Bits, Bytes, and Integers – Part 2

Computer Systems
3rd Lecture, 17 Aug, 2023

Borrowed from the Slides of Brian Railing (15-513)

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - **Representation: unsigned and signed; negation and addition**
  - Conversion, casting, extension, truncation
  - Multiplication, division, shifting
- Byte order in memory, pointers, strings

Encoding "Integers"

Unsigned

Signed (twos complement)

Given a bit vector x, w bits long…

$$\mathrm{B2U}(x) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

$$\mathrm{B2T}(x) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

Examples (w = 5)

±16	8	4	2	1
0	1	0	1	0

$$0 + 8 + 0 + 2 + 0 = 10$$

16	8	4	2	1
1	0	1	1	0
-16	8	4	2	1

$$16 + 8 + 0 + 2 + 0 = 26$$

$$-16 + 8 + 0 + 2 + 0 = -10$$

Negation: Complement & Increment

- **Negate through complement and increase**

  `~x + 1 == -x`

- **Why?**
  - `-x + x        == 0` *(by definition)*
  - `~x + x        == 1111…111 == -1`
  - `~x + x + 1 == 0`
  - `(~x+1) + x == 0`
  - `~x+1         == -x`

	x	1 0 0 1 1 1 0 1
+	~x	0 1 1 0 0 0 1 0
	−1	1 1 1 1 1 1 1 1

Example: x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 10010011
y	-15213	C4 93	11000100 10010011

Complement & Increment Examples

$x = 0$

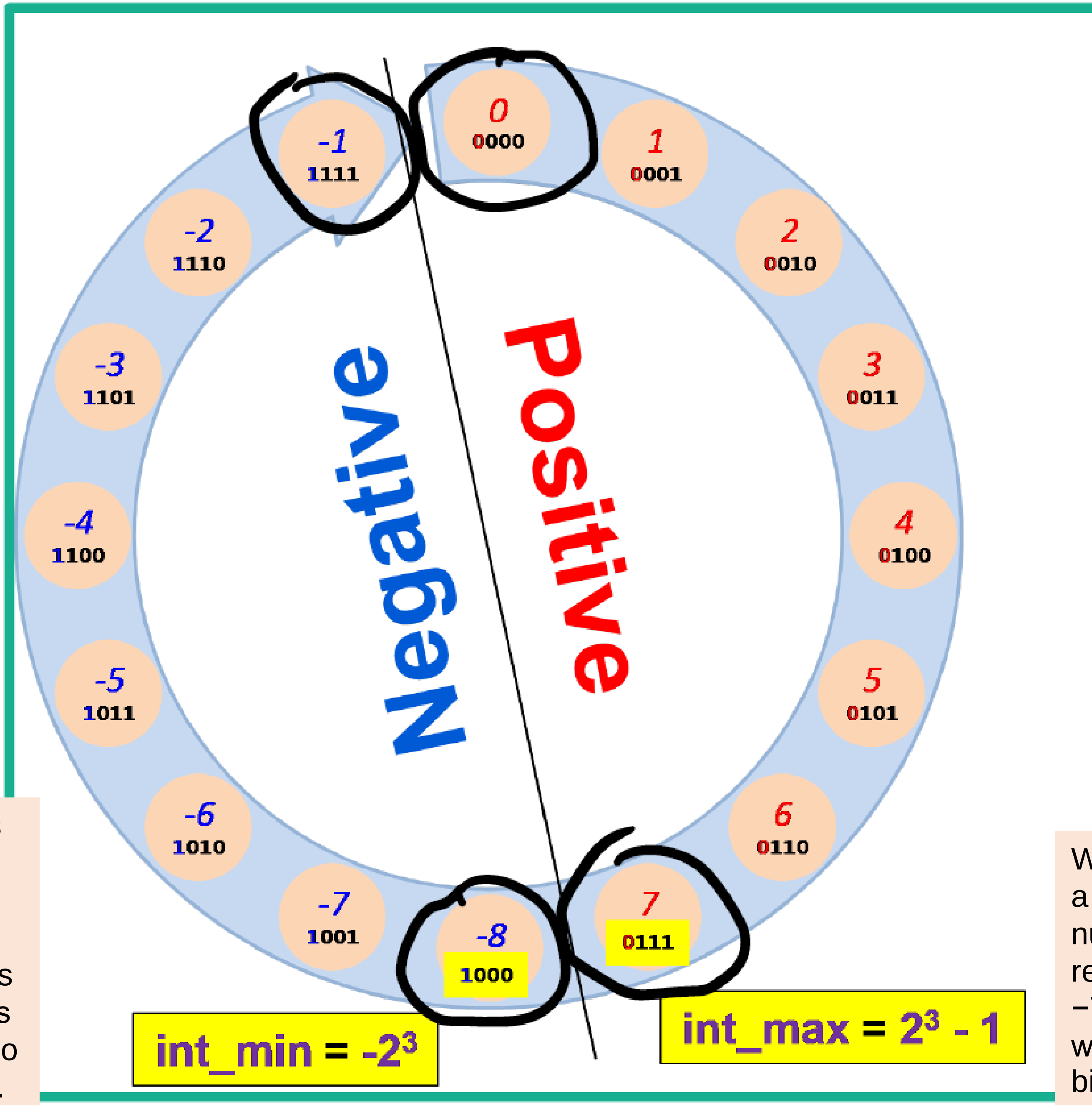
	Decimal	Hex	Binary
0	**0**	00 00	00000000 00000000
~0	**-1**	FF FF	11111111 11111111
~0+1	**0**	00 00	00000000 00000000

$x = T_{\min}$

	Decimal	Hex	Binary
x	**-32768**	80 00	10000000 00000000
~x	**32767**	7F FF	01111111 11111111
~x+1	**-32768**	80 00	10000000 00000000

Oops!
It's still
negative!

Eight negative values:
−1, − 2, …, − 8



Eight *non*-negative values:
0, 1, …, 7

Mathematicians would prefer it if a 4-bit signed number could represent values −8…8, but that's $2^4 + 1$ values, so they won't all fit.

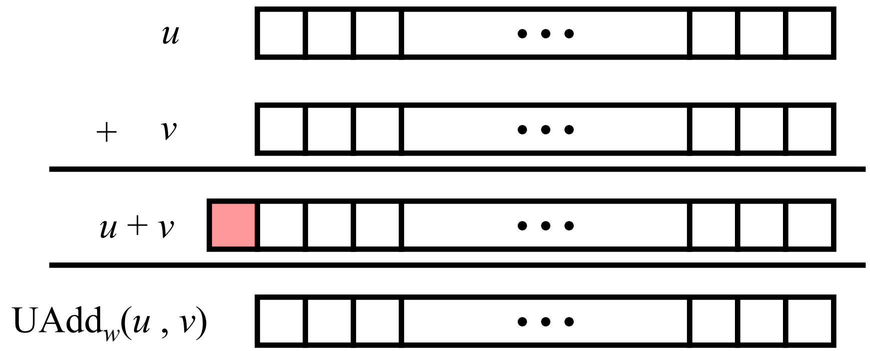
What if we made a 4-bit signed number only represent values −7…7? Then we wouldn't be using bit pattern `1000`…

Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



- Standard Addition Function
  - Ignores carry output
- Implements Modular Arithmetic

$$s \quad = \quad \mathrm{UAdd}_w(u\,,\,v) \quad = \quad u + v \mod 2^w$$

```
unsigned char      1110 1001      E9      233
                 + 1101 0101    + D5    + 213
                 ___________    ____    _____

                 ___________    ____    _____
```

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

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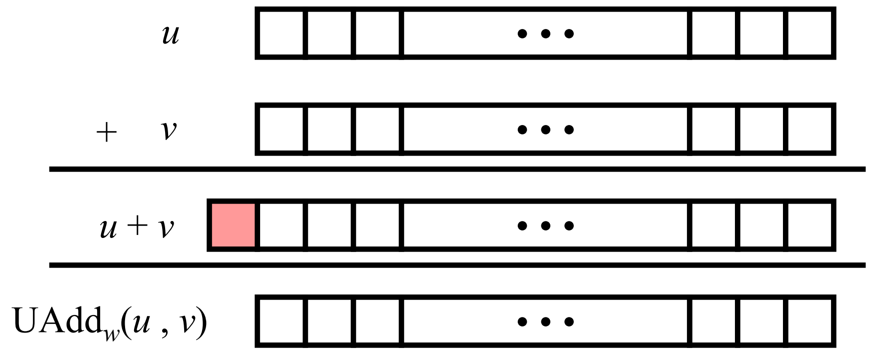

