THE UNIFORMIZATION THEOREM AND UNIVERSAL COVERS

PETAR YANAKIEV

ABSTRACT. This paper will deal with the consequences of the Uniformization Theorem, which is a major result in complex analysis and differential geometry. We will proceed by stating the theorem, which is that for any simply connected Riemann surface, there exists a biholomorphic map to one (and only one) of the following three: the Riemann sphere, the open unit disk, and the complex plane. After the theorem is stated and the basic terms defined, we will go into its consequences by introducing the notions of covering spaces and universal covers. By utilizing another fundamental result in differential geometry, the Gauss-Bonnet Theorem, we will be able to use the notions of curvature and genus in order to come up with a simple method using only the topological properties of compact, orientable Riemann surfaces in order to decide which of the three model geometries (the Riemann sphere, the open unit disk, or the complex plane) is a universal cover for it.

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1. BASICS

We begin by defining some basic terms.

Definition 1.1. A Riemann surface X is a Hausdorff topological space which has the following property: for each point x in X, there exists an open neighborhood containing x that is homeomorphic to the open unit disk in the complex plane. The maps that take these neighborhoods to the complex plane are called charts. Furthermore, the transition map between two charts Ψ_j and Ψ_t whose domains have a non-empty intersection, which is defined as the composition $\Psi_t(\Psi_j^{-1})$, is required to be holomorphic.

It is not difficult to see that the complex plane and open unit disk are Riemann surfaces. Take an arbitrary point a in the complex plane and consider the open ball of radius 1 around it. Then consider the map $f(x) = x - a$. This map is, of course, a homeomorphism. Additionally, we can see that, given another point b, a ball of radius 1 around b, and a chart $g(x) = x - b$, the resulting transition map, $f(g^{-1}) = y + b - a$, is clearly holomorphic. Thus, the complex plane is a Riemann

surface, and a similar argument establishes that the open unit disk is as well. The case of the Riemann sphere is slightly more complicated.

To construct the Riemann sphere, first take the unit sphere in $\mathbb{R}^3$. We associate each point on the unit sphere with coordinates (x, y, z) (with the exception of the north pole $(0, 0, 1)$), with a point in the complex plane using the following equation: $f(x, y, z) = (x + iy)/(1 - z)$. One can verify that this map covers all of $\mathbb{C}$ and that it is injective; the point at the north pole, we associate with infinity. Thus, the Riemann sphere is, simply put, a graphical representation of the extended complex plane.

Having defined the Riemann sphere, we now show it is a Riemann surface. Choose an arbitrary point on the sphere that is not $(0, 0, 1)$, and let the chart be $f(x, y, z) = (x + iy)/(1 - z)$. This map is a homeomorphism. At the point $(0, 0, 1)$, simply take the map, $g(x, y, z) = (x - iy)/(1 + z)$. This map is again clearly a homeomorphism. Thus, all we have to worry about is whether our transition maps are holomorphic.

To show that the transition maps are holomorphic, we will prove that the composition of g and f^{-1} is holomorphic. We first compute f^{-1}.

$f(x, y, z) = (x + iy)/(1 - z) = (z_1, z_2)$

$z_1 = x/(1 - z)$

$z_2 = y/(1 - z)$

Using these equations and the fact that $x^2 + y^2 + z^2 = 1$, we can conclude that

$z = (z_1^2 + z_2^2 \text{ - } 1)/(z_1^2 + z_2^2 + 1)$

And using the equations above, with some manipulation, we get that:

$x = 2z_1/(z_1^2 + z_2^2 + 1)$

and

$y = 2z_2/(z_1^2 + z_2^2 + 1)$

Thus we are able to express (x, y, z) as a function of $(z_1,\ z_2)$. Now we apply the map g. After some manipulation, we get that the composition of g and f^{-1} is equal to: $z_1/(z_1^2 + z_2^2)$ - $i\ (z_2/(z_1^2 + z_2^2))$.

Here we invoke a basic fact from complex analysis to show that the map is holomorphic.

Theorem 1.2. *Take a complex-valued map of the form $f(x, y) = u(x, y) + iv(x, y)$ (where u and v are C^1 on an open set in $\mathbb{C}$) whose first-order partial derivatives exist, are continuous, and satisfy the following differential equations (known as the Cauchy-Riemann equations):*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If the map satisfies all the requirements listed above, we can conclude that it is holomorphic.

