On the Continuum Hypotheses and the Correspondence of Infinite Sets with the Natural Numbers

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Dedicated to Maeby, my favorite kitty

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1 Introduction and Countability Theorems

Any "countable" set is equinumerous to a loose subset of the natural numbers, and there exists a bijection between them. Any finite set has this property, the main question being if all *infinite* sets do. In all infinite processes, one should think of the steps as executing simultaneously rather than separately and successively. $|S|$ and $C(S)$ denote the cardinality and countability of a set S respectively. Equations (2a) and (2b) describe what I will call "sum ring operators," "product ring operators," and collectively "ring operators." Most Theorems use " $\iff$ " instead of " $\implies$ " because Theorem 3.1 works as an inversion.

$$R \times S := \{(r, s) : r \in R \land s \in S\} \quad \text{(1a)} \qquad R \cdot S := \{r \cdot s : r \in R \land s \in S\} \quad \text{(1b)} \tag{1}$$

$$R(\vec{v}\,) := \left\{ \sum_{n=1}^{\dim \vec{v}} (a_n \cdot \vec{v}_n) : a_n \in R \right\} \quad \text{(2a)} \qquad R[\,\vec{v}\,] := \left\{ \prod_{n=1}^{\dim \vec{v}} (a_n + \vec{v}_n) : a_n \in R \right\} \quad \text{(2b)} \tag{2}$$

$$\aleph_0 := |\mathbb{N}| \tag{3}$$

$$C(S) := \begin{cases} 1, \ [S \text{ countable}] & (|S| \leqslant \aleph_0) \\ 0, \ [S \text{ uncountable}] & (|S| > \aleph_0) \end{cases} \tag{4}$$

Theorem 1. Singlular Unions of Countable Sets

If any sets R and S are countable, $R(a) \cup S(b)$ is countable for all a and b.

1. $C(R) \land C(S) \iff C(R\,(a) \cup S\,(b))\ \forall a, b$
2. $C(R) \land C(S) \iff C(R\,(a) \cup S[\,b\,])\ \forall a, b$
3. $C(R) \land C(S) \iff C(R[\,a\,] \cup S\,(b))\ \forall a, b$
4. $C(R) \land C(S) \iff C(R[\,a\,] \cup S[\,b\,])\ \forall a, b$

Proof. R and S being countable implies $R(a)$ and $S(b)$ are countable as well because ring operators keep cardinalities the same. Interlace the elements of $R(a)$ with those of $S(b)$, skipping over any duplicates. If either set is finite, append its elements to the beginning of the other set sans duplicates. Similar logic works using $R[\,a\,]$ or $S[\,b\,]$. ■

Theorem 2. Multiplications and Cartesian Products of Countable Sets

If any sets R and S are countable, both $R \times S$ and $R \cdot S$ are countable.

1. $C(R) \land C(S) \iff C(R \times S)$
2. $C(R) \land C(S) \iff C(R \cdot S)$

Proof. To create $R \times S$, put all possible ordered pairs into a table as shown. One can go along the diagonals, including every element in the output, and not "missing" any, implying a bijection between $R \times S$ and $\mathbb{N}$. The numbers and colons in the table are the output indices, the first 25 being labeled if in view. Similar logic works for $R \cdot S$. This method is also algorithmically viable (see Section 5). The output indices relate strongly to triangle numbers.

$1 : (R_1, S_1)$	$2 : (R_1, S_2)$	$4 : (R_1, S_3)$	$7 : (R_1, S_4)$	$\cdots$
$3 : (R_2, S_1)$	$5 : (R_2, S_2)$	$8 : (R_2, S_3)$	$12 : (R_2, S_4)$	$\cdots$
$6 : (R_3, S_1)$	$9 : (R_3, S_2)$	$13 : (R_3, S_3)$	$18 : (R_3, S_4)$	$\cdots$
$10 : (R_4, S_1)$	$14 : (R_4, S_2)$	$19 : (R_4, S_3)$	$25 : (R_4, S_4)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

Theorem 3. Subsets, Intersections, and Differences of Countable Sets
If any set R is countable, any loose subset S of R is also countable.

