On the Continuum Hypotheses and the Correspondence of Infinite Sets' Elements with the Natural Numbers

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Dedicated to Maeby, my favorite kitty

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1 Introduction and Countability Theorems

Any "countable" set is equinumerous to a improper subset of the natural numbers, and there exists a bijection between them. Any finite set has this property, the main question being if all *infinite* sets do. In all infinite processes, one should think of the steps as executing simultaneously rather than separately and successively. $|S|$ and $C(S)$ denote the cardinality and countability of a set S respectively. Equations (2a) and (2b) describe what will be called sum amd product "ring operators." Most theorems use " $\iff$ " instead of " $\implies$ " because Theorem 3.1 works as an inversion. Equation (1a) doesn't nest groups but flattens them.

$$R \times S := \{(r, s) : r \in R \land s \in S\} \quad \text{(1a)} \qquad R \cdot S := \{r \cdot s : r \in R \land s \in S\} \quad \text{(1b)} \tag{1}$$

$$R(\vec{v}\,) := \left\{ \sum_{n=1}^{\operatorname{len} \vec{v}} (r_n \cdot \vec{v}_n) : r_k \in R \right\} \quad \text{(2a)} \qquad R[\,\vec{v}\,] := \left\{ \prod_{n=1}^{\operatorname{len} \vec{v}} (r_n + \vec{v}_n) : r_k \in R \right\} \quad \text{(2b)} \tag{2}$$

$$\aleph_0 := |\mathbb{N}| \tag{3}$$

$$C(S) := \begin{cases} 1,\ [S \text{ countable}] & (|S| \leqslant \aleph_0) \\ 0,\ [S \text{ uncountable}] & (|S| > \aleph_0) \end{cases} \tag{4}$$

Theorem 1. Singlular Unions of Countable Sets
If any sets R and S are countable, $R(a) \cup S(b)$ is countable for all numbers a and b.

1. $C(R) \land C(S) \iff C(R\,(a) \cup S\,(b))\ \forall a, b$
2. $C(R) \land C(S) \iff C(R\,(a) \cup S[\,b\,])\ \forall a, b$
3. $C(R) \land C(S) \iff C(R[\,a\,] \cup S\,(b))\ \forall a, b$
4. $C(R) \land C(S) \iff C(R[\,a\,] \cup S[\,b\,])\ \forall a, b$

Proof. R and S being countable implies $R(a)$ and $S(b)$ are countable as well because ring operators keep cardinalities the same. (case 1) Interlace the elements of $R(a)$ with those of $S(b)$, skipping over any duplicates. If either set is finite, append its elements to the beginning of the other set sans duplicates. Similar logic works with the other cases. ■

Theorem 2. Cartesian Products and Element-wise Multiplications of Countable Sets
If any sets R and S are countable, both $R \times S$ and $R \cdot S$ are countable.

1. $C(R) \land C(S) \iff C(R \times S)$
2. $C(R) \land C(S) \iff C(R \cdot S)$

Proof. To create $R \times S$, put all possible ordered pairs into a table as shown. Going along the diagonals, including every element in the output, and not "missing" any, one can show there is a bijection between $R \times S$ and $\mathbb{N}$. The numbers and colons in the table are the output indices, the first 25 being labeled if in view. Similar logic works for $R \cdot S$. This method is also algorithmically viable (see Section 5), and the output indices relate strongly to triangle numbers.

$1 : (R_1, S_1)$	$2 : (R_1, S_2)$	$4 : (R_1, S_3)$	$7 : (R_1, S_4)$	$\cdots$
$3 : (R_2, S_1)$	$5 : (R_2, S_2)$	$8 : (R_2, S_3)$	$12 : (R_2, S_4)$	$\cdots$
$6 : (R_3, S_1)$	$9 : (R_3, S_2)$	$13 : (R_3, S_3)$	$18 : (R_3, S_4)$	$\cdots$
$10 : (R_4, S_1)$	$14 : (R_4, S_2)$	$19 : (R_4, S_3)$	$25 : (R_4, S_4)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

Theorem 3. Subsets, Intersections, and Differences of Countable Sets

If any set R is countable, any loose subset S of R is also countable.

