The Gamma Function, Its Inverse, and Its Relationship With Trigonometry

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1 Definitions

$$0 \in \mathbb{N} \tag{1}$$

$$\mathbb{N}_k := \{x + k : x \in \mathbb{N}\} \, \forall \, k \tag{2}$$

$$\tau := 2\pi \tag{3}$$

$$\Gamma(x) := \int_0^\infty \frac{t^{x-1}}{\exp t} \mathrm{d}t = (x-1)! \; \forall x > 0 \tag{4}$$

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x} \forall x > 1 \tag{5}$$

$$\eta(x) := \Gamma^{-1}(x) \tag{6}$$

$$\eta(\,\Gamma(x)\,) = \Gamma(\,\eta(x)\,) = x \tag{7}$$

(6) is the compositional inverse rather than the fractional inverse.
The set of all natural numbers, $\mathbb{N}$, is defined by equations (1) and (2).
(5) describes the Riemann Zeta Function.

2 Riemann Zeta Function

claim: $\zeta(x) = \dfrac{\zeta(1-x)\tau^x \sec\left(\dfrac{\pi x}{2}\right)}{2\,\Gamma(x)}$

This proof is important for part 3.

Proof:

know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\dfrac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) \quad$ (reference 1)

$$\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma\left(1-x\right) \zeta\left(1-x\right) \tag{8}$$

$$\zeta(x) = 2 \cdot 2^{x-1} \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) \tag{9}$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) \tag{10}$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x)\zeta(1-x) \tag{11}$$

$$\zeta(x) = \frac{2}{\tau^{1-x}} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x)\zeta(1-x) \tag{12}$$

$$\zeta(1-x) = \frac{2}{\tau^x} \cos\left(\frac{\pi x}{2}\right) \Gamma(x)\zeta(x) \tag{13}$$

$$\frac{\zeta(1-x)\tau^x}{2\cos\left(\dfrac{\pi x}{2}\right)\Gamma(x)} = \zeta(x) \tag{14}$$

$$\zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\dfrac{\pi x}{2}\right)}{2\,\Gamma(x)} \tag{15}$$

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3 Cosecant

claim: $\csc x$ can be written in terms of $\Gamma(x)$
Proof:
know: $\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x)$ (reference 1)

know: $\zeta(x) = \dfrac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)}$ (equation 15)

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\,\Gamma(x)} \tag{16}$$

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{2\,\Gamma(x)} \tag{17}$$

$$4 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{\Gamma(x)} \tag{18}$$

$$2 \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\pi}{\Gamma(x)} \tag{19}$$

$$\Gamma(1-x)\Gamma(x) \sin \pi x = \pi \tag{20}$$

$$\Gamma(1-x)\Gamma(x) = \pi \csc \pi x \tag{21}$$

$$\csc x = \frac{1}{\pi}\Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right) \tag{22}$$

$$\sin \pi x \neq 0 \Longrightarrow \zeta(1-x) \neq 0 \tag{23}$$

Therefore, the domains also match.

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4 Trig Functions In Terms of Cosecant

claim: The rest of the trig functions can be written in terms of cosecant
Proof:

$$\sin x = \frac{1}{\csc x} \tag{24}$$

$$\cos x = \frac{1}{\csc\left(\frac{\pi}{2} - x\right)} \tag{25}$$

$$\tan x = \frac{\csc\left(\frac{\pi}{2} - x\right)}{\csc x} \tag{26}$$

$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \tag{27}$$

$$\cot x = \frac{\csc x}{\csc\left(\frac{\pi}{2} - x\right)} \tag{28}$$

$$\sinh x = -\frac{i}{\csc x} \tag{29}$$

$$\cosh x = \frac{1}{\csc\left(\frac{\pi}{2} - ix\right)} \tag{30}$$

$$\tanh x = -\frac{i \csc\left(\frac{\pi}{2} - ix\right)}{\csc ix} \tag{31}$$

$$\operatorname{csch} x = i \csc ix \tag{32}$$

$$\operatorname{sech} x = \csc\left(\frac{\pi}{2} - ix\right) \tag{33}$$

$$\coth x = \frac{i \csc ix}{\csc\left(\frac{\pi}{2} - ix\right)} \tag{34}$$

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5 Trig Functions In Terms of Gamma

