On the Continuum Hypotheses and the Correspondence of Infinite Sets with the Natural Numbers

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Dedicated to Maeby, my favorite kitty

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1 Introduction and Countability Theorems

Any "countable" set is equinumerous to a improper subset of the natural numbers, and there exists a bijection between them. Any finite set has this property, the main question being if all *infinite* sets do. In all infinite processes, one should think of the steps as executing simultaneously rather than separately and successively. $|S|$ and $C(S)$ denote the cardinality and countability of a set S respectively. Equations (2a) and (2b) describe what will be called sum amd product "ring operators." Most theorems use " $\iff$ " instead of " $\implies$ " because Theorem 3.1 works as an inversion. Equation (1a) doesn't nest groups but flattens them.

$$R \times S := \{(r, s) : r \in R \wedge s \in S\} \quad \text{(1a)} \qquad R \cdot S := \{r \cdot s : r \in R \wedge s \in S\} \quad \text{(1b)} \tag{1}$$

$$R(\vec{v}\,) := \left\{ \sum_{n=1}^{\operatorname{len} \vec{v}} (r_n \cdot \vec{v}_n) : r_k \in R \right\} \quad \text{(2a)} \qquad R[\,\vec{v}\,] := \left\{ \prod_{n=1}^{\operatorname{len} \vec{v}} (r_n + \vec{v}_n) : r_k \in R \right\} \quad \text{(2b)} \tag{2}$$

$$\aleph_0 := |\mathbb{N}| \tag{3}$$

$$C(S) := \begin{cases} 1,\ [S \text{ countable}] & (|S| \leqslant \aleph_0) \\ 0,\ [S \text{ uncountable}] & (|S| > \aleph_0) \end{cases} \tag{4}$$

Theorem 1. Singlular Unions of Countable Sets

If any sets R and S are countable, $R(a) \cup S(b)$ is countable for all numbers a and b.

1. $C(R) \wedge C(S) \iff C(R\,(a) \cup S\,(b))\ \forall a, b$
2. $C(R) \wedge C(S) \iff C(R\,(a) \cup S[\,b\,])\ \forall a, b$
3. $C(R) \wedge C(S) \iff C(R[\,a\,] \cup S\,(b))\ \forall a, b$
4. $C(R) \wedge C(S) \iff C(R[\,a\,] \cup S[\,b\,])\ \forall a, b$

Proof. R and S being countable implies $R(a)$ and $S(b)$ are countable as well because ring operators keep cardinalities the same. (case 1) Interlace the elements of $R(a)$ with those of $S(b)$, skipping over any duplicates. If either set is finite, append its elements to the beginning of the other set sans duplicates. Similar logic works with the other cases. ■

Theorem 2. Cartesian Products and Element-wise Multiplications of Countable Sets

If any sets R and S are countable, both $R \times S$ and $R \cdot S$ are countable.

1. $C(R) \wedge C(S) \iff C(R \times S)$
2. $C(R) \wedge C(S) \iff C(R \cdot S)$

Proof. To create $R \times S$, put all possible ordered pairs into a table as shown. Going along the diagonals, including every element in the output, and not "missing" any, one can show there is a bijection between $R \times S$ and $\mathbb{N}$. The numbers and colons in the table are the output indices, the first 25 being labeled if in view. Similar logic works for $R \cdot S$. This method is also algorithmically viable (see Section 5), and the output indices relate strongly to triangle numbers.

$1:(R_1,S_1)$	$2:(R_1,S_2)$	$4:(R_1,S_3)$	$7:(R_1,S_4)$	$\cdots$
$3:(R_2,S_1)$	$5:(R_2,S_2)$	$8:(R_2,S_3)$	$12:(R_2,S_4)$	$\cdots$
$6:(R_3,S_1)$	$9:(R_3,S_2)$	$13:(R_3,S_3)$	$18:(R_3,S_4)$	$\cdots$
$10:(R_4,S_1)$	$14:(R_4,S_2)$	$19:(R_4,S_3)$	$25:(R_4,S_4)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

Theorem 3. Subsets, Intersections, and Differences of Countable Sets

If any set R is countable, any loose subset S of R is also countable.

