
ATAR Master

VCE Mathematical Methods

2021 Examination 2 (Technology-Active)

Questions & Marking Guide

Total: 80 marks

This document combines exam questions with detailed marking criteria.
Each question is followed by a marking guide showing the expected solution and mark allocation.

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Section A: Multiple Choice — 20 marks

Each question is worth 1 mark.

Question 1

1 mark

The period of the function with rule $y = \tan\left(\frac{\pi x}{2}\right)$ is

- A. 1
- B. 2
- C. 4
- D. 2π
- E. 4π

Marking Guide — Answer: B

- Period = $\frac{\pi}{\pi/2} = 2$.

Question 2

1 mark

The graph of $y = \log_e(x) + \log_e(2x)$, where $x > 0$, is identical, over the same domain, to the graph of

- A. $y = 2 \log_e(2x)$
- B. $y = \log_e(2x^2)$
- C. $y = \log_e(3x)$
- D. $y = \log_e(4x)$

Marking Guide — Answer: C

- $\log_e(x) + \log_e(2x) = \log_e(2x^2)$.

Question 3

1 mark

A box contains many coloured glass beads.

A random sample of 48 beads is selected and it is found that the proportion of blue-coloured beads in this sample is 0.125.

Based on this sample, a 95% confidence interval for the proportion of blue-coloured glass beads is

- A. (0.0314, 0.2186)
- B. (0.0465, 0.2035)
- C. (0.0018, 0.2482)
- D. (0.0896, 0.1604)
- E. (0.0264, 0.2136)

Marking Guide — Answer: E

- $\hat{p} = 0.125, n = 48$. $CI = 0.125 \pm 1.96\sqrt{\frac{0.125 \times 0.875}{48}}$.

Question 4

1 mark

The maximum value of the function $h : [0, 2] \rightarrow R$, $h(x) = (x - 2)e^x$ is

- A. $-e$
- B. 0
- C. 1
- D. 2
- E. e

Marking Guide — Answer: B

Question 5

1 mark

Consider the following four functional relations.

$$f(x) = f(-x) \quad -f(x) = f(-x) \quad f(x) = -f(x) \quad (f(x))^2 = f(x^2)$$

The number of these functional relations that are satisfied by the function $f : R \rightarrow R$, $f(x) = x$ is

- A. 0
- B. 1
- C. 2
- D. 3
- E. 4

Marking Guide — Answer: B

- Only $-f(x) = f(-x)$ is satisfied (odd function).

Question 6

1 mark

The probability of winning a game is 0.25.

The probability of winning a game is independent of winning any other game.

If Ben plays 10 games, the probability that he will win exactly four times is closest to

- A. 0.1460
- B. 0.2241
- C. 0.9219
- D. 0.0781
- E. 0.7759

Marking Guide — Answer: B

- $\Pr(X = 4) = \binom{10}{4}(0.25)^4(0.75)^6 \approx 0.1460...$ check options.

Question 7

1 mark

The tangent to the graph of $y = x^3 - ax^2 + 1$ at $x = 1$ passes through the origin.

The value of a is

- A. 1
- B. 2

Marking Guide — Answer: D

- Find tangent at $x = 1$ and set it to pass through $(0, 0)$.

Question 8

1 mark

The graph of the function f is shown below (a curve with a vertical asymptote at $x = a$, approaching from the left going to $-\infty$ and from the right increasing).

The graph corresponding to f' is

- A. Graph A (curve with horizontal asymptote, increasing from left)
- B. Graph B (curve with vertical asymptote, positive on both sides)
- C. Graph C (curve with vertical asymptote, negative left, positive right)
- D. Graph D (curve with vertical asymptote, positive left, positive right)
- E. Graph E (curve with vertical asymptote, positive decreasing)

Marking Guide — Answer: D

- Graph D shows the derivative consistent with the given function shape.

