# Integrating Functions with Countably-Many Jump Discontinuities

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September 2, 2024, 7:29pm MST

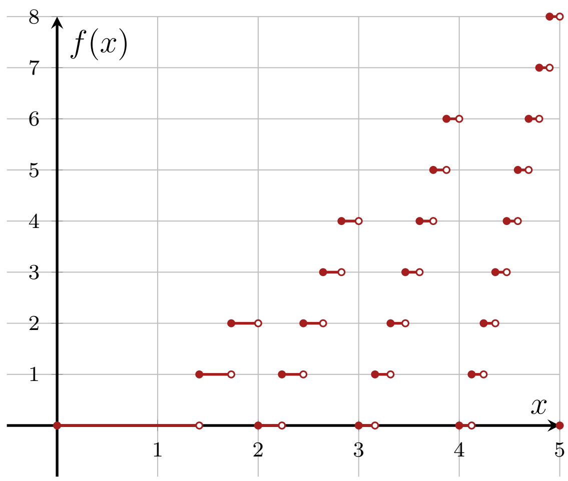
While applications don't immediately come to attention, the pursuit of knowledge and mathematical literacy is enough in itself. Godfrey Harold Hardy (1940), a 20th-century mathematician famously said "No discovery of mine has made, or is likely to make, directly or indirectly, for good or for ill, the least difference to ... the world" (A Mathematician's Apology, 1st ed., p. 49); his work is now used everywhere in critical fields like cryptography.

For the following process to work, the functions are limited to a countable amount of jump discontinuities, since if there is an uncountable amount, then by definition, the discontinuities can't be listed out, and step 2 won't work. It makes it much easier to assume that there are no discontinuities other than the jump discontinuities, but the main essence of the steps should be about the same in that case. This process pertains to step functions like $\lfloor x \rfloor$ and $\lceil x \rceil$. The notation used will be as follows: $f(x)$ is the full function with the step functions in it and $g(x)$ is the same function but with the step functions removed. For instance, if $f(x)$ is any of $x^2$, $\lfloor x \rfloor^2$, $\lfloor x^2 \rfloor$, or $x\lfloor x \rfloor$, then $g(x) = x^2$. Functions like $f(x) = \lfloor x \rfloor^2 - 2\lfloor x \rfloor$ are continuous analogs of discrete functions, so this process could introduce a way of integrating discrete functions.

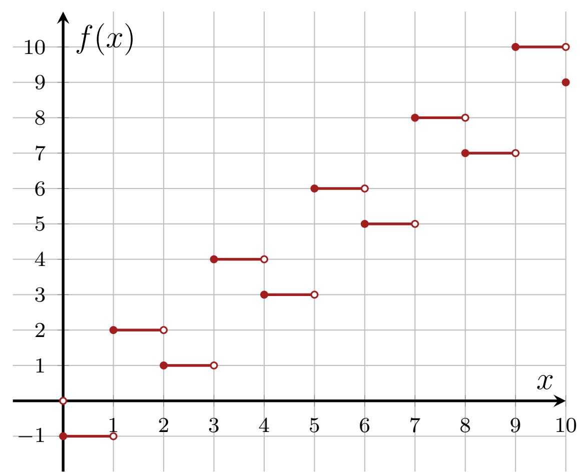
## Step 1: Define a Function to Integrate

You can use almost any function you want, but if it is too general steps 3 or 4 might not work. Some possible examples for $f(x)$ are $\lfloor x \rfloor$, $\lfloor \sin \pi x \rfloor$, $\lfloor x^2 - x + 1 \rfloor$, or $\left\lfloor \frac{1}{x} \right\rfloor$. Usually, if you have anything in terms of the ceiling, rounding, or truncating functions, you will want to convert them to the floor function with these formulas: $\operatorname{round}(x) = \left\lfloor x + \frac{1}{2} \right\rfloor$, $\lceil x \rceil = -\lfloor -x \rfloor$, and $\operatorname{truncate}(x) = \lfloor |x| \rfloor \operatorname{sgn}(x)$. Sometimes you will need to pay attention to the integral boundaries, as for most functions they can be 0 and $x$, but for $f(x) = \left\lfloor \frac{1}{x} \right\rfloor$, they should be 1 and $x$. The lower bound should be wherever the integral is zero. Don't use a naïve approach like $\int \lfloor x \rfloor \mathrm{d}x = x\lfloor x \rfloor + c$; while taking the derivative of both sides technically does give the same function, it is not correct because it is missing $-\dfrac{\lfloor x \rfloor^2 + \lfloor x \rfloor}{2}$. This is because $\lim\limits_{x \to a} \dfrac{\mathrm{d}\lfloor x \rfloor}{\mathrm{d}x}(a) = 0$ for any constant $a$, so $\lfloor x \rfloor$ can just be treated as a constant when taking derivatives. Here are a couple example graphs.

