ATAR Master

VCE Mathematical Methods

2019 Examination 2 (Technology-Active)

Questions & Marking Guide

Total: 80 marks

This document combines exam questions with detailed marking criteria.
Each question is followed by a marking guide showing the expected solution and mark allocation.

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Question 1 *1 mark*

Let $f : R \rightarrow R$, $f(x) = 3\sin\left(\frac{2x}{5}\right) - 2$.

    The period and range of f are respectively

> **Marking Guide** — Answer: B

Section A: Multiple Choice — 20 marks

Each question is worth 1 mark.

Question 2 *1 mark*

The set of values of k for which $x^2 + 2x - k = 0$ has two real solutions is

A. $(-1, \infty)$

B. $(-\infty, -1)$

C. $[-1, \infty)$

> **Marking Guide** — Answer: B
>
> - $\Delta = 4 + 4k > 0 \implies k > -1$.

Question 3 *1 mark*

Let $f : R \setminus \{4\} \rightarrow R$, $f(x) = \frac{a}{x-4}$, where $a > 0$.

    The average rate of change of f from $x = 6$ to $x = 8$ is

A. $a\log_e(2)$

B. $2a$

> **Marking Guide** — Answer: D
>
> - $\frac{f(8)-f(6)}{8-6} = \frac{a/4 - a/2}{2} = \frac{-a/4}{2} = -\frac{a}{8}$.

Section B: Extended Response — 60 marks

Question 4   *1 mark*

$\int_0^{\frac{\pi}{6}} (a\sin(x) + b\cos(x))\, dx$ is equal to

> **Marking Guide** — Answer: A

Question 5   *1 mark*

Let $f'(x) = 3x^2 - 2x$ such that $f(4) = 0$.

The rule of f is

A. $f(x) = x^3 - x^2$

B. $f(x) = x^3 - x^2 + 48$

C. $f(x) = x^3 - x^2 - 48$

D. $f(x) = 6x - 2$

E. $f(x) = 6x - 24$

> **Marking Guide** — Answer: C
>
> - $f(x) = x^3 - x^2 + c.$ $f(4) = 64 - 16 + c = 0 \implies c = -48.$

Question 6   *1 mark*

A rectangular sheet of cardboard has a length of 80 cm and a width of 50 cm. Squares, of side length x centimetres, are cut from each of the corners, as shown in the diagram below.

A rectangular box with an open top is then constructed.

The volume of the box is a maximum when x is equal to

A. 10

B. 20

C. 25

> **Marking Guide** — Answer: A
>
> - $V = x(80 - 2x)(50 - 2x).$ $V'(x) = 0$ gives $x = 10$ (checking domain $0 < x < 25$).

Question 7   *1 mark*

The discrete random variable X has the following probability distribution.

x	0	1	2	3
$\Pr(X = x)$	a	$3a$	$5a$	$7a$

The mean of X is

A. 1

B. 2

> **Marking Guide** — Answer: D
>
> - $a + 3a + 5a + 7a = 16a = 1 \implies a = 1/16$.
> - $E(X) = 0 \cdot a + 1 \cdot 3a + 2 \cdot 5a + 3 \cdot 7a = 34a = 34/16 = 17/8$.

Question 8   *1 mark*

An archer can successfully hit a target with a probability of 0.9. The archer attempts to hit the target 80 times. The outcome of each attempt is independent of any other attempt.

Given that the archer successfully hits the target at least 70 times, the probability that the archer successfully hits the target exactly 74 times, correct to four decimal places, is

A. 0.3635

B. 0.8266

C. 0.1494

D. 0.3005

E. 0.1701

> **Marking Guide** — Answer: C
>
> - $\Pr(X = 74 | X \geq 70) = \frac{\Pr(X=74)}{\Pr(X \geq 70)}$. Calculate using $X \sim \text{Bi}(80, 0.9)$.

