CH08-320201

Algorithms and Data Structures

ADS

Lecture 15

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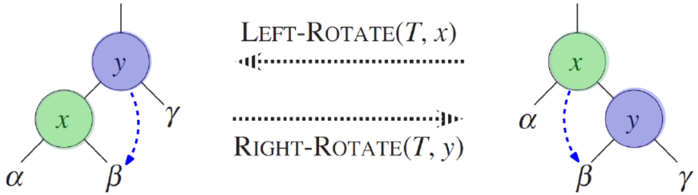
Operations

- Querying
  - Search/Minimum & Maximum/Successor & Predecessor
  - Just as in normal BST
  - $O(\lg n)$
- Modifying
  - Tree-Insert/Tree-Delete $\rightarrow O(\lg n)$
  - But, need to guarantee red-black tree properties:
    - must change color of some nodes
    - change pointer structure through rotation

Rotations (1)

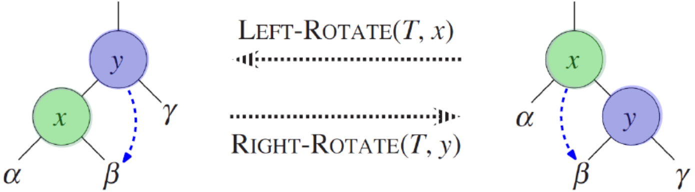
- *Right-Rotate*(T, y):
  - node y becomes right child of its left child x.
  - new left child of y is former right child of x.
- *Left-Rotate*(T, x):
  - node x becomes left child of its right child y.
  - new right child of x is former left child of y.



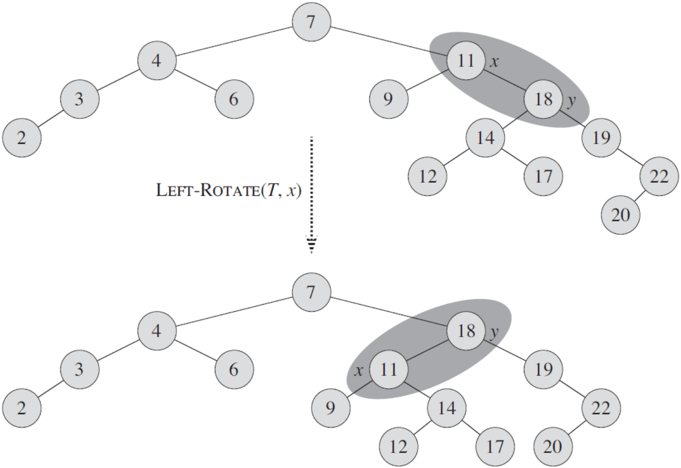
Rotations (2)

BST property is preserved:

- (left): $key(\alpha) \leq x.key \leq key(\beta) \leq y.key \leq key(\gamma)$
- (right): $key(\alpha) \leq x.key \leq key(\beta) \leq y.key \leq key(\gamma)$



Rotation: Example



Rotation Pseudocode

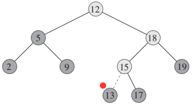
LEFT-ROTATE(T, x)

```
y = x.right                // set y
x.right = y.left           // turn y's left subtree into x's right subtree
if y.left ≠ T.nil
    y.left.p = x
y.p = x.p                  // link x's parent to y
if x.p == T.nil
    T.root = y
elseif x == x.p.left
    x.p.left = y
else x.p.right = y
y.left = x                 // put x on y's left
x.p = y
```

Time complexity: $O(1)$

Insertion

```
TREE-INSERT(T, z)
y = NIL
x = T.root
while x ≠ NIL
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.p = y
if y == NIL
    T.root = z
elseif z.key < y.key
    y.left = z
else y.right = z
```




```
RB-INSERT(T, z)
y = T.nil
x = T.root
while x ≠ T.nil
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.p = y
if y == T.nil
    T.root = z
elseif z.key < y.key
    y.left = z
else y.right = z
z.left = T.nil
z.right = T.nil
z.color = RED
RB-INSERT-FIXUP(T, z)
```

