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Quaternion Feedback Regulator for Large Angle Maneuvers of Underactuated Spacecraft

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***Abstract*— A quaternion feedback regulator is developed to yield three-axis rate stabilization of an underactuated rigid spacecraft's attitude with two body-fixed control torques and arbitrary inertia matrix. Stable control is achieved by properly combining a generalized inverse component for feedback linearization and an auxiliary input to stabilize the underactuated system. The proposed control yields local stability within a domain of attraction through perturbed feedback linearization. The generalized inverse is damped near a bound on an inherent singularity to provide piecewise-smooth control. Numerical simulation results for a detumbling to a desired orientation maneuver are presented for an asymmetric spacecraft with two bounded body-fixed control torques to demonstrate the capability of the proposed attitude control method. Although this control method is not intended to provide attitude maintenance for attitude tracking or in the presence of relatively large disturbance torques, it may prove widely applicable for detumbling and reorientation maneuvers of spacecraft with only two available control torques.**

I. INTRODUCTION

THE problem of stabilizing a fully actuated rigid spacecraft's attitude dynamics and kinematics has been studied over the years [1] - [4]. However, in space applications the question of controlling an underactuated spacecraft is particularly significant in view of possible actuator failures or when proposing minimally designed spacecraft systems. In particular, underactuated control could provide a key enabling technology in the Operationally Responsive Space construct where new designs that are smaller, faster and cheaper to manufacture and employ are envisioned to replace typical large monolithic spacecraft [5].

The investigation of stabilization of underactuated spacecraft kinematics and attitude dynamics began in [6] with Crouch's theoretical establishment of the necessary and sufficient conditions for the controllability of a rigid body's attitude with either gas thrusters or momentum exchange devices. He concluded that, for a spacecraft with momentum exchange devices, controllability is impossible with fewer than three, while for a spacecraft with independent paired jets, controllability is possible with two. Kerai later demonstrated in [7] that, by using geometric control theory, small-time local controllability of the rigid body equations assuming paired gas jets can indeed be achieved with only two control torques. Byrnes and Isidori proved in [8] that the full angular motion equations for a rigid spacecraft with only two controls cannot be asymptotically stabilized by smooth pure state feedback because they violate Brockett's theorem from [9] on nonholonomic underactuated systems. With this in mind, they proposed a smooth feedback controller to affect partial stabilization of the rigid body model resulting in a revolute constant-rate motion about the uncontrolled axis of rotation.

Later, Krishnan, McClamroch and Reyhanoglu proposed a hybrid control design in [10] that combined continuous time features with discrete event features to affect a discontinuous feedback control strategy to stabilize any equilibrium attitude of an underactuated spacecraft with two momentum wheel actuators in finite time under the restriction that the total angular momentum vector of the system is zero. This control methodology translates directly to a study of an underactuated axisymmetric spacecraft. Tsiotras, Corless, and Longuski in [11] and Tsiotras and Luo in [12] also dealt with control of underactuated axisymmetric spacecraft by proposing a time-invariant feedback control law to asymptotically stabilize the orientation of two of the three body-fixed axes. In addition to only providing for partial attitude stabilization of axisymmetric spacecraft, their discussion was limited to cases where the angular velocity about the unactuated axis is zero at the start of the maneuver. Tsiotras and Schleicher in [13] and Tsiotras and Doumtchenko in [14] relaxed the restriction on the symmetry of the spacecraft slightly to consider a nearly axisymmetric spacecraft by a small parameter and a set of time-invariant control laws are proposed to stabilize the angular velocity and attitude of a spacecraft about a certain axis by virtual control inputs of the two actuated angular rates.

The global asymptotic rate stabilization problem without concern for kinematics of a fully asymmetric underactuated rigid spacecraft was addressed by Coverstone-Carroll in [15] through the use of a Variable Structure Controller (VSC). Bajodah also addressed the rate-only stabilization problem for detumbling maneuvers in [16] through the use of singularly perturbed feedback linearization and generalized inverse control methodologies. Although both of these controllers prove to be robust to large initial angular velocities around all three axes in the presence of actuator torque limitations, they both require an additional controller

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to provide desired kinematic alignment after the detumbling maneuver. One such controller, as proposed in [15], is a simple linear controller that is used to perform a series of eigenaxis rotations which precludes smooth attitude tracking.

