# Induction

# **Proof by Contradiction**



To prove P is true, we assume P is false (i.e.  $\neg$  P is true), then using that hypothesis to derive a false hood or contradiction.

- If ¬ P  $\Rightarrow$  F is true, ∴ P is true

#### **Example**

Proof  $\sqrt{2}$  is irrational.

#### Solution

- Proof by contradiction
- Assume for purpose of contradiction that  $\sqrt{2}$  is rational.

$$\Rightarrow \sqrt{2} = \frac{a}{b}$$

$$\Rightarrow 2 = \frac{a^2}{b^2}$$

$$\Rightarrow 2b^2 = a^2$$

$$\therefore a \text{ is even } (\frac{2}{a})$$

$$\Rightarrow \frac{4}{a^2}$$

$$\Rightarrow \frac{4}{2b^2}$$

$$\Rightarrow \frac{2}{b^2}$$

$$\therefore b \text{ is even}$$

- If a, b are both even,  $\frac{a}{b}$  is not in lowest terms
- .: Contradiction

### Success

# Induction

### Important

Let P(n) be a predicate. If P(0) is true and  $\forall$  n  $\in$  N (P(n)  $\Rightarrow$  P(n + 1)) is true. Then  $\forall$  n  $\in$  N is true. If P(0), P(1)  $\Rightarrow$  P(2), P(2)  $\Rightarrow$  P(3) ..., then P(0), P(1), P(2) ... is true

#### Warning

Induction often does not give answer, and it just only proves or disproves the answer.

$$orall \, n \geq 0, 1+2+3+4+\ldots+n = rac{n(n+1)}{2} = \sum_{i=1}^n i = \sum_{1 \leq i \leq n}^{1 \leq i \leq n} i = \sum_{1 \leq i < n} i$$

#### Solution

Proof by Induction Let P(n) be the proposition,

$$P(n)=\sum_{i=1}^n i=\frac{n(n+1)}{2}$$

Base Case: P(0) is true,

$$\sum_{i=1}^0 i = \frac{0(0+1)}{2} = 0$$

Inductive Step: For  $n \ge 0$ , show  $P(n) \Rightarrow P(n + 1)$  is true.

Assume P(n) is true for purposes of induction. (i.e. assume 1+2+ ... +n = n (n + 1) / 2) and need to show 1+2+...+(n+1)=(n+1)(n+2)/2.

$$1+2+\ldots+(n+1) = \frac{(n+1)(n+2)}{2}$$

$$\Rightarrow \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

$$\Rightarrow \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2}$$

 $\therefore \forall \ n \ge 0$ , P(n)  $\Rightarrow$  P(n + 1) (proved)



$$\forall \ n \in N, \ \frac{3}{n^3-n}$$



Proof by Induction, Let P(n) be the proposition,

$$P(n) = \frac{3}{n^3 - n}$$

Base Condition: P(0),

$$P(0) = \frac{3}{0 - 0} = 0$$

Induction Step,  $\forall$   $n \ge 0$ , show P(n)  $\Rightarrow$  P(n + 1) is true Assume P(n) is true, i.e. P(n) =  $\frac{3}{n^3-n}$  and need to show,

$$P(n+1) = \frac{3}{(n+1)^3 - (n+1)}$$

$$= \frac{3}{n^3 + 3n^2 + 3n + 1 - (n-1)}$$

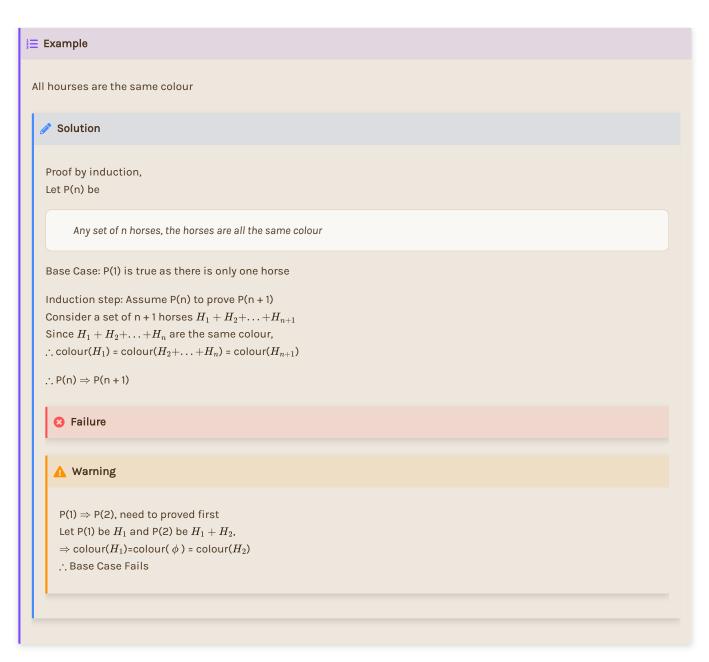
$$= \frac{3}{n^3 + 3n^2 + 2n}$$

$$= \frac{3}{n^3 - n + 3n^2 + 3n}$$

 $\therefore \forall \ n \geq 0, P(n) \Rightarrow P(n+1) \mbox{ (proved)}$ 



## **False Proof**



**Using Induction to find solution** 



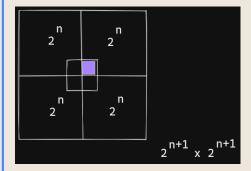
 $\forall$  n, there is way to tile a  $Z^n * Z^n$  region with a center square missing, using L shaped tiles (for bill).



Base Case: P(1) is true

Inductive Step: For  $n \ge 0$ , assume P(n) to verify, that P(n + 1) is true.

Consider a  $2^{n+1} * 2^{n+1}$ 



#### 🗯 Fail

### **b** Hint

When any induction hypothesis fails, making the hypothesis more stronger, makes it easier to proof.

 $\forall$  n, there is way to tile a  $Z^n*Z^n$  region with **any** square missing, using L shaped tiles (for bill).

(makes it much easier to proof)