Plot of $f(x) = \lfloor x^2 \rfloor - \lfloor x \rfloor^2$



Plot of $f(x) = \lfloor x \rfloor - (-1)^{\lfloor x \rfloor}$

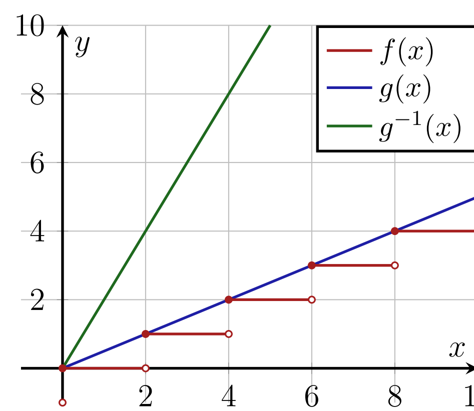


# Step 2: Split at the Jumps and Integrate Separately

Since $f(x)$ is not continuous, you need to figure out where the value of $f(x)$ changes, or where the $n$th jump discontinuity is. Then you split the integral by these boundaries and integrate each interval individually. Sometimes this can be slightly trickier if you have something like $f(x) = \sqrt{\lfloor x \rfloor}$ because not all the output values are at integers. You will benefit from using a different strategy for periodic functions like $\lfloor \cos x \rfloor$ to take advantage of their periodicity. Splitting at jumps is a similar method to how you integrate piecewise functions, although in piecewise functions you split the integral at the piece boundaries instead of just at jumps (Calculus, Larson & Edwards, 2010).

## Example 1

$$f(x) := \left\lfloor \frac{x}{2} \right\rfloor$$
$$g(x) = \frac{x}{2}$$
$$g^{-1}(x) = 2x$$
$$\int_{2n}^{2(n+1)} f(t)\mathrm{d}t = \int_{2n}^{2n+2} \left\lfloor \frac{t}{2} \right\rfloor \mathrm{d}t$$

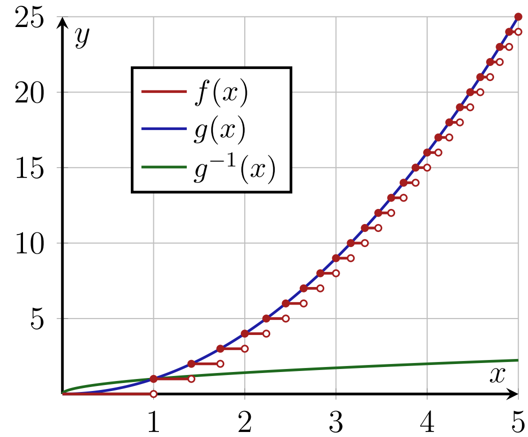



$f(x)$ on this interval will be the same value, so this is just the area of a rectangle. Ignore the discontinuities at the endpoints; they don't change the outcome of the integral.

$$\int_{2n}^{2n+2} \left\lfloor \frac{t}{2} \right\rfloor \mathrm{d}t = \left\lfloor \frac{2n}{2} \right\rfloor (2n + 2 - 2n) = \lfloor n \rfloor \cdot 2 = 2n$$

### Example 2

$$f(x) := \lfloor x^2 \rfloor$$
$$g(x) = x^2$$
$$g^{-1}(x) = \sqrt{x}$$




doing the same thing as in example 1, we get:

$$\int_{\sqrt{n}}^{\sqrt{n+1}} f(t)\mathrm{d}t = f(\sqrt{n})\left(\sqrt{n+1} - \sqrt{n}\right) = \lfloor x \rfloor \sqrt{1 + 2x - 2\sqrt{x^2 + x}}$$

$$\text{and in general, } \int_{g^{-1}(n)}^{g^{-1}(n+1)} f(t)\mathrm{d}t = f\left(g^{-1}(n)\right)\left[g^{-1}(n+1) - g^{-1}(n)\right]$$

This general formula only works if $f(x)$ is constant in between discontinuities. Make sure you use the correct area formula for your function. For instance, If you have $f(x) = x\lfloor x \rfloor$ then use the trapezoid area formula instead.

## Step 3: Find the Sum of Each Partial Integral

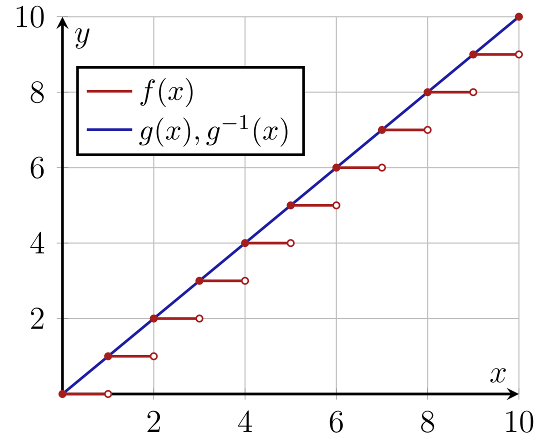
After you've figured out the closed form of the integral on each interval, you can use the additive interval property of integrals, where "if $f$ is integrable on the three closed intervals determined by $a$, $b$, and $c$, then $\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$" (Calculus, Larson & Edwards, 2010, p. 276). Sum up the separate integrals and get the closed form on the whole interval. The following summation is impossible for a general $f(x)$, so you need to have a specific function for this step.

$$\int_{g^{-1}(0)}^{g^{-1}(k)} f(t)\mathrm{d}t = \sum_{j=1}^{k-1} \int_{g^{-1}(j)}^{g^{-1}(j+1)} f(t)\mathrm{d}t$$

Oftentimes you can end up with something complicated as the closed form, like the general harmonic sequence, or a difference of Hurwitz Zeta functions. Other times the sum can just be impossible to get a closed form for.