Section B: Extended Response — 60 marks

Question 9

1 mark

Let $g(x) = x + 2$ and $f(x) = x^2 - 4$.

If h is the composite function given by $h : [-5, -1) \rightarrow \mathbb{R}$, $h(x) = f(g(x))$, then the range of h is

Marking Guide — Answer: A

Question 10

1 mark

Consider the functions $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{1-2x}$, defined over their maximal domains.

The maximal domain of the function $h = f + g$ is

A. $[-2, \infty)$

Marking Guide — Answer: D

Question 11

1 mark

If $\int_0^a f(x) dx = k$, then $\int_0^a (3f(x) + 2) dx$ is

A. $3k + 2a$

B. $3k$

C. $k + 2a$

D. $k + 2$

E. $3k + 2$

Marking Guide — Answer: A

- $\int_0^a (3f(x) + 2) dx = 3k + 2a$.

Question 12

1 mark

For a certain species of bird, the proportion of birds with a crest is known to be $\frac{3}{5}$.

Let \hat{P} be the random variable representing the proportion of birds with a crest in samples of size n for this specific bird.

The smallest sample size for which the standard deviation of \hat{P} is less than 0.08 is

A. 7

B. 27

C. 37

D. 38

E. 43

Marking Guide — Answer: D

- $\text{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}} < 0.08$. Solve for n .

Question 13

1 mark

The value of an investment, in dollars, after n months can be modelled by the function

$$f(n) = 2500 \times (1.004)^n$$

where $n \in \{0, 1, 2, \dots\}$.

The average rate of change of the value of the investment over the first 12 months is closest to

- A. \$1 per month
- B. \$1 per month
- C. \$1 per month
- D. \$1 per month
- E. \$1 per month

Marking Guide — Answer: C

- $\frac{f(12)-f(0)}{12} = \frac{2500(1.004^{12}-1)}{12}$.

Question 14

1 mark

A value of k for which the average value of $y = \cos(kx - \frac{\pi}{2})$ over the interval $[0, \pi]$ is equal to the average value of $y = \sin(x)$ over the same interval is

Marking Guide — Answer: E

- Compute average values and equate to find k .

Question 15

1 mark

Four fair coins are tossed at the same time.

The outcome for each coin is independent of the outcome for any other coin.

The probability that there is an equal number of heads and tails, given that there is at least one head, is

Marking Guide — Answer: D

- $\Pr(2H2T|\text{at least 1H}) = \frac{\binom{4}{2}/16}{15/16} = \frac{6}{15} = \frac{2}{5}$.

Question 16

1 mark

Let $\cos(x) = \frac{3}{5}$ and $\sin^2(y) = \frac{25}{169}$, where $x \in [\frac{3\pi}{2}, 2\pi]$ and $y \in [\frac{3\pi}{2}, 2\pi]$.

The value of $\sin(x) + \cos(y)$ is

Marking Guide — Answer: A

Question 17

1 mark

A discrete random variable X has a binomial distribution with a probability of success of $p = 0.1$ for n trials, where $n > 2$.

If the probability of obtaining at least two successes after n trials is at least 0.5, then the smallest possible value of n is

- A. 15
- B. 16
- C. 17
- D. 18
- E. 19

Marking Guide — Answer: E

- Find smallest n such that $\Pr(X \geq 2) \geq 0.5$ with $p = 0.1$.

Question 18

1 mark

Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = (2x - 1)(2x + 1)(3x - 1)$ and $g : (-\infty, 0) \rightarrow \mathbb{R}$, $g(x) = x \log_e(-x)$.

The maximum number of solutions for the equation $f(x - k) = g(x)$, where $k \in \mathbb{R}$, is

- A. 0
- B. 1
- C. 2
- D. 3
- E. 4

Marking Guide — Answer: D

- Analyse the number of intersections by considering graph translations.

Question 19

1 mark

Which one of the following functions is differentiable for all real values of x ?

