ATAR Master

VCE Mathematical Methods

2016 Examination 1 (Technology-Free)

Questions & Marking Guide

Total: 40 marks

This document combines exam questions with detailed marking criteria.
Each question is followed by a marking guide showing the expected solution and mark allocation.

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Question 1a *2 marks*

Let $y = \frac{\cos(x)}{x^2+2}$.

Find $\frac{dy}{dx}$.

Marking Guide — Answer: $\frac{dy}{dx} = \frac{-\sin(x)(x^2+2)-2x\cos(x)}{(x^2+2)^2}$

- Apply quotient rule: $\frac{dy}{dx} = \frac{-\sin(x)(x^2+2)-\cos(x)\cdot 2x}{(x^2+2)^2}$.
- Simplify: $\frac{dy}{dx} = \frac{-(x^2+2)\sin(x)-2x\cos(x)}{(x^2+2)^2}$.

Question 1b *2 marks*

Let $f(x) = x^2e^{5x}$.

Evaluate $f'(1)$.

Marking Guide — Answer: $f'(1) = 7e^5$

- Product rule: $f'(x) = 2xe^{5x} + 5x^2e^{5x} = xe^{5x}(2+5x)$.
- Evaluate: $f'(1) = e^5(2+5) = 7e^5$.

Question 2a *1 mark*

Let $f : \left(-\infty, \frac{1}{2}\right] \to R$, where $f(x) = \sqrt{1-2x}$.

Find $f'(x)$.

Marking Guide — Answer: $f'(x) = \frac{-1}{\sqrt{1-2x}}$

- $f'(x) = \frac{1}{2\sqrt{1-2x}} \cdot (-2) = \frac{-1}{\sqrt{1-2x}}$.

Question 2b *2 marks*

Find the angle θ from the positive direction of the x-axis to the tangent to the graph of f at $x = -1$, measured in the anticlockwise direction.

Marking Guide — Answer: $\theta = \pi - \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$

- $f'(-1) = \frac{-1}{\sqrt{3}}$.
- $\tan(\theta) = -\frac{1}{\sqrt{3}}$, and the gradient is negative, so $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Question 3a *3 marks*

Let $f : R \setminus \{1\} \to R$, where $f(x) = 2 + \frac{3}{x-1}$.

Sketch the graph of f. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation.

Marking Guide — Answer: Vertical asymptote: $x = 1$, Horizontal asymptote: $y = 2$, x-intercept: $\left(-\frac{1}{2}, 0\right)$, y-intercept: $(0, -1)$

- Vertical asymptote at $x = 1$.

- Horizontal asymptote at $y = 2$.
- x-intercept: $0 = 2 + \frac{3}{x-1} \implies x - 1 = -\frac{3}{2} \implies x = -\frac{1}{2}$. Point $\left(-\frac{1}{2}, 0\right)$.
- y-intercept: $f(0) = 2 + \frac{3}{-1} = -1$. Point $(0, -1)$.
- Correct shape: two branches, one in each region divided by $x = 1$.

Question 3b *2 marks*

Find the area enclosed by the graph of f, the lines $x = 2$ and $x = 4$, and the x-axis.

Marking Guide — Answer: $4 + 3\log_e(3)$

Question 4a *1 mark*

A paddock contains 10 tagged sheep and 20 untagged sheep. Four times each day, one sheep is selected at random from the paddock, placed in an observation area and studied, and then returned to the paddock.

What is the probability that the number of tagged sheep selected on a given day is zero?

Marking Guide — Answer: $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$

- $\Pr(\text{untagged}) = \frac{20}{30} = \frac{2}{3}$. Four independent selections: $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$.

Question 4b *1 mark*

What is the probability that at least one tagged sheep is selected on a given day?

Marking Guide — Answer: $1 - \frac{16}{81} = \frac{65}{81}$

- $\Pr(\text{at least one tagged}) = 1 - \Pr(\text{none tagged}) = 1 - \frac{16}{81} = \frac{65}{81}$.

