Point Rotation Formula Proof

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preface: $\tan^{-1}(y, x)$

pf:

goal: rotate (a, b) θ units around the point (h, k)
$A_1 := (a, b)$
$B_1 := (a, k)$
$C_1 := (h, k)$

shift everything by $(-h, -k)$
$A_2 := (a - h, b - k)$
$B_2 := (a - h, 0)$
$C_2 := (0, 0)$

$m\angle A_1B_1C_1 = m\angle A_2B_2C_2$

$t := \tan^{-1}(b - k, a - h)$ (the angle that (a, b) would have if (h, k) was the origin)

$\theta_2 := t + \theta$ (the new angle)
$(\cos\theta_2, \sin\theta_2)$ has the right angle with the distance scaled to 1

$d := \sqrt{(b - k)^2 + (a - h)^2}$ (the distance between (a, b) and (h, k))

the point would then be $(d\cos\theta_2, d\sin\theta_2)$

after shifting by (h, k) to counteract the shift from before it becomes $(h + d\cos\theta_2, k + d\sin\theta_2)$

$$= \begin{pmatrix} h + \cos\left(\tan^{-1}(b - k, a - h) + \theta\right)\sqrt{(b - k)^2 + (a - h)^2}, \\ k + \sin\left(\tan^{-1}(b - k, a - h) + \theta\right)\sqrt{(b - k)^2 + (a - h)^2} \end{pmatrix}$$

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