Marking Guide — Answer: E

- Check continuity and differentiability at $x = 0$ for each piecewise function.

Question 20

1 mark

Let A and B be two independent events from a sample space.

If $\Pr(A) = p$, $\Pr(B) = p^2$ and $\Pr(A) + \Pr(B) = 1$, then $\Pr(A' \cup B)$ is equal to

- A. $1 - p - p^2$
- B. $p^2 - p^3$

- C. $p - p^3$
 D. $1 - p + p^3$
 E. $1 - p - p^2 + p^3$

Marking Guide — Answer: E

- Use $\Pr(A' \cup B) = 1 - \Pr(A) + \Pr(A \cap B) = 1 - p + p \cdot p^2 = 1 - p + p^3 \dots$ with $p + p^2 = 1$.

1a

1 mark

A rectangular sheet of cardboard has a width of h centimetres. Its length is twice its width. Squares of side length x centimetres, where $x > 0$, are cut from each of the corners. The sides are then folded up to make a rectangular box with an open top.

A box is to be made from a sheet of cardboard with $h = 25$ cm.

Show that the volume, V_{box} , in cubic centimetres, is given by $V_{box}(x) = 2x(25 - 2x)(25 - x)$.

Marking Guide — Answer: Show: width $= h - 2x = 25 - 2x$, length $= 2h - 2x = 50 - 2x = 2(25 - x)$, height $= x$. $V = x(25 - 2x) \cdot 2(25 - x)$.

- Derive dimensions and multiply to obtain $V_{box}(x) = 2x(25 - 2x)(25 - x)$.

1b

1 mark

State the domain of V_{box} .

Marking Guide — Answer: $0 < x < 12.5$ (or $x \in (0, 12.5)$)

- Domain: $(0, 12.5)$ since $25 - 2x > 0 \Rightarrow x < 12.5$.

1c

1 mark

Find the derivative of V_{box} with respect to x .

Marking Guide — Answer: $V'_{box}(x) = 2(25 - 2x)(25 - x) + 2x(-2)(25 - x) + 2x(25 - 2x)(-1)$

- Differentiate $V_{box}(x) = 2x(25 - 2x)(25 - x)$ using product rule.

1d

3 marks

Calculate the maximum possible volume of the box and for which value of x this occurs.

Marking Guide — Answer: Solve $V'_{box}(x) = 0$ for x in domain, then evaluate V_{box} at that x .

- Set $V'_{box}(x) = 0$, solve, verify maximum, and compute V_{box} .

1e

2 marks

Waste minimisation is a goal when making cardboard boxes. Percentage wasted is based on the area of the sheet of cardboard that is cut out before the box is made.

Find the percentage of the sheet of cardboard that is wasted when $x = 5$.

Marking Guide — Answer: Wasted area = $4x^2 = 100$. Total area = $25 \times 50 = 1250$. Percentage = $\frac{100}{1250} \times 100 = 8\%$.

- Calculate $\frac{4x^2}{h \times 2h} \times 100\%$ with $h = 25$, $x = 5$.

1f.i

1 mark

Now consider a box made from a rectangular sheet of cardboard where $h > 0$ and the box's length is still twice its width.

Let V_{box} be the function that gives the volume of the box.

State the domain of V_{box} in terms of h .

Marking Guide — Answer: $0 < x < \frac{h}{2}$

- Domain: $(0, \frac{h}{2})$.

1f.ii

3 marks

Find the maximum volume for any such rectangular box, V_{box} , in terms of h .

Marking Guide — Answer: Express $V_{box}(x) = 2x(h - 2x)(h - x)$ and optimise in terms of h .

- Differentiate, solve $V'(x) = 0$, substitute back to get max volume in terms of h .

1g

2 marks

Now consider making a box from a square sheet of cardboard with side lengths of h centimetres.

Show that the maximum volume of the box occurs when $x = \frac{h}{6}$.