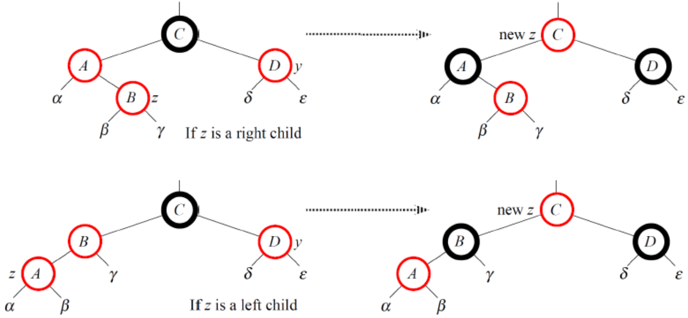
Fixing Red-Black Tree Properties

- We are inserting a **red** node to a valid red-black tree.
- Which properties may be violated?
  1. Duh: Cannot be violated. ✓
  2. RooB: Violated if inserted node is root. ✗
  3. LeaB: Inserted node is not a leaf, i.e., no violation. ✓
  4. BredB: Violated if parent of inserted node is red. ✗
  5. BH: Not affected by red nodes, i.e., no violation. ✓

Fixing BredB

- BredB for node z is violated, if $z.p$ is red.
- Then, $z.p.p$ is black. (BredB property)
- We need to consider different cases depending on the uncle y of z, i.e., the child of $z.p.p$ that is not $z.p$.
- There are 6 cases:
  - $z.p$ is left child of $z.p.p$
    - y is red (Case 1)
    - y is black
      - z is right child of $z.p$ (Case 2)
      - z is left child of $z.p$ (Case 3)
  - $z.p$ is right child of $z.p.p$
    - y is red (symmetric to Case 1)
    - y is black
      - z is right child of $z.p$ (symmetric to Case 2)
      - z is left child of $z.p$ (symmetric to Case 3)

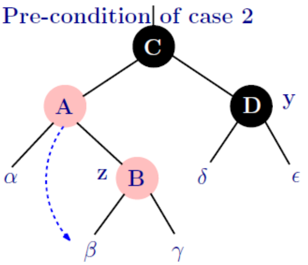
Case 1 (Red Uncle)

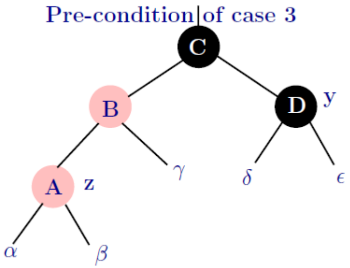



```
if z.p == z.p.p.left
    y = z.p.p.right
    if y.color == RED
        z.p.color = BLACK
        y.color = BLACK              Case 1
        z.p.p.color = RED
        z = z.p.p
```

There is a new problem, if *z.p.p.p* is red.
Algorithm needs to continue with *z.p.p*.

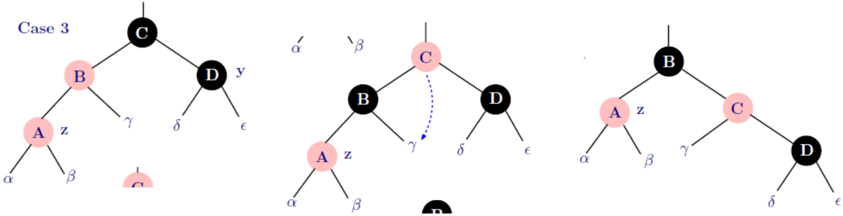
Case 2 (Black Uncle, z Right Child)





else if $z == z.p.right$
$\quad z = z.p$
$\quad$ LEFT-ROTATE(T, z) Case 2

Case 3 (Black Uncle, z Left Child)



$z.p.color = \text{BLACK}$
$z.p.p.color = \text{RED}$
$\text{RIGHT-ROTATE}(T, z.p.p)$