Question 4c *1 mark*

What is the probability that no tagged sheep are selected on each of six consecutive days?

Express your answer in the form $\left(\frac{a}{b}\right)^c$, where a, b and c are positive integers.

Marking Guide — Answer: $\left(\frac{2}{3}\right)^{24}$

- Each day: $\left(\frac{2}{3}\right)^4$. Six days: $\left(\left(\frac{2}{3}\right)^4\right)^6 = \left(\frac{2}{3}\right)^{24}$.

Question 5a.i *1 mark*

Let $f : (0, \infty) \to R$, where $f(x) = \log_e(x)$ and $g : R \to R$, where $g(x) = x^2 + 1$.

Find the rule for h, where $h(x) = f(g(x))$.

Marking Guide — Answer: $h(x) = \log_e(x^2 + 1)$

- $h(x) = f(g(x)) = f(x^2 + 1) = \log_e(x^2 + 1)$.

Question 5a.ii *2 marks*

State the domain and range of h.

Marking Guide — Answer: Domain: R, Range: $[0, \infty)$

- Domain: R (since $x^2 + 1 > 0$ for all $x \in R$).
- Range: $x^2 + 1 \geq 1$, so $\log_e(x^2 + 1) \geq \log_e(1) = 0$. Range $= [0, \infty)$.

Question 5a.iii *2 marks*

Show that $h(x) + h(-x) = f\left((g(x))^2\right)$.

Marking Guide — Answer: Proof as shown in marking guide

- $h(x)+h(-x) = \log_e(x^2+1)+\log_e((-x)^2+1) = \log_e(x^2+1)+\log_e(x^2+1) = 2\log_e(x^2+1) = \log_e((x^2+1)^2)$.
- $f((g(x))^2) = f((x^2+1)^2) = \log_e((x^2+1)^2)$.
- Therefore $h(x) + h(-x) = f((g(x))^2)$.

Question 5a.iv *2 marks*

Find the coordinates of the stationary point of h and state its nature.

Marking Guide — Answer: $(0, 0)$ is a local minimum

- $h'(x) = \frac{2x}{x^2+1}$. Set $h'(x) = 0$: $x = 0$.
- $h(0) = \log_e(1) = 0$. So stationary point is $(0, 0)$.
- For $x < 0$, $h'(x) < 0$; for $x > 0$, $h'(x) > 0$. So $(0, 0)$ is a local minimum.

Question 5b.i *2 marks*

Let $k : (-\infty, 0] \to R$, where $k(x) = \log_e(x^2 + 1)$.

Find the rule for k^{-1}.

Marking Guide — Answer: $k^{-1}(x) = -\sqrt{e^x - 1}$

- Let $y = \log_e(x^2 + 1)$. Then $e^y = x^2 + 1$, so $x^2 = e^y - 1$.
- Since $x \leq 0$: $x = -\sqrt{e^y - 1}$.
- $k^{-1}(x) = -\sqrt{e^x - 1}$.

Question 5b.ii *2 marks*

State the domain and range of k^{-1}.

Marking Guide — Answer: Domain: $[0, \infty)$, Range: $(-\infty, 0]$

- Domain of k^{-1} = Range of $k = [0, \infty)$.

Question 6a *2 marks*

Let $f : [-\pi, \pi] \to R$, where $f(x) = 2\sin(2x) - 1$.

Calculate the average rate of change of f between $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{6}$.