1. $C(R) \wedge S \subseteq R \implies C(S)$
2. $C(R) \wedge C(S) \implies C(R \cap S)$
3. $C(R) \wedge C(S) \implies C(R \setminus S)$

Proof. To create the subset, take each element in R that is not in S ($x \in R \setminus S$), move its index to the beginning of R, and remove it. This way, the new R has countability guaranteed. With applying this recursively, S must be countable. This also implies that set intersections and differences are countable because these operations return subsets of the inputs. ■

Theorem 4. Two Argument Ring Operators On Countable Sets
If any set R is countable, $R(a, b)$ and $R[a, b]$ are also countable for all a and b.

1. $C(R) \iff C(R(a, b))\ \forall a, b$
2. $C(R) \iff C(R[\,a, b\,])\ \forall a, b$

Proof. Create 2 new intermediate sets $S_a := \{a \cdot r : r \in R\}$ and $S_b := \{b \cdot r : r \in R\}$. $S := S_a \times S_b$ gives a set with the same cardinality as $R(a, b)$, only with ordered pairs instead of addition. S is countable via Theorem 2.1. Similar logic works for $R[a, b]$. ■

Theorem 5. Finite Natural Powers of Countable Sets
If any set R is countable, R^n is also countable for all natural numbers n.

1. $C(R) \iff C(R^n) \quad \forall n \in \mathbb{N}_0 < \infty$

Proof. R^n can be factored as $R \times R^{n-1}$ which is countable if R^{n-1} is countable. This can be applied recursively until it becomes countable if R^2 or $R \times R$ is countable, which is countable via Theorem 2.1. If $n = 0$, the output set is all the groups of zero elements from R, or just $\emptyset$, which is countable since $|\emptyset|$ is finite. ■

Theorem 6. Countably Infinite Unions of Countably Infinite Sets
If all sets R_i are countable, their union is also countable.

1. $C(R_i)\forall i \iff C\left(\bigcup_{i=1}^{\aleph_0} R_i\right)$

Proof. To create the union, give each set R_i a column in a table and put all its elements sequentially in that column going down. Do this for every R_i and using the same argument as in Theorem 2, one can show that this set bijects the naturals. Every column represents a different R_i and the rows represent a different $(R_i)_j$ or $R_{i,j}$. The numbers before the colons are the output index, the first 85 shown if in view. This is algorithmically viable (see Theorem 2).

$1 : R_{1,1}$	$2 : R_{2,1}$	$6 : R_{3,1}$	$7 : R_{4,1}$	$15 : R_{5,1}$	$16 : R_{6,1}$	$28 : R_{7,1}$	$\cdots$
$3 : R_{1,2}$	$5 : R_{2,2}$	$8 : R_{3,2}$	$14 : R_{4,2}$	$17 : R_{5,2}$	$27 : R_{6,2}$	$30 : R_{7,2}$	$\cdots$
$4 : R_{1,3}$	$9 : R_{2,3}$	$13 : R_{3,3}$	$18 : R_{4,3}$	$26 : R_{5,3}$	$31 : R_{6,3}$	$43 : R_{7,3}$	$\cdots$
$10 : R_{1,4}$	$12 : R_{2,4}$	$19 : R_{3,4}$	$25 : R_{4,4}$	$32 : R_{5,4}$	$42 : R_{6,4}$	$49 : R_{7,4}$	$\cdots$
$11 : R_{1,5}$	$20 : R_{2,5}$	$24 : R_{3,5}$	$33 : R_{4,5}$	$41 : R_{5,5}$	$50 : R_{6,5}$	$62 : R_{7,5}$	$\cdots$
$21 : R_{1,6}$	$23 : R_{2,6}$	$34 : R_{3,6}$	$40 : R_{4,6}$	$51 : R_{5,6}$	$61 : R_{6,6}$	$72 : R_{7,6}$	$\cdots$
$22 : R_{1,7}$	$35 : R_{2,7}$	$39 : R_{3,7}$	$52 : R_{4,7}$	$60 : R_{5,7}$	$73 : R_{6,7}$	$85 : R_{7,7}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

2 Countabilities of Common Sets Using the Theorems

2.1 Countability of the Integers

Claim: $\mathbb{Z}$ is a countable set.

$C(\mathbb{Z}) \equiv 1, \qquad |\mathbb{Z}| \equiv \aleph_0$

Proof. $R := \mathbb{N}_0$, $S := \mathbb{N}_1$. R and S are countable axiomatically (axiom of extensionality), being the naturals themselves. $\mathbb{Z} \equiv R(1) \cup S(-1)$. This is countable via Theorem 1.1. ■

2.2 Countability of the Rationals

Claim: $\mathbb{Q}$ is a countable set.