The problem of stabilizing both the kinematics and dynamics of an underactuated asymmetric spacecraft was most recently addressed in [17] by Casagrande, Astolfi, and Parisini who proposed a time-variant switching control law to effect global asymptotic stabilization of the closed-loop system. Although novel, the proposed law lacks detailed simulation results by considering only the case where the initial angular rates about two of the axes to include the unactuated axis are initially zero. Furthermore, real-world spacecraft with flexible parts, antennas, fuel slosh, etc., may preclude the use of time-variant control laws because they have the distinct potential of producing unacceptable transient response and might therefore lead to instability [14]. Finally, nonlinear tracking control of an axisymmetric spacecraft is addressed in [18] by developing a kinematic controller to determine the desired actuated angular rates which are in turn used as control inputs to the dynamic system through the use of standard back-stepping techniques. This method yields only asymptotic dynamic and kinematic stabilization results for an axisymmetric rigid body given the restriction that the angular rate about the unactuated axis is close to zero but it does yield bounded results otherwise.

The goal of this work is to extend the research into control algorithms for underactuated rigid spacecraft attitude control by proposing a novel time-invariant, piecewise-smooth quaternion feedback regulator based on generalized inverse methods to affect three-axis attitude stabilization of the error quaternion kinematics for an underactuated rigid spacecraft with arbitrary inertia matrix and two bounded body-fixed torques. The main contributions of this paper are twofold: first, it is shown that the method of generalized inverse control can be extended from a study of only rigid-body dynamics as considered in [16] to the full system of dynamics and quaternion kinematics without loss of generalization and without loss of realization. Second, a perturbed feedback linearizing control design is presented along with numerical simulation results for a detumbling to a desired orientation maneuver.

II. Underactuated Rigid Spacecraft Rotational Motion

In this section, the general case of an underactuated rigid spacecraft being rotated by two body-fixed control torques is considered. In order to simplify the problem, an ideal control torque is assumed. Two reference frames are established: an inertial frame $\mathcal{I}$ with orthogonal axes defined by the set of unit vectors $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$ and a body-fixed frame $\mathcal{B}$ with orthogonal axes defined by the set of unit vectors $\{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$.

A. Rotational Dynamics of the Rigid Spacecraft

The rotational motion of a rigid spacecraft about body-fixed axes with their origin located at the body's center of mass can be described using Euler's equations. For the general case, as presented in [19], where the control axes do not coincide with the principal axes of inertia and torquing devices, such as independent gas thrusters, are used to impart torques about these control axes, these become:

$$\dot{\boldsymbol{\omega}}_{BI} = -J^{-1}\omega_{BI}^{\times} J\boldsymbol{\omega}_{BI} + J^{-1}\mathbf{T} \tag{1}$$

where $\boldsymbol{\omega}_{BI}(t) \in \mathbb{R}^3, \forall \boldsymbol{\omega}_{BI} = \left[\omega_{BI,1}, \omega_{BI,2}, \omega_{BI,3}\right]^T$ is the angular velocity vector of the body-fixed frame $\mathcal{B}$ with respect to an inertial frame $\mathcal{I}$ expressed in $\mathcal{B}$ and $\mathbf{T}(t) \in \mathbb{R}^3, \forall \mathbf{T} = \left[{}^{B}T_1, {}^{B}T_2, {}^{B}T_3\right]^T$ is the control torque vector acting on $\mathcal{B}$. Furthermore, the notation $\omega_{BI}^{\times} \in \mathbb{R}^{3\times 3}$ denotes a standard skew-symmetric matrix and $J \in \mathbb{R}^{3\times 3}$ is the inertia matrix with respect to $\mathcal{B}$ with elements J_{ij} where *i* is the *i*-th column and *j* is the *j*-th row.

B. Attitude Error Parameterization via Quaternions

From [18] and [20], the error quaternion kinematic differential equation describing the error between a desired spacecraft body-fixed frame $\mathcal{B}_d$ and the spacecraft's body-fixed frame $\mathcal{B}$ becomes

$$\dot{\mathbf{q}}_{B_dB} = \frac{1}{2}\left(q_{B_dB,4} I_{3\times 3} + q_{B_dB}^{\times}\right)\boldsymbol{\omega}_{BI},\ \dot{q}_{B_dB,4} = -\frac{1}{2}\mathbf{q}_{B_dB}^{T}\boldsymbol{\omega}_{BI} \tag{2}$$

with $\boldsymbol{\omega}_{BI} = \boldsymbol{\omega}_{B_dB}$ considering a rate stabilization maneuver where $\boldsymbol{\omega}_{B_dI} = \mathbf{0}$. For simplicity, the error quaternion $\hat{\mathbf{q}}_{B_dB}(t) \in \mathbb{R}^3 \times \mathbb{R}$ will be represented by $\hat{\mathbf{q}}(t) \in \mathbb{R}^3 \times \mathbb{R}$, $\forall \hat{\mathbf{q}} = [\mathbf{q}, q_4]^T = [q_1, q_2, q_3, q_4]^T$ and $\boldsymbol{\omega}_{BI}$ will be represented by $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T$ for the remainder of the paper.