- Standard Addition Function
  - Ignores carry output
- Implements Modular Arithmetic

  $s \;=\; \text{UAdd}_w(u\,,\,v) \;=\; u + v \mod 2^w$

```
unsigned char      1110 1001        E9         233
                 +    1101 0101      + D5       + 213
                  1 1011 1110        1BE         446
                    1011 1110         BE         190
```

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
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A	10	1010
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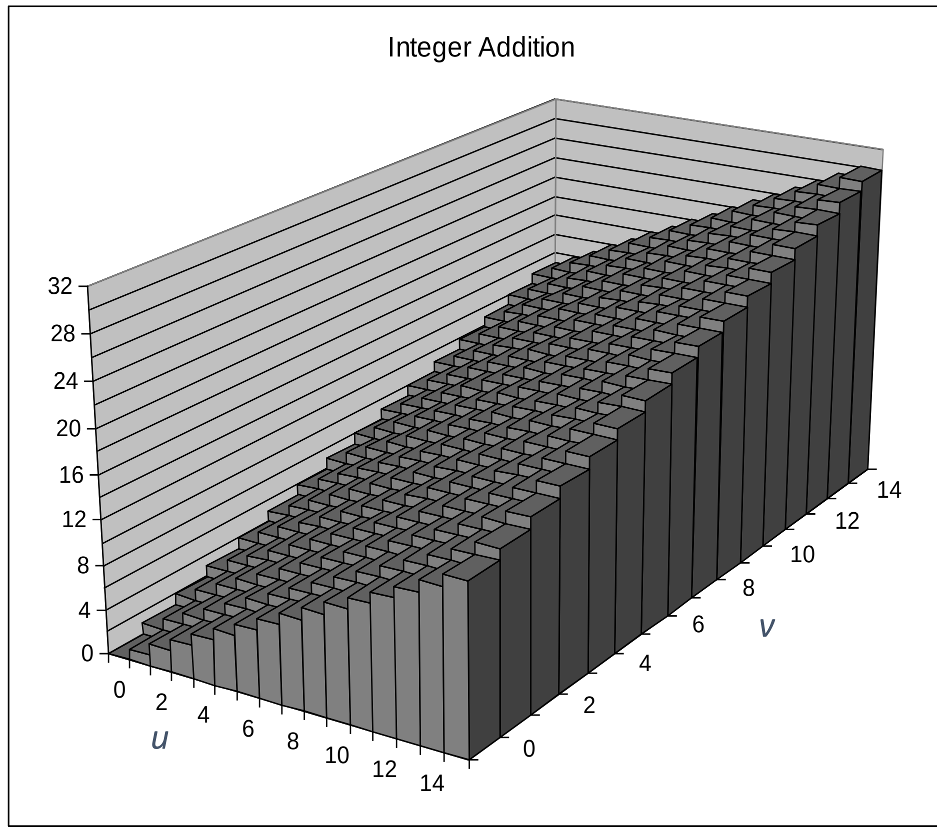
F	15	1111

Visualizing (Mathematical) Integer Addition

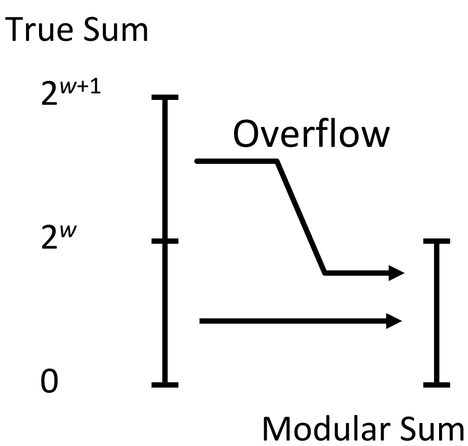
- Integer Addition
  - 4-bit integers *u*, *v*
  - Compute true sum $\text{Add}_4(u , v)$
  - Values increase linearly with *u* and *v*
  - Forms planar surface

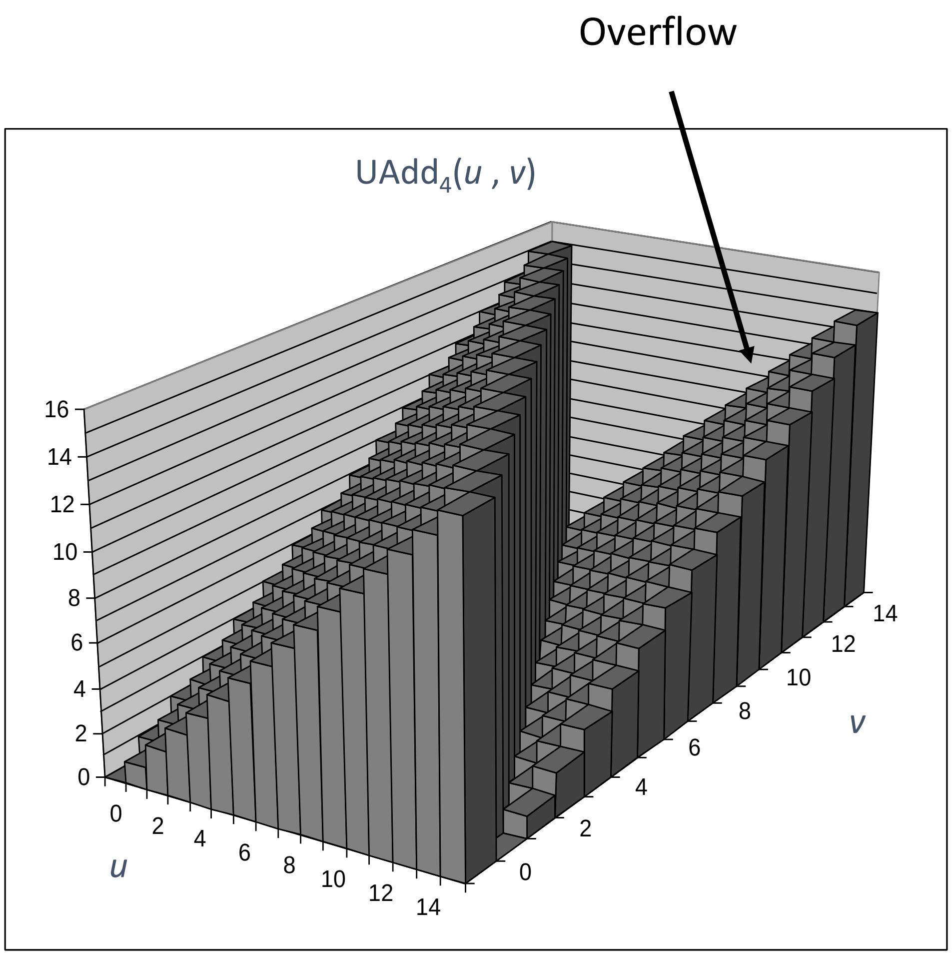
$\text{Add}_4(u , v)$



Visualizing Unsigned Addition

- Wraps Around
  - If true sum ≥ 2^w
  - At most once



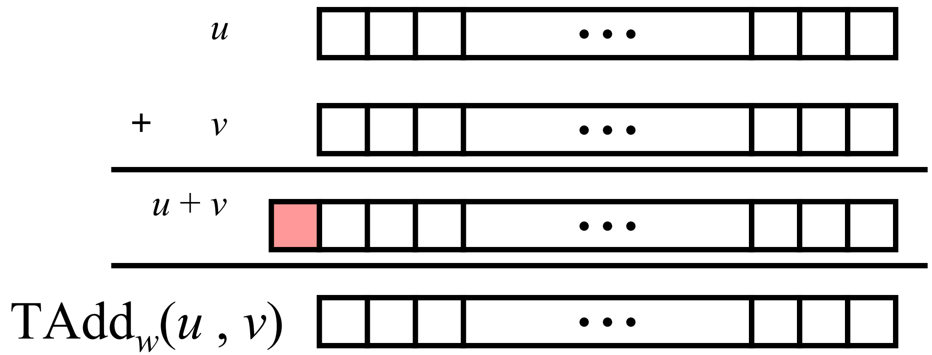


Two's Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



- TAdd and UAdd have Identical Bit-Level Behavior
  - Signed vs. unsigned addition in C:

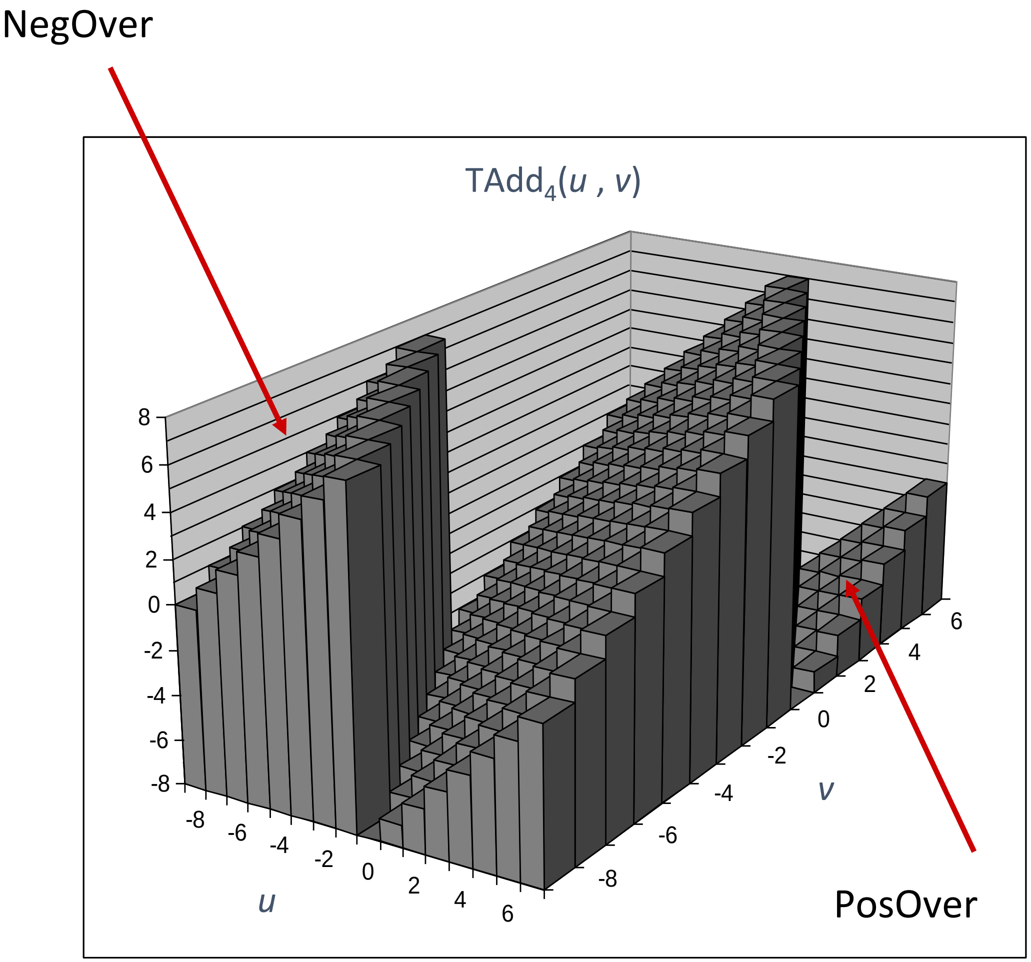
```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

  - Will give `s == t`

```
     1110 1001     E9     -23
+    1101 0101   + D5   + -43
   1 1011 1110    1BE     -66
     1011 1110     BE     -66
```

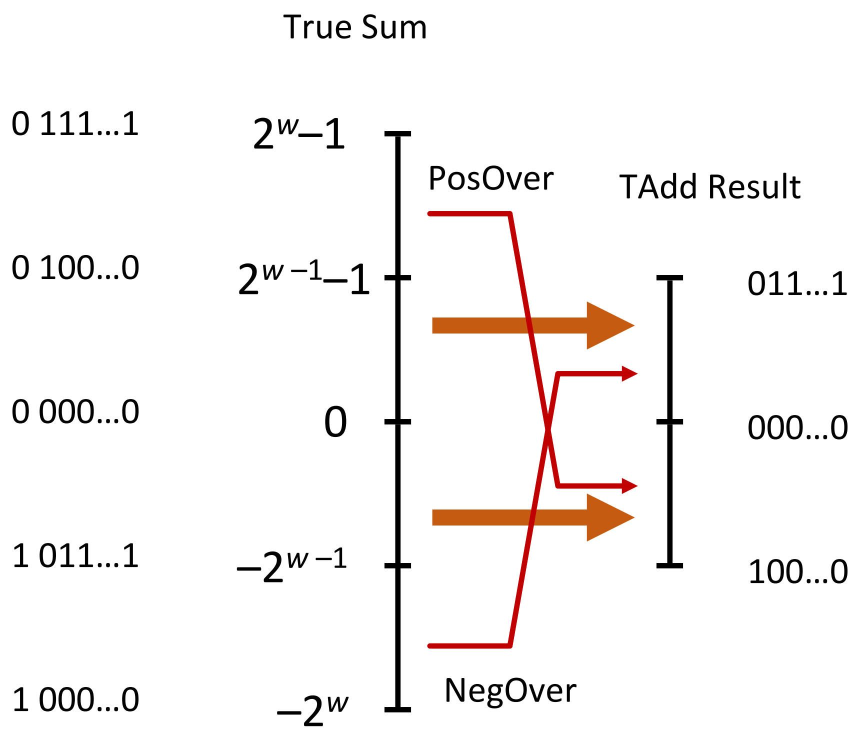
Visualizing 2's Complement Addition

- Values
  - 4-bit two's comp.
  - Range from -8 to +7
- Wraps Around
  - If sum $\geq 2^{w-1}$
    - Becomes negative
    - At most once
  - If sum $< -2^{w-1}$
    - Becomes positive
    - At most once



TAdd Overflow

- Functionality
  - True sum requires w+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer



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- Representing information as bits
- **Bit-level manipulations**
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Boolean Algebra

- **Developed by George Boole in 19th Century**
  - Algebraic representation of logic
    - Encode "True" as 1 and "False" as 0

And

- **A&B = 1 when both A=1 and B=1**

&	0	1
0	0	0
1	0	1

Or

- **A|B = 1 when either A=1 or B=1**

|	0	1
0	0	1
1	1	1

Not

- **~A = 1 when A=0**

~	
0	1
1	0

Exclusive-Or (Xor)

- **A^B = 1 when either A=1 or B=1, but not both**

^	0	1
0	0	1
1	1	0

General Boolean Algebras

- **Operate on Bit Vectors**
  - Operations applied bitwise

