CH08-320201

Algorithms and Data Structures

ADS

Lecture 8

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Heap as a Data Structure

- Heaps are a data structure that can be used for other purposes, as well.
- In particular, a max-heap is often used to build a max-priority queue.

Max-Priority Queues

Definition (priority queue):

- A priority queue is a data structure for maintaining a set S of elements, each with an associated value called a key.

Definition (max-priority queue):

- A max-priority queue is a priority queue that supports the following operations:
  - *Maximum*(S): return element from S with largest key.
  - *Extract-Max*(S): remove and return element from S with largest key.
  - *Increase-Key*(S, x, k): increase the value of the key of element x to k, where k is assumed to be larger or equal than the current key.
  - *Insert*(S, x): add element x to set S.

$Maximum(S)$

```
HEAP-MAXIMUM(A)
1  return A[1]
```

Time complexity: $O(1)$

Extract-Max(S)

HEAP-EXTRACT-MAX(A)

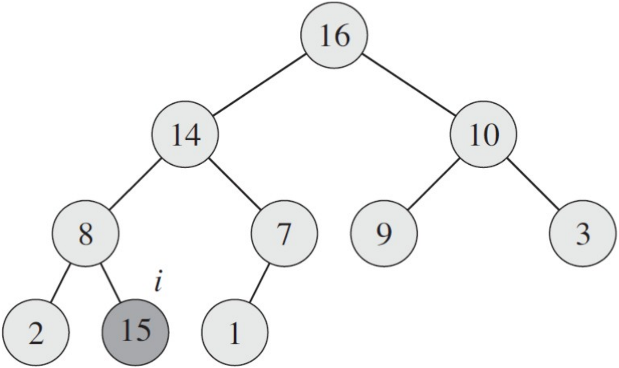
```
if A.heap-size < 1
    error "heap underflow"
max = A[1]
A[1] = A[A.heap-size]
A.heap-size = A.heap-size − 1
MAX-HEAPIFY(A, 1)
return max
```

Time complexity: $O(\lg n)$

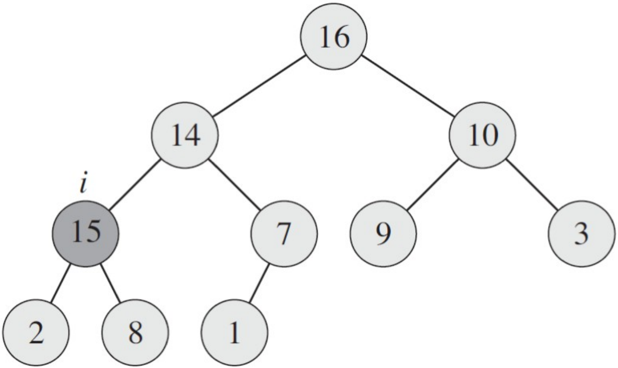
Increase-Key(S, x, k)

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
    error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
    exchange A[i] with A[PARENT(i)]
    i = PARENT(i)
```

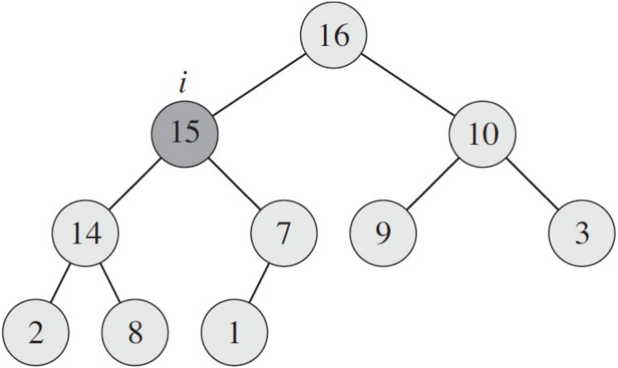
Increase-Key(S, x, k): Example (1)



Increase-Key(S, x, k): Example (2)



Increase-Key(S, x, k): Example (3)



Increase-Key(S, x, k)

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
    error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
    exchange A[i] with A[PARENT(i)]
    i = PARENT(i)
```

