ATAR Master

VCE Mathematical Methods

2023 Examination 1 (Technology-Free)

Questions & Marking Guide

Total: 40 marks

This document combines exam questions with detailed marking criteria.
Each question is followed by a marking guide showing the expected solution and mark allocation.

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Question 1a *2 marks*

Let $y = \frac{x^2-x}{e^x}$. Find and simplify $\frac{dy}{dx}$.

Marking Guide — Answer: $\frac{dy}{dx} = \frac{-x^2+3x-1}{e^x}$

- Quotient rule: $u = x^2 - x$, $v = e^x$, $u' = 2x - 1$, $v' = e^x$.
- $\frac{dy}{dx} = \frac{(2x-1)e^x-(x^2-x)e^x}{e^{2x}} = \frac{2x-1-x^2+x}{e^x}$.
- $= \frac{-x^2+3x-1}{e^x}$.

Question 1b *2 marks*

Let $f(x) = \sin(x)e^{2x}$. Find $f'\left(\frac{\pi}{4}\right)$.

Marking Guide — Answer: $f'\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}\, e^{\pi/2}$

- Product rule: $f'(x) = \cos(x)e^{2x} + 2\sin(x)e^{2x} = e^{2x}(\cos(x) + 2\sin(x))$.
- $f'\left(\frac{\pi}{4}\right) = e^{\pi/2}\left(\frac{\sqrt{2}}{2} + 2 \cdot \frac{\sqrt{2}}{2}\right) = e^{\pi/2} \cdot \frac{3\sqrt{2}}{2}$.

Question 2 *3 marks*

Solve $e^{2x} - 12 = 4e^x$ for $x \in R$.

Marking Guide — Answer: $x = \log_e(6)$

- Rearrange: $e^{2x} - 4e^x - 12 = 0$.
- Let $u = e^x$: $u^2 - 4u - 12 = 0$, $(u-6)(u+2) = 0$.
- $u = 6$ or $u = -2$. Since $e^x > 0$, discard $u = -2$.
- $e^x = 6 \Rightarrow x = \log_e(6)$.

Question 3a *3 marks*

Sketch the graph of $f(x) = 2 - \frac{3}{x-1}$ on the axes, labelling all asymptotes with their equations and axial intercepts with their coordinates.

Marking Guide — Answer: Vertical asymptote $x = 1$, horizontal asymptote $y = 2$, x-intercept $(\frac{5}{2}, 0)$, y-intercept $(0, 5)$.

- Vertical asymptote: $x = 1$.
- Horizontal asymptote: $y = 2$.
- x-intercept: $2 - \frac{3}{x-1} = 0 \Rightarrow x - 1 = \frac{3}{2} \Rightarrow x = \frac{5}{2}$.
- y-intercept: $f(0) = 2 - \frac{3}{-1} = 5$.
- Correct shape: two branches of a hyperbola.

Question 3b *1 mark*

Find the values of x for which $f(x) \leq 1$.

Marking Guide — Answer: $1 < x \leq 4$

- $2 - \frac{3}{x-1} \leq 1 \Rightarrow -\frac{3}{x-1} \leq -1 \Rightarrow \frac{3}{x-1} \geq 1$.
- For $x > 1$: $3 \geq x - 1$, so $x \leq 4$. Combined: $1 < x \leq 4$.

Question 4 *2 marks*

The graph of $y = x + \frac{1}{x}$ is shown over part of its domain.

Use two trapeziums of equal width to approximate the area between the curve, the x-axis and the lines $x = 1$ and $x = 3$.

Marking Guide — Answer: $\frac{31}{6}$

Question 5a *1 mark*

Evaluate $\int_0^{\pi/3} \sin(x)\,dx$.

Marking Guide — Answer: $\frac{1}{2}$

Question 5b *3 marks*

Hence, or otherwise, find all values of k such that $\int_0^{\pi/3} \sin(x)\,dx = \int_k^{\pi/2} \cos(x)\,dx$, where $-3\pi < k < 2\pi$.

Marking Guide — Answer: $k = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

Question 6a *1 mark*

From a sample of randomly selected households, an approximate 95% confidence interval for the proportion p of households having solar panels installed was $(0.04, 0.16)$.

Find the value of $\hat{p}$ that was used to obtain this confidence interval.

Marking Guide — Answer: $\hat{p} = 0.10$

- $\hat{p} = \frac{0.04+0.16}{2} = 0.10$.

Question 6b *2 marks*

Use $z = 2$ to approximate the 95% confidence interval.

Find the size of the sample from which this 95% confidence interval was obtained.

Marking Guide — Answer: $n = 100$

- Margin of error: $E = 0.16 - 0.10 = 0.06$.
- $E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$: $0.06 = 2\sqrt{\frac{0.10\times0.90}{n}}$.
- $0.03 = \sqrt{\frac{0.09}{n}}$, $0.0009 = \frac{0.09}{n}$, $n = 100$.

Question 6c *1 mark*

A larger sample with size four times the original is selected. The sample proportion is the same.

By what factor will the increased sample size affect the width of the confidence interval?

Marking Guide — Answer: Width is halved (factor of $\frac{1}{2}$).

