

# The Gamma Function, Its Inverse, and Its Relationship with Trigonometry

Daniel E. Janusch

December 15, 2022

## 1 Definitions

$$0 \in \mathbb{N} \quad (1a) \quad \tau := 2\pi \quad (1b) \quad (1)$$

$$\mathbb{N}_k := \{x + k : x \in \mathbb{N}\} \forall k \quad (2)$$

$$\text{sgn}(x) := \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \quad (3)$$

$$a \bmod_n b := a \bmod b + n \quad (4)$$

$$\exp(x) := \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{x^k}{k!} = e^x \quad \forall x \quad (5)$$

$$\Gamma(x) := \int_0^\infty \frac{t^{x-1}}{\exp t} dt = (x-1)! \quad \forall x > 0 \quad (6)$$

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x} \quad \forall x > 1 \quad (7)$$

$$\eta(x) := \Gamma^{-1}(x) \quad (8a) \quad \eta(\Gamma(x)) = \Gamma(\eta(x)) = x \quad (8b) \quad (8)$$

(8a) is the compositional inverse rather than the fractional inverse. The set of all natural numbers,  $\mathbb{N}$ , is defined in part by (1a) and (2). (7) describes the Riemann Zeta Function.  $x$  is assumed to be a real number for simplicity, but It probably doesn't matter.

## 2 Riemann Zeta Function

claim:  $\zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\Gamma(x)}$

This proof is important for part 3.

Proof:

know:  $\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x)$  (reference 1)

$$\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (9)$$

$$\zeta(x) = 2 \cdot 2^{x-1} \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (10)$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) \quad (11)$$

$$\zeta(x) = 2 \cdot \tau^{x-1} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x) \zeta(1-x) \quad (12)$$

$$\zeta(x) = \frac{2}{\tau^{1-x}} \cos\left(\frac{\pi}{2}(1-x)\right) \Gamma(1-x) \zeta(1-x) \quad (13)$$

$$\zeta(1-x) = \frac{2}{\tau^x} \cos\left(\frac{\pi x}{2}\right) \Gamma(x) \zeta(x) \quad (14)$$

$$\frac{\zeta(1-x)\tau^x}{2 \cos\left(\frac{\pi x}{2}\right) \Gamma(x)} = \zeta(x) \quad (15)$$

$$\zeta(x) = \frac{\zeta(1-x)\tau^x \sec\left(\frac{\pi x}{2}\right)}{2\Gamma(x)} \quad (16)$$

■

### 3 Cosecant

claim:  $\csc x$  can be written in terms of  $\Gamma(x)$

Proof:

know:  $\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x)$  (reference 1)

know:  $\zeta(x) = \frac{\zeta(1-x) \tau^x \sec\left(\frac{\pi x}{2}\right)}{2 \Gamma(x)}$  (equation 15)

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x) \zeta(1-x) = \frac{\zeta(1-x) \tau^x \sec\left(\frac{\pi x}{2}\right)}{2 \Gamma(x)} \quad (17)$$

$$2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{2 \Gamma(x)} \quad (18)$$

$$4 \cdot \tau^{x-1} \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\tau^x}{\Gamma(x)} \quad (19)$$

$$2 \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right) \Gamma(1-x) = \frac{\pi}{\Gamma(x)} \quad (20)$$

$$\Gamma(1-x) \Gamma(x) \sin \pi x = \pi \quad (21)$$

$$\Gamma(1-x) \Gamma(x) = \pi \csc \pi x \quad (22)$$

$$\csc x = \frac{1}{\pi} \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) \quad (23)$$

$$\sin \pi x \neq 0 \implies \zeta(1-x) \neq 0 \quad (24)$$

Therefore, due to (23), the domains also match.