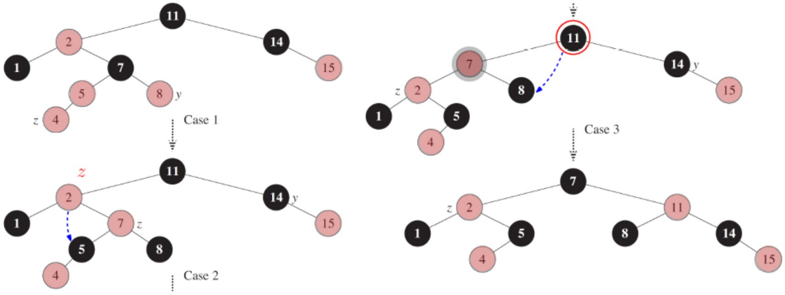
Case 3

Putting It All Together

- We need to put the 3 cases (and the 3 symmetric cases) together.
- Moreover, we need to propagate the considerations upwards (see Case 1).
- Finally, we have to fix RooB.

```
RB-INSERT-FIXUP(T, z)
while z.p.color == RED
    if z.p == z.p.p.left
        y = z.p.p.right
        if y.color == RED
            z.p.color = BLACK
            y.color = BLACK              Case 1
            z.p.p.color = RED
            z = z.p.p
        else if z == z.p.right
                z = z.p                  Case 2
                LEFT-ROTATE(T, z)
            z.p.color = BLACK
            z.p.p.color = RED            Case 3
            RIGHT-ROTATE(T, z.p.p)
    else (same as then clause
            with "right" and "left" exchanged)
T.root.color = BLACK
```

Insert Example

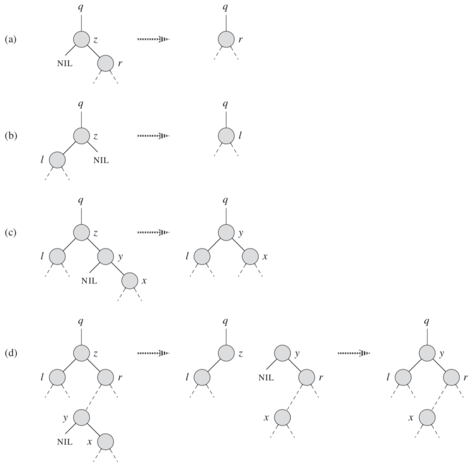


Time Complexity

- In worst case, we have to go all the way from the leaf to the root along the longest path within the tree.
- Hence, running time is $O(h) = O(\lg n)$ for the fixing of the red-black tree properties.
- Overall, running time for insertion is $O(h) = O(\lg n)$.
- Example for building up a red-black tree by iterated node insertion:
  `http://www.youtube.com/watch?v=vDHFF4wjWYU`

Deletion (Remember BST)

```
TREE-DELETE(T, z)
if z.left == NIL
    TRANSPLANT(T, z, z.right)
elseif z.right == NIL
    TRANSPLANT(T, z, z.left)
else y = TREE-MINIMUM(z.right)
    if y.p ≠ z
        TRANSPLANT(T, y, y.right)
        y.right = z.right
        y.right.p = y
    TRANSPLANT(T, z, y)
    y.left = z.left
    y.left.p = y
```



Deletion (RB) (1)

```
TREE-DELETE(T, z)
if z.left == NIL
    TRANSPLANT(T, z, z.right)
elseif z.right == NIL
    TRANSPLANT(T, z, z.left)
else y = TREE-MINIMUM(z.right)
    if y.p ≠ z
        TRANSPLANT(T, y, y.right)
        y.right = z.right
        y.right.p = y
    TRANSPLANT(T, z, y)
    y.left = z.left
    y.left.p = y
```

```
RB-DELETE(T, z)
y = z
y-original-color = y.color
if z.left == T.nil
    x = z.right
    RB-TRANSPLANT(T, z, z.right)
elseif z.right == T.nil
    x = z.left
    RB-TRANSPLANT(T, z, z.left)
else y = TREE-MINIMUM(z.right)
    y-original-color = y.color
    x = y.right
    if y.p == z
        x.p = y
    else RB-TRANSPLANT(T, y, y.right)
        y.right = z.right
        y.right.p = y
    RB-TRANSPLANT(T, z, y)
    y.left = z.left
    y.left.p = y
    y.color = z.color
if y-original-color == BLACK
    RB-DELETE-FIXUP(T, x)
```

