On the Continuum Hypothesis and the Correspondence of Infinite Sets with the Natural Numbers

Daniel E. Janusch

Dedicated to Maeby, my favorite kitty

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1 Background and Countability Theorems

If a set is "countable," there exists a bijection between it and loose subset of the natural numbers (equinumeral). Any finite set will have this property and the only question is if all *infinite* sets do. $|S|$ denotes the cardinality of any set S. In all processes with infinite steps, all the steps should be thought of as executing simultaneously rather than one by one in order.

$$R \times S := \{(r, s) : r \in R \wedge s \in S\} \quad \text{(1a)} \qquad R \cdot S := \{r \cdot s : r \in R \wedge s \in S\} \quad \text{(1b)} \tag{1}$$

$$R(\vec{v}\,) := \left\{ \sum_{n=1}^{\dim \vec{v}} a_n \vec{v}_n : a_n \in R \right\} \quad \text{(2a)} \qquad R[\vec{v}\,] := \left\{ \prod_{n=1}^{\dim \vec{v}} (a_n + \vec{v}_n) : a_n \in R \right\} \quad \text{(2b)} \tag{2}$$

Theorem 1. Unions of Countable Sets
If any sets R and S are countable, $R(a) \cup S(b)$ is countable for all a and b.

Proof. Since R and S are countable, $R(a)$ and $S(b)$ are as well because this just multiplies the already present elements by a constant. Interlace the elements of $R(a)$ with the elements of $S(b)$ skipping over any duplicates. If either of the sets runs out, meaning it is finite, stop interlacing and only pull elements from the infinite set. Similar logic works using $R[a]$ and $S[b]$ or combinations of those, $R(a)$, and $S(b)$. ■

Theorem 2. Multiplications and Cartesian Products of Countable Sets
If any sets R and S are countable, both $R \times S$ and $R \cdot S$ are countable.

Proof. Put ordered pairs using elements all the of R and S into a table as shown below and go along the diagonals getting every element in the product and not "missing" any. The numbers and colons are the output indices. The first 25 are labeled if in view. Similar logic works for $R \cdot S$. This is also algorithmically viable as shown in Section 4. The output indices relate strongly to triangle numbers. ■

$1 : (R_1, S_1)$	$2 : (R_1, S_2)$	$4 : (R_1, S_3)$	$7 : (R_1, S_4)$	$\cdots$
$3 : (R_2, S_1)$	$5 : (R_2, S_2)$	$8 : (R_2, S_3)$	$12 : (R_2, S_4)$	$\cdots$
$6 : (R_3, S_1)$	$9 : (R_3, S_2)$	$13 : (R_3, S_3)$	$18 : (R_3, S_4)$	$\cdots$
$10 : (R_4, S_1)$	$14 : (R_4, S_2)$	$19 : (R_4, S_3)$	$25 : (R_4, S_4)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Theorem 3. Subsets, Intersections, and Differences of Countable Sets

If any set R is countable, and a set S is a loose subset of R, S is also countable.

Proof. For each element in R but not in S ($x \in R \setminus S$), move its index to the beginning of R and then remove it. That way, the set is still guaranteed to be countable. Since every set along the way is countable, so is S. This also implies that set intersections and differences are countable because these operations return subsets of the original sets. ■

Theorem 4. Sum and Product "Rings" From Countable Sets

If any set R is countable, $R(a, b)$ and $R[a, b]$ are also countable for all a and b.

Proof. Create 2 new intermediate sets $S_a := \{a \cdot r : r \in R\}$ and $S_b := \{b \cdot r : r \in R\}$. $S_a \times S_b$ gives a set with the same countability as $R(a, b)$, only with ordered pairs instead of addition. This is countable via Theorem 2. Similar logic works for $R[a, b]$. They're not actually rings, it's just similar syntax for similar things. ■

Theorem 5. Finite Natural Powers of Countable Sets

If any set R is countable, R^n is also countable for all natural numbers ($\mathbb{N}_0$) n.

Proof. R^n can be factored as $R \times R^{n-1}$ which is countable if R^{n-1} is countable. This can be applied recursively until it becomes countable if R^2 or $R \times R$ is countable, which is countable via Theorem 2. If $n = 0$, the output set is all the groups of zero elements from R, or just $\emptyset$, which is countable since $|\emptyset|$ is finite. This theorem is unused in this document. ■

Theorem 6. Countably Infinite Unions of Countably Infinite Sets

If all sets R_i are countable, $\displaystyle\bigcup_{i=1}^{\aleph_0} R_i$ is also countable, where $\aleph_0 \equiv |\mathbb{N}| = \infty$.