1. $C(R) \wedge S \subseteq R \implies C(S)$
2. $C(R) \wedge C(S) \implies C(R \cap S)$
3. $C(R) \implies C(R \setminus S)$

Proof. Since subsets of $\mathbb{N}$ are countable and R is countable, all subsets of R must also be countable due to the bijection between R and $\mathbb{N}$. This also implies that set intersections and differences are countable because these operations return subsets of the left input. Case 3 doesn't require S to be countable because it is uncountable and R is countable, then it is just more likely that the output will be the empty set. ■

Theorem 4. Two-Argument Ring Operators on Countable Sets

If any set R is countable, $R(a, b)$ and $R[a, b]$ are also countable for any constants a, b.

1. $C(R) \iff C(R(a, b))\ \forall a, b$
2. $C(R) \iff C(R[\,a, b\,])\ \forall a, b$

Proof. $R(a, b)$ can be tabulated with axes n and k, where the corresponding table box is $aR_n + bR_k$. This is countable in the same way as Theorem 2. Similar logic works for $R[a, b]$.

$aR_1 + bR_1$	$aR_1 + bR_2$	$aR_1 + bR_3$	$aR_1 + bR_4$	$aR_1 + bR_5$	$\cdots$
$aR_2 + bR_1$	$aR_2 + bR_2$	$aR_2 + bR_3$	$aR_2 + bR_4$	$aR_2 + bR_5$	$\cdots$
$aR_3 + bR_1$	$aR_3 + bR_2$	$aR_3 + bR_3$	$aR_3 + bR_4$	$aR_3 + bR_5$	$\cdots$
$aR_4 + bR_1$	$aR_4 + bR_2$	$aR_4 + bR_3$	$aR_4 + bR_4$	$aR_4 + bR_5$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

Theorem 5. Finite Natural Powers of Countable Sets

If any set R is countable, R^n is also countable for all natural numbers n.

1. $C(R) \iff C(R^n)\ \ \forall n \in \mathbb{N}_0 < \infty$

Proof. R^n can be factored as $R \times R^{n-1}$ (eqn. 1a flattens outputs), which is countable if R^{n-1} is countable (Theorem 2.1). This can be applied recursively until it becomes countable if R^2 or $R \times R$ is countable, which is countable via Theorem 2.1. If $n = 0$, the output set is all the groups of zero elements from R, or just $\emptyset$, which is countable since $|\emptyset|$ is finite. ■

Theorem 6. Countably Infinite Unions of Countably Infinite Sets

If all sets R_i are countable, their union is also countable.

1. $C(R_i)\forall i \iff C\left(\bigcup_{i=1}^{\aleph_0} R_i\right)$

Proof. To create the union, give each set R_i a column in a table and put all its elements sequentially in that column going down. Do this for every R_i, and using the same method as in Theorem 2, one can show that this set bijects the naturals. Every column represents a different R_i and each row represents a different $(R_i)_j$ or $R_{i,j}$. The numbers before the colons are the output index, the first 85 shown if in view. This is algorithmically viable (see Section 5).

