Dmitriy Rivkin

AMS 232

Final Report

Introduction

This project is concerned with the optimization of large angle slews for reaction wheel controlled CubeSats. We successfully obtained minimum time and minimum energy numerical solutions in DIDO, and verified them using Pontryagin's Principle.

Dynamic Model

The system was modeled as four rotating rigid bodies with centers of mass fixed relative to their overall center of mass. Three of the bodies are the reaction wheels, and the fourth body is the body of the satellite. With no external torques, the dynamics are governed by the following equation:

$$\sum_k D^I\left(I_{B_k}^{B_k}\omega^{B_kI}\right) + D^I\left(I_C^C\omega^{CI}\right) = 0$$

Where:

k ranges from 1 to 4 (each value of k corresponds to one of the rigid bodies)

D^I is the rotational derivative w.r.t. the inertial frame

$I_{B_k}^{B_k}$ is the moment of inertia of body B_k about its own center of mass

ω^{B_kI} is the angular momentum of body frame B_k w.r.t. the inertial frame.

I_C^C is the total moment of inertia of all of the bodies about their mutual center of mass

ω^{CI} is the rate of rotation of the center of mass (C) frame w.r.t. the inertial frame. The center of mass frame has its origin at the center of mass, and its axes are parallel to the primary inertial axes of the reaction wheels (which mounted perpendicularly to each other).

We then write this equation in terms of rotational derivatives w.r.t. the C frame

$$\sum_k D^I\left(I_{B_k}^{B_k}\omega^{B_kI}\right) + D^I\left(I_C^C\omega^{CI}\right) = \sum_k \left[D^C\left(I_{B_k}^{B_k}\omega^{B_kI}\right) + \Omega^{CI}\left(I_{B_k}^{B_k}\omega^{B_kI}\right)\right] + D^C\left(I_C^C\omega^{CI}\right) + \Omega^{CI}\left(I_C^C\omega^{CI}\right) = 0$$

Where:

D^C is the rotational derivative w.r.t. the C frame

Ω^{CI} is the skew-symmetric form of ω^{CI} (multiplying by the skew symmetric form performs the cross product operation)

The rate of rotation of a body with respect to the inertial frame is the sum of its rate of rotation to the C frame plus the rate of rotation of the C frame w.r.t. the inertial frame:

$$\omega^{B_k I} = \omega^{B_k C} + \omega^{CI}$$

This form is convenient because for each of the wheels, $\omega^{B_k C}$ is the just the wheel speed (in a vector where the other two entries are zero), and for the satellite body it is equal to zero.

$D^C(\omega^{B_k C}) = \alpha^{B_k C}$ is then just the acceleration of each wheel, or 0 for the satellite body.

$D^C(\omega^{CI}) = \alpha^{CI}$ is the angular acceleration of the C frame with respect to the inertial frame.

Making these substitutions and solving for α^{CI}:

$$\alpha^{CI} = -\left(\sum_k I_{B_k}^{B_k} + I_C^C\right)^{-1} \left[\Omega^{CI}\left(\sum_k I_{B_k}^{B_k} + I_C^C\right)\omega^{CI} + \sum_k I_{B_k}^{B_k}\alpha^{B_k C} + \Omega^{CI}\left(\sum_k I_{B_k}^{B_k}\omega^{B_k C}\right)\right]$$

Adding the following equations:

$$\alpha^{B_k C} = u^k$$

which simply states the control input is the acceleration of each wheel, and

$$\dot{q} = \frac{1}{2} q \otimes q_\omega$$

produces the full state-space dynamic model for use in simulation, where

q is the quaternion expressing the orientation of the C frame w.r.t. the inertial frame,

$q_\omega = \begin{bmatrix} 0 \\ \omega^{CI} \end{bmatrix}$ is a quaternion constructed from the angular rate of the satellite, and

$\otimes$ is the quaternion multiplication operator.