1. $C(R) \wedge S \subseteq R \implies C(S)$
2. $C(R) \wedge C(S) \implies C(R \cap S)$
3. $C(R) \implies C(R \setminus S)$

Proof. Since subsets of $\mathbb{N}$ are countable and R is countable, all subsets of R must also be countable due to the bijection between R and $\mathbb{N}$. This also implies that set intersections and differences are countable because these operations return subsets of the left input. Case 3 doesn't require S to be countable because it is uncountable and R is countable, then it is just more likely that the output will be the empty set. ■

Theorem 4. Two-Argument Ring Operators on Countable Sets

If any set R is countable, $R(a,b)$ and $R[a,b]$ are also countable for any constants a,b.

1. $C(R) \iff C(R(a,b))\ \forall a,b$
2. $C(R) \iff C(R[\,a,b\,])\ \forall a,b$

Proof. $R(a,b)$ can be tabulated with axes n and k, where the corresponding table box is $aR_n + bR_k$. This is countable in the same way as Theorem 2. Similar logic works for $R[a,b]$.

aR_1+bR_1	aR_1+bR_2	aR_1+bR_3	aR_1+bR_4	aR_1+bR_5	$\cdots$
aR_2+bR_1	aR_2+bR_2	aR_2+bR_3	aR_2+bR_4	aR_2+bR_5	$\cdots$
aR_3+bR_1	aR_3+bR_2	aR_3+bR_3	aR_3+bR_4	aR_3+bR_5	$\cdots$
aR_4+bR_1	aR_4+bR_2	aR_4+bR_3	aR_4+bR_4	aR_4+bR_5	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

Theorem 5. Finite Natural Powers of Countable Sets

If any set R is countable, R^n is also countable for all natural numbers n.

1. $C(R) \iff C(R^n)\ \ \forall n \in \mathbb{N}_0 < \infty$

Proof. R^n can be factored as $R \times R^{n-1}$ (eqn. 1a flattens outputs), which is countable if R^{n-1} is countable (Theorem 2.1). This can be applied recursively until it becomes countable if R^2 or $R \times R$ is countable, which is countable via Theorem 2.1. If $n = 0$, the output set is all the groups of zero elements from R, or just $\emptyset$, which is countable since $|\emptyset|$ is finite. ■

Theorem 6. Countably Infinite Unions of Countably Infinite Sets

If all sets R_i are countable, their union is also countable.

1. $C(R_i)\forall i \iff C\left(\bigcup_{i=1}^{\aleph_0} R_i\right)$

Proof. To create the union, give each set R_i a column in a table and put all its elements sequentially in that column going down. Do this for every R_i, and using the same method as in Theorem 2, one can show that this set bijects the naturals. Every column represents a different R_i and each row represents a different $(R_i)_j$ or $R_{i,j}$. The numbers before the colons are the output index, the first 85 shown if in view. This is algorithmically viable (see Section 5).

$1 : R_{1,1}$	$2 : R_{2,1}$	$6 : R_{3,1}$	$7 : R_{4,1}$	$15 : R_{5,1}$	$16 : R_{6,1}$	$28 : R_{7,1}$	$\cdots$
$3 : R_{1,2}$	$5 : R_{2,2}$	$8 : R_{3,2}$	$14 : R_{4,2}$	$17 : R_{5,2}$	$27 : R_{6,2}$	$30 : R_{7,2}$	$\cdots$
$4 : R_{1,3}$	$9 : R_{2,3}$	$13 : R_{3,3}$	$18 : R_{4,3}$	$26 : R_{5,3}$	$31 : R_{6,3}$	$43 : R_{7,3}$	$\cdots$
$10 : R_{1,4}$	$12 : R_{2,4}$	$19 : R_{3,4}$	$25 : R_{4,4}$	$32 : R_{5,4}$	$42 : R_{6,4}$	$49 : R_{7,4}$	$\cdots$
$11 : R_{1,5}$	$20 : R_{2,5}$	$24 : R_{3,5}$	$33 : R_{4,5}$	$41 : R_{5,5}$	$50 : R_{6,5}$	$62 : R_{7,5}$	$\cdots$
$21 : R_{1,6}$	$23 : R_{2,6}$	$34 : R_{3,6}$	$40 : R_{4,6}$	$51 : R_{5,6}$	$61 : R_{6,6}$	$72 : R_{7,6}$	$\cdots$
$22 : R_{1,7}$	$35 : R_{2,7}$	$39 : R_{3,7}$	$52 : R_{4,7}$	$60 : R_{5,7}$	$73 : R_{6,7}$	$85 : R_{7,7}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

2 Countabilities of Common Sets

2.1 Countability of the Integers

Claim: $\mathbb{Z}$ is a countable set; $C(\mathbb{Z}) \equiv 1$.

