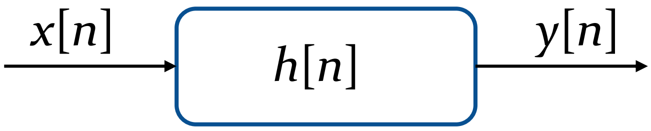
Lecture 7
z-Transform

Motivation

- Fourier Transform provides a frequency domain representation of discrete-time signal, but it may not exist for some sequences. (Reason?)
- Not easy for algebraic manipulations.
- z-transform used for:
  - Analysis of LTI systems
  - Solving difference equations
  - Determining system stability
  - Finding frequency response of stable systems

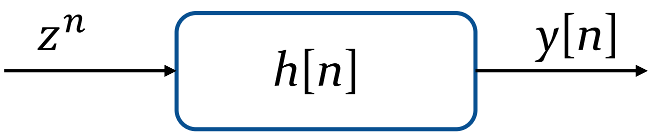
Eigen Functions of LTI Systems

- Consider an LTI system with impulse response $h[n]$:



- We already showed that $x[n] = e^{j\omega n}$ are eigen-functions
- What if $x[n] = z^n$, where z is a continuous complex variable $z = \mathrm{Re}(z) + j\mathrm{Im}(z)$?

Eigen Functions of LTI Systems



$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right)z^n = H(z)z^n$$

- $x[n] = z^n$ are also eigen-functions of LTI Systems
- $H(z)$ is called a z-transform transfer function
- $H(z)$ exists for larger class of $h[n]$ than $H(e^{j\omega})$

Definition

- z-Transform:

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where, z is a complex variable.

- **Example**

n	$n \le -1$	0	1	2	3	4	5	$n > 5$
$x[n]$	0	2	4	6	4	2	1	0

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$$

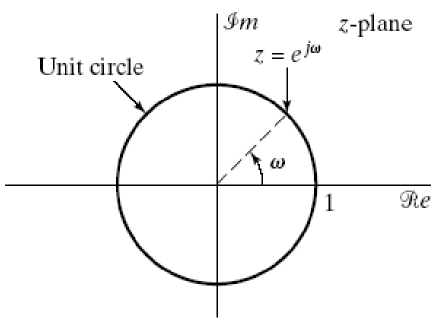
z-Transform vs. DTFT

- Let $z = re^{j\omega}$, then the expression reduces to

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n},$$

This can be interpreted as the Fourier Transform of the modified sequence $x[n]r^{-n}$.

- If $r = 1(\text{i.e.}, |z| = 1)$, the z-transform reduces to DTFT.
- The contour $|z| = 1$ is a circle in the z plan of unity radius, called **unit circle.**



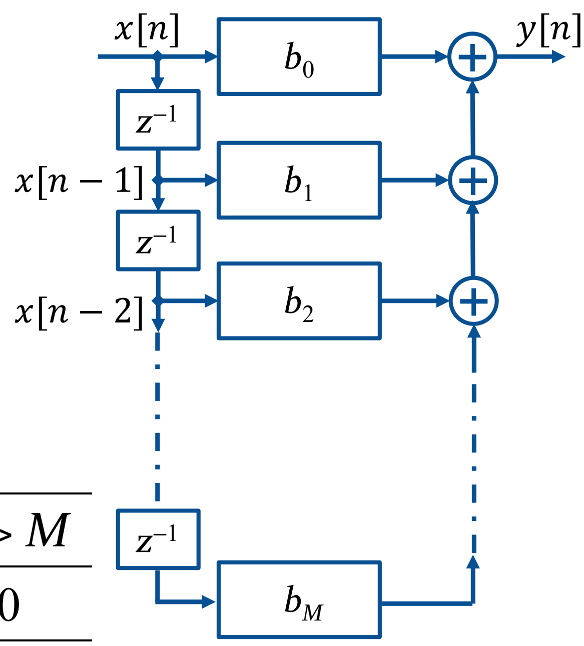
z-Transform and LTI system

- Consider a system of an unit delay system
$$y[n] = x[n-1]$$
- The impulse response of the unit delay is
$$h[n] = \delta[n-1]$$
- Its z-transform is
$$H(z) = z^{-1}$$
$$\longrightarrow \boxed{z^{-1}} \longrightarrow$$
- Similarly, delay of k samples: $h[n] = \delta[n-k]$
$$\longrightarrow \boxed{z^{-k}} \longrightarrow \qquad H(z) = z^{-k}$$

z-Transform of FIR System

- Consider a causal FIR LTI system
$$y[n] = \sum_{m=0}^{M} b_m x[n-m]$$
- Its impulse response is
$$h[n] = \sum_{m=0}^{M} b_m \delta[n-m]$$

n	$n<0$	0	1	2	…	M	$n>M$
$h[n]$	0	b_0	b_1	b_2	…	b_M	0



System Diagram of an FIR system

z-Transform of FIR System

- Take z-transform on both side of the input-output relation

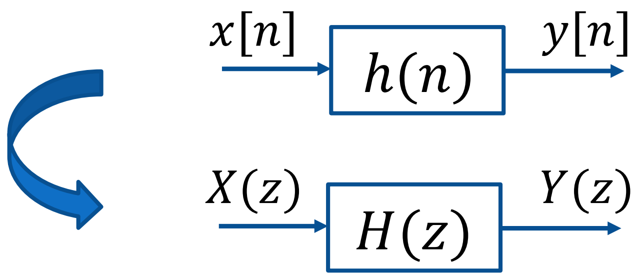
$$Y(z) = Z\{y[n]\} = Z\left\{\sum_{m=0}^{M} b_m x[n-m]\right\} = \sum_{n=-\infty}^{\infty}\sum_{m=0}^{M} b_m x[n-m]\, z^{-n}$$

$$= \sum_{m=0}^{M} b_m \sum_{n=-\infty}^{\infty} x[n-m]z^{-n} = \sum_{m=0}^{M} b_m\, z^{-m} \sum_{n=-\infty}^{\infty} x[n-m]z^{-(n-m)}$$

$$= \sum_{m=0}^{M} b_m\, z^{-m} Z\{x[n]\} = Z\{h[n]\}X(z) = H(z)X(z)$$

- The z-transform of the output of a FIR system is the product of the z-transform of the input signal and the z-transform of the impulse response.

