

On the Continuum Hypothesis and the Correspondence of Infinite Sets with the Natural Numbers

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Dedicated to Maeby, my favorite kitty

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1 Background and Countability Theorems

If a set is “countable,” there exists a bijection between it and loose subset of the natural numbers (equinumeral). Any finite set will have this property and the only question is if all *infinite* sets have it. $|S|$ denotes the cardinality of S for any set S . Iterated set multiplication of the first kind gives ordered groups with more than two elements, rather than nested ordered pairs. In all processes with infinite steps, all the steps should execute simultaneously.

$$R \times S := \{(r, s) : r \in R \wedge s \in S\} \quad (1a) \quad R \cdot S := \{r \cdot s : r \in R \wedge s \in S\} \quad (1b) \quad (1)$$

$$R(\vec{v}) := \left\{ \sum_{n=1}^{\dim \vec{v}} a_n \vec{v}_n : a_n \in R \right\} \quad (2a) \quad R[\vec{v}] := \left\{ \prod_{n=1}^{\dim \vec{v}} (a_n + \vec{v}_n) : a_n \in R \right\} \quad (2b) \quad (2)$$

Theorem 1. Unions of Countable Sets

If any pair of sets R and S are countable, $R(a) \cup S(b)$ is countable for all a and b .

Proof. Since R and S are countable, $R(a)$ and $S(b)$ are as well because this just multiplies the already present elements by a constant. Interlace the elements of $R(a)$ with the elements of $S(b)$ skipping over any duplicates. If either of the sets runs out, meaning it is finite, stop interlacing and only pull elements from the infinite set. Similar logic works using $R[a]$ and $S[b]$ or combinations of those, $R(a)$, and $S(b)$. ■

Theorem 2. Multiplications of Countable Sets

If any pair of sets R and S are countable, $R \times S$ and $R \cdot S$ are countable.

Proof. Put ordered pairs using elements all the of R and S into a table as shown below and go along the diagonals getting every element in the product and not “missing” any. The numbers and colons are the indices. The first 25 are labeled if in view. Similar logic works for $R \cdot S$. This is also algorithmically viable as shown in Section 4. ■

1 : (R_1, S_1)	2 : (R_1, S_2)	4 : (R_1, S_3)	7 : (R_1, S_4)	...
3 : (R_2, S_1)	5 : (R_2, S_2)	8 : (R_2, S_3)	12 : (R_2, S_4)	...
6 : (R_3, S_1)	9 : (R_3, S_2)	13 : (R_3, S_3)	18 : (R_3, S_4)	...
10 : (R_4, S_1)	14 : (R_4, S_2)	19 : (R_4, S_3)	25 : (R_4, S_4)	...
⋮	⋮	⋮	⋮	⋮

Theorem 3. Subsets, Intersections, and Differences of Countable Sets

If any set R is countable, and a set S is a loose subset of R , S is also countable.

Proof. For each element in R but not in S ($x \in R \setminus S$), move its index to the beginning of R and then remove it. That way, the set is still guaranteed to be countable. Since every set along the way is countable, so is S . This also implies that set intersections and differences are countable because these operations return subsets of the original sets. ■

Theorem 4. Sum and Product “Rings” From Countable Sets

If any set R is countable, $R(a, b)$ and $R[a, b]$ are also countable for all a and b .

Proof. Create 2 new intermediate sets $S_a := \{a \cdot r : r \in R\}$ and $S_b := \{b \cdot r : r \in R\}$. $S_a \times S_b$ gives a set with the same countability as $R(a, b)$, only with ordered pairs instead of addition. This is countable via Theorem 2. Similar logic works for $R[a, b]$. They’re not actually rings, it’s just similar syntax for similar things. ■

Theorem 5. Powers of Countable Sets

If any set R is countable, R^n is also countable for all natural numbers n (\mathbb{N}_0).

Proof. R^n can be factored as $R \times R^{n-1}$ which is countable if R^{n-1} is countable. R^{n-1} can be factored as $R \times R^{n-2}$ which is countable if R^{n-2} is countable, et cetera. This can be simplified continuously until it becomes countable if R^2 or $R \times R$ is countable, which is countable via Theorem 2. If $n = 0$, the output set is all the groups of zero elements from R , or just \emptyset , which is countable since $|\emptyset|$ is finite. This theorem is unused in this document. ■

Theorem 6. Countably Infinite Unions of Countable Sets

If all sets R_i are countable, $\bigcup_{i=1}^{\aleph_0} R_i$ is also countable, where $\aleph_0 \equiv |\mathbb{N}| = \infty$.

