The Uniformization Theorem

1 Introduction

The main purpose of this note is to discuss the following celebrated theorem and its consequences and applications:

Theorem 1. *A simply connected Riemann surface is biholomorphic to precisely one of the following Riemann surfaces:*

- *the unit open disk* $\mathbb{D}$*,*
- *the complex plane* $\mathbb{C}$*,*
- *the Riemann sphere* $\mathbb{CP}^1$*.*

Historically, this theorem first appeared in Riemann's dissertation but not in its most general form and not with a completely acceptable proof. The first rigorous proofs were given by Poincaré and Koebe at the beginning of the last century.

It should be mentioned that all Riemann surfaces are assumed to be connected. There is no ambiguity about complex structures we can equip the oriented sphere S^2 with. The only possibility is $\mathbb{CP}^1$ because the Riemann-Roch indicates that any genus 0 Riemann surface admits a non-trivial degree one meromorphic function and such a function is in fact an isomorphism onto $\mathbb{CP}^1$. The theorem may be formulated with the upper half plane $\mathbb{H} = \{z | \, \mathrm{Im} z > 0\}$ in place of $\mathbb{D}$ as there is a biholomorphic map $\mathbb{H} \to \mathbb{D} : z \mapsto \frac{z-\mathrm{i}}{z+\mathrm{i}}$.

Exercise 1. Explain that why the three Riemann surfaces in Theorem 1 are mutually non-isomorphic? Using the theorem, how many distinct complex structures exist on $\mathbb{R}^2$?

In the theory of Riemann surfaces, aside from algebraic or geometric viewpoints, there are occasions that one cannot avoid analysis! The *Riemann Existence Theorem* on existence of non-trivial meromorphic functions on a compact Riemann surface X, the finiteness of the dimension of the sheaf cohomology groups $H^i(X, L)$ for a holomorphic line bundle $L \to X$ and the most important one, the Uniformization Theorem 1 are examples of such results. Any proof of the Uniformization Theorem should invoke some serious analysis. The proof we are going to sketch is adopted from

[Donaldson, §10]. There are other proofs in literature too. For instance, the book Forster contains a much longer treatment, a proof which is based on constructing certain real harmonic functions by the so called *Perron method*. Check Wikipedia for references that prove the theorem through other approaches.

The note is organized as follows. We start with a weaker version of Theorem 1 in §2. There are two theorems that provide us with all we need from analysis. These are Theorems 3 (for compact Riemann surfaces) and 4 (for the non-compact case) that are addressed in §3 and §4. A proof of the Uniformization Theorem based on Theorem 4 is presented in §4. We exploit the Uniformization Theorem to obtain results about arbitrary Riemann surfaces in §5 and finally, §6 is devoted to various applications.

2 The Riemann Mapping Theorem

A simpler version of Theorem 1 is the *Riemann Mapping Theorem* which only deals with simply connected domains in the complex plane.

Theorem 2. *Let U be a proper simply connected domain in $\mathbb{C}$. Then for any arbitrary point $p \in U$ there is a unique biholomorphic map $f : U \to \mathbb{D}$ for which $f(p) = 0, f'(p) > 0$.*

Sketch of Proof. The Uniqueness part is easy as any automorphism $\mathbb{D} \to \mathbb{D}$ fixing the origin and with positive derivative at the origin is identity (cf. Proposition 1). It suffices to exhibit a $U \to \mathbb{D}$ and then in order to get desired conditions at p, one can modify it with an automorphism of $\mathbb{D}$. The key idea for constructing a biholomorphic map $f : U \to \mathbb{D}$ is to invoke *Montel's Theorem* which asserts that a locally uniformly bounded family of holomorphic functions has compact closure in the function space. We first introduce a family $\mathcal{F} := \{f : U \to \mathbb{D} \mid f \text{ holomorphic and injective}\}$. Since U is simply connected and proper, $\mathcal{F}$ is not empty because picking a point $a \in \mathbb{C} - U$ it is possible to define a branch $U \mapsto \mathbb{H} \xrightarrow{\cong} \mathbb{D}$ of $z \mapsto \sqrt{z-a}$. Setting $M = \sup_{f \in \mathcal{F}} |f'(p)|$, there is a subsequence $\{f_i\}_i$ of elements of $\mathcal{F}$ with $f_i'(p) \to M$. Montel's Theorem allows us to extract a subsequence $\{f_{i_k}\}_k$ which converges locally uniformly to another holomorphic function $f : U \to \mathbb{C}$. Now $|f'(p)| = M$ and f is injective by *Hurwitz's Theorem*. Hence $f \in \mathcal{F}$. The only thing left is to prove that this is surjective as well. This is achieved by showing that otherwise one can construct a new element of $\mathcal{F}$ out of f whose derivative at p has absolute value larger than M. Check [Krantz, chap. 0, §3] for details.

3 The Main Theorem for Compact Riemann Surfaces

On a Riemann surface X there is a natural second-order differential operator:

$$\Delta : \mathcal{A}(X) \to \mathcal{A}^{1,1}(X) \quad \Delta := 2\mathrm{i}\bar{\partial}\partial$$

In a holomorphic chart $(U, z = x + \mathrm{i}y)$, applying this operator to a C^∞ function f yields the $(1,1)$-form $-\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \mathrm{d}x \wedge \mathrm{d}y$. Hence at least locally Δ can be thought of as the usual Laplacian.

Exercise 2. Suppose the Riemann surface X is equipped with a Hermitian metric which in a local coordinate $z = x + \mathrm{i}y$ is given by $h.\mathrm{d}z \otimes \mathrm{d}\bar{z}$. We get a Riemannian oriented manifold which is therefore equipped with a Laplacian $\Delta f := -\mathrm{div}(\nabla f)$ that takes C^∞ functions to C^∞ functions. Show that in this local coordinates $\Delta f = -h(x,y)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$. How is this related to the differential operator above?

3.1 The Statement of The Main Theorem

The following so called "Main Theorem" provides us with all we need from analysis in the elementary theory of compact Riemann surfaces:

Theorem 3. *Let X be a compact Riemann surface and ρ a (real) C^∞ 2-form on X. Then the equation $\Delta f = \rho$ has a solution for a (real) smooth function f iff $\int_X \rho = 0$ and the solution is unique up to adding a constant.*[1]

Before sketching a proof, let us mention some immediate consequences:

- *The Dolbeault cohomology group $H^1(X, \mathcal{O})$ is naturally isomorphic with the space $\overline{\Omega(X)}$ of anti-holomorphic global 1-forms.*
  To see this, note that there is an embedding $\overline{\Omega(X)} \hookrightarrow H^1(X, \mathcal{O})$ taking any $\bar{\omega}$ to its Dolbeault class. Injectivity is due to tha fact that a non-zero global anti-holomorphic 1-form $\bar{\omega}$ cannot be $\bar{\partial}$-exact since $\bar{\omega} = \bar{\partial} f$ requires $\partial\bar{\partial} f$ and thus Δf to vanish which according to the theorem means that f is constant. To show that this map is in fact surjective, consider a $(0,1)$ form η. We are looking for a function f for which $\eta - \bar{\partial} f$ is anti-holomorphic or equivalently $\partial\left(\eta - \bar{\partial} f\right) = 0$. Hence it just suffices to solve $\Delta f = -2\mathrm{i}.\partial\eta$. This is possible as $\int_X \partial\eta = \int_X \mathrm{d}\eta$ vanishes due to Stokes' Theorem.

