PROJECT: FREE VORTEX DESIGN

Presented for the course:
MECH 535 – Turbomachinery & Propulsion

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December 10th, 2012

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Introduction

Flow through a turbomachine is three-dimensional. Centrifugal forces and forces exerted by pressure act on the fluid particles and change their course from travelling in parallel streams. This in turn alters the velocity profile of the flow. The flow and its thermophysical properties are therefore dependent on the position along the radius of a compressor or turbine blade. The Radial Equilibrium Equation takes into account the three-dimensional aspect of the flow.

The objective of this project is to use solve the Radial Equilibrium Equation for a free-vortex design and plot the 3-dimensional blade shape that results for the given whirl distribution. The finite-difference method will be used to solve the REE. Although the finite-element method is more powerful than the finite-difference method, the latter is simpler and yields similarly accurate results for rectilinear gas paths. Since the project deals with one rotor blade, fluctuations in the gas paths will be small and the finite-difference method will yield acceptable results.

Methodology

Step 1: Initialization

Aside from initializing all the given properties of the problem (i.e. number of stations, mass flow rate, hub and shroud radii, speed of rotation etc.), certain variables need to be initialized as matrices for updating.

An initial "guess for the stream function is made using the following equation:

$$\Psi(i,j) = \frac{\left({r_{i,j}}^2 - {r_h}^2\right)}{\left({r_s}^2 - {r_h}^2\right)}$$

The density matrix is populated with the given inlet stagnation density everywhere. In the case that the fluid is incompressible, the density will remain constant throughout; if compressible, the density matrix must be updated (see section below, "Compressibility.")

The stagnation pressure matrix is populated with the given stagnation pressure everywhere. The total enthalpy matrix is populated knowing the stagnation temperature and the specific heat, and the value is applied everywhere in the matrix. Entropy is set to zero everywhere, as is the $\omega_{i,j}$ ("RHS") value. Values for rC_w is given before and after the rotor in the free vortex design, and remain constant outside of the rotor (in the duct). In the rotor area, rC_w is linearly interpolated using the given values.

Step 2: Velocity Components

After the initialization of the variables, the iteration process starts, with the objective to converge to the solution.

First, the axial and radial components of the fluid velocity (C_z and C_r, respectively) are computed. These two velocity components can be written in terms of the stream function as follows:

$$C_z = \frac{m}{2\pi\rho r}\frac{\partial\Psi}{\partial r}$$

$$C_r = -\frac{m}{2\pi\rho r}\frac{\partial\Psi}{\partial z}$$

Rewriting these velocities in finite difference representation, the following is obtained:

$$C_{z_{i,j}} = \frac{m}{2\pi\rho_{i,j}r_{i,j}} \frac{[\Psi(i,j+1) - \Psi(i,j-1)]}{2\Delta r}$$

$$C_{r_{i,j}} = -\frac{m}{2\pi\rho_{i,j}r_{i,j}} \frac{[\Psi(i,j+1) - \Psi(i,j-1)]}{2\Delta z}$$

Boundary conditions

For the velocity components, forward and backward difference approximations are used for the duct inlet and outlet boundaries, respectively.

$$C_{z_{i,j}} = \frac{m}{2\pi\rho_{i,j}r_{i,j}} \frac{[-3\Psi(i,j)+4\Psi(i,j+1)-\Psi(i,j+2)]}{2\Delta r} \qquad \text{at hub}$$

$$C_{z_{i,j}} = \frac{m}{2\pi\rho_{i,j}r_{i,j}} \frac{[-3\Psi(i,j)+4\Psi(i,j-1)-\Psi(i,j-2)]}{2\Delta r} \qquad \text{at shroud}$$

$$C_{r_{i,j}} = -\frac{m}{2\pi\rho_{i,j}r_{i,j}} \frac{[-3\Psi(i,j)+4\Psi(i+1,j)-\Psi(i+2,j)]}{2\Delta r} \qquad \text{at duct inlet}$$

$$C_{r_{i,j}} = -\frac{m}{2\pi\rho_{i,j}r_{i,j}} \frac{[-3\Psi(i,j)+4\Psi(i-1,j)-\Psi(i-2,j)]}{2\Delta r} \qquad \text{at duct outlet}$$

Step 3: Compressibility

This step is ignored if the fluid is incompressible. In the case that the fluid is compressible through the compressor, the fluid density will vary and affect its velocity. Therefore, the density will have to be updated along with the other values in the computation. The ideal gas law is used to computer the densities at various positions along the length of the blade. The following relations are used to find the properties necessary to calculate the density:

$$C_{i,j}{}^2 = \left(\frac{(rC_w)_{i,j}}{r_{i,j}}\right)^2 + C_{z_{i,j}}{}^2 + C_{r_{i,j}}{}^2$$

$$h_{i,j} = H_{0_{i,j}} - \frac{C_{i,j}{}^2}{2}$$

$$7p_{i,j} = p_{0_{i,j}} \left(\frac{h_{i,j}}{H_{0_{i,j}}}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\rho_{i,j} = \frac{p_{i,j}}{Rh_{i,j}}$$

Step 4: Tracing the Thermodynamic Variables

The streamline of the flow needs to be traced from the stream function from station to station. For a value of a stream function at a particular station, the origin of that streamline is traced back to a point along the vertical station just before the station in question. Once the origin is located, the streamline can be linearly interpolated between the stations. The thermophysical properties of the flow can be evaluated at the location of the origin, which can be used in turn to re-estimate the stream function.