C. Underactuated Rigid Spacecraft with Two Control Torques

In considering an underactuated rigid spacecraft with only two independent control torques, the unactuated axis can be taken to be $\hat{b}_1$ without loss of generality and Eq. (1) can be rewritten as

$$\dot{\boldsymbol{\omega}} = F(\boldsymbol{\omega})\boldsymbol{\omega} + \boldsymbol{\tau} \tag{3}$$

where $\boldsymbol{\tau}(t) \in \mathbb{R}^3, \forall \boldsymbol{\tau} = J^{-1}\mathbf{T} = [0, \mathbf{u}]^T$ is the scaled control vector with $\mathbf{u}(t) \in \mathbb{U}^2, \forall \mathbf{u} = [u_2, u_3]^T$ and $F(\boldsymbol{\omega}) \in \mathbb{R}^{3\times 3}$ denotes the drift matrix

$$F(\boldsymbol{\omega}) = -J^{-1}\omega^{\times} J \tag{4}$$

In this manner, the full underactuated rigid spacecraft system of angular motion equations become

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{1}{2}\left(q_4 I_{3x3} + q^{\times}\right)\boldsymbol{\omega},\ \dot{q}_4 = -\frac{1}{2}\mathbf{q}^T\boldsymbol{\omega} \\ \dot{\boldsymbol{\omega}} &= F(\boldsymbol{\omega})\boldsymbol{\omega} + \boldsymbol{\tau} \end{aligned} \tag{5}$$

which can be written in control-affine form as

$$\dot{\mathbf{x}} = f(\mathbf{x}) + G(\mathbf{x})\mathbf{u} = F(\mathbf{x})\boldsymbol{\omega} + G\mathbf{u} \tag{6}$$

with state vector $\mathbf{x}(t) \in \mathbb{R}^6 \times \mathbb{R}, \forall \mathbf{x} = [\mathbf{q}, q_4, \boldsymbol{\omega}]^T$.

For convenience, Eq. (6) can be split into four coupled differential subsystems of concern; the first coupled subsystem combines the unactuated dynamics with the pseudo-unactuated vector component of the error quaternion, while the second considers the remaining two vector components, or pseudo-actuated vector components, of the error quaternion. The pseudo-unactuated vector component of the error quaternion is defined as the component numerically corresponding to the unactuated axis which in the considered case is q_1. The "pseudo" prefix is chosen to emphasize the nonlinear nature of the quaternion with respect to the body-fixed frame. Both of these systems are indirectly controlled through the action of the angular velocity components about the actuated body-fixed axes. The third subsystem considers the differential of the scalar component of the error quaternion while the fourth subsystem considers the actuated dynamics.

In order to compactly view the four subsystems of concern, it is useful to first partition Eq. (4) as done in [16] where

$$F(\boldsymbol{\omega}) = \begin{bmatrix} F_{11}(\boldsymbol{\omega}) & F_{12}(\boldsymbol{\omega}) \\ F_{21}(\boldsymbol{\omega}) & F_{22}(\boldsymbol{\omega}) \end{bmatrix} \tag{7}$$

for $F_{11}(\boldsymbol{\omega}) \in \mathbb{R}, F_{12}(\boldsymbol{\omega}) \in \mathbb{R}^{1\times 2}, F_{21}(\boldsymbol{\omega}) \in \mathbb{R}^{2\times 1}, F_{22}(\boldsymbol{\omega}) \in \mathbb{R}^{2\times 2}$, and then consider the drift matrix associated with the vector components of the error quaternion kinematic differential equations of Eq. (5) as

$$F(\hat{\mathbf{q}}) = \frac{1}{2}\left(q_4 I_{3\times 3} + q^{\times}\right) = \begin{bmatrix} F_{11}(\hat{\mathbf{q}}) & F_{12}(\hat{\mathbf{q}}) \\ F_{21}(\hat{\mathbf{q}}) & F_{22}(\hat{\mathbf{q}}) \end{bmatrix} \tag{8}$$

for $F_{11}(\hat{\mathbf{q}}) \in \mathbb{R}, F_{12}(\hat{\mathbf{q}}) \in \mathbb{R}^{1\times 2}, F_{21}(\hat{\mathbf{q}}) \in \mathbb{R}^{2\times 1}, F_{22}(\hat{\mathbf{q}}) \in \mathbb{R}^{2\times 2}$.