- *Denoting the first complex deRham cohomology group of X by $H^1_{dR}(X)$, the map $\Omega(X) \oplus \overline{\Omega(X)} \to H^1_{dR}(X) : (\omega_1, \bar{\omega}_2) \mapsto [\omega_1 + \bar{\omega}_2]$ is an isomorphism.*
  The map is injective since if $\omega_1 + \bar{\omega}_2 = \mathrm{d}f$ for some C^∞ function f, then $\bar{\partial} f = \bar{\omega}_2$ and $\partial f = \omega_1 \Leftrightarrow \bar{\partial}\bar{f} = \bar{\omega}_1$. Hence just like above, anti-holomorphic 1-froms $\bar{\omega}_1, \bar{\omega}_2$ are zero as they are $\bar{\partial}$-exact. This establishes the injectivity. To derive surjectivity, pick a d-closed 1-form α. Solving equations $\Delta f_1 = 2\mathrm{i}.\bar{\partial}(\alpha^{1,0})$ and $\Delta f_2 = -2\mathrm{i}.\partial(\alpha^{0,1})$ gives rise to C^∞ functions f_1, f_2 with 1-forms $\alpha^{1,0} - \partial f_1$ and $\alpha^{0,1} - \bar{\partial} f_2$ respectively $\bar{\partial}$-closed and ∂-closed. But $\mathrm{d}\alpha = 0$ implies $\bar{\partial}\alpha^{1,0} = -\partial\alpha^{0,1}$. Therefore $\bar{\partial}\partial f_1 = -\partial\bar{\partial} f_2$ or equivalently $2\mathrm{i}.\bar{\partial}\partial(f_1 - f_2) = \Delta(f_1 - f_2) = 0$. The theorem implies that $f_1 - f_2$ is constant and thus $\partial f_1 + \bar{\partial} f_2 = \mathrm{d}f_1$. Now:

$$\alpha = \underbrace{\left(\alpha^{1,0} - \partial f_1\right)}_{\text{holomorphic}} + \underbrace{\left(\alpha^{0,1} - \bar{\partial} f_2\right)}_{\text{anti-holomorphic}} + \underbrace{\mathrm{d}f_1}_{\text{exact}}$$

[1] The existence part is a special case of the $\partial\bar{\partial}$ *lemma* which asserts that on a compact Kähler manifold a form $\omega \in \mathcal{A}^n(X)$ which is $\partial, \bar{\partial}$ and d closed, is exact for one of these operators iff can be written as $\omega = \partial\bar{\partial}\alpha$ for some suitable global $(n-2)$-form $\alpha \in \mathcal{A}^{n-2}(X)$.

So the deRham class $[\alpha]$ is in the image of this map and we have established the so called *Hodge Decomposition* for compact Riemann surfaces: $H^1_{dR}(X) = \Omega(X) \oplus \overline{\Omega(X)}$.

- *The space $\Omega(X)$ of global holomorphic 1-forms and the Dolbeault cohomology group $H^1(X, \mathcal{O})$ are finite dimensional and in fact of dimension g when the compact Riemann surface X is of genus g*[2].
This is an immediate consequence of previous corollaries: the dimensions of $\Omega(X)$ and $\overline{\Omega(X)} \cong H^1(X, \mathcal{O})$ is half the dimension of the deRham group $H^1_{dR}(X)$ which is $2g$-dimensional. The equality of dimensions can also be established through constructing an isomorphism $\Omega(X)^* \cong H^1(X, \mathcal{O})$, i.e. the *Serre duality*. There is a non-degenerate pairing $\Omega(X) \oplus \overline{\Omega(X)} \to \mathbb{C} : (\omega_1, \bar{\omega}_2) \mapsto \int_X \omega_1 \wedge \bar{\omega}_2$ that identifies $\overline{\Omega(X)}$ with $\Omega(X)^*$. Composing it with $\Omega(X)^* \cong H^1(X, \mathcal{O})$, we arrive at a canonical isomorphism between $\Omega(X)^*$ and $H^1(X, \mathcal{O})$.

- The only hard task in establishing the Riemann-Roch for a compact Riemann surface X is to show that the dimension of $H^1(X, \mathcal{O})$ is finite, the fact that we just derived as a corollary to the Main Theorem. The rest is merely analyzing the long exact sequence in cohomology associated with certain short exact sequence of sheaves. See [Forster, §16] for details.

- Finally, we show that finite dimensionality of the Dolbeault cohomology group $H^1(X, \mathcal{O})$ may be used to prove existence of non-trivial meromorphic functions on the compact Riemann surface X. Denoting this dimension by k, pick $k+1$ distinct points $p_1, \ldots, p_{k+1} \in X$ and pairwise disjoint coordinate neighborhoods (U_i, z_i) with z_i a holomorphic chart centered at p_i. Next, choose bump functions $\beta_1, \ldots, \beta_{k+1}$ such that the support of β_i is included in U_i and this function is identically 1 over some smaller open neighborhood of p_i. We want to modify the C^∞ function $\frac{\beta_1}{z_1} + \cdots + \frac{\beta_{k+1}}{z_{k+1}}$ on $X - \{p_1, \ldots, p_{k+1}\}$ with a globally defined C^∞ function in order to get a meromorphic function on X. It suffices to notice that the classes of $(0,1)$-forms $\bar{\partial}\left(\frac{\beta_1}{z_1}\right), \ldots, \bar{\partial}\left(\frac{\beta_{k+1}}{z_{k+1}}\right)$ are linearly dependent in the k-dimensional space $H^1(X, \mathcal{O})$ and thus there is a $g : X \to \mathbb{C}$ with $\bar{\partial}\left(\frac{\beta_1}{z_1} + \cdots + \frac{\beta_{k+1}}{z_{k+1}} - g\right) = 0$. So $\frac{\beta_1}{z_1} + \cdots + \frac{\beta_{k+1}}{z_{k+1}} - g$ is a meromorphic function with simple poles at $p_1, \ldots, p_{k+1}$.

3.2 Sketch of Proof of "The Main Theorem"

The operator Δ commutes with complex conjugation and hence establishing the theorem for complex valued forms and functions implies it in the real case. The necessity of condition $\int_X \rho = 0$ and the uniqueness part are easy: $\Delta f = 2\mathrm{i}\bar{\partial}\partial f = \mathrm{d}\,(2\mathrm{i}.\partial f)$ is exact and hence its integral vanishes according to Stokes' Theorem. For the uniqueness part, one can invoke the maximal principle for harmonic functions to deduce that any $f : X \to \mathbb{C}$ in the of kernel Δ must be constant. The hard part is the sufficiency of the condition $\int_X \rho = 0$ for existence of a solution. This will be proved in several steps by an idea based on the *Dirichlet integral*.

