On the Continuum Hypotheses and the Correspondence of Infinite Sets' Elements with the Natural Numbers

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Dedicated to Maeby, my favorite kitty

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1 Introduction and Countability Theorems

Any "countable" set is equinumerous to an improper subset of the natural numbers, and as follows, there exists a bijection between them. Any finite set has this property, an important question being if all *infinite* sets do. In all infinite processes, one should think of the steps as executing simultaneously rather than separately and successively. $|S|$ and $C(S)$ denote the cardinality and countability of a set S respectively. Equations (2a) and (2b) describe what will be called "sum ring operators," "product ring operators," and collectively "ring operators." Most of the theorems use " $\iff$ " instead of " $\implies$ " because Theorem 3.1 works as an inversion. Equation (1a) doesn't nest groups but flattens them in the same way as R^n normally does. While sets don't have an indexing traditionally, it is easier to pretend they can to see if they are countable. R, S & T are sets. a & b are reals. n, k, i & j are naturals.

$$R \times S := \{(r, s) : r \in R \land s \in S\} \quad \text{(1a)} \qquad R \cdot S := \{r \cdot s : r \in R \land s \in S\} \quad \text{(1b)} \tag{1}$$

$$R(\vec{v}\,) := \left\{ \sum_{n=1}^{\dim \vec{v}} (r_n \cdot \vec{v}_n) : r_k \in R \right\} \quad \text{(2a)} \qquad R[\,\vec{v}\,] := \left\{ \prod_{n=1}^{\dim \vec{v}} (r_n + \vec{v}_n) : r_k \in R \right\} \quad \text{(2b)} \tag{2}$$

$$\aleph_0 := |\mathbb{N}| \tag{3}$$

$$C(S) := \begin{cases} 1, \text{ [}S\text{ countable]} & (|S| \leqslant \aleph_0) \\ 0, \text{ [}S\text{ uncountable]} & (|S| > \aleph_0) \end{cases} \tag{4}$$

Theorem 1. Singular Unions of Countable Sets

If any sets R and S are countable, $R(a) \cup S(b)$ is countable for all numbers a and b.

1. $C(R) \land C(S) \iff C(R\,(a) \cup S\,(b))\ \forall a, b$
2. $C(R) \land C(S) \iff C(R\,(a) \cup S[\,b\,])\ \forall a, b$
3. $C(R) \land C(S) \iff C(R[\,a\,] \cup S\,(b))\ \forall a, b$
4. $C(R) \land C(S) \iff C(R[\,a\,] \cup S[\,b\,])\ \forall a, b$

Proof. $R(a)$ and $S(b)$ exist due to the Axiom Schema of Replacement. They are countable because ring operators don't change cardinality. For case 1, Interlace the elements of $R(a)$ with those of $S(b)$, skipping over any duplicates. If either set is finite, append its elements to the beginning of the other set, excluding duplicates. The same logic works with the other 3 cases. ■

Theorem 2. Cartesian Products and Element-Wise Multiplications of Countable Sets
If any sets R and S are countable, both $R \times S$ and $R \cdot S$ are countable.

1. $C(R) \wedge C(S) \iff C(R \times S)$
2. $C(R) \wedge C(S) \iff C(R \cdot S)$

Proof. To construct $R \times S$, put all possible ordered pairs into a table as shown. Go along the diagonals adding each value to a new set T. Every element will be both included and indexed in T, and thus a bijection exists between T $(R \times S)$ and $\mathbb{N}$. Each table element is of the form "T-index : value," the first 25 being labeled if in view. The same core logic works for $R \cdot S$. This method is algorithmically viable (see Section 5), and the T-indices relate strongly to triangle numbers (see Section 6).

$1 : (R_1, S_1)$	$2 : (R_1, S_2)$	$4 : (R_1, S_3)$	$7 : (R_1, S_4)$	$\cdots$
$3 : (R_2, S_1)$	$5 : (R_2, S_2)$	$8 : (R_2, S_3)$	$12 : (R_2, S_4)$	$\cdots$
$6 : (R_3, S_1)$	$9 : (R_3, S_2)$	$13 : (R_3, S_3)$	$18 : (R_3, S_4)$	$\cdots$
$10 : (R_4, S_1)$	$14 : (R_4, S_2)$	$19 : (R_4, S_3)$	$25 : (R_4, S_4)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

Theorem 3. Subsets, Intersections, and Differences of Countable Sets
If a set R is countable, if any set S is an improper subset of R, is countable.

