Version2, with notes & errata

The Gamma Function, Its Inverse, And Its Relationship With Trigonometry

Daniel E. Janusch

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1 Definitions

$$0 \in \mathbb{N} \tag{1}$$

$$\mathbb{N}_k := \{x + k : x \in \mathbb{N}\} \quad k \in \mathbb{N} \tag{2}$$

$$\tau := 2\pi \tag{3}$$

$$\Gamma(x) := \int_0^\infty \frac{t^{x-1} dt}{\exp(t)} = (x-1)!\ \forall x > 0 \tag{4}$$

$$\zeta(x) := \sum_{n=\cancel{0} \leftarrow 1}^{\infty} \frac{1}{n^x} \forall x > 1 \tag{5}$$

$$\eta(x) := \Gamma^{-1}(x) \tag{6}$$

$$\Downarrow$$

$$\eta(\,\Gamma(x)\,) = \Gamma(\,\eta(x)\,) = x \tag{7}$$

(6) is the compositional inverse rather than the fractional inverse.
The set of all natural numbers, $\mathbb{N}$, is defined by (1) and (2).
(5) describes the Riemann Zeta Function.
~~"∋" means "such that".~~ unused

2 Riemann Zeta Function

claim: $\zeta(x) = \dfrac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)}$

This proof is important for part 3.

Proof:

know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x)$ (using analytic continuation)

$$\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma\left(1-x\right) \zeta\left(1-x\right) \tag{8}$$

$$\zeta(x) = 2 \cdot 2^{x-1} \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) \tag{9}$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) \tag{10}$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x)\zeta(1-x) \tag{11}$$

$$\zeta(x) = \frac{2}{\tau^{1-x}} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x)\zeta(1-x) \tag{12}$$

$$\zeta(1-x) = \frac{2}{\tau^{x}} \cos\left(\frac{\pi x}{2}\right) \Gamma(x)\zeta(x) \tag{13}$$

$$\frac{\zeta(1-x)\tau^x}{2\cos\left(\frac{\pi x}{2}\right)\Gamma(x)} = \zeta(x) \tag{14}$$

$$\zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)} \tag{15}$$

■

3 Cosecant

claim: $\csc x$ can be written in terms of $\Gamma(x)$
Proof:
know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x)$

know: $\zeta(x) = \dfrac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)}$ (eq. 15)

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)} \tag{16}$$

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{2\,\Gamma(x)} \tag{17}$$

$$4 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{\Gamma(x)} \tag{18}$$

$$2 \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\pi}{\Gamma(x)} \tag{19}$$

$$\Gamma(1-x)\Gamma(x) \sin \pi x = \pi \tag{20}$$

$$\Gamma(1-x)\Gamma(x) = \pi \csc \pi x \tag{21}$$

$$\csc x = \frac{1}{\pi}\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1 - \frac{x}{\pi}\right) \tag{22}$$

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$\sin \pi x \neq 0 \Longrightarrow \zeta(1-x) \neq 0$

Therefore the domains also match.

4 The Remaining Functions

claim: The rest of the trig functions can be written in terms of cosecant
Proof:

$$\sin x = \frac{1}{\csc x} \tag{23}$$

$$\cos x = \frac{1}{\csc\left(\frac{\pi}{2} - x\right)} \tag{24}$$

$$\tan x = \frac{\csc\left(\frac{\pi}{2} - x\right)}{\csc x} \tag{25}$$

$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \tag{26}$$

$$\cot x = \frac{\csc x}{\csc\left(\frac{\pi}{2} - x\right)} \tag{27}$$

$$\sinh x = -\frac{i}{\csc x} \tag{28}$$

$$\cosh x = \frac{1}{\csc\left(\frac{\pi}{2} - ix\right)} \tag{29}$$

$$\tanh x = -\frac{i\csc\left(\frac{\pi}{2} - ix\right)}{\csc ix} \tag{30}$$

$$\operatorname{csch} x = i\csc ix \tag{31}$$

$$\operatorname{sech} x = \csc\left(\frac{\pi}{2} - ix\right) \tag{32}$$

$$\coth x = \frac{i\csc ix}{\csc\left(\frac{\pi}{2} - ix\right)} \tag{33}$$

Therefore, all the trigonometric functions can be written in terms of $\Gamma(x)$.
■ based on (22) and (33)

5 Inverse Gamma Function

claim: $\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x))$
Proof:
know: $\Gamma(x)\Gamma(1-x) = \pi \csc \pi x$ (equation 21)

$$\pi \csc \pi x = \Gamma(x)\Gamma(1-x) \tag{34}$$

$$\pi \csc \pi\eta(x) = x\Gamma(1-\eta(x)) \tag{35}$$

$$\csc \pi\eta(x) = \frac{x}{\pi}\Gamma(1-\eta(x)) \tag{36}$$

$$\pi\eta(x) = \csc^{-1} \frac{x}{\pi}\Gamma(1-\eta(x)) \tag{37}$$

$$\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi}\Gamma(1-\eta(x)) \tag{38}$$

■

There are 2 conditions to where this $\eta(x)$ is defined.
defined means over the reals
$\eta(x)$ is not defined over the real numbers such that the following is true:

$$|x\,\Gamma(1-\eta(x))| < \pi \quad \text{or} \quad |x| < \sim \boxed{3.547373963...} \tag{39}$$

I assume that $\boxed{\text{this value}}$ is transcendental or at least algebraic.
$\eta(x)$ is guaranteed to be defined if (39) is false and x is negative.
defined around each integer
$\eta(x)$ is defined for all integers and around the integers.
The area ~~around the integer that is defined~~ gets smaller as x increases, though I do not know why this is the case.
If you are recursing $\eta(x)$ and you change the $\frac{x}{\pi}$ to $\frac{x^n}{\pi}$ in the innermost function, for some $n \in \mathbb{N}_2$, it either gets defined for more values, or there is a bug in Desmos' graphing calculations.
These extra values are just expanded ranges around the positive integers.
It doesn't change the rule described in (39), however.
See reference 2 for more information.
going a different way after (35) gives the following equation:

$$\eta(x) = 1 - \eta\left(\frac{\pi}{x} \csc \pi\eta(x)\right) \tag{40}$$

6 References

- https://en.wikipedia.org/wiki/Riemann_zeta_function (equation 8)
- https://www.desmos.com/calculator/ieu6ebnfn8 (for part 5, made by me)
- https://www.github.com/drizzt536/files/tree/main/TeX/gamma/ (me)

this file and
the LaTex
code