Mathematics behind Dobble

November 3, 2017

Agenda

- Introduction to Dobble card game
- Mathematics behind Dobble

Dobble card game

The facts

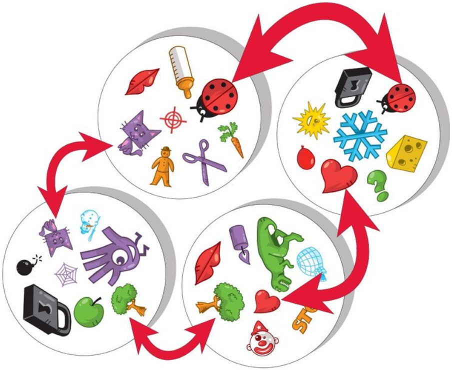
- Card game for kids
- Consists of 55 cards
- Each card has 8 different symbols
- There are 57 symbols in total

Dobble card game

How does it work?

Dobble card game

Main condition



Each pair of cards has exactly one common symbol

Main question

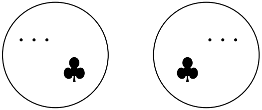
Given q symbols on each card:

- How many symbols are needed in total?
- How many cards can be made in total?
- How can such a card set be constructed?

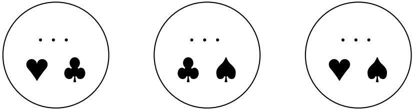
Example

Given 8 symbols on each card: how many symbols are needed?

- 2 cards: $2 \cdot 7 + 1 = 15$ symbols



- 3 cards: $3 \cdot 6 + 3 = 21$ symbols



How to construct a large set of cards?

Some mathematical background

A field

Definition: field

A field is a set of elements, along with two functions defined on that set:

- Addition function written as $a + b$
- Multiplication function written as $a \cdot b$

Each non-zero element of a field should have an inverse for the additional and multiplication.

Examples

- $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields
- $\mathbb{Z}$ is not a field: no inverse for the multiplication

Example: binary field

A field with two elements

- A field should have at least two elements:
  - Identity for the addition such that $a + 0 = a$
  - Identity for the multiplication such that $a \cdot 1 = a$
- The smallest field is a binary field with the following operations:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

This field is called $\mathbb{F}_2$.

Finite field

Definition

Take p a prime number, then $\mathbb{F}_p$ is a finite field, where

$$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{p-1}\}$$

The addition and multiplication functions are defined by integer modulo operations. Note that $\overline{n} = n \mod p$.

Examples

- In $\mathbb{F}_{11}$: $\overline{7} + \overline{4} = \overline{11} = \overline{0}$
- In $\mathbb{F}_{13}$: The multiplication inverse of $\overline{8}$ is $\overline{5}$ because

$$\overline{8} \cdot \overline{5} = \overline{8 \cdot 5} = \overline{40} = \overline{40 - 3 \cdot 13} = \overline{1}$$

- $\mathbb{F}_{12}$ is not a field because $\overline{3} \cdot \overline{4} = \overline{12} = \overline{0}$ while $\overline{3} \neq \overline{0}$ and $\overline{4} \neq \overline{0}$

Little theorem of Fermat

Little theorem of Fermat

If p is a prime number, then for any integer a:

$$a^p \equiv a \mod p$$

Equivalent theorem

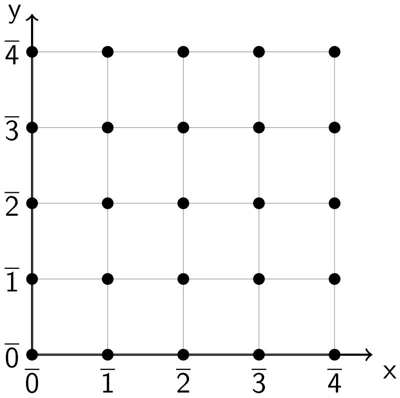
If a is not divisible by p then

$$a^{p-1} \equiv 1 \mod p$$

Finite geometry

Geometry over a finite field $\mathbb{F}_p$

- A geometric system that has only a finite number of points
- Example: finite affine plane of $\mathbb{F}_5$