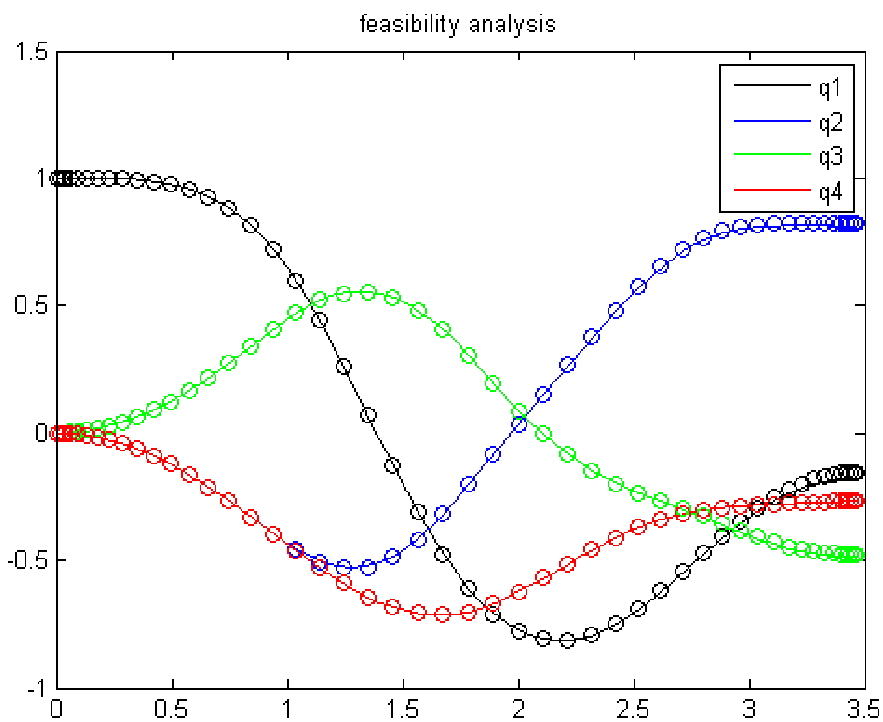
Minimum – Time Solution

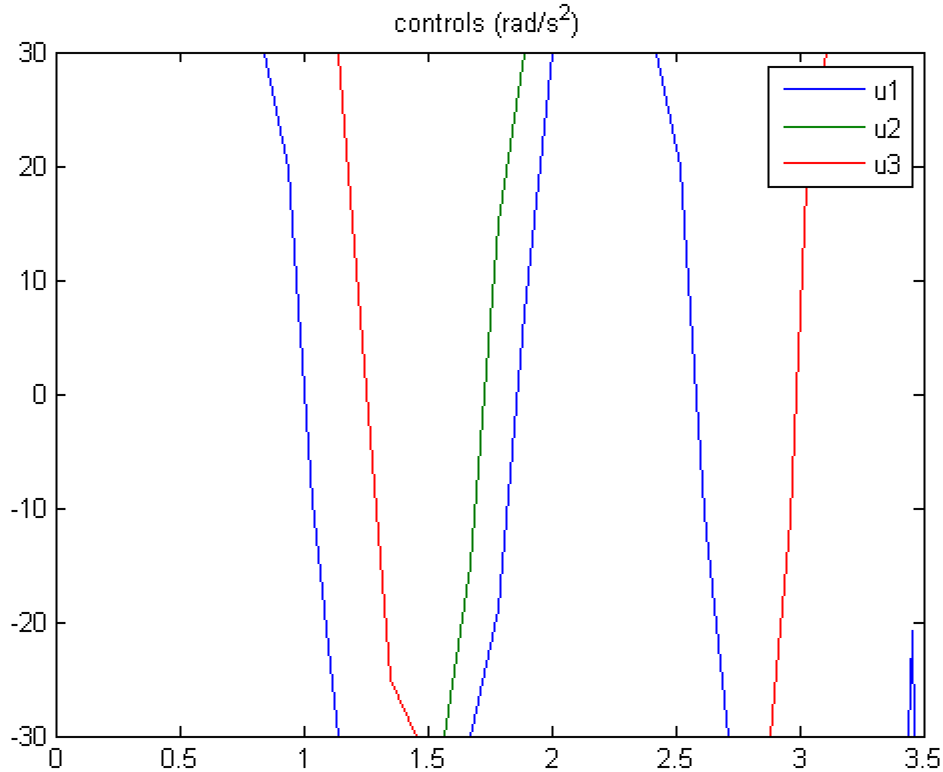
The cost function used to obtain the minimum time solution is simply t_f. In order to help DIDO converge to a solution for a reasonably high number of nodes, a "bootstrapping" method was used, where a low-node solution is first found and then used as a guess for a higher node solution. Using this method we obtained a solution for N=50 nodes.

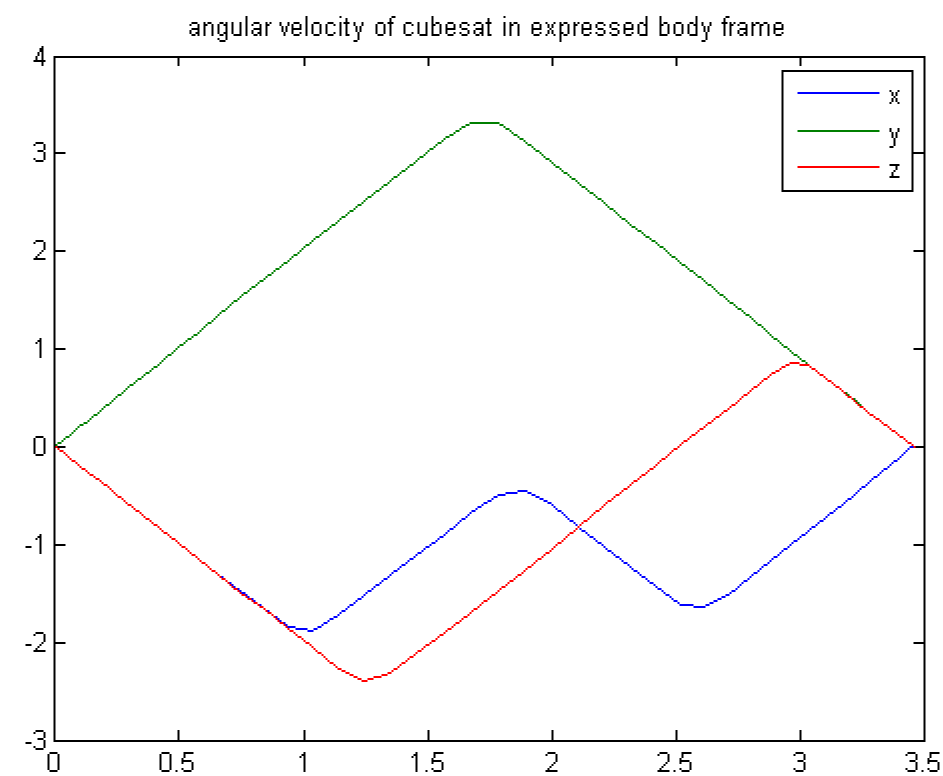
The resultant solution has a Bang-Bang form, which is to be expected of a minimum time solution. The solution is feasible, and satisfies the HMC (Hamiltonian Minimization Condition) and HVC(Hamiltonian Value Condition). However, we could not show that the co-states satisfy the $\dot{\lambda} = -\frac{\partial H}{\partial x}$ equation. This may have to do with the method used for integration: the derivative is first obtained symbolically using the symbolic toolbox, and then the derivative is computed by substituting into the symbolic equation. It is possible that the symbolic substitution is not very precise, and that the accumulated error results in a significant discrepancy between the costates returned by DIDO and the costates obtained by integrating the above equation. It is also possible that there is a mistake in the computation of the derivative, or that the solution is not really optimal. However, because the solution satisfies the HMC and HVC, we think that it really is optimal, and there is some problem with our verification of the costate dynamics.

