The Gamma Function, Its Inverse, and Its Relationship With Trigonometry

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1 Definitions

$$0 \in \mathbb{N} \quad \text{(1a)} \qquad \mathbb{N}_k := \{x + k : x \in \mathbb{N}\} \, \forall \, k \quad \text{(1b)} \tag{1}$$

$$\tau := 2\pi \tag{2}$$

$$\operatorname{sgn}(x) := \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases} = \frac{x}{|x|} \; \forall x \tag{3}$$

$$a \bmod_n b := a \bmod b + n \tag{4}$$

$$\exp(x) := \lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!} = e^x \; \forall x \tag{5}$$

$$\Gamma(x) := \int_0^\infty \frac{t^{x-1}}{\exp t} \mathrm{d}t = (x-1)! \; \forall x > 0 \tag{6}$$

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x} \forall x > 1 \tag{7}$$

$$\eta(x) := \Gamma^{-1}(x) \quad \text{(8a)} \qquad \Gamma(\,\eta(x)\,) = x \quad \text{(8b)} \tag{8}$$

(8a) is the compositional inverse rather than the fractional inverse. The set of all natural numbers, $\mathbb{N}$, is defined in part by (1a) and (1b). (7) describes the Riemann Zeta Function. x is assumed to be a real number everywhere for simplicity.

2 Riemann Zeta Function

claim: $\zeta(x) = \dfrac{\zeta(1-x)\tau^x \sec\dfrac{\pi x}{2}}{2\,\Gamma(x)}$

This proof is important for section 3.

Proof:

know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\dfrac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x)$ (reference 1)

$$\begin{aligned}
\zeta(x) &= 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) \\
\zeta(x) &= 2 \cdot 2^{x-1} \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) \\
\zeta(x) &= 2 \cdot \tau^{x-1} \sin\left(\frac{\pi}{2} x\right) \Gamma(1-x)\zeta(1-x) \\
\zeta(x) &= 2 \cdot \tau^{x-1} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x)\zeta(1-x) \\
\zeta(x) &= \frac{2}{\tau^{1-x}} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x)\zeta(1-x) \\
\zeta(1-x) &= \frac{2}{\tau^{x}} \cos\left(\frac{\pi}{2} x\right) \Gamma(x)\zeta(x) \\
\zeta(x) &= \frac{\zeta(1-x)\tau^x}{2\cos\left(\frac{\pi x}{2}\right)\Gamma(x)}
\end{aligned}$$

$$\zeta(x) = \frac{\zeta(1-x)\tau^x \sec\dfrac{\pi x}{2}}{2\,\Gamma(x)} \tag{9}$$

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3 Cosecant

claim: $\csc x$ can be written in terms of $\Gamma(x)$
Proof:
know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x)$ (reference 1)
know: $\zeta(x) = \dfrac{\zeta(1-x)\tau^x \sec \frac{\pi x}{2}}{2\,\Gamma(x)}$ (equation 9)

$$\begin{aligned}
2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) &= \frac{\zeta(1-x)\tau^x \sec \frac{\pi x}{2}}{2\,\Gamma(x)} \\
2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) &= \frac{\tau^x \sec \frac{\pi x}{2}}{2\,\Gamma(x)} \\
2 \cdot 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) &= \frac{\tau^x}{\Gamma(x)} \\
4 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) &= \frac{\tau^x}{\Gamma(x)} \\
2 \cdot 2 \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) &= \frac{\tau}{\Gamma(x)} \\
2 \sin(\pi x)\, \Gamma(1-x) &= \frac{\tau}{\Gamma(x)} \\
\Gamma(1-x)\Gamma(x) \sin \pi x &= \pi
\end{aligned}$$

$$\Gamma(1-x)\Gamma(x) = \pi \csc \pi x \tag{10}$$

$$\csc x = \frac{1}{\pi}\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1 - \frac{x}{\pi}\right) \tag{11}$$

$$\sin \pi x \neq 0 \implies \zeta(1-x) \neq 0 \tag{12}$$

(12) shows the domains also match.

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4 Trig Functions In Terms of Cosecant

The rest of the trig functions can be written in terms of cosecant

$$\sin x = \frac{1}{\csc x} \tag{13}$$

$$\cos x = \frac{1}{\csc\left(\frac{\pi}{2} - x\right)} \tag{14}$$

$$\tan x = \frac{\csc\left(\frac{\pi}{2} - x\right)}{\csc x} \tag{15}$$

$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \tag{16}$$

$$\cot x = \frac{\csc x}{\csc\left(\frac{\pi}{2} - x\right)} \tag{17}$$

$$\sinh x = -\frac{i}{\csc x} \tag{18}$$

$$\cosh x = \frac{1}{\csc\left(\frac{\pi}{2} - ix\right)} \tag{19}$$

$$\tanh x = -\frac{i\csc\left(\frac{\pi}{2} - ix\right)}{\csc ix} \tag{20}$$

$$\operatorname{csch} x = i\csc ix \tag{21}$$

$$\operatorname{sech} x = \csc\left(\frac{\pi}{2} - ix\right) \tag{22}$$

$$\coth x = \frac{i\csc ix}{\csc\left(\frac{\pi}{2} - ix\right)} \tag{23}$$

$$\csc x = \csc x \text{ by the reflexive property.} \tag{24}$$

5 Trig Functions In Terms of Gamma

These equations are just the previous ones with Gamma plugged in for csc.