By taking the unactuated state vector to be $\mathbf{x}_u(t) \in \mathbb{R}^2, \forall \mathbf{x}_u = [q_u, \omega_u]^T$ where $q_u = q_1$ and $\omega_u = \omega_1$, and the actuated angular velocity vector to be $\boldsymbol{\omega}_a(t) \in \mathbb{R}^2, \forall \boldsymbol{\omega}_a = [\omega_2, \omega_3]^T$, the unactuated subsystem becomes

$$\dot{\mathbf{x}}_u = \begin{bmatrix} 0 & F_{11}(\hat{\mathbf{q}}) \\ 0 & F_{11}(\boldsymbol{\omega}) \end{bmatrix} \mathbf{x}_u + \begin{bmatrix} F_{12}(\hat{\mathbf{q}}) \\ F_{12}(\boldsymbol{\omega}) \end{bmatrix} \boldsymbol{\omega}_a \tag{9}$$

Furthermore, by defining the pseudo-actuated error quaternion vector as $\mathbf{q}_a(t) \in \mathbb{R}^2, \forall \mathbf{q}_a = [q_2, q_3]^T$, the pseudo-actuated error quaternion subsystem becomes

$$\dot{\mathbf{q}}_a = \begin{bmatrix} \mathbf{0}_{2\times 1} & F_{21}(\hat{\mathbf{q}}) \end{bmatrix} \mathbf{x}_u + F_{22}(\hat{\mathbf{q}}) \boldsymbol{\omega}_a \tag{10}$$

The third coupled subsystem considers the differential of the scalar component of the error quaternion such that

$$\dot{q}_4 = -\frac{1}{2}\left(q_u \omega_u + \mathbf{q}_a^T \boldsymbol{\omega}_a\right) \tag{11}$$

The fourth and final coupled subsystem is the actuated subsystem and explicitly contains the control vector $\mathbf{u}$ such that

$$\dot{\boldsymbol{\omega}}_a = \begin{bmatrix} \mathbf{0}_{2\times 1} & F_{21}(\boldsymbol{\omega}) \end{bmatrix} \mathbf{x}_u + F_{22}(\boldsymbol{\omega}) \boldsymbol{\omega}_a + \mathbf{u} \tag{12}$$

III. LINEAR PARAMETERIZATION OF THE UNACTUATED RIGID SPACECRAFT SUBSYSTEM

The unactuated subsystem given in Eq. (9) is indirectly affected by the control vector $\mathbf{u}$ through $\boldsymbol{\omega}_a$. By defining the scalar function $\phi(\mathbf{x}_u): \mathbb{R}^2 \to \mathbb{R}$ as

$$\phi(\mathbf{x}_u) = \omega_u + a q_u \tag{13}$$

where $a > 0$ a scalar on q_u and $\phi(\mathbf{x}_u)$ continuous twice differentiable, the unactuated subsystem Eq. (9) can now be transformed into a stable, linear, second-order dynamic system similar to what is proposed in [4] and [16] where

$$\ddot{\phi}(\mathbf{x}_u) + 2\gamma\dot{\phi}(\mathbf{x}_u) + \gamma^2\phi(\mathbf{x}_u) = 0 \tag{14}$$

with the coefficient $\gamma > 0$.

The first and second time derivatives of $\phi(\mathbf{x}_u)$, along trajectories of the underactuated system in Eq. (6), are given by

$$\dot{\phi}(\mathbf{x}_u) = \left[\partial\phi(\mathbf{x}_u)/\partial\mathbf{x}\right]\dot{\mathbf{x}} = L_{\mathbf{f}}\phi(\mathbf{x}_u) \tag{15}$$

$$\ddot{\phi}(\mathbf{x}_u) = \left(\partial L_{\mathbf{f}}\phi(x_u)/\partial\mathbf{x}\right)\dot{\mathbf{x}} = L_{\mathbf{f}}^2\phi(\mathbf{x}_u) + L_G L_{\mathbf{f}}\phi(\mathbf{x}_u)\mathbf{u} \tag{16}$$

where $L_{\mathbf{f}}\phi(\mathbf{x}_u)$, $L_{\mathbf{f}}^2\phi(\mathbf{x}_u)$ and $L_G L_{\mathbf{f}}\phi(\mathbf{x}_u)$ represent the Lie derivatives, of $\phi(\mathbf{x}_u)$, along the direction of the vector fields defined by $\mathbf{f}(\mathbf{x})$ and G [21]. With $\dot{\phi}(\mathbf{x}_u)$ and $\ddot{\phi}(\mathbf{x}_u)$ given by Eqs. (15) and (16), Eq. (14) can then be expressed as the point-wise linear form

$$\mathbf{a}^T(\mathbf{x})\mathbf{u} = b(\mathbf{x}) \tag{17}$$

where the controls coefficient vector $\mathbf{a}(\mathbf{x}) \in \mathbb{R}^2$ is given by

$$\mathbf{a}(\mathbf{x}) = \left[L_G L_{\mathbf{f}}\phi(\mathbf{x}_u)\right]^T \tag{18}$$

and the scalar controls load $b(\mathbf{x})$ is given by

$$b(\mathbf{x}) = -L_{\mathbf{f}}^2\phi(\mathbf{x}_u) - 2\gamma L_{\mathbf{f}}\phi(\mathbf{x}_u) - \gamma^2\phi(\mathbf{x}_u) \tag{19}$$

The following theorem includes only minor changes from [16] to further expand them to include the full state $\mathbf{x}(t)$ containing both the dynamics and kinematics of the system. The proof of this flows directly from those presented in [16] due to the cascading nature of the controls and is formally defined with the supporting definitions and propositions in [20].