- Width $\propto \frac{1}{\sqrt{n}}$. If n is multiplied by 4, width is multiplied by $\frac{1}{\sqrt{4}} = \frac{1}{2}$.

Question 7a *1 mark*

Consider $f : (-\infty, 1] \to R$, $f(x) = x^2 - 2x$.

State the range of f.

Marking Guide — Answer: $[-1, \infty)$

- $f(x) = (x-1)^2 - 1$. Minimum at $x = 1$: $f(1) = -1$. As $x \to -\infty$, $f(x) \to \infty$.
- Range: $[-1, \infty)$.

Question 7b *2 marks*

Sketch the graph of the inverse function $y = f^{-1}(x)$ on the axes. Label any endpoints and axial intercepts with their coordinates.

Marking Guide — Answer: Reflection of f in $y = x$. Endpoint $(-1, 1)$, passes through $(0, 0)$.

- Correct reflection of f in $y = x$.
- Endpoint $(-1, 1)$ labelled (closed dot).
- Passes through $(0, 0)$.

Question 7c *2 marks*

Determine the equation and the domain for the inverse function f^{-1}.

Marking Guide — Answer: $f^{-1}(x) = 1 - \sqrt{x+1}$, domain $[-1, \infty)$.

- From $y = x^2 - 2x = (x-1)^2 - 1$: swap x and y: $x = (y-1)^2 - 1$.
- $(y-1)^2 = x + 1$, $y - 1 = \pm\sqrt{x+1}$.

Question 7d *2 marks*

Calculate the area of the regions enclosed by the curves of f, f^{-1} and $y = -x$.

Marking Guide — Answer: $\frac{1}{3}$

- Intersections: f and $y = -x$ at $(0, 0)$ and $(1, -1)$; f^{-1} and $y = -x$ at $(0, 0)$ and $(-1, 1)$; f and f^{-1} at $(0, 0)$.
- Area $= \int_{-1}^{0} (-x - f^{-1}(x))\, dx + \int_{0}^{1} (-x - f(x))\, dx$.
- $= \int_{-1}^{0} (-x - 1 + \sqrt{x+1})\, dx + \int_{0}^{1} (x - x^2)\, dx = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

Question 8a *1 mark*

The queuing time T (minutes) has PDF $f(t) = kt(16 - t^2)$ for $0 \le t \le 4$, and $f(t) = 0$ elsewhere.

Show that $k = \frac{1}{64}$.

Marking Guide — Answer: $k = \frac{1}{64}$

- $\int_0^4 kt(16 - t^2)\, dt = 1$.

Question 8b *2 marks*

Find $E(T)$.

Marking Guide — Answer: $E(T) = \frac{32}{15}$

- $E(T) = \int_0^4 t \cdot \frac{1}{64} t(16 - t^2)\, dt = \frac{1}{64} \int_0^4 (16t^2 - t^4)\, dt$.

Question 8c *3 marks*

What is the probability that a person has to queue for more than two minutes, given that they have already queued for one minute?

Marking Guide — Answer: $\frac{16}{25}$

- $\Pr(T > 2|T > 1) = \frac{\Pr(T>2)}{\Pr(T>1)}$.

Question 9a *1 mark*

Track 1 is $f(x) = a - x(x - 2)^2$. Track 2 is $g(x) = 12x + bx^2$.

Given that $f(0) = 12$ and $g(1) = 9$, verify that $a = 12$ and $b = -3$.

Marking Guide — Answer: $a = 12$, $b = -3$

- $f(0) = a - 0 = a = 12$ ✓.
- $g(1) = 12 + b = 9 \Rightarrow b = -3$ ✓.

Question 9b *2 marks*

Verify that $f(x)$ and $g(x)$ both have a turning point at P. Give the coordinates of P.

Marking Guide — Answer: $P = (2, 12)$

- $f(x) = 12 - x(x - 2)^2 = -x^3 + 4x^2 - 4x + 12$.
- $f'(x) = -3x^2 + 8x - 4 = -(3x - 2)(x - 2)$. $f'(2) = 0$ ✓.
- $g(x) = 12x - 3x^2$. $g'(x) = 12 - 6x$. $g'(2) = 0$ ✓.
- $f(2) = 12 - 2(0) = 12$, $g(2) = 24 - 12 = 12$. So $P = (2, 12)$.

Question 9c *3 marks*

A theme park is planned whose boundaries form triangle $\triangle OAB$ where O is the origin, A is at $(k, 0)$ and B is at $(k, g(k))$, where $k \in (0, 4)$.

Find the maximum possible area of the theme park, in km².

Marking Guide — Answer: $\frac{128}{9}$ km²

- Area $= \frac{1}{2} \times k \times g(k) = \frac{1}{2}k(12k - 3k^2) = 6k^2 - \frac{3}{2}k^3$.
- $A'(k) = 12k - \frac{9}{2}k^2 = k(12 - \frac{9}{2}k) = 0$.
- $k = 0$ or $k = \frac{8}{3}$. Since $k \in (0, 4)$, $k = \frac{8}{3}$.
- $A(\frac{8}{3}) = 6 \cdot \frac{64}{9} - \frac{3}{2} \cdot \frac{512}{27} = \frac{384}{9} - \frac{256}{9} = \frac{128}{9}$ km^2.