Seventeenth Annual Arkansas Undergraduate Mathematics Competition, February 29, 2020

No CALCULATORS, ELECTRONIC DEVICES, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team name and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Enjoy the problems!

* * Time control: three hours * *

1. Area of a triangle.

A right triangle has hypotenuse 10 and perimeter 25. What is is area?

2. Triangular numbers less than 2020.

A triangular number is a positive integer equal to the sum of the first n positive integers for some n. For example, $10 = 1 + 2 + 3 + 4$ is triangular. How many numbers less than 2020 are triangular?

3. Multiples of 7.

Given that a, b, c are integers such that $a^3 + b^3 + c^3 = 0$, prove that at least one of a, b, c is a multiple of 7.

4. Solve for b..

Suppose that a, b, c, d, e are positive real numbers satisfying $ab = 1$, $bc = 2$, $cd = 3$, $de = 4$ and $ea = 5$. Find the value of b, and express it in the form $\sqrt{r/s}$, where r and s are relatively prime positive integers.

5. Could this be rational?.

Determine whether $(39 + 4\sqrt{35})^{3/2} - (39 - 4\sqrt{35})^{3/2}$ is rational or irrational. If irrational, show it. If rational, express it as a fraction in lowest terms.

6. An inequality.

Prove that for all positive real numbers a, b, and c,

$$\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\geq\frac{2}{a}+\frac{2}{b}-\frac{2}{c}.$$

7. Sum of consecutive integers is a square .

Find the smallest integer n such that $n>2020$ and the sum of the integers from n to $n+2020$ inclusive is the square of an integer.

8. Side length of an equilateral triangle.

An equilateral triangle in the first quadrant has one vertex at $(0,0)$, another on the line $y=8$ and the third on the line $y=25$. Find the side length s of the triangle.

9. Squares in an arithmetic progression.

Show that if one term of an arithmetic progression of positive integers is a square, then infinitely many are squares.

10. Rational solutions.

Find all pairs (x,y) of rational numbers satisfying the equation $\sqrt{x}+\sqrt{2y}=\sqrt{3+\sqrt{5}}$.