We can verify that the map satisfies these criteria.

$\frac{\partial u}{\partial x} = (z_1^2 + z_2^2)2z_1/(z_1^2 + z_2^2)^2 = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -2z_2z_1/(z_1^2 + z_2^2)^2 = -\frac{\partial v}{\partial x}$

Because the partial derivatives satisfy the Cauchy-Riemann equations and are continuous, we are able to conclude that our transition map is holomorphic. And thus we can conclude that the Riemann sphere is in fact a Riemann surface.

Thus, we have shown that the Riemann sphere, the open unit disk, and the complex plane are Riemann surfaces. Now we look at what it means to be simply connected.

Definition 1.3. A topological space X is simply connected if and only if it is path-connected and any two paths f and g with the same beginning and endpoints are homotopic. Two paths are homotopic if and only if there exists a continuous map h that takes X x [0,1] to X and has the following property: $h(x, 0) = f(x)$, and $h(x, 1) = g(x)$

Informally, we can think of a simply-connected space as one which allows us to contract any loop to a single point all the while keeping it within our space; in even simpler terms, a simply connected space is one without any holes. To show that the complex plane, the open unit disk, and the Riemann sphere are simply-connected, we will show how to contract any loop to a single point.

Take a loop on the complex plane described by a map f and choose an arbitrary point x_0 on the loop. We want to show that the loop can be contracted to the point x_0. To do this, consider the following function:

$g(x, s) = (1 - s)(f(x) - f(x_0)) + f(x_0)$, where s is a number in the interval $[0, 1]$. When $s = 0$, we see that we simply have the map $f(x)$, and when $s = 1$, we can see that all we are left with is $f(x_0)$. A similar argument applies for the unit disk; the only difference in the case of the sphere is that our map becomes $g(x, s)/||g(x, s)||$, in order to make sure our loop stays on the surface of the sphere the whole time it is being contracted. We do have to be a little careful in the case of the sphere; it is possible that, having chosen a point p to which we want to contract our loop, our proposed contraction may require us to pass through the antipodal point q of p. If this is the case, i.e. if p and q are both on our loop, then preliminarily, after choosing either p or q as the point of contraction, we have to make sure that the antipodal point does not lie on our loop. To do this, we have to continuously stretch our loop slightly. Once this difficulty has been accounted for, we can conclude that the Riemann sphere is simply connected.

Thus, we have shown that the Riemann sphere, the open unit disk, and the complex plane are simply connected spaces. Now we define conformal equivalence.

Definition 1.4. Two Riemann Surfaces are conformally equivalent if and only if there exists a biholomorphic map between them. A map is biholomorphic if and only if it is a holomorphic bijection and its inverse is also holomorphic.

At this point, a natural question arises. Given that a Riemann surface is not necessarily embedded in $\mathbb{C}$, it is not clear what it means for a map between Riemann surfaces to be holomorphic. Here we slightly generalize the notion of a holomorphic map in order to rectify this problem.

Definition 1.5. Let X and Y be two Riemann surfaces. Let (U_t, Φ_t) and (V_j, Ψ_j) be the collection of charts on X and Y respectively. We say that a map $f : X \to Y$ is holomorphic if and only if the map $\Psi_j(f(\Phi_t^{-1})$ is holomorphic for all t and j where the intersection of $f(U_t)$ and V_j is non-empty.

It is easy to see that the Riemann sphere cannot be conformally equivalent to the open unit disk or the complex plane, because it is a compact set; a continuous

map takes compact sets to compact sets, and since neither the open unit disk nor the plane itself is compact, we know that neither can be conformally equivalent to the Riemann sphere. The argument is a little more complicated in the case of the disk and plane; here, we rely on an important result from complex analysis: Liouville's theorem, which we state without proof.

Theorem 1.6. *A bounded map that is holomorphic everywhere is constant.*

This theorem tells us that if we have a holomorphic map that takes the complex plane to the unit disk, it is necessarily constant, which means that the function cannot be a bijection. Thus, the open unit disk is not conformally equivalent to the complex plane.

With all these definitions in hand, we can now state the Uniformization theorem.

Theorem 1.7. *Every simply connected Riemann surface is conformally equivalent to one of the following three: the complex plane, the open unit disk, or the Riemann sphere.*

As we have shown, none of our model surfaces are conformally equivalent. Since the existence of a biholomorphic map is an equivalence relation, which implies that it obeys transitivity, we can conclude that any simply connected Riemann surface is conformally equivalent to only one of our three model surfaces. Thus, given an arbitrary simply connected Riemann surface, we can be sure that there exists a biholomorphic map from it to only one of our three model surfaces.