$$\sin x = \frac{\pi}{\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right)} \tag{25}$$

$$\cos x = \frac{\pi}{\Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right)} \tag{26}$$

$$\tan x = \frac{\Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right)}{\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right)} \tag{27}$$

$$\csc x = \frac{1}{\pi}\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right) \tag{28}$$

$$\sec x = \frac{1}{\pi}\Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right) \tag{29}$$

$$\cot x = \frac{\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right)} \tag{30}$$

$$\sinh x = -\frac{\pi i}{\Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right)} \tag{31}$$

$$\cosh x = \frac{\pi}{\Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right)} \tag{32}$$

$$\tanh x = -i\,\Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right)\Big/\,\Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right) \tag{33}$$

$$\operatorname{csch} x = \frac{i}{\pi}\Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right) \tag{34}$$

$$\operatorname{sech} x = \frac{1}{\pi}\Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right) \tag{35}$$

$$\coth x = i\,\Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right)\Big/\,\Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right) \tag{36}$$

6 Inverse Gamma Function

claim: $\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x))$
Proof:
know: $\Gamma(x)\,\Gamma(1-x) = \pi \csc \pi x$ (equation 10)

$$\pi \csc \pi x = \Gamma(x)\,\Gamma(1-x) \tag{37}$$

$$\pi \csc \pi\eta(x) = x\,\Gamma(1-\eta(x)) \tag{38}$$

$$\csc \pi\eta(x) = \frac{x}{\pi}\Gamma(1-\eta(x)) \tag{39}$$

$$\pi\eta(x) = \csc^{-1} \frac{x}{\pi}\Gamma(1-\eta(x)) \tag{40}$$

$$\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi}\Gamma(1-\eta(x)) \tag{41}$$

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There are 2 conditions to where this $\eta(x)$ is defined.
$\eta(x)$ is *not* defined (over the real numbers) if the following is true:

$$\Gamma\left(-\frac{1}{2}\right) \leq x \leq \Gamma\left(\frac{1}{2}\right) \tag{42}$$

$$|\eta(x)| \leq \frac{1}{2}$$

$\eta(x)$ is guaranteed to be defined if Equation (42) is false. $\eta(x)$ is defined for all integers and in small fields around each integer. The area defined around each integer gets smaller as x increases, though it is unclear why without further inspection. If $\eta(x)$ is being recursed and $\frac{x}{\pi}$ is changed to $\frac{x^n}{\pi}$ in the innermost function, for some $n \in \mathbb{N}_2$, either the fields get larger, or there is a bug in Desmos' graphing calculations. Reference 2 and Page 9 have much more information.
Going a different way after (38) gives the following equation:

$$\eta(x) = 1 - \eta\left(\frac{\pi}{x} \csc \pi\eta(x)\right) \implies \eta(x) = \eta(\pi \csc(\pi \csc \pi x) \sin \pi x) \tag{43}$$

This equation seems to be less useful because it has large derivatives everywhere when absolute-valued, which gets more severe as it is recursed.

7 Miscellaneous

(29) implies the following formula:

$$\Gamma\left(x+\frac{1}{2}\right)=\frac{\pi\sec\pi x}{\Gamma\left(\frac{1}{2}-x\right)} \tag{44}$$

Replacing x with $x+z$ and adjusting from gamma to factorial gives:

$$\left(x+\frac{1}{2}+z\right)!=\left[2\left(z \bmod 2\right)-1\right]\frac{\pi\sec\pi x}{\left(-\frac{3}{2}-x-z\right)!}\forall z\in\mathbb{Z} \tag{45}$$

This only simplifies things if z is an integer because $\sec(x+\tau)=\sec x$, and $\sec(x+\pi)=-\sec x$. rewriting (10) from gamma to factorial gives:

$$(x-1)!(-x)!=\pi\csc\pi x \tag{46}$$

$$(-x)!=\frac{\pi x\csc\pi x}{x!} \tag{47}$$

$$\left(x+\frac{1}{c}\right)!=\frac{\pi\csc\pi\frac{cx+1}{c}}{\left(-\frac{c+1}{c}-x\right)!}\forall c\in\mathbb{R} \tag{48}$$

$$x!=(x \bmod_n 1)!\left[\prod_{k=\operatorname{sgn}(|n-x|+n-x)}^{|\lfloor x-n\rfloor|-\operatorname{sgn}(|x-n|+x-n)}\left[x+k\operatorname{sgn}(n-x)\right]\right]^{\operatorname{sgn}(x-n)} \tag{49}$$

(49) is useful where $x!$ can be non self-dependantly defined for $x\in[n,n+1)$.