Proof. $R := \mathbb{N}_0$, $S := \mathbb{N}_1$. R and S are countable axiomatically (axiom of extensionality), being the naturals themselves. $\mathbb{Z} \equiv R(1) \cup S(-1)$; this is countable via Theorem 1.1. ■

2.2 Countability of the Rationals

Claim: $\mathbb{Q}$ is a countable set; $C(\mathbb{Q}) \equiv 1$.

Proof. $\mathbb{Q}$ has the same countability as $\mathbb{Z} \times \mathbb{N}_1$. $C(\mathbb{Z}) \equiv 1$ via Section 2.1 and $C(\mathbb{N}_1) \equiv 1$ axiomatically. Thus $C(\mathbb{Z} \times \mathbb{N}_1) \equiv 1$ via Theorem 2.1. This and Theorem 2 derive from Cantor's enumeration of countable collections of countable sets. ■

2.3 Countability of the Reals from Zero to One

Claim: $\{x \in \mathbb{R} : 0 \leqslant x < 1\}$ is a countable set; $C(\mathbb{R}_{0\leqslant x<1}) \equiv 1$.

Proof. Let R be a set where $\forall i \in \mathbb{N}_0$, $R_i :=$ ("0." + reversed digits of i), ie: $R_{2760} = 0.0672$ (see Section 6 for formula). R is countable by definition, and contains every possible sequence of digits in $[0, 1)$. The sequence trends towards different values with different sub-series; Section 2.5 explains this further. R is used in the next proof. ■

2.4 Countability of the Non-Negative Reals

Claim: $\mathbb{R}_{\geqslant 0}$ or equivalently $\{x \in \mathbb{R} : x \geqslant 0\}$, is a countable set; $C(\mathbb{R}_{\geqslant 0}) \equiv 1$.

Proof. $C(\mathbb{R}_{0 \leqslant x < 1}) \equiv 1$ via Section 2.3. $\mathbb{R}_{\geqslant 0} = \bigcup_{i=1}^{\aleph_0} S_i$ where $S_i := R[i-1]$. This is countable via Theorem 6; the following table illustrates this. $\forall x \in \mathbb{R}_{\geqslant 0}$, there exists a sequence of $s_i \in S_{\lceil x \rceil}$, with increasing indices, where $\lim_{i \in \mathbb{N} \to \infty} s_i = x$. This is because S_n contains every sequence of digits in $[n-1, n)$. S is used in the next proof.

0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.01	0.11	0.21	$\cdots$
1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	1.01	1.11	0.21	$\cdots$
2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	2.01	2.11	0.21	$\cdots$
3	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	3.01	3.11	0.21	$\cdots$
4	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	4.01	4.11	0.21	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

2.5 Countability of the Reals

Claim: $\mathbb{R}$ is a countable set which implies $2^{\aleph_0} = \aleph_0$.

Proof. $\mathbb{R}_{\geqslant 0}$ is countable via Section 2.4. $\mathbb{R}_{\geqslant 0}(1, -1) \equiv \mathbb{R}$. This is countable via Theorem 4.1. This conclusion can be further reinforced by the algorithmic methods in Section 5. ■

2.6 Countabilities of Miscellaneous Number Classes

2.6.1 Algebraic and Transcendental Numbers

Claim: $\mathbb{A}$ and $\mathbb{T}$ are countable sets over both $\mathbb{R}$ and $\mathbb{C}$.

Proof. $\mathbb{R}$ is a countable set via Section 2.5. Since the algebraic reals and transcendental reals are both subsets of the reals, they are countable via Theorem 3.1. They are also countable over the complex numbers using Section 2.7 and the same logic. ■

2.6.2 Imaginary Numbers

Claim: $\mathbb{I}$ is a countable set; $C(\mathbb{I}) \equiv 1$.

Proof. $\mathbb{R}$ is a countable set via Section 2.5. $\mathbb{R}(\sqrt{-1}) \cup \emptyset(1) \equiv \mathbb{I}$. This is countable via Theorem 1.1 becuase the union of any set and the null set is itself. ■

2.7 Countability of the Complex Numbers

Claim: $\mathbb{C}$ is a countable set; $C(\mathbb{C}) \equiv 1$.

Proof. $\mathbb{R}$ and $\mathbb{I}$ are countable sets via Sections 2.5 and 2.6.2 respectively. $\mathbb{R}(1, \sqrt{-1}) \equiv \mathbb{C}$ is countable via Theorem 4.1. $\mathbb{R} \times \mathbb{I} = \mathbb{C}$ is countable via Theorem 2.1. ■

3 Generalized Continuum Hypothesis

Claim: GCH has the same truth value as CH, and powersets don't change sizes of infinity.

