CH08-320201

Algorithms and Data Structures

ADS

Lecture 23

Dr. Kinga Lipskoch

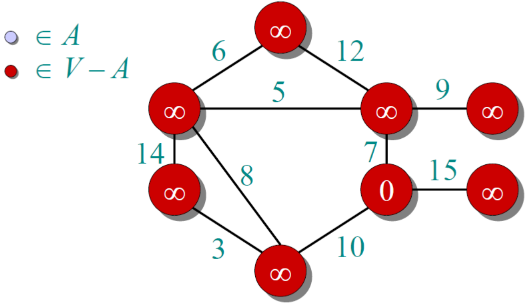
Spring 2019

Prim's Algorithm Pseudocode

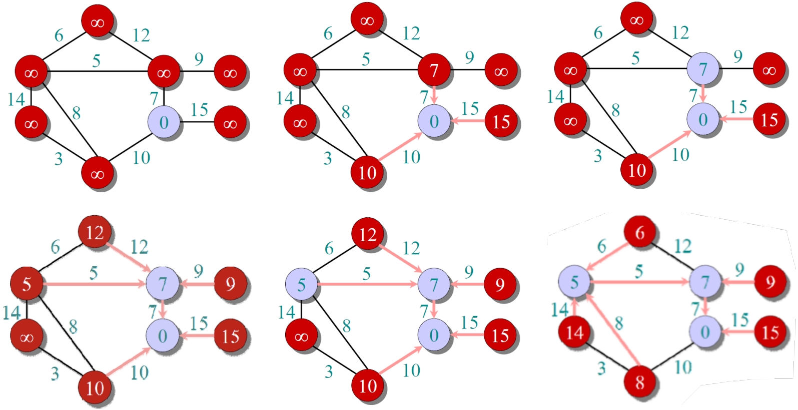
$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    for each $v \in Adj[u]$
      do if $v \in Q$ and $w(u, v) < key[v]$
        then $key[v] \leftarrow w(u, v)$
          $\pi[v] \leftarrow u$

- The output is provided by storing predecessors $\pi[v]$ of each node v.
- The set $\{(v, \pi[v]) | v \in V\}$ forms the MST.

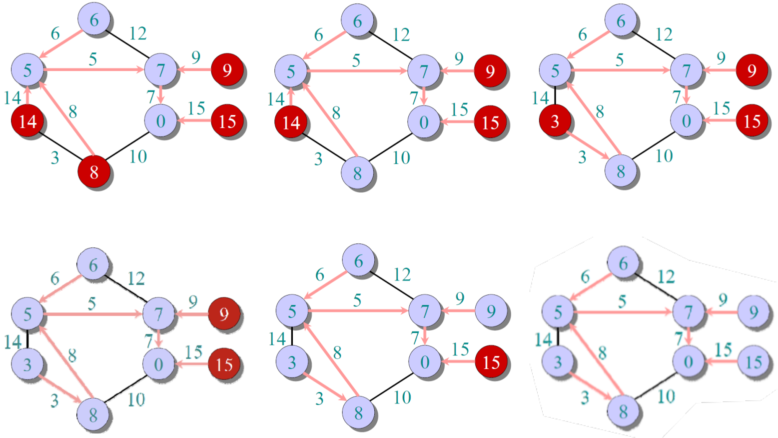
Example (1)



Example (2)


∞ ∞ ∞ ∞ ∞ 0 ∞ ∞
6 12 5 9 14 8 7 15 3 10
∞ ∞ 7 ∞ ∞ 0 15 10
6 12 5 9 14 8 7 15 3 10
∞ ∞ 7 ∞ ∞ 0 15 10
6 12 5 9 14 8 7 15 3 10
12 5 7 9 ∞ 0 15 10
6 12 5 9 14 8 7 15 3 10
12 5 7 9 ∞ 0 15 10
6 12 5 9 14 8 7 15 3 10
6 5 7 9 14 0 15 8
6 12 5 9 14 8 7 15 3 10

Example (3)


6
12
6
5
5
7
9
9
14
14
8
7
15
0
15
3
8
10
6
12
6
5
5
7
9
9
14
14
8
7
15
0
15
3
8
10
6
12
6
5
5
7
9
9
14
3
8
7
15
0
15
3
8
10
6
12
6
5
5
7
9
9
14
3
8
7
15
0
15
3
8
10
6
12
6
5
5
7
9
9
14
3
8
7
15
0
15
3
8
10
6
12
6
5
5
7
9
9
14
3
8
7
15
0
15
3
8
10

