# Induction

**Good Proofs**

- correct
- complete
- clear
- brief
- elegant (clever, work of art)
- well organized (using lemma)
- inorder (fact -> proof)

**Proof techniques not to use**

- Proof by example
- Proof by omission
- Proof by picture
- Too many notations

**Question**

| | | | | | | |
|---|---|---|---|---|---|---|
| A | B | C | | A | B | C |
| D | E | F | → | D | E | F |
| H | G | | | G | H | |

> No squence of legal moves to invert G and H and return all other letter to their original position

Rules: no diagonal moves

**Lemma 1**

(Lemma is true statement used in proving propositions)

| | | | | | | |
|---|---|---|---|---|---|---|
| A | B | C | | A | B | C |
| D | E | F | => | D | E | F |
| H | G | | | H | | G |

- A row move does not change the order of the items

**Proof**

In a row move, we move an item from cell `i` into an adjacent `i+1` or `i-1`.
Nothing else moves, hence the order of the items is preserved

**Lemma 2**

| | | | | | | |
|---|---|---|---|---|---|---|
| A | B | C | | A | B | C |
| D | E | | => | D | E | G |
| F | H | G | | F | H | |

- A Column move changes the relative order of precisely 2 pairs of items

**Proof**

- In a column move, we move an item in cell `i` to a blank spot in cell `i-3` or `i+3`.
- When a item moves 3 positions, it changes relative order with 2 other items

| | | | | | | |
|---|---|---|---|---|---|---|
| A | B | C | | A | B | C |
| F | D | G | => | D | E | F |
| E | H | | | G | H | |

- A pair of letter L1 and L2 is an inversion
- (D, F), (E, F), (F, G) inversions.

**Lemma 3**

During a move, the number of inversions can only increase by 2 or decrease by 2 or stay the same.

**Proof**

- Row move: no changes (by lemma 1)
- Column move: 2 pairs change order (by lemma 2)

Case A: Both paires are inorder ⇒ +2 inversions

Case B: Both paires are inverted ⇒ -2 inversions

Case C: One pair is inverted ⇒ inversion stays the same

**Corollary**

(corollary is a proposition followed from a lemma)

- During a move the parity (fact of being even/odd) of the number of inversions does not change.

**Proof**

- adding or substracting to does not change the parity.

**Lemma 4**

- In every state reachable from

| | | |
|---|---|---|
| A | B | C |
| D | E | F |
| H | G | |

**Proposition**

P(n): After any sequence of n moves from the start state,
the parity of number of inversion is odd

- Base Case: n = 0 (no moves are made) (start)
- Inductive Step: ∀ n ≥ 0, Show P(n) ⇒ P(n+1)
  Consider any sequence of n + 1 moves $M_1, M_2, M_3, .., M_n$
- Let the parity after moves $M_1, ..., M_n$ is odd
- By Corollary 1, we know parity of number of inversions does not change during $M_{n+1}$
- ∴ P(n) ⇒ P(n + 1)

**Conclusion**

- The parity of the number of inversions in desired state is even.
- By Lemma 4, the desired state cannot be reached from its start state.

| | | | | | | |
|---|---|---|---|---|---|---|
| A | B | C | | A | B | C |
| D | E | F | $\not\rightarrow$ | D | E | F |
| H | G | | | G | H | |

# Strong Induction Axiom

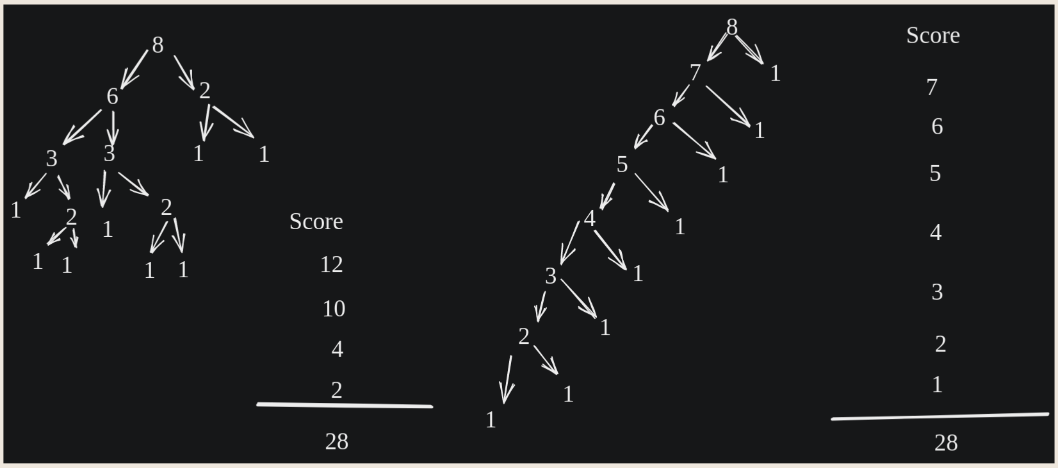
**Important**

- Let P(n) be any predicate, If P(0) is true,
  $\forall$ n $\Rightarrow$ P(0) & P(1) & P(2) & ... & P(n) $\Rightarrow$ P(n + 1) is true, then $\forall$ n, P(n) is true.

- Here it can be assumed that P(0) & P(1) & P(2) & ... & P(n) are all true,
  to prove P(n + 1) is true

**Question**

All strategies for the n-block game produces the same score S(n).




**Solution**

Proof by Strong induction

- By Inductive Hypothesis, P(n): n blocks in a stack have the same score, i.e. $n(n+1)/2$
- Base Case: P(1) = 0
- Inductive Step: Assume P(1),P(2), ... , P(n) is true, to prove P(n +
  - $n+1$ is splitted into $k$ and $n+1-k$

$$
\begin{aligned}
Score &= k(n+1-k) + P(k) + P(n+1-k) \\
&= k(n+1-k) + \frac{k(k-1)}{2} + \frac{(n+1-k)(n-k)}{2} \\
&= kn - k^2 + k + \frac{k^2 - k + n^2 + n - 2kn - k + k^2}{2} \\
&\Rightarrow Score = \frac{n^2+n}{2} = \frac{n(n+1)}{2}
\end{aligned}
$$

**Success**