Marking Guide — Answer: For square sheet: $V = x(h - 2x)^2$. $V' = (h - 2x)^2 - 4x(h - 2x) = (h - 2x)(h - 6x) = 0$. So $x = \frac{h}{6}$.

- Derive $V = x(h - 2x)^2$, differentiate and show $x = \frac{h}{6}$ gives maximum.

2a

1 mark

Four rectangles of equal width are drawn and used to approximate the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$. The heights of the rectangles are the values of the graph of $y = x^2$ at the right endpoint of each rectangle.

State the width of each of the rectangles shown above.

Marking Guide — Answer: $\frac{1}{4}$

- Width = $\frac{1-0}{4} = \frac{1}{4}$.

2b

1 mark

Find the total area of the four rectangles shown above.

Marking Guide — Answer: $\frac{1}{4} \left[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2 \right] = \frac{15}{32}$

- $\text{Sum} = \frac{1}{4} \left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1 \right) = \frac{15}{32}.$

2c

2 marks

Find the area between the graph of $y = x^2$, the x -axis and the line $x = 1$.

Marking Guide — Answer: $\int_0^1 x^2 dx = \frac{1}{3}$

2d

1 mark

The graph of f is shown below (a curve from roughly $(-3, 2)$ rising to about $(-1, 8)$ then decreasing through $(0, 0)$ and down to about $(2, -8)$ then rising).

Approximate $\int_{-2}^2 f(x) dx$ using four rectangles of equal width and the right endpoint of each rectangle.

Marking Guide — Answer: Width = 1. Sum = $1 \times [f(-1) + f(0) + f(1) + f(2)]$. Read values from graph.

- Use right endpoints $x = -1, 0, 1, 2$ with width 1, read f values from graph.

2e

1 mark

Parts of the graphs of $y = x^2$ and $y = \sqrt{x}$ are shown below.

Find the area of the shaded region.

Marking Guide — Answer: $\int_0^1 (\sqrt{x} - x^2) dx = \frac{1}{3}$

2f

4 marks

The graph of $y = x^2$ is transformed to the graph of $y = ax^2$, where $a \in (0, 2]$.

Find the values of a such that the area defined by the region(s) bounded by the graphs of $y = ax^2$ and $y = \sqrt{x}$ and the lines $x = 0$ and $x = a$ is equal to $\frac{1}{3}$. Give your answer correct to two decimal places.

Marking Guide — Answer: Set up integral and solve for a .

- Set $\int_0^a |\sqrt{x} - ax^2| dx = \frac{1}{3}$ and solve for a numerically.

3a

2 marks

Let $q(x) = \log_e(x^2 - 1) - \log_e(1 - x)$.

State the maximal domain and the range of q .

Marking Guide — Answer: Domain: $(1, \infty)$; Range: R

- Need $x^2 - 1 > 0$ and $1 - x > 0$; simplify $q(x) = \log_e(x + 1)$ on appropriate domain.

3b.i

1 mark

Find the equation of the tangent to the graph of q when $x = -2$.

Marking Guide — Answer: Find $q(-2)$ and $q'(-2)$, write tangent equation.

- Evaluate $q(-2)$ and $q'(-2)$, then $y - q(-2) = q'(-2)(x + 2)$.

3b.ii

1 mark

Find the equation of the line that is perpendicular to the graph of q when $x = -2$ and passes through the point $(-2, 0)$.

Marking Guide — Answer: Gradient $= -\frac{1}{q'(-2)}$, passing through $(-2, 0)$.

- Perpendicular gradient $= -1/q'(-2)$; line: $y = m(x + 2)$.

3c

1 mark

Let $p(x) = e^{-2x} - 2e^{-x} + 1$.

Explain why p is not a one-to-one function.

Marking Guide — Answer: $p(x) = (e^{-x} - 1)^2$ which has a minimum and is not monotonic.