[2] So the three notions of genus coincide: the usual *topological genus* which is half the first Betti number $\dim_{\mathbb{C}} H^1_{dR}(X)$, the *arithmetic genus* which is $\dim_{\mathbb{C}} H^1(X, \mathcal{O})$ and at last the *geometric genus* which is defined to be $\dim_{\mathbb{C}} \Omega(X)$.

Step 1. **The Dirichlet Norm on $C^\infty(X)$.** Define the *Dirichlet inner product* on the space $C^\infty(X)$ of complex-valued smooth functions on X by:

$$\langle f, g\rangle_D = 2\mathrm{i}\int_X \partial f \wedge \overline{\partial g}$$ [3]

This is a Hermitian product on the space $C^\infty(X)/\mathbb{C}$ where $\mathbb{C}$ denotes the subspace of constant functions. The motivation behind this definition is:

$$\int_X \psi.\Delta\bar{\phi} = -\int_X \psi.\partial\left(2\mathrm{i}\bar{\partial}\bar{\phi}\right) = 2\mathrm{i}\int_X \partial\psi \wedge \overline{\partial\phi} = \langle\psi, \phi\rangle_D$$ [4]

Thus, fixing ρ, a solution $\Delta\phi = \rho$ satisfies $\forall\psi : \int_X \psi.\bar{\rho} = \langle\psi, \phi\rangle_D$. But the left hand side can be considered as a functional $\hat{\rho} : C^\infty(X)/\mathbb{C} \to \mathbb{C} : \psi \mapsto \int_X \psi.\bar{\rho}$ which is well-defined due to $\int_X \bar{\rho} = 0$. Therefore, any solution ϕ to $\Delta\phi = \rho$ is in fact a *weak solution*, an element that represents the functional $\hat{\rho} : C^\infty(X)/\mathbb{C} \to \mathbb{C}$. Now the proof divides into two parts: existence of a weak solution and then proving that it is a solution in usual sense.

Step 2. **Existence of a Weak Solution.** One has to show that the functional $\hat{\rho}$ on the Hermitian space $(C^\infty(X)/\mathbb{C}, \langle.\rangle_D)$ is bounded. Since locally the Dirichlet norm of a function ϕ is like the L^2 norm of $|\nabla\phi|$, we need a *Poincaré type inequality* that gives a bound on a function in terms of its derivatives. Next, $\hat{\rho}$ extends to a continuous functional on the Hilbert space which is the completion of $(C^\infty(X)/\mathbb{C}, \langle.\rangle_D)$ and then invoking the *Riesz Representation Theorem* implies $\hat{\rho} = \langle -, \phi\rangle_D$ for a suitable ϕ in this Hilbert space.

Step 3. **The Regularity Argument.** This weak solution ϕ determines a Cauchy sequence $\{\phi_i\}_i$ of elements of $(C^\infty(X)/\mathbb{C}, \langle.\rangle_D)$ with the property that $\forall\psi \in C^\infty(X) : \lim_{i\to\infty}\int_X \psi.\Delta\bar{\phi}_i = \int_X \psi.\bar{\rho}$ or equivalently $\lim_{i\to\infty}\int_X \phi_i.\Delta\bar{\psi} = \int_X \bar{\psi}.\rho$. This Cauchy sequence converges to a function ϕ on X which is locally in L^2. Hence $\int_X \phi.\Delta\psi = \int_X \psi.\rho$ for any compactly supported smooth function ψ on X. We have to deduce from this property that ϕ is smooth and satisfies $\Delta\phi = \rho$. The argument is local and hence, considering a chart, everything reduces to showing that for a domain Ω in $\mathbb{C}$, a function $\phi \in \mathrm{L}^2(\Omega)$ satisfying $\int_\Omega \phi.\Delta\psi = \int_\Omega \psi.\rho$ for any C^∞ compactly supported function ψ on Ω is C^∞ and furthermore $\Delta\phi = \rho$. This is a version of *Weyl's Lemma*.

4 Proof of The Uniformization Theorem

Let us discuss a proof of Theorem 1 which is presented in [Donaldson, §10] and employs an analogue of the "Main Theorem" 3 for simply connected non-compact Riemann surfaces.

Theorem 4. *Let X be a non-compact simply connected Riemann surface. For a (real) smooth compactly supported 2-form ρ on X there exits a smooth (real) function ϕ on X which tends to 0 at infinity and satisfies $\Delta\phi = \rho$.*

[3] We have added a positive multiple of i as the coefficient in order to have a positive definite product. This is due to the fact that in any holomorphic local chart $z = x + \mathrm{i}y$, $\mathrm{i}\mathrm{d}z \wedge \mathrm{d}\bar{z} = 2\mathrm{d}x \wedge \mathrm{d}y$ is a volume form compatible with the orientation.

[4] This identity yields an alternative proof for $\ker\Delta = \mathbb{C} \subset C^\infty(X)$: if $\Delta\phi = \Delta\bar{\phi} = 0$, then putting $\psi = \phi$ results in $\langle\phi, \phi\rangle_D = 0$ which implies that ϕ is constant.

It should be mentioned that that a function $\phi : X \to \mathbb{C}$ is said to tend to c at infinity if for any $\epsilon > 0$ there is a compact subset K of X with $\forall x \in X - K : |\phi(x) - c| < \epsilon$.

Accepting this theorem, we can achieve the primary goal of this section, that is to give a proof of the Uniformization Theorem. First, let X be a compact Riemann surface of genus 0. Pick an arbitrary point $p \in X$. Writing the Riemann-Roch for the divisor p indicates that $l(p) \geq 1-g+\deg(p) = 2$, which means that there exists a non-constant meromorphic function $f : X \to \mathbb{CP}^1$ with $(f) \geq -p$, i.e. f has only one pole which is a simple pole at p. Hence $f : X \to \mathbb{CP}^1$ is of degree one and thus an isomorphism.

Now let us concentrate on the non-compact case. Given a simply connected non-compact Riemann surface X, the idea is to construct a suitable holomorphic embedding $X \to \mathbb{CP}^1$ and utilize it to deduce that X must be isomorphic with either $\mathbb{C}$ or $\mathbb{D}$.

Fix a point p of X and a holomorphic coordinate (U, z) centered at it where U is a small open neighborhood of p with $\bar{U}$ compact. Let β be a bump function whose support is contained in U and is identically 1 in a neighborhood of p. Thus $A := \bar{\partial}\left(\frac{\beta}{z}\right)$ is a globally defined compactly supported $(0,1)$-form. Since $\partial A = \mathrm{d}A$ is exact, $\int_X \partial A = 0$ by Stokes' Theorem. So we can apply Theorem 4 to $-2\mathrm{i}\partial A$ and deduce that $\Delta\phi = -2\mathrm{i}\partial\bar{\partial}\phi = -2\mathrm{i}\partial A$ for a function ϕ which tends to 0 at infinity. Thus the $(0,1)$-form $\bar{\partial}\phi - A = \bar{\partial}\left(\phi - \frac{\beta}{z}\right)$ is ∂-closed and therefore d-closed. Any closed 1-form on X is exact because X is simply connected. We deduce that there is a C^∞ function $g : X \to \mathbb{C}$ with $\bar{\partial}\left(\phi - \frac{\beta}{z}\right) = \mathrm{d}g$. Consequently, functions $\phi - \frac{\beta}{z} - g$ and g are holomorphic and anti-holomorphic, respectively. This implies $f := \phi - \frac{\beta}{z} - g - \bar{g}$ is holomorphic as well. The imaginary part of this function is the same as that of $\phi - \frac{\beta}{z}$ which itself away from p (that is outside of $\bar{U}$) coincides with the imaginary part of ϕ that tends to infinity due to the way we chose ϕ. Hence f is a holomorphic on $X - \{p\}$ with a simple pole at p and its imaginary part tends to 0. This function thus can be considered as a holomorphic map $f : X \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. The last property of f that is going to be used in the proof is the fact that it is locally biholomorphic around p because the pole at p is simple.