2. UNIVERSAL COVERS

Definition 2.1. Let P be a topological space. A covering space of P is a pair (X, Φ) where X is a topological space and Φ is a continuous map which takes X onto P and has the following property: for each point p in P, there exists an open neighborhood U around it, whose pre-image is a disjoint union of open sets in X, each one of which is homeomorphic to U.

Definition 2.2. A universal cover is a simply connected covering space.

Now, we shall prove the following lemma:

Lemma 2.3. *The universal cover of a Riemann surface is a Riemann surface.*

Proof. Let X be a Riemann surface. Let M be the universal cover of X We want to show that M is also a Riemann surface. First take an arbitrary point m in M. We know there exists a continuous map f that takes M onto X. Consider the point $f(m)$. We know that X is a Riemann surface, so we know that there exists a homeomorphic map g between some open neighborhood U (contained in X) of $f(m)$ and the open unit disk. Furthermore, we know that there exists some open neighborhood V around $f(m)$ which is homeomorphic to an open subset of M containing m. We take the intersection of U and V, which is an open neighborhood. This open neighborhood will be homeomorphic to an open subset of M which contains m. And it will be homeomorphic to an open subset of the unit disk denoted W. Thus we have an open set containing m that is homeomorphic to an open subset of the unit disk, with our map being simply $g(f)$. Consider $g(f(m))$. We can find a ball around $g(f(m))$ entirely contained in W. The pre-image of this ball will be an open set (which we call A) which contains m. And this open set is

homeomorphic to an open ball contained in the disk, which in turn means that it is homeomorphic to the disk itself.

Thus we have shown that for any point, we can find an open neighborhood around it which is homeomorphic to the unit disk. Take two points a and b, where U_a is taken to the unit disk by ϕ and U_b is taken to the unit disk by ψ. We want to show that $\phi(\psi^{-1})$ is holomorphic. Note that ϕ is just the composition of some map r and f, where r is a homeomorphic map taking an open neighborhood of $f(a)$ to the open unit disk. The same is true for ψ, except there we replace r with p, and p takes an open neighborhood of $f(b)$ to the unit disk. Since we can assume that the intersection of U_a and U_b is non-empty, we can conclude that the intersection of $f(U_a)$ and $f(U_b)$ is also non-empty. Since $\psi = p(f)$, $\psi^{-1} = f^{-1}(p^{-1})$. And we know that $\phi = r(f)$. Consequently $\phi(\psi^{-1}) = r(p^{-1})$. Because X is a Riemann surface, we know that the map $r(p^{-1})$ is holomorphic. Thus we have shown that M is a Riemann surface. $\square$

It is worth noting that nowhere did we use the fact that M is simply connected. Thus, we have a slightly more general fact: any covering space of a Riemann surface is a Riemann surface.

From the previous proof, and the fact that, by definition, the universal cover is simply connected, we can conclude that the universal cover of a Riemann surface is conformally equivalent to one of our three choice surfaces.

To summarize, we start with a topological space and we consider its universal covering; we put a Riemann surface structure on both the space itself and its universal covering. Consequently, the universal covering is now conformally equivalent to one of our model surfaces. The goal of the next few sections is to ultimately come up with an easy way of telling us, given a topological space with a Riemann surface structure on it, which of our three model surfaces will be its universal cover.

3. Model Geometries

Here, we will briefly discuss the complete, constant curvature geometries of the complex plane, the open unit disk, and the sphere. We will do so with the aim of developing an intuitive view of what curvature is. Once we have built up this notion of curvature, we will introduce the Gauss-Bonnet theorem, the genus, and the Euler characteristic, and use these notions to show that the genus of a compact, orientable Riemann surface, thanks to its curvature, entirely determines what its universal cover will be.

We will approach the subject of curvature in an intuitive way by considering what triangles look like in each of our three spaces. The complex plane is the simplest case, since it is just standard Euclidean geometry. Between any two points, there is a unique shortest path, which is, of course, the straight line. The angles of triangles in Euclidean space always add up to π radians; also, similar triangles exist, which is to say, two triangles can have the same angle measurements but entirely different areas.

