)^\top M^* \psi(s')$

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Linear mixture models (Model-based) :$\mathbb{P}(s' \mid s, a) = \sum_{j=1}^{d} \theta_j^* \mathbb{P}_j(s' \mid s, a)$

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Generalized linear function approximation (Model-free)

- Optimism: `LSVI-UCB with GLM` $\widetilde{\mathcal{O}}(d^{3/2}H\sqrt{T})$ Wang et al. (2021)

  ↳ But, this is not GLM approximation of the transition model

Limitation of Linear Transition Model

Proposition (Limited admissible features)

For an arbitrary set of features, a linear transition model cannot induce a proper probability distribution over next states.

- Difficult to ensure $\sum_{s'} \hat{P}(s' \mid s, a) = 1$

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Proposition (Dependence on state space)

`UC-MatrixRL` (Yang and Wang, 2020) based on the linear model has the regret of $\widetilde{\mathcal{O}}(|\mathcal{S}| dH^2 \sqrt{T})$.

- Potentially leading to serious deterioration of the performances

Multinomial Logistic Transition model

Motivation

- State transition in MDP is essentially categorical distribution.
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MNL Transition model

$$P(s' \mid s, a) = \frac{\exp(\varphi(s, a, s')^\top \theta^*)}{\sum_{\widetilde{s} \in \mathcal{S}_{s,a}} \exp(\varphi(s, a, \widetilde{s})^\top \theta^*)}$$

- $\varphi(s, a, s') \in \mathbb{R}^d$: given feature vector
- $\theta^* \in \mathbb{R}^d$: Unknown transition core parameter
- $\mathcal{S}_{s,a} := \{s' \in \mathcal{S} : P(s' \mid s, a) \neq 0\}$: set of reachable states from (s, a)

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Can we design a provably efficient RL algorithm for the multinomial logistic transition model?

Upper Confidence Model-based RL for MNL

Algorithm Upper Confidence Model-based RL for MNL (UCRL-MNL)

Initialize $A_1 = \lambda I_d$, $\hat{\theta}_1 = \mathbf{0} \in \mathbb{R}^d$
for episode $k = 1, ..., K$ **do**
  Construct optimistic value functions for $(s, a) \in \mathcal{S} \times \mathcal{A}$ and $h \in [H]$

$$\hat{Q}_{k,H+1}(s,a) = 0 \quad \text{and} \quad \hat{V}_{k,h}(s) = \min\{\max_a \hat{Q}_{k,h}(s,a), H\}$$

$$\hat{Q}_{k,h}(s,a) = r(s,a) + \sum_{s' \in \mathcal{S}_{s,a}} \frac{\exp(\varphi(s,a,s')^\top \hat{\theta}_k)\hat{V}_{k,h+1}(s')}{\sum_{\tilde{s} \in \mathcal{S}_{s,a}} \exp(\varphi(s,a,\tilde{s})^\top \hat{\theta}_k)} + 2H\beta_k \max_{s' \in \mathcal{S}_{s,a}} \|\varphi(s,a,s')\|_{A_k^{-1}}$$

  for horizon $h = 1, \cdots, H$ **do**
    Select $a_{k,h} = \operatorname{argmax}_{a \in \mathcal{A}} \hat{Q}_{k,h}(s_{k,h}, a)$ and observe $s_{k,h+1}$
  end for
  Update $A_{k+1} = A_k + \sum_{h \leq H} \sum_{s' \in \mathcal{S}_{k,h}} \varphi_{k,h,s'} \varphi_{k,h,s'}^\top$
  Compute $\hat{\theta}_{k+1}$ using the ridge penalized MLE
end for

Regret Analysis

Regularity assumptions (standard in previous literature)

- (Bounded feature & parameter) $\|\varphi(s, a, s')\|_2 \leq L_\varphi$, $\|\theta^*\|_2 \leq L_\theta$
- (Non-singular Fisher info. matrix) $\inf_{\theta \in \mathbb{R}^d} p_{k,h}(s', \theta) p_{k,h}(s'', \theta) > 0$

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Lemma (Concentration of $\hat{\theta}_k$ and Optimsim)

For $\beta_k = \widetilde{\mathcal{O}}(\sqrt{d})$, $\theta^* \in \mathcal{C}_k = \left\{\theta \in \mathbb{R}^d : \|\theta - \hat{\theta}_k\|_{A_k} \leq \beta_k\right\}$ and $\hat{Q}_{k,h}(s, a) \geq Q^*_h(s, a)$ with high probability.