Transfer Function



$$y[n] = x[n] \circledast h[n]$$

$$Y(z) = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{m=0}^{M} b_m\, z^{-m}$$

is called the **z-transform transfer function** (or system function) of a LTI FIR system

Transfer Function and Impulse Response

- When the input $x[n] = \delta[n]$, the z-transform of the impulse response satisfies :

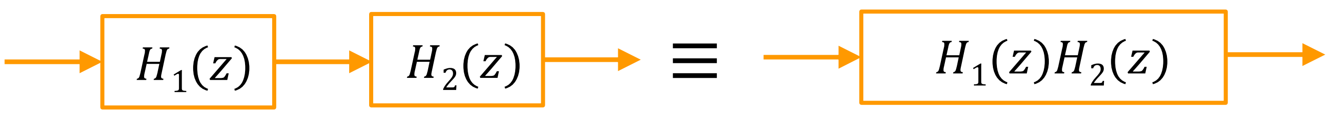
$$Z\{h[n]\} = H(z)Z\{\delta[n]\}.$$

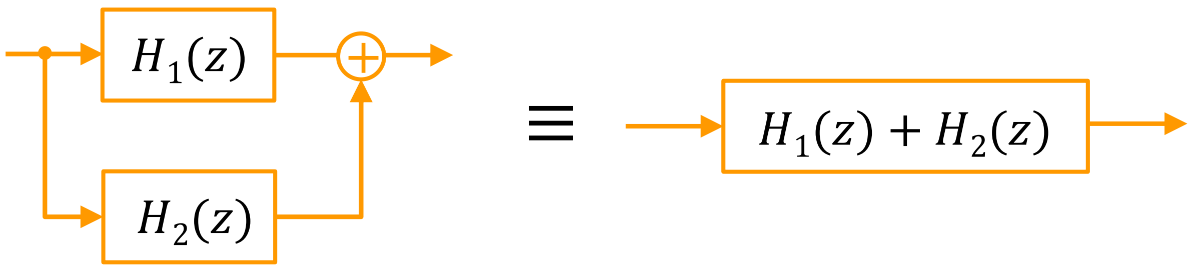
- Since the z-transform of the unit impulse $\delta[n]$ is equal to one, we have

$$Z\{h[n]\} = H(z)$$

- That is, the z-transform transfer function $H(z)$ is the z-transform of the impulse response $h[n]$.

Cascade & Parallel Connection





Example

- Consider an FIR system

$$y[n] = 6x[n] - 5x[n-1] + x[n-2]$$

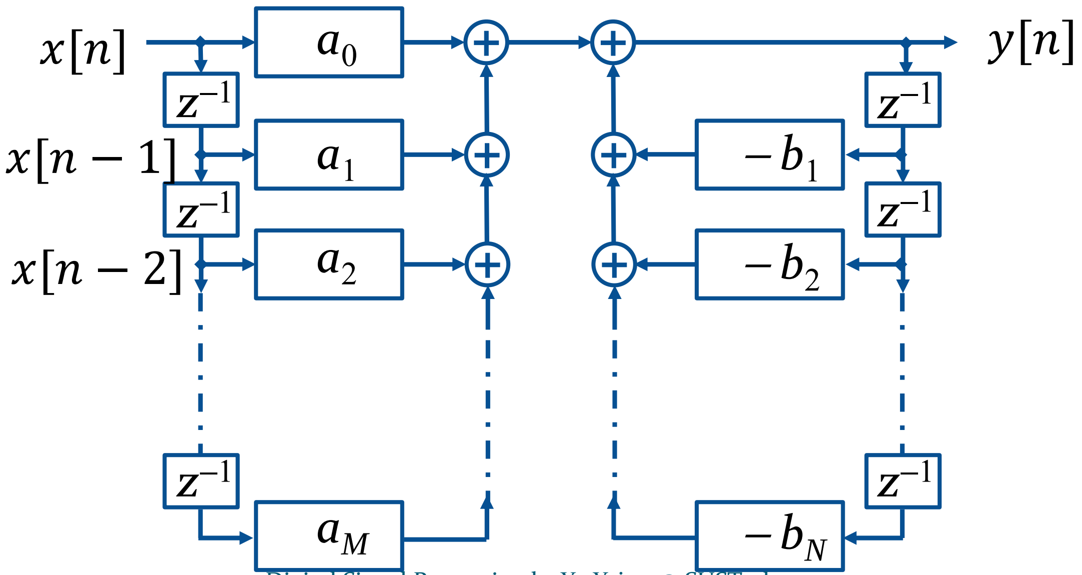
- So, the impulse response is $h[n] = \{6, -5, 1\}, 0 \le n \le 2$
- The z-transform transfer function is:

$$H(z) = 6 - 5z^{-1} + z^{-2}$$
$$= (3 - z^{-1})(2 - z^{-1}) = 6\frac{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}{z^2}$$

z-transform of Difference Equation

$$\sum_{m=0}^{N} b_m y[n-m] = \sum_{m=0}^{M} a_m x[n-m]$$

- Revisit system diagram for normalized $b_0 = 1$



- Take z-transform on both sides of the input-output relation

$$\sum_{m=0}^{N} b_m Y(z) z^{-m} = \sum_{m=0}^{M} a_m X(z) z^{-m}$$

- We have:

$$\frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} a_m z^{-m}}{\sum_{m=0}^{N} b_m z^{-m}} \triangleq H(z)$$