Proof. Let S be a countably infinite set; it cannot be finite or this proof does not work. S^i is countable via Theorem 5. $P := \mathcal{P}(S)$. Let R_k denote the set of elements in P with k sub-elements; these are countable via Theorem 5. Observe that P has $\aleph_0^0$ elements with 0 sub-elements, $\aleph_0^1$ elements with 1 sub-element, et cetera, and has generally $\aleph_0^k$ elements with k sub-elements, for $k \leqslant \aleph_0$. Imagine a k-dimensioned table of integers and listing them in a k-dimensional spiral, or recursively combining two table dimensions into one using the Theorem 2 table method until the table gets to one dimension. $\bigcup_{i=1}^{\aleph_0} R_{i-1}$ is countable via Theorem 6. This contradicts Cantor's Diagonal Argument (see Section 4) and implies that uncountable sets cannot be created through powersets of countable sets, and either cannot be created at all, or need a much stronger method to create them. ■

4 Cantor's Diagonal Argument

According to Georg Cantor in 1891[references 1, 2], If someone is trying to list all the real numbers, they can always find a number that is not in the list using his "Diagonal Argument". This argument basically accomplishes the same as the following, though for reals instead of naturals: Suppose someone is trying to make a set S with every natural number. They first add zero to the set and the set is $\{0\}$, then they could say, "one isn't in the set." When they add one and have $\{0, 1\}$, they can say "two isn't in the set," then "three isn't in the set," "four isn't in the set," et cetera. No matter how many natural numbers they add, they can always find one not in it; $\max(S) + 1$. This seems to be implying that there is not a bijection between the natural numbers and themselves, which is clearly wrong. Every set is bijective onto itself via the identity function. $f(x) := x$ for $f : X \mapsto X$.

5 Enumerating the Continuum Algorithmically

The following Node JS code prints out real numbers to stdout delimited by a comma-space pair. With infinite time and memory, it will have printed every real number. The problems are that it prints out both positive and negative zero, and the functions return strings. Both of these are easily resolvable though. There is a reference to this source code and the C version in Section 8 References 6 and 5. The subscripts are just so it looks nicer. If `process.stdout.write` is replaced with `console.log`, then it will work in vanilla JavaScript in versions beginning with ECMAScript 6. The downside being `console.log` adds a trailing newline character at the end of each call, which isn't ideal for printing a large quantity of small strings each using an individual function call. The complexity of finding the nth real number with this method is less than or equal to $\mathcal{O}(n \log^2 n)$, emphasis on "less", because after over 16 million iterations on my machine, it hadn't slowed down noticably.

```
function isqrt(n) { // ⌊√n⌋ for non-negative big integers n.
```

```
   if (n < 2) return n;
   var x0, x1 = n / 2n;

   do x0 = x1,
      x1 = x0 + n / x0 >> 1n; // Newton's method for f(x) = x^2+a
   while ( x1 < x0 );

   return x0;
} // code continued
// iterate over the natural numbers (up to 0.5*2^2^30)
for (var runningIndex = 0n ;;) {
   let positive = true, currentIndex = runningIndex++;

   // if the current index is odd, return a negation the previous index's value
   if (currentIndex % 2n) positive = false, currentIndex--;
   currentIndex /= 2n; // divide index by 2 so all integers can be reached

   const c = isqrt(1n + 8n*currentIndex) // intermediate value
   // the index the next comments refer to
      , u = (c + c % 2n) / 2n - 1n
   // the largest triangle number with an integer index that is less ...
   // than or equal to the current index.  but subtracted from the current index
      , k = currentIndex - u * (u + 1n) / 2n;

   // generate real number from indices k and u-k and print it
   process.stdout.write(`${positive ? "" : "-"}${k}.` +
      `${u - k}`.split("").reverse().join("") + ", "
   );
}
```

6 Further Algorithms

The axioms of Zermelo-Fraenkel Set theory required to prove the algorithmic viability of continuum enumeration are as follows: Axiom Schema of Specification, Axiom of Union, Axiom of Infinity, and Axiom of Choice (for creating sets of ordered *pairs*). Without the Axiom of Choice, Theorems 2 doesn't work, although the algorithm from Section 5 is separate from set theory and assumes no axioms. The algorithm uses string manipulation to reverse the numbers and find their lengths, but the following equations do the same thing mathematically. length() and reverse() are only defined for natural number inputs.