```
  01101001      01101001      01101001
& 01010101    | 01010101    ^ 01010101    ~ 01010101
  01000001      01111101      00111100      10101010
```

- **All of the Properties of Boolean Algebra Apply**

Example: Representing & Manipulating Sets

Representation

- Width w bit vector represents subsets of {0, …, w–1}
- $a_j = 1$ if $j \in A$

    - 01101001 { 0, 3, 5, 6 }
    - *76543210*

    - 01010101 { 0, 2, 4, 6 }
    - *76543210*

Operations

&	Intersection	01000001	{ 0, 6 }
|	Union	01111101	{ 0, 2, 3, 4, 5, 6 }
^	Symmetric difference	00111100	{ 2, 3, 4, 5 }
~	Complement	10101010	{ 1, 3, 5, 7 }

Bit-Level Operations in C

- **Operations &, |, ~, ^ Available in C**
  - Apply to any “integral” data type
    - long, int, short, char, unsigned
  - View arguments as bit vectors
  - Arguments applied bit-wise
- **Examples (Char data type)**
  - ~0x41 →
  - ~0x00 →
  - 0x69 & 0x55 →
  - 0x69 | 0x55 →

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
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A	10	1010
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Bit-Level Operations in C

- **Operations &, |, ~, ^ Available in C**
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  - Arguments applied bit-wise
- **Examples (Char data type)**
  - ~0x41 → 0xBE
    - $\sim 0100\ 0001_2 \rightarrow 1011\ 1110_2$
  - ~0x00 → 0xFF
    - $\sim 0000\ 0000_2 \rightarrow 1111\ 1111_2$
  - 0x69 & 0x55 → 0x41
    - $0110\ 1001_2\ \&\ 0101\ 0101_2 \rightarrow 0100\ 0001_2$
  - 0x69 | 0x55 → 0x7D
    - $0110\ 1001_2\ |\ 0101\ 0101_2 \rightarrow 0111\ 1101_2$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Contrast: Logic Operations in C

- **Contrast to Bi… vel Operators**
  - **Logic Operatio… | !**
    - View 0 as "Fa…
    - Any…
    - Alw…
    - Ear…
- **Examp…**
  - !0x41
  - !0x00 → 0x01
  - !!0x41→ 0x01
  - 0x69 && 0x55 → 0x01
  - 0x69 || 0x55 → 0x01
  - p && *p (avoids null pointer access)

Watch out for && vs. & (and || vs. |)… one of the more common oopsies in C programming

Logical versus Bitwise

x	!x	!!x	!!x == x
-1	0	1	No
0	1	0	Yes
1	0	1	Yes
2	0	1	No

x	~x	~~x	~~x == x
-1	0	-1	Yes
0	-1	0	Yes
1	-2	1	Yes
2	-3	2	Yes

`!!x != x`

`~~x == x`

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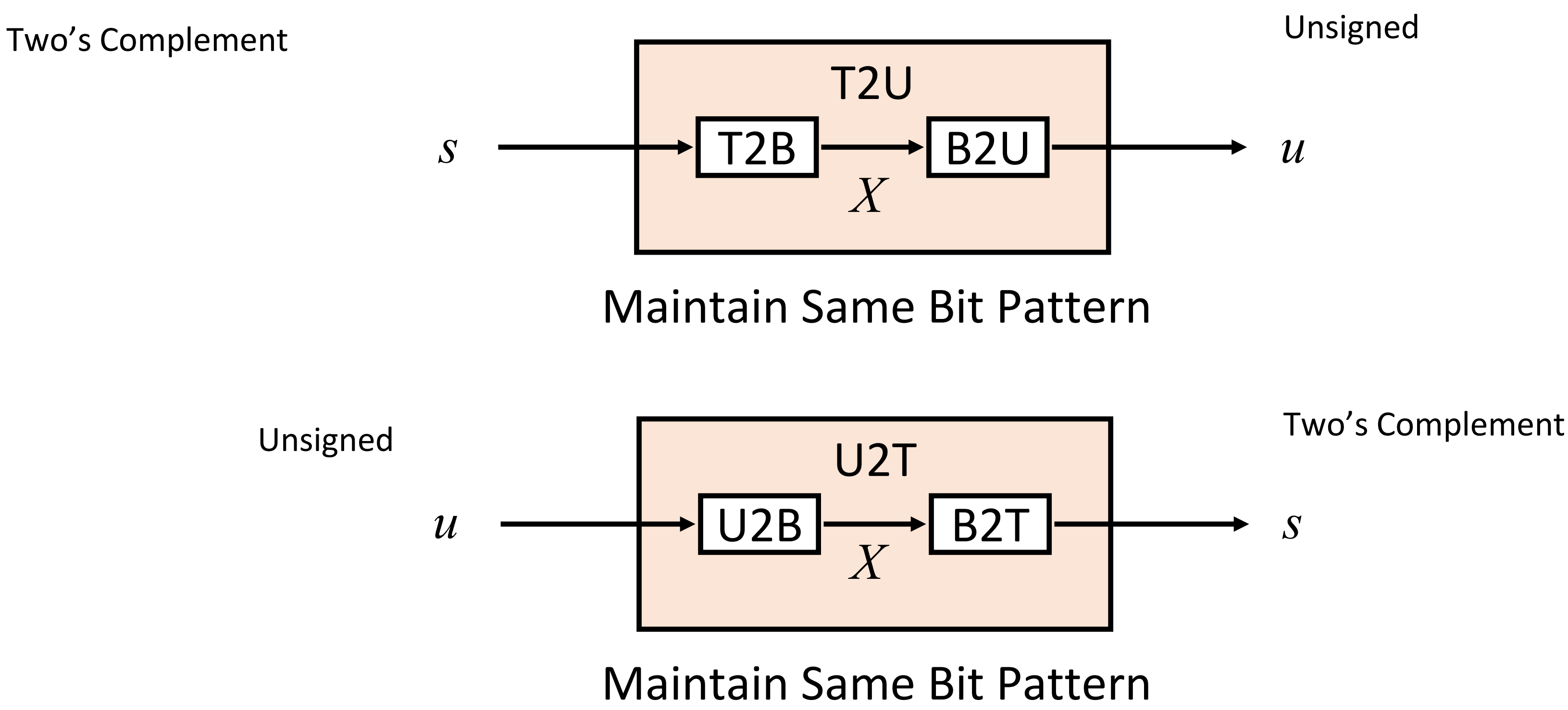
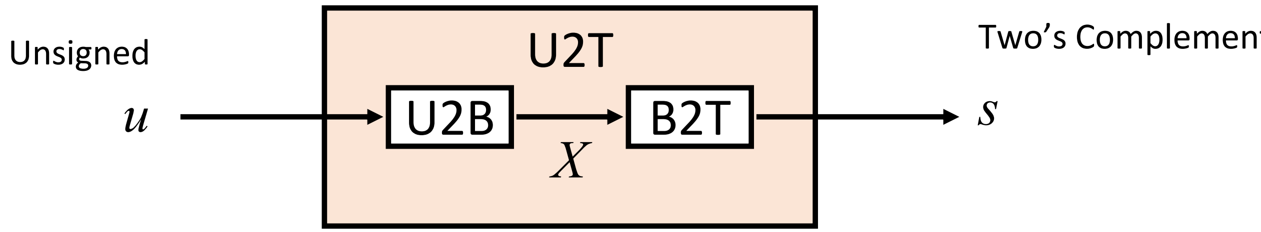
Mapping Between Signed & Unsigned

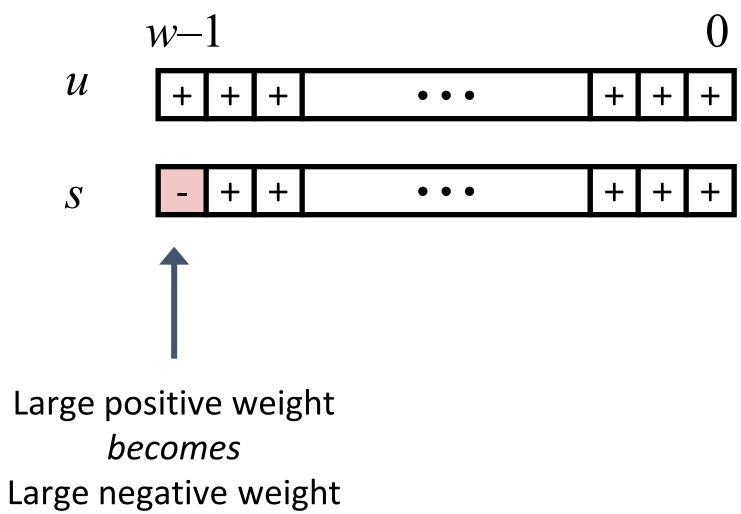


- Mappings between unsigned and two's complement numbers:
  Keep bit representations and reinterpret

