

The Gamma Function, Its Inverse, And Its Relationship With Trigonometry

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1 Definitions

$$0 \in \mathbb{N} \tag{1}$$

$$\mathbb{N}_k := \{x + k : x \in \mathbb{N}\} \tag{2}$$

$$\tau := 2\pi \tag{3}$$

$$\Gamma(x) := \int_0^\infty \frac{t^{x-1} dt}{\exp(t)} = (x-1)! \quad \forall x > 0 \tag{4}$$

$$\zeta(x) := \sum_{n=0}^\infty \frac{1}{n^x} \quad \forall x > 1 \tag{5}$$

$$\eta(x) := \Gamma^{-1}(x) \tag{6}$$

$$\eta(\Gamma(x)) = \Gamma(\eta(x)) = x \tag{7}$$

(6) is the compositional inverse rather than the fractional inverse.

The set of all natural numbers, \mathbb{N} , is defined by (1) and (2).

(5) describes the Riemann Zeta Function.

“ \ni ” means “such that”.

2 Riemann Zeta Function

$$\text{claim: } \zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\Gamma(x)}$$

This proof is important for part 3.

Proof:

$$\text{know: } \zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x)$$

$$\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (8)$$

$$\zeta(x) = 2 \cdot 2^{x-1} \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (9)$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (10)$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x) \zeta(1-x) \quad (11)$$

$$\zeta(x) = \frac{2}{\tau^{1-x}} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x) \zeta(1-x) \quad (12)$$

$$\zeta(1-x) = \frac{2}{\tau^x} \cos\left(\frac{\pi x}{2}\right) \Gamma(x) \zeta(x) \quad (13)$$

$$\frac{\zeta(1-x)\tau^x}{2 \cos\left(\frac{\pi x}{2}\right) \Gamma(x)} = \zeta(x) \quad (14)$$

$$\zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\Gamma(x)} \quad (15)$$

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3 Cosecant

claim: $\csc x$ can be written in terms of $\Gamma(x)$

Proof:

know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x)$

know: $\zeta(x) = \frac{\zeta(1-x) \tau^x \sec\left(\frac{\pi x}{2}\right)}{2 \Gamma(x)}$

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) = \frac{\zeta(1-x) \tau^x \sec\left(\frac{\pi x}{2}\right)}{2 \Gamma(x)} \quad (16)$$

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{2 \Gamma(x)} \quad (17)$$

$$4 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{\Gamma(x)} \quad (18)$$

$$2 \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\pi}{\Gamma(x)} \quad (19)$$

$$\Gamma(1-x) \Gamma(x) \sin \pi x = \pi \quad (20)$$

$$\Gamma(1-x) \Gamma(x) = \pi \csc \pi x \quad (21)$$

$$\csc x = \frac{1}{\pi} \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) \quad (22)$$

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4 The Remaining Functions

claim: The rest of the trig functions can be written in terms of cosecant

Proof:

$$\sin x = \frac{1}{\csc x} \quad (23)$$

$$\cos x = \frac{1}{\csc\left(\frac{\pi}{2} - x\right)} \quad (24)$$

$$\tan x = \frac{\csc\left(\frac{\pi}{2} - x\right)}{\csc x} \quad (25)$$

$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \quad (26)$$

$$\cot x = \frac{\csc x}{\csc\left(\frac{\pi}{2} - x\right)} \quad (27)$$

$$\sinh x = -\frac{i}{\csc x} \quad (28)$$

$$\cosh x = \frac{1}{\csc\left(\frac{\pi}{2} - ix\right)} \quad (29)$$

$$\tanh x = -\frac{i \csc\left(\frac{\pi}{2} - ix\right)}{\csc ix} \quad (30)$$

$$\operatorname{csch} x = i \csc ix \quad (31)$$

$$\operatorname{sech} x = \csc\left(\frac{\pi}{2} - ix\right) \quad (32)$$

$$\operatorname{coth} x = \frac{i \csc ix}{\csc\left(\frac{\pi}{2} - ix\right)} \quad (33)$$

Therefore, all the trigonometric functions can be written in terms of $\Gamma(x)$.

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5 Inverse Gamma Function

claim: $\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x))$

Proof:

know: $\Gamma(x)\Gamma(1-x) = \pi \csc \pi x$ (equation 21)

$$\pi \csc \pi x = \Gamma(x)\Gamma(1-x) \quad (34)$$

$$\pi \csc \pi \eta(x) = x\Gamma(1-\eta(x)) \quad (35)$$

$$\csc \pi \eta(x) = \frac{x}{\pi} \Gamma(1-\eta(x)) \quad (36)$$

$$\pi \eta(x) = \csc^{-1} \frac{x}{\pi} \Gamma(1-\eta(x)) \quad (37)$$

$$\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi} \Gamma(1-\eta(x)) \quad (38)$$

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There are 2 conditions to where $\eta(x)$ is defined.

$\eta(x)$ is not defined over the real numbers such that the following is true:

$$|x \Gamma(1 - \eta(x))| < \pi \quad \text{or} \quad |x| < \sim 3.547373963... \quad (39)$$

I assume that this value is transcendental or at least algebraic.

$\eta(x)$ is guaranteed to be defined if (39) is false and x is negative.

$\eta(x)$ is defined for all integers and around the integers.

The area around the integer that is defined gets smaller as x increases, though I do not know why this is the case.

If you are recursing $\eta(x)$ and you change the $\frac{x}{\pi}$ to $\frac{x^n}{\pi}$,

for some $n \in \mathbb{N}_2$, it either gets defined for more values,

or there is a bug in Desmos' graphing calculations.

These extra values are just expanded ranges around the positive integers.

It doesn't change the rule described in (39), however.

See reference 2 for more information.

going a different way after (35) gives the following equation:

$$\eta(x) = 1 - \eta\left(\frac{\pi}{x} \csc \pi \eta(x)\right) \quad (40)$$

6 References

- https://en.wikipedia.org/wiki/Riemann_zeta_function (equation 8)
- <https://www.desmos.com/calculator/ieu6ebnfn8> (for part 5, made by me)
- <https://www.github.com/drizzt536/files/tree/main/TeX/gamma/> (me)