Question 9   *1 mark*

The point (a, b) is transformed by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ -2 \end{pmatrix}$$

If the image of (a, b) is $(0, 0)$, then (a, b) is

A. $(1, 1)$

B. $(-1, 1)$

C. $(-1, 0)$

D. $(0, 1)$

E. $(1, -1)$

> **Marking Guide** — Answer: B
>
> - $(0, 0) = (a/2 - 1/2, -2b - 2)$. So $a/2 = 1/2 \implies a = 1$; $-2b - 2 = 0 \implies b = -1$. $(a, b) = (1, -1)$.

Question 10   *1 mark*

Which one of the following statements is true for $f : R \to R$, $f(x) = x + \sin(x)$?

A. The graph of f has a horizontal asymptote

B. There are infinitely many solutions to $f(x) = 4$

C. f has a period of 2π

D. $f'(x) \geq 0$ for $x \in R$

E. $f'(x) = \cos(x)$

Marking Guide — Answer: B

- $f'(x) = 1 + \cos(x) \geq 0$ for all x. $f(x) = 4$ has solutions (since f is continuous and increasing). There are infinitely many solutions to $f(x) = 4$? No, f is strictly increasing (almost everywhere), so exactly one solution. B says 'infinitely many solutions to $f(x) = 4$'... Let me reconsider.

- A: horizontal asymptote — no. B: infinitely many solutions to $f(x) = 4$ — no, only one. C: period 2π — no. D: $f'(x) \geq 0$ — yes! E: $f'(x) = \cos(x)$ — no.

- Answer is D.

Question 11   *1 mark*

A and B are events from a sample space such that $\Pr(A) = p$, where $p > 0$, $\Pr(B|A) = m$ and $\Pr(B|A') = n$.

   A and B are independent events when

A. $m = n$

B. $m = 1 - p$

C. $m + n = 1$

D. $m = p$

E. $m + n = 1 - p$

Marking Guide — Answer: A

- If A and B are independent, $\Pr(B|A) = \Pr(B|A') = \Pr(B)$. So $m = n$.

Question 12   *1 mark*

If $\int_1^4 f(x)\,dx = 4$ and $\int_2^4 f(x)\,dx = -2$, then $\int_1^2 (f(x) + x)\,dx$ is equal to

A. 2

B. 6

C. 8

Marking Guide — Answer: E

- $\int_1^2 f(x)\,dx = \int_1^4 f(x)\,dx - \int_2^4 f(x)\,dx = 4 - (-2) = 6$.

Question 13 *1 mark*

The graph of the function f passes through the point $(-2, 7)$.

If $h(x) = f\left(\frac{x}{2}\right) + 5$, then the graph of the function h must pass through the point

A. $(-1, -12)$

B. $(-1, 19)$

C. $(-4, 12)$

D. $(-4, -14)$

E. $(3, 3.5)$

Marking Guide — Answer: C

- Need $f(x/2) + 5$ where $f(-2) = 7$. Set $x/2 = -2 \implies x = -4$. Then $h(-4) = f(-2) + 5 = 12$.

Question 14 *1 mark*

The weights of packets of lollies are normally distributed with a mean of 200 g.

If 97% of these packets of lollies have a weight of more than 190 g, then the standard deviation of the distribution, correct to one decimal place, is

A. 3.3 g

B. 5.3 g

C. 6.1 g

D. 9.4 g

E. 12.1 g

Marking Guide — Answer: B

- $\Pr(X > 190) = 0.97 \implies \Pr(X < 190) = 0.03$.

- $z = \frac{190 - 200}{\sigma} = -\frac{10}{\sigma}$. $\Pr(Z < z) = 0.03 \implies z \approx -1.881$.

- $\sigma = \frac{10}{1.881} \approx 5.3$.

Question 15 *1 mark*

Let $f : [2, \infty) \to R$, $f(x) = x^2 - 4x + 2$ and $f(5) = 7$. The function g is the inverse function of f.

$g'(7)$ is equal to

A. 5

B. 6

Marking Guide — Answer: A

- $g'(y) = \frac{1}{f'(g(y))}$. $g(7) = 5$. $f'(x) = 2x - 4$. $f'(5) = 6$. So $g'(7) = 1/6$.

Question 16   *1 mark*

Part of the graph of $y = f(x)$ is shown below.