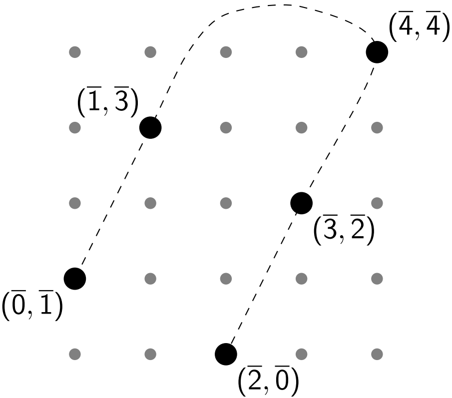


Finite geometry

Lines in a finite affine field

Example in $\mathbb{F}_5$: $y = \overline{2} \cdot x + \overline{1}$

x	y
$\overline{0}$	$\overline{1}$
$\overline{1}$	$\overline{3}$
$\overline{2}$	$\overline{0}$
$\overline{3}$	$\overline{2}$
$\overline{4}$	$\overline{4}$



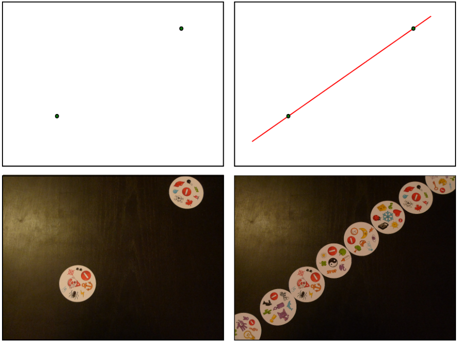
Dobble and geometry

Geometry

Through any two points, there is exactly one line.

Dobble

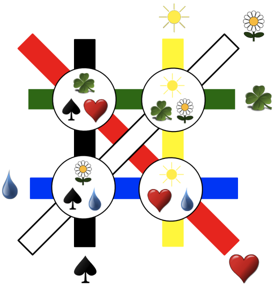
Two cards have exactly one common symbol.



Dobble game with 4 symbols

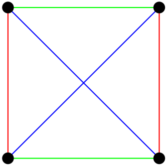
A plane in $\mathbb{F}_2 = \{\overline{0}, \overline{1}\}$ consists of

- 4 points: $(\overline{0}, \overline{0})$, $(\overline{1}, \overline{0})$, $(\overline{0}, \overline{1})$ and $(\overline{1}, \overline{1})$
- 6 lines:
  - 2 vertical lines: $x = \overline{0}$ and $x = \overline{1}$
  - 2 horizontal lines: $y = \overline{0}$ and $y = \overline{1}$
  - 2 others: $y = x$ and $y = x + \overline{1}$



Projective geometry

- Not very pair of lines has an intersection point:

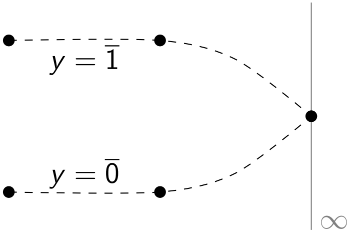


Parallel lines:

  - $x = \overline{0}$ and $x = \overline{1}$
  - $y = \overline{0}$ and $y = \overline{1}$
  - $y = x$ and $y = x + \overline{1}$

Projective geometry

- Affine geometry is extended with points at infinity.
- Two *parallel* lines intersect at infinity.



- There are no parallel lines in projective geometry

Projective geometry

Points

- Euclidean plane: each point represented as a pair: (x, y)
- Projective plane: each point represented as a triple: $(x : y : z)$ where x, y and z are not all 0.
  - When $z \neq 0$, the point represented is the point $(x/z, y/z)$ in the Euclidean plane
  - An Euclidean point (x, y) maps to the projective point $(x : y : 1)$
  - When $z = 0$, the point represented is a point at infinity
- Note that $(x : y : z)$ and $(x/z : y/z : 1)$ represent the same point for $z \neq 0$

Projective geometry

Lines

- A line is represented as follows

$$ax + by + cz = 0$$

- For Euclidean geometry (where $z = 1$), this line corresponds with

$$ax + by + c = 0$$

- The line $z = 0$ contains all points at infinity

- A point is represented by $(x : y : z) = (\alpha x : \alpha y : \alpha z)$ where $\alpha \neq 0$ and x,y,z not all 0.
- A line is represented by $[a : b : c] = [\alpha a : \alpha b : \alpha c]$ where $\alpha \neq 0$ and a,b,c not all 0.