Complexity Analysis (1)

$\Theta(V)$ total:

$Q \leftarrow V$

$key[v] \leftarrow \infty$ for all $v \in V$

$key[s] \leftarrow 0$ for some arbitrary $s \in V$

$|V|$ times:

while $Q \neq \emptyset$

  do $u \leftarrow$ EXTRACT-MIN(Q)

  $degree(u)$ times:

    for each $v \in Adj[u]$

      do if $v \in Q$ and $w(u, v) < key[v]$

        then $\boxed{key[v] \leftarrow w(u, v)}$

          $\pi[v] \leftarrow u$

$\Theta(E)$ implicit DECREASE-KEY's.

Notation Θ(V) means Θ(|V|).

Complexity Analysis (2)

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

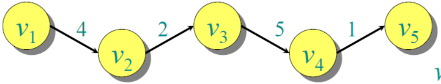
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
min-heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
array	$O(V)$	$O(1)$	$O(V^2)$

Definition: Path

- Consider a directed graph $G = (V, E)$, where each edge $e \in E$ is assigned a non-negative weight $w : E \to \mathbb{R}^+$.
- A path is a sequence of vertices in the graph, where two consecutive vertices are connected by a respective edge.
- The weight of a path $p = (v_1, ..., v_k)$ is defined by

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

- Example:



$w(p) = 12$

Definition: Shortest Path

- A shortest path from a vertex u to a vertex v in a graph G is a path of minimum weight.
- The weight of a shortest path from u to v is defined as $\delta(u, v) = \min\{w(p) : p \text{ is a path from } u \text{ to } v\}$.
- Note that $\delta(u, v) = \infty$, if no path from u to v exists.
- Why of interest?
  One example is finding a shortest route in a road network.

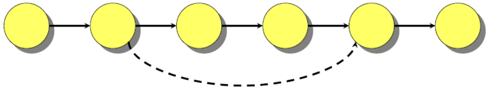
Optimal Substructure

Theorem:

A subpath of a shortest path is a shortest path.

Proof:

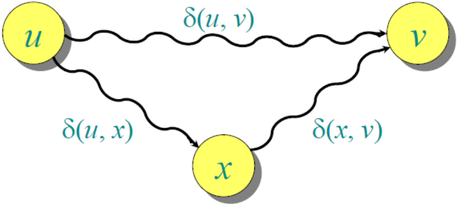
- Let $p = (v_1, ..., v_k)$ be a shortest path and $q = (v_i, ..., v_j)$ a subpath of p.
- Assume that q is not a shortest path.
- Then, there exists a shorter path from v_i to v_j than q.
- But then, there is also a shorter path from v_1 to v_k than p. Contradiction.



Triangle Inequality

Theorem:
For all $u, v, x \in V$, we have that $\delta(u, v) \leq \delta(u, x) + \delta(x, v)$.

Proof:



(Single-Source) Shortest Paths

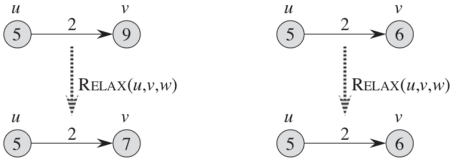
Problem:
Given a source vertex $s \in V$, find for all $v \in V$ the shortest-path weights $\delta(s, v)$.

Idea: Greedy approach.

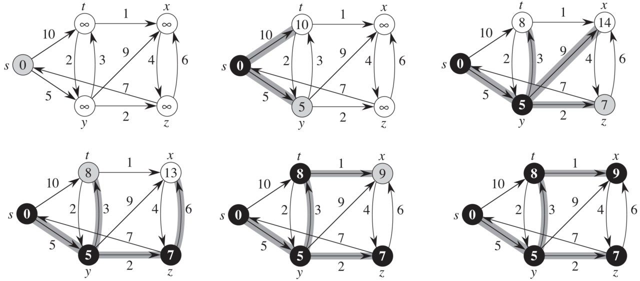
1. Maintain a set S of vertices whose shortest-path distances from s are known.
2. At each step, add to S the vertex $v \in V \setminus S$ whose distance estimate from s is minimal.
3. Update the distance estimates of vertices adjacent to v.