$1 : R_{1,1}$	$2 : R_{2,1}$	$6 : R_{3,1}$	$7 : R_{4,1}$	$15 : R_{5,1}$	$16 : R_{6,1}$	$28 : R_{7,1}$	$\cdots$
$3 : R_{1,2}$	$5 : R_{2,2}$	$8 : R_{3,2}$	$14 : R_{4,2}$	$17 : R_{5,2}$	$27 : R_{6,2}$	$30 : R_{7,2}$	$\cdots$
$4 : R_{1,3}$	$9 : R_{2,3}$	$13 : R_{3,3}$	$18 : R_{4,3}$	$26 : R_{5,3}$	$31 : R_{6,3}$	$43 : R_{7,3}$	$\cdots$
$10 : R_{1,4}$	$12 : R_{2,4}$	$19 : R_{3,4}$	$25 : R_{4,4}$	$32 : R_{5,4}$	$42 : R_{6,4}$	$49 : R_{7,4}$	$\cdots$
$11 : R_{1,5}$	$20 : R_{2,5}$	$24 : R_{3,5}$	$33 : R_{4,5}$	$41 : R_{5,5}$	$50 : R_{6,5}$	$62 : R_{7,5}$	$\cdots$
$21 : R_{1,6}$	$23 : R_{2,6}$	$34 : R_{3,6}$	$40 : R_{4,6}$	$51 : R_{5,6}$	$61 : R_{6,6}$	$72 : R_{7,6}$	$\cdots$
$22 : R_{1,7}$	$35 : R_{2,7}$	$39 : R_{3,7}$	$52 : R_{4,7}$	$60 : R_{5,7}$	$73 : R_{6,7}$	$85 : R_{7,7}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

2 Countabilities of Common Sets

2.1 Countability of the Integers

Claim: $\mathbb{Z}$ is a countable set; $C(\mathbb{Z}) \equiv 1$.

Proof. $R := \mathbb{N}_0$, $S := \mathbb{N}_1$. R and S are countable axiomatically (axiom of extensionality), being the naturals themselves. $\mathbb{Z} \equiv R(1) \cup S(-1)$; this is countable via Theorem 1.1. ■

2.2 Countability of the Rationals

Claim: $\mathbb{Q}$ is a countable set; $C(\mathbb{Q}) \equiv 1$.

Proof. $\mathbb{Q}$ has the same countability as $\mathbb{Z} \times \mathbb{N}_1$. $C(\mathbb{Z}) \equiv 1$ via Section 2.1 and $C(\mathbb{N}_1) \equiv 1$ axiomatically. Thus $C(\mathbb{Z} \times \mathbb{N}_1) \equiv 1$ via Theorem 2.1. This and Theorem 2 derive from Cantor's enumeration of countable collections of countable sets. ■

2.3 Countability of the Reals from Zero to One

Claim: $\{x \in \mathbb{R} : 0 \leqslant x < 1\}$ is a countable set; $C(\mathbb{R}_{0\leqslant x<1}) \equiv 1$.

Proof. Let R be a set where $\forall i \in \mathbb{N}_0$, $R_i :=$ ("0." + reversed digits of i), ie: $R_{2760} = 0.0672$ (see Section 6 for formula). R is countable by definition, and contains every possible sequence of digits in $[0, 1)$. The sequence trends towards different values with different sub-series; Section 2.5 explains this further. R is used in the next proof. ■

2.4 Countability of the Non-Negative Reals

Claim: $\mathbb{R}_{\geqslant 0}$ or equivalently $\{x \in \mathbb{R} : x \geqslant 0\}$, is a countable set; $C(\mathbb{R}_{\geqslant 0}) \equiv 1$.

Proof. $C(\mathbb{R}_{0 \leqslant x < 1}) \equiv 1$ via Section 2.3. $\mathbb{R}_{\geqslant 0} = \bigcup_{i=1}^{\aleph_0} S_i$ where $S_i := R[i-1]$. This is countable via Theorem 6; the following table illustrates this. $\forall x \in \mathbb{R}_{\geqslant 0}$, there exists a sequence of $s_i \in S_{\lceil x \rceil}$, with increasing indices, where $\lim_{i \in \mathbb{N} \to \infty} s_i = x$. This is because S_n contains every sequence of digits in $[n-1, n)$. S is used in the next proof.

