Trigonometric Functions in Terms of Gamma

Daniel E. Janusch

November 16, 2022

1 Definitions

Inverse trig functions don't count, but hyperbolic trig functions do.
"$\ni$" means "such that".
"$\wedge$" means "and".
$0 \in \mathbb{N}_0$

$$\mathbb{N}_k := \{x + k : x \in \mathbb{N}_0\} \tag{1}$$

$$\tau := 2\pi \tag{2}$$

$$\Gamma(x) := \int_0^\infty t^x e^{-t} dt = (x-1)!\ \forall x > -1 \tag{3}$$

$$\zeta(x) := \sum_{n=0}^{\infty} \frac{1}{n^x} \forall x > 1 \tag{4}$$

2 Riemann Zeta Function

claim: $\zeta(x) = \dfrac{\zeta(1-x)\tau^x \sec\left(\dfrac{\pi x}{2}\right)}{2\,\Gamma(x)}$

Proof:

know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\dfrac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x)$

$$\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma\left(1-x\right) \zeta\left(1-x\right) \tag{5}$$

$$\zeta(x) = 2 \cdot 2^{x-1} \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) \tag{6}$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) \tag{7}$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x)\zeta(1-x) \tag{8}$$

$$\zeta(x) = \frac{2}{\tau^{1-x}} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x)\zeta(1-x) \tag{9}$$

$$\zeta(1-x) = \frac{2}{\tau^{x}} \cos\left(\frac{\pi x}{2}\right) \Gamma(x)\zeta(x) \tag{10}$$

$$\frac{\zeta(1-x)\tau^x}{2\cos\left(\frac{\pi x}{2}\right)\Gamma(x)} = \zeta(x) \tag{11}$$

$$\zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)} \tag{12}$$

■

3 Cosecant

claim: $\csc x = \frac{1}{\pi}\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right)$
Proof:
know: $\zeta(x) = 2^x\pi^{x-1}\sin\left(\frac{\pi x}{2}\right)\Gamma(1-x)\zeta(1-x)$

know: $\zeta(x) = \dfrac{\zeta(1-x)\tau^x\sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)}$

$$2^x\pi^{x-1}\sin\left(\frac{\pi x}{2}\right)\Gamma(1-x)\zeta(1-x) = \frac{\zeta(1-x)\tau^x\sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)} \tag{13}$$

$$2^x\pi^{x-1}\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\Gamma(1-x) = \frac{\tau^x}{2\,\Gamma(x)} \tag{14}$$

$$4\cdot\tau^{x-1}\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\Gamma(1-x) = \frac{\tau^x}{\Gamma(x)} \tag{15}$$

$$2\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\Gamma(1-x) = \frac{\pi}{\Gamma(x)} \tag{16}$$

$$\sin(\pi x)\Gamma(1-x)\Gamma(x) = \pi \tag{17}$$

$$\Gamma(1-x)\Gamma(x) = \pi\csc(\pi x) \tag{18}$$

$$\csc x = \frac{1}{\pi}\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right) \ni \zeta(1-x) \neq 0 \wedge \sin\pi x \neq 0 \tag{19}$$

$x \neq 2n+1 \ni n \in \mathbb{N}_1 \wedge x \neq k \ni k \in \mathbb{Z} \implies x \neq \mathbb{Z}$ (same domain as $\csc\pi x$)

$\blacksquare$

4 The remaining Functions

$$\sin x = \frac{1}{\csc x} \tag{20}$$

$$\cos x = \frac{1}{\csc\left(\frac{\pi}{2} - x\right)} \tag{21}$$

$$\tan x = \frac{\csc\left(\frac{\pi}{2} - x\right)}{\csc x} \tag{22}$$

$$\csc x = \csc x \tag{23}$$

$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \tag{24}$$

$$\cot x = \frac{\csc x}{\csc\left(\frac{\pi}{2} - x\right)} \tag{25}$$

$$\sinh x = -\frac{i}{\csc x} \tag{26}$$

$$\cosh x = \frac{1}{\csc\left(\frac{\pi}{2} - ix\right)} \tag{27}$$

$$\tanh x = -i\frac{\csc\left(\frac{\pi}{2} - ix\right)}{\csc ix} \tag{28}$$

$$\operatorname{csch} x = i \csc ix \tag{29}$$

$$\operatorname{sech} x = \csc\left(\frac{\pi}{2} - ix\right) \tag{30}$$

$$\coth x = \frac{i \csc ix}{\csc\left(\frac{\pi}{2} - ix\right)} \tag{31}$$