■

## 4 Trig Functions In Terms of Cosecant

claim: The rest of the trig functions can be written in terms of cosecant

Proof:

$$\sin x = \frac{1}{\csc x} \quad (25)$$

$$\cos x = \frac{1}{\csc\left(\frac{\pi}{2} - x\right)} \quad (26)$$

$$\tan x = \frac{\csc\left(\frac{\pi}{2} - x\right)}{\csc x} \quad (27)$$

$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \quad (28)$$

$$\cot x = \frac{\csc x}{\csc\left(\frac{\pi}{2} - x\right)} \quad (29)$$

$$\sinh x = -\frac{i}{\csc x} \quad (30)$$

$$\cosh x = \frac{1}{\csc\left(\frac{\pi}{2} - ix\right)} \quad (31)$$

$$\tanh x = -\frac{i \csc\left(\frac{\pi}{2} - ix\right)}{\csc ix} \quad (32)$$

$$\operatorname{csch} x = i \csc ix \quad (33)$$

$$\operatorname{sech} x = \csc\left(\frac{\pi}{2} - ix\right) \quad (34)$$

$$\coth x = \frac{i \csc ix}{\csc\left(\frac{\pi}{2} - ix\right)} \quad (35)$$

■

## 5 Trig Functions In Terms of Gamma

These equations are just the previous ones with Gamma plugged in for csc.

$$\sin x = \pi / \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) \quad (36)$$

$$\cos x = \pi / \Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right) \quad (37)$$

$$\tan x = \Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right) / \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) \quad (38)$$

$$\csc x = \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) / \pi \quad (39)$$

$$\sec x = \Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right) / \pi \quad (40)$$

$$\cot x = \Gamma\left(\frac{x}{\pi}\right) \Gamma\left(1 - \frac{x}{\pi}\right) / \Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right) \quad (41)$$

$$\sinh x = -\pi i / \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) \quad (42)$$

$$\cosh x = \pi / \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) \quad (43)$$

$$\tanh x = -i \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) / \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) \quad (44)$$

$$\operatorname{csch} x = i \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) / \pi \quad (45)$$

$$\operatorname{sech} x = \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) / \pi \quad (46)$$

$$\coth x = i \Gamma\left(\frac{ix}{\pi}\right) \Gamma\left(1 - \frac{ix}{\pi}\right) / \Gamma\left(\frac{1}{2} - \frac{ix}{\pi}\right) \Gamma\left(\frac{1}{2} + \frac{ix}{\pi}\right) \quad (47)$$

where “/” means divide everything on the left by everything on the right.

## 6 Inverse Gamma Function

claim:  $\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x))$

Proof:

know:  $\Gamma(x) \Gamma(1 - x) = \pi \csc \pi x$  (equation 21)

$$\pi \csc \pi x = \Gamma(x) \Gamma(1 - x) \quad (48)$$

$$\pi \csc \pi \eta(x) = x \Gamma(1 - \eta(x)) \quad (49)$$

$$\csc \pi \eta(x) = \frac{x}{\pi} \Gamma(1 - \eta(x)) \quad (50)$$

$$\pi \eta(x) = \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x)) \quad (51)$$

$$\eta(x) = \frac{1}{\pi} \csc^{-1} \frac{x}{\pi} \Gamma(1 - \eta(x)) \quad (52)$$

■

There are 2 conditions to where this  $\eta(x)$  is defined.

$\eta(x)$  is **not** defined (over the real numbers) such that the following is true:

$$|x \Gamma(1 - \eta(x))| < \pi \quad \text{or} \quad |x| < 3.547373963... \quad (V) \quad (53)$$

$V$  is probably transcendental.  $\eta(x)$  is guaranteed to be defined if equation (52) is false and  $x$  is negative.  $\eta(x)$  is defined for all integers and in small fields around each integer. The area defined around each integer gets smaller as  $x$  increases, though it is unclear why at this time. If  $\eta(x)$  is being recursed and  $\frac{x}{\pi}$  is changed to  $\frac{x^n}{\pi}$  in the innermost function, for some  $n \in \mathbb{N}_2$ , either the fields get larger, or there is a bug in Desmos' graphing calculations. See reference 2 for more information. going a different way after (48) gives the following equation:

$$\eta(x) = 1 - \eta\left(\frac{\pi}{x} \csc \pi \eta(x)\right) \implies \eta(x) = \eta(\pi \csc(\pi \csc \pi x) \sin \pi x) \quad (54)$$

This equation seems to be less useful because it has large derivatives everywhere when absolute valued, which gets more severe as it is recursed.