0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.01	0.11	0.21	$\cdots$
1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	1.01	1.11	0.21	$\cdots$
2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	2.01	2.11	0.21	$\cdots$
3	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	3.01	3.11	0.21	$\cdots$
4	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	4.01	4.11	0.21	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

2.5 Countability of the Reals

Claim: $\mathbb{R}$ is a countable set which implies $2^{\aleph_0} = \aleph_0$.

Proof. $\mathbb{R}_{\geqslant 0}$ is countable via Section 2.4. $\mathbb{R}_{\geqslant 0}(1, -1) \equiv \mathbb{R}$. This is countable via Theorem 4.1. This conclusion can be further reinforced by the algorithmic methods in Section 5. ■

2.6 Countabilities of Miscellaneous Number Classes

2.6.1 Algebraic and Transcendental Numbers

Claim: $\mathbb{A}$ and $\mathbb{T}$ are countable sets over both $\mathbb{R}$ and $\mathbb{C}$.

Proof. $\mathbb{R}$ is a countable set via Section 2.5. Since the algebraic reals and transcendental reals are both subsets of the reals, they are countable via Theorem 3.1. They are also countable over the complex numbers using Section 2.7 and the same logic. ■

2.6.2 Imaginary Numbers

Claim: $\mathbb{I}$ is a countable set; $C(\mathbb{I}) \equiv 1$.

Proof. $\mathbb{R}$ is a countable set via Section 2.5. $\mathbb{R}(\sqrt{-1}) \cup \emptyset(1) \equiv \mathbb{I}$. This is countable via Theorem 1.1 becuase the union of any set and the null set is itself. ■

2.7 Countability of the Complex Numbers

Claim: $\mathbb{C}$ is a countable set; $C(\mathbb{C}) \equiv 1$.

Proof. $\mathbb{R}$ and $\mathbb{I}$ are countable sets via Sections 2.5 and 2.6.2 respectively. $\mathbb{R}(1, \sqrt{-1}) \equiv \mathbb{C}$ is countable via Theorem 4.1. $\mathbb{R} \times \mathbb{I} = \mathbb{C}$ is countable via Theorem 2.1. ■

3 Generalized Continuum Hypothesis

Claim: GCH has the same truth value as CH, and powersets don't change sizes of infinity.

Proof. Let S be a countably infinite set; it cannot be finite or this proof does not work. $P := \mathcal{P}(S)$. Let R_k denote the set of elements in P with k sub-elements (S^k); these are countable via Theorem 5. Observe that P has $\aleph_0^0$ elements with 0 sub-elements, $\aleph_0^1$ elements with 1 sub-element, et cetera, and generally $\aleph_0^k$ elements with k sub-elements, for $k \leqslant \aleph_0$. Imagine a k-dimensioned table of integers and listing them in a k-dimensional spiral, or recursively combining two table dimensions into one using the Theorem 2 table method until the table gets to one dimension. $P = \displaystyle\bigcup_{i=1}^{\aleph_0} R_{i-1}$ is countable via Theorem 6. This contradicts Cantor's Diagonal Argument (see Section 4) and implies that uncountable sets cannot be created through powersets of countable sets, and either cannot be created at all, or need a much stronger method to create them. ∎

4 Cantor's Diagonal Argument

According to Georg Cantor in 1891[ref 1, 2], If someone is trying to list all the real numbers, they can always find a number that is not in the list using his "Diagonal Argument" which accomplishes the same as the following, though for reals instead of naturals: suppose someone is trying to make a set S with every natural number ($\mathbb{N}_1$). They could first skip 1 and add 2 to S. Then they add 3, then 4, 5, 6, et cetera forever. No matter how many natural numbers they add, they can always find one not in it; $\max(S)+1$. This seems to be implying that there is not a bijection between the natural numbers and themselves since in the limit towards infinity time, they've used up all infinity natural numbers indexing S and yet not all of them are in S. Well of course, because they *skipped* 1. Due to this possibility of skipping elements, the diagonal argument doesn't always accurately depict the countability of sets, and it can just as easily skip elements in the set of real numbers.