Theorem: *Given a non-singular zero actuated state Jacobian $\mathcal{J}_0(\mathbf{x})$ from [20] of the controls coefficient $\mathbf{a}(\mathbf{x}) = \mathbf{0}$ along $f(\mathbf{x}) = F(\mathbf{x})\boldsymbol{\omega}$ for all $\mathbf{x}(t) \in \mathbb{R}^6 \times \mathbb{R}$, the infinite set of control laws that globally realize the unactuated dynamics by the underactuated rigid spacecraft angular motion equations given by Eq. (6) is given by*

$$\mathbf{u} = \bar{\mathbf{u}} + P(\mathbf{x})\mathbf{y} \tag{20}$$

where $\bar{\mathbf{u}} \in \mathbb{R}^2$ is

$$\bar{\mathbf{u}} = \mathbf{a}^+(\mathbf{x}) b(\mathbf{x}) \tag{21}$$

and $\mathbf{a}^+(\mathbf{x}) \in \mathbb{R}^2$ *represents the Moore-Penrose or generalized inverse of the controls coefficient* $\mathbf{a}(\mathbf{x}) \in \mathbb{R}^2$ *so that*

$$\mathbf{a}^+(\mathbf{x}) = \begin{cases} \mathbf{a}(\mathbf{x}) / \|\mathbf{a}(\mathbf{x})\|^2 & \|\mathbf{a}(\mathbf{x})\| \neq 0 \\ \mathbf{0} & \|\mathbf{a}(\mathbf{x})\| = 0 \end{cases} \tag{22}$$

and $P(\mathbf{x}) \in \mathbb{R}^{2\times 2}$ *represents the null-projector of* $\mathbf{a}(\mathbf{x})$ *such that*

$$P(\mathbf{x}) = I_{2\times 2} + \mathbf{a}^+(\mathbf{x}) \mathbf{a}^T(\mathbf{x}) \tag{23}$$

and $\mathbf{y} \in \mathbb{R}^2$ *is an arbitrarily selected null-control vector.*

Eq. (20) consists of two parts. The first part, given by $\bar{\mathbf{u}}$, is termed the particular solution and acts specifically on the range-space of the generalized inverse of the controls coefficient $\mathbf{a}^+(\mathbf{x})$. The second part, given by $P(\mathbf{x})\mathbf{y}$, is termed the auxiliary solution and resides in the orthogonal complement subspace, or the null-space of the controls coefficient, in $\mathbb{R}^3$ with the null-control vector $\mathbf{y}$ being projected onto this space by means of the null-projector $P(\mathbf{x})$.

As discussed in [4] and [16], the null-control vector $\mathbf{y} \in \mathbb{R}^2$ is not fully arbitrary and should be chosen to yield stability of the closed-loop system of equations. One such null-control vector will be presented in the following section that is perturbed feedback linearizing in nature, yielding local stability and guaranteed boundedness. This null-control vector provides a key contribution to the body of knowledge with respect to underactuated rigid body attitude control.

IV. Development of the Quaternion Feedback Regulators and Stability Analysis

In this section, the stability of the closed-loop system of equations given in Eqs. (9) through (12) is addressed for the proposed null-control vector.

A. Damped Controls Coefficient Generalized Inverse and Null-Projector

From the definition of the generalized inverse of $\mathbf{a}(\mathbf{x})$ given by Eq. (22), a singularity on the closed-loop stability of the system exits. However, given Proposition 1 in [20], if the linear unactuated dynamics Eq. (14) are globally realizable by the underactuated rigid spacecraft system of angular motion equations given in Eq. (6), then

$$\lim_{\phi(\mathbf{x}_u)\to 0} \mathbf{a}(\mathbf{x}) = \mathbf{0} \tag{24}$$

In order to properly place a bound on the control input to allow for the control law derivations that follow, the damped controls coefficient generalized inverse and its corresponding damped controls coefficient null-projector are used as presented in [4]. The damped controls coefficient generalized inverse is formulated by considering an arbitrarily small damping coefficient β_1 to provide a bound on the generalized inverse as the squared Euclidean norm of the controls coefficient tends to zero. Thus, the damped controls coefficient generalized inverse $\mathbf{a}_d^+(\mathbf{x}) \in \mathbb{R}^2$ is

$$\mathbf{a}_d^+(\mathbf{x}) = \begin{cases} \mathbf{a}(\mathbf{x}) / \|\mathbf{a}(\mathbf{x})\|^2 & \|\mathbf{a}(\mathbf{x})\| \geq \beta_1 \\ \mathbf{a}(\mathbf{x}) / \beta_1^2 & \|\mathbf{a}(\mathbf{x})\| < \beta_1 \end{cases} \tag{25}$$