Next step is to investigate f. Denote the open upper and lower half planes in $\mathbb{C}$ by $\mathbb{H}^+$ and $\mathbb{H}^-$ respectively and their preimages by $f^{-1}(\mathbb{H}^\pm) = X^\pm$. We claim that each of maps $f|_{X^\pm} : X^\pm \to \mathbb{H}^\pm$ is proper. There is a simple argument for this: in any compact subset K of $\mathbb{H}^+$ or $\mathbb{H}^-$ the imaginary parts of points have finite non-zero infimum and supremum while according to properties of f, in a subset of X with non-compact closure there is a sequence of points $\{a_n\}_{n=1}^\infty$ with $\mathrm{Im}f(a_n) \to 0$, the fact that implies the compactness of $\overline{f^{-1}(K)} = f^{-1}(K)$.

We conclude that $f|_{X^\pm} : X^\pm \to \mathbb{H}^\pm$ are branched covering maps and there is a well-defined notion of degree for them [5]. The claim is these degrees are one or equivalently maps $f|_{X^\pm}$ are injective. If $f : X^+ \to \mathbb{H}^+$ (respectively $f : X^- \to \mathbb{H}^-$) is not injective, then for any $n \in \mathbb{N}$ the fiber $f^{-1}(n\mathrm{i})$ (resp. $f^{-1}(-n\mathrm{i})$) contains either more than one point or a critical point. All of these fibers are included in a compact subset of X as the imaginary part of f tends to 0 at infinity and

[5] A *branched covering of Riemann surfaces* is a holomorphic map $p : X \to Y$ which restricts to a finite-sheeted covering away from some closed discrete subset. That is, there exists a closed discrete subset B of Y such that p induces a holomorphic covering $X - p^{-1}(B) \to Y - B$ of finite degree, say a m-sheeted covering. The number of points counted with multiplicities in any fiber $p^{-1}(y)$ is precisely m. An elementary proposition indicates that a necessary and sufficient condition for a holomorphic map $p : X \to Y$ to be a branched cover is being proper, i.e. the preimage of any compact subset is compact. Check [Forster, §4] for more details.

so points of these fibers must accumulate somewhere as $n \to \infty$. The only possible accumulation point is the unique point in $f^{-1}(\infty)$ which is p. We arrive at a contradiction because p is a simple pole and so small enough neighborhoods of it neither contain critical points nor intersect a fiber at more than one point. Therefore, the maps $f|_{X^\pm} : X^\pm \to \mathbb{H}^\pm$ are injective and in fact bijective[6]. It is not hard to deduce that f is injective as well. The map $f : X \to \mathbb{CP}^1$ is non-constant holomorphic and hence an open map. Suppose $a \neq b$ are in the same fiber of f and separate them with open neighborhoods U_1, U_2. The open neighborhood $f(U_1) \cap f(U_2)$ of $f(a) = f(b) \in \mathbb{CP}^1$ must intersect at least one of $\mathbb{H}^+$ or $\mathbb{H}^-$ non-trivially and so there are points of disjoint subsets $U_1 \cap \mathbb{H}^+$, $U_2 \cap \mathbb{H}^+$ or $U_1 \cap \mathbb{H}^-$, $U_2 \cap \mathbb{H}^-$ which are in the same fiber and this contradicts the injectivity of $f|_{\mathbb{H}^\pm}$ established before.

Finally, we finish the proof: $f : X \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is an embedding (being open and injective) whose image is open and contains $f(X^+) \sqcup f(X^-) \sqcup f(p) = \mathbb{H}^+ \sqcup \mathbb{H}^- \sqcup \{\infty\} = \mathbb{C} \cup \{\infty\} - \mathbb{R}$. The complement $\mathbb{CP}^1 - f(X)$ is a compact subset of $\mathbb{R}$ and must be connected, otherwise $f(X)$, and hence X which is biholomorphic with it, cannot be simply connected[7]. Therefore, this complement is either a single point or a closed interval $[a, b]$. We conclude that the Riemann surface X is isomorphic to either $\mathbb{CP}^1 -$ point or $\mathbb{CP}^1 - [a, b], (a < b)$ which are $\mathbb{C}$ and $\mathbb{D}$, respectively.

Exercise 3. Fill in the details, show that the complement of an interval of positive length in $\mathbb{CP}^1$ is biholomorphic to $\mathbb{D}$. (Hint: Reduce to the case of $\mathbb{C} - (-\infty, 0]$ and use a suitable branch of $z \mapsto \sqrt{z}$ to construct an isomorphism from this domain onto the upper half plane $\mathbb{H}$.)

5 Uniformization for Arbitrary Riemann Surfaces

The main way that one can utilize Theorem 1 to obtain results about an arbitrary Riemann surface X is to consider the universal covering map $\tilde{X} \to X$ where according to this theorem there are only three choices for the simply connected Riemann surface $\tilde{X}$.

5.1 Spherical, Euclidean and Hyperbolic Riemann Surfaces

To investigate the Riemann surface X, we should think about the way it can be recovered from its universal cover $\tilde{X}$. The surface X is just the quotient of $\tilde{X}$ under the deck transformation group $\mathrm{Deck}\left(\tilde{X} \to X\right)$, the group which can be naturally identified with $\pi_1(X)$ whose action on $\tilde{X}$ is obviously free. But given a subgroup Γ of the group $\mathrm{Aut}(\tilde{X})$ of biholomorphic automorphisms of $\tilde{X}$, in order to have a reasonable quotient space $\tilde{X}/\Gamma$ more than being just fixed point free is necessary. Going back to the action of $\mathrm{Deck}\left(\tilde{X} \to X\right)$ on $\tilde{X}$, the fact that $\tilde{X} \to X$ is a

[6] Any branched covering of Riemann surfaces is surjective. The image must be both open because of holomorphicity and also closed because an easy point-set topology argument indicates that proper maps between locally compact Hausdorff spaces are closed. Connectedness now implies the surjectivity.

