The Gamma Function, Its Inverse, and Its Relationship with Trigonometry

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1 Definitions

$$0 \in \mathbb{N} \quad \text{(1a)} \qquad \mathbb{N}_k := \{x + k : x \in \mathbb{N}\}\,\forall\, k \quad \text{(1b)} \tag{1}$$

$$\tau := 2\pi \tag{2}$$

$$\operatorname{sgn}(x) := \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases} = \frac{x}{|x|}\ \forall x \tag{3}$$

$$a \bmod_n b := a \bmod b + n \tag{4}$$

$$\exp(x) := \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} = e^x\ \forall x \tag{5}$$

$$\Gamma(x) := \int_0^\infty \frac{t^{x-1}}{\exp t}\mathrm{d}t = (x-1)!\ \forall x > 0 \tag{6}$$

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x} \forall x > 1 \tag{7}$$

$$\eta(x) := \Gamma^{-1}(x) \quad \text{(8a)} \qquad \eta(\,\Gamma(x)\,) = \Gamma(\,\eta(x)\,) = x \quad \text{(8b)} \tag{8}$$

(8a) is the compositional inverse rather than the fractional inverse. The set of all natural numbers, $\mathbb{N}$, is defined in part by (1a) and (1b). (7) describes the Riemann Zeta Function. x is assumed to be a real number everywhere for simplicity.

2 Riemann Zeta Function

claim: $\zeta(x) = \dfrac{\zeta(1-x)\tau^x \sec\left(\dfrac{\pi x}{2}\right)}{2\,\Gamma(x)}$

This proof is important for section 3.

Proof:

know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\dfrac{\pi x}{2}\right)\Gamma(1-x)\zeta(1-x) \quad \text{(reference 1)}$

$$\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right)\Gamma\left(1-x\right)\zeta\left(1-x\right) \tag{9}$$

$$\zeta(x) = 2 \cdot 2^{x-1} \pi^{x-1} \sin\left(\frac{\pi x}{2}\right)\Gamma(1-x)\zeta(1-x) \tag{10}$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right)\Gamma(1-x)\zeta(1-x) \tag{11}$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \cos\left(\frac{\pi}{2}(1-x)\right)\Gamma(1-x)\zeta(1-x) \tag{12}$$

$$\zeta(x) = \frac{2}{\tau^{1-x}} \cos\left(\frac{\pi}{2}(1-x)\right)\Gamma(1-x)\zeta(1-x) \tag{13}$$

$$\zeta(1-x) = \frac{2}{\tau^{x}} \cos\left(\frac{\pi x}{2}\right)\Gamma(x)\zeta(x) \tag{14}$$

$$\frac{\zeta(1-x)\tau^x}{2\cos\left(\frac{\pi x}{2}\right)\Gamma(x)} = \zeta(x) \tag{15}$$

$$\zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)} \tag{16}$$

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3 Cosecant

claim: $\csc x$ can be written in terms of $\Gamma(x)$
Proof:
know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x)$ (reference 1)

know: $\zeta(x) = \dfrac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)}$ (equation 16)

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)} \tag{17}$$

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{2\,\Gamma(x)} \tag{18}$$

$$4 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{\Gamma(x)} \tag{19}$$

$$2 \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\pi}{\Gamma(x)} \tag{20}$$

$$\Gamma(1-x)\Gamma(x) \sin \pi x = \pi \tag{21}$$

$$\Gamma(1-x)\Gamma(x) = \pi \csc \pi x \tag{22}$$

$$\csc x = \frac{1}{\pi} \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) \tag{23}$$

$$\sin \pi x \neq 0 \Longrightarrow \zeta(1-x) \neq 0 \tag{24}$$

Therefore, due to (24), the domains also match.