$C(\mathbb{Q}) \equiv 1$

Proof. $R := \mathbb{Z}$, $S := \mathbb{N}_1$. $\mathbb{Q}$ has the same countability as the set of all the ordered pairs of integers with naturals. $R \times S$ defines all of these pairs. R is countable via Section 2.1 and S is countable axiomatically. $R \times S$ is thus countable via Theorem 2.1. This argument and Theorem 2 derives from Cantor's enumeration of countable collections of countable sets. ■

2.3 Countability of the Reals from Zero to One

Claim: $\{x \in \mathbb{R} : 0 \leqslant x < 1\}$ is a countable set. $\qquad C(\mathbb{R}_{0\leqslant x<1}) \equiv 1$

Proof. Let R be a countable set where, for any $i \in \mathbb{N}_0$, R_i can be defined to be the digits of i reversed with "0." at the beginning (see Section 6 for formula). For example, $R_{246} = 0.642\overline{0}$. This set is countable because it was defined to be countable, and it contains every real number in the range because every possible sequence of digits is in it and indexed. The sequence trends upwards, asymptotically approaching one, though it fluctuates wildly along the way. Section 2.5 explains this further. R is used throughout the sections about $\mathbb{R}$. ■

2.4 Countability of the Non-Negative Reals

Claim: $\mathbb{R}_{\geqslant 0}$ or equivalently $\{x \in \mathbb{R} : x \geqslant 0\}$, is a countable set. $\quad C(\mathbb{R}_{\geqslant 0}) \equiv 1$

Proof. $C(\mathbb{R}_{0\leqslant x<1}) = 1$ via Section 2.3. $\mathbb{R}_{\geqslant 0} = \bigcup_{i=1}^{\aleph_0} S_i$ where $S_i := R[i-1]$. This is countable via Theorem 6.1. The following table illustrates this. S is used in the next proof.

$0.\overline{0}$	$1.\overline{0}$	$2.\overline{0}$	$3.\overline{0}$	$4.\overline{0}$	$5.\overline{0}$	$\cdots$
$0.1\overline{0}$	$1.1\overline{0}$	$2.1\overline{0}$	$3.1\overline{0}$	$4.1\overline{0}$	$5.1\overline{0}$	$\cdots$
$0.2\overline{0}$	$1.2\overline{0}$	$2.2\overline{0}$	$3.2\overline{0}$	$4.2\overline{0}$	$5.2\overline{0}$	$\cdots$
$0.3\overline{0}$	$1.3\overline{0}$	$2.3\overline{0}$	$3.3\overline{0}$	$4.3\overline{0}$	$5.3\overline{0}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$0.7\overline{0}$	$1.7\overline{0}$	$2.7\overline{0}$	$3.7\overline{0}$	$4.7\overline{0}$	$5.7\overline{0}$	$\cdots$
$0.8\overline{0}$	$1.8\overline{0}$	$2.8\overline{0}$	$3.8\overline{0}$	$4.8\overline{0}$	$5.8\overline{0}$	$\cdots$
$0.9\overline{0}$	$1.9\overline{0}$	$2.9\overline{0}$	$3.9\overline{0}$	$4.9\overline{0}$	$5.9\overline{0}$	$\cdots$
$0.01\overline{0}$	$1.01\overline{0}$	$2.01\overline{0}$	$3.01\overline{0}$	$4.01\overline{0}$	$5.01\overline{0}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

■

2.5 Countability of the Reals

Claim: $\mathbb{R}$ is a countable set which implies $2^{\aleph_0} \equiv \aleph_0$.

Proof. $\mathbb{R}_{\geqslant 0}$ is countable via Section 2.4. $\mathbb{R}_{\geqslant 0}(1, -1) \equiv \mathbb{R}$. This is countable via Theorem 4.1. This conclusion can be further reinforced by the algorithmic methods in Section 5. For all real numbers x, there exists at least one sequence of elements in an S_i, each element with a higher index than the last, where the limit of the sequence equals $|x|$. This is because every sequence of digits in $[i-1, i)$ is contained by S_i. ■

2.6 Countabilities of Miscellaneous Number Classes

2.6.1 Algebraic and Transcendental Numbers

Claim: $\mathbb{A}$ and $\mathbb{T}$ are countable sets over $\mathbb{R}$ and $\mathbb{C}$.