Deletion (RB) (2)

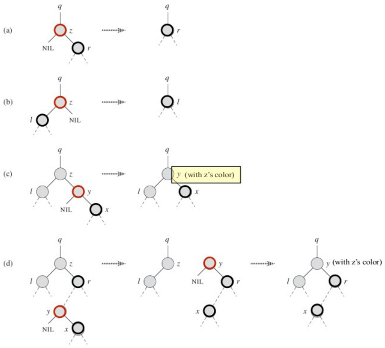
- node y
  - either removed (a/b)
  - or moved in the tree (c/d)
  - y-original-color
- node x
  - the node that moves into y's original position
  - x.p points to y's original parent (since it moves into y's position, note special case in 12/13)

RB-DELETE(T, z)

```
1  y = z
2  y-original-color = y.color
3  if z.left == T.nil
4      x = z.right
5      RB-TRANSPLANT(T, z, z.right)
6  elseif z.right == T.nil
7      x = z.left
8      RB-TRANSPLANT(T, z, z.left)
9  else y = TREE-MINIMUM(z.right)
10     y-original-color = y.color
11     x = y.right
12     if y.p == z
13         x.p = y
14     else RB-TRANSPLANT(T, y, y.right)
15         y.right = z.right
16         y.right.p = y
17     RB-TRANSPLANT(T, z, y)
18     y.left = z.left
19     y.left.p = y
20     y.color = z.color
21 if y-original-color == BLACK
22     RB-DELETE-FIXUP(T, x)
```

Deletion (RB) (3)

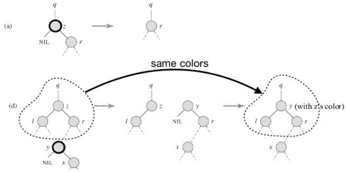
- y-original-color == red




```
RB-DELETE(T, z)
y = z
y-original-color = y.color
if z.left == T.nil
    x = z.right
    RB-TRANSPLANT(T, z, z.right)
elseif z.right == T.nil
    x = z.left
    RB-TRANSPLANT(T, z, z.left)
else y = TREE-MINIMUM(z.right)
    y-original-color = y.color
    x = y.right
    if y.p == z
        x.p = y
    else RB-TRANSPLANT(T, y, y.right)
        y.right = z.right
        y.right.p = y
    RB-TRANSPLANT(T, z, y)
    y.left = z.left
    y.left.p = y
    y.color = z.color
if y-original-color == BLACK
    RB-DELETE-FIXUP(T, x)
```

Deletion (RB) (4)

- y-original-color == red
  - no problem
- y-original-color == black
  - violations might occur (2,4,5)
  - main idea to fix
    - x gets an "**extra black**" & needs to get rid of it
  - 4 cases

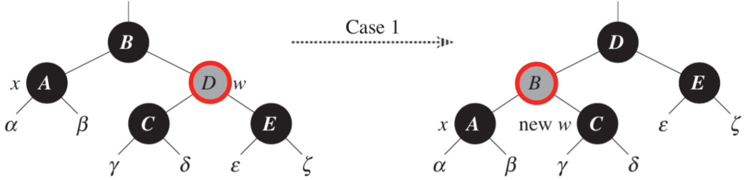



```
RB-DELETE(T, z)
y = z
y-original-color = y.color
if z.left == T.nil
    x = z.right
    RB-TRANSPLANT(T, z, z.right)
elseif z.right == T.nil
    x = z.left
    RB-TRANSPLANT(T, z, z.left)
else y = TREE-MINIMUM(z.right)
    y-original-color = y.color
    x = y.right
    if y.p == z
        x.p = y
    else RB-TRANSPLANT(T, y, y.right)
        y.right = z.right
        y.right.p = y
    RB-TRANSPLANT(T, z, y)
    y.left = z.left
    y.left.p = y
    y.color = z.color
if y-original-color == BLACK
    RB-DELETE-FIXUP(T, x)
```

Fixing Red-Black Tree Properties (1)

Case 1: x's sibling w is red.