These equations are just the previous ones with Gamma plugged in for csc.

$$\sin x = \pi \Big/ \Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right) \tag{35}$$

$$\cos x = \pi \Big/ \Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right) \tag{36}$$

$$\tan x = \Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right) \Big/ \Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right) \tag{37}$$

$$\csc x = \Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right) \Big/ \pi \tag{38}$$

$$\sec x = \Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right) \Big/ \pi \tag{39}$$

$$\cot x = \Gamma\left(\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right) \Big/ \Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right) \tag{40}$$

$$\sinh x = -\pi i \Big/ \Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right) \tag{41}$$

$$\cosh x = \pi \Big/ \Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right) \tag{42}$$

$$\tanh x = -i\,\Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right) \Big/ \Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right) \tag{43}$$

$$\operatorname{csch} x = i\,\Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right) \Big/ \pi \tag{44}$$

$$\operatorname{sech} x = \Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right) \Big/ \pi \tag{45}$$

$$\coth x = i\,\Gamma\left(\frac{ix}{\pi}\right)\Gamma\left(1-\frac{ix}{\pi}\right) \Big/ \Gamma\left(\frac{1}{2}-\frac{ix}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{\pi}\right) \tag{46}$$

where "/" means divide everything on the left by everything on the right.

6 Inverse Gamma Function

claim: $\eta(x) = \frac{1}{\pi}\csc^{-1}\frac{x}{\pi}\Gamma(1-\eta(x))$
Proof:
know: $\Gamma(x)\,\Gamma(1-x) = \pi\csc\pi x$ (equation 21)

$$\pi\csc\pi x = \Gamma(x)\,\Gamma(1-x) \tag{47}$$

$$\pi\csc\pi\eta(x) = x\,\Gamma(1-\eta(x)) \tag{48}$$

$$\csc\pi\eta(x) = \frac{x}{\pi}\Gamma(1-\eta(x)) \tag{49}$$

$$\pi\eta(x) = \csc^{-1}\frac{x}{\pi}\Gamma(1-\eta(x)) \tag{50}$$

$$\eta(x) = \frac{1}{\pi}\csc^{-1}\frac{x}{\pi}\Gamma(1-\eta(x)) \tag{51}$$

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There are 2 conditions to where this $\eta(x)$ is defined.
$\eta(x)$ is **not** defined (over the real numbers) such that the following is true:

$$|x\,\Gamma(1-\eta(x))| < \pi \quad \text{or} \quad |x| < 3.547373963...\;\; (V) \tag{52}$$

V is probably transcendental. $\eta(x)$ is guaranteed to be defined if equation (52) is false and x is negative. $\eta(x)$ is defined for all integers and in small fields around each integer. The area defined around each integer gets smaller as x increases, though it is unclear why at this time. If $\eta(x)$ is being recursed and $\frac{x}{\pi}$ is changed to $\frac{x^n}{\pi}$ in the innermost function, for some $n \in \mathbb{N}_2$, either the fields get larger, or there is a bug in Desmos' graphing calculations. See reference 2 for more information. going a different way after (48) gives the following equation:

$$\eta(x) = 1-\eta\left(\frac{\pi}{x}\csc\pi\eta(x)\right) \Longrightarrow \eta(x) = \eta(\pi\csc(\pi\csc\pi x)\sin\pi x) \tag{53}$$

This equation seems to be less useful because it has large absolute derivatives which gets more severe as it is recursed.

7 References

- https://en.wikipedia.org/wiki/Riemann_zeta_function#Riemann's_functional_equation
  Link for equation 8

- https://www.desmos.com/calculator/t51xnveadf
  Extra information for part 6, made by me

- https://www.github.com/drizzt536/files/tree/main/TeX/gamma
  The files for the most recent version of this pdf and the LaTeX code