- $H(z)$ is the z-transform transfer function of the LTI system defined by the linear constant-coefficient difference equation.
- The multiplication rule still holds: $Y(z) = H(z)X(z)$, i.e.,

$$Z\{y[n]\} = H(z)Z\{x[n]\}$$

Rational z-transform

- The transfer function of a difference equation (or a generally infinite impulse response (IIR) system) is a **rational form** $H(z) = P(z)/D(z)$.
- Since LTI systems are often realized by difference equations, the rational form is the most common and useful form of z-transforms.
- LTI system with z-transforms represented as a rational function of z^{-1}

$$H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}$$

where the degree of $P(z)$ is M, and that of $D(z)$ is N. The degree of the system is the larger one of M and N.

Alternate representations:

- A ratio of two polynomials in z,

$$H(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N}$$

- A product of second order rational z-transforms,

$$= \frac{p_0}{d_0} \cdot \frac{\prod_{l=1}^{M/2} (1 + p_{1l} z^{-1} + p_{2l} z^{-2})}{\prod_{l=1}^{N/2} (1 + d_{1l} z^{-1} + d_{2l} z^{-2})}$$

- Factorized form，

$$= \frac{p_0}{d_0} \cdot \frac{\prod_{l=1}^{M} (1 - \xi_l z^{-1})}{\prod_{l=1}^{N} (1 - \lambda_l z^{-1})} = z^{(N-M)} \frac{p_0}{d_0} \cdot \frac{\prod_{l=1}^{M} (z - \xi_l)}{\prod_{l=1}^{N} (z - \lambda_l)}$$

- For the $\boldsymbol{z}$-transform of General Difference Equation

$$\sum_{m=0}^{N} b_m Y(z) z^{-m} = \sum_{m=0}^{M} a_m X(z) z^{-m}$$

- When b_0 is normalized to 1, and $b_m = 0$ for $m = 1\ \ldots\ N$, the difference equation degenerates to an FIR system we have investigated before.

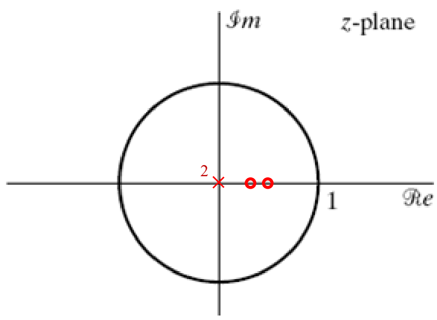
$$H(z) = \frac{Y(z)}{X(z)} = \sum_{m=0}^{M} a_m z^{-m}$$

- It can still be represented by a rational form of the variable z as

$$H(z) = \frac{\sum_{m=0}^{M} a_m z^{(M-m)}}{z^M}$$

Poles and Zeros

- The **pole** of a z-transform $X(z)$ are the values of z for which $X(z) = \infty$.
- The **zero** of a z-transform $X(z)$ are the values of z for which $X(z) = 0$.
- When $X(z) = P(z)/D(z)$ is a rational form, and both $P(z)$ and $D(z)$ are polynomials of z, the poles of $X(z)$ are the roots of $D(z)$, and the zeros are the roots of $P(z)$, respectively.



Examples

- Zeros of a system function
  - The system function of the FIR system $y[n] = 6x[n] - 5x[n-1] + x[n-2]$ has been shown as

$$H(z) = 6\frac{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}{z^2}$$

- The zeros of this system are 1/3 and 1/2, and the pole is 0.
- Since 0 and 0 are double roots of $D(z)$, the pole is a second-order pole.

- In most practical cases, the complex poles and zeros of z-transforms occur as complex conjugate pairs, and simple poles and zeros (i.e., poles or zeros of order 1) are real.
- In such cases, rational z-transform are ratios of polynomials with real coefficients.
- For example, let $z = a_i \pm jb_i$ be a pair of complex conjugate poles of the rational z-transform $H(z)$, where a_i and b_i are real, i.e.,

$$H(z) = \frac{Y(z)}{(z - a_i - jb_i)(z - a_i + jb_i)}$$

$$= \frac{Y(z)}{(z - a_i)^2 + {b_i}^2} = \frac{Y(z)}{z^2 - 2a_i z + ({a_i}^2 + {b_i}^2)}$$

Region of Convergence (ROC)

- **ROC:** the set $\mathcal{R}$ of values of z for which a sequence's z-transform converges, i.e., $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ converges.
- Since z-transform of $x[n]$ is equivalent to DTFT of $x[n]r^{-n}$, if $x[n]r^{-n}$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty,$$

the z-transform of $x[n]$ uniformly converges.