Proof. Give each set R_i a column in a table and put all its elements in that column going down. Do this for every set and using the same argument as in Theorem 2, one can show that every element is indexed and none are skipped. Every column represents a different R_i and the rows represent a different $(R_i)_j$ or $R_{i,j}$. The numbers before the colons are the output index. The first 85 are shown if in view. This is algorithmically viable, even though this table has a slightly different numbering than that of Theorem 2. ■

$1 : R_{1,1}$	$2 : R_{2,1}$	$6 : R_{3,1}$	$7 : R_{4,1}$	$15 : R_{5,1}$	$16 : R_{6,1}$	$28 : R_{7,1}$	$\cdots$
$3 : R_{1,2}$	$5 : R_{2,2}$	$8 : R_{3,2}$	$14 : R_{4,2}$	$17 : R_{5,2}$	$27 : R_{6,2}$	$30 : R_{7,2}$	$\cdots$
$4 : R_{1,3}$	$9 : R_{2,3}$	$13 : R_{3,3}$	$18 : R_{4,3}$	$26 : R_{5,3}$	$31 : R_{6,3}$	$43 : R_{7,3}$	$\cdots$
$10 : R_{1,4}$	$12 : R_{2,4}$	$19 : R_{3,4}$	$25 : R_{4,4}$	$32 : R_{5,4}$	$42 : R_{6,4}$	$49 : R_{7,4}$	$\cdots$
$11 : R_{1,5}$	$20 : R_{2,5}$	$24 : R_{3,5}$	$33 : R_{4,5}$	$41 : R_{5,5}$	$50 : R_{6,5}$	$62 : R_{7,5}$	$\cdots$
$21 : R_{1,6}$	$23 : R_{2,6}$	$34 : R_{3,6}$	$40 : R_{4,6}$	$51 : R_{5,6}$	$61 : R_{6,6}$	$72 : R_{7,6}$	$\cdots$
$22 : R_{1,7}$	$35 : R_{2,7}$	$39 : R_{3,7}$	$52 : R_{4,7}$	$60 : R_{5,7}$	$73 : R_{6,7}$	$85 : R_{7,7}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

2 Applications of the Theorems

2.1 Countability of the Integers

Claim: $\mathbb{Z}$ is a countable set.

Proof. Let $R = \mathbb{N}_0$, $S = \mathbb{N}_1$. R and S are countable axiomatically (axiom of extensionality), being the naturals themselves. $\mathbb{Z} \equiv R \cup S(-1)$. This is countable via Theorem 1. ∎

2.2 Countability of the Rationals

Claim: $\mathbb{Q}$ is a countable set.

Proof. The rationals are basically just ordered pairs of integers and naturals. $R := \mathbb{Z}$, $S := \mathbb{N}_1$. $R \times S$ defines all of these pairs. R is countable via Section 2.1 and S is countable axiomatically. $R \times S$ is thus countable via Theorem 2. This argument and Theorem 2 derives from Cantor's argument for the rationals. ∎

2.3 Countability of the Reals from Zero to One

Claim: $\{x \in \mathbb{R} : 0 \leqslant x < 1\}$ is a countable set.

Proof. Let R be some set. Each R_i can be defined to be the digits of i reversed with "0." at the beginning and infinite optional trailing zeros at the end, for any $i \in \mathbb{N}_0$. For example, $R_{246} = 0.642\overline{0}$ and $R_0 = 0.0\overline{0} = 0$. This set is countable because it was defined to be countable; each natural number corresponds to a single element. This set contains every real number in the range because every possible sequence of digits is in it. The sequence trends upwards, asymptotically approaching 1, though it fluctuates wildly along the way. ∎

2.4 Countability of the Non-Negative Reals

Claim: $\mathbb{R}_{\geqslant 0}$ or equivalently $\{x \in \mathbb{R} : x \geqslant 0\}$ is a countable set.

Proof. The set of real numbers from zero to one is countable via Section 2.3. Let R be the same set used in that proof. $\mathbb{R}_{\geqslant 0} = \bigcup_{i=1}^{\aleph_0} S_i$ where $S_i := R[i-1]$. This is countable via Theorem 6. The following table illustrates this. ∎