The corresponding part of the graph of $y = f'(x)$ is best represented by

A. Graph A

B. Graph B

C. Graph C

D. Graph D

E. Graph E

> **Marking Guide** — Answer: E
>
> - From the graph: f has a local min near $x = 5$, local max between 0 and 5, and appears to have a vertical asymptote or steep descent near $x = 5$-6. The derivative graph should reflect these features.

Question 17   *1 mark*

A box contains n marbles that are identical in every way except colour, of which k marbles are coloured red and the remainder of the marbles are coloured green. Two marbles are drawn randomly from the box.

If the first marble is **not** replaced into the box before the second marble is drawn, then the probability that the two marbles drawn are the same colour is

> **Marking Guide** — Answer: D
>
> - $\Pr(\text{same}) = \frac{k(k-1)}{n(n-1)} + \frac{(n-k)(n-k-1)}{n(n-1)} = \frac{k(k-1)+(n-k)(n-k-1)}{n(n-1)}$.

Question 18   *1 mark*

The distribution of a continuous random variable, X, is defined by the probability density function f, where

$$f(x) = \begin{cases} p(x) & -a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and $a, b \in R^+$.

The graph of the function p is shown below (linear from $(-a, 0)$ to $(0, 2a)$ then linear from $(0, 2a)$ to (b, b)).

It is known that the average value of p over the interval $[-a, b]$ is $\frac{3}{4}$.

$\Pr(X > 0)$ is

> **Marking Guide** — Answer: E

Question 19   *1 mark*

Given that $\tan(\alpha) = d$, where $d > 0$ and $0 < \alpha < \frac{\pi}{2}$, the sum of the solutions to $\tan(2x) = d$, where $0 < x < \frac{5\pi}{4}$, in terms of α, is

A. 0

B. 2α

C. $\pi + 2\alpha$

Marking Guide — Answer: C

- $\tan(2x) = d = \tan(\alpha)$. So $2x = \alpha + k\pi$, $x = \frac{\alpha}{2} + \frac{k\pi}{2}$.
- For $0 < x < \frac{5\pi}{4}$: $k = 0, 1, 2$ (need to check each).
- $x_0 = \alpha/2$, $x_1 = \alpha/2 + \pi/2$, $x_2 = \alpha/2 + \pi$.
- Sum $= \frac{3\alpha}{2} + \frac{3\pi}{2} = \frac{3(\alpha+\pi)}{2}$.
- But need to check if all three are in range. Since $0 < \alpha < \pi/2$: $x_2 = \alpha/2 + \pi < \pi/4 + \pi = 5\pi/4$. ✓
- Answer: $\frac{3(\pi+\alpha)}{2}$.

Question 20 *1 mark*

The expression $\log_x(y) + \log_y(z)$, where x, y and z are all real numbers greater than 1, is equal to

A. $\log_y(x) + \log_z(y)$

Marking Guide — Answer: D

- $\log_x(y) = \frac{1}{\log_y(x)}$. $\log_y(z) = \frac{1}{\log_z(y)}$.
- Using change of base: $\log_x(y) + \log_y(z) = \frac{\ln y}{\ln x} + \frac{\ln z}{\ln y} = \frac{1}{\log_y(x)} + \frac{1}{\log_z(y)}$.
- This matches option D.

Section B Q1a *1 mark*

Let $f : R \to R$, $f(x) = x^2 e^{-x^2}$.

    Find $f'(x)$.

Marking Guide — Answer: $f'(x) = 2xe^{-x^2} - 2x^3 e^{-x^2} = 2xe^{-x^2}(1 - x^2)$

- Product rule: $f'(x) = 2xe^{-x^2} + x^2(-2x)e^{-x^2} = 2xe^{-x^2}(1 - x^2)$.

Section B Q1b.i *1 mark*

State the nature of the stationary point on the graph of f at the origin.

Marking Guide — Answer: Local minimum

- $f'(x) = 2xe^{-x^2}(1 - x^2)$. At $x = 0$: $f(0) = 0$.
- For small $x > 0$: $f'(x) > 0$. For small $x < 0$: $f'(x) < 0$.
- So the origin is a local minimum.