Proof. Give each set R_i a column in a table and put all its elements in that column going down. Do this for every set and using the same argument as in Theorem 2, one can show that every element is indexed and none are skipped. Every column represents a different R_i and the rows represent a different $(R_i)_j$ or $R_{i,j}$. The numbers before the colons are the output index. The first 85 are shown if in view. This is algorithmically viable, even though this table has a slightly different numbering than that of Theorem 2. ■

1 : $R_{1,1}$	2 : $R_{2,1}$	6 : $R_{3,1}$	7 : $R_{4,1}$	15 : $R_{5,1}$	16 : $R_{6,1}$	28 : $R_{7,1}$...
3 : $R_{1,2}$	5 : $R_{2,2}$	8 : $R_{3,2}$	14 : $R_{4,2}$	17 : $R_{5,2}$	27 : $R_{6,2}$	30 : $R_{7,2}$...
4 : $R_{1,3}$	9 : $R_{2,3}$	13 : $R_{3,3}$	18 : $R_{4,3}$	26 : $R_{5,3}$	31 : $R_{6,3}$	43 : $R_{7,3}$...
10 : $R_{1,4}$	12 : $R_{2,4}$	19 : $R_{3,4}$	25 : $R_{4,4}$	32 : $R_{5,4}$	42 : $R_{6,4}$	49 : $R_{7,4}$...
11 : $R_{1,5}$	20 : $R_{2,5}$	24 : $R_{3,5}$	33 : $R_{4,5}$	41 : $R_{5,5}$	50 : $R_{6,5}$	62 : $R_{7,5}$...
21 : $R_{1,6}$	23 : $R_{2,6}$	34 : $R_{3,6}$	40 : $R_{4,6}$	51 : $R_{5,6}$	61 : $R_{6,6}$	72 : $R_{7,6}$...
22 : $R_{1,7}$	35 : $R_{2,7}$	39 : $R_{3,7}$	52 : $R_{4,7}$	60 : $R_{5,7}$	73 : $R_{6,7}$	85 : $R_{7,7}$...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

2 Applications of the Theorems

2.1 Countability of the Integers

Claim: \mathbb{Z} is a countable set.

Proof. Let $R = \mathbb{N}_0$, $S = \mathbb{N}_1$. R and S are countable axiomatically, being the naturals themselves. $\mathbb{Z} \equiv R \cup S(-1)$. This is countable via Theorem 1. ■

2.2 Countability of the Rationals

Claim: \mathbb{Q} is a countable set.

Proof. The rationals are basically just ordered pairs of integers and naturals. $R := \mathbb{Z}$, $S := \mathbb{N}_1$. $R \times S$ defines all of these pairs. R is countable via Section 2.1 and S is countable axiomatically. $R \times S$ is thus countable via Theorem 2. This argument and Theorem 2 derives from Cantor's argument for the rationals. ■

2.3 Countability of the Reals from Zero to One

Claim: $\{x \in \mathbb{R} : 0 \leq x < 1\}$ is a countable set.

Proof. Let R be some set. Each R_i can be defined to be the digits of i reversed with "0." at the beginning and infinite optional trailing zeros at the end, for any $i \in \mathbb{N}_0$. For example, $R_{246} = 0.642\bar{0}$ and $R_0 = 0.0\bar{0} = 0$. This set is countable because it was defined to be countable; each natural number corresponds to a single element. This set contains every real number in the range because every possible sequence of digits is in it. The sequence trends upwards, asymptotically approaching 1, though it fluctuates wildly along the way. ■

2.4 Countability of the Non-Negative Reals

Claim: $\mathbb{R}_{\geq 0}$ or equivalently $\{x : x \in \mathbb{R} \wedge x \geq 0\}$ is a countable set.

Proof. The set of real numbers from zero to one is countable via Section 2.3. Let R be the same set used in that proof. $\mathbb{R}^+ = \bigcup_{i=1}^{\aleph_0} S_i$ where $S_i := R[i - 1]$. This is countable via Theorem 6. The following table illustrates this. ■