1. $C(R) \wedge S \subseteq R \implies C(S)$
2. $C(R) \wedge C(S) \implies C(R \cap S)$
3. $C(R) \implies C(R \setminus S)$

Proof. Since subsets of $\mathbb{N}$ are countable and R bijects $\mathbb{N}$, all subsets of R must also be countable due to the Axiom Schema of Replacement. This implies that set intersections and differences are countable since these operations return subsets of the left input. Case 3 doesn't require S to be countable because its elements cannot appear in the output set. ■

Theorem 4. Two-Argument Ring Operators on Countable Sets
If any set R is countable, $R(a, b)$ and $R[a, b]$ are also countable for any constants a, b.

1. $C(R) \iff C(R(a, b))\ \forall a, b$
2. $C(R) \iff C(R[\,a, b\,])\ \forall a, b$

Proof. $R(a, b)$ can be tabulated with axes n and k, where the corresponding table box is $aR_n + bR_k$. This is countable in the same way as Theorem 2 except with addition as the combining operator instead of multiplication. Similar logic works for $R[a, b]$.

$aR_1 + bR_1$	$aR_1 + bR_2$	$aR_1 + bR_3$	$aR_1 + bR_4$	$aR_1 + bR_5$	$\cdots$
$aR_2 + bR_1$	$aR_2 + bR_2$	$aR_2 + bR_3$	$aR_2 + bR_4$	$aR_2 + bR_5$	$\cdots$
$aR_3 + bR_1$	$aR_3 + bR_2$	$aR_3 + bR_3$	$aR_3 + bR_4$	$aR_3 + bR_5$	$\cdots$
$aR_4 + bR_1$	$aR_4 + bR_2$	$aR_4 + bR_3$	$aR_4 + bR_4$	$aR_4 + bR_5$	$\cdots$
$aR_5 + bR_1$	$aR_5 + bR_2$	$aR_5 + bR_3$	$aR_5 + bR_4$	$aR_5 + bR_5$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

Theorem 5. Finite Natural Powers of Countable Sets

If any set R is countable, R^n is also countable for all natural numbers n.

1. $C(R) \iff C(R^n) \ \ \forall n \in \mathbb{N}_0 < \infty$

Proof. R^n can be factored as $R \times R^{n-1}$ (eqn. 1a flattens outputs), which is countable if R^{n-1} is countable (Theorem 2.1). This can be applied recursively until it becomes countable if R^2 or $R \times R$ is countable, which is countable via Theorem 2.1. If $n = 0$, the output set is all the groups of zero elements ($\emptyset$) from R; this is countable since $|\emptyset|$ is finite. ■

Theorem 6. Countably Infinite Unions of Countably Infinite Sets

If all sets R_n are countable, their union is also countable.

1. $C(R_n)\forall n \iff C\left(\bigcup_{n=1}^{\aleph_0} R_n\right)$

Proof. To create the union, give each set R_n a column in a table T and put all its elements sequentially in that column going down. Do this for every R_n, and using the same method as in Theorem 2, one can show that T bijects $\mathbb{N}$. Every column represents a different R_n and each row represents a different $(R_n)_k$ or $R_{n,k}$. This is algorithmically viable (see Section 5). Refer to Theorem 2 for more information on the algorithm or the table format.

$1 : R_{1,1}$	$2 : R_{2,1}$	$6 : R_{3,1}$	$7 : R_{4,1}$	$15 : R_{5,1}$	$16 : R_{6,1}$	$28 : R_{7,1}$	$\cdots$
$3 : R_{1,2}$	$5 : R_{2,2}$	$8 : R_{3,2}$	$14 : R_{4,2}$	$17 : R_{5,2}$	$27 : R_{6,2}$	$30 : R_{7,2}$	$\cdots$
$4 : R_{1,3}$	$9 : R_{2,3}$	$13 : R_{3,3}$	$18 : R_{4,3}$	$26 : R_{5,3}$	$31 : R_{6,3}$	$43 : R_{7,3}$	$\cdots$
$10 : R_{1,4}$	$12 : R_{2,4}$	$19 : R_{3,4}$	$25 : R_{4,4}$	$32 : R_{5,4}$	$42 : R_{6,4}$	$49 : R_{7,4}$	$\cdots$
$11 : R_{1,5}$	$20 : R_{2,5}$	$24 : R_{3,5}$	$33 : R_{4,5}$	$41 : R_{5,5}$	$50 : R_{6,5}$	$62 : R_{7,5}$	$\cdots$
$21 : R_{1,6}$	$23 : R_{2,6}$	$34 : R_{3,6}$	$40 : R_{4,6}$	$51 : R_{5,6}$	$61 : R_{6,6}$	$72 : R_{7,6}$	$\cdots$
$22 : R_{1,7}$	$35 : R_{2,7}$	$39 : R_{3,7}$	$52 : R_{4,7}$	$60 : R_{5,7}$	$73 : R_{6,7}$	$85 : R_{7,7}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

2 Countabilities of Common Sets

2.1 Countability of the Integers

Claim: $\mathbb{Z}$ is a countable set; $C(\mathbb{Z}) \equiv 1$.