[7] We are using a basic observation that every component of the complement of a connected subset of S^2 is simply connected and conversely the complement of a closed disconnected subset has a non-simply connected component.

covering map requires this action to be *properly discontinuous*: any two distinct points admit open neighborhoods U, U' with $\gamma U \cap U' = \emptyset$ for any $\gamma \in \Gamma$ except for finitely many elements. It turns out that being free and properly discontinuous is sufficient for an action to have a nice orbit space[8]: *for any complex manifold Y and any subgroup G of $\mathrm{Aut}(Y)$ whose action on Y is free and properly discontinuous, there is a unique complex structure on the quotient space Y/G for which the quotient map $Y \to Y/G$ is locally biholomorphic*. This is due to the fact that these two constraints on the action imply that the quotient map is locally homeomorphism and hence sufficiently small charts of Y can be projected to define a complex atlas for Y/G. Notice that the definition of properly continuous action forces such a subgroup of the automorphism group to be discrete (in the usual compact-open topology). It seems that to exploit the power of the Uniformization Theorem one has to investigate the automorphism groups of simply connected Riemann surfaces $\mathbb{CP}^1, \mathbb{C}, \mathbb{D}$. This is the content of the following proposition:

Proposition 1.

- $\mathrm{Aut}\left(\mathbb{CP}^1\right) \cong \mathrm{PGL}_2(\mathbb{C})$ *where an element of* $\mathrm{PGL}_2(\mathbb{C})$ *acts on* $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ *by the corresponding linear-fractional transformation.*
- $\mathrm{Aut}(\mathbb{C}) \cong \mathbb{C} \rtimes \mathbb{C}^*$ *is the group of affine transformations $z \mapsto az + b$ where $a \in \mathbb{C}^*, b \in \mathbb{C}$.*
- *For the unit open disk $\mathbb{D}$ the group of automorphism is the group of Möbius transformations $z \mapsto \mathrm{e}^{\mathrm{i}\theta}\frac{z-a}{1-\bar{a}z}$ which indicates that* $\mathrm{Aut}(\mathbb{D}) \cong \mathrm{PSU}(1,1)$. *If we consider the upper half plane $\mathbb{H}$ instead, we get* $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R})$ *where the group* $\mathrm{PSL}_2(\mathbb{R})$ *acts by linear-fractional transformations.*

The first statement is an immediate consequence of the fact that rational maps are the only meromorphic functions on $\mathbb{CP}^1$ and such a map is an isomorphism iff its degree is one. For the second assertion, consider an automorphism $f : \mathbb{C} \to \mathbb{C}$. The point ∞ cannot be an essential singularity of f due to the *Casorati-Weierstrass Theorem* and so f possesses a unique pole at ∞ which indicates that f is a polynomial. The degree must be one due to injectivity and f is therefore linear. Finally, to classify automorphisms of $\mathbb{D}$, by composing an automorphism $f : \mathbb{D} \to \mathbb{D}$ with $z \mapsto \frac{z-f(0)}{1-\overline{f(0)}z}$, WLOG we may assume $f(0) = 0$ and then *Schwarz Lemma* implies that f is a rotation.

Therefore, we are looking for subgroups of the automorphism group that act properly discontinuous. In the case of $\mathbb{CP}^1$ any such a group is trivial (in fact any non-trivial Möbius transformation has a fixed point) otherwise $\mathbb{CP}^1$ will appear in non-trivial covering maps which contradicts its simply connectedness. It is not hard to classify discrete subgroups of the group $\mathrm{Aut}(\mathbb{C})$ of affine transformations of plane:

Exercise 4. Show that a discrete subgroup of the group of affine transformations of $\mathbb{C}$ is a group of translations $z \mapsto z + c$ where c varies either in a cyclic subgroup of $\mathbb{C}$ or in a lattice.

[8] It turns out that in dimension one even when the action is not free the quotient space may admit a complex structure. In [Miranda, p. 78] it is proved that the quotient of any Riemann surface to a finite subgroup of its automorphisms can be equipped with a unique complex structure for which the quotient map is holomorphic. This is not true in higher dimensions. For example the quotient of $\mathbb{C}^2$ modulo $(z_1, z_2) \mapsto (-z_1, -z_2)$ does not have even manifold structure around the origin.

In the hyperbolic case, it can be proved that a subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ acts properly discontinuous on the upper half plane $\mathbb{H}$ iff it is discrete. For a proof of this proposition check the first two sections of Milne's notes on modular forms which also contains an elementary discussion on quotients of topological spaces under group actions. Combining this with the classification of simply connected Riemann surfaces in Theorem 1, we arrive at the following:

Theorem 5. *According to their universal cover, Riemann surfaces can be divided into three different categories:*

- *Spherical Riemann Surfaces $\star$ The Riemann sphere is the unique Riemann surface whose universal cover is $\mathbb{CP}^1$.*
- *Euclidean Riemann Surfaces $\star$ These are Riemann surfaces whose universal cover is $\mathbb{C}$ and consist of the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}-\{0\}$ and complex tori (compact Riemann surfaces of genus 1).*
- *Hyperbolic Riemann Surfaces $\star$ The rest (including all compact Riemann surfaces of genus greater than 1) have $\mathbb{D}$ (or equivalently $\mathbb{H}$) as their universal cover and can be realized as $X=\mathbb{D}/\Gamma$ (respectively $X=\mathbb{H}/\Gamma$) where $\Gamma\cong\pi_1(X)$ is a discrete subgroup of $\mathrm{PSU}(1,1)$ (resp. of $\mathrm{PSL}_2(\mathbb{R})$) which acts freely on $\mathbb{D}$ (resp. on $\mathbb{H}$).*

The hyperbolic case is of primary interest. Discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ that we encountered in this case are called *Fuchsian groups*. They might have fixed points in their action (the so called *elliptic orbits*) e.g. the subgroup $\mathrm{PSL}_2(\mathbb{Z})$ which has the element $z\mapsto\frac{-1}{z}$ that fixes the point i. For a Fuchsian group Γ that does not act freely, the quotient map $\mathbb{H}\to\mathbb{H}/\Gamma$ is not a covering map anymore but is a branched covering of Riemann surfaces. The proposition below determines which Fuchsian groups are fixed point free:

Proposition 2. *A Fuchsian group acts freely iff it is torsion free.*

Exercise 5. In this exercise we are going to realize $\mathbb{H}$ as the universal covering of the annulus. $\mathbb{A}_r:=\{z\in\mathbb{C}\mid 1<|z|<r\}$ where $r>1$. First, check that the exponential map $z\mapsto \mathrm{e}^{-\mathrm{i}z}$ carries the horizontal strip $\{z\in\mathbb{C}\mid 0<\mathrm{Im}z<\mathrm{Ln}\,r\}$ onto $\mathbb{A}_r$. Next, in order to identify the strip with $\mathbb{H}$, show that $z\mapsto \mathrm{i}.\mathrm{e}^{\left(\pi.\frac{z-\frac{\mathrm{Ln}\,r}{2}}{\mathrm{Ln}\,r}\right)}$ bijects this horizontal strip onto the upper half plane. Next, verify that the deck transformation group of the universal covering map $\mathbb{H}\to\mathbb{A}_r$ just constructed is generated by the automorphism $z\mapsto \mathrm{e}^{\frac{2\pi^2}{\mathrm{Ln}\,r}}.z$ and these cyclic subgroups of $\mathrm{PSL}_2(\mathbb{R})$ are not conjugate for different values of $r>1$. Conclude that annuli are hyperbolic Riemann surfaces and the ratio of radii of an annulus is a holomorphic invariant[9].