Proof. $\mathbb{R}$ is a countable set via Section 2.5. Since the algebraic reals and transcendental reals are both subsets of the reals, they are countable via Theorem 3.1. They are also countable over the complex numbers using Section 2.7 and the same logic. ■

2.6.2 Imaginary Numbers

Claim: $\mathbb{I}$ is a countable set. $\quad C(\mathbb{I}) \equiv 1$

Proof. $\mathbb{R}$ is a countable set via Section 2.5. $\mathbb{R}(\sqrt{-1}) \cup \emptyset(1) \equiv \mathbb{I}$. This is countable via Theorem 1.1 becuase the union of any set and the null set is itself. ■

2.7 Countability of the Complex Numbers

Claim: $\mathbb{C}$ is a countable set.

Proof. $\mathbb{R}$ is a countable set via Section 2.5. $\mathbb{R}(1, \sqrt{-1}) \equiv \mathbb{C}$. This is countable via Theorem 4.1. The identity used here stems from the rectangular form of complex numbers. ■

3 Generalized Continuum Hypothesis

Claim: GCH has the same truth value as CH, and powersets don't change sizes of infinity.

Proof. Let S be a countable set. $R_i := S^{i-1}$, and is countable via Theorem 5.1. $Q := \bigcup_{i=1}^{\aleph_0} R_i$ is countable via Theorem 6.1. $P := \mathcal{P}(S)$. $n := |S|$. $|P| = |Q|$. This is because P has n^0 elements with 0 sub-elements, n^1 elements with 1 sub-element, n^2 elements with 2 sub-elements, and has generally n^k elements with k sub-elements, for $k \leqslant n$.

S^0 has n^0 elements, S^1 has n^1 elements, and generally S^k has n^k elements, and when taking the union of all of S^i, where $0 \leqslant i \leqslant n$, the resulting set has the same cardinality as P. This expands to countably infinite sets as well, replacing n with $\aleph_0$. Since Q is countable, P is countable as well, meaning $|S| \equiv |\mathcal{P}(S)|$ for all countable sets S.

There is also no reason "uncountable" sets can't be used for S, but the powerset would end up having the same cardinality as S anyway. This is because Theorems 5.1 and 6.1 can be adapted to biject to whatever set is inputed to the theorem, instead of $\mathbb{N}$, This implies that uncountable sets cannot be created through powersets of countable sets, and either cannot be created at all, or need a much stronger method to create them. ■

4 Cantor's Diagonal Argument

According to Georg Cantor in 1891[references 1, 2], If someone is trying to list all the real numbers, they can always find a number that is not in the list using his "Diagonal Argument". This argument basically accomplishes the same as the following, though for reals instead of naturals: Suppose someone is trying to make a set S with every natural number. They first add zero to the set and the set is $\{0\}$, then they could say, "one isn't in the set." When they add one and have $\{0, 1\}$, they can say "two isn't in the set," then "three isn't in the set," "four isn't in the set," et cetera. No matter how many natural numbers they add, they can always find one not in it; $\max(S) + 1$. This seems to be implying that there is not a bijection between the natural numbers and themselves, which is clearly wrong. Every set is bijective onto itself via the identity function. $f(x) := x$ for $f : X \mapsto X$.

5 Enumerating the Continuum Algorithmically

The following Node JS code prints out real numbers to stdout delimited by a comma-space pair. With infinite time and memory, it will have printed every real number. The problems are that it prints out both positive and negative zero, and the functions return strings.

Both of these are easily resolvable though. There is a reference to this source code and the C version in Section 8 References 6 and 5. The subscripts are just so it looks nicer. If `process.stdout.write` is replaced with `console.log`, then it will work in vanilla JavaScript in versions beginning with ECMAScript 6. The downside being `console.log` adds a trailing newline character at the end of each call, which isn't ideal for printing a large quantity of small strings each using an individual function call. The complexity of finding the nth real number with this method is less than or equal to $\mathcal{O}\left(n \log^2 n\right)$, emphasis on "less", because after over 16 million iterations on my machine, it hadn't slowed down noticably.

```
function isqrt(n) { // ⌊√n⌋ for non-negative big integers n.
   if (n < 2) return n;
   var x0, x1 = n / 2n;

   do x0 = x1,
      x1 = x0 + n / x0 >> 1n; // Newton's method for f(x) = x^2+a
   while ( x1 < x0 );

   return x0;
} // code continued
// iterate over the natural numbers (up to 0.5*2^2^30)
for (var runningIndex = 0n ;;) {
   let positive = true, currentIndex = runningIndex++;

   // if the current index is odd, return a negation the previous index's value
   if (currentIndex % 2n) positive = false, currentIndex--;
   currentIndex /= 2n; // divide index by 2 so all integers can be reached

   const c = isqrt(1n + 8n*currentIndex) // intermediate value
   // the index the next comments refer to
      , u = (c + c % 2n) / 2n - 1n
   // the largest triangle number with an integer index that is less ...
   // than or equal to the current index.  but subtracted from the current index
      , k = currentIndex - u * (u + 1n) / 2n;

   // generate real number from indices k and u-k and print it
   process.stdout.write(`${positive ? "" : "-"}${k}.` +
      `${u - k}`.split("").reverse().join("") + ", "
   );
}
```