Proof. $R := \mathbb{N}_0$, $S := \mathbb{N}_1$. R and S are countable axiomatically (axiom of extensionality), being the naturals themselves. $\mathbb{Z} \equiv R(1) \cup S(-1)$; this is countable via Theorem 1.1. ■

2.2 Countability of the Rationals

Claim: $\mathbb{Q}$ is a countable set; $C(\mathbb{Q}) \equiv 1$.

Proof. $\mathbb{Q}$ has the same countability as $\mathbb{Z} \times \mathbb{N}_1$. $C(\mathbb{Z}) \equiv 1$ via Section 2.1 and $C(\mathbb{N}_1) \equiv 1$ axiomatically. Thus $C(\mathbb{Z} \times \mathbb{N}_1) \equiv 1$ via Theorem 2.1. This and Theorem 2 derive from Cantor's enumeration of countable collections of countable sets. ■

2.3 Countability of the Reals from Zero to One

Claim: $\{x \in \mathbb{R} : 0 \leqslant x < 1\}$ is a countable set; $C(\mathbb{R}_{0\leqslant x<1}) \equiv 1$.

Proof. Let R be a set where $\forall n \in \mathbb{N}_0$, $R_n :=$ ("0." + reversed digits of n), ie: $R_{2760} = 0.0672$ (see Section 6 for formula). R is countable by definition, and contains every possible sequence of digits in $[0, 1)$. The sequence trends towards different values with different sub-series; Section 2.5 explains this further. R is used in the next proof. ■

2.4 Countability of the Non-Negative Reals

Claim: $\mathbb{R}_{\geqslant 0}$ or equivalently $\{x \in \mathbb{R} : x \geqslant 0\}$, is a countable set; $C(\mathbb{R}_{\geqslant 0}) \equiv 1$.

Proof. $C(\mathbb{R}_{0\leqslant x<1}) \equiv 1$ via Section 2.3. $\mathbb{R}_{\geqslant 0} = \bigcup_{n=1}^{\aleph_0} S_n$ where $S_n := R[n-1]$. This is countable via Theorem 6; the following table illustrates this. $\forall x \in \mathbb{R}_{\geqslant 0}$, there exists a sequence of $s_n \in S_{\lceil x \rceil}$, with increasing indices, where $\lim\limits_{n\in\mathbb{N}\to\infty} s_n = x$. This is because S_n contains every sequence of digits in $[n-1, n)$. S is used in the next proof.

0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.01	0.11	0.21	$\cdots$
1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	1.01	1.11	0.21	$\cdots$
2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	2.01	2.11	0.21	$\cdots$
3	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	3.01	3.11	0.21	$\cdots$
4	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	4.01	4.11	0.21	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

2.5 Countability of the Reals

Claim: $\mathbb{R}$ is a countable set, which implies $2^{\aleph_0} = \aleph_0$.

Proof. $\mathbb{R}_{\geqslant 0}$ is countable via Section 2.4. $\mathbb{R}_{\geqslant 0}(1, -1) \equiv \mathbb{R}$. This is countable via Theorem 4.1. This conclusion can be further reinforced by the algorithmic methods in Section 5. ■

2.6 Countabilities of Miscellaneous Number Classes

2.6.1 Algebraic and Transcendental Numbers

Claim: $\mathbb{A}$ and $\mathbb{T}$ are countable sets over both $\mathbb{R}$ and $\mathbb{C}$.

Proof. $\mathbb{R}$ is a countable set via Section 2.5. Since the algebraic reals and transcendental reals are both subsets of the reals, they are countable via Theorem 3.1. They are also countable over the complex numbers using Section 2.7 and the same logic. ■

2.6.2 Imaginary Numbers

Claim: $\mathbb{I}$ is a countable set; $C(\mathbb{I}) \equiv 1$.

Proof. $\mathbb{R}$ is a countable set via Section 2.5. $\mathbb{R}(\sqrt{-1}) \cup \emptyset(1) \equiv \mathbb{I}$. This is countable via Theorem 1.1 becuase the union of any set and the null set is itself. ■

2.7 Countability of the Complex Numbers

Claim: $\mathbb{C}$ is a countable set; $C(\mathbb{C}) \equiv 1$.