Loss Coefficient

Aerodynamic losses are considered when calculating total pressure during the calculation of the thermophysical properties. They account for the pressure loss across the blade. These losses occur only between the leading edge and trailing edge of the blade, and are linearly interpolated from 0 to 0.03 from edge-to-edge along the z-axis of the compressor.

Step 5: Radial Equilibrium Equation

The Radial Equilibrium Equation (REE) relates the thermodynamic variables of the fluid (i.e. enthalpy, temperature, and entropy) and the velocity components of the fluid. A stream function, Ψ, is derived to be a function of the velocity components of the fluid, and the REE can be arranged in terms of stream function. The REE can therefore be expressed as:

$$\frac{\partial}{\partial z}\left(\frac{1}{\rho r}\frac{\partial \Psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial \Psi}{\partial r}\right) = -\frac{2\pi}{m C_z}\left[\frac{C_\theta}{r}\frac{\partial}{\partial r}(r C_\theta) + T\frac{\partial s}{\partial r} + \frac{\partial H_0}{\partial r}\right]$$

Using finite differences, the above REE transforms into the following

$$-\left[\frac{\Psi(i,j)}{\rho r_{i+\frac{1}{2},j}} + \frac{\Psi(i,j)}{\rho r_{i-\frac{1}{2},j}} + \frac{\Psi(i,j)}{\rho r_{i,j+\frac{1}{2}}} + \frac{\Psi(i,j)}{\rho r_{i,j-\frac{1}{2}}}\right] + \left[\frac{\Psi(i+1,j)}{\rho r_{i+\frac{1}{2},j}} + \frac{\Psi(i-1,j)}{\rho r_{i-\frac{1}{2},j}} + \frac{\Psi(i,j+1)}{\rho r_{i,j+\frac{1}{2}}} + \frac{\Psi(i,j-1)}{\rho r_{i,j-\frac{1}{2}}}\right] = -\Delta z^2 \omega_{i,j}$$

where

$$\omega_{i,j} = -\frac{2\pi}{2\Delta r m C_{z\,i,j}}\left[\frac{C_{\theta-i,j}}{r_{i,j}}\left[(rC_\theta)_{i,j+1}-(rC_\theta)_{i,j-1}\right]+T\left[S_{i,j+1}-S_{i,j-1}\right] + \left[H_{0\,i,j+1}-H_{0\,i,j-1}\right]\right]$$

Finally, this can be rewritten as the follow to solve for the stream function, and is used as the governing equation for this problem:

$$\Psi(i,j) = A_{i,j}\left[B_{i,j}+\Delta z^2\omega_{i,j}\right]$$

$$A_{i,j} = 1\Big/\left[\frac{1}{\rho_{i+\frac{1}{2},j}r_{i+\frac{1}{2},j}}+\frac{1}{\rho_{i-\frac{1}{2},j}r_{i-\frac{1}{2},j}}+\frac{1}{\rho_{i,j+\frac{1}{2}}r_{i,j+\frac{1}{2}}}+\frac{1}{\rho_{i,j-\frac{1}{2}}r_{i,j-\frac{1}{2}}}\right]$$

$$B_{i,j} = \left[\frac{\Psi(i+1,j)}{\rho_{i+\frac{1}{2},j}r_{i+\frac{1}{2},j}}+\frac{\Psi(i-1,j)}{\rho_{i-\frac{1}{2},j}r_{i-\frac{1}{2},j}}+\frac{\Psi(i,j+1)}{\rho_{i,j+\frac{1}{2}}r_{i,j+\frac{1}{2}}}+\frac{\Psi(i,j-1)}{\rho_{i,j-\frac{1}{2}}r_{i,j-\frac{1}{2}}}\right]$$

Boundary Conditions

A Dirichlet boundary condition is automatically satisfied at the bottom and top limits since the values for the stream function are set to 0 and 1 at the hub and shroud, respectively. At the duct inlet, conditions remain constant. At the duct outlet, the Neumann boundary condition is used and substituted into Eq(1):

$$B_{i,j} = \left(\frac{2\Psi(N-1,j)}{\rho_{i+\frac{1}{2},j}r_{i+\frac{1}{2},j}}+\frac{\Psi(i,j+1)}{\rho_{i,j+\frac{1}{2}}r_{i,j+\frac{1}{2}}}+\frac{\Psi(i,j-1)}{\rho_{i,j-\frac{1}{2}}r_{i,j-\frac{1}{2}}}\right)$$