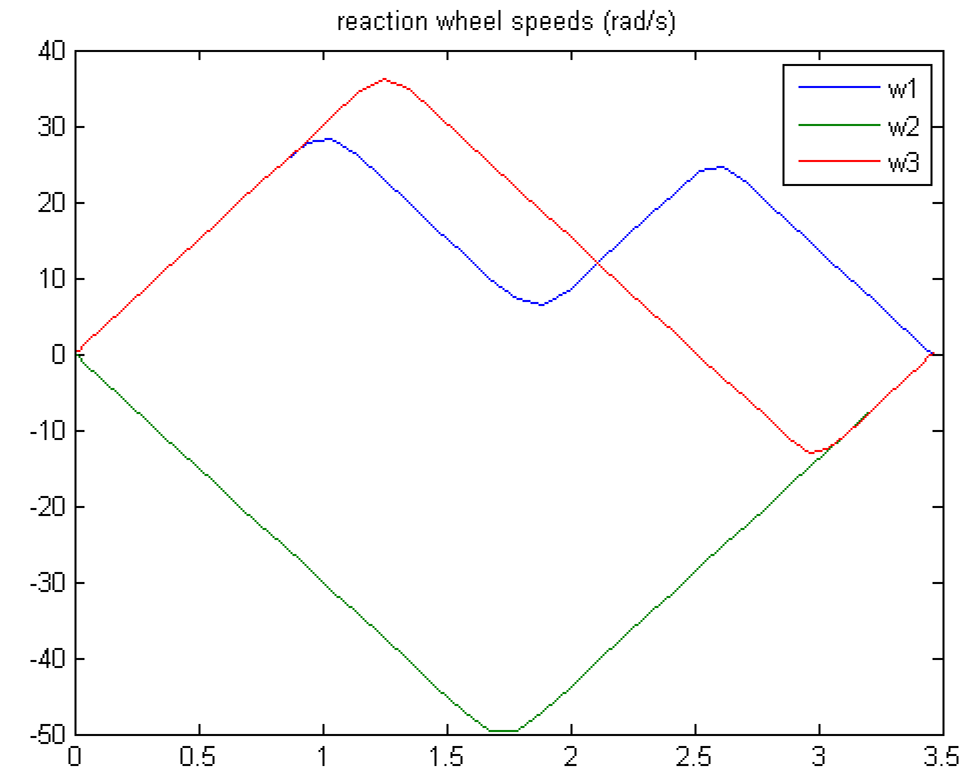
Minimum Time Results

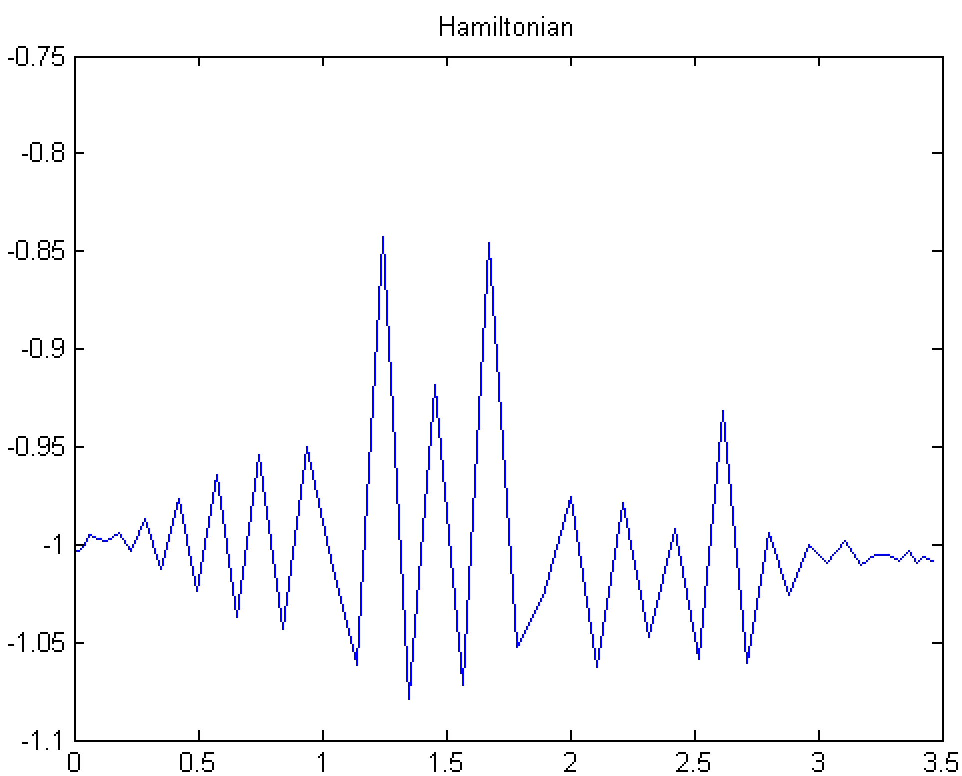
These results were obtained for a large angle slew (from (yaw,pitch,roll) = (0,0,0) to (-pi/2, pi/5, pi)) for a 1U CubeSat, whose properties are described in the appendix.

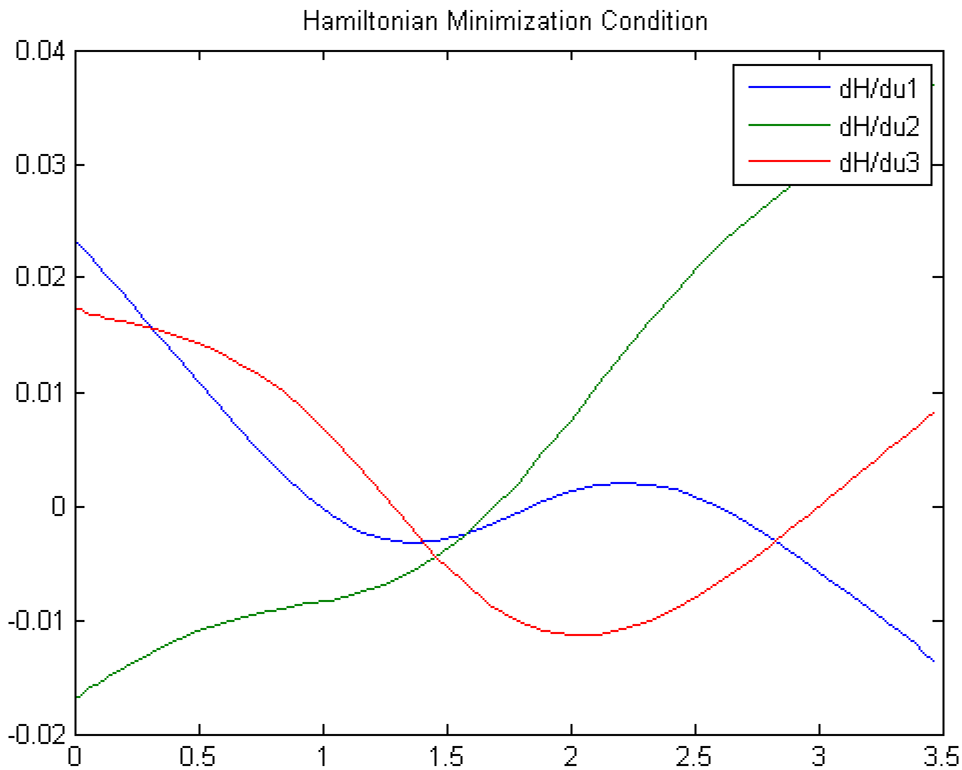


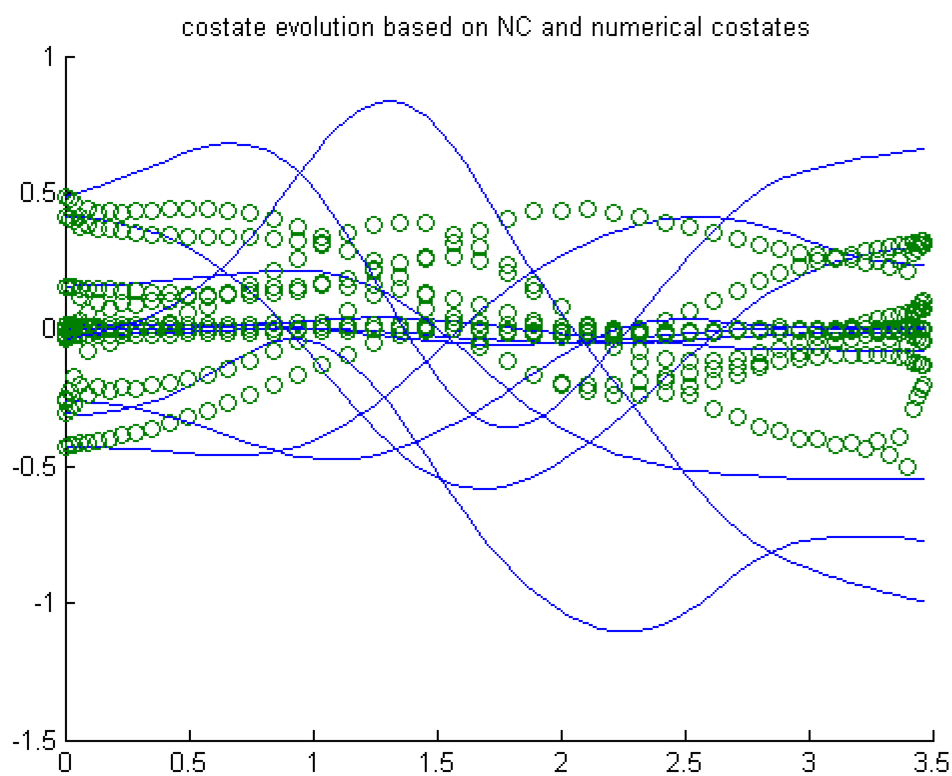

controls (rad/s^2)
30
20
10
0
-10
-20
-30
0
0.5
1
1.5
2
2.5
3
3.5
u1
u2
u3


angular velocity of cubesat in expressed body frame
4
3
2
1
0
-1
-2
-3
0
0.5
1
1.5
2
2.5
3
3.5
x
y
z