Relation between Signed & Unsigned



Maintain Same Bit Pattern

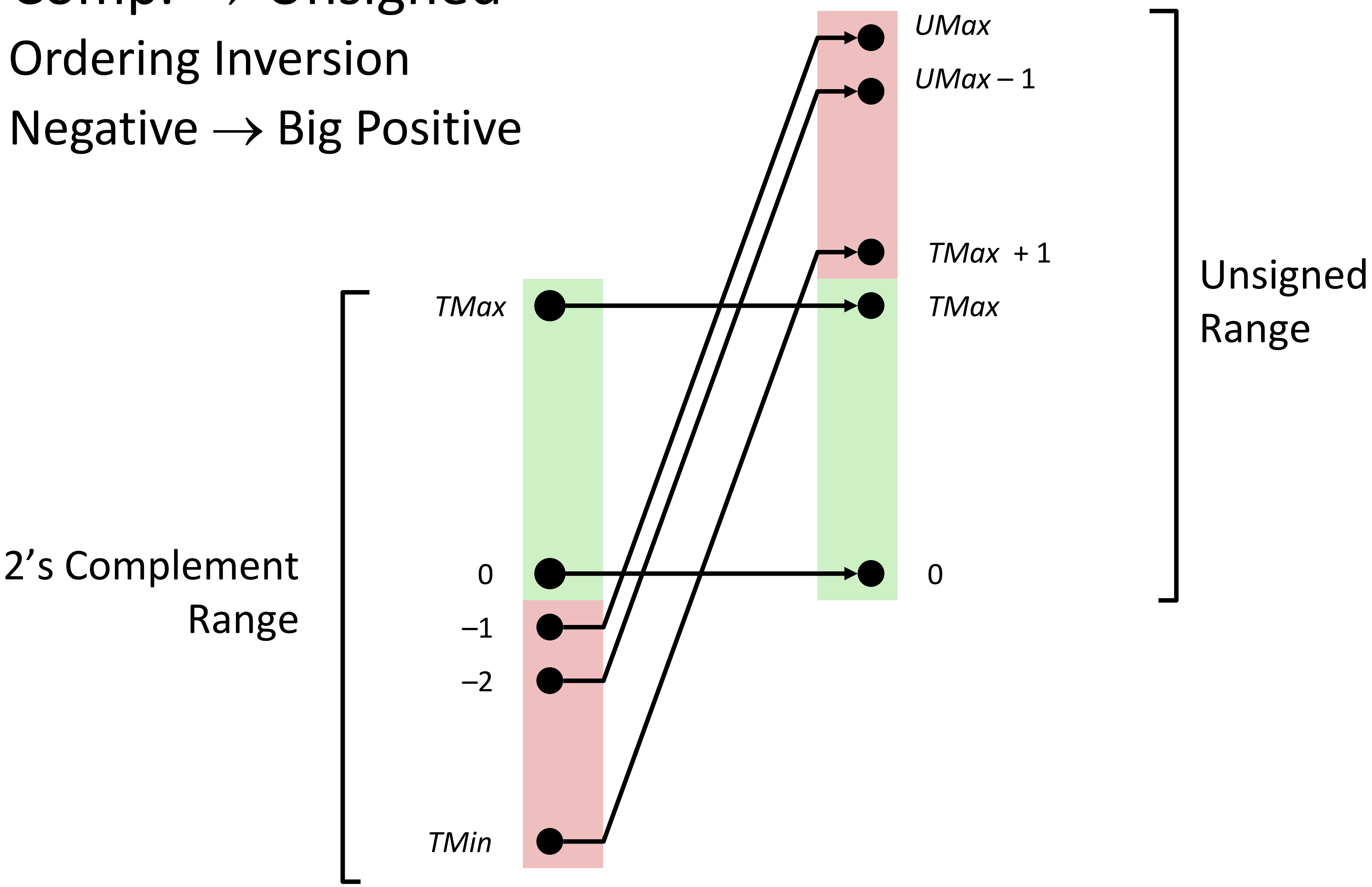


Mapping Signed $\leftrightarrow$ Unsigned

Bits	Signed		Unsigned
0000	0		0
0001	1		1
0010	2		2
0011	3		3
0100	4	$\overset{=}{\longleftrightarrow}$	4
0101	5		5
0110	6		6
0111	7		7
1000	−8		8
1001	−7		9
1010	−6		10
1011	−5	$\overset{\pm 16}{\longleftrightarrow}$	11
1100	−4		12
1101	−3		13
1110	−2		14
1111	−1		15

Conversion Visualized

- 2’s Comp. $\rightarrow$ Unsigned
  - Ordering Inversion
  - Negative $\rightarrow$ Big Positive



Signed vs. Unsigned in C

- Constants
  - By default are considered to be signed integers
  - Unsigned if have "U" as suffix

    ```
    0U, 4294967259U
    ```

- Casting
  - Explicit casting between signed & unsigned same as U2T and T2U

    ```
    int tx, ty;
    unsigned ux, uy;
    tx = (int) ux;
    uy = (unsigned) ty;
    ```

  - Implicit casting also occurs via assignments and procedure calls

    ```
    tx = ux;                int fun(unsigned u);
    uy = ty;                uy = fun(tx);
    ```

Casting Surprises

- Expression Evaluation
  - If there is a mix of unsigned and signed in single expression, ***signed values implicitly cast to unsigned***
  - Including comparison operations **<, >, ==, <=, >=**
  - Examples:

Constant 1	Constant 2	Relation	Evaluation
`0`	`0U`	`==`	Unsigned
`-1`	`0`	`<`	Signed
`-1`	`0U`	`>`	Unsigned
`INT_MAX`	`INT_MIN`	`>`	Signed
`(unsigned)INT_MAX`	`INT_MIN`	`<`	Unsigned
`-1`	`-2`	`>`	Signed
`(unsigned)-1`	`-2`	`>`	Unsigned
`INT_MAX`	`((unsigned)INT_MAX) + 1`	`<`	Unsigned

Summary
Casting Signed ↔ Unsigned: Basic Rules

- Bit pattern is maintained
- But reinterpreted
- Can have unexpected effects: adding or subtracting 2^w
- Expression containing signed and unsigned int
  - `int` is cast to `unsigned`!!

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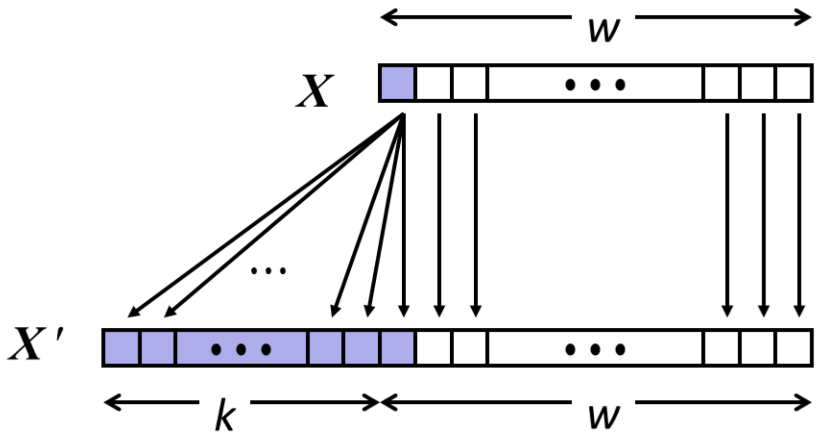
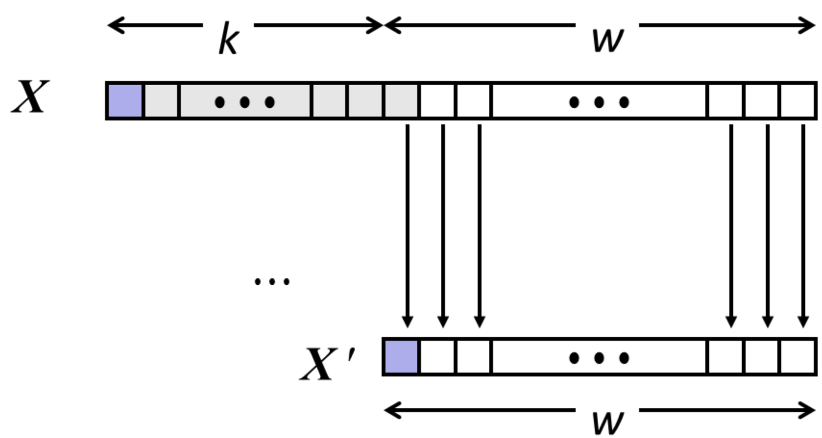
Sign Extension and Truncation

- **Sign Extension**



Make k copies of sign bit

- **Truncation**



Chop off k highest bits

Sign Extension: Simple Example

Positive number

	-16	8	4	2	1
10 =	0	1	0	1	0

	-32	16	8	4	2	1
10 =	0	0	1	0	1	0

Negative number

	-16	8	4	2	1
−10 =	1	0	1	1	0

	-32	16	8	4	2	1
−10 =	1	1	0	1	1	0

Truncation: Simple Example

No sign change

	-16	8	4	2	1
2 =	0	0	0	1	0

	-8	4	2	1
2 =	0	0	1	0

2 mod 16 = 2

	-16	8	4	2	1
-6 =	1	1	0	1	0

	-8	4	2	1
-6 =	1	0	1	0

-6 mod 16 = 26U mod 16 = 10U = -6

Sign change

	-16	8	4	2	1
10 =	0	1	0	1	0

