Induction

Good Proofs

- correct
- complete
- clear
- brief
- elegant (clever, work of art)
- well organized (using lemma)
- inorder (fact -> proof)

Proof techniques not to use

- Proof by example
- Proof by omission
- Proof by picture
- Too many notations

Question

A	B	C		A	B	C
D	E	F	→	D	E	F
H	G			G	H	

> No squence of legal moves to invert G and H and return all other letter to their original position

Rules: no diagonal moves

Lemma 1

(Lemma is true statement used in proving propositions)

A	B	C		A	B	C
D	E	F	=>	D	E	F
H	G			H		G

- A row move does not change the order of the items

Proof

In a row move, we move an item from cell `i` into an adjacent `i+1` or `i-1` .
Nothing else moves, hence the order of the items is preserved

Lemma 2

A	B	C		A	B	C
D	E		=>	D	E	G
F	H	G		F	H	

- A Column move changes the relative order of precisely 2 pairs of items

Proof

- In a column move, we move an item in cell `i` to a blank spot in cell `i-3` or `i+3` .
- When a item moves 3 positions, it changes relative order with 2 other items

A	B	C		A	B	C
F	D	G	=>	D	E	F
E	H			G	H	

- A pair of letter L1 and L2 is an inversion
- (D, F), (E, F), (F, G) inversions.

Lemma 3

During a move, the number of inversions can only increase by 2 or decrease by 2 or stay the same.

Proof

- Row move: no changes (by lemma 1)
- Column move: 2 pairs change order (by lemma 2)

Case A: Both paires are inorder $\Rightarrow$ +2 inversions

Case B: Both paires are inverted $\Rightarrow$ -2 inversions

Case C: One pair is inverted $\Rightarrow$ inversion stays the same

Corollary

(corollary is a proposition followed from a lemma)

- During a move the parity (fact of being even/odd) of the number of inversions does not change.

Proof

- adding or substracting to does not change the parity.

Lemma 4

- In every state reachable from

A	B	C
D	E	F
H	G	

Proposition

P(n): After any sequence of n moves from the start state,
the parity of number of inversion is odd

- Base Case: n = 0 (no moves are made) (start)
- Inductive Step: $\forall$ n $\geq$ 0, Show P(n) $\Rightarrow$ P(n+1)
  Consider any sequence of n + 1 moves $M_1, M_2, M_3, .., M_n$
- Let the parity after moves $M_1, ..., M_n$ is odd
- By Corollary 1, we know parity of number of inversions does not change during M_{n+1}
- $\therefore$ P(n) $\Rightarrow$ P(n + 1)

> **Conclusion**
>
> - The parity of the number of inversions in desired state is even.
> - By Lemma 4, the desired state cannot be reached from its start state.
>
> | | | | | | | |
> |---|---|---|---|---|---|---|
> | A | B | C | | A | B | C |
> | D | E | F | → | D | E | F |
> | H | G | | | G | H | |

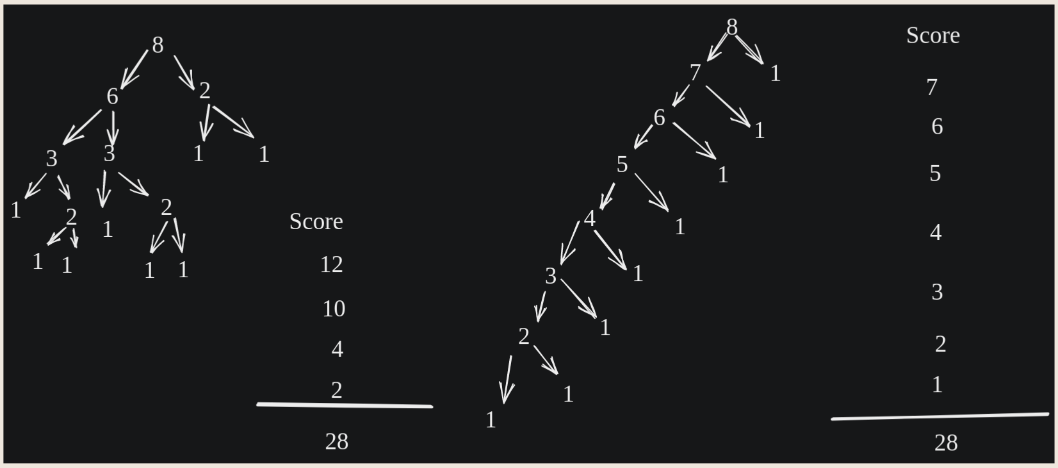
Strong Induction Axiom

> **Important**
>
> - Let P(n) be any predicate, If P(0) is true,
> $\forall$ n $\Rightarrow$ P(0) & P(1) & P(2) & ... & P(n) $\Rightarrow$ P(n + 1) is true, then $\forall$ n, P(n) is true.

- Here it can be assumed that P(0) & P(1) & P(2) & ... & P(n) are all true,
  to prove P(n + 1) is true

Question

All strategies for the n-block game produces the same score S(n).



Solution

Proof by Strong induction

- By Inductive Hypothesis, P(n): n blocks in a stack have the same score, i.e. $n(n+1)/2$
- Base Case: P(1) = 0
- Inductive Step: Assume P(1),P(2), ... , P(n) is true, to prove P(n +
  - $n+1$ is splitted into k and $n+1-k$

$$\begin{aligned} Score &= k(n+1-k) + P(k) + P(n+1-k) \\ &= k(n+1-k) + \frac{k(k-1)}{2} + \frac{(n+1-k)(n-k)}{2} \\ &= kn - k^2 + k + \frac{k^2 - k + n^2 + n - 2kn - k + k^2}{2} \\ \Rightarrow Score &= \frac{n^2+n}{2} = \frac{n(n+1)}{2} \end{aligned}$$

Success