ROC examples

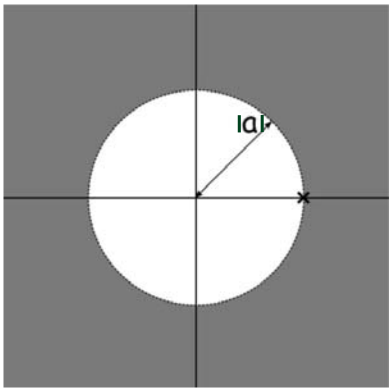
- Example 1: Right-sided sequence $x[n] = a^n\mu[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

recall: $1 + x + x^2 + \cdots = \frac{1}{1-x}$, if $|x| < 1$

- So, $X(z) = \frac{1}{1-az^{-1}}$, for $|az^{-1}| < 1$

- ROC $= \{z\colon |z| > |a|\}$



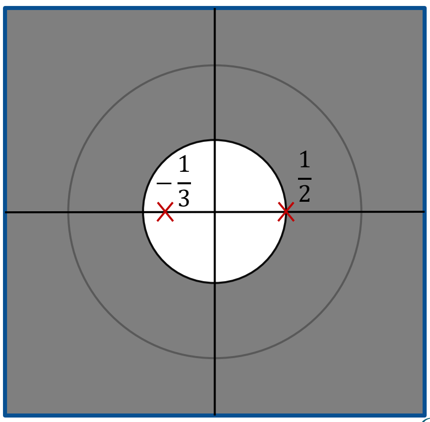
ROC examples

- Example 2: $x[n] = \left(\frac{1}{2}\right)^n \mu[n] + \left(-\frac{1}{3}\right)^n \mu[n]$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}},$$

- ROC $= \left\{z\colon |z| > \frac{1}{2}\right\} \cap \left\{z\colon |z| > \frac{1}{3}\right\}$

$$= \left\{z\colon |z| > \frac{1}{2}\right\}$$



ROC examples

- Example 3: Left sided sequence $x[n] = -a^n\mu[-n-1]$

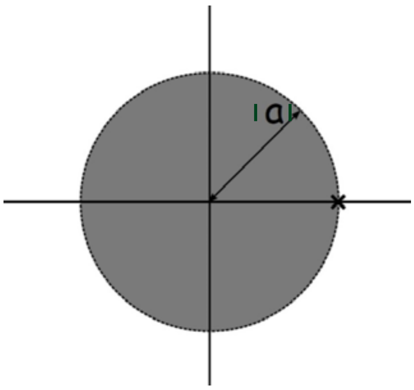
$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$
$$= \sum_{m=1}^{\infty} -a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m$$

- If $|a^{-1}z| < 1$, i.e., $|z| < |a|$,

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}}$$

ROC examples

- Example 3 continued.
- Expression is the same as that of Example 1!
- ROC $= \{z\colon\ |z| < |a|\}$ is different



- Different sequences may have the same z-transform expression.
- The z-transform without ROC does not uniquely define a sequence!

ROC examples

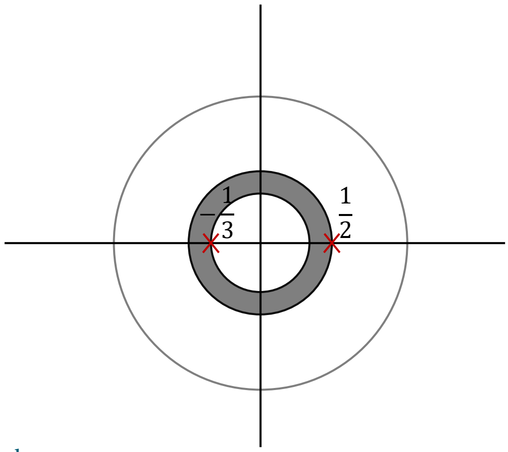
- Example 4: $x[n] = -\left(\frac{1}{2}\right)^n \mu[-n-1] + \left(-\frac{1}{3}\right)^n \mu[n]$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}},$$

Expression Same as that of Example 2

- ROC $= \left\{z:\ |z| < \frac{1}{2}\right\} \cap \left\{z:\ |z| > \frac{1}{3}\right\}$

$$= \left\{z:\ \frac{1}{3} < |z| < \frac{1}{2}\right\}$$



ROC examples

- Example 5: $x[n] = \left(\frac{1}{2}\right)^n \mu[n] - \left(-\frac{1}{3}\right)^n \mu[-n-1]$

$$\text{ROC} = \left\{z:\ |z| > \frac{1}{2}\right\} \cap \left\{z:\ |z| < \frac{1}{3}\right\} = \emptyset$$

- Example 6: $x[n] = a^n$, two sided $a \neq 0$

$$\text{ROC} = \{z:\ |z| > a\} \cap \{z:\ |z| < a\} = \emptyset$$

ROC Examples

- Example 7: Finite sequence $x[n] = a^n \mu[n]\mu[-n+M-1]$

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n}$$

Finite, always converges

$$= \frac{1-a^M z^{-M}}{1-az^{-1}} = \frac{1}{z^{M-1}} \cdot \frac{z^M - a^M}{z-a}$$

Zero cancels pole

There are M roots of $z^M = a^M, z_k = ae^{j\frac{2\pi k}{M}}$. The root of $k = 0$ cancels the pole at $z = a$. Thus there are M–1 zeros, $z_k = ae^{j\frac{2\pi k}{M}}, k = 1, \ldots, M$, and a ($M$–1)$^{\text{th}}$ order pole at zero.

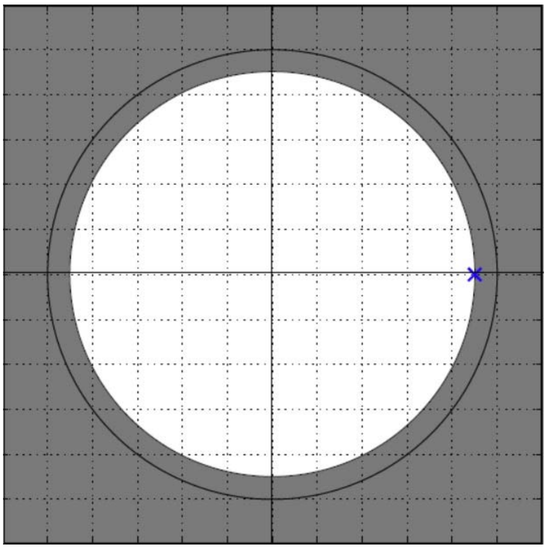
$$X(z) = \prod_{k=1}^{M-1}\left(1 - ae^{j\frac{2\pi k}{M}} z^{-1}\right)$$