[9] Another way to realize $\frac{2\pi^2}{\mathrm{Ln}\,r}$ as a conformal invariant of $\mathbb{A}_r$ is showing that the *Poincaré metric* (to be introduced in §6) on this annulus has a unique simple closed geodesic whose length is $\frac{2\pi^2}{\mathrm{Ln}\,r}$.

5.2 On The Group $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{Aut}(\mathbb{D})$

It might be a good idea to include a subsection on the group of conformal self-maps of the disk or equivalently of the upper half plane. Proposition 1 implies that any automorphism of $\mathbb{D}$ extends to a homeomorphism $\bar{\mathbb{D}} \to \bar{\mathbb{D}}$ carrying the boundary $\partial\mathbb{D}$ to itself. A non-identity automorphism $\bar{\mathbb{D}} \to \bar{\mathbb{D}}$ either has just one interior fixed point (the *elliptic* case where the automorphism is conjugate with a rotation $z \mapsto \mathrm{e}^{\mathrm{i}\theta}z$ around the origin), or just one fixed point on $\partial\mathbb{D}$ (the *parabolic* case where the automorphism is conjugate with $z \mapsto \frac{(2\mathrm{i}-1)z+1}{-z+(2\mathrm{i}+1)}$) or two fixed point on $\partial\mathbb{D}$ (the *hyperbolic* case where the automorphism is conjugate with $z \mapsto \frac{(k+1)z+(k-1)}{(k-1)z+(k+1)}$ $(k > 0)$). In the group $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R})$ these cases correspond to matrices $A \neq \pm I_2$ with $|\mathrm{tr}(A)| < 2$, $|\mathrm{tr}(A)| = 2$ and $|\mathrm{tr}(A)| > 2$, respectively. Two non-identity automorphisms commute iff they have the same fixed point set. Therefore, any non-identity element of $\mathrm{Aut}(\mathbb{D})$ (or equivalently $\mathrm{Aut}(\mathbb{H})$) lies in a unique maximal abelian subgroup which is the 1-parameter subgroup containing that element[10]. Let us explain this in more details for the action of $\mathrm{PSL}_2(\mathbb{R})$ on the upper half plane $\mathbb{H}$. A matrix $A \neq \pm I_2$ from this group is either elliptic, parabolic or hyperbolic. The corresponding automorphism of $\mathbb{H}$ respectively has precisely a unique fixed point in $\mathbb{H}$, a unique fixed point in $\mathbb{R} \cup \{\infty\}$ or two fixed points in $\mathbb{R} \cup \{\infty\}$. In the elliptic case A is conjugate with $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and the maximal abelian subgroup containing this matrix is the subgroup of rotation matrices. In the parabolic case A is conjugate with $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ which is a member of the maximal abelian subgroup $\left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$. Finally, when A is hyperbolic, A and the corresponding maximal abelian subgroup are conjugate with $\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}$ and $\left\{ \begin{bmatrix} t & 0 \\ 0 & \frac{1}{t} \end{bmatrix} \middle| t \in \mathbb{R} - \{0\} \right\}$.

Exercise 6. With the help of properties mentioned above show that $\mathrm{PSL}_2(\mathbb{R})$ does not contain any discrete subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Conclude that a Riemann surface of genus 1 must be in the form of $\mathbb{C}/\Lambda$ for some suitable lattice Λ in $\mathbb{C}$, the fact that briefly mentioned in Theorem 5. Also explain why the universal cover of a compact Riemann surface of genus greater than 1 cannot be $\mathbb{C}$.

6 Applications and Further Results

This section is devoted to various wonderful corollaries of Theorem 1 and Theorem 5.

6.1 The Theorems of Picard

The Uniformization Theorem provides us with an elegant proof of the following very sharp generalization of Liouville's Theorem:

[10]Hence the group $\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{PSU}(1,1)$ has this very interesting property that the commuting relation on the set of non-identity elements is transitive!

Theorem 6. Picard's Little Theorem. *A non-constant entire function attains all values in $\mathbb{C}$ except at most one value.*

To obtain this theorem, suppose $f : \mathbb{C} \to \mathbb{C} - \{0,1\}$ is an entire function. Since $\mathbb{C}$ is simply connected, f lifts to a holomorphic map $\tilde{f}$ form $\mathbb{C}$ to the universal cover of $\mathbb{C} - \{0,1\}$. Theorem 1 implies that this universal cover is either $\mathbb{C}$ or $\mathbb{D}$. In the latter case, we are done because any holomorphic map $\tilde{f} : \mathbb{C} \to \mathbb{D}$ must be constant according to Liouville's Theorem. Hence it suffices to show that $\mathbb{D}$ is the universal cover of $\mathbb{C} - \{0,1\}$, for example through constructing a covering map $\mathbb{D} \to \mathbb{C} - \{0,1\}$. This can be done by invoking the *Schwarz Reflection Principle* (see [Lang, p. 335]) or realizing $\mathbb{C} - \{0,1\}$ as a quotient of $\mathbb{H}$ modulo a torsion free Fuchsian subgroup (it can be proved that $\mathbb{H}/\Gamma(2) \cong \mathbb{C} - \{0,1\}$, cf. [Milne, p. 39]).

There is also a geometric approach to this theorem using the concept of *curvature*. For this proof and also a very nice geometric treatment of functions of one complex variable, consult Krantz.

We finish with another theorem due to Picard that is a strengthening of both Casorati-Weierstrass Theorem and Picard's Little Theorem:

Theorem 7. Picard's Great Theorem. *On any deleted open neighborhood of an essential singularity of a holomorphic function, the function omits at most one complex value from its image.*

A proof based on Montel's Theorem can be found in [Krantz, chap 2, §4].