$$\operatorname{length} n := 1 + \lfloor \log_b |n| \rfloor = \lceil \log_b(|n| + 1) \rceil, \text{ such that } \ln 0 \leqslant 0 \tag{5}$$

$$\operatorname{reverse} n := \operatorname{sgn}(n) \sum_{k=0}^{\operatorname{length} n} \left(\frac{1}{b} \left\lfloor \frac{|n|}{b^k} \right\rfloor \bmod 1 \right) b^{\operatorname{length} n - k} \tag{6}$$

$$R_{i,j} := p \cdot \left(i + \frac{\operatorname{reverse} j}{b^{\operatorname{length} j}} \right) \tag{7}$$

Where i is `k`, p is `(positive ? 1 : -1)`, j is `(u - k)`, and b is the base. i and j can be swapped for a different enumeration (see Reference 7). An interesting formula used in the algorithm in Section 5 is for the greatest triangle number $t = T(k) \leqslant n$, where $n, k \in \mathbb{N}_0$.

$$T(n) := \frac{n^2 + n}{2} = \sum_{j=1}^{n} j \tag{8}$$

$$t = T\left(\left\lceil \frac{\lfloor \sqrt{8n+1} \rfloor}{2} \right\rceil - 1 \right) = T\left(\left\lfloor \frac{\sqrt{8n+1}}{2} \right\rceil - 1 \right) \tag{9}$$

Because of the bijection, there is also an inverse of the Section 5 algorithm.

```
function inverse(string) { // assume valid input
   const match = /^(-?)(\d+)\.(\d+)$/.exec(string)
      , i = BigInt(match[2]) // the integer part
      , j = BigInt(match[3]); // the decimal part
   return i*(i+1n) + (2n*i+j)*(j+1n) + BigInt(match[1] === "-");
}
```

7 Conclusion

Since there exists a bijection between any infinite set and its powerset (GCH), there is no set with a cardinality strictly or loosely in between the naturals and reals (CH), because they are the same cardinality. This also implies that $\aleph_n = \beth_n = \beth_0$ for all natural numbers n, which makes sense intuitively because $2^\infty = \infty$. Also if any sequence of natural numbers is concatenated, it will always create a new natural number, meaning every element in $\mathcal{P}(\mathbb{N}_0)$ corresponds to an element in $\mathbb{N}_0$. This all could have disasterous consequences for set theory or mathematics as a whole because CH is provably undecidable in some models and yet provably decidable. This means either Zermelo-Fraenkel Set Theory with the axiom of choice (ZHC) is unsound, the aforementioned models are invalid, or at least one of the decidability proofs is invalid. The entire foundation of aleph numbers, beth numbers, and sizes of infinities could be entirely flawed. CH implies the Gimel Hypothesis is true according to Reference 9, and impliesWetzel's problem is false according to Reference 10. A truth value for CH or GCH is not asserted here bacause the specifics of CH are unclear, and they are important for its truth value. If the reals have to be bigger than the naturals, it is false, but if it just requires that there is no set in between them, it is true.

8 References

1. https://en.wikipedia.org/wiki/Cantor's_diagonal_argument
   Wikipedia page with background for Section 4

2. https://www.digizeitschriften.de/dms/img/?PID=GDZPPN002113910physid=phys84navi
   Georg Cantor's 1891 article with the diagonal argument. Same source as on wikipedia.

3. https://en.wikipedia.org/wiki/Continuum_hypothesis
   Wikipedia page with elaboration on Section 2.5

4. https://www.github.com/drizzt536/files/tree/main/TeX/continuum
   The files for the most recent public version of this pdf and the LaTeX code

5. https://raw.githubusercontent.com/drizzt536/files/main/JavaScript/continuum.js
   The raw JavaScript source code for the Section 5

6. https://raw.githubusercontent.com/drizzt536/files/main/C/continuum.c
   The raw C source code for the Section 5

7. https://www.desmos.com/calculator/wndesoagsd
   Desmos with real number enumeration formulas for Section 6

8. https://github.com/drizzt536/files/blob/main/TeX/continuum/real-enum-formula/formula.pdf
   pdf with a fully-expanded real number enumeration formula for Section 6

9. https://en.wikipedia.org/wiki/Gimel_function#The_gimel_hypothesis
   Gimel Hypothesis Wikipedia page

10. https://en.wikipedia.org/wiki/Wetzel's_problem
    Wikipedia page for Wetzel's Problem

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