reaction wheel speeds (rad/s)
w1
w2
w3
40
30
20
10
0
-10
-20
-30
-40
-50
0
0.5
1
1.5
2
2.5
3
3.5


Hamiltonian
-0.75
-0.8
-0.85
-0.9
-0.95
-1
-1.05
-1.1
0
0.5
1
1.5
2
2.5
3
3.5


Hamiltonian Minimization Condition
dH/du1
dH/du2
dH/du3
0.04
0.03
0.02
0.01
0
-0.01
-0.02
0
0.5
1
1.5
2
2.5
3
3.5


costate evolution based on NC and numerical costates
1
0.5
0
-0.5
-1
-1.5
0
0.5
1
1.5
2
2.5
3
3.5

Minimum Energy Solution

In this section we strive to minimize the energy required to perform the same large angle slew as in the time optimal section, but this time the final time is constrained to ten seconds. There are a number of sources of energy losses associated with a slew:

1. When the wheel is spun up, electrical energy is converted into mechanical energy. When the wheel is spun down, this energy is not recovered, but dissipated as heat. The energy dissipated here is proportional to $\int |\alpha\omega| dt$, where α is the acceleration of the momentum wheel and ω is the angular velocity of the momentum wheel. It should be noted that since there are three momentum wheels, the total energy is the sum of this cost function over each of the three wheels.

2. Frictional losses, which are proportional to $\int \omega^2 dt$.

3. Resistive losses in the motor coils, which are proportional to $\int \alpha^2 dt$.

4. Other losses including switching and hysteresis losses (not considered here).

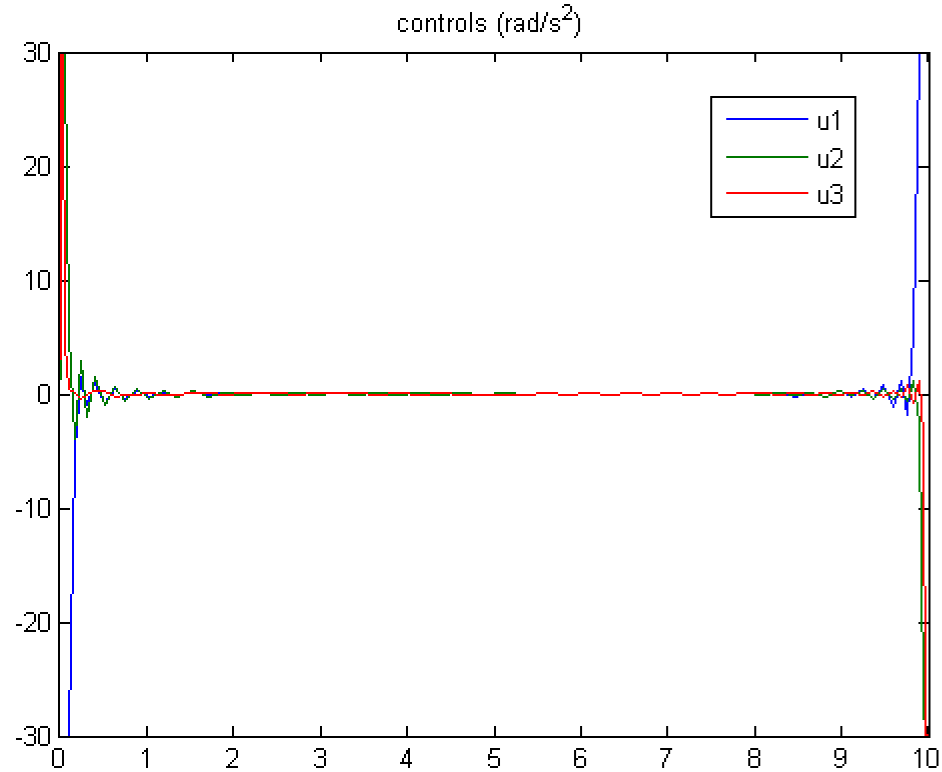
While DIDO performs well finding optimal solutions to cost functions 2 and 3, cost function 1 is not so easily optimized, so let us leave it for the time being and focus on optimizing the second cost function.

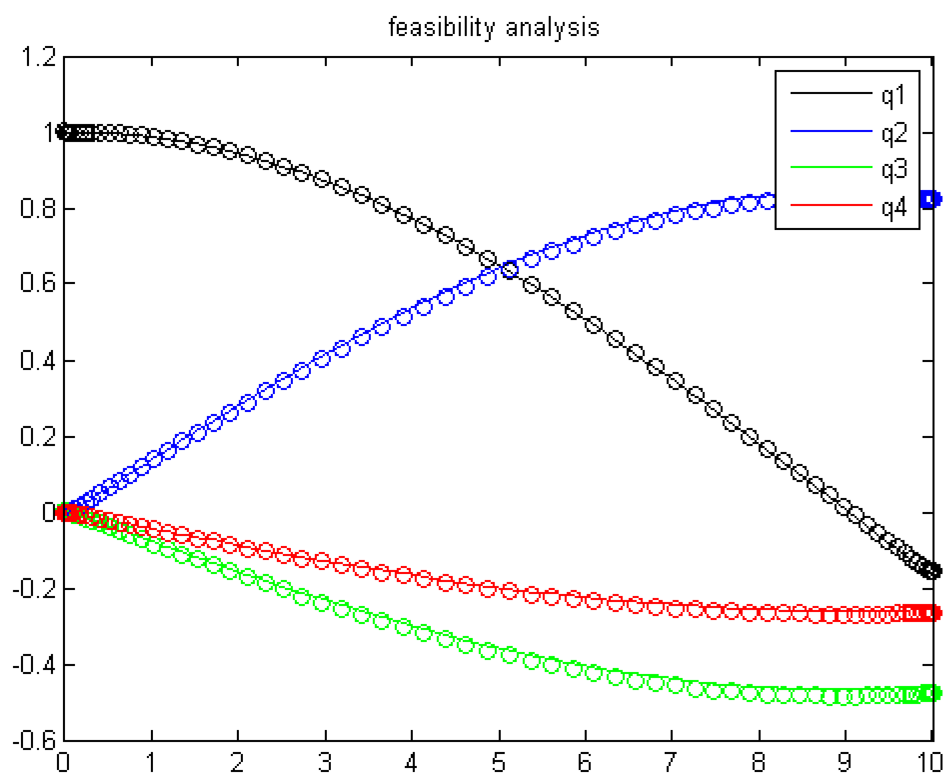
$\int \boldsymbol{\omega^2 dt}$ Solution