6 Further Algorithms

The axioms of Zermelo-Fraenkel Set theory required to prove the algorithmic viability of continuum enumeration are as follows: Axiom Schema of Specification, Axiom of Union, Axiom of Infinity, and Axiom of Choice (for creating sets of ordered pairs). Without the Axiom of Choice, Theorems 2, 4, and 6 don't work, although the algorithm from Section 5 is separate from set theory and assumes no axioms. The algorithm uses string manipulation to reverse the numbers and find their lengths, but the following equations do the same thing mathematically. length() and reverse() are only defined for natural number inputs.

$$\operatorname{length} n := 1 + \lfloor \log_b n \rfloor = \lceil \log_b (n+1) \rceil \tag{5}$$

$$\operatorname{reverse} n := \sum_{k=0}^{\operatorname{length} n} \left(\left\lfloor \frac{n}{b^k} \right\rfloor \bmod b \right) b^{\operatorname{length} n - k - 1} \tag{6}$$

$$R_{i,j} := p \left(i + \frac{\operatorname{reverse} j}{b^{\operatorname{length} j}} \right) \tag{7}$$

Where i is `k`, p is `(positive ? 1 : -1)`, j is `(u - k)`, and b is the base. An interesting formula used in the algorithm in Section 5 is for the greatest triangle number $t = T(k)$ less than or equal to a natural number n, with a natural number index k.

$$T(n) := \frac{n^2 + n}{2} = \sum_{j=1}^{n} j \tag{8}$$

$$t = T\left(\left\lceil \frac{\lfloor \sqrt{8n+1} \rfloor}{2} \right\rceil - 1 \right) = T\left(\left\lfloor \frac{\sqrt{8n+1}}{2} \right\rceil - 1 \right) \tag{9}$$

Because of the bijection, there is also an inverse of the Section 5 algorithm.

```
function inverse(string) { // assume valid input
   const match = /^(-?)(\d+)\.(\d)$/.exec(string)
      , i = BigInt(match[2]) // the x coordinate index, or the integer part
      , j = BigInt(match[3]); // the y coordinate index, or the decimal part
   return i*(i+1n) + (2n*i+j)*(j+1n) + BigInt(match[1] === "-");
}
```

7 Conclusion

Since there exists a bijection between any infinite set and its powerset, there is no set with a cardinality strictly or loosely in between the naturals and reals (CH and GCH), because they are the same cardinality. This also implies that $\beth_n = \beth_0$ for all natural numbers n, which makes sense intuitively because $2^\infty = \infty$. Also if any sequence of natural numbers is concatenated, it will always create a new natural number, meaning every element in $\mathcal{P}(\mathbb{N}_0)$ corresponds to an element in $\mathbb{N}_0$. This all could have disasterous consequences for set theory or mathematics as a whole because CH is provably undecidable in some models and yet provably decidable. This means either Zermelo-Fraenkel Set Theory with the axiom of choice (ZHC) is unsound, the aforementioned models are invalid, or at least one of the proofs is invalid. The entire foundation of aleph and beth numbers and sizes of infinities could be deeply flawed to the core. CH implies the Gimel Hypothesis is true according to Reference 7, and Wetzel's problem is false according to Reference 8. A truth value for CH or GCH is not asserted here bacause the specifics of CH are unclear, and they can change its truth value. If the reals have to be bigger than the naturals, it is false, but if it just needs there to be no set in between them, it is true.

8 References

1. https://en.wikipedia.org/wiki/Cantor's_diagonal_argument
   Wikipedia page with background for Section 4

2. https://www.digizeitschriften.de/dms/img/?PID=GDZPPN002113910physid=phys84navi
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