- ROC = $\{z: |z| > 0\}$

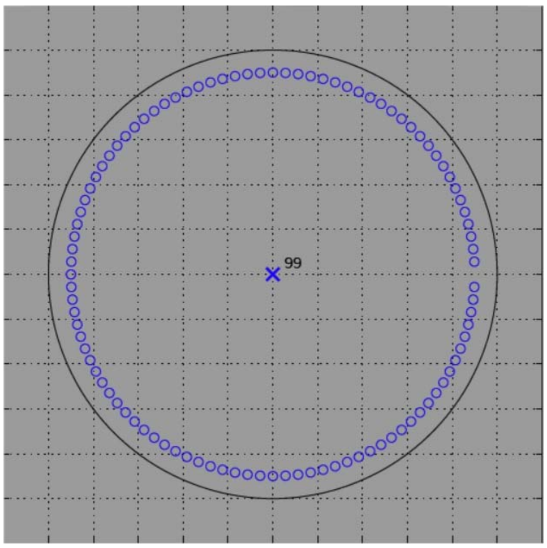
ROC examples

- Example 7 continued:

Infinite Sequence



Finite Sequence

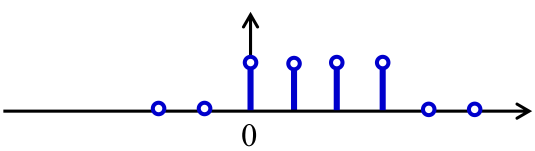


Properties of ROC

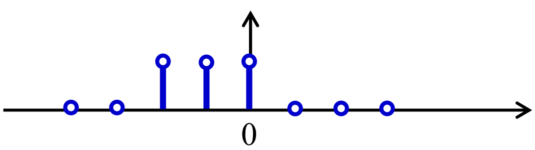
- For right-sided sequences: ROC extends outward from the outermost pole to infinity
  - Examples 1, 2
- For left-sided: ROC inwards from the inner most pole to the original point.
  - Example 3
- For two-sided: ROC either is a ring - or do not exist
  - Examples 4, 5, 6

Properties of ROC

- For finite duration sequences, ROC is the entire *z*-plane, except possibly z=0, z=∞



$X(z) = 1 + z^{-1} + z^{-2} + z^{-3}$ ROC excludes z = 0



$X(z) = 1 + z^{1} + z^{2}$ ROC excludes z = ∞

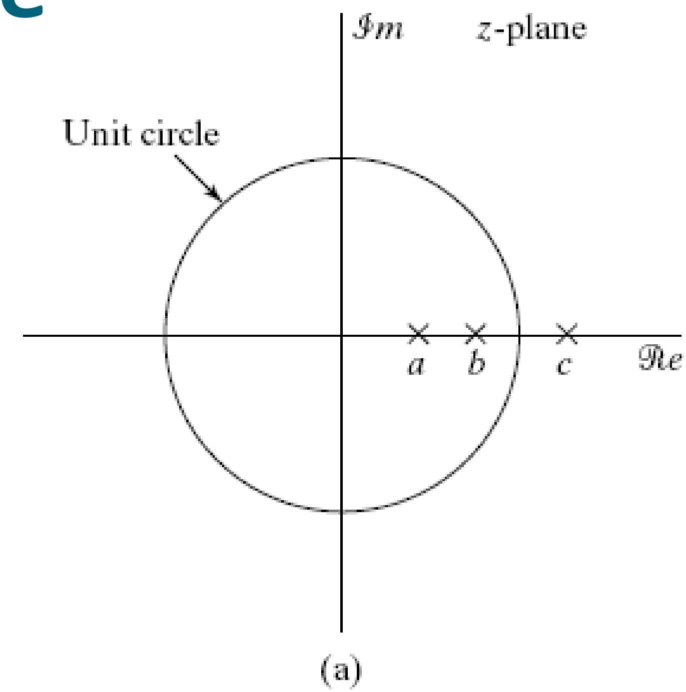
Properties of ROC

- In general, ROC of a z-transform is in a form:
  $R_{x^-} < |z| < R_{x^+}$, an annular region