	-8	4	2	1
-6 =	1	0	1	0

10 mod 16 = 10U mod 16 = 10U = -6

	-16	8	4	2	1
-10 =	1	0	1	1	0

	-8	4	2	1
6 =	0	1	1	0

-10 mod 16 = 22U mod 16 = 6U = 6

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- Representing information as bits
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed; negation
  - Conversion, casting
  - Extension, truncation, shifting
  - **Addition, multiplication**
- Representations in memory, pointers, strings

Shifting

- Left Shift: `x << y`
  - Shift bit-vector `x` left `y` positions
  - Throw away extra bits on left
  - Fill with 0's on right
  - Equivalent to multiplying by 2^y
- Right Shift: `x >> y`
  - Shift bit-vector `x` right `y` positions
  - Throw away extra bits on right
  - Two kinds:
    - "Logical": Fill with 0's on left
    - "Arithmetic": Replicate most significant bit on left
  - *Almost* equivalent to dividing by 2^y
- Undefined Behavior (in C)
  - Shift amount < 0 or ≥ word size

Argument x	**01100010**
<< 3	00010000
Logical >> 2	00011000
Arithmetic >> 2	00011000

Argument x	**10100010**
<< 3	00010000
Logical >> 2	00101000
Arithmetic >> 2	11101000

Multiplication

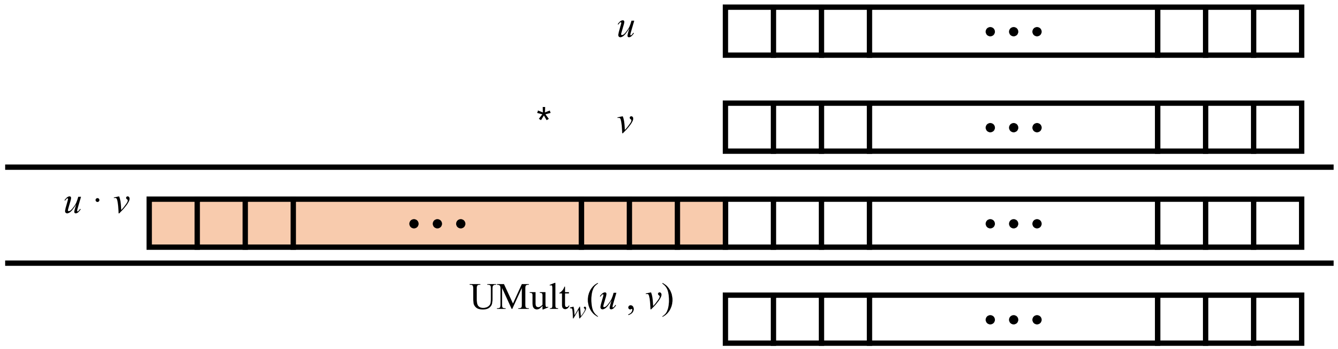
- Goal: Computing Product of *w*-bit numbers *x*, *y*
  - Either signed or unsigned
- But, exact results can be bigger than *w* bits
  - Unsigned: up to 2*w* bits
    - Result range: $0 \le x * y \le (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
  - Two's complement min (negative): Up to 2*w*-1 bits
    - Result range: $x * y \ge (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
  - Two's complement max (positive): Up to 2*w* bits, but only for $(TMin_w)^2$
    - Result range: $x * y \le (-2^{w-1})^2 = 2^{2w-2}$
- So, maintaining exact results…
  - would need to keep expanding word size with each product computed
  - is done in software, if needed
    - e.g., by "arbitrary precision" arithmetic packages

Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



- Standard Multiplication Function
  - Ignores high order *w* bits
- Implements Modular Arithmetic

$$\mathrm{UMult}_w(u, v) = u \cdot v \bmod 2^w$$

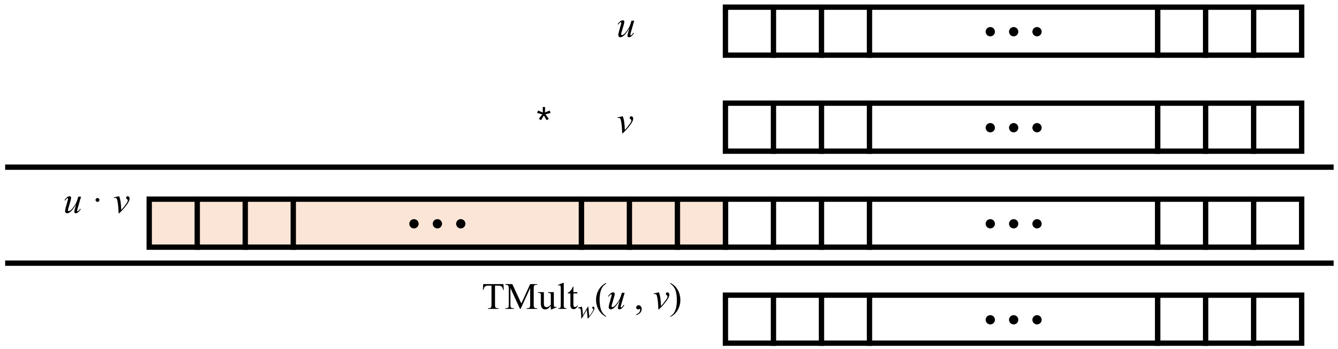
	1110 1001	E9	233
*	1101 0101	* D5	* 213
	1100 0001 1101 1101	C1DD	49629
	1101 1101	DD	221

Signed Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits

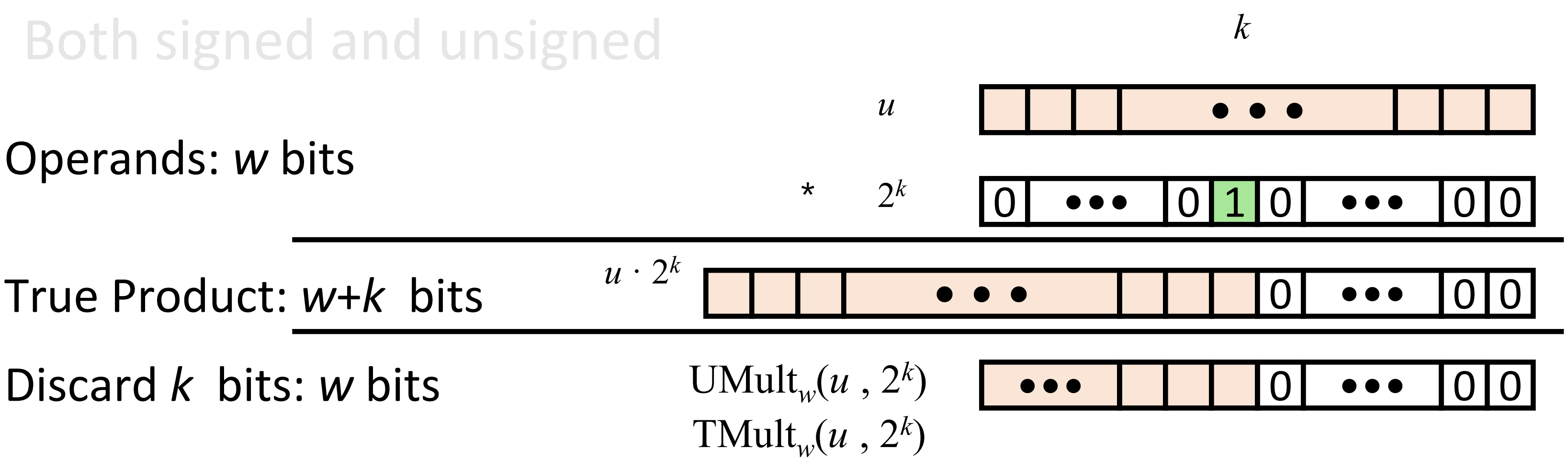


- Standard Multiplication Function
  - Ignores high order *w* bits
  - Some of which are different for signed vs. unsigned multiplication
  - Lower bits are the same

