$\boxed{\text{2013 AUMC Solutions}}$

1. An integral.

The value is $\boxed{e^2+2-2/e}$. We have

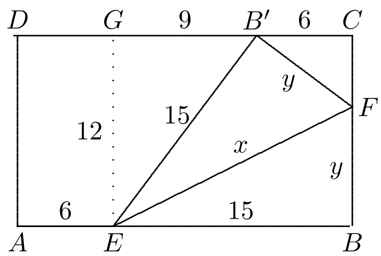
$$\begin{aligned}
\int_{1/e}^{e^2} |\ln x| dx &= \int_{1/e}^{1} (-\ln x)dx + \int_{1}^{e^2} (\ln x)dx \\
&= (-x\ln x + x)\Big|_{1/e}^{1} + (x\ln x - x)\Big|_{1}^{e^2} \\
&= 1 - \left(-\frac{1}{e}(-1) + \frac{1}{e}\right) + (2e^2 - e^2) - (-1) \\
&= e^2 + 2 - 2/e.
\end{aligned}$$

2. An arithmetic progression of 2013 terms.

Let d be the common difference in the arithmetic progression. Rationalize denominators to get

$$\begin{aligned}
&\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{2012}}+\sqrt{a_{2013}}} \\
&\quad = \frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2} + \frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3} + \cdots + \frac{\sqrt{a_{2012}}-\sqrt{a_{2013}}}{a_{2012}-a_{2013}} \\
&\quad = -\frac{1}{d}(\sqrt{a_1}-\sqrt{a_2}+\sqrt{a_2}-\sqrt{a_3}+\cdots+\sqrt{a_{2012}}-\sqrt{a_{2013}}) \\
&\quad = -\frac{1}{d}(\sqrt{a_1}-\sqrt{a_{2013}}) = -\frac{1}{d}\frac{a_1-a_{2013}}{\sqrt{a_1}+\sqrt{a_{2013}}} \\
&\quad = -\frac{1}{d}\frac{-2012d}{\sqrt{a_{2012}}+\sqrt{a_{2013}}} = \frac{2012}{\sqrt{a_1}+\sqrt{a_{2013}}}.
\end{aligned}$$

3. Length of a crease.



We'll show that $\boxed{|EF| = 15\sqrt{5}/2}$. Let $x = |EF|$ and $y = |BF| = |B'F|$. Let G be the point where the line through E parallel to AD crosses CD. Then $|GE| = |AD| = 12$, $|EB'| = |EB| = 15$, so $|GB'| = 9$ by the Pythagorean Theorem. Then $|B'C| = 6$, and from the right triangle $B'CF$, $6^2 + (12-y)^2 = y^2$, giving $y = 15/2$. Finally, from the right triangle EBF we get $x^2 = 15^2 + (15/2)^2$, and $x = 15\sqrt{5}/2$.

4. Roots are three consecutive integers.

They are $\boxed{(p, q) = (12, 60) \text{ and } (-12, -60)}$. If the roots are $n - 1$, n and $n + 1$, then

$$47 = (n - 1)n + (n - 1)(n + 1) + n(n + 1) = 3n^2 - 1,$$

so $n^2 = 16$ and $n = \pm 4$. Then $q = (n - 1)n(n + 1)$ and $p = (n - 1) + n + (n + 1) = 3n$, so with $n = 4$ we get $(p, q) = (12, 60)$, and with $n = -4$, $(p, q) = (-12, -60)$.

5. A polynomial of degree 2013.

The remainder is $\boxed{x^2 + 4x}$. We know that the remainder is a polynomial of the form $ax^2 + bx + c$, so we may write

$$x^{2013} + x^{1013} + x^{513} + x^{113} + x^2 = (x^3 - x)Q(x) + ax^2 + bx + c \tag{1}$$

for some partial quotient $Q(x)$ and remainder $ax^2 + bx + c$. Substituting successively $x = 1$, $x = 0$ and $x = -1$ into (1), we obtain the three equations $5 = a + b + c$, $0 = c$, and $-3 = a - b + c$. The solution of this system is $a = 1$, $b = 4$, $c = 0$, so our remainder is as claimed.