0. $\bar{0}$	1. $\bar{0}$	2. $\bar{0}$	3. $\bar{0}$	4. $\bar{0}$	5. $\bar{0}$...
0.1 $\bar{0}$	1.1 $\bar{0}$	2.1 $\bar{0}$	3.1 $\bar{0}$	4.1 $\bar{0}$	5.1 $\bar{0}$...
0.2 $\bar{0}$	1.2 $\bar{0}$	2.2 $\bar{0}$	3.2 $\bar{0}$	4.2 $\bar{0}$	5.2 $\bar{0}$...
0.3 $\bar{0}$	1.3 $\bar{0}$	2.3 $\bar{0}$	3.3 $\bar{0}$	4.3 $\bar{0}$	5.3 $\bar{0}$...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	...
0.7 $\bar{0}$	1.7 $\bar{0}$	2.7 $\bar{0}$	3.7 $\bar{0}$	4.7 $\bar{0}$	5.7 $\bar{0}$...
0.8 $\bar{0}$	1.8 $\bar{0}$	2.8 $\bar{0}$	3.8 $\bar{0}$	4.8 $\bar{0}$	5.8 $\bar{0}$...
0.9 $\bar{0}$	1.9 $\bar{0}$	2.9 $\bar{0}$	3.9 $\bar{0}$	4.9 $\bar{0}$	5.9 $\bar{0}$...
0.01 $\bar{0}$	1.01 $\bar{0}$	2.01 $\bar{0}$	3.01 $\bar{0}$	4.01 $\bar{0}$	5.01 $\bar{0}$...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

2.5 Countability of the Reals

Claim: \mathbb{R} is a countable set which implies $|\mathcal{P}(\mathbb{N})| \equiv |\mathbb{N}|$.

Proof. $\mathbb{R}_{\geq 0}$ is countable via Section 2.4. $\mathbb{R}_{\geq 0}(1, -1) \equiv \mathbb{R}$. This is countable via Theorem 4. This conclusion can be further reinforced by the algorithmic methods later on. For each real number x , there exists at least 1 sequence of elements in R , each element with a higher index than the last, where the limit of the sequence equals x . This is because every sequence of digits is contained by R . ■

2.6 Miscellaneous Number Classes

2.6.1 Algebraic and Transcendental Real Numbers

Claim: \mathbb{A} and \mathbb{T} are countable sets.

Proof. \mathbb{R} is a countable set via Section 2.5. Since the algebraic reals and transcendental reals are both subsets of the reals, they are countable via Theorem 3. They are also countable over the complex numbers using Section 2.7 and the same logic. ■

2.6.2 Imaginary Numbers

Claim: \mathbb{I} is a countable set.

Proof. \mathbb{R} is a countable set via Section 2.5. $\mathbb{R}(\sqrt{-1}) \equiv \mathbb{I}$. This is countable via Theorem 1, or more precisely $R(\sqrt{-1}) \cup \emptyset$ is countable. The union of any set and the null set is itself. ■

2.7 Countability of the Complex Numbers

Claim: \mathbb{C} is a countable set.

Proof. \mathbb{R} is a countable set via Section 2.5. $\mathbb{R}(1, \sqrt{-1}) \equiv \mathbb{C}$. This is countable via Theorem 4. The identity used here stems from the rectangular form of complex numbers. ■

3 Addressing Cantor's Diagonal Argument

According to Georg Cantor in 1891, If someone is trying to list all the real numbers, they can always find a number that is not in the list using his “Diagonal Argument”. This argument is basically the same as the following, though for reals instead of naturals: Suppose someone is trying to make a set S of every natural number. They first add zero to the set and the set is $\{0\}$, then they could say, “one isn’t in the set.” When they add one and have $\{0, 1\}$, they can say “two isn’t in the set,” “then three isn’t in the set,” “four isn’t in the set,” et cetera. No matter how many natural numbers they add, they can always find one not in it; $\max(S) + 1$. Using Cantor’s same logic, this seems to be saying that there is not a bijection between the natural numbers and themselves, which is clearly wrong. Every set is bijective onto itself via the identity function. $f(x) := x$. $f : X \mapsto X$.

4 Enumerating the Real Numbers Algorithmically

The following Node JS code prints out real numbers to the console delimited by a comma followed by a space. With infinite time and memory, it will have printed every real number. The only problems are that it prints out both 0 and -0 , and the functions return strings. Both of these are easily resolvable though. There is a reference to this source code and the C version in Section 6. The subscripts are just so it looks nicer. If `process.stdout.write` is replaced with `console.log`, then it will work in vanilla JavaScript in versions beginning with ECMAScript 6, although `console.log` adds a trailing newline character at the end of each call, which isn't ideal for printing a large quantity of small strings each using an individual function call. The complexity of finding the n th real number with this method is less than or equal to $\mathcal{O}(n \log^2 n)$, but it is probably less, since after over 16 million iterations on my machine, it hadn't slowed down noticeably.

```
function isqrt(n) { //  $\lfloor \sqrt{n} \rfloor$  for non-negative big integers n.
  if (n < 2) return n;
  var x0, x1 = n / 2n;

  do
    x0 = x1,
    x1 = x0 + n / x0 >> 1n; // Newton's method for  $f(x) = x^2 + a$ 
  while ( x1 < x0 );

  return x0;
}
// iterate over the natural numbers (up to  $0.5 \cdot 2^{2^{30}}$ )
for (var runningIndex = 0n ;;) {
  let positive = true, currentIndex = runningIndex++;

  // if the current index is odd, return a negation the previous index's value
  if (currentIndex % 2n) positive = false, currentIndex--;
  currentIndex /= 2n; // divide index by 2 so all integers can be reached

  const c = isqrt(1n + 8n*currentIndex) // intermediate value
  // the index the next comments refer to
  , u = (c + c % 2n) / 2n - 1n
  // the largest triangle number with an integer index that is less ...
  // than or equal to the current index. but subtracted from the current index
  , k = currentIndex - u * (u + 1n) / 2n;

  // generate real number from indices k and u-k and print it
  process.stdout.write(`${positive ? "" : "-"}${k}.` +
    `${u - k}`.split("").reverse().join("") + ", "
  );
}
```