- Allows us to work with the *estimated* value function in stead of *unknown optimal* value function

Regret Analysis

Lemma (Value iteration error per step)

$$\hat{Q}_{k,h}(s_{k,h}, a_{k,h}) - \left[r(s_{k,h}, a_{k,h}) + P_h \hat{V}_{k,h+1}(s_{k,h}, a_{k,h}) \right] \leq 2H\beta_k \max_{s' \in \mathcal{S}_{k,h}} \|\varphi_{k,h,s'}\|_{A_k^{-1}}$$

- Hence, the regret under the optimistic policy can be controlled.

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- Hence, the regret under the optimistic policy can be controlled.

Theorem (Regret of `UCRL-MNL`)

For $\beta_k = \mathcal{O}(\sqrt{d})$, with high probability, the cumulative regret of the `UCRL-MNL` policy π is upper-bounded by

$$\textbf{Regret}_\pi(K) = \widetilde{\mathcal{O}}(d\sqrt{H^3 T} + H\sqrt{T})$$

- Applied to any feature representation of state-action and parameter
- Cumulative regret, sublinear in $T \Rightarrow$ converges to optimality
- First theoretical guarantee for RL with MNL function approximation

Numerical Experiments: RiverSwim Envrionment

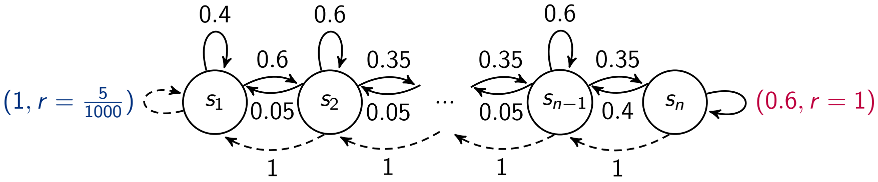
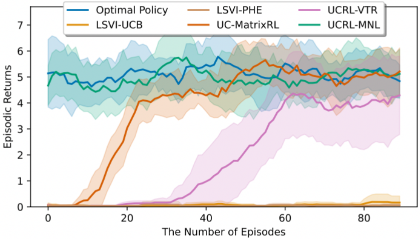


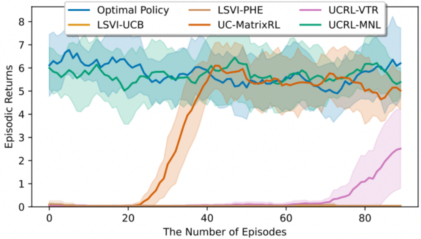
Figure: RiverSwim environment with n states

- Small reward on the left-most state
- Large reward on the right-most state
- Challenging for myopic policies. The environment requires *deeper exploration* to solve.

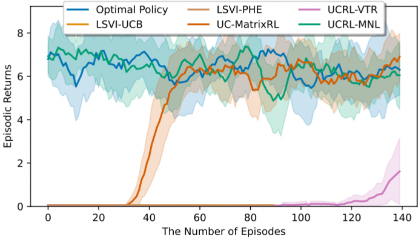
Numerical Experiments: Results



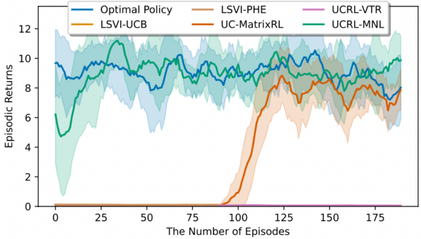
RiverSwim: nState=6, epLen=24



RiverSwim: nState=8, epLen=32



RiverSwim: nState=10, epLen=40



RiverSwim: nState=20, epLen=80

Summary

- MNL function approximation: a new model for provable RL
  - Natural function approximation for transition probabilities
- Propose a RL algorithm, UCRL-MNL, under this new model
  - Achieves the provable guarantees on regret performance
- Superior numerical performances compared to existing methods
- Attains both theoretical and practical efficiency

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