

The Gamma Function, Its Inverse, and Its Relationship with Trigonometry

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1 Definitions

$$0 \in \mathbb{N} \tag{1}$$

$$\mathbb{N}_k := \{x + k : x \in \mathbb{N}\} \forall k \tag{2}$$

$$\tau := 2\pi \tag{3}$$

$$\Gamma(x) := \int_0^\infty \frac{t^{x-1}}{\exp t} dt = (x-1)! \forall x > 0 \tag{4}$$

$$\zeta(x) := \sum_{n=1}^\infty \frac{1}{n^x} \forall x > 1 \tag{5}$$

$$\eta(x) := \Gamma^{-1}(x) \tag{6}$$

$$\eta(\Gamma(x)) = \Gamma(\eta(x)) = x \tag{7}$$

(6) is the compositional inverse rather than the fractional inverse.
The set of all natural numbers, \mathbb{N} , is defined by equations (1) and (2).
(5) describes the Riemann Zeta Function.

2 Riemann Zeta Function

$$\text{claim: } \zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\Gamma(x)}$$

This proof is important for part 3.

Proof:

$$\text{know: } \zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (\text{reference 1})$$

$$\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (8)$$

$$\zeta(x) = 2 \cdot 2^{x-1} \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (9)$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (10)$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x) \zeta(1-x) \quad (11)$$

$$\zeta(x) = \frac{2}{\tau^{1-x}} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x) \zeta(1-x) \quad (12)$$

$$\zeta(1-x) = \frac{2}{\tau^x} \cos\left(\frac{\pi x}{2}\right) \Gamma(x) \zeta(x) \quad (13)$$

$$\frac{\zeta(1-x)\tau^x}{2 \cos\left(\frac{\pi x}{2}\right) \Gamma(x)} = \zeta(x) \quad (14)$$

$$\zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\Gamma(x)} \quad (15)$$

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3 Cosecant

claim: $\csc x$ can be written in terms of $\Gamma(x)$

Proof:

know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x)$ (reference 1)

know: $\zeta(x) = \frac{\zeta(1-x) \tau^x \sec\left(\frac{\pi x}{2}\right)}{2 \Gamma(x)}$ (equation 15)

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) = \frac{\zeta(1-x) \tau^x \sec\left(\frac{\pi x}{2}\right)}{2 \Gamma(x)} \quad (16)$$

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{2 \Gamma(x)} \quad (17)$$

$$4 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{\Gamma(x)} \quad (18)$$

$$2 \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\pi}{\Gamma(x)} \quad (19)$$

$$\Gamma(1-x) \Gamma(x) \sin \pi x = \pi \quad (20)$$

$$\Gamma(1-x) \Gamma(x) = \pi \csc \pi x \quad (21)$$

$$\csc x = \frac{1}{\pi} \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) \quad (22)$$

$$\sin \pi x \neq 0 \implies \zeta(1-x) \neq 0 \quad (23)$$

Therefore, the domains also match.

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4 Trig Functions In Terms of Cosecant

claim: The rest of the trig functions can be written in terms of cosecant

Proof:

$$\sin x = \frac{1}{\csc x} \quad (24)$$

$$\cos x = \frac{1}{\csc\left(\frac{\pi}{2} - x\right)} \quad (25)$$

$$\tan x = \frac{\csc\left(\frac{\pi}{2} - x\right)}{\csc x} \quad (26)$$

$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \quad (27)$$

$$\cot x = \frac{\csc x}{\csc\left(\frac{\pi}{2} - x\right)} \quad (28)$$

$$\sinh x = -\frac{i}{\csc x} \quad (29)$$

$$\cosh x = \frac{1}{\csc\left(\frac{\pi}{2} - ix\right)} \quad (30)$$

$$\tanh x = -\frac{i \csc\left(\frac{\pi}{2} - ix\right)}{\csc ix} \quad (31)$$

$$\operatorname{csch} x = i \csc ix \quad (32)$$

$$\operatorname{sech} x = \csc\left(\frac{\pi}{2} - ix\right) \quad (33)$$

$$\coth x = \frac{i \csc ix}{\csc\left(\frac{\pi}{2} - ix\right)} \quad (34)$$

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5 Trig Functions In Terms of Gamma

These equations are just the previous ones with Gamma plugged in for csc.

$$\sin x = \pi / \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) \quad (35)$$

$$\cos x = \pi / \Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right) \quad (36)$$

$$\tan x = \Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right) / \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) \quad (37)$$

$$\csc x = \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) / \pi \quad (38)$$

$$\sec x = \Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right) / \pi \quad (39)$$

$$\cot x = \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) / \Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right) \quad (40)$$

$$\sinh x = -\pi i / \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) \quad (41)$$

$$\cosh x = \pi / \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) \quad (42)$$

$$\tanh x = -i \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) / \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) \quad (43)$$

$$\operatorname{csch} x = i \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) / \pi \quad (44)$$

$$\operatorname{sech} x = \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) / \pi \quad (45)$$

$$\operatorname{coth} x = i \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) / \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) \quad (46)$$

where “/” means divide everything on the left by everything on the right.

6 Inverse Gamma Function

claim: $\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x))$

Proof:

know: $\Gamma(x) \Gamma(1 - x) = \pi \csc \pi x$ (equation 21)

$$\pi \csc \pi x = \Gamma(x) \Gamma(1 - x) \quad (47)$$

$$\pi \csc \pi \eta(x) = x \Gamma(1 - \eta(x)) \quad (48)$$

$$\csc \pi \eta(x) = \frac{x}{\pi} \Gamma(1 - \eta(x)) \quad (49)$$

$$\pi \eta(x) = \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x)) \quad (50)$$

$$\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x)) \quad (51)$$

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There are 2 conditions to where this $\eta(x)$ is defined.

$\eta(x)$ is **not** defined (over the real numbers) such that the following is true:

$$|x \Gamma(1 - \eta(x))| < \pi \quad \text{or} \quad |x| < 3.547373963... \quad (V) \quad (52)$$

V is probably transcendental. $\eta(x)$ is guaranteed to be defined if equation (52) is false and x is negative. $\eta(x)$ is defined for all integers and in small fields around each integer. The area defined around each integer gets smaller as x increases, though it is unclear why at this time. If $\eta(x)$ is being recursed and $\frac{x}{\pi}$ is changed to $\frac{x^n}{\pi}$ in the innermost function, for some $n \in \mathbb{N}_2$, either the fields get larger, or there is a bug in Desmos' graphing calculations. See reference 2 for more information. going a different way after (48) gives the following equation:

$$\eta(x) = 1 - \eta\left(\frac{\pi}{x} \csc \pi \eta(x)\right) \implies \eta(x) = \eta(\pi \csc(\pi \csc \pi x) \sin \pi x) \quad (53)$$

This equation seems to be less useful because it has large absolute derivatives which gets more severe as it is recursed.

7 References

- https://en.wikipedia.org/wiki/Riemann_zeta_function#Riemann's_functional_equation
Link for equation 8
- <https://www.desmos.com/calculator/t51xnveadf>
Extra information for part 6, made by me
- <https://www.github.com/drizzt536/files/tree/main/TeX/gamma>
The files for the most recent version of this pdf and the LaTeX code