- $p(x) = (e^{-x} - 1)^2 \geq 0$ with minimum at $x = 0$; not one-to-one as it takes the same value for different x .

3d

1 mark

Find the gradient of the tangent to the graph of p at $x = a$.

Marking Guide — Answer: $p'(a) = -2e^{-2a} + 2e^{-a}$

- $p'(x) = -2e^{-2x} + 2e^{-x}$; evaluate at $x = a$.

3e

3 marks

The line $y = x + 2$ and the tangent to the graph of p at $x = a$ intersect with an acute angle of θ between them.

Find the value(s) of a for which $\theta = 60$. Give your answer(s) correct to two decimal places.

Marking Guide — Answer: Use $\tan(60) = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$ with $m_1 = 1$ and $m_2 = p'(a)$.

- Set $\tan(60) = \left| \frac{1 - p'(a)}{1 + p'(a)} \right|$ and solve for a numerically.

3f

3 marks

Find the x -coordinate of the point of intersection between the line $y = x + 2$ and the graph of p , and hence find the area bounded by $y = x + 2$, the graph of p and the x -axis, both correct to three decimal places.

Marking Guide — Answer: Solve $e^{-2x} - 2e^{-x} + 1 = x + 2$ numerically, then integrate.

- Find intersection numerically, then compute area using definite integral.

4a

1 mark

A teacher coaches their school's table tennis team. The teacher has an adjustable ball machine. The speed, measured in metres per second, of the balls shot by the ball machine is a normally distributed

random variable W .

The teacher sets the ball machine with a mean speed of 10 metres per second and a standard deviation of 0.8 metres per second.

Determine $\Pr(W \geq 11)$, correct to three decimal places.

Marking Guide — Answer: $\Pr(W \geq 11)$

- $\Pr(W \geq 11) = \Pr\left(Z \geq \frac{11-10}{0.8}\right)$.

4b

1 mark

Find the value of k , in metres per second, which 80% of ball speeds are below. Give your answer in metres per second, correct to one decimal place.

Marking Guide — Answer: Find k such that $\Pr(W < k) = 0.80$.

- $k = 10 + 0.8 \times z_{0.80}$.

4c

2 marks

The teacher adjusts the height setting for the ball machine. With the new height setting, 8% of balls do not land on the table.

Let \hat{P} be the random variable representing the sample proportion of balls that do not land on the table in random samples of 25 balls.

Find the mean and the standard deviation of \hat{P} .

Marking Guide — Answer: Mean = 0.08, sd = $\sqrt{\frac{0.08 \times 0.92}{25}}$

- Mean = $p = 0.08$; sd = $\sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.08 \times 0.92}{25}}$.

4d

2 marks

Use the binomial distribution to find $\Pr(\hat{P} > 0.1)$, correct to three decimal places.

Marking Guide — Answer: $\Pr(\hat{P} > 0.1) = \Pr(X > 2.5) = \Pr(X \geq 3)$ where $X \sim \text{Bin}(25, 0.08)$.

- $\hat{P} > 0.1 \Leftrightarrow X > 2.5 \Leftrightarrow X \geq 3$. Compute $1 - \Pr(X \leq 2)$.

4e

1 mark

The teacher can also adjust the spin setting on the ball machine. The spin, measured in revolutions per second, is a continuous random variable X with the probability density function

$$f(x) = \begin{cases} \frac{x}{500} & 0 \leq x < 20 \\ \frac{50-x}{750} & 20 \leq x \leq 50 \\ 0 & \text{elsewhere} \end{cases}$$

Find the maximum possible spin applied by the ball machine, in revolutions per second.

Marking Guide — Answer: 50 revolutions per second

- Maximum spin is 50 (upper bound of the pdf support).

4f

2 marks

Find the median spin, in revolutions per second, correct to one decimal place.

Marking Guide — Answer: Solve for median m from $\int_0^m f(x) dx = 0.5$.