Transform to Case 2, 3, or 4 by left rotation and changing colors of nodes B and D.

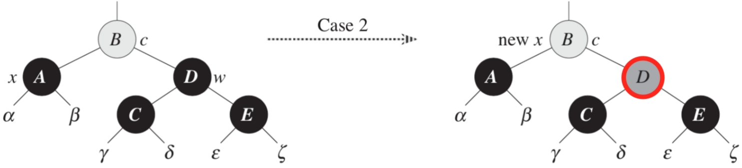


x = node with extra black
w = x's sibling

```
if w.color == RED
    w.color = BLACK
    x.p.color = RED
    LEFT-ROTATE(T, x.p)
    w = x.p.right
```

Fixing Red-Black Tree Properties (2)

Case 2: x's sibling w is black and the children of w are black.
Set color of w to red and propagate upwards.

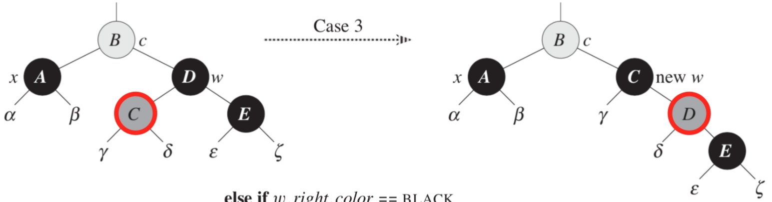


x = node with extra black
w = x's sibling
c = color of the node

if $w.left.color$ == BLACK and $w.right.color$ == BLACK
    $w.color$ = RED
    $x = x.p$

Fixing Red-Black Tree Properties (3)

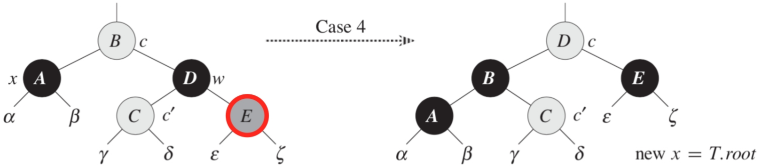
Case 3: x's sibling w is black and the left child of w is red, while the right child of w is black.
Transform to Case 4 by right rotation and changing colors of nodes C and D.




```
else if w.right.color == BLACK
        w.left.color = BLACK
        w.color = RED
        RIGHT-ROTATE(T, w)
        w = x.p.right
```

Fixing Red-Black Tree Properties (4)

Case 4: x's sibling w is black and the right child of w is red. Perform a left-rotate and change colors of B, D, and E. Then, the loop terminates.



$w.color = x.p.color$
$x.p.color = \text{BLACK}$
$w.right.color = \text{BLACK}$
$\text{LEFT-ROTATE}(T, x.p)$

Fixing Red-Black Tree Properties (5)

```
RB-DELETE-FIXUP(T, x)
while x ≠ T.root and x.color == BLACK
    if x == x.p.left
        w = x.p.right
        if w.color == RED
            w.color = BLACK                              // case 1
            x.p.color = RED                              // case 1
            LEFT-ROTATE(T, x.p)                          // case 1
            w = x.p.right                                // case 1
        if w.left.color == BLACK and w.right.color == BLACK
            w.color = RED                                // case 2
            x = x.p                                      // case 2
        else if w.right.color == BLACK
                w.left.color = BLACK                     // case 3
                w.color = RED                            // case 3
                RIGHT-ROTATE(T, w)                       // case 3
                w = x.p.right                            // case 3
            w.color = x.p.color                          // case 4
            x.p.color = BLACK                            // case 4
            w.right.color = BLACK                        // case 4
            LEFT-ROTATE(T, x.p)                          // case 4
            x = T.root                                   // case 4
    else (same as then clause with "right" and "left" exchanged)
x.color = BLACK
```

Time complexity: $O(h) = O(\lg n)$

Conclusion

Modifying operations on red-black trees can be executed in $O(\lg n)$ time.