5 Further Algorithms and Conclusion

The axioms of Zermelo-Fraenkel Set theory required to prove the countability of the continuum are as follows: Axiom Schema of Specification, Axiom of Union, Axiom of Replacement, Axiom of Infinity, and Axiom of Choice. Without the Axiom of Choice, theorems 2, 4, and 6 don't work, which are all used, although the algorithm is separate from set theory and uses no axioms. The algorithm from Section 4 uses string manipulation to reverse the numbers and find their lengths. The following equations can do the same thing without it.

$$\text{length } n := 1 + \lfloor \log_{10} n \rfloor = \lceil \log_{10}(n + 1) \rceil \quad (3)$$

$$\text{reverse } n := \sum_{k=0}^{\text{length } n} \left(\left\lfloor \frac{n}{10^k} \right\rfloor \bmod 10 \right) 10^{\text{length } n - k - 1} \quad (4)$$

$$R_{i,j} := p \left(i + \frac{\text{reverse } j}{10^{\text{length } j}} \right) \quad (5)$$

Where i is k , p is $(\text{positive} ? 1 : -1)$, and j is $(u - k)$. length and reverse are only defined for positive integer inputs. An interesting formula used in the algorithm in Section 4 is for the greatest triangle number $t = T(k)$ less than or equal to a natural number n with a natural number index k .

$$T(n) := \frac{n^2 + n}{2} = \sum_{j=1}^n j \quad (6)$$

$$\text{sgn } x := \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \quad (7)$$

$$f(n) := \frac{\lfloor \sqrt{8n+1} \rfloor + \lfloor \sqrt{8n+1} \rfloor \bmod 2}{2} - 1 \quad (8)$$

$$t = T \circ f(n) \quad (9)$$

Because of the bijection, there is also an inverse of Section 4's algorithm.

```
function inverse(string) { // assume valid input
  const match = /^(?)(\d+)\.(\d)$/.exec(string)
  , i = BigInt(match[2]) // the x coordinate index, or the integer part
  , j = BigInt(match[3]); // the y coordinate index, or the decimal part
  return i*(i+1n) + (2n*i+j)*(j+1n) + BigInt(match[1] === "-");
}
```

Conclusion: Since there exists a bijection between the natural numbers and the Real numbers, there is no set with a cardinality strictly or loosely in between the naturals and reals (CH), because they are secretly the same. This also implies that $2^{\aleph_0} = \aleph_0$ because $|\mathbb{R}| \equiv |\mathcal{P}(\mathbb{N}_0)|$. This makes sense because $2^\infty = \infty$ as well. There may not be different sizes of infinities after all. Also if one takes any sequence of natural numbers and concatenates them, it will always create a new natural number, meaning every element in $\mathcal{P}(\mathbb{N})$ corresponds to a natural number. It makes sense why CH is independent of ZFC and can't be disproved within it, because it is true either way.

6 References

- https://en.wikipedia.org/wiki/Cantor's_diagonal_argument
Wikipedia page with elaboration on Section 3
- <https://www.digizeitschriften.de/dms/img/?PID=GDZPPN002113910physid=phys84navi>
Georg Cantor's 1891 article with the diagonal argument. Same source as on wikipedia.
- https://en.wikipedia.org/wiki/Continuum_hypothesis
Wikipedia page with elaboration on Section 2.5
- <https://www.github.com/drizzt536/files/tree/main/TeX/continuum>
The files for the most recent public version of this pdf and the \LaTeX code
- <https://raw.githubusercontent.com/drizzt536/files/main/JavaScript/continuum.js>
The raw JavaScript source code for the Section 4
- <https://raw.githubusercontent.com/drizzt536/files/main/C/continuum.c>
The raw C source code for the Section 4

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