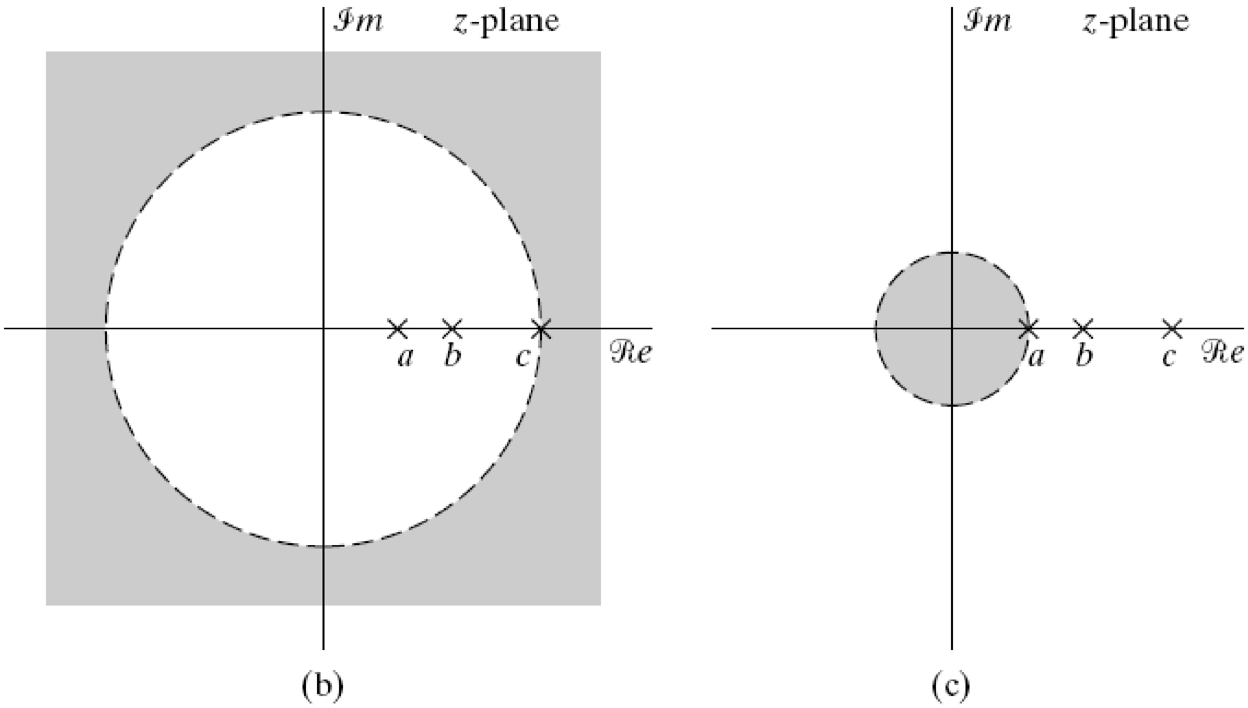
- The DTFT $X(e^{j\omega})$ of $x[n]$ absolutely convergent iff the ROC of the z-transform $X(z)$ of $x[n]$ includes the unit circle.

- ROC can't contain poles

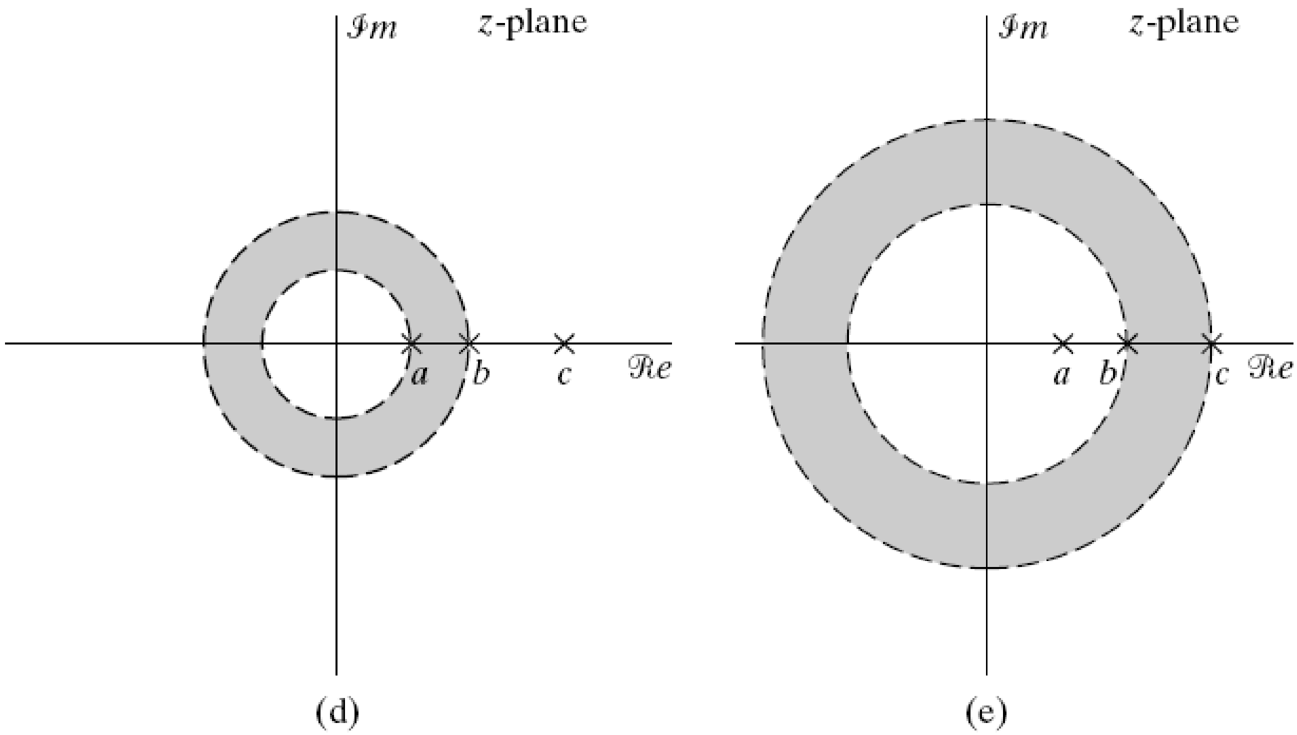
Example



(a) A system with three poles.



Different possibilities of the ROC. (b) ROC to a right-sided sequence. (c) ROC to a left-sided sequence.



Different possibilities of the ROC. (d) ROC to a two-sided sequence. (e) ROC to another two-sided sequence.

ROC for LTI System

- Consider the transfer function $H(z)$ of a linear system:
  - If the system is stable, the impulse response $h(n)$ is absolutely summable and therefore has a Fourier transform, then the ROC must include the unit circle.
  - If the system is causal, then the impulse response $h(n)$ is right-sided, and thus the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $H(z)$ to (and possibly include) $z = \infty$.
  - Therefore, **a stable causal LTI system has all poles inside unit circle**.

