Fourth Annual Arkansas Undergraduate Mathematics Competition, February 24, 2007

No CALCULATORS, COMPUTERS, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. Find the integer n.

Is there an integer n satisfying the following equation? If so, find it and show that it satisfies the equation. If not, show why not.

$$\frac{1+3+5+\cdots+(2n-1)}{2+4+6+\cdots+2n} = \frac{2007}{2008}.$$

2. Length of a segment.

In the square $ABCD$, the line from point E on side $\overline{CD}$ to point G on side $\overline{BC}$ is perpendicular to the diagonal $\overline{AC}$ and intersects it at F. If $AF = EG = 20$, find length DE.

3. How many slips of paper?

Starting with 18 sheets of paper, some of them are selected and each is cut into 18 pieces. Then some of the smaller pieces are selected and each of those is cut into 18 pieces. This process is continued for a time, and when it is finally stopped, the total number of pieces (which are not necessarily all the same size) is more than 1990 but less than 2020. What is the exact number? Justify your answer.

4. An integral.

Evaluate the following integral, showing all work:

$$\int_0^4 \frac{dx}{|x-1|+|x-2|}$$

5. A probability.

Suppose that the real number x is chosen uniformly at random in the interval $(200, 300)$. Given that $\lfloor \sqrt{x} \rfloor = 15$, find the probability that $\lfloor \sqrt{100x} \rfloor = 150$. Express your answer, if possible, as a rational fraction in lowest terms. (For every real number t, $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t.)

6. Divisible by 2007.

Let $A = (1)(5)\cdots(2001)(2005) = \prod_{n=0}^{501}(4n+1)$ and $B = (2)(6)\cdots(2002)(2006) = \prod_{n=0}^{501}(4n+2)$.

Prove that $B - A$ is divisible by 2007.

7. Remainders.

Let $P(x)$ be a polynomial with real coefficients. Suppose that the remainder on division of $P(x)$ by $(x - 2007)$ is 1 and the remainder on division of $P(x)$ by $(x - 2006)$ is 2. What is the remainder on division of $P(x)$ by $(x - 2007)(x - 2006)$?

8. Secant-cosecant inequality.

If $0 < A < \frac{\pi}{2}$, prove that

$$\sec A + \sec\frac{A}{2} + \sec\frac{A}{3} + \csc A + \csc\frac{A}{2} + \csc\frac{A}{3} > \sec A \csc A + \sec\frac{A}{2}\csc\frac{A}{2} + \sec\frac{A}{3}\csc\frac{A}{3}.$$

9. An extremum?

Let $f(x, y) = (x^2 + 2y)(x^2 + y)$. Determine, with proof, whether the function f has a local minimum, a local maximum, or neither, at $(0, 0)$.

10. Sums in pairs.

On each of five slips of paper a positive integer is written. The sums of the integers taken in pairs are (in no particular order) 21, 25, 32, 13, 28, 40, 25, 33, 18, 37. What are the integers on the five slips? (Justify your answer.)