Dijkstra's Algorithm

```
d[s] := 0
for each v ∊ V\{s}
  d[v] := infinity
S := Ø
Q := V   // min-priority queue maintaining V \ S.
while Q != Ø
    u := Extract-Min(Q)
    S := S U {u}
    for each v ∊ Adj[u]
        if d[v] > d[u] + w(u,v)      // *****
        then d[v] := d[u] + w(u,v)   // Relaxation
             pi[v] := u               // *****
```



Example Dijkstra's Algorithm




```
while Q != Ø
    u := Extract-Min(Q)
    S := S U {u}
    for each v ∈ Adj[u]
        if d[v] > d[u] + w(u,v)
        then d[v] := d[u] + w(u,v)
             pi[v] := u
```

S = {s, y, z, t, x}

Correctness of Dijkstra's Algorithm

Correctness can be shown in 3 steps:

(i) $d[v] \geq \delta(s, v)$ at all steps (for all v)

(ii) $d[v] = \delta(s, v)$ after relaxation from u,

(iii) if (u, v) on shortest path (for all v) algorithm terminates with $d[v] = \delta(s, v)$

Correctness (i)

Lemma:

- Initializing $d[s] = 0$ and $d[v] = \infty$ for all $v \in V \setminus \{s\}$ establishes $d[v] \geq \delta(s, v)$ for all $v \in V$.
- This invariant is maintained over any sequence of relaxation steps.

Proof:

Suppose the Lemma is not true, then let v be the first vertex for which $d[v] < \delta(s, v)$ and let u be the vertex that caused $d[v]$ to change by $d[v] = d[u] + w(u, v)$. Then,

$$
\begin{aligned}
d[v] &< \delta(s, v) && \text{supposition} \\
&\leq \delta(s, u) + \delta(u, v) && \text{triangle inequality} \\
&\leq \delta(s,u) + w(u, v) && \text{sh. path} \leq \text{specific path} \\
&\leq d[u] + w(u, v) && v \text{ is first violation}
\end{aligned}
$$

Contradiction.

Correctness (ii)

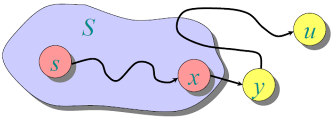
Lemma:

- Let u be v's predecessor on a shortest path from s to v.
- Then, if $d[u] = \delta(s, u)$, we have $d[v] = \delta(s, v)$ after the relaxation of edge (u, v).

Proof:

- Observe that $\delta(s, v) = \delta(s, u) + w(u, v)$.
- Suppose that $d[v] > \delta(s, v)$ before relaxation (else: done).
- Then, $d[v] > \delta(s, v) = \delta(s, u) + w(u, v) = d[u] + w(u, v)$ (if clause in the algorithm).
- Thus, the algorithm sets $d[v] = d[u] + w(u, v) = \delta(s, v)$.

Correctness (iii)



Theorem:
Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.

Proof:

- It suffices to show that $d[v] = \delta(s, v)$ for every $v \in V$ when v is added to S.
- Suppose u is the first vertex added to S with $d[u] > \delta(s, u)$.
- Let y be the first vertex in $V \setminus S$ along the shortest path from s to u, and let x be its predecessor.
- Then, $d[x] = \delta(s, x)$ and $d[y] = \delta(s, y) \leq \delta(s, u) < d[u]$.
- But we chose u such that $d[u] \leq d[y]$. Contradiction.

Complexity Analysis

$|V|$ times

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$

$degree(u)$ times

    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$

- Similar to Prim's minimum spanning tree algorithm, we get the computation time

$$\Theta(V \cdot T_{\text{EXTRACT-MIN}} + E \cdot T_{\text{DECREASE-KEY}})$$

- Hence, depending on what data structure we use, we get the same computation times as for Prim's algorithm.

Unweighted Graphs

- Suppose that we have an unweighted graph, i.e., the weights $w(u, v) = 1$ for all $(u, v) \in E$.
- Can we improve the performance of Dijkstra's algorithm?
- Observation: The vertices in our data structure Q are processed following the FIFO principle.
- Hence, we can replace the min-priority queue with a queue.
- This leads to a breadth-first search.

