ATAR Master

VCE Mathematical Methods

2025 Examination 1 (Technology-Free)

Questions & Marking Guide

Total: 40 marks

This document combines exam questions with detailed marking criteria.
Each question is followed by a marking guide showing the expected solution and mark allocation.

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Question 1a *1 mark*

Let $y = x^2 \cos(x)$. Find $\frac{dy}{dx}$.

Marking Guide — Answer: $2x\cos(x) - x^2\sin(x)$

- Apply product rule: $u = x^2, v = \cos(x)$.
- $\frac{dy}{dx} = 2x\cos(x) - x^2\sin(x)$.
- A – brackets not required around argument, either order, clearly only 2 terms.

Question 1b *2 marks*

Let $f(x) = 6\sqrt{x+1} + 5$. Find the gradient of the tangent to $y = f(x)$ at $x = 8$.

Marking Guide — Answer: 1

- Write $f(x) = 6(x+1)^{1/2} + 5$.
- M – attempt at derivative: $f'(x) = \frac{3}{\sqrt{x+1}}$.
- A – $f'(8) = \frac{3}{\sqrt{9}} = \frac{3}{3} = 1$.

Question 2 *2 marks*

Let $g(x)$ be a function defined for $x > -\frac{3}{2}$ so that $g'(x) = \frac{1}{2x+3}$ and $g(1) = 0$. Find $g(x)$.

Marking Guide — Answer: $g(x) = \frac{1}{2}\log_e(2x+3) - \frac{1}{2}\log_e(5)$

- M – integrate to get $g(x) = \frac{1}{2}\log_e(2x+3) + c$, must have $+c$.
- A – apply $g(1) = 0$: $c = -\frac{1}{2}\log_e(5)$. Any correct form, base e or ln.

Question 3a *1 mark*

Let $f : [0, 2\pi] \to R, f(x) = 2\cos(2x) + 1$.

State the range of f.

Marking Guide — Answer: $[-1, 3]$

Question 3b *3 marks*

Let $f : [0, 2\pi] \to R, f(x) = 2\cos(2x) + 1$.

Solve $f(x) = 0$ for x.

Marking Guide — Answer: $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

- $2\cos(2x) + 1 = 0 \implies \cos(2x) = -\frac{1}{2}$.
- M – see base angle of $\frac{\pi}{3}$.
- M – see 2 correct results for $2x$.
- A – 4 correct values only: $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

Question 3c *2 marks*

Let $f : [0, 2\pi] \to R, f(x) = 2\cos(2x) + 1$.

Sketch the graph of $y = f(x)$ for $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ on the axes provided. Label the endpoints with their coordinates.

Marking Guide — Answer: Graph from $(\frac{\pi}{2}, -1)$ to $(\frac{3\pi}{2}, -1)$ with correct shape

- M – accurately represents correct shape with symmetry and correct range, may extend beyond domain.
- A – correct endpoints labelled: $(\frac{\pi}{2}, -1)$ and $(\frac{3\pi}{2}, -1)$, graph restricted to domain.

Question 4a *2 marks*

The probability distribution for the discrete random variable X is given in the table below, where k is a positive real number.

| x | 0 | 1 | 2 | 3 | |—|—|—|—|—| | Pr$(X = x)$ | $\frac{4}{k}$ | $\frac{2k}{75}$ | $\frac{k}{75}$ | $\frac{2}{k}$ |

Show that $k = 10$ or $k = 15$.

Marking Guide — Answer: $k = 10$ or $k = 15$

- Sum of probabilities $= 1$: $\frac{4}{k} + \frac{2k}{75} + \frac{k}{75} + \frac{2}{k} = 1$.
- $\frac{6}{k} + \frac{3k}{75} = 1 \implies \frac{6}{k} + \frac{k}{25} = 1$.
- M – multiply through: $150 + k^2 = 25k$.
- A – $k^2 - 25k + 150 = 0 \implies (k - 10)(k - 15) = 0$.

Question 4b.i *1 mark*

Let $k = 15$.

Find $\Pr(X > 1)$.

Marking Guide — Answer: $\frac{1}{3}$

- With $k = 15$: $\Pr(X = 2) = \frac{15}{75} = \frac{1}{5}$, $\Pr(X = 3) = \frac{2}{15}$.
- $\Pr(X > 1) = \frac{1}{5} + \frac{2}{15} = \frac{3+2}{15} = \frac{5}{15} = \frac{1}{3}$.

Question 4b.ii *1 mark*

Let $k = 15$.