6.2 The Automorphism Groups of Compact Riemann Surfaces

Let X be a compact Riemann surface and $\mathrm{Aut}(X)$ the group of its (biholomorphic) automorphisms $f : X \to X$. When $g(X) > 1$, X can be thought of as $\mathbb{H}/\Gamma$ for a Fuchsian subgroup Γ of $\mathrm{Aut}(\mathbb{H})$. Such an automorphism lifts to an automorphism of the universal cover $\mathbb{H}$. But an automorphism of $\mathbb{H}$ descends to an automorphism of $\mathbb{H}/\Gamma$ iff it belongs to the normalizer $N(\Gamma)$ of the subgroup Γ. Therefore: $\mathrm{Aut}(X) \cong N(\Gamma)/\Gamma$. We claim that $N(\Gamma)$ is Fuchsian as well. Otherwise, there is a sequence $\{t_i\}_i$ of non-identity elements of $N(\Gamma)$ converging to the identity element. Pick two non-commuting elements α, β form the non-abelian group $\Gamma \cong \pi_1(X)$. Then we have limits $t_i \alpha t_i^{-1} \to \alpha$ and $t_i \beta t_i^{-1} \to \beta$ in the discrete group Γ which means t_i commutes with both α and β for sufficiently large i. So α, β lie in the maximal abelian subgroup containing the non-identity element t_i which contradicts $\alpha\beta \neq \beta\alpha$. Since $N(\Gamma)$ is Fuchsian, its action on $\mathbb{H}$ is properly discontinuous. It is easy to see that this requires the image under $\mathbb{H} \to \mathbb{H}/\Gamma$ of any $N(\Gamma)$-orbit to be a closed discrete subset. Next, we will prove that the index $[N(\Gamma) : \Gamma] = |\mathrm{Aut}(X)|$ is finite. Assume the contrary: one can choose an infinite sequence$\{\gamma_i\}_i$ of elements of $N(\Gamma)$ with $\forall i \neq j : \gamma_i^{-1}\gamma_j \notin \Gamma$. Pick a point $x \in \mathbb{H}$ which is not fixed under the action of the countable group $N(\Gamma)$. The orbits of points $\gamma_i(x) \in \mathbb{H}$ comprise an infinite subset of $X = \mathbb{H}/\Gamma$ which is, as mentioned above, closed and discrete. The existence of such a subset contradicts the compactness of X. We have proved:

Theorem 8. *The automorphism group of any compact Riemann surface of genus greater than* 1 *is finite*[11].

This is in stark contrast with the genus 0 case where the automorphism group is a complex Lie group of dimension three (cf. 1) and the genus 1 case where the automorphism group of $\mathbb{C}/\Lambda$ is one dimensional containing all maps $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ induced by translations of $\mathbb{C}$.

6.3 Metrics of Constant Negative Curvature

Theorem 5 allows us to introduce certain metrics on Riemann surfaces via considering some suitable metrics on their universal covers. There are metric of constant curvature on simply connected Riemann surfaces:

$$\frac{2|\mathrm{d}z|}{1+|z|^2} \text{ on } \mathbb{CP}^1 \quad |\mathrm{d}z| \text{ on } \mathbb{C} \quad \frac{2|\mathrm{d}z|}{1-|z|^2} \text{ on } \mathbb{D} \, (\text{or } \frac{|\mathrm{d}z|}{\mathrm{Im}z} \text{ on } \mathbb{H})$$

It is not hard to verify that for a conformal metric $\rho(z)|\mathrm{d}z|$ on a domain U of $\mathbb{C}$ ($\rho > 0$ is C^2) the curvature is given by $\kappa(z) = -\frac{\Delta(\mathrm{Ln}(\rho(z)))}{\rho(z)^2}$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the usual Laplacian. Using this formula, it is easy to check that the metrics on $\mathbb{CP}^1$, $\mathbb{C}$ or $\mathbb{D}$ which we just encountered have constant curvatures equal to 1, 0 and -1, respectively.

Exercise 7. The formula $\kappa = -\frac{\Delta(\mathrm{Ln}\rho)}{\rho^2}$ for curvature can be derived with the help of *Cartan structural equations* in dimension two, cf. [Krantz, appendix]. Here is an alternative approach: equip the holomorphic tangent bundle of the domain U with the Hermitian metric $\rho(z)^2\mathrm{d}z \otimes \mathrm{d}\bar{z}$. Let ∇ be the corresponding Chern connection[12]. Verify that $\nabla^2(\frac{\partial}{\partial z}) = 2.\bar{\partial}\left(\frac{\partial\rho}{\rho}\right) \otimes \frac{\partial}{\partial z}$. Now check that the curvature form $\Theta = 2.\bar{\partial}\left(\frac{\partial\rho}{\rho}\right)$ and the $(1,1)$-form ω associated with this metric are related by $\mathrm{i}.\Theta = -\frac{\Delta(\mathrm{Ln}\rho)}{\rho^2}.\omega$.

For hyperbolic Riemann surfaces the universal cover $\mathbb{D}$ (or $\mathbb{H}$) may be equipped with the *Poincaré metric* $\frac{2|\mathrm{d}z|}{1-|z|^2}$ (or $\frac{|\mathrm{d}z|}{\mathrm{Im}z}$). Biholomorphic automorphisms of this domain (which are characterized in Proposition 1) keep this metric invariant and in fact it can be proved that up to a positive multiple the Poincaré metric is the unique Riemannian metric on $\mathbb{D}$ which is invariant under all conformal self-maps $\mathbb{D} \to \mathbb{D}$, cf. [Krantz, chap. 1]. Therefore, in a universal covering map $\mathbb{D} \to \mathbb{D}/\Gamma$ the Poincaré metric on $\mathbb{D}$ is invariant under the action of Γ and thus this metric descends to a metric on the quotient, the unique metric for which the covering map is a local isometry. This metric on $\mathbb{D}/\Gamma$ is also called the Poincaré metric. Since $\mathbb{D}$ equipped with the Poincaré metric is complete and

[11] There is an effective bound for the order of the automorphism group in terms of the genus: $|\mathrm{Aut}(X)| \leq 84\,(g(X)-1)$. See [Miranda, p.82] for a proof.

[12] A holomorphic vector bundle $E \to M$ equipped with a Hermitian metric admits a connection ∇ compatible with the metric which furthermore satisfies $\nabla^{0,1} = \bar{\partial}$. Such a connection is unique and is called the *Chern connection*. When E is the holomorphic tangent bundle $T_M \cong T_{M,\mathbb{R}}$, the Chern connection of a Kähler metric on M (so any Hermitian metric in the case of Riemann surfaces) coincides with the Levi-Civita connection of the associated Riemannian metric on $T_{M,\mathbb{R}} \to M$.

of constant curvature -1, the same is true for the quotient. A metric on $\mathbb{D}/\Gamma$ with such properties is unique: any other metric of this type is pulled back via $\mathbb{D} \to \mathbb{D}/\Gamma$ to a complete Riemannian metric on $\mathbb{D}$ whose curvature is -1. A theorem due to Hopf asserts that for any number real K there is one and only one simply connected complete surface of constant curvature K, up to isometry. So $\mathbb{D}$ equipped with the preceding metric is isometric to $\mathbb{D}$ equipped with the Poincaré metric. Such an isometry must be a holomorphic automorphism of $\mathbb{D}$ and we know that conformal self-maps of $\mathbb{D}$ preserve the Poincaré matric. Therefore the metric obtained from the pullback and thus the original metric on $\mathbb{D}/\Gamma$ are both Poincaré metrics on hyperbolic surfaces. We have proved that every hyperbolic Riemann surface has a preferred Riemannian (in fact Hermitian) metric:

Theorem 9. *Every hyperbolic Riemann surface admits a unique metric which is complete and of constant curvature -1.*

Similarly, there is a metric of zero curvature unique up to a multiplicative constant on any Euclidean Riemann surface: these surfaces are quotients of $\mathbb{C}$ to discrete subgroups of the group of translations $z \mapsto z+c$. The Euclidean metric $|dz|$ is of curvature zero and up to a positive constant is the unique metric on $\mathbb{C}$ invariant under translations.