Results

3D Blade Shape

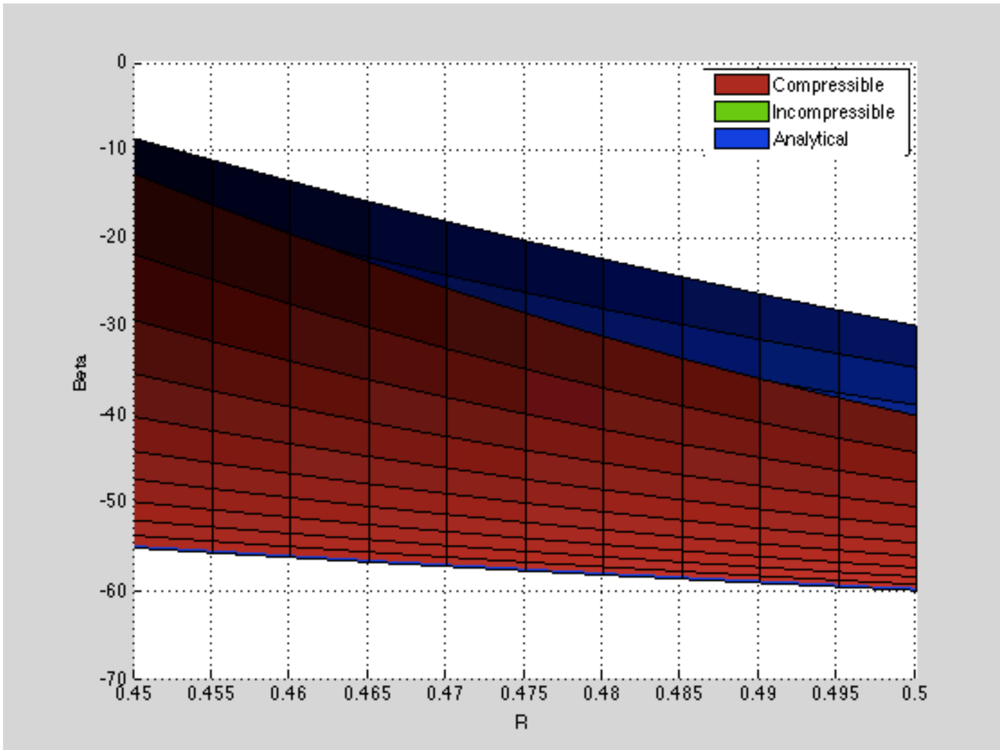


Figure 1: 3D Blades Plot View 1

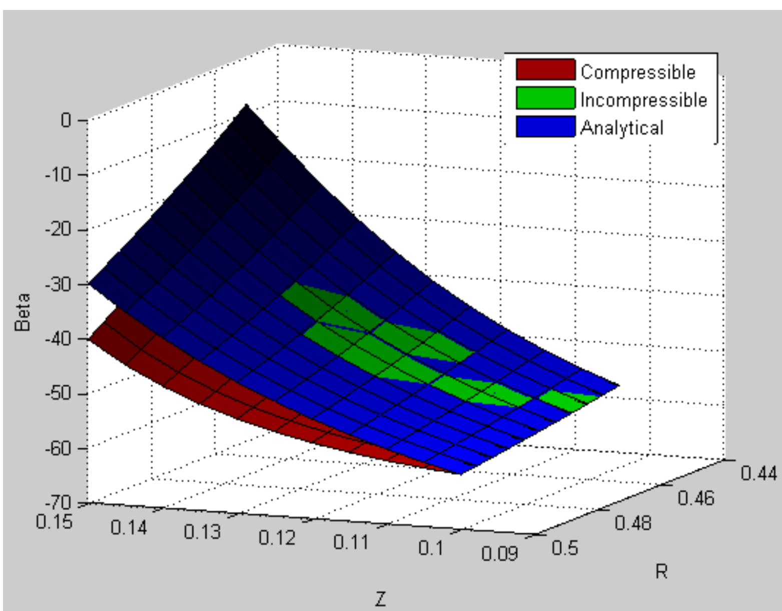


Figure 2: 3D Blades Plot View 2

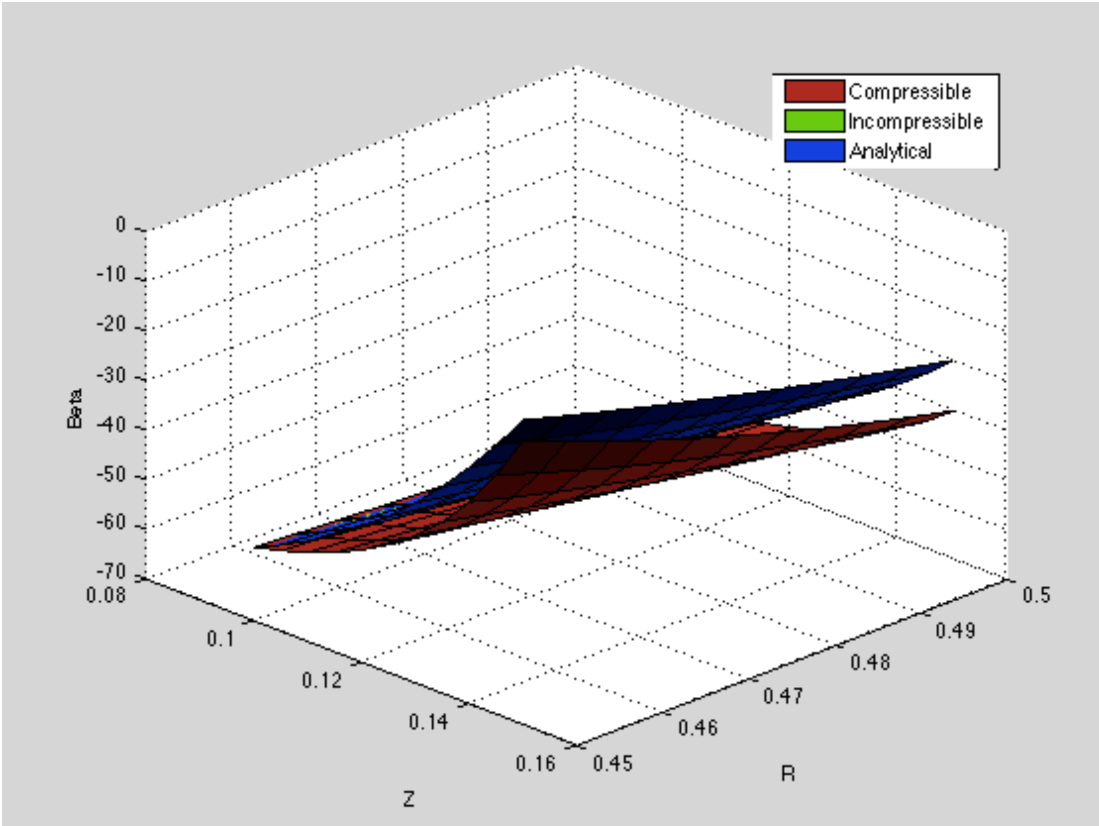


Figure 3: 3D Blades Plot Isometric View

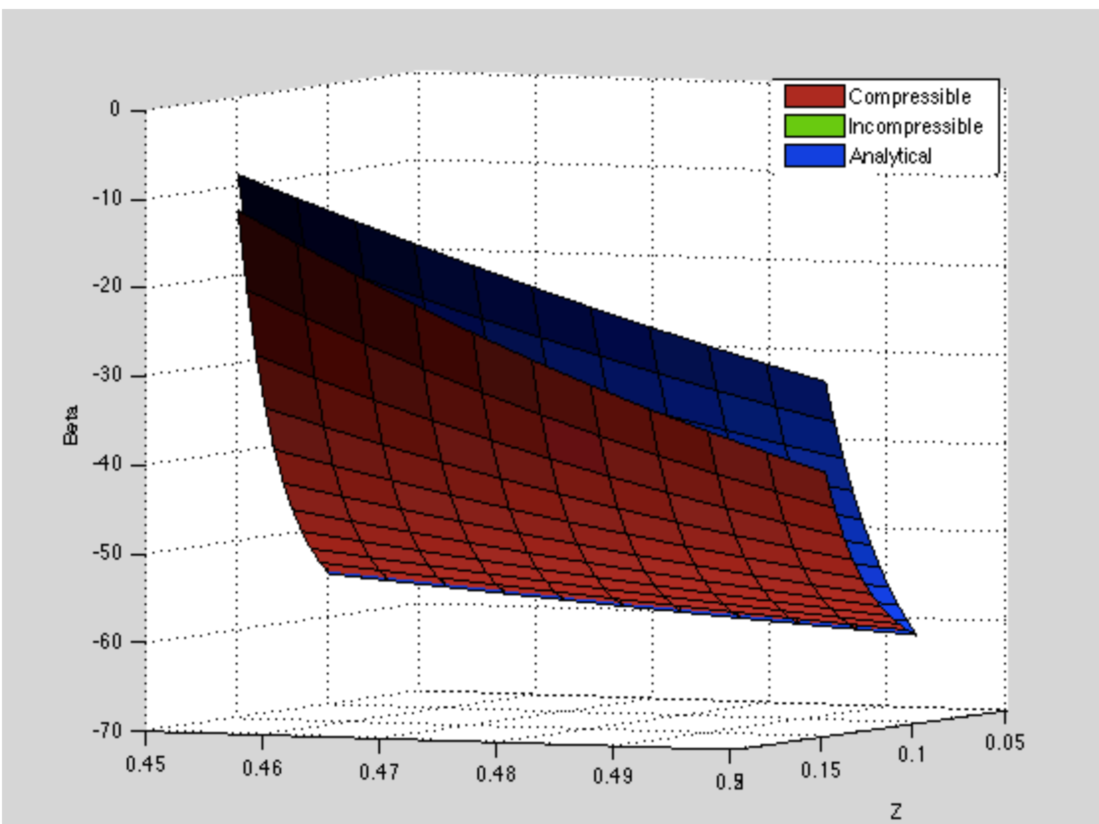


Figure 4: 3D Blades Plot View 4

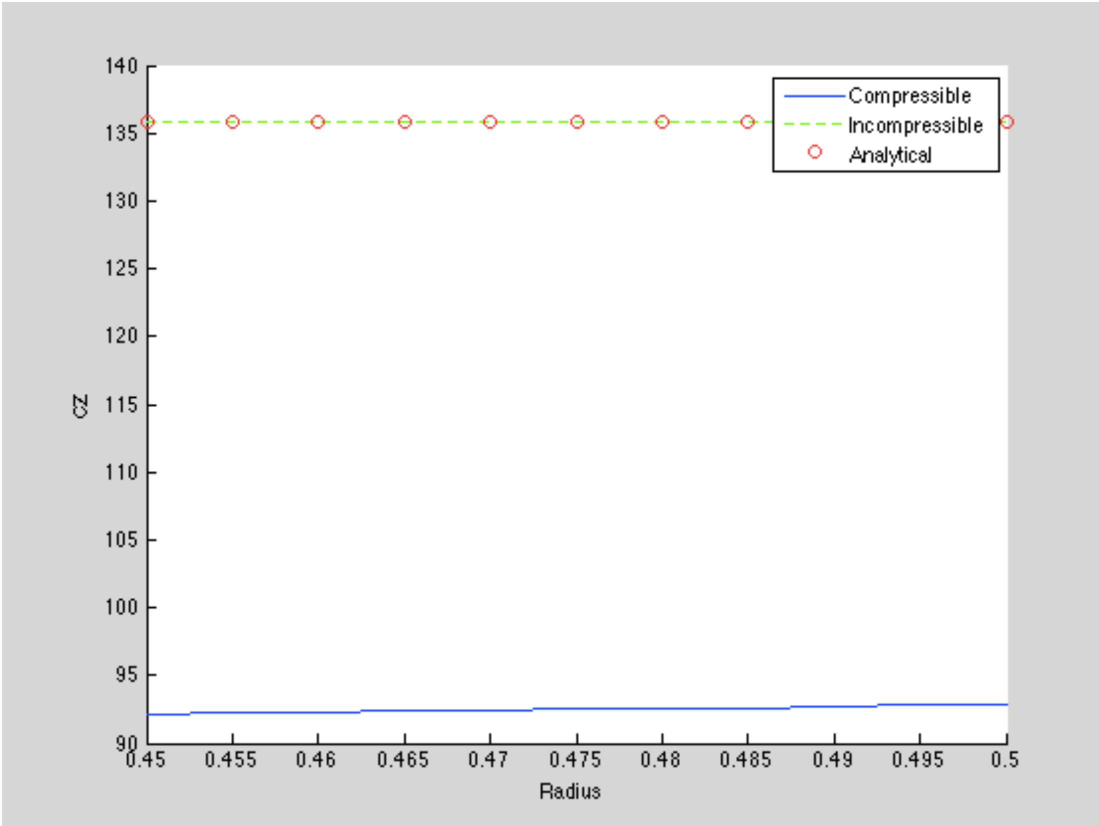


Figure 5: Cz vs Radius

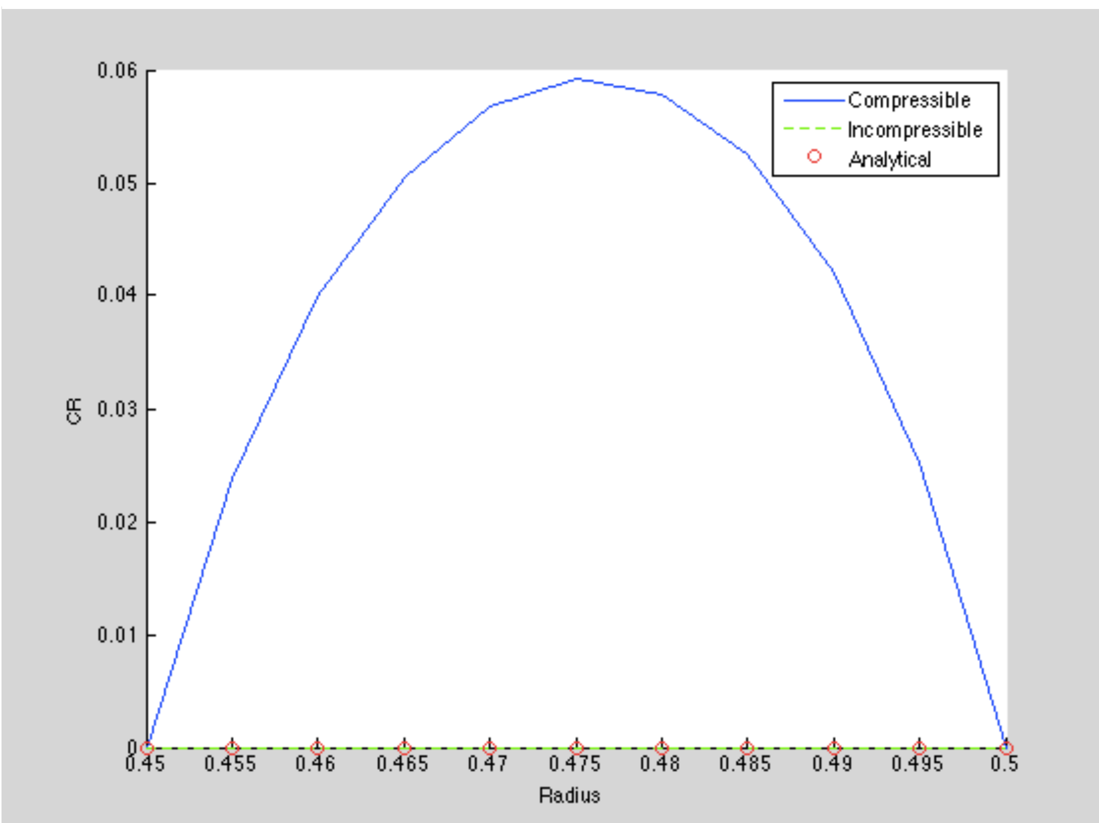


Figure 6: Cr vs Radius

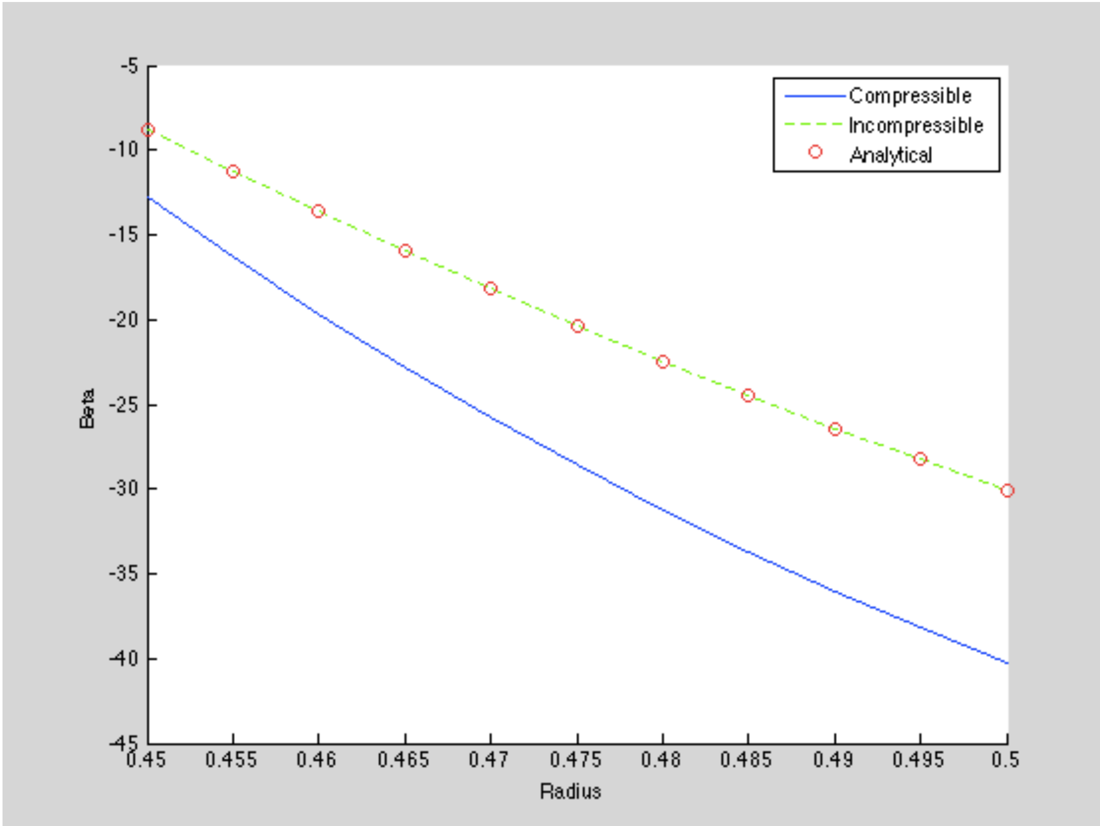


Figure 7: β vs Radius

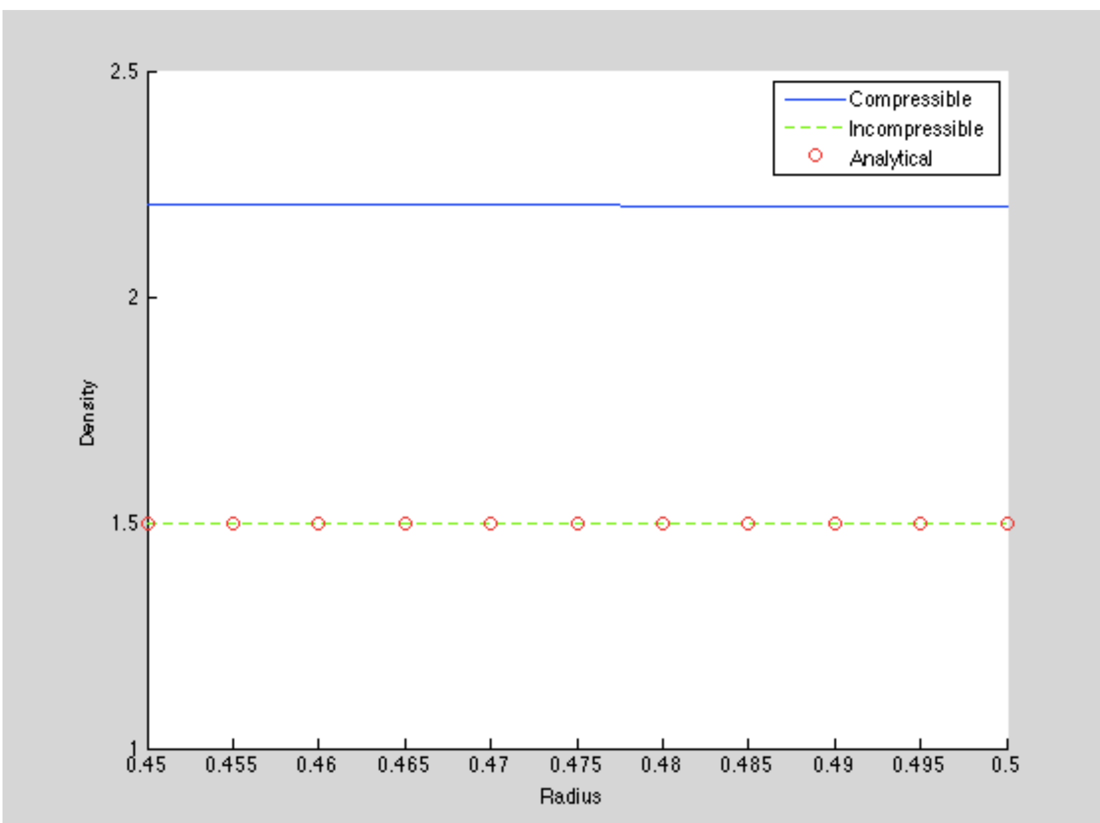


Figure 8: ρ vs Radius

Analysis

Variation of CZ versus Radius at the trailing edge

The value of CZ is constant from hub to shroud for all 3 cases, with slight variations in the compressible case. This is to be expected, since the shape of the blade is designed to ensure that the work done is uniform from root to tip. The value of CZ is lower for the compressible