Find $E(X)$.

Marking Guide — Answer: $\frac{6}{5}$ or 1.2

- Probs: $\frac{4}{15}, \frac{6}{15}, \frac{3}{15}, \frac{2}{15}$.
- $E(X) = 0 \cdot \frac{4}{15} + 1 \cdot \frac{6}{15} + 2 \cdot \frac{3}{15} + 3 \cdot \frac{2}{15} = \frac{6+6+6}{15} = \frac{18}{15} = \frac{6}{5}$.

Question 5a *2 marks*

Solve $e^{2x} - 8e^x + 7 = 0$ for x.

Marking Guide — Answer: $x = 0$ or $x = \log_e(7)$

- Let $u = e^x$: $u^2 - 8u + 7 = 0$.
- M – $(u-1)(u-7) = 0$, so $e^x = 1$ or $e^x = 7$.
- A – $x = 0$ or $x = \log_e(7)$.

Question 5b *2 marks*

Let $g(x) = e^{2x} - 8e^x + 7$, where $x \in R$. The function $g(x)$ has exactly one stationary point, a local minimum.

Find the largest value of a such that when g is restricted to the domain $(-\infty, a]$, it has an inverse function.

Marking Guide — Answer: $a = \log_e(4)$

- Find stationary point: $g'(x) = 2e^{2x} - 8e^x = 0$.
- M – $2e^x(e^x - 4) = 0 \implies e^x = 4$.

Question 6a *1 mark*

Consider the binomial random variable $X \sim \text{Bi}(6, \frac{1}{4})$.

Find $\text{var}(X)$.

Marking Guide — Answer: $\frac{9}{8}$

- $\text{var}(X) = np(1-p) = 6 \times \frac{1}{4} \times \frac{3}{4} = \frac{18}{16} = \frac{9}{8}$.

Question 6b *2 marks*

Consider the binomial random variable $X \sim \text{Bi}(6, \frac{1}{4})$.

Determine $\Pr(X \geq 5)$. Give your answer in the form $\frac{a}{2^b}$, where $a, b \in Z$.

Marking Guide — Answer: $\frac{19}{2^{12}}$

- $\Pr(X = 5) = \binom{6}{5}(\frac{1}{4})^5(\frac{3}{4})^1 = \frac{18}{4^6}$.
- $\Pr(X = 6) = (\frac{1}{4})^6 = \frac{1}{4^6}$.
- M – correct binomial expressions.
- A – $\Pr(X \geq 5) = \frac{19}{4^6} = \frac{19}{2^{12}}$.

Question 7a *1 mark*

Let $f : R \to R, f(x) = x^3 - x^2 - 16x - 20$.

Verify that $x = 5$ is a solution of $f(x) = 0$.

Marking Guide — Answer: $f(5) = 125 - 25 - 80 - 20 = 0$

- Substitute: $f(5) = 125 - 25 - 80 - 20 = 0$ ✓.

Question 7b *2 marks*

Express $f(x)$ in the form $(x + d)^2(x - 5)$, where $d \in R$.

Marking Guide — Answer: $f(x) = (x+2)^2(x-5)$, so $d = 2$

- Divide $f(x)$ by $(x-5)$: $f(x) = (x-5)(x^2+4x+4)$.
- M – polynomial division or factor theorem.
- A – $(x-5)(x+2)^2$, so $d = 2$.

Question 7c *1 mark*

Consider the graph of $y = f(x)$, as shown.

Complete the coordinate pairs of all axial intercepts of $y = f(x)$.

Marking Guide — Answer: $(-2, 0)$, $(5, 0)$, $(0, -20)$

- x-intercepts at $x = -2$ (touch) and $x = 5$ (cross).
- y-intercept: $f(0) = -20$.
- All three coordinate pairs: $(-2, 0)$, $(5, 0)$, $(0, -20)$.

Question 7d.i *1 mark*

Let $g : R \to R, g(x) = x + 2$.

State the coordinates of the stationary point of inflection for the graph of $y = f(x)g(x)$.

Marking Guide — Answer: $(-2, 0)$

- $f(x)g(x) = (x+2)^2(x-5)(x+2) = (x+2)^3(x-5)$.
- Stationary point of inflection at $x = -2$ (triple root).
- Coordinates: $(-2, 0)$.

Question 7d.ii *1 mark*

Write down the values of x for which $f(x)g(x) \geq 0$.