$$\lim_{x\to\infty}\int_0^x\int_0^x\exp\left[\frac{1}{2}\ln\frac{4t}{u}-t-u\right]\mathrm{d}^2tu=\pi \tag{50}$$

claim: $\sin(x) = -\sin(-x)$:
Proof:

$$(47)\ x \mapsto -x \implies (-x)! = \frac{-\pi x \csc(-\pi x)}{x!} \tag{51}$$

$$(-x)! = \frac{\pi x \csc \pi x}{x!} \qquad (47)$$

$$\frac{\pi x \csc \pi x}{x!} = \frac{-\pi x \csc(-\pi x)}{x!} \implies \csc x = -\csc(-x) \implies \sin(x) = -\sin(-x) \tag{52}$$

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claim: $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$
Proof:

$$x! = \frac{(x+1)!}{x+1} \implies \left(-\frac{1}{2}\right)! = 2\left(\frac{1}{2}\right)! \tag{53}$$

$$(-x)! = \frac{\pi x \csc \pi x}{x!} \implies \left(-\frac{1}{2}\right)! = \frac{\pi}{2\left(\frac{1}{2}\right)!} \tag{54}$$

The formula for (54) is just (47). (53) and (54) imply the following:

$$2\left(\frac{1}{2}\right)! = \frac{\pi}{2\left(\frac{1}{2}\right)!} \implies \left(\frac{1}{2}\right)!^2 = \frac{\pi}{4} \implies \left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2} \tag{55}$$

Using this strategy only works if $(c_1 - x) = (c_2 + x)$ for some integers c_1 and c_2. This limits it to $(z + \frac{1}{2})$ for integers z. Also, you can only apply (47) an odd number of times or the factorials will cancel and leave a trigonometric equation. If you apply (47) twice, you get back to where you started because everything new cancels.
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$$\left(\frac{1}{2} + n\right)! = \frac{\sqrt{\pi}}{2} \prod_{k=\operatorname{sgn}(n+|n|)}^{|n| - \operatorname{sgn}(n+|n|) + \operatorname{sgn} n} \left[\frac{1}{2} + k \operatorname{sgn} n\right]^{\operatorname{sgn} n} \quad \forall n \in \mathbb{Z} \tag{56}$$

$$\Gamma\left(\frac{1}{\pi}\csc^{-1}\frac{\Gamma(11)}{\pi}\right)-\Gamma(11)\approx\int_0^1 \ln\ln\frac{1}{x}\ \mathrm{d}x \tag{57}$$

$$\lim_{x\to\infty}\left[\Gamma(x)-\Gamma\left(\frac{1}{\pi}\csc^{-1}\frac{\Gamma(x)}{\pi}\right)\right]=\gamma \tag{58}$$

$$x\approx\Gamma\left(\frac{1}{\pi}\csc^{-1}\frac{x}{\pi}\right) \tag{59}$$

$$x\approx\Gamma\circ\eta_1(x)\ni\eta_0(x)=0 \tag{60}$$

(57) is a specific case of (58). (59) and (60) are the same equation with different syntax. Equation (61) clarifies what the n^{th} $\eta(x)$ means. Equations (57) and (58) don't work on Desmos after around $x=11$ because it doesn't work well with large numbers since it is based in JavaScript. It does work well in Mathematica, however.

$$\eta_n(x):=\frac{1}{\pi}\csc^{-1}\left[\frac{x}{\pi}\Gamma(1-\eta_{n-1}(x))\right] \tag{61}$$

Where $\eta_0(x)$ can be any function. Section 6 uses $\eta_0(x):=x$, but $\eta_0(x)=-|x^n|$ or $\eta_0(x):=c$ for some constant c both seem to work well, though the powers of x work better; also being better for integer powers n. It is important to note that $\eta(x)$ only returns values in the range $\left[-\frac{1}{2},\frac{1}{2}\right]$. This implies that while $\Gamma(\,\eta(x)\,)=x$ is always true, $\eta(\,\Gamma(x)\,)=x$ is not necessarily true, only being valid for small x. The following table is for the approximate best constant values, (method: eyeballing the graph), and is split by n mod 4. It is just for a rought approximation.

$\eta_n(x)$	best η_0	$\eta_n(x)$	best η_0	$\eta_n(x)$	best η_0	$\eta_n(x)$	best η_0
0	0	1	0.15	2	0.42	3	0.42
4	0.4	5	0.4	6	0.4	7	0.4
8	0.394	9	0.394	10	0.39	11	0.39
12	0.387	13	0.387	14	0.385	15	0.3855
16	0.3835	17	0.3835	18	0.382	19	0.382
20	0.3808	21	0.3805	22	0.3797	23	0.3797
24	0.3786	25	0.3786	26	0.3777	27	0.3777
28	0.3771	29	0.3772	30	0.3767	31	0.3766
32	0.3761	33	0.376	34	0.3754	35	0.3754

8 References

1. https://en.wikipedia.org/wiki/Riemann_zeta_function#Riemann's_functional_equation
   Link for Equation 9

2. https://www.desmos.com/calculator/qnnylllsvt
   Extra information and graph for Section 6

3. https://www.github.com/drizzt536/files/tree/main/TeX/gamma
   The files for the most recent version of this PDF and the LaTeX code.