- If $m < 20$: $\int_0^m \frac{x}{500} dx = \frac{m^2}{1000} = 0.5$, so $m = \sqrt{500}$. If $m \geq 20$: use both pieces.

4g

3 marks

Find the standard deviation of the spin, in revolutions per second, correct to one decimal place.

Marking Guide — Answer: Compute $E(X)$ and $E(X^2)$, then $\text{sd} = \sqrt{E(X^2) - [E(X)]^2}$.

- Calculate $E(X)$ and $\text{Var}(X)$ using the piecewise pdf, then take square root.

4h

2 marks

The teacher adjusts the spin setting so that the median spin becomes 30 revolutions per second. This will transform the original probability density function f to a new probability density function g , where $g(x) = af\left(\frac{x}{b}\right)$.

Find the values of a and b for which the new median spin is 30 revolutions per second, giving your answer correct to two decimal places.

Marking Guide — Answer: Find a and b such that the transformed pdf has median 30.

- Use the transformation $g(x) = af(x/b)$ and the constraint that median scales by b , with $a = 1/b$.

5a

1 mark

Part of the graph of $f: R \rightarrow R$, $f(x) = \sin\left(\frac{x}{2}\right) + \cos(2x)$ is shown.

State the period of f .

Marking Guide — Answer: 4π

- Period of $\sin(x/2)$ is 4π ; period of $\cos(2x)$ is π . LCM = 4π .

5b

1 mark

State the minimum value of f , correct to three decimal places.

Marking Guide — Answer: Find minimum of $\sin(x/2) + \cos(2x)$ using CAS.

- Use CAS to find the minimum value of $f(x) = \sin(x/2) + \cos(2x)$.

5c

1 mark

Find the smallest positive value of h for which $f(h - x) = f(x)$.

Marking Guide — Answer: Find axis of symmetry; smallest positive h .

- $f(h - x) = f(x)$ means the graph is symmetric about $x = h/2$. Find smallest positive h .

5d

1 mark

Consider the set of functions of the form $g_a : \mathbb{R} \rightarrow \mathbb{R}$, $g_a(x) = \sin\left(\frac{x}{a}\right) + \cos(ax)$, where a is a positive integer.

State the value of a such that $g_a(x) = f(x)$ for all x .

Marking Guide — Answer: $a = 2$

- $g_2(x) = \sin(x/2) + \cos(2x) = f(x)$.

5e.i

1 mark

Find an antiderivative of g_a in terms of a .

Marking Guide — Answer: $-a \cos\left(\frac{x}{a}\right) + \frac{1}{a} \sin(ax) + c$

- $\int g_a(x) dx = -a \cos(x/a) + \frac{1}{a} \sin(ax) + c$.

5e.ii

3 marks

Use a definite integral to show that the area bounded by g_a and the x -axis over the interval $[0, 2a\pi]$ is equal above and below the x -axis for all values of a .

Marking Guide — Answer: Show $\int_0^{2a\pi} g_a(x) dx = 0$.

- Evaluate $\int_0^{2a\pi} g_a(x) dx$ and show it equals 0 for all positive integer a .

5f

1 mark

Explain why the maximum value of g_a cannot be greater than 2 for all values of a and why the minimum value of g_a cannot be less than -2 for all values of a .

Marking Guide — Answer: $|\sin(x/a)| \leq 1$ and $|\cos(ax)| \leq 1$, so $-2 \leq g_a(x) \leq 2$.

- Since $-1 \leq \sin(x/a) \leq 1$ and $-1 \leq \cos(ax) \leq 1$, we have $-2 \leq g_a(x) \leq 2$.

5g

1 mark

Find the greatest possible minimum value of g_a .

Marking Guide — Answer: Find the value of a that maximises the minimum of g_a .

- As $a \rightarrow \infty$, the minimum of g_a approaches -1 . Greatest possible minimum $= -1$.