## 7 Miscellaneous

(39) gives the following formulas after some manipulation:

$$\Gamma\left(x + \frac{1}{2}\right) = \frac{\pi \sec \pi x}{\Gamma\left(\frac{1}{2} - x\right)} \quad (55)$$

replacing  $x$  with  $x + z$  and adjusting from gamma to factorial gives:

$$\left(x + \frac{1}{2} + z\right)! = (2(z \bmod 2) - 1) \frac{\pi \sec \pi x}{\left(-\frac{3}{2} - x - z\right)!} \forall z \in \mathbb{Z} \quad (56)$$

This only simplifies things if  $z$  is an integer because  $\sec(x + \tau) = \sec x$ , and  $\sec(x + \pi) = -\sec x$ . rewriting (21) from gamma to factorial gives:

$$(x - 1)!(-x)! = \pi \csc \pi x \quad (57)$$

$$(-x)! = \frac{\pi x \csc \pi x}{x!} \quad (58)$$

$$\left(x + \frac{1}{c}\right) = \frac{-\pi \csc \frac{cx + 1}{c} \pi}{\left(-\frac{c + 1}{c} - x\right)!} \forall c \in \mathbb{R} \quad (59)$$

$$x! = (x \bmod_n 1) \left[ \prod_{k=\text{sgn}(|n-x|+n-x)}^{\lfloor |x-n| \rfloor + \text{sgn}(n-x-|n-x|)} (x + \text{sgn}(n-x)k) \right]^{\text{sgn}(x-n)} \quad (60)$$

(60) is useful where  $x!$  is non self-dependantly defined for  $x \in [n, n + 1]$ .

Proof sine is odd using gamma:

claim:  $\sin(x) = -\sin(-x)$ :

Proof:

$$(57) \quad x \mapsto -x \implies (-x)! = \frac{-\pi x \csc(-\pi x)}{x!} \quad (61)$$

$$(-x)! = \frac{\pi x \csc \pi x}{x!} \quad (57)$$

$$\frac{\pi x \csc \pi x}{x!} = \frac{-\pi x \csc(-\pi x)}{x!} \implies \csc x = -\csc(-x) \implies \sin(x) = -\sin(-x) \quad (62)$$

■

Proof for the value of  $\frac{1}{2}$  factorial

claim:  $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$

Proof:

$$x! = \frac{(x+1)!}{x+1} \implies \left(-\frac{1}{2}\right)! = 2 \left(\frac{1}{2}\right) \quad (63)$$

$$(-x)! = \frac{\pi x \csc \pi x}{x!} \implies \left(-\frac{1}{2}\right)! = \frac{\pi}{2 \left(\frac{1}{2}\right)} \quad (64)$$

The formula for (63) comes from (57) and is trivial to convert.

(62) and (63) imply the following:

$$2 \left(\frac{1}{2}\right) = \frac{\pi}{2 \left(\frac{1}{2}\right)} \implies 4 \left(\frac{1}{2}\right)^2 = \pi \implies \left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \quad (65)$$

using this strategy only works if  $(c_1 - x) = (c_2 + x)$  for some integers  $c_1$  and  $c_2$ . this limits it to  $(z + \frac{1}{2})$  for integers  $z$ . Also, you can only apply (63) an odd number of times or the factorials will cancel and leave a trigonometric equation, which could be interesting later. If you apply (63) twice, you get back to where you started because everything new cancels.

■



## 8 References

- [https://en.wikipedia.org/wiki/Riemann\\_zeta\\_function#Riemann's\\_functional\\_equation](https://en.wikipedia.org/wiki/Riemann_zeta_function#Riemann's_functional_equation)  
Link for equation 8
- <https://www.desmos.com/calculator/t51xnveadf>  
Extra information for part 6, made by me
- <https://www.github.com/drizzt536/files/tree/main/TeX/gamma>  
The files for the most recent version of this pdf and the LaTeX code