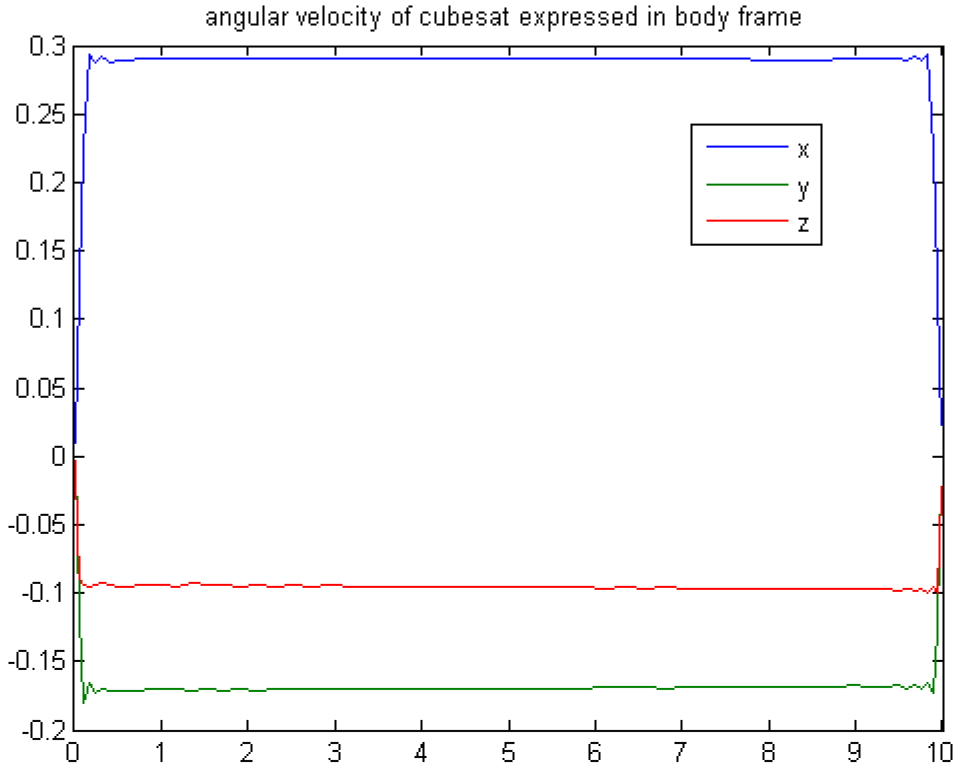
Using DIDO, we were able to compute a solution for this cost function. The solution is Bang-Off-Bang, where the satellite accelerates as quickly as possible to some constant speed which will allow it to get to its destination in time, and then maintains a constant velocity until it gets near the end, at which point it decelerates as quickly as possible and comes to rest at the desired location. The solution is shown to be feasible and satisfy the HMC and the HVC, but could not be shown to satisfy the costate dynamic equations. This supports the hypothesis that there is something wrong with our method for verifying the costate equations, because not only does this solution satisfy the HMC and HVC, but makes a lot of sense intuitively as well.

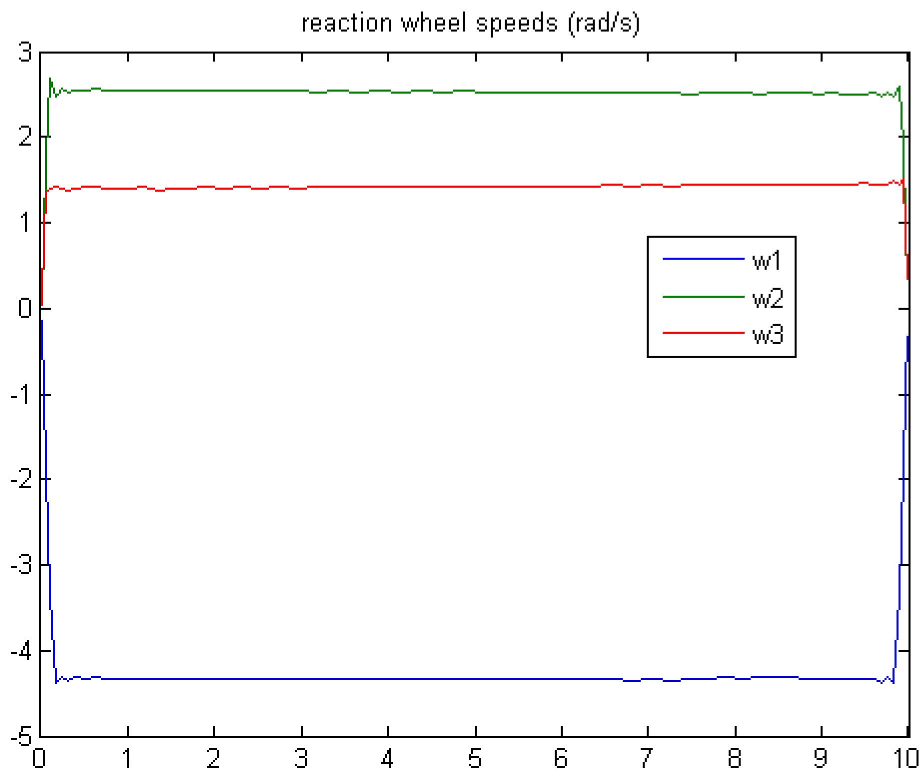
Results

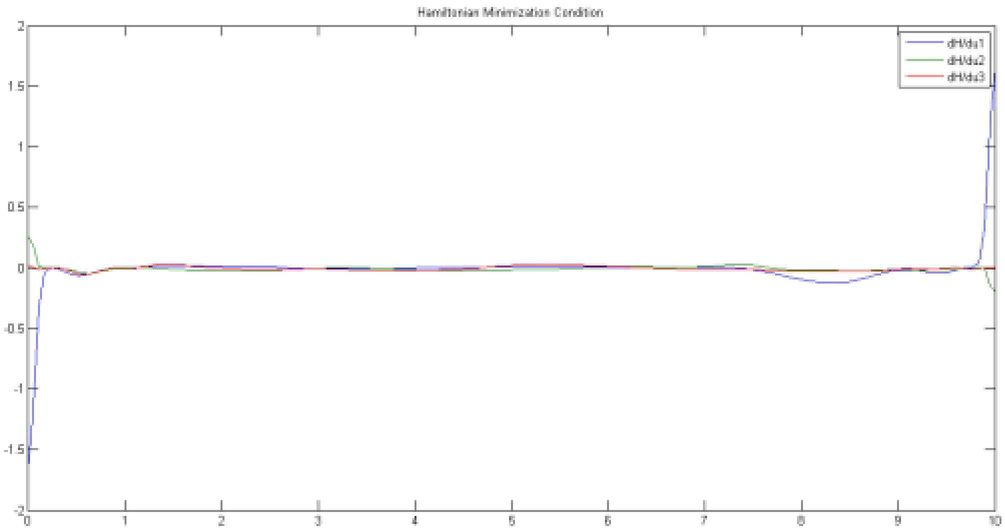
The system and the slew are the same as in the time optimal case, but the final time is fixed at 10 seconds.