where scalar β_1 is a positive damping coefficient. Bounding the generalized inverse of $\mathbf{a}(\mathbf{x})$ in this manner smoothes the infinite set of control laws presented in Theorem 1. Finally, from [4] it can be shown that $\mathbf{a}_d^+(\mathbf{x})$ pointwise converges to $\mathbf{a}^+(\mathbf{x})$ as β_1 vanishes and that

$$\|\mathbf{a}_d^+(\mathbf{x})\| \leq 1/\beta_1 \tag{26}$$

Likewise, the damped controls coefficient null-projector is modified from Eq. (25) such that

$$P_d(\mathbf{x}) = I_{2\times 2} - \mathbf{a}_d^+(\mathbf{x}) \mathbf{a}^T(\mathbf{x}) \tag{27}$$

Eqs. (25) and (27) imply that

$$P_d(\mathbf{x}) = \begin{cases} I_{2\times 2} - \left[\mathbf{a}(\mathbf{x})\mathbf{a}^T(\mathbf{x})\right] / \|\mathbf{a}(\mathbf{x})\|^2 & \|\mathbf{a}(\mathbf{x})\| \geq \beta_1 \\ I_{2\times 2} - \left[\mathbf{a}(\mathbf{x})\mathbf{a}^T(\mathbf{x})\right] / \beta_1^2 & \|\mathbf{a}(\mathbf{x})\| < \beta_1 \end{cases} \tag{28}$$

and from Eq. (24), during the steady-state phase of response the damped controls coefficient null-projector tends to

$$\lim_{\phi(\mathbf{x}_u)\to 0} P_d(\mathbf{x}) = I_{2\times 2} \tag{29}$$

such that the auxiliary part of the infinite set of control laws expressed in Eq. (20) converges to the null-control vector $\mathbf{y}$. Construction of these control laws with Eqs. (25) and (27) yields the damped control vector $\mathbf{u}_d$ defined as

$$\mathbf{u}_d = \bar{\mathbf{u}}_d + P_d(\mathbf{x})\mathbf{y} \tag{30}$$

where $\bar{\mathbf{u}}_d \in \mathbb{R}^2$ is given by

$$\bar{\mathbf{u}}_d = \mathbf{a}_d^+(\mathbf{x}) b(\mathbf{x}) \tag{31}$$

Substitution of the damped control vector given in Eq. (30) into Eq. (12) yields the closed-loop actuated subsystem

$$\dot{\boldsymbol{\omega}}_a = \left[\mathbf{0} \quad F_{21}(\boldsymbol{\omega})\right]\mathbf{x}_u + F_{22}(\boldsymbol{\omega})\boldsymbol{\omega}_a + \bar{\mathbf{u}}_d + P_d(\mathbf{x})\mathbf{y} \tag{32}$$

B. Feedback-Linearizing Quaternion Feedback Regulator

Let the null-control vector $\mathbf{y}$ be chosen as

$$\mathbf{y} = -F_{21}(\boldsymbol{\omega})\omega_u - F_{22}(\boldsymbol{\omega})\boldsymbol{\omega}_a - d\boldsymbol{\omega}_a - k\mathbf{q}_a \tag{33}$$

where d, k are positive scalar gains to be determined. In this manner, substitution of Eq. (33) into Eq. (27) forms the class of control laws

$$\mathbf{u}_d = \bar{\mathbf{u}}_d + P_d(\mathbf{x})\left[-F_{21}(\boldsymbol{\omega})\omega_u - F_{22}(\boldsymbol{\omega})\boldsymbol{\omega}_a - d\boldsymbol{\omega}_a - k\mathbf{q}_a\right] \tag{34}$$

Substitution of Eq. (34) into Eq. (32) yields the close-loop actuated subsystem

$$\begin{aligned} \dot{\boldsymbol{\omega}}_a &= F_{21}(\boldsymbol{\omega})\omega_u + F_{22}(\boldsymbol{\omega})\boldsymbol{\omega}_a \\ &+ \bar{\mathbf{u}}_d + P_d(\mathbf{x}) \begin{bmatrix} -F_{21}(\boldsymbol{\omega})\omega_u - F_{22}(\boldsymbol{\omega})\boldsymbol{\omega}_a \\ -d\boldsymbol{\omega}_a - k\mathbf{q}_a \end{bmatrix} \end{aligned} \tag{35}$$

which converges by Eqs. (26) and (29) to the linear dynamical system [16]

$$\dot{\boldsymbol{\omega}}_a = -d\boldsymbol{\omega}_a - k\mathbf{q}_a \tag{36}$$

Furthermore, the linear dynamical system given by Eq. (36) is upper bounded by selection of the damping coefficient β_1 forming a domain of attraction in $\mathbf{x}$ whereby $\|\mathbf{a}(\mathbf{x})\| \geq \beta_1$ due to the damped controls coefficient generalized inverse defined by Eq. (25). Therefore, the linear dynamical system given by Eq. (36) coupled with the remaining equations of motion given by Eqs. (9) through (11) form a feedback linearizing transformation from the underactuated rigid spacecraft angular motion equations given by Eq. (6) over a domain of attraction in $\mathbf{x}$. For a full discussion of the following conjecture, the interested reader is directed to [20].