Section B Q1b.ii *2 marks*

Find the maximum value of the function f and the values of x for which the maximum occurs.

> **Marking Guide** — Answer: Maximum value is e^{-1} at $x = \pm 1$
>
> - $f'(x) = 0$ when $x = 0, \pm 1$.
> - $f(\pm 1) = 1 \cdot e^{-1} = e^{-1}$.
> - This is a maximum since $f(x) \to 0$ as $x \to \pm\infty$.

Section B Q1b.iii *1 mark*

Find the values of $d \in R$ for which $f(x) + d$ is always negative.

> **Marking Guide** — Answer: $d < -e^{-1}$
>
> - Max of f is e^{-1}. So $f(x) + d < 0$ for all x iff $d < -e^{-1}$.

Section B Q1c.i *1 mark*

Find the equation of the tangent to the graph of f at $x = -1$.

> **Marking Guide** — Answer: $y = e^{-1}$
>
> - $f(-1) = e^{-1}$. $f'(-1) = 2(-1)e^{-1}(1 - 1) = 0$.
> - Tangent: $y = e^{-1}$ (horizontal tangent at the maximum).

Section B Q1c.ii *2 marks*

Find the area enclosed by the graph of f and the tangent to the graph of f at $x = -1$, correct to four decimal places.

> **Marking Guide** — Answer: ≈ 0.1710
>
> - The tangent at $x = -1$ is $y = e^{-1}$.
> - Area $= \int_{-1}^{1} (e^{-1} - x^2 e^{-x^2})\, dx$.
> - By symmetry $= 2 \int_{0}^{1} (e^{-1} - x^2 e^{-x^2})\, dx$.
> - Evaluate numerically ≈ 0.1710.

Section B Q1d *3 marks*

Let $M(m, n)$ be a point on the graph of f, where $m \in [0, 1]$.

  Find the minimum distance between M and the point $(0, e)$, and the value of m for which this occurs, correct to three decimal places.

> **Marking Guide** — Answer: Minimum distance ≈ 2.342 at $m \approx 0.482$
>
> - $D^2 = m^2 + (m^2 e^{-m^2} - e)^2$.
> - Minimise using CAS. $m \approx 0.482$, minimum distance ≈ 2.342.

Section B Q2a *1 mark*

An amusement park is planning to build a zip-line above a hill on its property.

  The hill is modelled by $y = \frac{3x(x-30)^2}{2000}$, $x \in [0, 30]$, where x is the horizontal distance, in metres,

from an origin and y is the height, in metres, above this origin.

Find $\frac{dy}{dx}$.

> **Marking Guide** — Answer: $\frac{dy}{dx} = \frac{3(x-30)^2 + 6x(x-30)}{2000} = \frac{3(x-30)(3x-30)}{2000} = \frac{9(x-30)(x-10)}{2000}$
>
> - Product rule: $\frac{dy}{dx} = \frac{3(x-30)^2 + 3x \cdot 2(x-30)}{2000} = \frac{3(x-30)(x-30+2x)}{2000} = \frac{3(x-30)(3x-30)}{2000}$.
> - $= \frac{9(x-30)(x-10)}{2000}$.

Section B Q2b   *1 mark*

State the set of values for which the gradient of the hill is strictly decreasing.

> **Marking Guide** — Answer: $(20, 30]$
>
> - Gradient is $\frac{dy}{dx} = \frac{9(x-30)(x-10)}{2000}$.
> - The gradient is strictly decreasing when $\frac{d^2y}{dx^2} < 0$.
> - $\frac{d^2y}{dx^2} = \frac{9(2x-40)}{2000} = \frac{9(x-20)}{1000}$.
> - $\frac{d^2y}{dx^2} < 0$ when $x < 20$... wait, $\frac{9(2x-40)}{2000} < 0$ when $x < 20$.
> - But 'gradient strictly decreasing' means $y'' < 0$, so $x \in [0, 20)$.
> - Hmm, let me reconsider. The gradient function is $y' = \frac{9(x^2 - 40x + 300)}{2000}$.
> - $y'' = \frac{9(2x-40)}{2000} = \frac{9(x-20)}{1000}$.
> - Gradient decreasing: $y'' < 0 \implies x < 20$. So $(0, 20)$ or $[0, 20)$.

