Twelfth Annual Arkansas Undergraduate Mathematics Competition, February 28, 2015

No CALCULATORS, COMPUTERS, CELL-PHONES, BOOKS, NOTES or **NON-TEAM-MEMBERS** may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team name and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

*   *   Time control: three hours   *   *

1. Coin weights.

There are two piles, one containing 9 identical gold coins and the other 11 identical silver coins. The two piles weigh the same. One coin is taken from each pile and put into the other. It is now found that the pile of mainly gold coins weighs 13 grams less than the pile of mainly silver coins. Find the weight of a silver coin and of a gold coin.

2. A 2015-term sum.

Evaluate the following sum as a rational number in lowest terms:

$$\frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\left(\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right)+\cdots+\left(\frac{1}{2016}+\frac{2}{2016}+\cdots+\frac{2015}{2016}\right).$$

Justify your answer.

3. A rational square root.

Let a, b and c be distinct rational numbers. Prove that

$$\sqrt{\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}}$$

is a rational number.

4. The same sum.

The set A consists of ten distinct integers between 1 and 50 inclusive. Prove that there are two different 5-element subsets of A having the same sum.

5. A differential equation with only the zero solution.

Let f be a function which satisfies the differential equation

$$f''(x) + f'(x)\cos^3 x - f(x) = 0.$$

Suppose that there are two real numbers a and b with $a < b$ such that $f(a) = f(b) = 0$. Prove that $f(x) = 0$ for all x in $[a, b]$.

6. 2015 integers, mostly composite.

Let $S = \{10^k + 1 : k = 1, 2, 3, \ldots, 2015\}$. Prove that more than 99% of the numbers in S are composite.

7. Limit of a function.

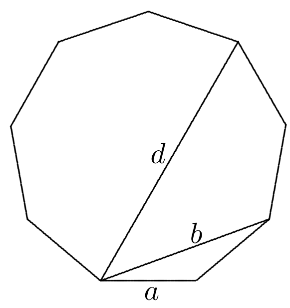
Let $f\colon [0, \infty) \to R$ be a function such that for every $a \geq 0$, the sequence $\{f(a+n)\}$ converges to 0 as $n \to \infty$. Does $\lim_{x\to\infty} f(x)$ necessarily exist? Justify your answer.

8. An even function.

Let a and b be given real numbers, with $0 < a < b$. Suppose that P is a polynomial of degree 6 with real coefficients, such that $P(-a) = P(a)$, $P(-b) = P(b)$, and $P'(0) = 0$. Prove that $P(-x) = P(x)$ for all real x.

9. Diagonals of a regular nonagon.

In the regular nonagon at the right, a, b and d are, respectively, the lengths of a side, a shortest diagonal and a longest diagonal. Show that $d = a + b$.



10. Distance less than 1/2015?.

Does there exist a point A on the graph of $y = x^3$ and a point B on the graph of $y = x^3 + |x| + 1$ such that the distance between A and B is less than 1/2015?