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4 Trig Functions In Terms of Cosecant

claim: The rest of the trig functions can be written in terms of cosecant
Proof:

$$\sin x = \frac{1}{\csc x} \tag{25}$$

$$\cos x = \frac{1}{\csc\left(\frac{\pi}{2} - x\right)} \tag{26}$$

$$\tan x = \frac{\csc\left(\frac{\pi}{2} - x\right)}{\csc x} \tag{27}$$

$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \tag{28}$$

$$\cot x = \frac{\csc x}{\csc\left(\frac{\pi}{2} - x\right)} \tag{29}$$

$$\sinh x = -\frac{i}{\csc x} \tag{30}$$

$$\cosh x = \frac{1}{\csc\left(\frac{\pi}{2} - ix\right)} \tag{31}$$

$$\tanh x = -\frac{i \csc\left(\frac{\pi}{2} - ix\right)}{\csc ix} \tag{32}$$

$$\operatorname{csch} x = i \csc ix \tag{33}$$

$$\operatorname{sech} x = \csc\left(\frac{\pi}{2} - ix\right) \tag{34}$$

$$\coth x = \frac{i \csc ix}{\csc\left(\frac{\pi}{2} - ix\right)} \tag{35}$$

$\csc x = \csc x$ by the reflexive property.

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5 Trig Functions In Terms of Gamma

These equations are just the previous ones with Gamma plugged in for csc.

$$\sin x = \frac{\pi}{\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right)} \tag{36}$$

$$\cos x = \frac{\pi}{\Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right)} \tag{37}$$

$$\tan x = \frac{\Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right)}{\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right)} \tag{38}$$

$$\csc x = \frac{1}{\pi}\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right) \tag{39}$$

$$\sec x = \frac{1}{\pi}\Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right) \tag{40}$$

$$\cot x = \frac{\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right)} \tag{41}$$

$$\sinh x = -\pi i \Big/ \Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right) \tag{42}$$

$$\cosh x = \pi \Big/ \Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right) \tag{43}$$

$$\tanh x = -i\,\Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right) \Big/ \Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right) \tag{44}$$

$$\operatorname{csch} x = \frac{i}{\pi}\Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right) \tag{45}$$

$$\operatorname{sech} x = \frac{1}{\pi}\Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right) \tag{46}$$

$$\coth x = i\,\Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right) \Big/ \Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right) \tag{47}$$

where "/" means divide everything on the left by everything on the right.

6 Inverse Gamma Function

claim: $\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi}\Gamma(1 - \eta(x))$
Proof:
know: $\Gamma(x)\,\Gamma(1 - x) = \pi \csc \pi x$ (equation 22)

$$\pi \csc \pi x = \Gamma(x)\,\Gamma(1 - x) \tag{48}$$

$$\pi \csc \pi\eta(x) = x\,\Gamma(1 - \eta(x)) \tag{49}$$

$$\csc \pi\eta(x) = \frac{x}{\pi}\Gamma(1 - \eta(x)) \tag{50}$$

$$\pi\eta(x) = \csc^{-1} \frac{x}{\pi}\Gamma(1 - \eta(x)) \tag{51}$$

$$\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi}\Gamma(1 - \eta(x)) \tag{52}$$

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There are 2 conditions to where this $\eta(x)$ is defined.
$\eta(x)$ is **not** defined (over the real numbers) such that the following is true:

$$|x\,\Gamma(1 - \eta(x))| < \pi \quad \text{or} \quad |x| < 3.547373963... \;\; (V) \tag{53}$$

V is probably transcendental. $\eta(x)$ is guaranteed to be defined if equation (53) is false and x is negative. $\eta(x)$ is defined for all integers and in small fields around each integer. The area defined around each integer gets smaller as x increases, though it is unclear why without further inspection. If $\eta(x)$ is being recursed and $\frac{x}{\pi}$ is changed to $\frac{x^n}{\pi}$ in the innermost function, for some $n \in \mathbb{N}_2$, either the fields get larger, or there is a bug in Desmos' graphing calculations. See reference 2 for more information.
Going a different way after (49) gives the following equation:

$$\eta(x) = 1 - \eta\left(\frac{\pi}{x} \csc \pi\eta(x)\right) \implies \eta(x) = \eta(\pi \csc(\pi \csc \pi x) \sin \pi x) \tag{54}$$

This equation seems to be less useful because it has large derivatives everywhere when absolute valued, which gets more severe as it is recursed.