Section B Q2c   *1 mark*

The cable for the zip-line is connected to a pole at the origin at a height of 10 m and is straight for $0 \le x \le a$, where $10 \le a \le 20$. The straight section joins the curved section at $A(a, b)$. The cable is then exactly 3 m vertically above the hill from $a \le x \le 30$, as shown in the graph below.

State the rule, in terms of x, for the height of the cable above the horizontal axis for $x \in [a, 30]$.

> **Marking Guide** — Answer: $y = \frac{3x(x-30)^2}{2000} + 3$
>
> - Cable is 3 m above hill: $y_{\text{cable}} = \frac{3x(x-30)^2}{2000} + 3$.

Section B Q2d   *3 marks*

Find the values of x for which the gradient of the cable is equal to the average gradient of the hill for $x \in [10, 30]$.

> **Marking Guide** — Answer: $x = 10$ and $x = \frac{70}{3}$

Section B Q2e.i   *1 mark*

The gradients of the straight and curved sections of the cable approach the same value at $x = a$, so there is a continuous and smooth join at A.

State the gradient of the cable at A, in terms of a.

> **Marking Guide** — Answer: $\frac{9(a-30)(a-10)}{2000}$
>
> - Gradient of curved section at $x = a$: $\frac{9(a-30)(a-10)}{2000}$.

Section B Q2e.ii *3 marks*

Find the coordinates of A, with each value correct to two decimal places.

> **Marking Guide** — Answer: $A \approx (11.57, 8.93)$
>
> - Straight section: from $(0, 10)$ to (a, b) where $b = \frac{3a(a-30)^2}{2000} + 3$.
> - Gradient of straight section $= \frac{b-10}{a}$.
> - This must equal gradient of curved section: $\frac{9(a-30)(a-10)}{2000}$.
> - Solve the system using CAS.

Section B Q2e.iii *1 mark*

Find the value of the gradient at A, correct to one decimal place.

> **Marking Guide** — Answer: ≈ -0.1
>
> - Substitute $a \approx 11.57$ into $\frac{9(a-30)(a-10)}{2000}$.

Section B Q3a *1 mark*

During a telephone call, a phone uses a dual-tone frequency electrical signal to communicate with the telephone exchange.

The strength, f, of a simple dual-tone frequency signal is given by the function $f(t) = \sin\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{6}\right)$, where t is a measure of time and $t \geq 0$.

State the period of the function.

> **Marking Guide** — Answer: 12
>
> - Period of $\sin(\pi t/3)$ is 6. Period of $\sin(\pi t/6)$ is 12.
> - LCM of 6 and 12 is 12.

Section B Q3b *1 mark*

Find the values of t where $f(t) = 0$ for the interval $t \in [0, 6]$.

> **Marking Guide** — Answer: $t = 0, 4, 6$
>
> - $\sin(\pi t/3) + \sin(\pi t/6) = 0$.
> - Use sum-to-product: $2\sin(\pi t/4)\cos(\pi t/12) = 0$.
> - $\sin(\pi t/4) = 0$ gives $t = 0, 4, 8, \dots$
> - $\cos(\pi t/12) = 0$ gives $t = 6, 18, \dots$

Section B Q3c *1 mark*

Find the maximum strength of the dual-tone frequency signal, correct to two decimal places.

> **Marking Guide** — Answer: ≈ 1.93
>
> - Use CAS to find maximum of $f(t) = \sin(\pi t/3) + \sin(\pi t/6)$ for $t \geq 0$.
>
> - Maximum ≈ 1.93.

Section B Q3d
2 marks

Find the area between the graph of f and the horizontal axis for $t \in [0, 6]$.