BFS Algorithm

```
d[s] := 0
for each v ϵ V\{s}
  d[v] := infinity
Enqueue (Q,s)
while Q != Ø
  u := Dequeue(Q)
  for each v ϵ Adj[u]
      if d[v] = infinity
      then d[v] := d[u] + 1
           pi[v] :=u
           Enqueue(Q,v)
```

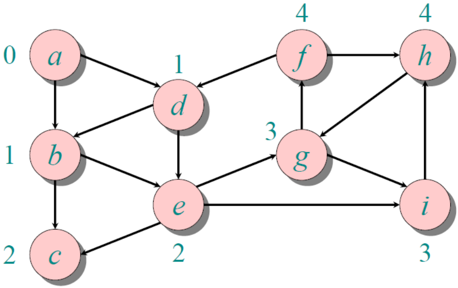
Analysis: BFS Algorithm

Correctness:

- The FIFO queue Q mimics the min-priority queue in Dijkstra's algorithm.
- Invariant:
  If v follows u in Q, then $d[v] = d[u]$ or $d[v] = d[u] + 1$.
- Hence, we always dequeue the vertex with smallest d.

Time complexity:
$O(|V|T_{Dequeue} + |E|T_{Enqueue}) = O(|V| + |E|)$

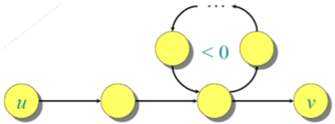
Example: BFS Algorithm



Q: a b d c e g i f h

Negative Weights

- We had postulated that all weights are nonnegative.
- How can we extend the algorithm to also handle negative entries?
- The problems are caused by negative weight cycles.



- Goal: Find shortest-path lengths from a source vertex $s \in V$ to all vertices $v \in V$ or determine the existence of a negative-weight cycle.

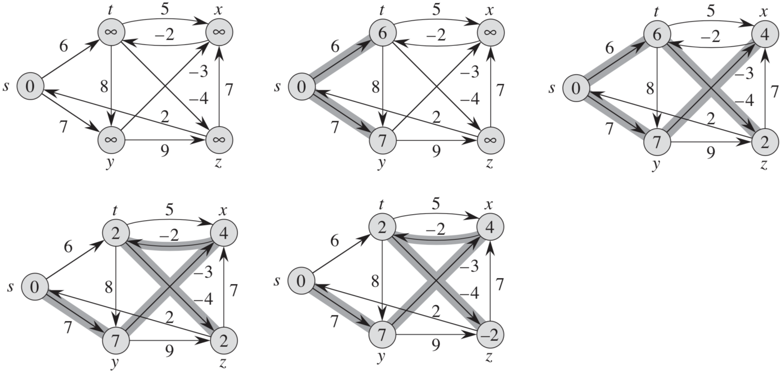
Bellmann-Ford Algorithm

```
d[s] := 0
for each v ϵ V\{s}
  d[v] := infinity
for i:=1 to |V|-1
    for each (u,v) ϵ E
        if d[v] > d[u] + w(u,v)
        then  d[v] := d[u] + w(u,v)
              pi[v] :=u

for each (u,v) ϵ E
  if d[v] > d[u] + w(u,v)
    report existence of negative-weight cycle
```

Time complexity: $O(|V| \cdot |E|)$

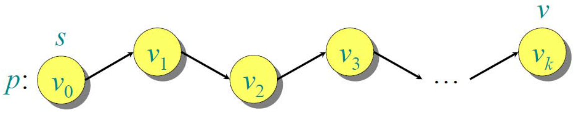
Example: Bellman-Ford Algorithm




```
for i:=1 to |V|-1
    for each (u,v) ∈ E
        if d[v] > d[u] + w(u,v)
        then  d[v] := d[u] + w(u,v)
              pi[v] :=u
```

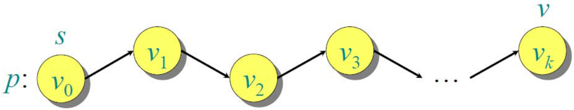
Bellmann-Ford Algorithm: Correctness (1)

Theorem:
If $G = (V, E)$ contains no negative-weight cycles, then the Bellman-Ford algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.
Proof:
Let $v \in V$ be any vertex.
Consider a shortest path $p = (v_0, ..., v_k)$ from s to v.
Then, $\delta(s, v_i) = \delta(s, v_i - 1) + w(v_i - 1, v_i)$ for $i = 1, ..., k$.