6. Integer values of a quotient.

(a) There are $\boxed{14}$ such integers, and (b) the largest is $\boxed{n = 871}$. By long division we find that

$$\frac{(n - 1)^2}{n + 29} = n - 31 + \frac{900}{n + 29}.$$

This is an integer if, and only if, $n + 29$ is a divisor of 900. Now, $900 = 2^2 3^2 5^2$, which has $3^3 = 27$ divisors (namely, $2^r 3^s 5^t$ with r, s, t chosen from $\{0, 1, 2\}$). As n is required to be positive, $n + 29 \geq 30$, and $30 \cdot 30 = 900$, so of the 27 divisors, 13 are smaller than 30 and 13 are larger. Thus there are 14 positive integers n such that $n + 29$ is a divisor of 900. The largest is the one for which $n + 29 = 900$, namely $n = 871$.

7. A quotient of tangents.

We show that $\boxed{\frac{\tan a}{\tan b} = \frac{m}{n}}$. We have

$$\frac{\sin(a+b)}{\sin(a-b)} = \frac{\sin a \cos b + \cos a \sin b}{\sin a \cos b - \cos a \sin b} = \frac{\frac{\sin a \cos b}{\cos a \sin b} + 1}{\frac{\sin a \cos b}{\cos a \sin b} - 1} = \frac{u+1}{u-1}$$

where $u = \frac{\tan a}{\tan b}$. Then

$$1 + \frac{2n}{m-n} = \frac{m+n}{m-n} = \frac{u+1}{u-1} = 1 + \frac{2}{u-1},$$

so that $\frac{2}{u-1} = \frac{2n}{m-n}$, and thus $u - 1 = \frac{m-n}{n} = \frac{m}{n} - 1$, and $\frac{\tan a}{\tan b} = u = \frac{m}{n}$.

8. m+n=2013.

$\boxed{\text{No such integer exists}}$. The fact that a has n digits means that $10^{n-1} \le a < 10^n$. Then $10^{3n-3} \le a^3 < 10^{3n}$, so $m \in \{3n-2, 3n-1, 3n\}$ and $m + n \in \{4n-2, 4n-1, 4n\}$. Thus $m + n$ is 2, 3 or 0 mod 4. But 2013 is 1 mod 4, so $m + n$ cannot be 2013.

9. Consecutive positive terms.

The largest possible number of consecutive positive terms is $\boxed{5}$. The sequence starting 1,2,3,1,1 has 5 consecutive positive terms. We show that there cannot be six consecutive positive terms. It suffices to show that the first six terms cannot all be positive. We have $a_{n+2} = a_n - a_{n-1}$. If $a_4 \le a_3$, then $a_6 = a_4 - a_3 \le 0$. If $a_4 > a_3$ and a_1 and a_2 are positive, then $a_2 - a_1 = a_4 > a_3$, so $a_3 < a_2 - a_1 < a_2$, and $a_5 = a_3 - a_2 < 0$. Thus at least one of any six consecutive terms is non-positive.

10. Rational or irrational?

We show that $\boxed{N \text{ is rational}}$ by showing that the sequence of final digits is periodic; i.e., that for some k, T_{n+k} and T_n always have the same final digit. Consider that

$$\begin{aligned} T_{n+k} - T_n &= \frac{(n+k)(n+k+1)}{2} - \frac{n(n+1)}{2} \\ &= \frac{n^2 + (2k+1)n + k(k+1) - n^2 - n}{2} = \frac{2kn + k(k+1)}{2}. \end{aligned}$$

If $k = 20$ we have $T_{n+20} - T_n = 20n + 210 \equiv 0 \pmod{10}$, so T_{n+20} and T_n always have the same final digit. Thus the decimal expansion of N is repeating, and N is rational.