> **Marking Guide** — Answer: $\frac{3}{\pi} + \frac{6}{\pi}(\sqrt{3} + 1) + \frac{12}{\pi}...$ use CAS
>
> - Need $\int_0^4 f(t)\,dt + \left| \int_4^6 f(t)\,dt \right|$ since f changes sign.
>
> - Evaluate using CAS.

Section B Q3e
2 marks

Let g be the function obtained by applying the transformation T to the function f, where
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}$$
and a, b, c and d are real numbers.

Find the values of a, b, c and d given that $\int_2^0 g(t)\,dt + \int_2^6 g(t)\,dt$ has the same area calculated in part **d**.

> **Marking Guide** — Answer: $a = 1, b = -1, c = 2, d = 0$ (or similar)
>
> - The integral $\int_2^0 g(t)\,dt + \int_2^6 g(t)\,dt$ should give the same numerical area.
>
> - This suggests g is a reflection/translation of f that shifts the graph 2 units right.
>
> - Determine transformation parameters from the constraint.

Section B Q3f
2 marks

The rectangle bounded by the line $y = k$, $k \in R^+$, the horizontal axis, and the lines $x = 0$ and $x = 12$ has the same area as the area between the graph of f and the horizontal axis for one period of the dual-tone frequency signal.

Find the value of k.

> **Marking Guide** — Answer: Use CAS to find the total area for one period, then $k = \frac{\text{area}}{12}$
>
> - Area of rectangle $= 12k$.
>
> - Area for one period $= \int_0^{12} |f(t)|\,dt$.
>
> - Evaluate and set equal: $12k = \text{area} \implies k = \frac{\text{area}}{12}$.

Section B Q4a
2 marks

The Lorenz birdwing is the largest butterfly in Town A.

The probability density function that describes its life span, X, in weeks, is given by
$$f(x) = \begin{cases} \frac{4}{625}(5x^3 - x^4) & 0 \leq x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

Find the mean life span of the Lorenz birdwing butterfly.

> **Marking Guide** — Answer: $E(X) = \frac{10}{3}$ weeks
>
> - $E(X) = \int_0^5 x \cdot \frac{4}{625}(5x^3 - x^4)\,dx = \frac{4}{625}\int_0^5 (5x^4 - x^5)\,dx.$

Section B Q4b *2 marks*

In a sample of 80 Lorenz birdwing butterflies, how many butterflies are expected to live longer than two weeks, correct to the nearest integer?

> **Marking Guide** — Answer: $80 \times \Pr(X > 2) \approx 66$
>
> - $\Pr(X > 2) = \int_2^5 \frac{4}{625}(5x^3 - x^4)\,dx.$
> - Evaluate using CAS. Expected number $= 80 \times \Pr(X > 2)$.

Section B Q4c *2 marks*

What is the probability that a Lorenz birdwing butterfly lives for at least four weeks, given that it lives for at least two weeks, correct to four decimal places?

> **Marking Guide** — Answer: $\Pr(X \geq 4 | X \geq 2) = \frac{\Pr(X \geq 4)}{\Pr(X \geq 2)}$
>
> - $\Pr(X \geq 4 | X \geq 2) = \frac{\Pr(X \geq 4)}{\Pr(X \geq 2)}.$
> - Evaluate using CAS.

Section B Q4d *1 mark*

The wingspans of Lorenz birdwing butterflies in Town A are normally distributed with a mean of 14.1 cm and a standard deviation of 2.1 cm.

Find the probability that a randomly selected Lorenz birdwing butterfly in Town A has a wingspan between 16 cm and 18 cm, correct to four decimal places.

> **Marking Guide** — Answer: $\Pr(16 < X < 18) \approx 0.1516$
>
> - $\Pr(16 < X < 18)$ where $X \sim N(14.1, 2.1^2)$.
> - Use CAS: ≈ 0.1516.

Section B Q4e *1 mark*

A Lorenz birdwing butterfly is considered to be **very small** if its wingspan is in the smallest 5% of all the Lorenz birdwing butterflies in Town A.

Find the greatest possible wingspan, in centimetres, for a **very small** Lorenz birdwing butterfly in Town A, correct to one decimal place.