Conjecture. *For the controller defined by Eq. (41), if the zero-actuated state Jacobian* $\mathcal{J}_0(\mathbf{x})$ *given in [20] is nonsingular, then the positive scalar feedback gains* $k, d \in \mathbb{R}$ *can be chosen to stabilize the closed-loop underactuated rigid spacecraft system of angular motion equations given by Eqs. (9) through (12) by bounded, piecewise-smooth, time-invariant state feedback control to an arbitrarily small vicinity of the equilibrium of the system defined by the damping coefficient* β_1 *within a domain of attraction in* $\mathbf{x}(t)$.

V. Design Example

As an illustration, consider an asymmetric rigid spacecraft with principal moments of inertia (in kgm^2) of $J_{11} = 32.5, J_{22} = 25$ and $J_{33} = 12.5$. This corresponds to a 30 kg rigid rectangular spacecraft with uniform mass distribution and sides of length 1x2x3 m. A nominal available torque about each control axis is considered to be limited to 1 Nm. For the considered inertia matrix and quaternion feedback regulator, the closed loop system of equations given in Eqs. (9) through (11) and Eq. (35) become

$$\begin{bmatrix} \dot{q}_1 \\ \dot{\omega}_1 \end{bmatrix} = \frac{1}{2J_{11}} \begin{bmatrix} J_{11}(q_4\omega_1 + q_2\omega_3 - q_3\omega_2) \\ 2(J_{22} - J_{33})\omega_2\omega_3 \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} q_3\omega_1 + q_4\omega_2 - q_1\omega_3 \\ -q_2\omega_1 + q_1\omega_2 + q_4\omega_3 \end{bmatrix}\right) \tag{38}$$

$$\dot{q}_4 = -\frac{1}{2}(q_1\omega_1 + q_2\omega_2 + q_3\omega_3) \tag{39}$$

$$\begin{bmatrix} \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{pmatrix} \dfrac{1}{J_{22}J_{33}} \begin{bmatrix} J_{33}(J_{33} - J_{11})\omega_1\omega_3 \\ J_{22}\omega_2(J_{11}\omega_1 - J_{22}\omega_3) \end{bmatrix} + \dfrac{\mathbf{a}_d^+(\mathbf{x})b(\mathbf{x})}{J_{22}J_{33}} \\ -\dfrac{P_d(\mathbf{x})}{J_{22}J_{33}} \begin{bmatrix} J_{33}(J_{33} - J_{11})\omega_1\omega_3 - d\omega_2 - kq_2 \\ J_{22}\omega_2(J_{11}\omega_1 - J_{22}\omega_3) - d\omega_3 - kq_3 \end{bmatrix} \end{pmatrix} \tag{40}$$

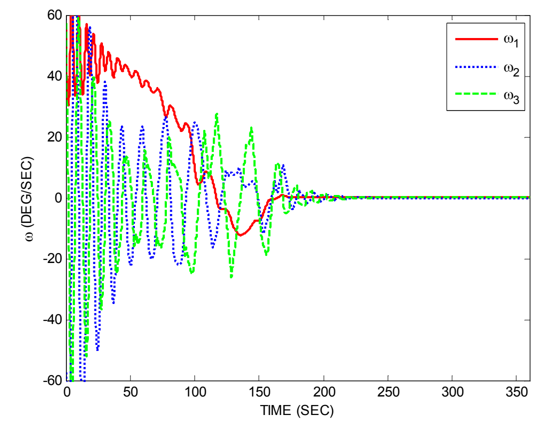
The control coefficient $\mathbf{a}(\mathbf{x})$ given by Eq. (18) becomes

$$\mathbf{a}(\mathbf{x}) = \frac{1}{2J_{11}} \begin{bmatrix} -J_{11}q_3 + (J_{22} - J_{33})\omega_3 \\ J_{11}q_2 + (J_{22} - J_{33})\omega_2 \end{bmatrix} \tag{41}$$

