MECH309
Projects in Numerical Methods

Due date: March 26, 2012

Instructions:

- Up to 3 students working together.
- Write a report (word, LateX, LibreOffice...) with comments, equations, figures, and algorithms and send a pdf file to mathias.legrand@mcgill.ca. It should not exceed 2 Mb. Family names must be part of the file name.
- For problem 2, send your audio files (before time-stretching, after method 1 and after method 2) as wave files to mathias.legrand@mcgill.ca as well. They should not exceed 2 Mb each. Family names must be part of the file name.
- Always mention "Mech 309 — Projects — Family names" in the subject of your email.

1 Basins of attraction

Nonlinear dynamical systems feature subsets of the state space to which trajectories originating from initial conditions tend as time increases. These subsets are called attractors and can be fixed points, periodic or quasi-periodic limit cycles or strange attractors. A given dynamical system may have more than one attractor. It is then of high interest to numerically describe their *basin of attraction* which is the set of initial conditions leading to long-time behavior that approaches that attractor. The structure of such basins of attraction may be quite intricate.

In what follows, this concept is used to compare the hidden behavior of numerical methods derived to find solutions to a set of nonlinear equations. To this end, consider the two equations:

$$\begin{cases} x^2 + y^2 - 4 = 0 \\ \dfrac{(x-1)^2}{9} + \dfrac{(y-1)^2}{4} - 1 = 0 \end{cases} \tag{1}$$

An example is given in Figure 1 which shows the basins of attraction of the Newton's method use to solve $z^3 - 1 = 0$ defined in the complex plane. Denoting $z = x + \mathrm{i}y$, this equation can be seen as a system of two nonlinear equations in (x, y) and has three solutions (x_1, y_1), (x_2, y_2), and (x_3, y_3).

1.1 Fixed-point iteration

Through two fixed-point iterations, numerically find the two solutions (x_1, y_1) and (x_2, y_2) to system (1) up to 8 digits. For each of them:

- Prove existence of the fixed-points.
- Can you prove uniqueness as shown in class?

1.2 Newton's method

The respective basins of attraction of (x_1, y_1) and (x_2, y_2) using Newton's method can be described by scanning a set of initial conditions $(x^{(0)}, y^{(0)})$ and by using a color scale to illustrate how fast Newton's method converges to one of the solutions:

- Write an algorithm which uses Newton's method to find a solution to system (1).

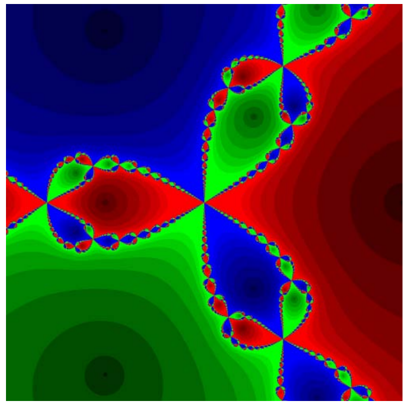


Figure 1: Basins of attraction for Newton's method used to solve $z^3 - 1 = 0$. Initial guesses in dark areas converge faster than initial guesses in light areas to their respective solution.

- Embed this algorithm into two *For* loops such that the space $(x^{(0)}, y^{(0)}) \subset [-3;3] \times [-3;3]$ is scanned with two steps Δx and Δy of your choice:
  - If $(x^{(0)}, y^{(0)})$ leads to (x_1, y_1), use a red scale.
  - If $(x^{(0)}, y^{(0)})$ leads to (x_2, y_2), use a green scale.
  - Use a blue color otherwise.

  Make sure that the final plot clearly shows how many Newton's iterations are required to converge to a solution within a given tolerance (*i.e.* 8 digits for instance).
- Comment. Could you find an equation of the boundary separating the two basins of attraction?

1.3 Broyden's method

Adapt the algorithms developed for Newton's method to Broyden's method:

- Illustrate the sensitivity of the basins of attraction to the initial guess of the Broyden's matrix **A**. Try the identity matrix as well as the Jacobian matrix as initial guesses.
- Comment.

Hints Both Newton's and Broyden's methods involve systems of linear equations and your strategies should make use of the gaussian elimination with partial pivoting.

Useful Matlab commands imshow; imwrite; helpdesk; plot; pcolor;

2 Audio time-stretching

In this project, we consider that audio time-stretching is a numerical technique which is capable of changing the duration of an audio signal. Consider a monophonic audio sample which lasts no more than 10 seconds with a sample rate of 44,100 Hz (which is the current professional standard in audio devices). This monophonic audio sample can numerically be viewed as a vector with $44{,}100 \times d$ coordinates where d is a positive integer denoting the duration of the sample in seconds.

2.1 Interpolation or approximation?

Among the numerical techniques dedicated to the interpolation and approximation of discrete data, which one would you use to reduce the speed of your sample by a factor $\alpha \in [1;2]$ (2 means that your new sample is twice as long as the original one while 1 means that your sample is left unchanged)?

- Explain your choice.
- Provide a general algorithm which implements the selected approach.
- What is the main drawback of this strategy?