controls (rad/s2)
u1
u2
u3
30
20
10
0
-10
-20
-30
0
1
2
3
4
5
6
7
8
9
10

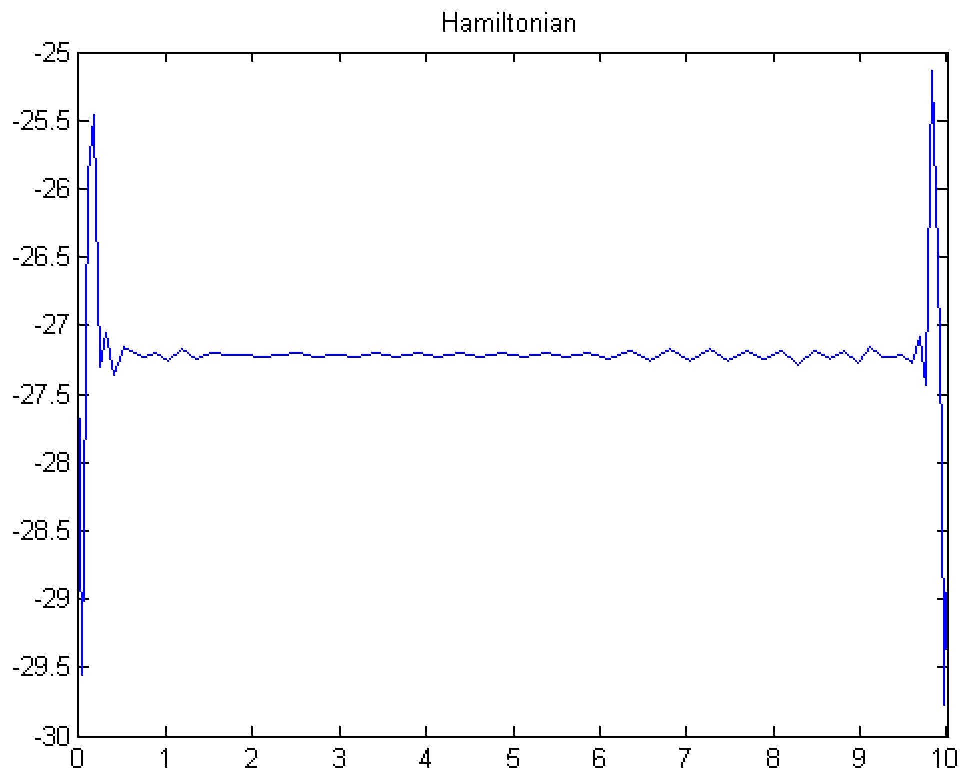

feasibility analysis
q1
q2
q3
q4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
1
2
3
4
5
6
7
8
9
10


angular velocity of cubesat expressed in body frame
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
0
1
2
3
4
5
6
7
8
9
10
x
y
z

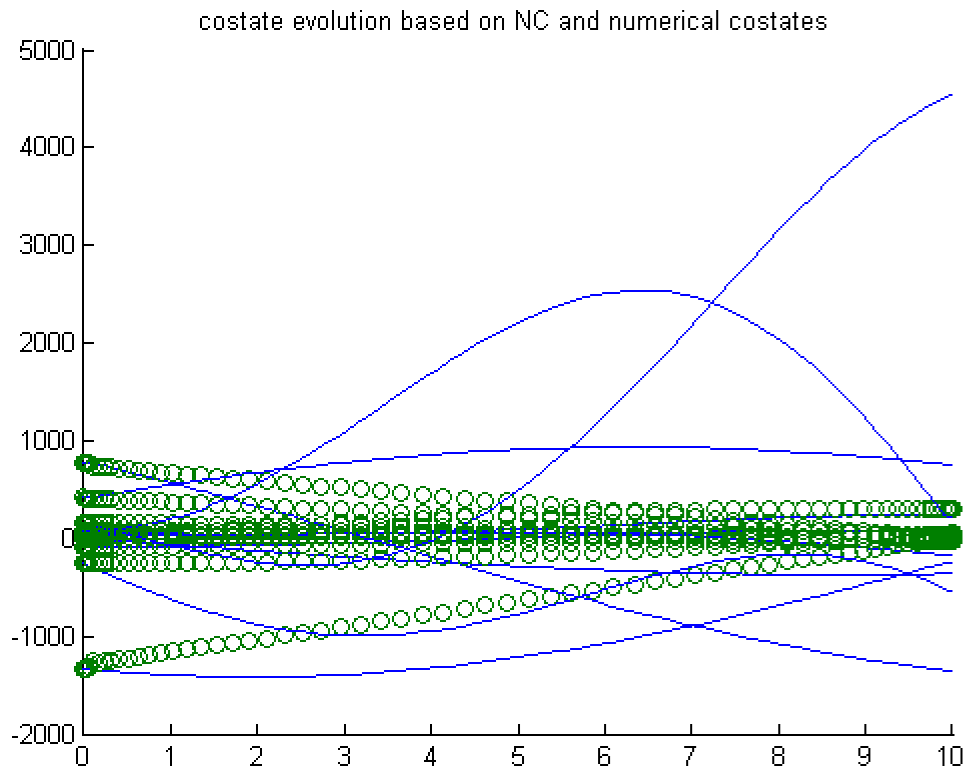

reaction wheel speeds (rad/s)
3
2
1
0
-1
-2
-3
-4
-5
0
1
2
3
4
5
6
7
8
9
10
w1
w2
w3



Hamiltonian Minimization Condition
dH/du1
dH/du2
dH/du3



Hamiltonian



$\int |\alpha\omega| dt$ Solution

As mentioned before, obtaining a numerical solution to this cost function is difficult. For one, it is not smooth because of the absolute value. We overcame this obstacle as follows:

First, we defined two "pseudo-controls" for each wheel. We constrained one of them to be positive, and the other one to be negative. We then added path constraints so that their sum would always be equal to the velocity, and also so that only one them could be non-zero at a time. The implementation of the first path constraint, but the second was implemented by specifying that

$(pc1^2 + pc2^2) - pc1^2 - pc2^2 = 0$. The only way for this condition to be satisfied is for only one of the pseudo-controls to be active at a time. By implementing these constraints, we made it so that when the velocity of the wheel was positive, the positive pseudo-control would equal the speed, and the negative one would be zero. When the speed was negative, the opposite would be true.