Properties of the *z*-transform

Property	Sequence		z-Transform	ROC
	$x[n]$	$\leftrightarrow$	$X(z)$	$\mathcal{R}_x$
Conjugate	$x^*[n]$	$\leftrightarrow$	$X^*(z^*)$	$\mathcal{R}_x$
Time shifting	$x[n-n_d]$	$\leftrightarrow$	$z^{-n_d}X(z)$	$\mathcal{R}_x$ except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$r^n x[n]$	$\leftrightarrow$	$X\left(\frac{z}{r}\right)$	$\lvert r\rvert\mathcal{R}_x$
Differentiation of $X(z)$	$nx[n]$	$\leftrightarrow$	$-z\frac{dX(z)}{dz}$	$\mathcal{R}_x$ except possibly the point $z = 0$ or ∞
Time-reversal	$x[-n]$	$\leftrightarrow$	$X(z^{-1})$	$1/\mathcal{R}_x$
Convolution	$x[n] \circledast y[n]$	$\leftrightarrow$	$X(z)Y(z)$	Includes $\mathcal{R}_x \cap \mathcal{R}_y$

Commonly Used z-transform Pairs

Sequence		z-Transform	ROC
$\delta[n]$	$\leftrightarrow$	1	All values of z
$\mu[n]$	$\leftrightarrow$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-\mu[-n-1]$	$\leftrightarrow$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	$\leftrightarrow$	z^{-m}	All z, except 0 (if m>0) or ∞ (if m<0)
$\alpha^n\mu[n]$	$\leftrightarrow$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-\alpha^n\mu[-n-1]$	$\leftrightarrow$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
$n\alpha^n\mu[n]$	$\leftrightarrow$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$

Commonly Used z-transform Pairs

Sequence		z-Transform	ROC
$-n\alpha^n\mu[-n-1]$	$\leftrightarrow$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
$(n+1)\alpha^n\mu[n]$	$\leftrightarrow$	$\dfrac{1}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-(n+1)\alpha^n\mu[-n-1]$	$\leftrightarrow$	$\dfrac{1}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
$(r^n\cos\omega_0 n)\mu[n]$	$\leftrightarrow$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$(r^n\sin\omega_0 n)\mu[n]$	$\leftrightarrow$	$\dfrac{1-(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$\begin{cases} a^n, 0\le n\le N-1 \\ 0, \quad \text{otherwise}\end{cases}$	$\leftrightarrow$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Example

- Determine the z-transform and its ROC of the causal sequence
$$x[n] = (r^n \cos\omega_0 n)\mu[n]$$
- We can express $x[n] = v[n] + v^*[n]$, where
$$v[n] = \frac{1}{2} r^n e^{j\omega_0 n}\mu[n] = \frac{1}{2}\alpha^n \mu[n]$$
- The z-transform of $v[n]$ is given by
$$V(z) = \frac{1}{2}\cdot\frac{1}{1-\alpha z^{-1}} = \frac{1}{2}\cdot\frac{1}{1-re^{j\omega_0}z^{-1}}, |z| > |\alpha| = |r|$$

- Using the conjugate property, we obtain the z-transform of $v^*[n]$ as
$$V^*(z^*) = \frac{1}{2}\cdot\frac{1}{1-\alpha^* z^{-1}} = \frac{1}{2}\cdot\frac{1}{1-re^{-j\omega_0}z^{-1}}, |z| > |r|$$

- Finally, using the linear property, we get
$$X(z) = \frac{1}{2}\cdot\frac{1}{1-re^{j\omega_0}z^{-1}} + \frac{1}{2}\cdot\frac{1}{1-re^{-j\omega_0}z^{-1}}$$
$$= \frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}, |z| > |r|$$

Inversion of the z-Transform

- In general, by contour integral

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz$$

  where C is any counterclockwise contour encircling the point $z = 0$ in the ROC.
- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Partial fraction expansion
  - Power series expansion
  - Residue theorem

By Inspection

- Eg. If we need to find the inverse z-transform of

$$X(z) = \frac{1}{1 - 0.5z^{-1}}, \qquad |z| < 0.5$$

- From the transform pair we see that

$$x[n] = 0.5^n \mu[n] \text{ or } x[n] = -0.5^n \mu[-n-1]$$

- Since ROC is $|z| < 0.5$, the sequence is left-sided. Therefore,

$$x[n] = -0.5^n \mu[-n-1]$$

By Partial Fraction Expansion

- If $X(z)$ is the rational form with

$$X(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{M} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

- If $M \geq N$, then $X(z)$ can be expressed as

$$X(z) = \sum_{l=0}^{M-N} \eta_l z^{-l} + \frac{P_1(z)}{D(z)}$$

where the degree of $P_1(z)$ is less than N.

- The rational function $\frac{P_1(z)}{D(z)}$ is called a called a proper fraction.

- To develop the proper fraction of $\frac{P_1(z)}{D(z)}$ from $X(z)$, a long division of $P(z)$ by $D(z)$ should be carried out in a reversed order until the remainder polynomial $P_1(z)$ is of lower degree than that of the denominator $D(z)$.
- Example: consider

$$X(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

- By long division in a revered order, we arrive at

$$X(z) = -3.5 + 1.5z^{-1} + \frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