Time complexity: $O(\lg n)$

Insert(S, x)

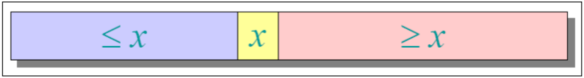
MAX-HEAP-INSERT(A, key)

$A.heap\text{-}size = A.heap\text{-}size + 1$
$A[A.heap\text{-}size] = -\infty$
HEAP-INCREASE-KEY$(A, A.heap\text{-}size, key)$

Time complexity: $O(\lg n)$

Quicksort: Divide & Conquer

1. Divide:
   Partition the array into two subarrays around a pivot x such that the elements in lower subarray $\leq x \leq$ the elements in upper subarray.



2. Conquer:
   Recursively sort the two subarrays
3. Combine:
   Nothing to be done.

Key observation: Linear-time partitioning subroutine.

Quicksort: Divide (1)

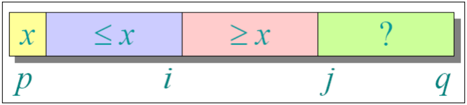
In the literature there are two popular division (partition) methods:

- Nico Lomuto's partition used in the textbook of Cormen and other textbooks
- C.A.R. Hoare's partition
- Hoare's partition is more efficient than Lomuto's partition because it does three times fewer swaps on average, but the time complexities are the same

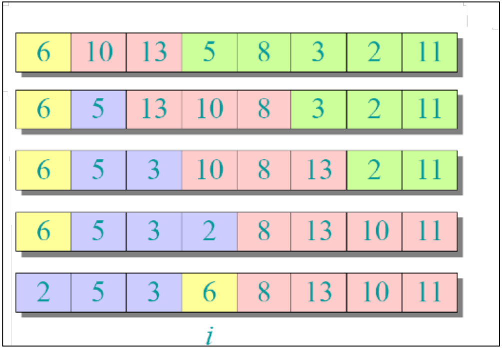
Quicksort: Divide (2)

```
Partition(A, p, q)      //A[p. . q]
  x = A[p]              //pivot = A[p]
  i = p
  for j = p + 1 to q
    if A[j] <= x then
      i = i + 1
      exchange A[i] and A[j]
  exchange A[p] and A[i]
  return i
```

Invariant:



Quicksort: Partition Example



Quicksort: Divide Complexity

```
Partition(A, p, q)      //A[p. . q]
  x = A[p]              //pivot = A[p]
  i = p
  for j = p + 1 to q
    if A[j] <= x then
      i = i + 1
      exchange A[i] and A[j]
  exchange A[p] and A[i]
  return i
```

Time complexity:

For $n = q - p + 1$ elements: $T(n) = \Theta(n)$

Quicksort: Conquer

```
QuickSort(A, p, r)
  if p < r
    q = Partition(A, p, r)
    QuickSort(A, p, q - 1)
    QuickSort(A, q + 1, r)
```

Initial call: `QuickSort(A, 1, n)`

Runtime Analysis (1)

- Assume all input elements are distinct.
  - In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ be the worst-case running time for n elements.
- Worst case:
  - Input sorted or reverse sorted.
  - Partition around min or max element.
  - One side of partition always has no elements.

Runtime Analysis (2)

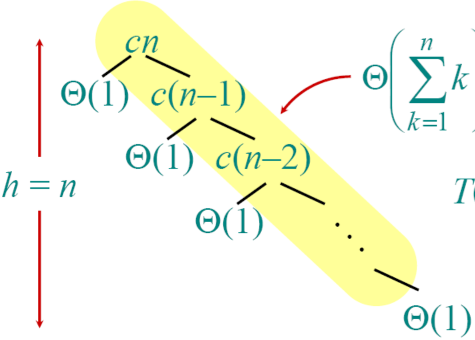
Worst case:

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \qquad \textit{\textbf{(arithmetic series)}} \end{aligned}$$