Proof. $\mathbb{R}$ and $\mathbb{I}$ are countable sets via Sections 2.5 and 2.6.2 respectively. $\mathbb{R}(1, \sqrt{-1}) \equiv \mathbb{C}$ is countable via Theorem 4.1. $\mathbb{R} \times \mathbb{I} = \mathbb{C}$ is countable via Theorem 2.1. ■

3 Generalized Continuum Hypothesis

Claim: GCH has the same truth value as CH, and powersets don't change sizes of infinity.

Proof. Let S be a countably infinite set; it cannot be finite or this proof does not work. $P := \mathcal{P}(S)$. Let R_k denote the set of elements in P with k sub-elements (S^k); these are countable via Theorem 5. Observe that P has $\aleph_0^0$ elements with 0 sub-elements, $\aleph_0^1$ elements with 1 sub-element, et cetera, and generally $\aleph_0^k$ elements with k sub-elements, for $k \leqslant \aleph_0$. Imagine a k-dimensioned table of integers and listing them in a k-dimensional spiral, or recursively combining two table dimensions into one using the Theorem 2 table method until the table gets to one dimension. $P = \bigcup_{i=1}^{\aleph_0} R_{i-1}$ is countable via Theorem 6. This contradicts Cantor's Diagonal Argument (see Section 4) and implies that uncountable sets cannot be created through powersets of countable sets, and either cannot be created at all, or need a much stronger method to create them. ■

4 Cantor's Diagonal Argument

According to Georg Cantor in 1891[ref 1, 2], If someone is trying to list all the real numbers, they can always find a number that is not in the list using his "Diagonal Argument" which accomplishes the same as the following, though for reals instead of naturals: suppose someone is trying to make a set S with every natural number ($\mathbb{N}_1$). They could first skip 1 and add 2 to S. Then they add 3, then 4, 5, 6, et cetera forever. No matter how many natural numbers they add, they can always find one not in it; $\max(S)+1$. This seems to be implying that there is not a bijection between the natural numbers and themselves since in the limit towards infinity, they've used up all infinity natural numbers indexing S and yet not all of them are in S. Well of course, because they *skipped* 1. Due to this possibility of skipping elements, the diagonal argument cannot always accurately depict the countability of sets, and it can just as easily skip elements in the set of real numbers.

5 Enumerating the Continuum Algorithmically

The following JavaScript code prints out real numbers to stdout, delimited by a comma-space pair, up to the maximum allowed big integer index. It prints out both ± 0, though this is not really a problem and is easily resolvable. The more verbatim version as well as the C version of the source code can be found at References 5 and 6. The complexity of finding the n^{th} real number with this method is around $\Theta(\log n)$, assuming basic operations such as bit shifts, comparisons, addition, and multiplication run in $\mathcal{O}(1)$. isqrt(n) is less than $\mathcal{O}(\log_2 n)$, and contains most of the complexity of the formula. The code didn't start to slow down noticably on my machine until around $n = 10^{100}$, and at $n = 10^{10{,}000}$ only took 2.4 seconds each (rather than 0.2ms at the start).

```
const write = globalThis.toString().slice(8, -1).toLowerCase() === "global" ?
    s => process.stdout.write.call(process.stdout, s) : // NodeJS
    console.log; // Browser

function isqrt(n) { // floored square root
    if (n < 2n) return n;

    var cur = n >> 1n, prev; // current, previous

    do [cur, prev] = [cur + n / cur >> 1n, cur];
    while (cur < prev);

    return prev;
}

for (var i = 0n ;;) {
    const k = 1n + isqrt(1n + (i << 2n)) >> 1n // T(k-1) < i/2 <= T(k)
        , t = i + k*(1n - k) >> 1n; // i/2 - T(k-1)

    write( // coordinates: (k, k-t-1)
        `${i++ % 2n ? "-" : ""}${t}.` + // integer part
        `${k - t - 1n}`.split("").reverse().join("") // fractional part
    )
}
```

6 The Algorithm Mathematically

The axioms of Zermelo-Fraenkel Set theory required to prove the algorithmic viability of continuum enumeration are as follows: Axiom Schema of Specification (for creating Cartesian products), Axiom of Union, and Axiom of Infinity. The previous algorithm uses string manipulation to reverse integers and find their lengths, but the following formulas for length() and reverse() do the same thing mathematically. They are used to find the nth real number.