The sphere is different in a number of ways. Our notion of straight line has to be modified.

Definition 3.1. The intersection of the sphere with a plane passing through the center of the sphere is called a great circle.

Given any two points on the sphere, there is at least one great circle containing both of them. If the two points are antipodal, meaning that there exists a straight line passing through the center of the sphere which contains both of them, the number of such great circles is infinite. Otherwise, the great circle is unique.

The shortest distance between two points on the sphere is along the great circle which joins them. Given two points a and b on the sphere, and a great circle that connects them, there will be two paths along that circle which will take one from a to b. In many cases, a unique shortest path will exist, and it will simply be the shorter of the two paths along the great circle. This is not always the case: consider antipodal points like the north and south poles. However, for our purposes, our definition of triangle on the sphere will exclude such difficulties. The picture below shows the difference between great and small circles:
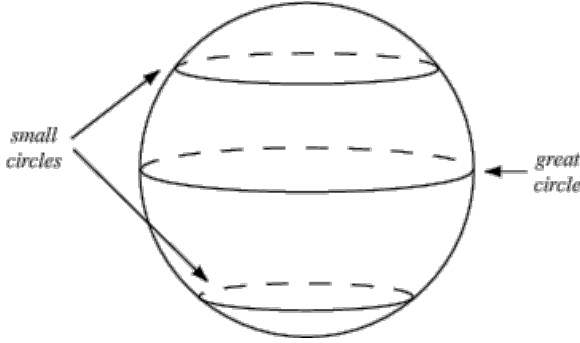

We define "small" triangles on the sphere by taking three points, no two of which are antipodal, and considering the shorter segments of the great circles that connect them. What is noteworthy about such triangles is that their area will be entirely determined by the sum of the interior angles, and that the sum of the interior angles will always be greater than π radians. For our purposes, we will identify the notion of "positive" curvature with a space where triangles behave like they do on the sphere, i.e. the sum of their interior angles is greater than π radians. Below is an illustration of what triangles on the sphere look like:
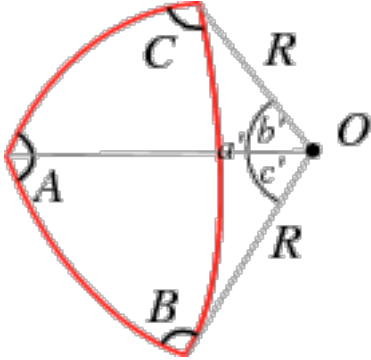
Things are a little different when we start to do hyperbolic geometry on the open unit disk. In this case, the straight line between two points is a circular arc which is perpendicular to the boundary of the open unit disk, and if continued past the disk's boundary, ultimately forms a complete circle. The sum of the interior angles of triangles in this space is always strictly less than π radians, and once again, the area of a triangle is entirely determined by the sum of the angles. Similarly to

before, we identify negative curvature with the behavior of triangles in our space, which in this case simply refers to the fact that the sum of the interior angles will always be less than π radians. Below is a picture of a hyperbolic triangle:
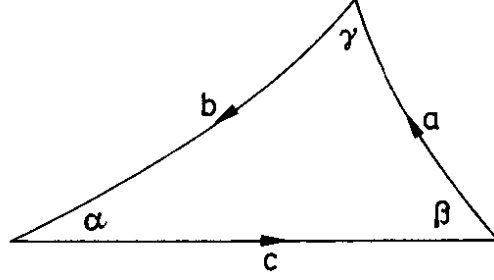
This brief introduction to these three geometries will prove to be quite useful in the next section of the paper, in which we will try to come up with a method using only the topological properties of a Riemann surface in order to determine which of our three model spaces is conformally equivalent to it.

4. Gauss-Bonnet

First, we will prove a pair of lemmas. We know that, given one of our three model geometries, we can construct a genus g compact, orientable Riemann surface if we mod out by isometries. What this means is that, given our space and a group of isometries G, we assert that two elements a and b from our space are equivalent if there exists an isometry g contained in G such that $g(a) = b$. We will add the following requirement for our isometries; for each point a in our space, there exists an open set U such that the intersection of U and $g(U)$ is empty for all isometries except the identity. This additional requirement will help us show that the covering map from one of our three choice spaces to the quotient space described above is a local isometry; having this requirement for our isometries does not at all prevent us from being able to construct a compact, orientable Riemann surface of genus g from one of our model spaces.