Runtime Analysis (3)

Worst-case recursion tree:

$$T(n) = T(0) + T(n-1) + cn$$



$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta(n^2)$$

$$T(n) = \Theta(n) + \Theta(n^2)$$
$$= \Theta(n^2)$$

Runtime Analysis (4)

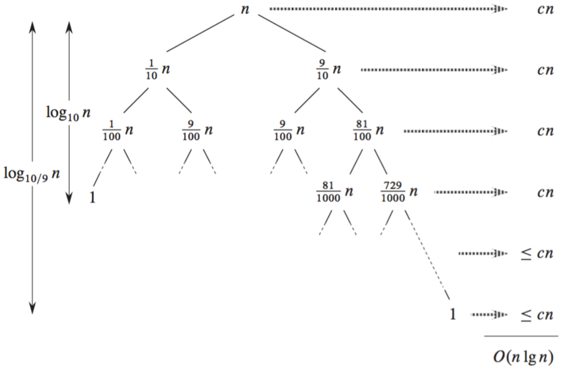
Best case:

- In best case partition splits the array evenly.
- $T(n) = 2T(n/2) + \Theta(n) = \Theta(n \lg n)$
- This is the same as Merge Sort.

Runtime Analysis (5)

What if the split is $1/10 : 9/10$?

$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$



Runtime Analysis

- What if we alternate between lucky and unlucky choices
  - $L(n) = 2U(n/2) + \Theta(n)$ lucky
  - $U(n) = L(n-1) + \Theta(n)$ unlucky
- Solving:
  - $L(n) = 2(L(n/2-1) + \Theta(n/2)) + \Theta(n)$
    $= 2L(n/2-1) + \Theta(n)$
    $= \Theta(n \lg n)$
- How can we make sure that this is usually happening?

Randomized Quicksort (1)

- Idea: Partition around a random element.
- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.

Randomized Quicksort (2)

```
RANDOMIZED-PARTITION(A, p, r)
    i = RANDOM(p, r)
    exchange A[p] with A[i]
    return PARTITION(A, p, r)
```

```
RANDOMIZED-QUICKSORT(A, p, r)
    if p < r
        q = RANDOMIZED-PARTITION(A, p, r)
        RANDOMIZED-QUICKSORT(A, p, q - 1)
        RANDOMIZED-QUICKSORT(A, q + 1, r)
```

Randomized Quicksort (3)

- Let $T(n)$ be the random variable for the running time of the randomized quicksort on an input of size n (assuming random numbers are independent).

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n{-}k{-}1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

- For $k = 0, 1, ..., n - 1$, define indicator random variable
- $E[X_k] = Pr\{X_k = 1\} = 1/n$, since all splits are equally likely (assuming elements are distinct).

Randomized Quicksort (4)

Recurrence:

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right)$$

Randomized Quicksort (5)

Calculating expectations:

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

Randomized Quicksort (6)

- Use substitution method to solve recurrence.
- Guess: $E[T(n)] = \Theta(n \lg n)$.
- Prove: $E[T(n)] \leq an \lg n$ for constant $a > 0$.
- Use:

$$\sum_{k=2}^{n-1} k \lg k \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$$

(proof by induction)

Randomized Quicksort (7)

Proof:

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\begin{aligned} E[T(n)] &\le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n) \\ &= \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n) \\ &= an\lg n - \left(\frac{an}{4} - \Theta(n)\right) \\ &\le an\lg n, \end{aligned}$$

if a is chosen large enough.

Quicksort: Conclusion

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is often the best practical choice because its expected runtime is $\Theta(n \lg n)$ and the constant is quite small.
- Quicksort is typically over twice as fast as MergeSort.
- Quicksort is an in-situ sorting algorithm (debatable).
- Quicksort has a worst-case runtime of $\Theta(n^2)$ when the array is already sorted.
- Visualization Randomized Quicksort: http://www.sorting-algorithms.com/quick-sort