5 Enumerating the Continuum Algorithmically

The following NodeJS code prints out real numbers to stdout up to the maximum allowed big integer delimited by a comma-space pair, the problem being that it prints out both ± 0, though this is easily resolvable. The more verbatim version as well as the C version of the source code are at References 5 and 6. The complexity of finding the n^{th} real number with this method is around $\Theta(\log n)$, assuming basic operations such as bit shifts, comparisons, addition, and multiplication run in $\mathcal{O}(1)$. isqrt(n) is less than $\mathcal{O}(\log_2 n)$, and contains most of the complexity of the formula. The code didn't start to slow down noticably until around 10^{100}, and at $10^{10{,}000}$ only took 2.4 seconds (rather than 0.2ms at the start).

```
const write = globalThis.toString().slice(8, -1).toLowerCase() === "global" ?
   s => process.stdout.write.call(process.stdout, s) :  // NodeJS
   console.log; // Browser
```

```
function isqrt(n) { // floored square root
   if (n < 2n) return n;

   var cur = n >> 1n, prev; // current, previous

   do [cur, prev] = [cur + n / cur >> 1n, cur];
   while (cur < prev);

   return prev;
}

for (var i = 0n ;;) {
   const k = 1n + isqrt(1n + (i << 2n)) >> 1n // T(k-1) < i/2 ≤ T(k)
      , t = i + k*(1n - k) >> 1n; // i/2 - T(k-1)

   write( // coordinates:  (k, k-t-1)
      `${i++ % 2n ? "-" : ""}${t}.`+ // integer part
      `${k - t - 1n}`.split("").reverse().join("") // fractional part
   )
}
```

6 Further Algorithms

The axioms of Zermelo-Fraenkel Set theory required to prove the algorithmic viability of continuum enumeration are as follows: Axiom Schema of Specification, Axiom of Union, Axiom of Infinity, and Axiom of Choice (for creating sets of ordered *pairs*, which is not the controversial part). Without the Axiom of Choice, Theorem 2 doesn't work, although the algorithm from Section 5 is separate from set theory and assumes no axioms. The previous algorithm uses string manipulation to reverse the numbers and find their lengths, but the following formulas for length() and reverse() do the same thing mathematically.

$$\operatorname{length} n := 1 + \lfloor \log_b |n| \rfloor = \lceil \log_b(|n| + 1) \rceil \ni \ln 0 \leqslant 0 \tag{5}$$

$$\operatorname{reverse} n := \operatorname{sgn}(n) \sum_{k=0}^{\operatorname{length} n} \left(\left\lfloor \frac{|n|}{b^k} \right\rfloor \bmod b \right) b^{\operatorname{length} n - k - 1} \tag{6}$$

$$R_{i,j} := s \cdot \left(i + \frac{\operatorname{reverse} j}{b^{\operatorname{length} j}} \right) \tag{7}$$

Where i is the integer part, s is the sign, j is the reversed decimal part, and b is the base. i and j can be swapped for a different enumeration (and inverse) (see Reference 7). A formula for the greatest triangle number $t = T(k) \leqslant n$, where $n, k \in \mathbb{N}_0$ (used in Section 5):

$$T(n) := \frac{n^2+n}{2} = \sum_{j=1}^{n} j \tag{8}$$

$$t = T\left(\left\lceil \frac{\lfloor\sqrt{8n+1}\rfloor}{2} \right\rceil - 1\right) = T\left(\left\lfloor \frac{\sqrt{8n+1}}{2} \right\rceil - 1\right) \tag{9}$$

Because of the bijection, there is also an inverse of the Section 5 algorithm. The simplicity and symmetry of the inverse is striking due to the complexity of the original function.

```
function inverse(string) {
   const match = /^-??(\d+)\.(\d+)$/.exec(string)
      , x = BigInt(match[1]) // integer part
      , y = BigInt(match[3].split("").reverse().join("")); // decimal part
   return (x+y)**2n + 3n*x + y + BigInt(string[0] === "-");
}
```

