Proof by Contradiction

Important

To prove P is true, we assume P is false (i.e. $\neg$ P is true), then using that hypothesis to derive a false hood or contradiction.

- If $\neg$ P $\Rightarrow$ F is true, $\therefore$ P is true

Example

Proof $\sqrt{2}$ is irrational.

Solution

- Proof by contradiction
- Assume for purpose of contradiction that $\sqrt{2}$ is rational.

$$\begin{aligned} \Rightarrow \sqrt{2} &= \frac{a}{b} \\ \Rightarrow 2 &= \frac{a^2}{b^2} \\ \Rightarrow 2b^2 &= a^2 \\ \therefore a\ is\ even\ (\frac{2}{a}) \\ \Rightarrow \frac{4}{a^2} \\ \Rightarrow \frac{4}{2b^2} \\ \Rightarrow \frac{2}{b^2} \\ \therefore b\ is\ even \end{aligned}$$

- If a, b are both even, $\therefore$ $\frac{a}{b}$ is not in lowest terms
- $\therefore$ Contradiction

Success

Induction

Inference Rule

A rule (modus ponens) says that a proof of P together with proof of P $\Rightarrow$ Q is a proof of Q.

Important

Let P(n) be a predicate. If P(0) is true and $\forall$ n $\in$ N (P(n) $\Rightarrow$ P(n + 1)) is true. Then $\forall$ n $\in$ N is true. If P(0), P(1) $\Rightarrow$ P(2), P(2) $\Rightarrow$ P(3) ..., then P(0), P(1), P(2) ... is true

Warning

Induction often does not give answer, and it just only proves or disproves the answer.

Example

$$\forall\, n \geq 0, 1+2+3+4+...+n = \frac{n(n+1)}{2} = \sum_{i=1}^{n} i = \sum^{1 \leq i \leq n} i = \sum_{1 \leq i \leq n} i$$

Solution

Proof by Induction
Let P(n) be the proposition,

$$P(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Base Case: P(0) is true,

$$\sum_{i=1}^{0} i = \frac{0(0+1)}{2} = 0$$

Inductive Step: For n $\geq$ 0, show P(n) $\Rightarrow$ P(n + 1) is true.
Assume P(n) is true for purposes of induction. (i.e. assume 1+2+ ... +n = n (n + 1) / 2)
and need to show 1+2+ ... + (n + 1) = (n + 1)(n + 2) / 2.

$$\begin{aligned} 1+2+...+(n+1) &= \frac{(n+1)(n+2)}{2} \\ \Rightarrow \frac{n(n+1)}{2} + (n+1) &= \frac{(n+1)(n+2)}{2} \\ \Rightarrow \frac{n^2+3n+2}{2} &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

$\therefore \forall$ n $\geq$ 0, P(n) $\Rightarrow$ P(n + 1) (proved)

Success

Example

$$\forall\, n \in N,\ \frac{3}{n^3 - n}$$

Solution

Proof by Induction,
Let P(n) be the proposition,

$$P(n) = \frac{3}{n^3 - n}$$

Base Condition: P(0),

$$P(0) = \frac{3}{0 - 0} = 0$$

Induction Step, $\forall$ n $\geq$ 0, show P(n) $\Rightarrow$ P(n + 1) is true
Assume P(n) is true, i.e. P(n) = $\frac{3}{n^3 - n}$
and need to show,

$$\begin{aligned} P(n+1) &= \frac{3}{(n+1)^3 - (n+1)} \\ &= \frac{3}{n^3 + 3n^2 + 3n + 1 - (n-1)} \\ &= \frac{3}{n^3 + 3n^2 + 2n} \\ &= \frac{3}{n^3 - n + 3n^2 + 3n} \end{aligned}$$

$\therefore \forall$ n $\geq$ 0, P(n) $\Rightarrow$ P(n + 1) (proved)

Success

False Proof

Example

All hourses are the same colour

Solution

Proof by induction,
Let P(n) be

> *Any set of n horses, the horses are all the same colour*

Base Case: P(1) is true as there is only one horse

Induction step: Assume P(n) to prove P(n + 1)
Consider a set of n + 1 horses $H_1 + H_2 + ... + H_{n+1}$
Since $H_1 + H_2 + ... + H_n$ are the same colour,
$\therefore$ colour(H_1) = colour($H_2 + ... + H_n$) = colour(H_{n+1})

$\therefore$ P(n) $\Rightarrow$ P(n + 1)

Failure

Warning

P(1) $\Rightarrow$ P(2), need to proved first
Let P(1) be H_1 and P(2) be $H_1 + H_2$,
$\Rightarrow$ colour(H_1)=colour(ϕ) = colour(H_2)
$\therefore$ Base Case Fails

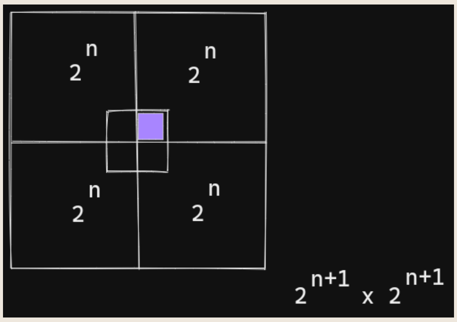
Using Induction to find solution

Example

$\forall$ n, there is way to tile a $Z^n * Z^n$ region with a center square missing, using L shaped tiles (for bill).

Solution

Base Case: P(1) is true
Inductive Step: For n $\geq$ 0, assume P(n) to verify, that P(n + 1) is true.
Consider a $2^{n+1} * 2^{n+1}$



Fail

Hint

When any induction hypothesis fails, making the hypothesis more stronger, makes it easier to proof.

$\forall$ n, there is way to tile a $Z^n * Z^n$ region with **any** square missing, using L shaped tiles (for bill).

(makes it much easier to proof)