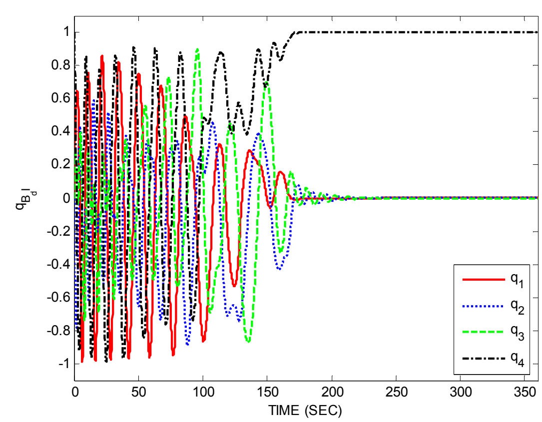
with the determinant of the associated $\mathcal{J}_0(\mathbf{x})$ given in [20] being non-zero for all $\mathbf{x}(t) \in \mathbb{R}^6 \times \mathbb{R}$.

For the numerical simulations that follow, a fourth-order Runge-Kutta numerical integration scheme is used to integrate Eqs. (37) through (40) with a fixed time step of .1s. In order to accommodate exact zero initial conditions on $\boldsymbol{\omega}$ and $\mathbf{q}$ without failure of the regulator, the measured state is perturbed to $1\mathrm{x}10^{-4}$ in these situations.

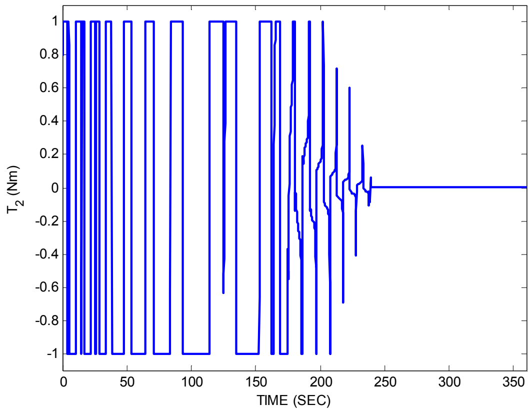
A three-axis detumbling and reorientation maneuver is considered. The quaternions describing the initial and final orientation of $\mathcal{B}_d$ with respect to $\mathcal{I}$ are $\hat{\mathbf{q}}(t_0) = [0,0,0,1]^T$ while the initial angular velocity vector is $\boldsymbol{\omega}(t_0) = [1,-1,1]^T\ rad/s$. The gains are selected as $\gamma = .70$, $a = 1.25$, $d = 7.5$, $k = 3$, $\beta_1 = 1x10^{-3}$. Fig. 1a and b show the time histories of the angular rates and quaternion, demonstrating attitude stabilization within 250 s. The failure to smoothly converge to the origin is due to the limited control torque. Fig. 1c and d show the time histories of the available control torques, demonstrating a reasonable and achievable control profile.



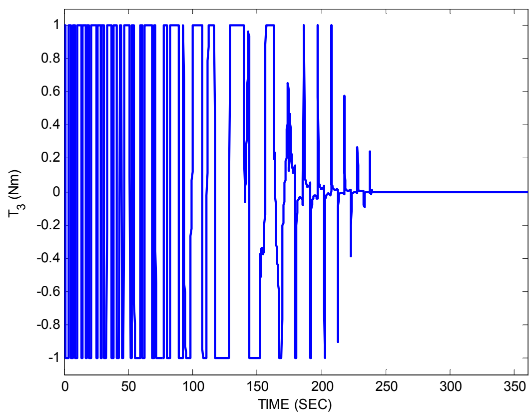
a)



b)



c)



d)

Fig. 1. Time histories of the a) quaternion, b) angular rates, c) control torque about the $\hat{b}_2$ axis, and d) control torque about the $\hat{b}_3$ axis for a 3-axis detumbling and reorientation maneuver from large initial angular rates about each axis with nominal available control torque.

VI. CONCLUSION

Three-axis stabilization of a rigid spacecraft's attitude with only two control torques is considered. A control design methodology is presented that involves perturbed feedback linearization and yields stability of the system within a domain of attraction. In order to overcome the potential for the control laws to produce numerical instability as the generalized inverse of the controls coefficient becomes singular, the generalized inverse is damped near a bound on the singularity, thus providing piecewise-smooth control and resulting in stabilization near to the zero error state. As such, the controller does not contradict the conjecture by Byrnes and Isidori in [8] that no smooth, time-invariant state feedback controller can be found to locally asymptotically drive the system to the zero error state. However, given the arbitrary selection of damping coefficients, the region about the zero error state can be reduced to an insignificant sized ball. Numerical simulation results of a 3-axis detumbling to a desired orientation maneuver are presented for an underactuated asymmetric spacecraft with two, bounded, body-fixed torques. Although the proposed attitude control method is not intended to provide attitude maintenance or for attitude tracking in the presence of relatively large disturbance torques, it may prove widely applicable to detumbling and reorientation maneuvers of spacecraft with only two available control torques.

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