Bellmann-Ford Algorithm: Correctness (2)

Initially, $d[v_0] = 0 = \delta(s, v_0)$.
According to our Lemma from Dijkstra's algorithm we have $d[v] \geq \delta(s, v)$, i.e., $d[v_0]$ is not changed.
After the 1^{st} pass, we have $d[v_1] = \delta(s, v_1)$.
After the 2^{nd} pass, we have $d[v_2] = \delta(s, v_2)$.
...
After the k^{th} pass, we have $d[v_k] = \delta(s, v_k)$.
Since G has no negative-weight cycles, p is a simple path, i.e., it has $\leq |V| - 1$ edges.

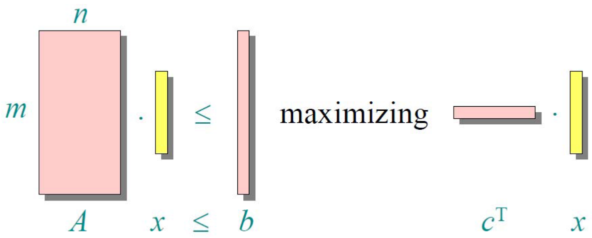


Detecting Negative-Weight Cycles

Corollary:
If a value $d[v]$ fails to converge after $|V| - 1$ passes, there exists a negative-weight cycle in G reachable from s.

Excurse: Linear Programming

Linear programming problem:
Let A be matrix of size $m \times n$, b a vector of size m, and c a vector of size n.
Find a vector x of size n that maximizes $c^T x$ subject to $Ax \leq b$, or determine that no such solution exists.



Example: Difference Constraints

Linear programming example, where each row of A contains exactly one 1 and one -1, other entries are 0.

$$\left.\begin{array}{l} x_1 - x_2 \le 3 \\ x_2 - x_3 \le -2 \\ x_1 - x_3 \le 2 \end{array}\right\} \; x_j - x_i \le w_{ij}$$

Goal: Find 3-vector x that satisfies these inequations.
Solution: $x_1 = 3$, $x_2 = 0$, $x_3 = 2$.
Build constraint graph (matrix A of size $|E| \times |V|$):

$$x_j - x_i \le w_{ij}$$

Case 1: Unsatisfiable Constraints

Theorem:
If the constraint graph contains a negative-weight cycle, then the constraints are unsatisfiable.

Proof:
Suppose we have a negative-weight cycle:

$$v_1 \to v_2 \to \cdots \to v_k \to v_1.$$

Then,

$$\begin{aligned}
x_2 - x_1 &\le w_{12} \\
x_3 - x_2 &\le w_{23} \\
&\vdots \\
x_k - x_{k-1} &\le w_{k-1,k} \\
x_1 - x_k &\le w_{k1}
\end{aligned}$$

Summing the inequations delivers: $LHS = 0$, $RHS < 0$.
Hence, no x exists that satisfies the inequations.

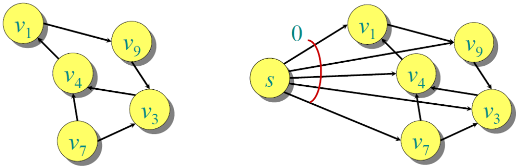
Case 2: Satisfiable Constraints (1)

Theorem:

If no negative-weight cycle exists in the constraint graph, then the constraints are satisfiable.

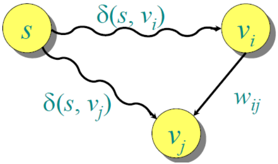
Proof:

Add a vertex s with a 0-weight edge to all vertices. Note that this does not introduce a negative-weight cycle.



Case 2: Satisfiable Constraints (2)

Show that the assignments $x_i = \delta(s, v_i)$ for $i = 1, \ldots, n$ solve the constraints.
Consider any constraint $x_j - x_i \leq w_{ij}$.
Then, consider the shortest path from s to v_j and v_i.
The triangle inequality delivers $\delta(s, v_j) \leq \delta(s, v_i) + w_{ij}$.
Since $x_i = \delta(s, v_i)$ and $x_j = \delta(s, v_j)$, constraint $x_j - x_i \leq w_{ij}$ is satisfied.



Bellmann-Ford for Linear Programming

Corollary:
The Bellman-Ford algorithm can solve a system of m difference constraints on n variables in $O(mn)$ time.

Remark:
Single-source shortest paths is a simple linear programming problem.