

**Fifth Annual Arkansas Undergraduate
Mathematics Competition, February 23, 2008**

No CALCULATORS, COMPUTERS, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. A function of two integers.

Let f be a function such that for all integers m and n ,

$$f(m, 1) = m$$

and

$$f(m, n) = f(m + 1, n - 1).$$

Find $f(2008, 1776)$, and justify your answer.

2. A real product.

Find all complex numbers z such that $(z + 1)(\bar{z} - 2i)$ is real.

3. Difference of two terms.

The arithmetic progression $a_1, a_2, a_3, \dots, a_{2008}$ has sum $a_1 + a_2 + \dots + a_{2008} = S$, while the even-numbered terms have sum $a_2 + a_4 + \dots + a_{2008} = T$. If $2T - S = 251$, find $a_{2008} - a_{2006}$. (You should be able to give a numerical answer.)

4. A multiple of 7.

Prove that $2008^{8002} - 8002^{2008}$ is divisible by 7.

5. A global minimum.

Show that the function f defined on the real numbers by

$$f(x) = \sqrt[3]{3 \sin x - 4 \cos x - 10} \sqrt[3]{3 \sin x + 4 \cos x - 10}$$

has a global minimum, and find its minimum value.

6. Probability of one white and one black.

There are 2009 urns, each containing 2008 marbles. For $0 \leq k \leq 2008$, the k -th urn has k black marbles and $2008 - k$ white ones. An urn is selected at random, and two marbles are drawn simultaneously from it. Find the probability that one is white and the other is black.

7. Limit of a quotient.

Given that the function f is differentiable at a , and that $f(a) = r$ and $f'(a) = s$, show that

$$\lim_{x \rightarrow a} \frac{xf(a) - af(x)}{x - a}$$

exists and find its value in terms of r and s . Note that L'Hospital's Rule is not applicable because f is not known to be differentiable throughout a neighborhood of a ; only at the point a .

8. Tangent of $x + y$.

If $\tan x + \tan y = 1004$ and $\cot x + \cot y = 2008$, find the value of $\tan(x + y)$, and justify your answer.

9. Distance from the fourth corner.

The point P lies interior to the rectangle $ABCD$. (B and D are opposite corners.) If the lengths AP , BP , and CP are, respectively, 8, 12, and 15, find the length DP .

10. No real roots.

Prove that the equation

$$x^6 - x^5 + x^4 - x^3 + x^2 - x + \frac{3}{4} = 0$$

has no real roots.