MA 345 Exact Equations Method Justification

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$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0 \tag{1}$$

Given an exact ODE (1), where $M_y = N_x$, the following equation holds for the solution:

$$\int M\mathrm{d}x \uplus \int N\mathrm{d}y = c \tag{2}$$

where $a \uplus b$ means add and ignore duplicates. Let $f(x,y)$ be the solution to the differential equation.

$M_1(x) + M_2(x,y) = \displaystyle\int M(x,y)\mathrm{d}x$, where $M_1(x)$ has all the terms with only x, and $M_2(x,y)$ has all the terms that can't be separated further. $N_1(y) + N_2(x,y) = \displaystyle\int N(x,y)\mathrm{d}y$, where $N_1(y)$ has all the terms with only y, and $N_2(x,y)$ has all the terms that can't be separated further.

$$f(x,y) = \int M(x,y)\mathrm{d}x + g(y) = M_1(x) + M_2(x,y) + g(y) \tag{3}$$

$$f(x,y) = \int N(x,y)\mathrm{d}y + h(x) = N_1(y) + N_2(x,y) + h(x) \tag{4}$$

Given (3) and (4), by the transitive property, $M_1(x)+M_2(x,y)+g(y) = N_1(y)+N_2(x,y)+h(x)$. Each side has a term of just x, a term of just y, and a term with both. Separating gives the following three equations:

$$M_1(x) = h(x) \tag{5}$$

$$M_2(x,y) = N_2(x,y) \tag{6}$$

$$g(y) = N_1(y) \tag{7}$$

Equation (3) can be rearranged into $\displaystyle\int N(x,y)\mathrm{d}y - N_2(x,y) = N_1(y) = g(y)$. Substituting this into Equation (2) gives:

$$f(x,y) = \int M(x,y)\mathrm{d}x + \left[\int N(x,y)\mathrm{d}y - N_2(x,y)\right] \tag{8}$$

This is then equivalent to the following by equations (1) and (6):

$$f(x,y) = [M_1(x) + N_2(x,y)] - N_2(x,y) + \int N(x,y)\mathrm{d}y \tag{9}$$

But this is just the sum of the integrals with the duplicates (M_2 and N_2) subtracted off. Thus, the original equation holds (also because the integral of 0 is a constant).

NOTE: stopping after equation 7 is sufficient proof for on exams, so long as the functions are defined concretely.