case at the trailing edge because there are losses present in the rotor. The other two cases have ω=0, meaning that there is no loss in pressure and hence no loss in CZ across the rotor blade. The absolute velocity, on the other hand, increases across the rotor for all 3 cases, because the whirl component of velocity increases across the rotor no matter what.

Variation of CR versus Radius at the trailing edge

For Case 2 and 3, CR is assumed to be 0, and this reflects itself in their plots.
For Case 1, CR is very close to 0 and displays a parabolic shape. Close to the hub and shroud it is 0, while at the mean-line it is maximized.

Variation of Density versus Radius at the trailing edge

Both Case 2 and 3 have constant density everywhere in the duct. This is normal, since both cases were assumed to be incompressible. Case 1 has a nearly constant density from hub to shroud at the trailing edge, though it is slightly higher at the hub. The density increases across the rotor blade, and stays constant before and after it (although this cannot be seen from the 2D plot).

Variation of β versus Radius at the trailing edge

The results show that the blade is more curved at the hub than at the tip, which is to be expected. The blade shapes for Cases 2 and 3 match exactly. There is a difference between Case 1 and the two other Cases. The compressible case has a greater blade curvature distribution at the trailing edge than the other 2 cases.

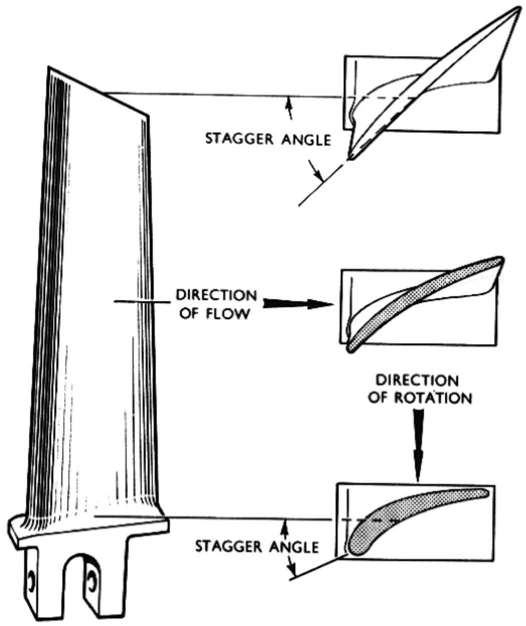


Figure 9: Change in Stagger Angle in the Radial Direction

5. Using the results of this design code (given a whirl distribution, find blade angles) ***if you were to*** *reverse the problem to an analysis mode (blade angle now is known from numerical solution; find the corresponding whirl distribution) would any differences in the results occur; i.e. would the distribution of