

The Gamma Function, Its Inverse, and Its Relationship with Trigonometry

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1 Definitions

$$0 \in \mathbb{N} \quad (1a) \quad \mathbb{N}_k := \{x + k : x \in \mathbb{N}\} \forall k \quad (1b) \quad (1)$$

$$\tau := 2\pi \quad (2)$$

$$\text{sgn}(x) := \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases} = \frac{x}{|x|} \forall x \quad (3)$$

$$a \bmod_n b := a \bmod b + n \quad (4)$$

$$\exp(x) := \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{x^k}{k!} = e^x \forall x \quad (5)$$

$$\Gamma(x) := \int_0^\infty \frac{t^{x-1}}{\exp t} dt = (x-1)! \forall x > 0 \quad (6)$$

$$\zeta(x) := \sum_{n=1}^\infty \frac{1}{n^x} \forall x > 1 \quad (7)$$

$$\eta(x) := \Gamma^{-1}(x) \quad (8a) \quad \eta(\Gamma(x)) = \Gamma(\eta(x)) = x \quad (8b) \quad (8)$$

(8a) is the compositional inverse rather than the fractional inverse. The set of all natural numbers, \mathbb{N} , is defined in part by (1a) and (1b). (7) describes the Riemann Zeta Function. x is assumed to be a real number everywhere for simplicity.

2 Riemann Zeta Function

claim: $\zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\Gamma(x)}$

This proof is important for section 3.

Proof:

know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x)$ (reference 1)

$$\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (9)$$

$$\zeta(x) = 2 \cdot 2^{x-1} \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (10)$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (11)$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x) \zeta(1-x) \quad (12)$$

$$\zeta(x) = \frac{2}{\tau^{1-x}} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x) \zeta(1-x) \quad (13)$$

$$\zeta(1-x) = \frac{2}{\tau^x} \cos\left(\frac{\pi x}{2}\right) \Gamma(x) \zeta(x) \quad (14)$$

$$\frac{\zeta(1-x)\tau^x}{2 \cos\left(\frac{\pi x}{2}\right) \Gamma(x)} = \zeta(x) \quad (15)$$

$$\zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\Gamma(x)} \quad (16)$$

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3 Cosecant

claim: $\csc x$ can be written in terms of $\Gamma(x)$

Proof:

know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x)$ (reference 1)

know: $\zeta(x) = \frac{\zeta(1-x) \tau^x \sec\left(\frac{\pi x}{2}\right)}{2 \Gamma(x)}$ (equation 16)

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) = \frac{\zeta(1-x) \tau^x \sec\left(\frac{\pi x}{2}\right)}{2 \Gamma(x)} \quad (17)$$

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{2 \Gamma(x)} \quad (18)$$

$$4 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{\Gamma(x)} \quad (19)$$

$$2 \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\pi}{\Gamma(x)} \quad (20)$$

$$\Gamma(1-x) \Gamma(x) \sin \pi x = \pi \quad (21)$$

$$\Gamma(1-x) \Gamma(x) = \pi \csc \pi x \quad (22)$$

$$\csc x = \frac{1}{\pi} \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) \quad (23)$$

$$\sin \pi x \neq 0 \implies \zeta(1-x) \neq 0 \quad (24)$$

Therefore, due to (24), the domains also match.

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4 Trig Functions In Terms of Cosecant

claim: The rest of the trig functions can be written in terms of cosecant

Proof:

$$\sin x = \frac{1}{\csc x} \quad (25)$$

$$\cos x = \frac{1}{\csc \left(\frac{\pi}{2} - x \right)} \quad (26)$$

$$\tan x = \frac{\csc \left(\frac{\pi}{2} - x \right)}{\csc x} \quad (27)$$

$$\sec x = \csc \left(\frac{\pi}{2} - x \right) \quad (28)$$

$$\cot x = \frac{\csc x}{\csc \left(\frac{\pi}{2} - x \right)} \quad (29)$$

$$\sinh x = -\frac{i}{\csc x} \quad (30)$$

$$\cosh x = \frac{1}{\csc \left(\frac{\pi}{2} - ix \right)} \quad (31)$$

$$\tanh x = -\frac{i \csc \left(\frac{\pi}{2} - ix \right)}{\csc ix} \quad (32)$$

$$\operatorname{csch} x = i \csc ix \quad (33)$$

$$\operatorname{sech} x = \csc \left(\frac{\pi}{2} - ix \right) \quad (34)$$

$$\coth x = \frac{i \csc ix}{\csc \left(\frac{\pi}{2} - ix \right)} \quad (35)$$

$\csc x = \csc x$ by the reflexive property.

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5 Trig Functions In Terms of Gamma

These equations are just the previous ones with Gamma plugged in for csc.

$$\sin x = \frac{\pi}{\Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right)} \quad (36)$$

$$\cos x = \frac{\pi}{\Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right)} \quad (37)$$

$$\tan x = \frac{\Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right)}{\Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right)} \quad (38)$$

$$\csc x = \frac{1}{\pi} \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) \quad (39)$$

$$\sec x = \frac{1}{\pi} \Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right) \quad (40)$$

$$\cot x = \frac{\Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right)}{\Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right)} \quad (41)$$

$$\sinh x = -\pi i / \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) \quad (42)$$

$$\cosh x = \pi / \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) \quad (43)$$

$$\tanh x = -i \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) / \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) \quad (44)$$

$$\operatorname{csch} x = \frac{i}{\pi} \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) \quad (45)$$

$$\operatorname{sech} x = \frac{1}{\pi} \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) \quad (46)$$

$$\coth x = i \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) / \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) \quad (47)$$

where “/” means divide everything on the left by everything on the right.