Proper fraction

- **Simple pole:** in most practical cases, the rational z-transform of interest $X(z)$ is a proper fraction with simple poles.
- Let the poles of $X(z)$ be at $z = \lambda_k, 1 \leq k \leq N$
- A **partial-fraction** expansion of $X(z)$ is of the form

$$X(z) = \sum_{l=1}^{N} \left(\frac{\rho_l}{1 - \lambda_l z^{-1}} \right)$$

- The constants ρ_l in the partial-fraction expansion are called the **residue**, and are given by

$$\rho_l = (1 - \lambda_l z^{-1}) \, X(z) \Big|_{z=\lambda_l}$$

- Assume that each term of the sum in partial-fraction expansion has an ROC given by $|z| > |\lambda_l|$, and thus has an inverse transform of the form $\rho_l(\lambda_l)^n \mu[n]$.
- Therefore, the inverse transform $x[n]$ of $X(z)$ is given by

$$x[n] = \sum_{l=1}^{N} \rho_l (\lambda_l)^n \mu[n]$$

Example

- Let the z-transform $H(z)$ of a causal system $h[n]$ is given by

$$H(z) = 1 + \frac{z(z+2)}{(z-0.2)(z+0.6)} = 1 + \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

- The second term is a proper fraction. A partial-fraction expansion of $H(z)$ is then of form

$$H(z) = 1 + \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}}$$

- And

$$\rho_1 = (1-0.2z^{-1})\frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}\bigg|_{z=0.2} = 2.75$$

$$\rho_2 = (1+0.6z^{-1})\frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}\bigg|_{z=-0.6} = -1.75$$

- Thus, we have

$$H(z) = 1 + \frac{2.75}{1-0.2z^{-1}} + \frac{-1.75}{1+0.6z^{-1}}$$

- Since it is given that $h[n]$ is causal, the inverse transform of the above is given by

$$h[n] = \delta[n] + 2.75(0.2)^n\mu[n] - 1.75(-0.6)^n\mu[n]$$

Another example

- Find the inverse z-transform of

$$X(z) = \frac{(1+z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})}, |z| > 1$$

- Since both the numerator and denominator are of degree 2, a constant term exists.

$$X(z) = B_0 + \frac{A_1}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{A_2}{(1-z^{-1})}$$

- B_0 can be determined by the fraction of the coefficients of z^{-2}. $B_0 = \frac{1}{\frac{1}{2}} = 2$.

- Therefore, $X(z) = 2 + \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})} = 2 + \frac{A_1}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{A_2}{(1-z^{-1})}$

$$A_1 = \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})} \cdot \left(1-\frac{1}{2}z^{-1}\right)\Bigg|_{z=\frac{1}{2}} = -9$$

$$A_2 = \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})} \cdot (1-z^{-1})\Big|_{z=1} = 8$$

$$X(z) = 2 - \frac{9}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{8}{(1-z^{-1})}$$

- From the ROC, the solution is right-handed. So

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n \mu[n] + 8\mu[n]$$

By Power Series Expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

- We can determine any particular value of the sequence by finding the coefficient of the appropriate power of z^{-1}.

Example: Finite-length Sequence

- Find the inverse z-transform of
$$X(z) = z^2(1 - 0.5z^{-1})(1 + z^{-1})(1 - z^{-1})$$

- By directly expand $X(z)$, we have
$$X(z) = z^2 - 0.5z - 1 + 0.5z^{-1}$$

- Thus,
$$x[n] = \delta[n+2] - 0.5\delta[n+1] - \delta[n] + 0.5\delta[n-1]$$

Example: Rational z-Transform

- If a rational z-transform is expressed as a ratio of polynomials in z^{-1}, the power series expansion can be obtained by long division.
- Consider

$$H(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$$

- The long division of the numerator by the denominator yields

$$H(z) = 1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} - 0.2224z^{-4} + \cdots$$

- Thus, $h[n] = \delta[n] + 1.6\delta[n-1] - 0.52\delta[n-2] + 0.4\delta[n-3] - 0.2224\delta[n-4] + \cdots$

Frequency Response from Transfer Function

- z-transform transfer function

$$H(z) = H_{\mathrm{re}}(z) + jH_{\mathrm{im}}(z) = |H(z)|e^{j\arg H(z)}$$

where $\arg H(z) = \tan^{-1}\frac{H_{\mathrm{im}}(z)}{H_{\mathrm{re}}(z)}$

- If the ROC of $H(z)$ includes the unit circle, the frequency response $H(e^{j\omega})$ of the LTI digital system can be obtained by evaluating $H(z)$ on the unit circle:

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

- For a real coefficient transfer function

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega})$$
$$= H(z)H(z^{-1})\Big|_{z=e^{j\omega}}$$

Stability Condition in Terms of Pole Locations

- **A stable causal LTI system has all poles inside unit circle**.
  - A causal LTI FIR digital filter with bounded impulse response coefficients is always stable, as all its poles are at the origin in the z-plane.
  - A causal LTI IIR digital filter may or may not be stable.
  - An originally stable IIR filter characterized by infinite precision coefficients and with all poles inside the unit circle may become unstable after implementation due to the unavoidable quantization of all coefficients.

Example

- Analyze the stability of the causal system

$$H(z) = \frac{1}{1 - 1.845z^{-1} + 0.850586z^{-2}}$$

and the system implemented by keeping 2 digits after the decimal points of the coefficients.

A: the poles of the systems are the roots of

$$1 - 1.845z^{-1} + 0.850586z^{-2}$$
$$= z^{-2}(z^2 - 1.845z^1 + 0.850586)$$

We have, $z_p = \frac{1.845 \pm \sqrt{1.845^2 - 4\times 0.850586}}{2} = 0.943$, or 0.902

Both poles are in the unit circle, and the system is stable.

- If the system is implemented by keeping 2 digits after the decimal points of the coefficients, the transfer function becomes

$$H(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

The root of the denominator is

$$z_p = \frac{1.85 \pm \sqrt{1.85^2 - 4 \times 0.85}}{2} = 1, \text{or } 0.85$$

i.e., one pole is on the unit circle. So the system becomes unstable.