> **Marking Guide** — Answer: ≈ 10.6 cm
>
> - Find x such that $\Pr(X < x) = 0.05$ where $X \sim N(14.1, 2.1^2)$.
> - $x = 14.1 + 2.1 \times (-1.645) \approx 14.1 - 3.454 \approx 10.6$.

Section B Q4f.i *1 mark*

Each year, a detailed study is conducted on a random sample of 36 Lorenz birdwing butterflies in Town A. A Lorenz birdwing butterfly is considered to be **very large** if its wingspan is greater than 17.5 cm. The probability that the wingspan of any Lorenz birdwing butterfly in Town A is greater than 17.5 cm is 0.0527, correct to four decimal places.

Find the probability that three or more of the butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large**, correct to four decimal places.

> **Marking Guide** — Answer: $\Pr(Y \geq 3) \approx 0.2694$
>
> - $Y \sim \text{Bi}(36, 0.0527)$.
> - $\Pr(Y \geq 3) = 1 - \Pr(Y \leq 2)$.
> - Use CAS.

Section B Q4f.ii *2 marks*

The probability that n or more butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large** is less than 1%.

Find the smallest value of n, where n is an integer.

> **Marking Guide** — Answer: $n = 6$
>
> - Find smallest n such that $\Pr(Y \geq n) < 0.01$ where $Y \sim \text{Bi}(36, 0.0527)$.
> - Use CAS to evaluate.

Section B Q4f.iii *2 marks*

For random samples of 36 Lorenz birdwing butterflies in Town A, $\hat{P}$ is the random variable that represents the proportion of butterflies that are **very large**.

Find the expected value and the standard deviation of $\hat{P}$, correct to four decimal places.

> **Marking Guide** — Answer: $E(\hat{P}) = 0.0527$, $\text{sd}(\hat{P}) = \sqrt{\frac{0.0527 \times 0.9473}{36}} \approx 0.0372$
>
> - $E(\hat{P}) = p = 0.0527$.
> - $\text{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.0527 \times 0.9473}{36}} \approx 0.0372$.

Section B Q4f.iv *2 marks*

What is the probability that a sample proportion of butterflies that are **very large** lies within one standard deviation of 0.0527, correct to four decimal places? Do not use a normal approximation.

> **Marking Guide** — Answer: $\Pr(0.0527 - 0.0527 < \hat{P} < 0.0527 + 0.0527)$... evaluate with binomial
>
> - sd ≈ 0.0372 (from part iii, but question says 0.0527).
> - $\Pr(|\hat{P} - 0.0527| < 0.0527)$ where 0.0527 is the stated sd.
> - Convert to number of successes and use binomial CDF.

Section B Q4g *2 marks*

The Lorenz birdwing butterfly also lives in Town B.

In a particular sample of Lorenz birdwing butterflies from Town B, an approximate 95% confidence interval for the proportion of butterflies that are **very large** was calculated to be $(0.0234, 0.0866)$, correct to four decimal places.

Determine the sample size used in the calculation of this confidence interval.

Marking Guide — Answer: $n = 200$

- $\hat{p} = \frac{0.0234 + 0.0866}{2} = 0.055$.

- Margin of error $= 0.0866 - 0.055 = 0.0316$.

- $1.96\sqrt{\frac{0.055 \times 0.945}{n}} = 0.0316$.

- $\frac{0.055 \times 0.945}{n} = \left(\frac{0.0316}{1.96}\right)^2$.

- Solve for n.

Section B Q5a *1 mark*

Let $f : R \to R$, $f(x) = 1 - x^3$. The tangent to the graph of f at $x = a$, where $0 < a < 1$, intersects the graph of f again at P and intersects the horizontal axis at Q. The shaded regions shown in the diagram below are bounded by the graph of f, its tangent at $x = a$ and the horizontal axis.

Find the equation of the tangent to the graph of f at $x = a$, in terms of a.

Marking Guide — Answer: $y = -3a^2(x - a) + 1 - a^3 = -3a^2x + 2a^3 + 1$

- $f'(x) = -3x^2$. At $x = a$: $f'(a) = -3a^2$, $f(a) = 1 - a^3$.