6 Inverse Gamma Function

claim: $\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x))$

Proof:

know: $\Gamma(x) \Gamma(1 - x) = \pi \csc \pi x$ (equation 22)

$$\pi \csc \pi x = \Gamma(x) \Gamma(1 - x) \quad (48)$$

$$\pi \csc \pi \eta(x) = x \Gamma(1 - \eta(x)) \quad (49)$$

$$\csc \pi \eta(x) = \frac{x}{\pi} \Gamma(1 - \eta(x)) \quad (50)$$

$$\pi \eta(x) = \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x)) \quad (51)$$

$$\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x)) \quad (52)$$

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There are 2 conditions to where this $\eta(x)$ is defined.

$\eta(x)$ is **not** defined (over the real numbers) such that the following is true:

$$|x \Gamma(1 - \eta(x))| < \pi \quad \text{or} \quad |x| < 3.547373963... \quad (V) \quad (53)$$

V is probably transcendental. $\eta(x)$ is guaranteed to be defined if equation (53) is false and x is negative. $\eta(x)$ is defined for all integers and in small fields around each integer. The area defined around each integer gets smaller as x increases, though it is unclear why without further inspection. If $\eta(x)$ is being recursed and $\frac{x}{\pi}$ is changed to $\frac{x^n}{\pi}$ in the innermost function, for some $n \in \mathbb{N}_2$, either the fields get larger, or there is a bug in Desmos' graphing calculations. See reference 2 for more information. Going a different way after (49) gives the following equation:

$$\eta(x) = 1 - \eta\left(\frac{\pi}{x} \csc \pi \eta(x)\right) \implies \eta(x) = \eta(\pi \csc(\pi \csc \pi x) \sin \pi x) \quad (54)$$

This equation seems to be less useful because it has large derivatives everywhere when absolute valued, which gets more severe as it is recursed.

7 Miscellaneous

(40) implies the following formula:

$$\Gamma\left(x + \frac{1}{2}\right) = \frac{\pi \sec \pi x}{\Gamma\left(\frac{1}{2} - x\right)} \quad (55)$$

replacing x with $x + z$ and adjusting from gamma to factorial gives:

$$\left(x + \frac{1}{2} + z\right)! = [2(z \bmod 2) - 1] \frac{\pi \sec \pi x}{\left(-\frac{3}{2} - x - z\right)!} \forall z \in \mathbb{Z} \quad (56)$$

This only simplifies things if z is an integer because $\sec(x + \tau) = \sec x$, and $\sec(x + \pi) = -\sec x$. rewriting (22) from gamma to factorial gives:

$$(x - 1)!(-x)! = \pi \csc \pi x \quad (57)$$

$$(-x)! = \frac{\pi x \csc \pi x}{x!} \quad (58)$$

$$\left(x + \frac{1}{c}\right)! = \frac{\pi \csc \frac{cx + 1}{c} \pi}{\left(-\frac{c + 1}{c} - x\right)!} \forall c \in \mathbb{R} \quad (59)$$

$$x! = (x \bmod_n 1)! \left[\prod_{k=\text{sgn}(|n-x|+n-x)}^{|\lfloor x-n \rfloor| + \text{sgn}(n-x-|n-x|)} (x + \text{sgn}(n-x)k) \right]^{\text{sgn}(x-n)} \quad (60)$$

(60) is useful where $x!$ is non self-dependantly defined for $x \in [n, n + 1]$. equation relating to factorials:

$$\lim_{x \rightarrow \infty} \int_0^x \int_0^x \exp \left[\frac{1}{2} \ln \frac{4t}{u} - t - u \right] d^2 t u = \pi \quad (61)$$

Proof sine is odd using gamma:

claim: $\sin(x) = -\sin(-x)$:

Proof:

$$(57) \quad x \mapsto -x \implies (-x)! = \frac{-\pi x \csc(-\pi x)}{x!} \quad (62)$$

$$(-x)! = \frac{\pi x \csc \pi x}{x!} \quad (58)$$

$$\frac{\pi x \csc \pi x}{x!} = \frac{-\pi x \csc(-\pi x)}{x!} \implies \csc x = -\csc(-x) \implies \sin(x) = -\sin(-x) \quad (63)$$

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Proof for the value of $\frac{1}{2}$ factorial

claim: $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$

Proof:

$$x! = \frac{(x+1)!}{x+1} \implies \left(-\frac{1}{2}\right)! = 2 \left(\frac{1}{2}\right)! \quad (64)$$

$$(-x)! = \frac{\pi x \csc \pi x}{x!} \implies \left(-\frac{1}{2}\right)! = \frac{\pi}{2 \left(\frac{1}{2}\right)!} \quad (65)$$

The formula for (65) is just (58).

(64) and (65) imply the following:

$$2 \left(\frac{1}{2}\right)! = \frac{\pi}{2 \left(\frac{1}{2}\right)!} \implies 4 \left(\frac{1}{2}\right)!^2 = \pi \implies \left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2} \quad (66)$$

using this strategy only works if $(c_1 - x) = (c_2 + x)$ for some integers c_1 and c_2 . this limits it to $(z + \frac{1}{2})$ for integers z . Also, you can only apply (65) an odd number of times or the factorials will cancel and leave a trigonometric equation, which could be interesting later. If you apply (65) twice, you get back to where you started because everything new cancels.

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8 References

- https://en.wikipedia.org/wiki/Riemann_zeta_function#Riemann's_functional_equation
Link for equation 9
- <https://www.desmos.com/calculator/t51xnveadf>
Extra information for section 6, made by me
- <https://www.github.com/drizzt536/files/tree/main/TeX/gamma>
The files for the most recent version of this pdf and the LaTeX code