Duality principle

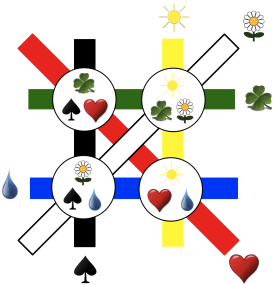
The role of points and lines are interchangeable

Examples

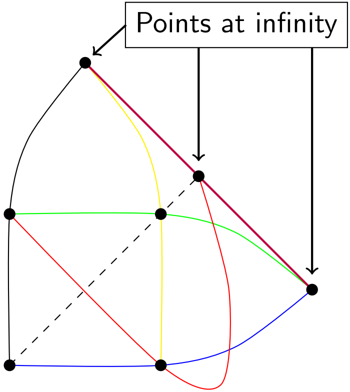
- Any two distinct points define a unique line.
- Any two distinct lines define a unique point.

Back to Dobble

Euclidean plane



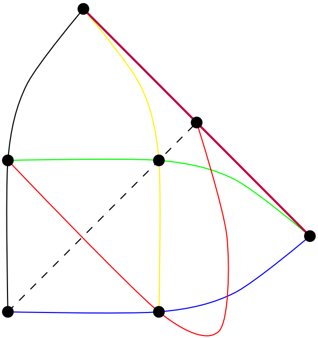
Projective plane



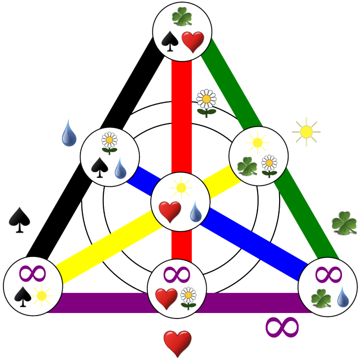
Back to Dobble

Another representation of the projective plane

Projective plane

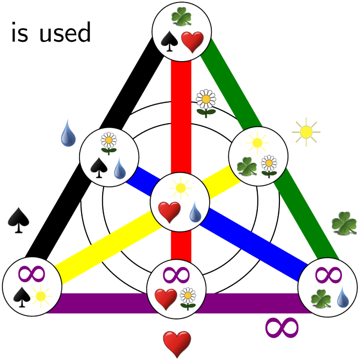


Fano representation



Summary of our mini Dobble

- The finite field $\mathbb{F}_2$ of two elements is used
- Each card consists of 3 symbols
- There are 7 symbols in total
- There are 7 cards in total



Dobble

Number of symbols

- The number of symbols on a card it determined by the number of lines intersecting a single point
- In $\mathbb{F}_p$, there are $p + 1$ lines going through a each point
- In $\mathbb{F}_p$, there are $p^2 + p + 1$ lines in total
  - $y = ax + b$ $\quad p \cdot p$ lines
  - $x = a$ $\quad p$ lines
  - 1 line at infinity

The real Dobble

- The real Dobble has $7 + 1 = 8$ symbols on each card
- Therefore it is constructed using field $\mathbb{F}_7$
- In $\mathbb{F}_7$, there are $7^2 + 7 + 1 = 57$ lines.
  Therefore, there are 57 symbols in Dobble
- Due to the duality, there are also 57 cards.
  But in the actual game, there are only 55 cards.

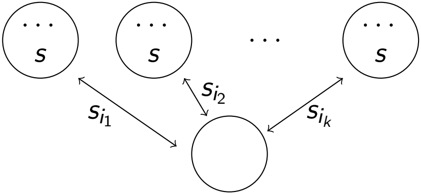
Summary

- A card set can be constructed using projective geometry over a finite field $\mathbb{F}_p$
- There are as many cards as symbols due to the duality
- In the real Dobble, two cards are missing (55 vs 57)

Dobble game

Is this a complete set?

- Assume each card has q symbols
- There are m symbols in total: $S = \{s_1, s_2, \ldots, s_m\}$
- Take a fixed symbol s and assume there are k cards containing s
- Take another card which does not contain s, it should have a different symbol with every card



- All s_{i_j} are different because otherwise a card will have at least two symbols in common. Therefore $k \leq q$

Dobble game

Is this a complete set?

- Take a fixed card with symbols $\{s_{i_1}, \ldots, s_{i_q}\}$
- Assume there are
  - $k_1 - 1$ other cards with symbol s_{i_1}
  - $\ldots$
  - $k_q - 1$ other cards with symbol s_{i_q}
- Since every card should have one symbol in common with the initial card. Therefore it is a full set.

$$\sum_{j=1}^{q} (k_j - 1) = n - 1$$

- Using $k \leq$ q: $n \leq q(q-1) + 1$
- For the real Dobble: $n \leq 57$ since $q = 8$

References

1. Dobble et La Geometrie Finie - Maxime Bourrigan
http://images.math.cnrs.fr/Dobble-et-la-geometrie-finie.html