Ninth Annual Arkansas Undergraduate Mathematics Competition, February 25, 2012

No CALCULATORS, COMPUTERS, CELL-PHONES, BOOKS, NOTES or **NON-TEAM-MEMBERS** may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. Color of the last ball.

In a large urn there are 2012 red balls and 2013 blue balls. Beside the urn is a box with a large supply of red balls. The following procedure is performed repeatedly: Two balls are drawn at random from the urn:

(i) If both are red, one is put back and the other is discarded.

(ii) If both are blue, both are discarded and a red ball from the box is put into the urn.

(iii) If one is red and the other blue, the blue one is put back into the urn and the red one is discarded.

After the prodedure has been performed 4024 times, just one ball remains in the urn. What color is it? Explain

2. The coefficient of x.

Let $P_0(x) = x^3 + 178x^2 + 15x - 39$, and for integers $n \geq 1$, let $P_n(x) = P_{n-1}(x - n)$. Find the coefficient of x in the polynomial $P_{15}(x)$.

3. Area of the bounded region.

The (x, y)-plane is partitioned by the graph of $x^2 + 4xy + 64|y| = 256$ into several regions, one of which is bounded. Sketch this bounded region and find its area.

4. An inequality.

Show that, for all $x > 0$,

$$x(2 + \cos x) > 3 \sin x.$$

5. A quotient less than 2012.

Let A be the product of the squares of the first 503 positive even integers and B the product of the squares of the first 503 positive odd integers. Show that $\dfrac{A}{B} < 2012$.

6. Some five of the 2012 have a sum at least 50.

Suppose that the 2012 (not necessarily distinct) positive numbers $x_1, x_2, x_3, \ldots, x_{2012}$ have sum 20120. Show that there must be some five of the numbers having sum at least 50.

7. Probability that $3^m + 3^n$ is divisible by 5.

Integers m and n are chosen independently and at random (uniform distribution) from the set $\{1, 2, 3, \ldots, 50\}$ of the first 50 positive integers. Find the probability that $3^m + 3^n$ is divisible by 5.

8. Three intersection points.

The graphs of $y = x^3 + x^2 - 4x + 1$ and $2x + 3y = 6$ intersect in three points (x_1, y_1), (x_2, y_2), (x_3, y_3). Determine, with proof, the sums $x_1 + x_2 + x_3$ and $y_1 + y_2 + y_3$.

9. A sum of 2012 reciprocals.

Let a_n be the integer nearest to $\sqrt{n}$. Find, with proof,the value of

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2012}}.$$

10. Divergent integrals.

Show that if $f(x)$ and $g(x)$ are positive for $x \geq 1$ and $\int_1^\infty f(x)$ diverges, then at least one of the integrals

$$\int_1^\infty f(x)g(x)dx \qquad \text{and} \qquad \int_1^\infty \frac{f(x)}{g(x)}dx$$

diverges.