Things are different in the spherical case where the spherical metric $\frac{2|\mathrm{d}z|}{1+|z|^2}$ on $\mathbb{CP}^1$ is far from unique. This metric is invariant only under elements of $\mathrm{SO}(3)$ (which embeds in $\mathrm{Aut}\left(\mathbb{CP}^1\right) = \mathrm{PGL}_2(\mathbb{C})$ as the maximal compact subgroup $\mathrm{PSU}(2)$) and hence by applying elements of $\mathrm{PGL}_2(\mathbb{C})$ one gets a 3-dimensional family of Hermitian metrics on the Riemann sphere all of constant curvature $+1$.

Corollary 1. *Any Riemann surface admits a complete conformal metric with constant curvature which is negative, zero or positive in hyperbolic, Euclidean and spherical cases respectively.*

Exercise 8. By writing down the universal covering map verify that $-\frac{|\mathrm{d}z|}{|z|.\mathrm{Ln}|z|}$ is the Poincaré metric on the punctured disk $\mathbb{D} - \{0\}$.

6.4 Rado Theorem

The first consequence is the fact that since $\tilde{X} = \mathbb{CP}^1$ or $\mathbb{C}$ or $\mathbb{D}$ is second countable, the same is true for X. This is the so called *Rado Theorem*[13]:

Theorem 10. *Any Riemann surface (any connected complex manifold of dimension one) has a countable basis for its topology.*

6.5 The Dimension of The Moduli Space of Compact Riemann Surfaces

Here we present a simple heuristic argument based on the Uniformization Theorem 5 for counting the dimension of the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g. When $g = 0$

[13] In some of other approaches, e.g. that of Forster, first the Rado Theorem is established and then is used to give a proof of uniformization.

we only have $\mathbb{CP}^1$ so let us concentrate on positive genera.

For $g = 1$ every compact Riemann surface is isomorphic with a $\mathbb{C}/\Lambda$ and isomorphisms $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ can be transformed to $[z] \mapsto [\alpha z]$ where $\alpha \in \mathbb{C} - \{0\}$ and $\alpha.\Lambda = \Lambda'$. After a suitable change of coordinates, any lattice in $\mathbb{C}$ can be written as $\mathbb{Z} + \mathbb{Z}\tau$ where $\tau \in \mathbb{H}$. For two lattices in this form $\alpha.(\mathbb{Z} + \mathbb{Z}\tau) = \mathbb{Z} + \mathbb{Z}\tau'$ means $\tau' = \frac{a\tau+b}{c\tau+d}$, $\alpha = \frac{1}{c\tau+d}$ where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Therefore, $\mathcal{M}_1$ can be described as $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ which is one dimensional.

Fix $g \geq 2$. Every Riemann surface X of genus g may be described as $\mathbb{H}/\Gamma$ where Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ isomorphic to the fundamental group of the compact orientable surface of genus g:

$$\Gamma \cong \langle A_1, \ldots, A_g, B_1, \ldots, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A_g^{-1} B_g^{-1} = I_2 \rangle$$

Since $\dim_{\mathbb{R}} \mathrm{PSL}_2(\mathbb{R}) = 3$ there are $6g$ degrees of freedom in choosing matrices $A_1, \ldots, A_g, B_1, \ldots, B_g$ and the one relation imposed on them reduces the number of needed parameters to $6g - 3$. But it might be the case that two different subgroups Γ and Γ' both with the desired presentation give rise to isomorphic surfaces $\mathbb{H}/\Gamma$, $\mathbb{H}/\Gamma'$. Since such an isomorphism lifts to an automorphism of the universal cover $\mathbb{H}$, this is the case iff Γ, Γ' are conjugate by an element of $\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R})$. In order to taking into account conjugation via elements of the three-dimensional group $\mathrm{PSL}_2(\mathbb{R})$, three other degrees of freedom must be subtracted which yields $\dim_{\mathbb{R}} \mathcal{M}_g = 6g - 6$.

6.6 Proving the Riemann Existence Theorem via Uniformization

Again, let us go back to the non-trivial analytic question of existence of non-constant meromorphic functions on compact Riemann surfaces. Having established the Riemann-Roch as a result of our Main Theorem 3, this question has already been answered. But describing a compact Riemann surface X as $\mathbb{C}/\Lambda$ in genus 1 and $\mathbb{H}/\Gamma$ in higher genera suggests a different approach for constructing meromorphic functions on X, that is presenting meromorphic functions on the universal cover ($\mathbb{C}$ or $\mathbb{H}$) which are invariant under the group action. To get invariance under the group action we construct such functions by certain summations over orbits.

In genus 1 this is the classical construction of the *Weierstrass $\wp$-function* associated with a lattice Λ in $\mathbb{C}$:

Proposition 3. *For a lattice Λ in $\mathbb{C}$ the series*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

converges to a meromorphic function on $\mathbb{C}$ which is doubly periodic with respect to Λ (convergence is uniform in compact subsets of $\mathbb{C} - \Lambda$). The poles of this function are points of Λ and every pole is of order 2.

When $g(X) \geq 2$, X can be realized as $\mathbb{H}/\Gamma$ for a suitable torsion free Fuchsian group $\Gamma < \mathrm{PSL}_2(\mathbb{R})$. There are associated *Poincaré sreies*, meromorphic functions on $\mathbb{H}$ which are *modular forms* for Γ. The proposition below introduces such series:

Proposition 4. *For any $a \in \mathbb{H}$ and $k \in \mathbb{N}$ the series*

$$Q_{k,a}(z) := \sum_{\gamma \in \Gamma} \frac{\gamma'(z)^{2k}}{\gamma(z) - a}$$

defines a meromorphic function on $\mathbb{H}$ which satisfies:

$$\forall z \in \mathbb{H}, \gamma \in \Gamma : Q_{k,a}(\gamma(z)) = \frac{1}{\gamma'(z)^{2k}} Q_{k,a}(z)$$

Taking this proposition for granted, for two points $a, b \in \mathbb{H}$ in distinct Γ-orbits the quotient $\frac{Q_a(z)}{Q_b(z)}$ is invariant under the action of Γ and hence descends to a meromorphic function on the compact Riemann surface $X = \mathbb{H}/\Gamma$ which vanishes at the orbit of b and has a pole at the orbit of a. We just proved that the meromorphic functions on X separate points.

6.7 Higher Dimensions

It is not possible to formulate an analogue of the Riemann Mapping Theorem 2 for domains in $\mathbb{C}^n$ where $n > 1$. This is due to the fact that there are domains in $\mathbb{C}^2$ which are simply connected and even contractible and also homeomorphic nevertheless, conformally they are distinct. The classical example of such domains is the unit open ball $\mathbb{B} := \left\{(z_1, z_2) \in \mathbb{C}^2 \middle| |z_1|^2 + |z_2|^2 < 1\right\}$ and the *bidisc* $\mathbb{D} \times \mathbb{D} = \left\{(z_1, z_2) \in \mathbb{C}^2 \middle| |z_1|, |z_2| < 1\right\}$. These are not biholomorphic because they have different automorphism groups, cf. [Krantz, chap. 4, §2].