Thus, given our model space U, with a constant curvature metric d on it, we propose the following metric d' on the base space X: for two points $[X]$ and $[Y]$ in X, the distance d' between $[X]$ and $[Y]$ is the infimum of the set d(a,b), where a belongs to $[X]$ and b belongs to $[Y]$.

Lemma 4.1. *The definition of distance above satisfies the requirements of a metric.*

Proof. Both symmetry and the triangle inequality are simple.

Assume that $d'([X], [Y]) = 0$. We will show $[X] = [Y]$.

Take x' from $[X]$ and y' from $[Y]$.

$d(x', y') = d(f(a), g(b))$, where f and g are isometries. Let p be the inverse map of f; p is also an isometry. Thus, $d(f(a), g(b)) = d(a, p(g(b)))$. So we can conclude that $d'([X], [Y])$ is the infimum of d(a,y), where y is just a point in $[Y]$. We can fix a, in other words, and look at the infimum of distances from a to points in $[Y]$.

We know there exists an element y_n in $[Y]$ such that $d(a, y_n) < 1/n$.

Take an arbitrary open set around $[X]$. Call it O'. We know there exists an open set O around $[X]$ such that the pre-image of O is a disjoint union of open sets homeomorphic to O. Consider the intersection of O' and O. Call this set O''. O''

clearly contains [X] and is open. Consider $F^{-1}(O'')$. This set will of course be a disjoint union of open sets, and one of these open sets will have to contain the point a. Then an open ball around a will exist which is contained in that same open set. And for sufficiently large n, all y_n will be found in that ball. And the map F will take that ball to O", which is a subset of O'. Which means that [Y] will be found in O'.

We chose an arbitrary open set containing [X] and showed that [Y] has to be in it. Since our space is required to be Hausdorff, we know that if [X] is not equal to [Y], there have to exist two disjoint open sets, one of which contains [X] and one of which contains [Y]. We have shown that this is impossible. Thus we conclude that [X] = [Y]. □

Lemma 4.2. *With the metric defined above, the covering map that takes a point to its equivalence class is a local isometry.*

Proof. Here we invoke the requirement we added above regarding our isometries, and we use it to prove an additional fact. Given a point a in U, we can find an open ball around a such that there do no exist two equivalent elements within it. We know there exists an open set U containing a such that the intersection of U and $g(U)$ is empty for all isometries g that are not the identity. Since the set is open, we can take a ball of radius r around a which will be contained in that set. Assume that there exist two not equal, equivalent points c and c' in our ball. If this is the case, then there exists an isometry g which is not the identity for which it is true that $g(c) = c'$. Which means that the intersection of our ball B with the set $g(B)$ cannot be empty, since c' will be in both. Since this is clearly a contradiction, we conclude that there do not exist two equivalent points in our ball of radius r.

Choose an arbitrary point a in U. Take a ball of radius r around a, where r is chosen in the manner above. Then consider the ball of radius $r/8$ around a. Choose two arbitrary points b and c from the ball. We know that $d(b, c)$ cannot be less than $d'([B], [C])$, by the definition of our metric. So let's assume that $d(b, c) > d'([B], [C])$. Then there must exist a point c' in [C] such that $d(b, c') < d(b, c)$. Take that point and consider the following:

$d(c, c') \leq d(c', b) + d(b, c) \leq r/2$.

But this clearly implies that we have a point c' in our ball which is equivalent to c. Since we constructed our ball in order for this to be impossible, we have a contradiction. Which means that our map has to be a local isometry.

□

In general, we have come up with a way with which, given a covering space with a constant curvature metric, we can put a metric on a Riemann surface, such that our map is a local isometry. And once we have this fact, we can conclude that "small" triangles will look identical on both our covering space and our Riemann surface, which is to say that the curvature on both will be the same sign. This fact, when combined with the following theorem, is very powerful.