Marking Guide — Answer: $x = -2$ or $x \geq 5$

- $y = (x+2)^3(x-5)$. Quartic, positive leading coefficient.
- Zero at $x = -2$ (inflection, touches) and $x = 5$ (crosses).
- $y \geq 0$ when $x = -2$ or $x \geq 5$.

Question 8a *3 marks*

Consider $f(x) = \begin{cases} \frac{3}{8}(4-3x) & 0 \leq x \leq \frac{4}{3} \\ 0 & \text{otherwise} \end{cases}$.

The continuous random variable X has probability density function $f(x)$.

Find k such that $\Pr(X > k) = \frac{9}{16}$.

Marking Guide — Answer: $k = \frac{1}{3}$

- $\Pr(X > k) = \int_k^{4/3} \frac{3}{8}(4-3x)\,dx = \frac{9}{16}$.
- Note: $f(x)$ is linear from $\frac{3}{2}$ at $x=0$ to 0 at $x=\frac{4}{3}$. The region from k to $\frac{4}{3}$ forms a triangle.
- Area $= \frac{1}{2}(\frac{4}{3}-k)\cdot f(k) = \frac{1}{2}(\frac{4}{3}-k)\cdot\frac{3}{8}(4-3k) = \frac{9}{16}(\frac{4}{3}-k)^2$.
- Solve $\frac{9}{16}(\frac{4}{3}-k)^2 = \frac{9}{16}$, so $(\frac{4}{3}-k)^2 = 1$.
- $\frac{4}{3}-k = \pm 1$, giving $k = \frac{1}{3}$ or $k = \frac{7}{3}$.
- Since $0 \le k \le \frac{4}{3}$, $k = \frac{1}{3}$.

Question 8b *2 marks*

The function $h(x)$ is a transformation of $f(x)$ such that $h(x) = mf(x) + n$, where m and n are real numbers.

Find $\int_0^{4/3} h(x)\,dx$ in terms of m and n.

Marking Guide — Answer: $m + \frac{4n}{3}$

Question 9a *3 marks*

Consider the functions $f : R \setminus \{1\} \to R, f(x) = \frac{w^2}{(x-1)^2}$ and $g : R \to R, g(x) = (x-w)^2$, where $w \in R$.

If $w = -3$, find the four solutions to $f(x) = g(x)$.

Marking Guide — Answer: $x = -1-\sqrt{7}, -1+\sqrt{7}, -2, 0$

- $\frac{9}{(x-1)^2} = (x+3)^2 \implies \left(\frac{3}{x-1}\right)^2 = (x+3)^2$.
- Take square root: $(x-1)(x+3) = \pm 3$.
- Case 1: $x^2+2x-3 = 3 \implies x^2+2x-6 = 0 \implies x = -1\pm\sqrt{7}$.
- Case 2: $x^2+2x-3 = -3 \implies x^2+2x = 0 \implies x(x+2) = 0 \implies x = 0, -2$.
- M – setting up squared equation correctly.
- M – solving one case.
- A – all four solutions.

Question 9b.i *2 marks*

Consider the case where $w > 0$.

Find, in terms of w, the coordinates of the minimum point of the graph of $y = (x-1)(x-w)$.

Marking Guide — Answer: $\left(\frac{w+1}{2}, -\frac{(w-1)^2}{4}\right)$

- Axis of symmetry: $x = \frac{1+w}{2}$.
- M – find x-coordinate of vertex.
- $y = (\frac{w+1}{2}-1)(\frac{w+1}{2}-w) = \frac{w-1}{2}\cdot\frac{1-w}{2} = -\frac{(w-1)^2}{4}$.
- A – coordinates $\left(\frac{w+1}{2}, -\frac{(w-1)^2}{4}\right)$.

Question 9b.ii *2 marks*

Hence, or otherwise, find the positive values of w for which $f(x) = g(x)$ has exactly three solutions.

Marking Guide — Answer: $w = 3 - 2\sqrt{2}$ or $w = 3 + 2\sqrt{2}$

- $f(x) = g(x) \implies (x-1)(x-w) = \pm w$.
- The parabola $y = (x-1)(x-w)$ intersects $y = w$ (2 solutions always for $w > 0$) and $y = -w$.
- For exactly 3 total solutions, $y = -w$ must be tangent to the parabola (1 solution).
- Set vertex value equal to $-w$: $-\frac{(w-1)^2}{4} = -w$.
- $(w-1)^2 = 4w \implies w^2 - 6w + 1 = 0$.
- A $-$ $w = 3 \pm 2\sqrt{2}$. Both positive.