### Example

$$f(x) := \lfloor x \rfloor$$
$$g(x) = x$$
$$g^{-1}(x) = x$$




$$\int_0^k \lfloor t \rfloor \mathrm{d}t = \sum_{j=0}^{k-1} \int_j^{j+1} \lfloor t \rfloor \mathrm{d}t = \sum_{j=1}^{k-1} j \cdot (j+1-j) = \frac{k^2-k}{2}$$

## Step 4: Integrate up to $x$

To integrate up to $x$, you can once again use the additive interval property of integrals. The first integral (on the right) is from step 3, and the second is similar to in step 2. $g^{-1}(f(x))$ is the greatest $x$-value less than $x$ with a jump discontinuity. The formula for the second integral is not always correct; use the correct area formula for the section. It is the same process as in step 2 but using a subset of the interval.

$$\int_{g^{-1}(0)}^{x} f(t)\mathrm{d}t = \int_{g^{-1}(0)}^{g^{-1}(f(x))} f(t)\mathrm{d}t + \int_{g^{-1}(f(x))}^{x} f(t)\mathrm{d}t$$
$$\int_{g^{-1}(f(x))}^{x} f(t)\mathrm{d}t = f(x)\left[x - g^{-1}(f(x))\right]$$

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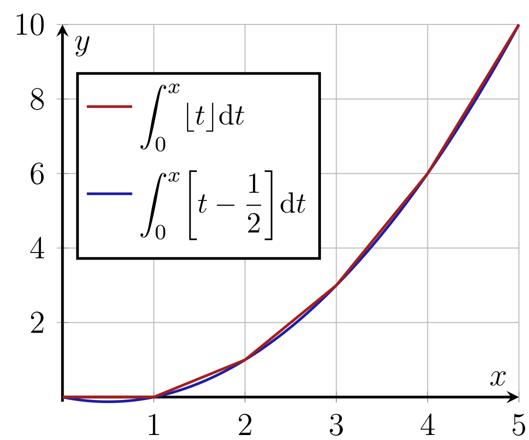

$$\int_{g^{-1}(0)}^{x} f(t)\mathrm{d}t = \int_{0}^{\lfloor x \rfloor} \lfloor t \rfloor \mathrm{d}t + \int_{\lfloor x \rfloor}^{x} \lfloor t \rfloor \mathrm{d}t = \frac{\lfloor x \rfloor^2 - \lfloor x \rfloor}{2} + (x - \lfloor x \rfloor)\lfloor x \rfloor$$

Once you have this, you have found the principle indefinite integral. Add the constant of integration for the general answer.

## Table of Results

Assume the bounds are 0 and $x$ unless explicitly mentioned.

$$\int x \bmod c \,\mathrm{d}x = \frac{c^2}{2}\left(\left\lfloor \frac{x}{c} \right\rfloor + \frac{x}{c} \operatorname{mod}^2 1\right) \tag{1}$$

$$\int \lfloor x \rfloor \mathrm{d}x = \frac{1}{2}\lfloor x \rfloor(2x - \lfloor x \rfloor - 1) \tag{2}$$

$$\int \lceil x \rceil \mathrm{d}x = \frac{1}{2}\lceil x \rceil(2x - \lceil x \rceil + 1) \tag{3}$$

$$\int \lfloor x \rceil \mathrm{d}x = \frac{1}{2}(\lfloor x \rceil - 1)(2x - \lfloor x \rceil - 1) \ni \lfloor x \rceil = \mathrm{round}(x) \tag{4}$$

$$\int \lceil x \rfloor \mathrm{d}x = \frac{1}{2}\lfloor x \rfloor(2x - \lfloor x \rfloor - 1) + \min(0, x) \ni \lceil x \rfloor = \mathrm{truncate}(x) \tag{5}$$

$$\int f \mathrm{d}x = \left[b + \frac{b}{2a}\right] f - \frac{|b|n}{2a} + \frac{\zeta\left(-\frac{1}{2}, \frac{b^2}{4a} + 1 + f - n\right) - \zeta\left(-\frac{1}{2}, \frac{b^2}{4a} + 1\right)}{\sqrt{a}} \tag{6}$$

$$\ni f(x) = \left\lfloor ax^2 + bx + n \right\rfloor$$

## References


- Hardy, G. H. (1940). A Mathematician's Apology (1st ed.). University of Alberta Mathematical Sciences Society.
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`http://teacherpress.ocps.net/cynthiaandrews/files/2013/06/Calculus-9th-Edition-by-Ron-Larson-Bruce-H.-Edwards.pdf`