Marking Guide — Answer: $\frac{f(\pi/6)-f(-\pi/3)}{\pi/6-(-\pi/3)} = \frac{0-(-1-1)}{\pi/2} = \frac{2}{\pi/2} = \frac{4}{\pi}$

- $f\left(\frac{\pi}{6}\right) = 2\sin\left(\frac{\pi}{3}\right) - 1 = 2 \cdot \frac{\sqrt{3}}{2} - 1 = \sqrt{3} - 1.$
- $f\left(-\frac{\pi}{3}\right) = 2\sin\left(-\frac{2\pi}{3}\right) - 1 = 2 \cdot \left(-\frac{\sqrt{3}}{2}\right) - 1 = -\sqrt{3} - 1.$
- Average rate $= \frac{(\sqrt{3}-1)-(-\sqrt{3}-1)}{\frac{\pi}{6}+\frac{\pi}{3}} = \frac{2\sqrt{3}}{\frac{\pi}{2}} = \frac{4\sqrt{3}}{\pi}.$

Question 6b

3 marks

Calculate the average value of f over the interval $-\frac{\pi}{3} \leq x \leq \frac{\pi}{6}$.

Marking Guide — Answer: $\frac{1}{\pi/2}\int_{-\pi/3}^{\pi/6}(2\sin(2x)-1)\,dx = \frac{2}{\pi}\left(\frac{1}{2} - \frac{\pi}{6}\right) = \frac{3-\pi}{3\pi}$

- Average value $= \frac{1}{\pi/6-(-\pi/3)}\int_{-\pi/3}^{\pi/6}(2\sin(2x)-1)\,dx = \frac{2}{\pi}\int_{-\pi/3}^{\pi/6}(2\sin(2x)-1)\,dx.$

Question 7a

2 marks

A company produces motors for refrigerators. There are two assembly lines, Line A and Line B. 5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty. In one hour, 40 motors are produced from Line A and 50 motors are produced from Line B. At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

What is the probability that the selected motor is faulty? Express your answer in the form $\frac{1}{b}$, where b is a positive integer.

Marking Guide — Answer: $\frac{1}{15}$

- $\Pr(\text{faulty}) = \frac{40}{90} \times 0.05 + \frac{50}{90} \times 0.08 = \frac{2}{90} + \frac{4}{90} = \frac{6}{90} = \frac{1}{15}.$

Question 7b

1 mark

The selected motor is found to be faulty.

What is the probability that it was assembled on Line A? Express your answer in the form $\frac{1}{c}$, where c is a positive integer.

Marking Guide — Answer: $\frac{1}{3}$

- $\Pr(A|\text{faulty}) = \frac{\Pr(\text{faulty}\cap A)}{\Pr(\text{faulty})} = \frac{2/90}{6/90} = \frac{2}{6} = \frac{1}{3}.$

Question 8a

2 marks

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} -4x\log_e(x) & 0 < x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Show by differentiation that $\frac{x^k}{k^2}(k\log_e(x) - 1)$ is an antiderivative of $x^{k-1}\log_e(x)$, where k is a positive real number.

Marking Guide — Answer: Proof by differentiation as shown in marking guide

- Let $F(x) = \frac{x^k}{k^2}(k\log_e(x) - 1)$.
- $F'(x) = \frac{kx^{k-1}}{k^2}(k\log_e(x) - 1) + \frac{x^k}{k^2} \cdot \frac{k}{x}$.
- $= \frac{x^{k-1}}{k}(k\log_e(x) - 1) + \frac{x^{k-1}}{k} = \frac{x^{k-1}}{k}(k\log_e(x) - 1 + 1) = x^{k-1}\log_e(x)$.

Question 8b.i *2 marks*

Calculate $\Pr\left(X > \frac{1}{e}\right)$.

Marking Guide — Answer: $\frac{2}{e^2}$

Question 8b.ii *2 marks*

Hence, explain whether the median of X is greater than or less than $\frac{1}{e}$, given that $e > \frac{5}{2}$.

Marking Guide — Answer: The median is greater than $\frac{1}{e}$

- $\Pr\left(X > \frac{1}{e}\right) = \frac{e^2-3}{e^2} = 1 - \frac{3}{e^2}$.
- Since $e > \frac{5}{2}$, $e^2 > \frac{25}{4}$, so $\frac{3}{e^2} < \frac{12}{25} < \frac{1}{2}$.
- Therefore $\Pr\left(X > \frac{1}{e}\right) > \frac{1}{2}$, so the median is greater than $\frac{1}{e}$.