Theorem 4.3. *Given a compact Riemann surface M without boundary, $\int_M K dA$ = $2\pi\chi$, where K is the curvature and χ is the Euler characteristic. The Euler characteristic of a topological space is defined as $V - E + F$, where V is the number of vertices, E is the number of edges, and F is the number of faces for a triangulation in the space.*

First we will compute the Euler characteristics of the torus and an arbitrary genus g surface. We start with the torus; one of the standard constructions of the torus takes a rectangle and glues sides together in the manner indicated by the picture below.
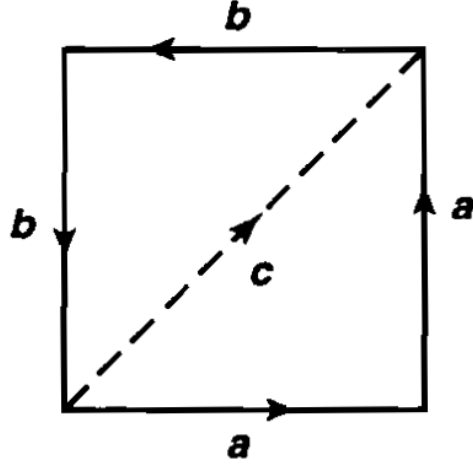
Having triangulated the torus, we end up with 1 vertex, 3 edges, and 2 faces. $1 - 3 + 2 = 0$. We can check this answer by using the relationship that the Euler characteristic is equal to $2 - 2g$, where g is the genus of our surface. Since $0 = 2 - 2g$, that implies $g = 1$, which is true for the torus, as it only possesses one hole.

A similar argument applies in the case of a genus 2 surface. Pictured below is a standard construction of such a surface:
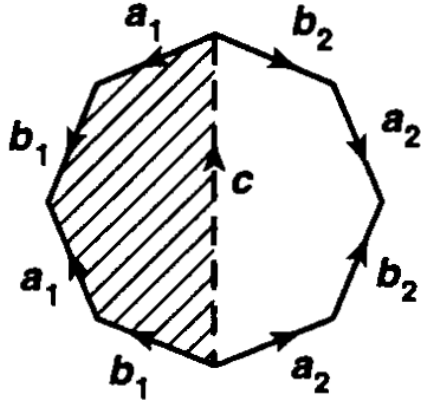
In the picture above, the edge a_1 on the lined side will be identified with the other a_1 that is also on the lined side.

We triangulate by drawing a point in the middle of the figure and extending a line from it to each one of our vertices. We end up with 2 vertices (we have to remember to count the one in the center of the figure), 12 edges (8 of them come from each vertex to the center of the figure, and 4 of them are the result of our identifying a_1 with a_1 and b_1 with b_1), and 8 faces. $2 - 12 + 8 = -2$. An Euler characteristic of -2 implies a genus of 2, which is exactly what we needed.

For a genus 1 figure, the torus, we began with a 4-sided figure; for a genus 2 figure, our construction began with an 8-sided figure. For a genus g figure, the pattern is maintained: the construction starts with a 4g-sided figure, and sides are similarly identified together.

The reader can note and verify for herself that the Euler characteristic of the sphere is equal to 2.

The Gauss-Bonnet theorem is a very powerful result. Thanks to it, we can use the topological information of a Riemann surface in order to find out which of our three model geometries is conformally equivalent to its universal cover (and thus, is also a universal cover of it).

This is quite unexpected. We start off with a Riemann surface. We can see that the universal cover of our Riemann surface is another Riemann surface. This new Riemann surface is simply connected, which means it is conformally equivalent to one of our three model geometries. But it's unclear, without Gauss-Bonnet and the arguments that preceded it, how we can tell which of the three model geometries is the one conformally equivalent to our universal cover. It turns out, however, that the answer to this is quite simple, if we assume that our Riemann surface satisfies a few conditions like compactness and orientability. Assuming that our surface is compact and orientable, all we need to do is look at its genus–that is, how many holes it has. If the genus is 0, then the integral of the curvature will be positive, which will imply that the curvature of the universal cover is positive, which will imply the universal cover is the Riemann sphere. If the genus is 1, then the integral of the curvature will be 0, which will indicate that the universal cover is the complex plane. And in all other cases, the universal cover will be the open unit disk in $\mathbb{C}$.

In other words, we have found a way to use purely topological information in order to come up with information about the holomorphic structure on a given Riemann surface. This is a quite unexpected result, which does not immediately follow from the statement of the Uniformization Theorem. All the Uniformization Theorem tells us is that, given a simply connected Riemann surface, one of the three model geometries is conformally equivalent to it. However, we have no way of knowing which of the the model geometries is equivalent to our surface. The approach outlined here, using Gauss-Bonnet, provides us with a partial solution to this problem.

5. Acknowledgments

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[5] Eric W. Weisstein. Great Circle. http://mathworld.wolfram.com/GreatCircle.html. Accessed 20 August 2011.