the whirl at the trailing edge be different from the one specified at the beginning of problem (hint: consider errors that occur from discretization of a continuum).*

From the governing equation for the through-flow in terms of stream function,

$$\frac{\partial}{\partial z}\left(\frac{1}{\rho r}\frac{\partial \Psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial \Psi}{\partial r}\right) = -\frac{2\pi}{mC_z}\left[\frac{C_\theta}{r}\frac{\partial}{\partial r}(rC_\theta) + T\frac{\partial s}{\partial r} - \frac{\partial H_o}{\partial r}\right]$$

the first term on the RHS of the equation needs to be grouped with the LHS in **analysis mode** (will be discussed in Question 6). In this case, the values of C_w need to be replaced by these equations:

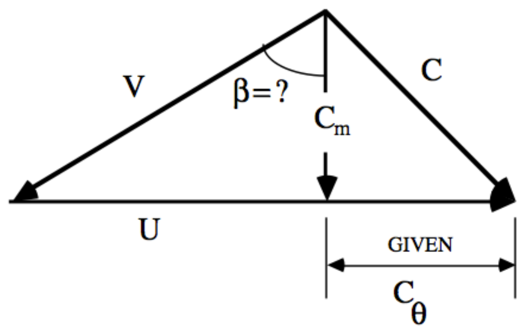
$$V_{\theta_2} = V_m \cdot \left[\tan\beta_2\right]_{\text{given}}$$

$$C_{\theta_2} = V_{\theta_2} + \Omega r_2$$

Furthermore, they will need to be replaced by terms involving the stream function and will, thus, be involving extra discretization steps as C_w involves C_r and C_z. That being said, the final distribution obtained from reversing the problem will involve more calculation steps and higher errors although the distribution should be close to the original one from the design problem.

6. Would computing time be similar between design and analysis or is it inherent that the analysis problem should take longer? Discuss, by showing which terms of the REE are unknown in each case and by using the velocity triangles.

For design mode, the swirl at the leading edge and trailing edge is known. In order words, the relation between rC_w *vs* r is specified at both locations. Thus, the REE is solved in order to obtain the distribution of C_m *vs* r. Once this is obtained, the blade angle β can be determined. This can be observed from the following velocity triangle.