$$\operatorname{length}_b x := \max(0, 1 + \lfloor \log_b |x| \rfloor) \ni \ln 0 \leqslant 0 \tag{5}$$

$$\operatorname{reverse}_b x := \operatorname{sgn}(x) \sum_{m=0}^{\operatorname{length}_b x} \left(\left\lfloor \frac{|x|}{b^m} \right\rfloor \bmod b \right) b^{\operatorname{length}_b x - m - 1} \tag{6}$$

$$R(i, j) := i + \frac{\operatorname{reverse}_{10} j}{10^{\operatorname{length}_{10} j}} \tag{7}$$

Where i is the integer part, j is the reversed decimal part, and b is the base. i and j can be swapped for a different enumeration (and inverse) (see Reference 7). t is the greatest triangle number $T(k) \leqslant n$, where $n, k \in \mathbb{N}_0$.

$$T(x) := \frac{x^2 + x}{2} = \sum_{j=1}^{x} j \tag{8}$$

$$n := \left\lfloor \frac{N}{2} \right\rfloor \tag{9}$$

$$k := \left\lceil \frac{\lfloor \sqrt{8n+1} \rfloor}{2} \right\rceil - 1 = \left\lfloor \frac{\sqrt{8n+1} - 1}{2} \right\rfloor \tag{10}$$

$$t := T(k) \tag{11}$$

$$u := n - t \tag{12}$$

$$\mathbb{R}_N \; = (2[(N+1) \bmod 2] - 1) R(u, k - u) \tag{13}$$

For each integer k, This cycles through the ordered pairs of positive (i, j) where $i + j = k$ and it makes a unique real number out of each pair. Because of the bijection it creates, there is also an inverse of the (Section 5) algorithm. The simplicity and near-symmetry of the inverse is striking due to the complexity of the original function.

```
function inverse(string) {
    const [ipart, fpart] = string.replace("-", "").split(".")
        , x = BigInt( ipart )
        , y = BigInt( fpart.split("").reverse().join("") );

    return (x+y)**2n + 3n*x + y + BigInt(string[0] === "-");
}
```

The following is the same formula in math form. ∞ can be replaced with the max precision.

$$x(t) := \lfloor |t| \rfloor \tag{14}$$

$$y(t) := \operatorname{reverse}_{10} \left((|t| \bmod 1) \, 10^{\sum_{n=0}^{\infty} \operatorname{sgn}(10^n |t| \bmod 1)} \right) 10^{\sum_{n=1}^{\infty} \left(1 - \operatorname{sgn} \lfloor 10^n (|t| \bmod 1) \rfloor \right)} \tag{15}$$

$$\operatorname{inverse}(t) := [x(t) + y(t)]^2 + 3\, x(t) + y(t) + 1 - \operatorname{sgn}(1 + \operatorname{sgn} t) \tag{16}$$

7 Further Evidence

TODO: finish this.

- $\lim_{n\to\infty} x_n = x \Longrightarrow \lim_{n\to\infty} \text{inverse}(x_n) = N$ (some 10-adic integer index unique to x).
- x_n approaches x at the same rate that inverse(x_n) approaches N. So for each extra (decimal) digit that x_n gains, inverse(x_n) also gains one (integer) digit. It can gain more if the digits after it are zero, but never less.
- $\text{inverse}(10^n) = \begin{cases} 100^n + 3 \cdot 10^n, \ n \geq 0 \\ \dfrac{1}{100^{n+1}} + \dfrac{1}{10^{n+1}}, \ n < 0 \end{cases}$
- $\text{inverse}(b^n) = b^{2n} + 3 \cdot b^n \forall n \geq 0 \wedge b \geq 2$

8 Conclusion

Since there exists a bijection between any infinite set and its powerset (GCH), there is no set with a cardinality strictly or loosely in between the naturals and reals (CH), because they are the same cardinality. This also implies that $\aleph_n = \beth_n = \beth_0$ for all natural numbers n, which makes sense intuitively since $2^\infty = \infty$. Also if any sequence of natural numbers is concatenated, it will always create a new natural number, meaning every element in $\mathcal{P}(\mathbb{N}_0)$ corresponds to an element in $\mathbb{N}_0$. This all could have disasterous consequences for set theory because CH is provably undecidable in some models and yet provably decidable. This means either Zermelo-Fraenkel Set Theory with the axiom of choice (ZHC) is unsound, the aforementioned models are invalid, or all of the decidability proofs are invalid. The entire foundation of aleph numbers, beth numbers, and sizes of infinities could be entirely flawed. CH implies the Gimel Hypothesis is true according to Reference 9, and implies Wetzel's problem is false according to Reference 10. A truth value for CH or GCH is not asserted here bacause the specifics of CH are unclear, and they are important for its truth value. If the reals have to be bigger than the naturals, it is false, but if it just requires that there is no set in between them, it is true.

9 References

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