$\boxed{\text{2020 AUMC Solutions}}$

1. Area of a triangle.

The area is $\boxed{125/4}$. Let the legs be a and b. Then $a+b = 25-10 = 15$ and $225 = (a+b)^2 = a^2+b^2+2ab = 10^2+2ab$, so $2ab = 125$, and the area is $ab/2 = 125/4$.

2. Triangular numbers less than 2020.

We show that there are $\boxed{63}$ triangular numbers among the positive integers less than 2020. The n-th triangular number T_n is $1+2+\cdots+n = n(n+1)/2$. Then $T_{63} = 63\cdot 64/2 = 2016$ and $T_{64} = 64\cdot 65/2 = 2080$. So, the first 63 triangular numbers are less than 2020 and all others are larger than 2020.

3. Multiples of 7.

Look at the equation modulo 7. The cubes modulo 7 are 0, 1 and -1. The equation $a^3+b^3+c^3 \equiv 0 \pmod 7$ is not possible if each of a, b c is from $\{-1,1\}$ mod 7.

4. Solve for b.

We show that $\boxed{b = \sqrt{8/15}}$. We have $(abcde)^2 = (ab)(bc)(cd)(de)(ea) = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$, so $(abcde) = \sqrt{120}$ Then $c = abc = \sqrt{120}/de = \sqrt{120}/4$, so

$$b = \frac{bc}{c} = \frac{2\cdot 4}{\sqrt{120}} = \frac{4}{\sqrt{30}} = \sqrt{\frac{16}{30}} = \sqrt{\frac{8}{15}}.$$

5. Could this be rational?.

It is not only rational, but an integer. We will show that the value is $\boxed{436}$. We first look for a positive square root of $39+4\sqrt{35}$ of the form $a+b\sqrt{35}$ where a and b are integers. If

$$(a+b\sqrt{35})^2 = a^2+35b^2+2ab\sqrt{35} = 39+4\sqrt{35},$$

then $ab = 2$ and $a^2+35b^2 = 39$. This is satisfied by $a=2,\ b=1$. Thus

$$(39+4\sqrt{35})^{3/2} = (39+4\sqrt{35})(2+\sqrt{35}) = 78+4\cdot 35+47\sqrt{35} = 218+47\sqrt{35}.$$

Similarly we find $\sqrt{39-4\sqrt{35}} = \sqrt{35}-2$, and

$$(39-4\sqrt{35})^{3/2} = (39-4\sqrt{35})(\sqrt{35}-2) = -218+47\sqrt{35},$$

so $(39+4\sqrt{35})^{3/2} - (39-4\sqrt{35})^{3/2} = 436.$

6. An inequality.

The following are equivalent:

$$\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\geq\frac{2}{a}+\frac{2}{b}-\frac{2}{c};$$

$$\frac{a^2+b^2+c^2}{abc}\geq\frac{2bc+2ac-2ab}{abc};$$

$$a^2+b^2+c^2\geq 2bc+2ac-2ab;$$

$$a^2+b^2+c^2-2bc-2ac+2ab\geq 0;$$

$$(a+b-c)^2\geq 0.$$

The last inequality is obviously true, and therefore, so is the first.

7. Sum of consecutive integers is a square.

We show that the smallest such integer is $\boxed{n=7074}$. The sum of the integers from n to $n+2020$ is

$$\frac{n+(n+2020)}{2}(2021)=(n+1010)(2021).$$

As 2021 has no square factors, this will be a square if and only if $n+1010=2021k^2$ for some integer k. With $k=1$, we have $n=1011$, not greater than 2020. But with $k\geq 2$, we have $n+1010\geq 4\cdot 2021=8084$ and $n\geq 7074$, so the smallest value of n is 7074

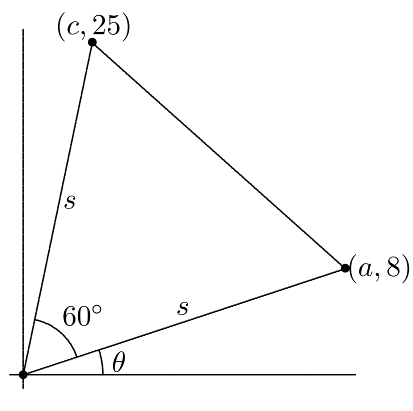
8. Side length of an equilateral triangle.

The side length is $\boxed{\sqrt{652}}$. Let $(a,8)$ and $(c,25)$ be the other two vertices, and let θ be the angle which the side from $(0,0)$ to $(a,8)$ makes with the positive x-axis. Then

$$\sin(\theta+60^\circ)=\sin\theta\cos 60^\circ+\cos\theta\sin 60^\circ;$$

i.e.,

$$\frac{25}{s}=\frac{8}{s}\cdot\frac{1}{2}+\frac{a}{s}\cdot\frac{\sqrt{3}}{2}.$$



This gives us $a\sqrt{3}=42$, so $a=14\sqrt{3}$. Then $s^2=a^2+64=652$, so $s=\sqrt{652}$.

9. Squares in an arithmetic progression.

Let r^2 be one term of the A.P., and let d be the common difference. Note that d must be positive in order that all terms are positive. We want to show that for some integer $m > 0$, $r^2 + md$ is a square; i.e., that $r^2 + md = s^2$ for some integer s. So we want $md = s^2 - r^2 = (s-r)(s+r)$. If we choose $s = r + d$ and $m = s + r$, we have $md = (s+r)(s-r) = s^2 - r^2$, and $r^2 + md = s^2$, as desired. This shows that for every square appearing in the A.P. there is another larger square term, so there are infinitely many.

10. Rational solutions.

The solutions are $\boxed{(x, y) = (5/2, 1/4) \text{ and } (x, y) = (1/2, 5/4)}$. Squaring both members of the given equation gives

$$x + 2y + 2\sqrt{2xy} = 3 + \sqrt{5}, \tag{1}$$

so $2\sqrt{2xy} = r + \sqrt{5}$ for some rational number r. Squaring again gives $8xy = r^2 + 2r\sqrt{5} + 5$, showing that $2r\sqrt{5}$ is rational. With r rational, this implies that $r = 0$. Thus, $2\sqrt{2xy} = \sqrt{5}$, and $xy = 5/8$. From (1) then we have $x + 2y = 3$. Substituting $y = 5/8x$, we get $x + 5/4x = 3$; $4x^2 - 12x + 5 = 0$, giving $x = 5/2$ or $x = 1/2$. Thus the solutions are $(x, y) = (5/2, 1/4)$ and $(x, y) = (1/2, 5/4)$.