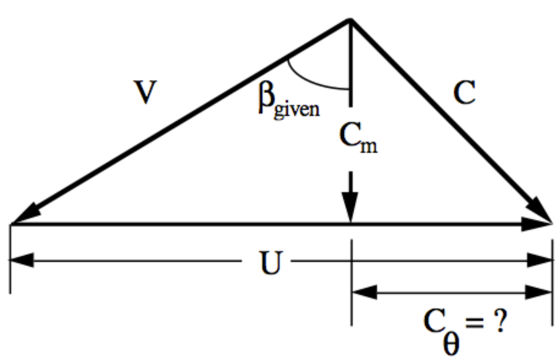


From the REE governing the through-flow in terms of stream function given below,

$$\frac{\partial}{\partial z}\left(\frac{1}{\rho r}\frac{\partial \Psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial \Psi}{\partial r}\right) = -\frac{2\pi}{mC_z}\left[\frac{C_\theta}{r}\frac{\partial}{\partial r}(rC_\theta) + T\frac{\partial s}{\partial r} - \frac{\partial H_o}{\partial r}\right]$$

the LHS and the two last terms of the RHS are still unknown and need to be solved through the iterative process.

For analysis mode, the blade angle β is known. In other words, the relation between C_m and C_w is specified at all r. Thus, the REE is solved in order to obtain the distribution of C_m *vs* r and C_w *vs* r. This can be observed from the following velocity triangle.



In this case, the first term on the RHS of the REE is also unknown and needs to be grouped with the LHS.

Intuitively, the computing time should take longer due to the fact that the analysis mode involves more calculation steps as the blade angle β only specifies the ratio between C_m and C_w.

Conclusion

- Incompressible and analytical cases were in excellent agreement
- CZ, Density constant from hub to shroud for the 3 cases
- Blade more twisted at hub, less twisted at shroud.
- Blade more staggered at shroud, less staggered at shroud
- Density increases across rotor for compressible case
- CZ is lowered when there are losses in the rotor
- Compressible flow with losses requires a more curved blade than incompressible flow
- Design problems require less computation time and are easier to perform than analysis problems.

Appendix

1. Project Description

PROJECT 1: FREE -VORTEX DESIGN

We shall numerically repeat the example on pages 157-160 of Dixon, for which the following data is given:

$$
\begin{aligned}
&\text{Before rotor,} \quad rC_{\theta_1} && = 39.3\,\mathrm{m}^2/\mathrm{s}\ \text{(constant with radius: free vortex)} \\
&\text{After rotor,} \quad rC_{\theta_2} && = 117.8\,\mathrm{m}^2/s\ \text{(constant with radius: free vortex)} \\
&N && = 6000\ \text{rpm} \\
&r_{shroud} && = 0.50\ \text{m} \\
&r_{hub} && = 0.45\ \text{m} \\
&\text{mass flow (m)} && = 30.4\ \text{kg/s} \\
&T_o\ \text{(inlet)} && = 288^\circ K \\
&\rho_o\ \text{(inlet)} && = 1.5\ \text{kg/m}^3 \\
&\varpi\text{, at blade trailing edge} && = 0.03
\end{aligned}
$$

By definition, ϖ=0 at L.E., hence ϖ at any x location between L.E. and T.E. is to be linearly interpolated in z.

We shall solve the REE equation for the above conditions, for a rotor placed in a duct. Choose the axial width of the rotor to have an aspect ratio, height/chord, of unity. Place the inlet and exit conditions 2 blade widths away from the leading and trailing edges, respectively. Take 9 intra-blade stations in each blade row, excluding the leading and trailing edges (i.e. 10 stations from L.E to T/E.). Use the same subdivisions in the radial direction, so that $\Delta z/\Delta r$=1.

2. Derivation of analytical solution

Franck Ah-Kane

Part 3 : analytical solution 04/12/2012

Quasi-1D assumption, i.e $C_r = 0$, $C_x =$ constant and also $\rho = \rho_0$

Given:
- $rC_{\theta_2} = 117.8 \frac{m^2}{s}$
- $N = 6000$ rpm
- $r_s = 0.50$ m
- $r_h = 0.45$ m
- $\dot{m} = 30.4 \frac{kg}{s}$
- $\rho_0 = 1.5 \frac{kg}{m^3}$

① $\Omega = 2\pi \frac{N}{60} = 628.319 \frac{rad}{s}$

$U = \Omega \cdot r$

② find C_x: $\dot{m} = \rho C_x A$

$C_x = \frac{\dot{m}}{\rho A} = 135.812 \frac{m}{s}$

③ Solve for β :

at the trailing edge, $rC_{\theta_2} = 117.8 \frac{m^2}{s}$

$C_{\theta_2} = \frac{117.8}{r} \left(\frac{m^2}{s}\right)$

$$\left.\begin{aligned} V_\theta &= C_x \tan\beta \\ V_\theta &= U - C_W \end{aligned}\right\}$$

combining these two,

$$\beta(r) = \tan^{-1}\left(\frac{\Omega r - \frac{117.8}{r}}{C_x}\right)$$

So the β at the trailing edge as a function of the radius is given by

$$\boxed{\beta(r) = \tan^{-1}\left(\frac{\Omega r - \frac{117.8}{r}}{C_x}\right)}$$