7 Miscellaneous

(40) implies the following formula:

$$\Gamma\left(x+\frac{1}{2}\right)=\frac{\pi\sec\pi x}{\Gamma\left(\frac{1}{2}-x\right)} \tag{55}$$

replacing x with $x+z$ and adjusting from gamma to factorial gives:

$$\left(x+\frac{1}{2}+z\right)!=\left[2\left(z \bmod 2\right)-1\right]\frac{\pi\sec\pi x}{\left(-\frac{3}{2}-x-z\right)!}\forall z\in\mathbb{Z} \tag{56}$$

This only simplifies things if z is an integer because $\sec(x+\tau)=\sec x$, and $\sec(x+\pi)=-\sec x$. rewriting (22) from gamma to factorial gives:

$$(x-1)!(-x)!=\pi\csc\pi x \tag{57}$$

$$(-x)!=\frac{\pi x\csc\pi x}{x!} \tag{58}$$

$$\left(x+\frac{1}{c}\right)!=\frac{\pi\csc\frac{cx+1}{c}\pi}{\left(-\frac{c+1}{c}-x\right)!}\forall c\in\mathbb{R} \tag{59}$$

$$x!=\left(x \bmod_n 1\right)!\left[\prod_{k=\operatorname{sgn}(|n-x|+n-x)}^{|\lfloor x-n\rfloor|+\operatorname{sgn}(n-x-|n-x|)}\left[x+k\operatorname{sgn}\left(n-x\right)\right]\right]^{\operatorname{sgn}(x-n)} \tag{60}$$

(60) is useful where $x!$ is non self-dependantly defined for $x\in[n,n+1]$. equation relating to factorials:

$$\lim_{x\to\infty}\int_0^x\int_0^x\exp\left[\frac{1}{2}\ln\frac{4t}{u}-t-u\right]\mathrm{d}^2tu=\pi \tag{61}$$

claim: $\sin(x) = -\sin(-x)$:
Proof:

$$(57)\ x \mapsto -x \Longrightarrow (-x)! = \frac{-\pi x \csc(-\pi x)}{x!} \tag{62}$$

$$(-x)! = \frac{\pi x \csc \pi x}{x!} \qquad (58)$$

$$\frac{\pi x \csc \pi x}{x!} = \frac{-\pi x \csc(-\pi x)}{x!} \Longrightarrow \csc x = -\csc(-x) \Longrightarrow \sin(x) = -\sin(-x) \tag{63}$$

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claim: $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$
Proof:

$$x! = \frac{(x+1)!}{x+1} \Longrightarrow \left(-\frac{1}{2}\right)! = 2\left(\frac{1}{2}\right)! \tag{64}$$

$$(-x)! = \frac{\pi x \csc \pi x}{x!} \Longrightarrow \left(-\frac{1}{2}\right)! = \frac{\pi}{2\left(\frac{1}{2}\right)!} \tag{65}$$

The formula for (65) is just (58). (64) and (65) imply the following:

$$2\left(\frac{1}{2}\right)! = \frac{\pi}{2\left(\frac{1}{2}\right)!} \Longrightarrow 4\left(\frac{1}{2}\right)!^2 = \pi \Longrightarrow \left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2} \tag{66}$$

Using this strategy only works if $(c_1 - x) = (c_2 + x)$ for some integers c_1 and c_2. This limits it to $(z + \frac{1}{2})$ for integers z. Also, you can only apply (65) an odd number of times or the factorials will cancel and leave a trigonometric equation. If you apply (65) twice, you get back to where you started because everything new cancels.
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$$\left(\frac{1}{2} + n\right)! = \frac{\sqrt{\pi}}{2} \prod_{k = \operatorname{sgn}(n+|n|)}^{|n| - \operatorname{sgn}(n+|n|) + \operatorname{sgn} n} \left(\frac{1}{2} + k \operatorname{sgn} n\right)^{\operatorname{sgn} n} \quad \forall n \in \mathbb{Z} \tag{67}$$

8 References

- https://en.wikipedia.org/wiki/Riemann_zeta_function#Riemann's_functional_equation
Link for equation 9

- https://www.desmos.com/calculator/t51xnveadf
Extra information for section 6, made by me

- https://www.github.com/drizzt536/files/tree/main/TeX/gamma
The files for the most recent version of this pdf and the LaTeX code