$0.\overline{0}$	$1.\overline{0}$	$2.\overline{0}$	$3.\overline{0}$	$4.\overline{0}$	$5.\overline{0}$	$\cdots$
$0.1\overline{0}$	$1.1\overline{0}$	$2.1\overline{0}$	$3.1\overline{0}$	$4.1\overline{0}$	$5.1\overline{0}$	$\cdots$
$0.2\overline{0}$	$1.2\overline{0}$	$2.2\overline{0}$	$3.2\overline{0}$	$4.2\overline{0}$	$5.2\overline{0}$	$\cdots$
$0.3\overline{0}$	$1.3\overline{0}$	$2.3\overline{0}$	$3.3\overline{0}$	$4.3\overline{0}$	$5.3\overline{0}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$0.7\overline{0}$	$1.7\overline{0}$	$2.7\overline{0}$	$3.7\overline{0}$	$4.7\overline{0}$	$5.7\overline{0}$	$\cdots$
$0.8\overline{0}$	$1.8\overline{0}$	$2.8\overline{0}$	$3.8\overline{0}$	$4.8\overline{0}$	$5.8\overline{0}$	$\cdots$
$0.9\overline{0}$	$1.9\overline{0}$	$2.9\overline{0}$	$3.9\overline{0}$	$4.9\overline{0}$	$5.9\overline{0}$	$\cdots$
$0.01\overline{0}$	$1.01\overline{0}$	$2.01\overline{0}$	$3.01\overline{0}$	$4.01\overline{0}$	$5.01\overline{0}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

2.5 Countability of the Reals

Claim: $\mathbb{R}$ is a countable set which implies $|\mathcal{P}(\mathbb{N})| \equiv |\mathbb{N}|$.

Proof. $\mathbb{R}_{\geqslant 0}$ is countable via Section 2.4. $\mathbb{R}_{\geqslant 0}(1, -1) \equiv \mathbb{R}$. This is countable via Theorem 4. This conclusion can be further reinforced by the algorithmic methods later on. For each real number x, there exists at least 1 sequence of elements in R, each element with a higher index than the last, where the limit of the sequence equals x. This is because every sequence of digits is contained by R. ■

2.6 Countabilities of Miscellaneous Number Classes

2.6.1 Algebraic and Transcendental Numbers

Claim: $\mathbb{A}$ and $\mathbb{T}$ are countable sets.

Proof. $\mathbb{R}$ is a countable set via Section 2.5. Since the algebraic reals and transcendental reals are both subsets of the reals, they are countable via Theorem 3. They are also countable over the complex numbers using Section 2.7 and the same logic. ■

2.6.2 Imaginary Numbers

Claim: $\mathbb{I}$ is a countable set.

Proof. $\mathbb{R}$ is a countable set via Section 2.5. $\mathbb{R}(\sqrt{-1}) \equiv \mathbb{I}$. This is countable via Theorem 1, or more precisely $R(\sqrt{-1}) \cup \emptyset$ is countable. The union of any set and the null set is itself. ■

2.7 Countability of the Complex Numbers

Claim: $\mathbb{C}$ is a countable set.

Proof. $\mathbb{R}$ is a countable set via Section 2.5. $\mathbb{R}(1, \sqrt{-1}) \equiv \mathbb{C}$. This is countable via Theorem 4. The identity used here stems from the rectangular form of complex numbers. ■

3 Addressing Cantor's Diagonal Argument

According to Georg Cantor in 1891, If someone is trying to list all the real numbers, they can always find a number that is not in the list using his "Diagonal Argument". This argument is basically the same as the following, though for reals instead of naturals: Suppose someone is trying to make a set S with every natural number. They first add zero to the set and the set is $\{0\}$, then they could say, "one isn't in the set." When they add one and have $\{0, 1\}$, they can say "two isn't in the set," then "three isn't in the set," "four isn't in the set," et cetera. No matter how many natural numbers they add, they can always find one not in it; $\max(S) + 1$. Using Cantor's same logic, this seems to be implying that there is not a bijection between the natural numbers and themselves, which is clearly wrong. Every set is bijective onto itself via the identity function. $f(x) := x$ for $f : X \mapsto X$.