```
                    1110 1001        E9        -23
*                   1101 0101     *  D5     *  -43
 0000 0011 1101 1101               03DD        989
                    1101 1101        DD        -35
```

Power-of-2 Multiply with Shift

- Operation
  - **`u << k`** gives $u * 2^k$
  - Both signed and unsigned



- Examples
  - **`u << 3          ==  u * 8`**
  - **`(u << 5) - (u << 3)  ==   u * 24`**
  - Most machines shift and add faster than multiply
    - Compiler generates this code automatically

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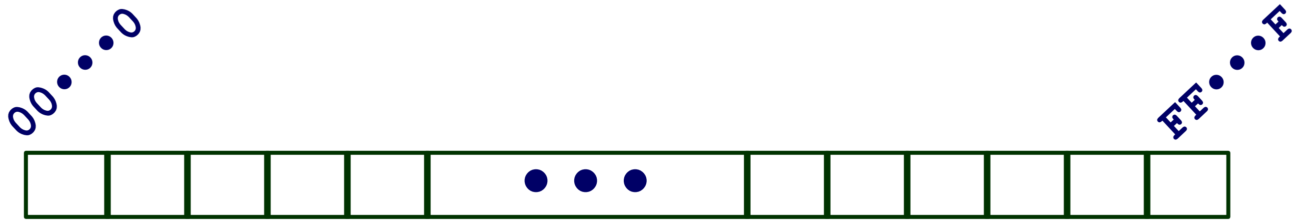
Byte-Oriented Memory Organization



- Programs refer to data by address
  - Imagine all of RAM as an enormous array of bytes
  - An address is an index into that array
    - A pointer variable stores an address
- System provides a private *address space* to each "process"
  - A process is an instance of a program, being executed
  - An address space is one of those enormous arrays of bytes
  - Each program can see only its own code and data within its enormous array
  - We'll come back to this later ("virtual memory" classes)

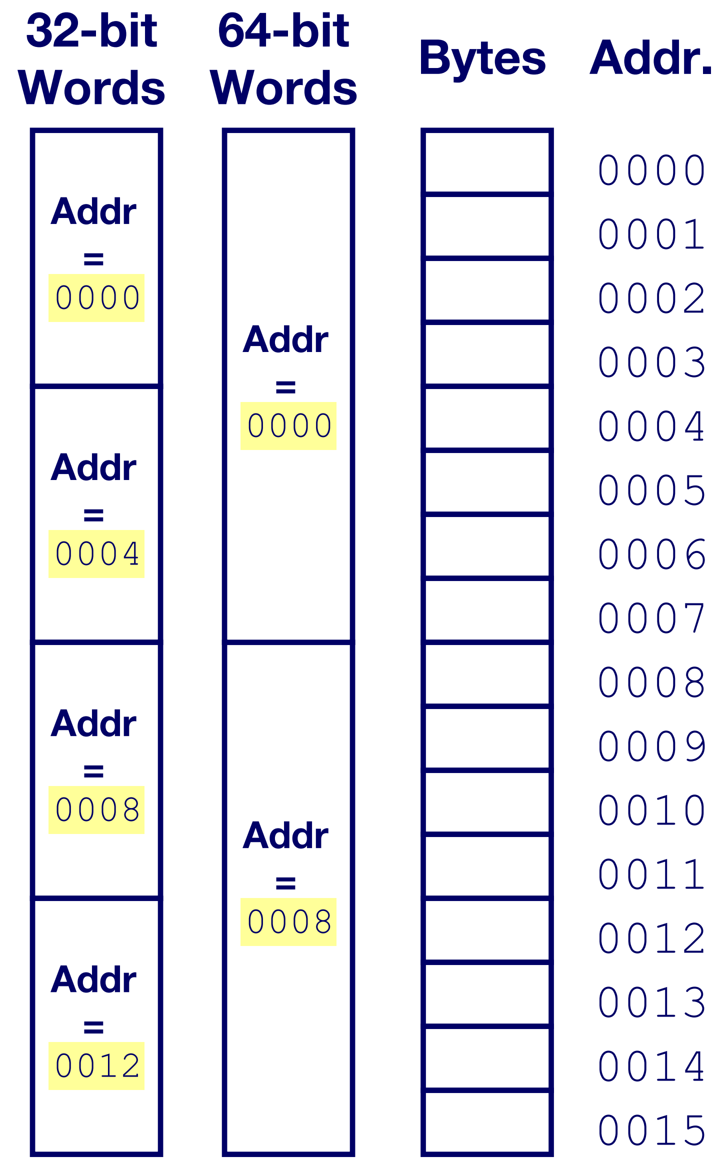
Machine Words

- **Any given computer has a "Word Size"**
  - Nominal size of integer-valued data
    - and of addresses
  - Until recently, most machines used 32 bits (4 bytes) as word size
    - Limits addresses to 4GB (2^{32} bytes)
  - Increasingly, machines have 64-bit word size
    - Potentially, could have 16 EB (exabytes) of addressable memory
    - That's 18.4×10^{18} bytes
  - Machines still support multiple data formats
    - Fractions or multiples of word size
    - Always integral number of bytes

Yes, both of these numbers are correct.
This discrepancy is known as the Great Storage Industry Marketing Lie.
Ask me about it after class if you really want to know.

Addresses *Always* Specify Byte Locations

- Address of a word is address of the first byte in the word
- Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



Example Data Representations

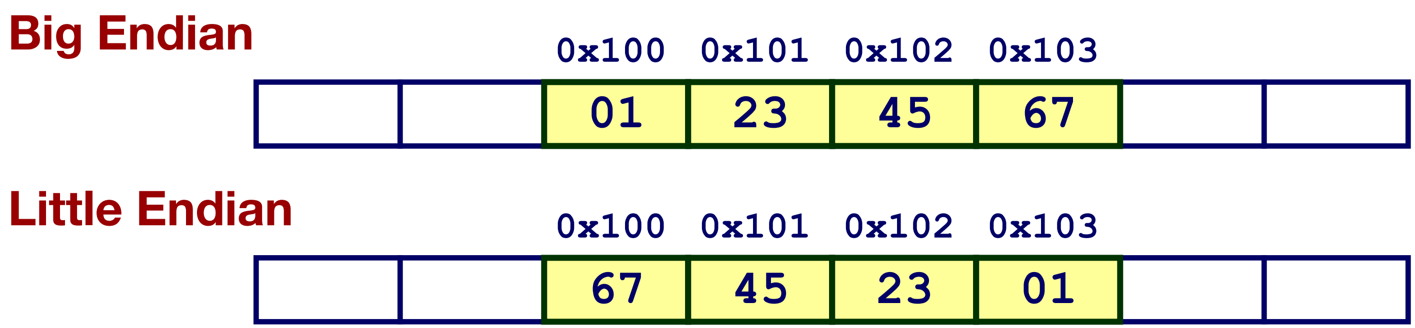
C Data Type	Typical 32-bit	Typical 64-bit	x86-64
`char`	1	1	1
`short`	2	2	2
`int`	4	4	4
`long`	4	8	8
`float`	4	4	4
`double`	8	8	8
pointer	4	8	8

Byte Ordering

- **So, how are the bytes within a multi-byte word ordered in memory?**
- **Conventions**
  - Big Endian: Sun, PPC Mac, *network packet headers*
    - Least significant byte has highest address
  - Little Endian: *x86*, ARM processors running Android, iOS, and Windows
    - Least significant byte has lowest address

Byte Ordering Example

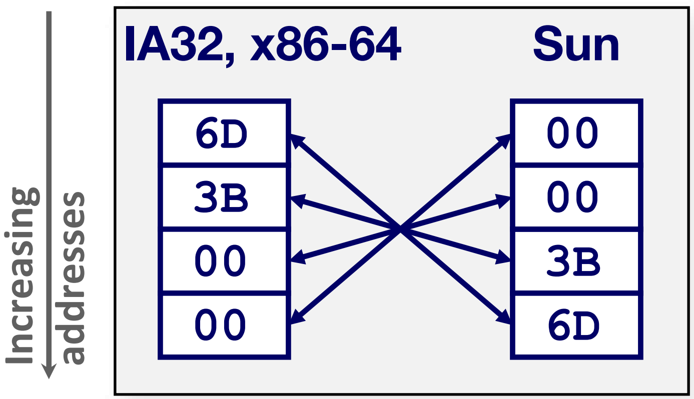
- **Example**
  - Variable x has 4-byte value of 0x01234567
  - Address given by &x is 0x100



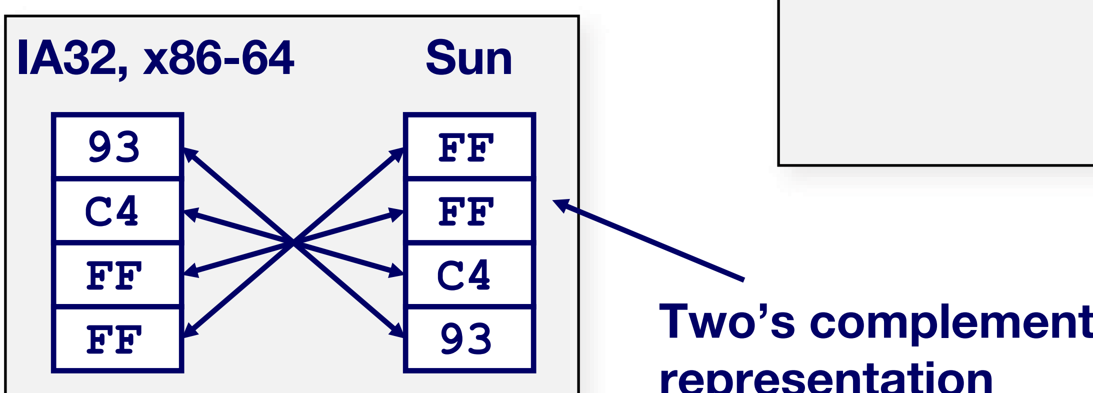
Representing Integers

Decimal:	15213			
Binary:	0011	1011	0110	1101
Hex:	3	B	6	D

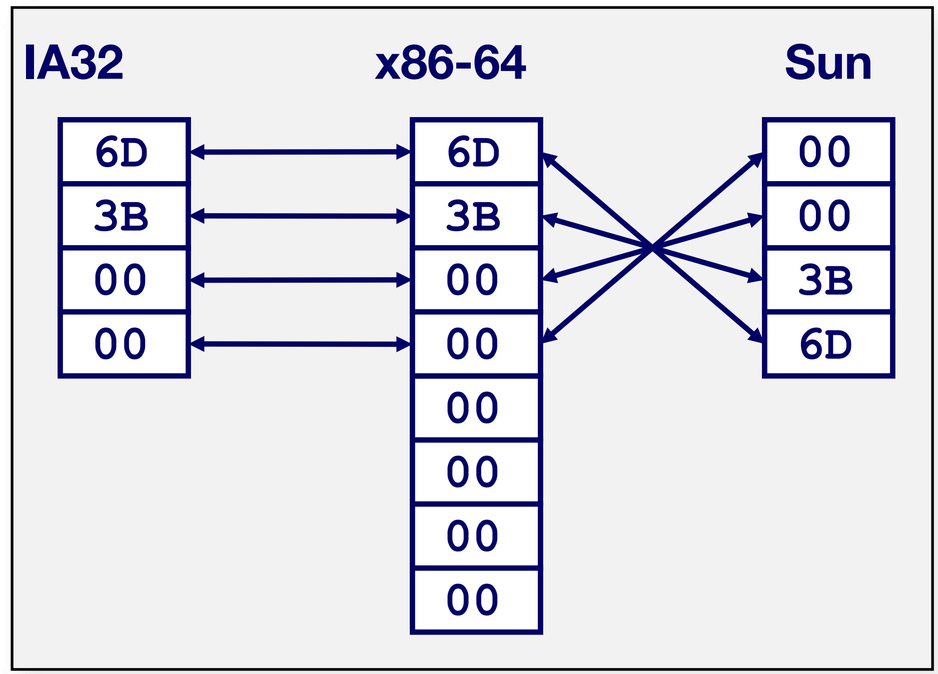
`int A = 15213;`



`int B = -15213;`



`long int C = 15213;`



Examining Data Representations

- **Code to Print Byte Representation of Data**
  - Casting pointer to unsigned char * allows treatment as a byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, size_t len){
  size_t i;
  for (i = 0; i < len; i++)
    printf(”%p\t0x%.2x\n",start+i, start[i]);
  printf("\n");
}
```