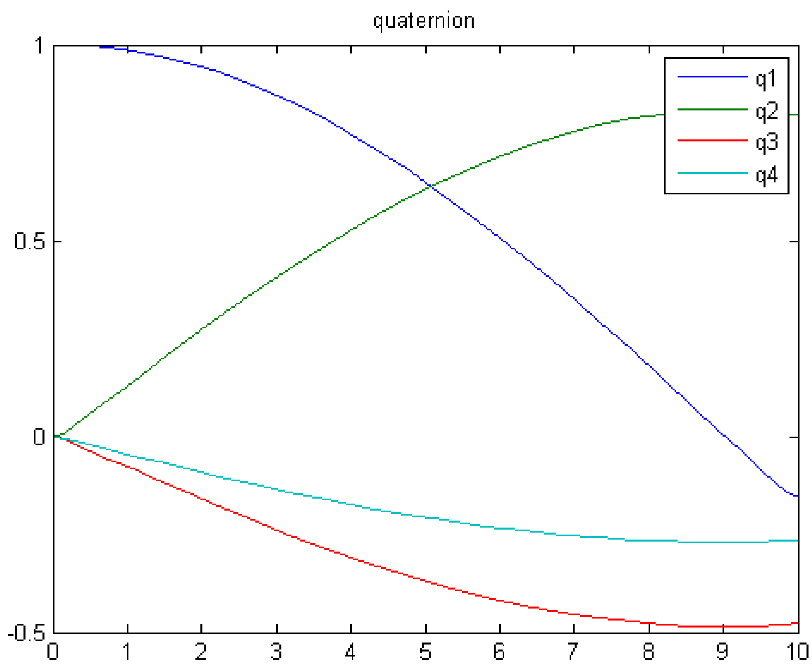
Then, the actual controls for each wheel were split in two, a positive and negative part. The total control on each wheel was the sum of the two. These were also constrained as described above so that only one would be active at a time. This constraint may not be strictly necessary, but it helps limit the search space.

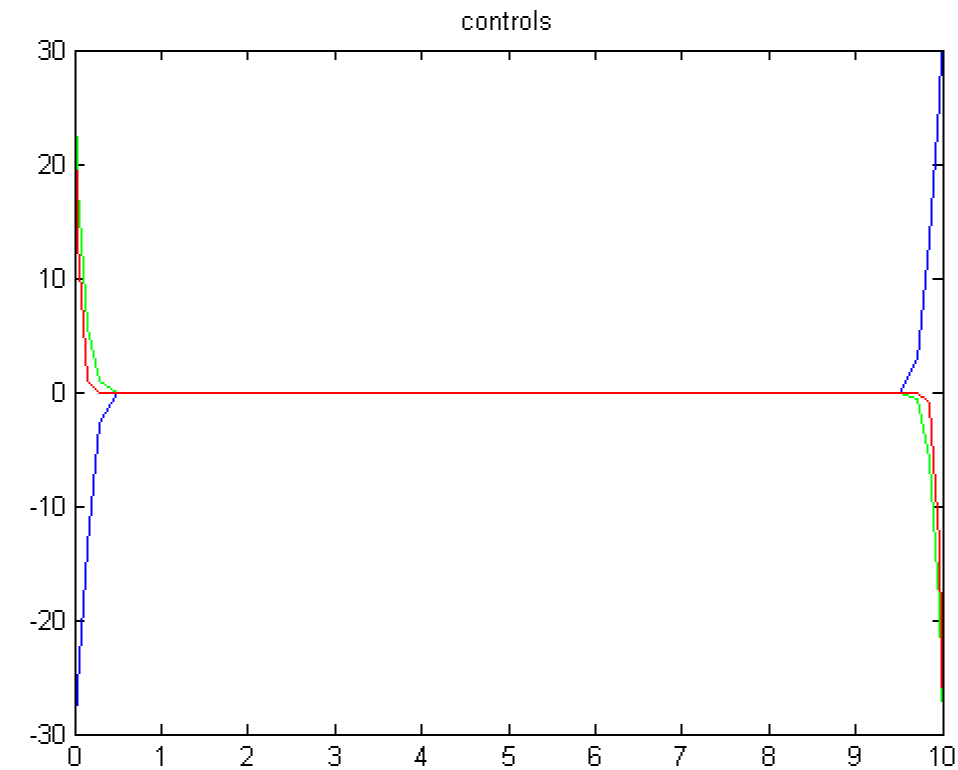
Once we had the pseudo-controls which reflected the velocity of the wheel and the controls broken up into positive and negative parts, the cost function could be implemented smoothly as:

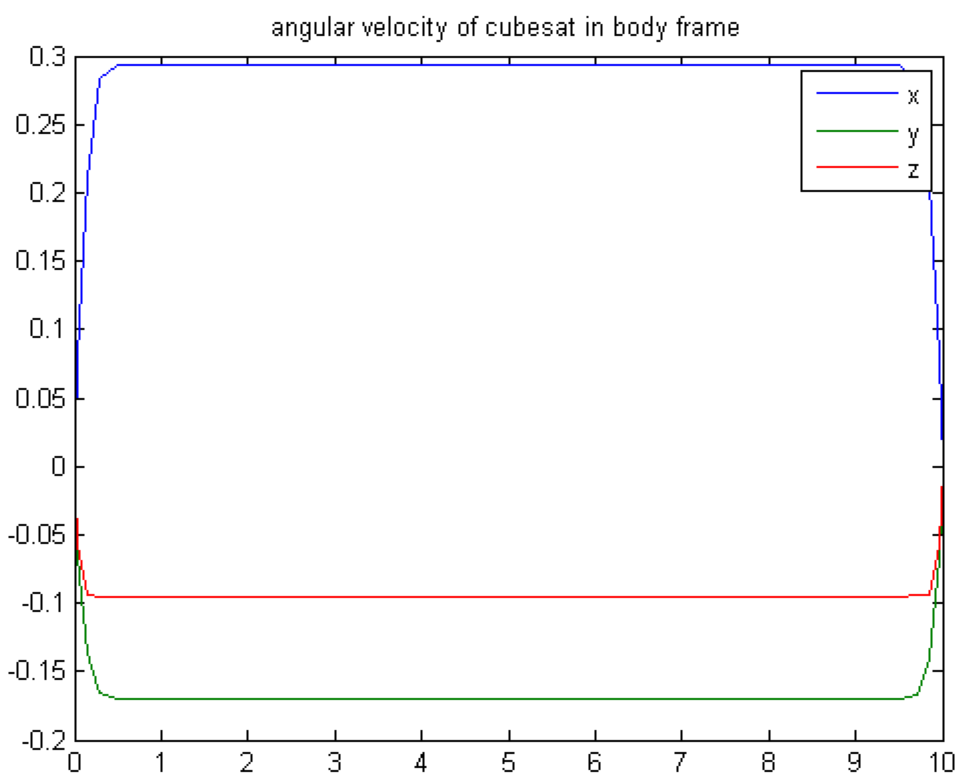
$$\int (pc1 * c1 + pc2 * c2)\, dt$$

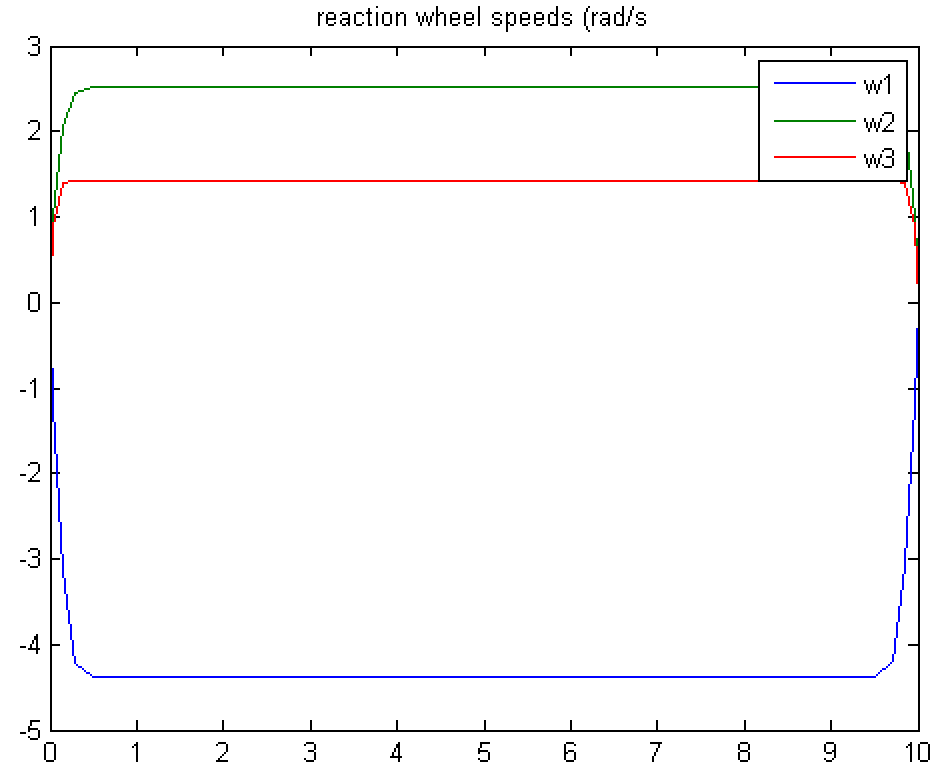
We implemented this formulation and tried to obtain a solution (without making an initial guess), but a solution could not be had. However, the $\int \omega^2 dt$ solution was the same as we intuitively expected for $\int \omega^2 dt$, so we ran DIDO again, using the $\int \omega^2 dt$ solution as an initial guess, and DIDO very quickly returned more or less exactly the same solution, seeming to verify that the same solution minimized the two cost functions. We have not yet checked the HMC, but the HVC looks like it is satisfied. If it is indeed true that both cost functions are minimized by the same control, then this is very good news, because together they account for the majority of energetic losses in the control system.

Results

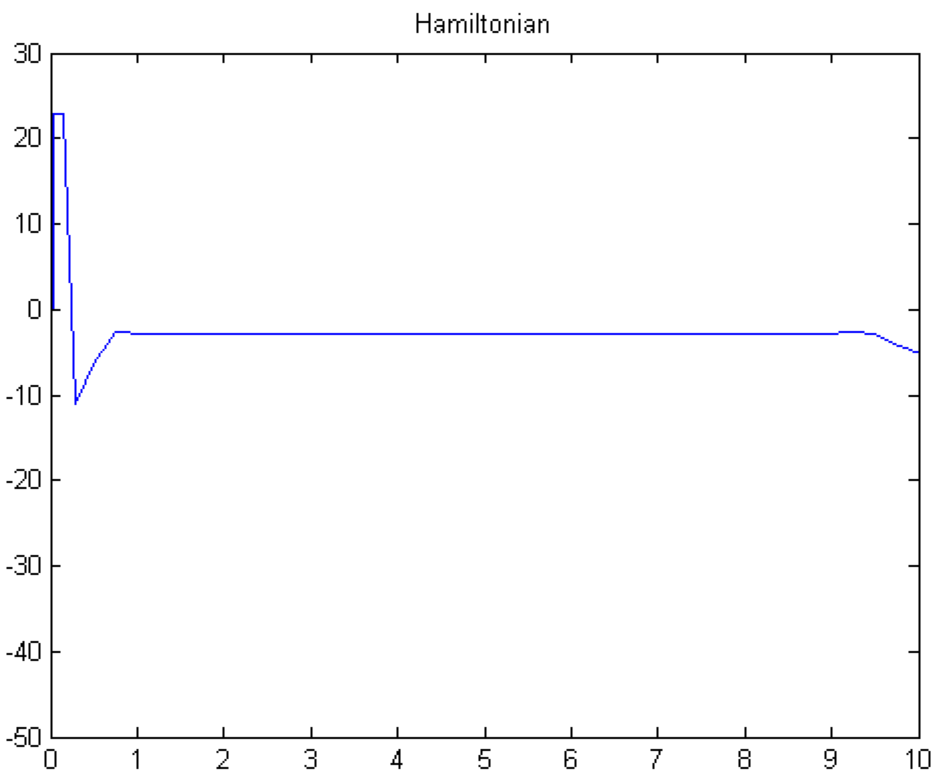



controls
30
20
10
0
-10
-20
-30
0
1
2
3
4
5
6
7
8
9
10


angular velocity of cubesat in body frame
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
0
1
2
3
4
5
6
7
8
9
10
x
y
z



reaction wheel speeds (rad/s
w1
w2
w3
3
2
1
0
-1
-2
-3
-4
-5
0
1
2
3
4
5
6
7
8
9
10



Hamiltonian
30
20
10
0
-10
-20
-30
-40
-50
0
1
2
3
4
5
6
7
8
9
10

Appendix: Description of Satellite Model

The satellite body was modeled as 10 x 10 x 10 cm box with 700 grams evenly distributed. Each wheel weighed 100 grams, had a diameter of approximately 9cm, and was located near the outer wall of the CubeSat. The total mass of the system was 1kg. The motors used can provide a torque of up to 3mNm and spin up to 6000 rpm. The pertinent MOI (Moment of Inertia) matrices are provided below (all expressed in the C frame):

Total MOI of the 4 bodies w.r.t. their overall center of mass:

```
sumMOI = [0.00152828000000000,2.11600000000000e-05,2.11600000000000e-05;
    2.11600000000000e-05,0.00152828000000000,2.11600000000000e-05;
    2.11600000000000e-05,2.11600000000000e-05,0.00152828000000000;];
```

MOI Matrices for each wheel:

```
Iw1 = [0.000101000000000000,0,0;
    0,5.07000000000000e-05,0;
    0,0,5.07000000000000e-05;];

Iw2 = [5.07000000000000e-05,0,0;
    0,0.000101000000000000,0;
    0,0,5.07000000000000e-05;];

Iw3 = [5.07000000000000e-05,0,0;
    0,5.07000000000000e-05,0;
    0,0,0.000101000000000000;];
```