4 Enumerating the Continuum Algorithmically

The following Node JS code prints out real numbers to stdout delimited by a comma-space pair. With infinite time and memory, it will have printed every real number. The only problems are that it prints out both 0 and -0, and the functions return strings. Both of these are easily resolvable though. There is a reference to this source code and the C version in Section 6. The subscripts are just so it looks nicer. If `process.stdout.write` is replaced with `console.log`, then it will work in vanilla JavaScript in versions beginning with ECMAScript 6. The downside being `console.log` adds a trailing newline character at the end of each call, which isn't ideal for printing a large quantity of small strings each using an individual function call. The complexity of finding the nth real number with this method is less than or equal to $\mathcal{O}\left(n \log^2 n\right)$, emphasis on "less", because after over 16 million iterations on my machine, it hadn't slowed down noticably.

```
function isqrt(n) { // ⌊√n⌋ for non-negative big integers n.
   if (n < 2) return n;
   var x₀, x₁ = n / 2n;

   do
      x₀ = x₁,
      x₁ = x₀ + n / x₀ >> 1n; // Newton's method for f(x) = x^2+a
   while ( x₁ < x₀ );

   return x₀;
}
// iterate over the natural numbers (up to 0.5*2^2^30)
for (var runningIndex = 0n ;;) {
   let positive = true, currentIndex = runningIndex++;

   // if the current index is odd, return a negation the previous index's value
   if (currentIndex % 2n) positive = false, currentIndex--;
   currentIndex /= 2n; // divide index by 2 so all integers can be reached

   const c = isqrt(1n + 8n*currentIndex) // intermediate value
   // the index the next comments refer to
      , u = (c + c % 2n) / 2n - 1n
   // the largest triangle number with an integer index that is less ...
   // than or equal to the current index.  but subtracted from the current index
      , k = currentIndex - u * (u + 1n) / 2n;

   // generate real number from indices k and u-k and print it
   process.stdout.write(`${positive ? "" : "-"}${k}.` +
      `${u - k}`.split("").reverse().join("") + ", "
   );
}
```

5 Further Algorithms and Conclusion

The axioms of Zermelo-Fraenkel Set theory required to prove the algorithmic viability of continuum enumeration are as follows: Axiom Schema of Specification, Axiom of Union, Axiom of Infinity, and Axiom of Choice. Without the Axiom of Choice, Theorems 2, 4, and 6 don't work, although the algorithm from Section 4 is separate from set theory and assumes no axioms. The algorithm uses string manipulation to reverse the numbers and find their lengths, but the following equations do the same thing mathematically.

$$\operatorname{length} n := 1 + \lfloor \log_{10} n \rfloor = \lceil \log_{10}(n+1) \rceil \tag{3}$$

$$\operatorname{reverse} n := \sum_{k=0}^{\operatorname{length} n} \left(\left\lfloor \frac{n}{10^k} \right\rfloor \bmod 10 \right) 10^{\operatorname{length} n - k - 1} \tag{4}$$

$$R_{i,j} := p \left(i + \frac{\operatorname{reverse} j}{10^{\operatorname{length} j}} \right) \tag{5}$$

Where i is `k`, p is `(positive ? 1 : -1)`, and j is `(u - k)`. length() and reverse() are only defined for natural number inputs. An interesting formula used in the algorithm in Section 4 is for the greatest triangle number $t = T(k)$ less than or equal to a natural number n, with a natural number index k.

$$T(n) := \frac{n^2 + n}{2} = \sum_{j=1}^{n} j \tag{6}$$

$$t = T\left(\left\lfloor \frac{\lfloor \sqrt{8n+1} \rfloor}{2} \right\rfloor - 1 \right) \tag{7}$$

Because of the bijection, there is also an inverse of the Section 4 algorithm.

```
function inverse(string) { // assume valid input
   const match = /^(-?)(\d+)\.(\d)$/.exec(string)
      , i = BigInt(match[2]) // the x coordinate index, or the integer part
      , j = BigInt(match[3]); // the y coordinate index, or the decimal part
   return i*(i+1n) + (2n*i+j)*(j+1n) + BigInt(match[1] === "-");
}
```

Conclusion: Since there exists a bijection between the natural numbers and the real numbers, there is no set with a cardinality strictly or loosely in between the naturals and reals (CH), because they are the same. This also implies that $2^{\aleph_0} = \aleph_0$. This makes sense because $2^\infty = \infty$ as well. Also if one takes any sequence of natural numbers and concatenates them, it will always create a new natural number, meaning every element in $\mathcal{P}(\mathbb{N}_0)$ corresponds to a natural number. This could have disasterous consequences for set theory because CH is provably undecidable in some models and yet provably decidable. This means either ZHC is unsound, or at least one of the proofs is invalid. The entire foundation of aleph numbers and sizes of infinities cound be deeply flawed to the core.

6 References

- https://en.wikipedia.org/wiki/Cantor’s_diagonal_argument
  Wikipedia page with elaboration on Section 3

- https://www.digizeitschriften.de/dms/img/?PID=GDZPPN002113910physid=phys84navi
  Georg Cantor’s 1891 article with the diagonal argument. Same source as on wikipedia.

- https://en.wikipedia.org/wiki/Continuum_hypothesis
  Wikipedia page with elaboration on Section 2.5

- https://www.github.com/drizzt536/files/tree/main/TeX/continuum
  The files for the most recent public version of this pdf and the LaTeX code

- https://raw.githubusercontent.com/drizzt536/files/main/JavaScript/continuum.js
  The raw JavaScript source code for the Section 4

- https://raw.githubusercontent.com/drizzt536/files/main/C/continuum.c
  The raw C source code for the Section 4

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