Printf directives:

%p: Print pointer

%x: Print Hexadecimal

`show_bytes` Execution Example

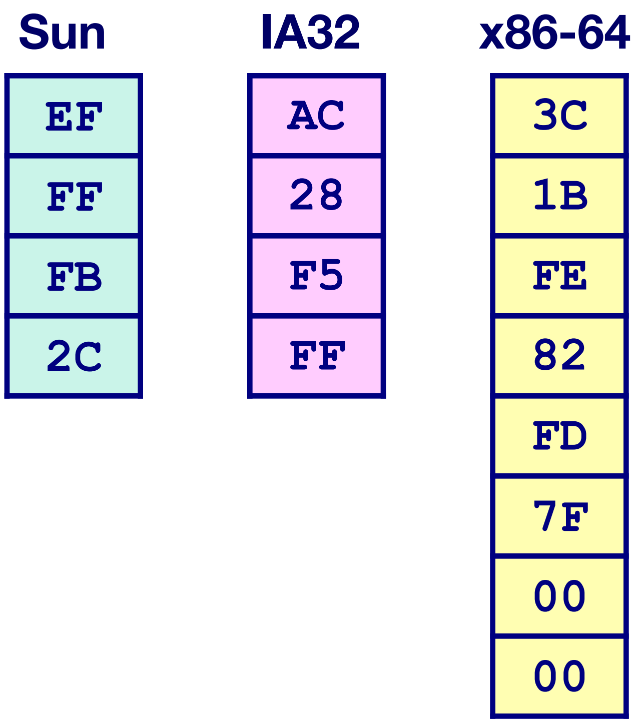
```
int a = 15213;
printf("int a = 15213;\n");
show_bytes((pointer) &a, sizeof(int));
```

Result (Linux x86-64):

```
int a = 15213;
0x7fffb7f71dbc    6d
0x7fffb7f71dbd    3b
0x7fffb7f71dbe    00
0x7fffb7f71dbf    00
```

Representing Pointers

```
int B = -15213;
int *P = &B;
```



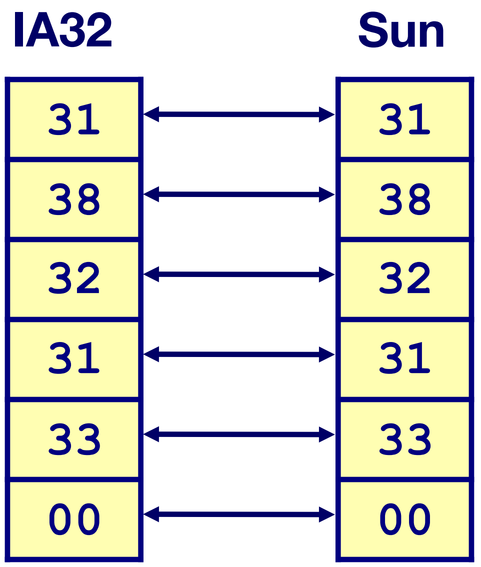
Different compilers & machines assign different locations to objects

Even get different results each time run program

Representing Strings

```
char S[6] = "18213";
```

- **Strings in C**
  - Represented by array of characters
  - Each character encoded in ASCII format
    - Standard 7-bit encoding of character set
    - Character “0” has code 0x30
      - Digit *i* has code 0x30+*i*
  - String should be null-terminated
    - Final character = 0
- **Compatibility**
  - Byte ordering not an issue



Representing x86 machine code

- x86 machine code is a sequence of *bytes*
  - Grouped into variable-length instructions, which look like strings…
  - But they contain embedded little-endian numbers…
- Example Fragment

Address	Instruction Code	Assembly Rendition
`8048365:`	`5b`	`pop %ebx`
`8048366:`	`81 c3 ab 12 00 00`	`add $0x12ab,%ebx`
`804836c:`	`83 bb 28 00 00 00 00`	`cmpl $0x0,0x28(%ebx)`

- Deciphering Numbers
  - Value: `0x12ab`
  - Pad to 32 bits: `0x000012ab`
  - Split into bytes: `00 00 12 ab`
  - Reverse: `ab 12 00 00`