Direct Access Table

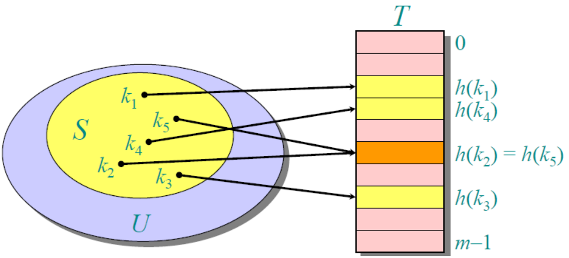
- The idea of a direct access table is that objects are directly accessed via their key.
- Assuming that keys are out of $U = \{0, 1, ..., m-1\}$.
- Moreover, assume that keys are distinct.
- Then, we can set up an array $T[0..m-1]$ with

$$T[k] = \begin{cases} x & \text{if } x \in K \text{ and } key[x] = k \\ \text{NIL} & \text{otherwise.} \end{cases}$$

- Time complexity: With this set-up, we can have the dynamic-set operations (Search, Insert, Delete, ...) in $\Theta(1)$.
- Problem: m is often large. For example, for 64-bit numbers we have $18,446,744,073,709,551,616$ different keys.

Hash Function

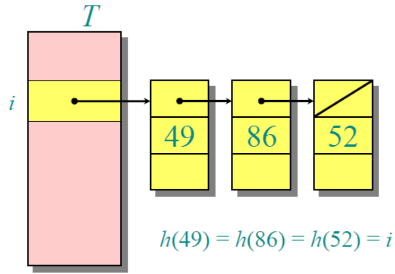
- Use a function h that maps U to a smaller set $\{0, 1, ..., m-1\}$.



- Such a function is called a hash function.
- The table T is called a hash table.
- If two keys are mapped to the same location, we have a collision.

Resolving Collisions

- Collisions can be resolved by storing the colliding mappings in a (singly-)linked list.



- Worst case: All keys are mapped to the same location. Then, access time is $\Theta(n)$.

Average Case Analysis (1)

- Assumption (simple uniform hashing): Each key is equally likely to be hashed to any slot of the table, independent of where other keys are hashed.
- Let n be the number of keys.
- Let m be the number of slots.
- The load factor $\alpha = n/m$ represents the average number of keys per slot.

Average Case Analysis (2)

Theorem:
In a hash table in which collisions are resolved by chaining, an unsuccessful search takes average-case time $\Theta(1 + \alpha)$ under the assumption of simple uniform hashing.
Proof:

- Any key k not already stored in the table is equally likely to hash to any of the m slots.
- The expected time to search unsuccessfully for a key k is the expected time to search to the end of list $T[h(k)]$.
- Expected length of the list is $E[n_{h(k)}] = \alpha$.
- Time for computing $h(k) = O(1) \Rightarrow$ overall time $\Theta(1 + \alpha)$.

Average Case Analysis (3)

- Runtime for unsuccessful search:
  The expected time for an unsuccessful search is $\Theta(1+\alpha)$ including applying the hash function and accessing the slot and searching the list.
- What does this mean?
  - $m \sim n$, i.e., if $n = O(m) \Rightarrow \alpha = n/m = O(m)/m = O(1)$
  - Thus, search time is $O(1)$
- A successful search has the same asymptotic bound.