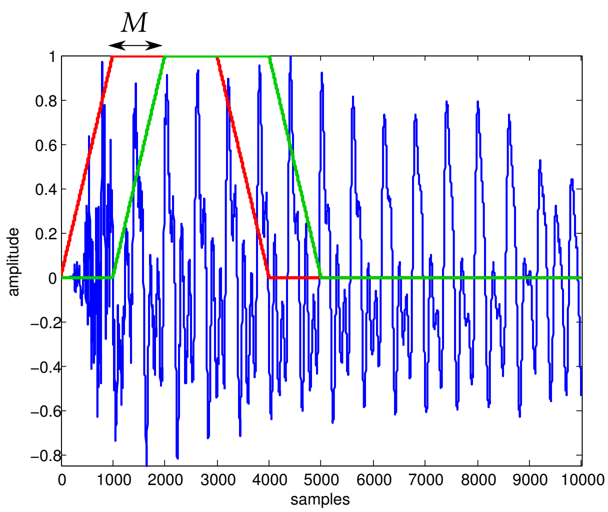
Hints This problem is quite open and there is no “correct" solution. Therefore, make sure to clearly explain your strategy.

2.2 Granular synthesis

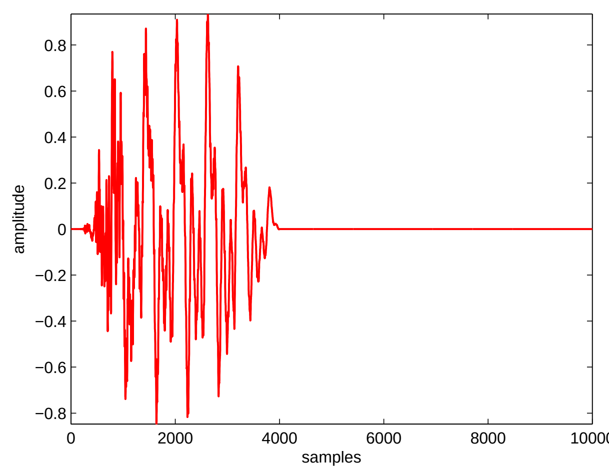
A direct interpolation or approximation of a digital audio signal may be an intuitive and easy technique but it does not provide the best results. A more sophisticated method relies on the decomposition of an acoustic signal into short-duration acoustic elements, called grains. The extraction of a grain is performed by multiplying the signal by a continuous and positive window function which is non-zero for a given duration only. Denoting n the sample of interest[1], the multiplying windows $w[n]$ considered in this work are as simple as possible. They are piecewise and described by three portions and respective parameters N_1, N_2, and N_3:

$$w[n] = \begin{cases} (n - N_1)/N_2 & \text{if } N_1 \leq n < N_1 + N_2 \\ 1 & \text{if } N_1 + N_2 \leq n < N_1 + N_2 + N_3 \\ (N_1 + 2N_2 + N_3 - n)/N_2 & \text{if } N_1 + N_2 + N_3 \leq n < N_1 + 2N_2 + N_3 \end{cases} \tag{2}$$

The resulting operation is depicted in Figure 2 for two windows. These grains can then be spaced and



(a) Initial signal and two windows



(b) Typical resulting grain

Figure 2: Grain extraction

recombined to alter the time-scale of the signal. This is illustrated in Figure 3. Note that these windows are defined in such a way that their sum during the recombination stage is always unity except for the first portion of the first window and the third portion of the last window.

- Define a typical window by choosing parameters N_1, N_2, and N_3 and plot it. The other windows will be a translation of the latter.
- Assume a constant phase shift of M samples (illustrated in Figure 2a) between successive windows during the extraction of the grains . Write and implement a general algorithm able to time-stretch

[1] As opposed to the previous use of the term *sample*, it here indicates *one* coordinate of the discrete data.

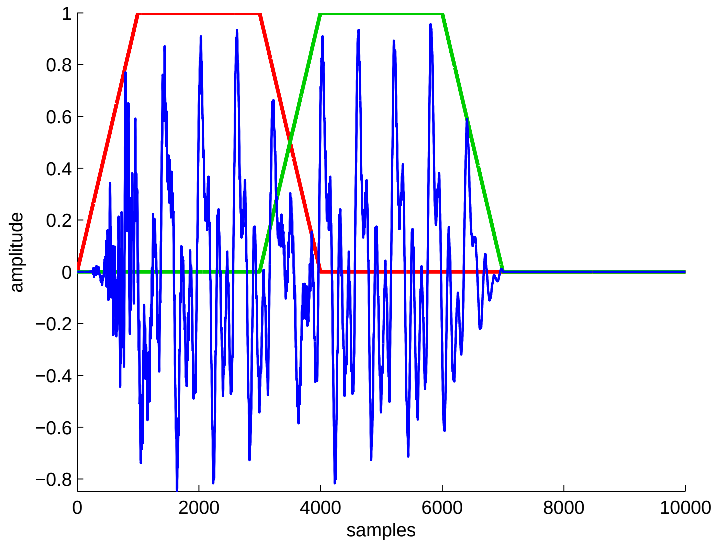


Figure 3: Grain recombination. Notice the new position of the green window compared to Figure 2a

an audio signal of a factor $\alpha \in [1;2]$ through the described methodology. The four parameters M, N_1, N_2, and N_3 should be controllable by the user.

- Compare to the previous approach. What is its main advantage?
- Quickly describe how would you reduce the duration of the sample?

Useful Matlab commands wavread; wavwrite; sound; soundsc; helpdesk; plot;