7 Second Enumeration

There are more enumerations than just what was previously mentioned. A (mostly unrigorous) proof will be presented for the second of the following equations, both of which derive from a similar idea. Something similar works for any natural number base (Reference 9: Python code, Reference 10: desmos graph), instead of just 2, although it works best with prime bases.

$$\lim_{n,k\in\mathbb{N}_1\to\aleph_0} \left\{ k\left(\frac{2i+1}{2^{\lfloor \log_2 i \rfloor}} - 3\right) : i \in \mathbb{N}_1 < 2^n \right\} = \mathbb{R} \tag{10}$$

$$\bigcup_{n=1}^{\aleph_0} \frac{2n+1}{2^{1+\lfloor \log_2 n \rfloor}} \equiv \bigcup_{n=1}^{\aleph_0} \frac{2n+1}{2^{\mathrm{length}_2(n)}} = \mathbb{R}_{(1,2)} \tag{11}$$

Proof. This proof will first prove that $S_{\aleph_0}$ is equivalent to $\mathbb{R}_{[0,1)}$, and then proves that it must be countable.

$$\begin{aligned}
&S_0 := \{0\} \\
&S_n := S_{n-1} \cup \left\{ x + \frac{1}{2^n} : x \in S_{n-1} \right\} = \frac{\{k \in \mathbb{N} : 0 \le k < 2^n\}}{2^n} \\
&S_{\aleph_0} := \lim_{n\in\mathbb{N}\to\aleph_0} S_n = \mathbb{R}_{[0,1)}
\end{aligned}$$

$S_{\aleph_0}$ must equal [0,1) because 0 and 1 are its bounds and it has the same cardinality as the real numbers. It starts with 0 and only adds to it conditionally so its elements never go below 0, and $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, meaning it can't extend past 1. $|S_n| = 2^n$ which implies $|S_{\aleph_0}| = 2^{\aleph_0}$, the same cardinality as $|\mathcal{P}(\mathbb{N})|$ or $|\mathbb{R}|$. $S_{\aleph_0}$ is countable via Cantor's Enumeration of Countable

Collections of Countable sets, since there are countably infinitely many countable processess involved in constructing $S_{\aleph_0}$. The "countable collection" refering to the indices "n," and the "processes" being the evaluation of S_n given S_{n-1}, each of which processes is itself finite, and thus countable. The same results arise when starting with $S_0 = \{1\}$ and subtracting subsequent inverse powers of 2 rather than adding. ■

8 Conclusion

Since there exists a bijection between any infinite set and its powerset (GCH), there is no set with a cardinality strictly or loosely in between the naturals and reals (CH), because they are the same cardinality. This also implies that $\aleph_n = \beth_n = \beth_0$ for all natural numbers n, which makes sense intuitively because $2^\infty = \infty$. Also if any sequence of natural numbers is concatenated, it will always create a new natural number, meaning every element in $\mathcal{P}(\mathbb{N}_0)$ corresponds to an element in $\mathbb{N}_0$. This all could have disasterous consequences for set theory because CH is provably undecidable in some models and yet provably decidable. This means either Zermelo-Fraenkel Set Theory with the axiom of choice (ZHC) is unsound, the aforementioned models are invalid, or all of the decidability proofs are invalid. The entire foundation of aleph numbers, beth numbers, and sizes of infinities could be entirely flawed. CH implies the Gimel Hypothesis is true according to Reference 11, and implies Wetzel's problem is false according to Reference 12. A truth value for CH or GCH is not asserted here bacause the specifics of CH are unclear, and they are important for its truth value. If the reals have to be bigger than the naturals, it is false, but if it just requires that there is no set in between them, it is true.

9 References

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