Choosing a Hash Function (1)

- What makes a good hash function?
  - The goal for creating a hash function is to distribute the keys as uniformly as possible to the slots.
- Division method
  - Define hashing function $h(k) = k \bmod m$.
  - Deficiency: Do not pick an m that has a small divisor d, as a prevalence of keys with the same modulo d can negatively effect uniformity.
  - Example: if m is a power of 2, the hash function only depends on a few bits: If $k = 1011000111011010$ and $m = 2^6$, then $h(k) = 011010$.

Choosing a Hash Function (2)

- Division method (continue)
  - blueCommon choice: Pick m to be a prime not too close to a power of 2 or 10 and not otherwise prominently used in computing environments.
  - Example: $n = 2000$; we are OK with average 3 elements in our collision chain $\Rightarrow m = 701$ (a prime number close to 2000/3), $h(k) = k \bmod 701$.

Choosing a Hash Function (3)

- Multiplication method
  - Assume all keys are integers, $m = 2^r$, and the computer uses w-bit words.
  - Define hash function $h(k) = (A \cdot k \bmod 2^w) >> (w - r)$, where "$>>$" is the right bit-shift operator and A is an odd integer with $2^{w-1} < A < 2^w$.
  - Note that these operations are faster than divisions.
  - Example: $m = 2^3 = 8$ and $w = 7$.

$$
\begin{array}{rl}
 1011001 & = A \\
\times \qquad\qquad 1101011 & = k \\
\hline
1001010\underbrace{011}_{h(k)}0011 &
\end{array}
$$

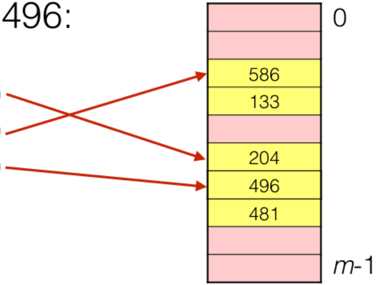
Resolving Collisions by Open Addressing

- No additional storage is used.
- Only store one element per slot.
- Insertion probes the table systematically until an empty slot is found.
- The hash function depends on the key and the probe number, i.e., $h : U \times \{0, 1, ..., m-1\} \to \{0, 1, ..., m-1\}$.
- The probe sequence $< h(k, 0), h(k, 1), ..., h(k, m-1) >$ should be a permutation of $\{0, 1, ..., m-1\}$.

Insert Example

- Insert key k = 496:

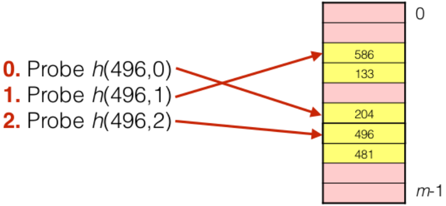
0. Probe $h(496,0)$
1. Probe $h(496,1)$
2. Probe $h(496,2)$




```
HASH-INSERT(T, k)
i = 0
repeat
    j = h(k, i)
    if T[j] == NIL
        T[j] = k
        return j
    else i = i + 1
until i == m
error "hash table overflow"
```

Search Example

```
HASH-SEARCH(T, k)
i = 0
repeat
    j = h(k, i)
    if T[j] == k
        return j
    i = i + 1
until T[j] == NIL or i == m
return NIL
```



- Search key $k = 496$
  - Search uses the same probe sequence, terminating successfully if it finds the key and unsuccessfully if it encounters an empty slot (or made it all the way through the list)
- What about delete?
  - Have a special node type: DELETED
  - Note though: search times no longer depend on load factor α
  - Chaining more commonly used when keys must also be deleted

Probing Strategies (1)

Linear probing:

- Given an ordinary hash function $h'(k)$, linear probing uses the hash function $h(k,i) = (h'(k) + i) \bmod m$.
- This is a simple computation.
- However, it may suffer from primary clustering, where long runs of occupied slots build up and tend to get longer.
  - empty slot preceded by i full slots gets filled next with probability $(i+1)/m$

Probing Strategies (2)

Quadratic probing:

- Quadratic probing uses the hash function $h(k, i) = (h'(k) + c_1 \cdot i + c_2 \cdot i^2) \bmod m$.
- Offset by amount that depends on quadratic manner, works much better than linear probing
- But, it may still suffer from secondary clustering: If two keys have initially the same value, then they also have the same probe sequence
- In addition c_1, c_2, and m need to be constrained to make full use of the hash table

Probing Strategies (3)

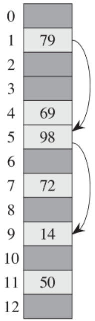
Double hashing:

- Given two ordinary hash functions $h_1(k)$ and $h_2(k)$, double hashing uses the hash function $h(k, i) = (h_1(k) + i \cdot h_2(k)) \bmod m$.
- The initial probe goes to position $T[h_1(k)]$; successive probe positions are offset by $h_2(k)$ $\rightarrow$ the initial probe position, the offset, or both, may vary
- This method generates excellent results, if $h_2(k)$ is "relatively prime" to the hash-table size m,

Probing Strategies (4)

Double hashing (continue):

- e.g., by making m a power of 2 and design $h_2(k)$ to only produce odd numbers.
- or let m be prime and design h_2 such that it always returns a positive integer less than m, e.g. let m' be slightly less than m:
  $h_1(k) = k \bmod m$
  $h_2(k) = 1 + (k \bmod m')$



$h_1(k) = k \bmod 13$
$h_2(k) = 1 + (k \bmod 11)$

—> $k=14$; $h_1(k)=1$, $h_2(k)=4$

—> $k=27$; $h_1(k)=1$, $h_2(k)=6$

Analysis of Open Addressing (1)

Theorem:

- Assume uniform hashing, i.e., each key is likely to have any one of the $m!$ permutations as its probe sequence.
- Given an open-addressed hash table with load factor $\alpha = n/m < 1$.
- The expected number of probes in an unsuccessful search is, at most, $1/(1-\alpha)$.

Analysis of Open Addressing (2)

Proof:

- At least, one probe is always necessary.
- With probability n/m, the first probe hits an occupied slot, i.e., a second probe is necessary.
- With probability $(n-1)/(m-1)$, the second probe hits an occupied slot, i.e., a third probe is necessary.
- With probability $(n-2)/(m-2)$, the third probe hits an occupied slot, i.e., a fourth probe is necessary.
- ...

Analysis of Open Addressing (3)

Given that $\frac{n-i}{m-i} < \frac{n}{m} = \alpha$ for $i = 1, 2, \ldots, n$.

$$
\begin{aligned}
1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\cdots\left(1+\frac{1}{m-n+1}\right)\cdots\right)\right)\right)
&\le 1+\alpha(1+\alpha(1+\alpha(\cdots(1+\alpha)\cdots))) \\
&\le 1+\alpha+\alpha^2+\alpha^3+\cdots \\
&= \sum_{i=0}^{\infty}\alpha^i \\
&= \frac{1}{1-\alpha}.
\end{aligned}
$$

Analysis of Open Addressing (4)

- The successful search takes less number of probes
pected number is $1/\alpha \ln(1/(1-\alpha))$.
- We conclude that if α is constant, then accessing an open-addressed hash table takes constant time.
- For example, if the table is half full, the expected number of probes is $1/(1-0.5) = 2$.
- Or, if the table is 90% full, the expected number of probes is $1/(1-0.9) = 10$.

Summary

- Dynamic sets with queries and modifying operations.
- Array: Random access, search in $O(\lg n)$, but modifying operations $O(n)$.
- Stack: LIFO only. Operations in $O(1)$.
- Queue: FIFO only. Operations in $O(1)$.
- Linked list: Modifying operations in $O(1)$, but search $O(n)$.
- BST: All operations in $O(h)$.
- Red-black trees: All operations in $O(\lg n)$.
- Heap: All operations in $O(\lg n)$.
- Hash tables: Operations in $O(1)$, but additional storage space.