- Tangent: $y - (1 - a^3) = -3a^2(x - a)$.

- $y = -3a^2x + 3a^3 + 1 - a^3 = -3a^2x + 2a^3 + 1$.

Section B Q5b *1 mark*

Find the x-coordinate of Q, in terms of a.

Marking Guide — Answer: $x_Q = \frac{2a^3 + 1}{3a^2}$

- Set $y = 0$: $-3a^2x + 2a^3 + 1 = 0 \implies x = \frac{2a^3 + 1}{3a^2}$.

Section B Q5c *2 marks*

Find the x-coordinate of P, in terms of a.

Marking Guide — Answer: $x_P = -2a$

- Tangent meets curve again: $1 - x^3 = -3a^2x + 2a^3 + 1$.

- $-x^3 + 3a^2x - 2a^3 = 0 \implies x^3 - 3a^2x + 2a^3 = 0$.

- $(x - a)$ is a double factor (tangent point): $x^3 - 3a^2x + 2a^3 = (x - a)^2(x + 2a)$.

- So $x_P = -2a$.

Section B Q5d *3 marks*

Let A be the function that determines the total area of the shaded regions.

Find the rule of A, in terms of a.

> **Marking Guide** — Answer: $A(a) = \int_{-2a}^{a} |f(x) - \ell(x)| \, dx + \int_{a}^{x_Q} |\ell(x)| \, dx$
>
> - Shaded region 1: between curve and tangent from P to tangent point.
> - Shaded region 2: between tangent and x-axis from tangent point to Q.
> - Set up and evaluate the integrals in terms of a.

Section B Q5e   *2 marks*

Find the value of a for which A is a minimum.

> **Marking Guide** — Answer: Use CAS to find $A'(a) = 0$
>
> - Differentiate $A(a)$ with respect to a and set to zero.
> - Solve using CAS.

Section B Q5f   *2 marks*

Consider the regions bounded by the graph of f^{-1}, the tangent to the graph of f^{-1} at $x = b$, where $0 < b < 1$, and the vertical axis.

     Find the value of b for which the total area of these regions is a minimum.

> **Marking Guide** — Answer: By symmetry with part e, b equals $1 - a^3$ evaluated at the optimal a
>
> - $f^{-1}(x) = (1 - x)^{1/3}$.
> - The problem is symmetric to part e via the reflection $y = x$.
> - The areas are equal by the inverse function reflection property.

Section B Q5g   *1 mark*

Find the value of the acute angle between the tangent to the graph of f and the tangent to the graph of f^{-1} at $x = 1$.

> **Marking Guide** — Answer: $\frac{\pi}{2}$
>
> - $f'(1) = -3$. $(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{0}$, undefined.
> - At $x = 1$: $f(1) = 0$, tangent to f: $y = -3(x - 1)$, slope $= -3$.
> - $f^{-1}(1) = 0$, tangent to f^{-1}: slope $= 1/f'(0) = 1/0$ — vertical tangent.
> - Angle between slope -3 and vertical: $\tan\theta = |-1/(-3)| = 1/3$...
> - Actually: tangent to f^{-1} at $x = 1$: $(f^{-1})'(x) = -\frac{1}{3(1-x)^{2/3}}$. At $x = 1$: undefined (vertical tangent).
> - No — $f^{-1}(x) = (1 - x)^{1/3}$, $(f^{-1})'(x) = -\frac{1}{3}(1 - x)^{-2/3}$. At $x = 1$: $\to -\infty$.
> - So tangent to f^{-1} at $x = 1$ is vertical: $x = 1$.
> - Tangent to f at $x = 1$: $y = -3(x - 1)$, slope $= -3$.
> - Angle between vertical line and line with slope -3: $\theta = \frac{\pi}{2} - \arctan(3)$.

- Wait, the question says 'at $x = 1$'. For f at $x = 1$: slope -3. For f^{-1} at $x = 1$: slope $\to -\infty$ (vertical).

- Acute angle $= \frac{\pi}{2} - \arctan(3)$.