2007 Arkansas Mathematics Competition Solutions

1. Find the integer n.

The integer $\boxed{n = 2007}$ is the unique solution to the equation. The sum of the arithmetic progression in the numerator is

$$\frac{1 + (2n-1)}{2} \cdot n,$$

and that in the denominator is

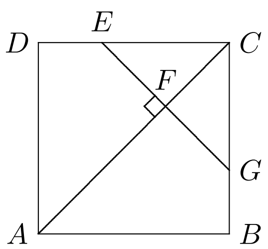
$$\frac{2 + 2n}{2} \cdot n.$$

Then the equation simplifies to

$$\frac{n}{n+1} = \frac{2007}{2008},$$

with solution $n = 2007$.

2. Length of a segment.



The length $\boxed{DE \text{ is } 5\sqrt{2}}$. We have $AF = EG = 20$. By the symmetry with respect to the diagonal $\overline{AC}$ we see that FCE is a $45°$ angle, and therefore $FC = FE = 10$. Then $AC = 30$, $DC = 15\sqrt{2}$, $EC = 10\sqrt{2}$ and so $DE = 5\sqrt{2}$.

3. How many slips of paper?

There are exactly $\boxed{2007}$ pieces. Each time one of the pieces is cut into 18 smaller pieces, the total number of pieces increases by 17. Therefore the total number at the end is of the form $18 + 17n$, where n is an integer. The only number of this form between 1990 and 2020 is $2007 = 18 + (17)(117)$.

4. An integral.

The value is $\boxed{1 + (1/2)\ln 15}$. For $x \leq 1$, $|x-1| + |x-2| = -x+1-x+2 = 3-2x$. For $1 < x < 2$, $|x-1| + |x-2| = x - 1 + (-x+2) = 1$. For $2 < x < 4$, $|x-1| + |x-2| = (x-1) + (x-2) = 2x-3$. Thus

$$\begin{aligned}
\int_0^4 \frac{dx}{|x-1|+|x-2|} &= \int_0^1 \frac{dx}{3-2x} + \int_1^2 dx + \int_2^4 \frac{dx}{2x-3} \\
&= -\frac{1}{2}\ln(3-2x)\Big|_0^1 + 1 + \frac{1}{2}\ln(2x-3)\Big|_2^4 \\
&= -\frac{1}{2}(\ln 1 - \ln 3) + 1 + \frac{1}{2}(\ln 5 - \ln 1) \\
&= 1 + \frac{1}{2}\ln 15.
\end{aligned}$$

5. A probability.

The probability is $\boxed{301/3100}$. On the one hand,

$$\lfloor \sqrt{x} \rfloor = 15 \iff 15 \leq \sqrt{x} < 16 \iff 225 \leq x < 256.$$

On the other hand,

$$\begin{aligned}
\lfloor \sqrt{100x} \rfloor = \lfloor 10\sqrt{x} \rfloor = 150 &\iff 150 \leq 10\sqrt{x} < 151 \\
&\iff 15 \leq \sqrt{x} < 15.1 \\
&\iff 225 \leq x < 228.01.
\end{aligned}$$

Thus the conditional probability is $\frac{3.01}{31} = \frac{301}{3100}$.

6. Divisible by 2007.

Modulo 2007, $1 \equiv -2006$, $5 \equiv -2002$, etc, so

$$A = (1)(5)\cdots(2001)(2005) \equiv (-2006)(-2002)\cdots(-6)(-2) = (-1)^{502}B = B,$$

and thus $A - B \equiv 0 \pmod{2007}$.

(Alternatively, note that $2007 = (9)(223)$ and both 9 and 669 occur as factors in A, while 18 and 446 occur in B. Thus, A and B are, in fact, separately divisible by 2007.)

7. Remainders.

The remainder is $\boxed{(-x+2008)}$. From what is given we know that there are polynomials $Q_1(x)$ and $Q_2(x)$ such that

$$P(x) = (x-2007)Q_1(x) + 1 \tag{1}$$

and

$$P(x) = (x-2006)Q_2(x) + 2. \tag{2}$$

Upon multiplying (1) by $(x-2006)$ and (2) by $(x-2007)$ and subtracting, we obtain

$$\begin{aligned} P(x) &= (x-2007)(x-2006)[Q_1(x)-Q_2(x)] + (x-2006) - 2(x-2007) \\ &= (x-2007)(x-2006)[Q_1(x)-Q_2(x)] + (-x+2008). \end{aligned}$$

A simple example of such a polynomial is

$$\begin{aligned} P(x) &= (x-2007)(-x+2005) + 1 \\ &= (x-2006)(-x+2006) + 2 \\ &= (x-2007)(x-2006)(-1) + (-x+2008). \end{aligned}$$

8. Secant-cosecant inequality.

It suffices to prove that $\sec x + \csc x > \sec x \csc x$ for every acute angle x. Consider the right triangle with x as one of its acute angles and legs a and b, hypotenuse c. Then

$$\sec x + \csc x = \frac{c}{a} + \frac{c}{b} = \frac{c(a+b)}{ab}$$

and

$$\sec x \csc x = \frac{c}{a}\frac{c}{b} = \frac{c^2}{ab}.$$

Because $a+b > c$ we immediately have

$$\sec x + \csc x = \frac{c(a+b)}{ab} > \frac{c^2}{ab} = \sec x \csc x.$$

9. An extremum?

It has $\boxed{\text{neither}}$ a maximum nor a minimum. We see that $f(0,0) = 0$, but in every neighborhood of $(0,0)$ there are points (x_1, y_1) and (x_2, y_2) where $f(x_1, y_1) > 0$ and $f(x_2, y_2) < 0$. For example, for $y_1 = 0$ and arbitrary $x_1 \neq 0$ we have $f(x_1, 0) = x_1^4 > 0$, and for points (x_2, y_2) on the curve $y = (-3/4)x^2$ with $x_2 \neq 0$ we have

$$f(x_2, y_2) = \left(x_2^2 - \frac{3}{2}x_2^2\right)\left(x_2^2 - \frac{3}{4}x_2^2\right) = \left(-\frac{1}{2}x_2^2\right)\left(\frac{1}{4}x_2^2\right) = -\frac{1}{8}x_2^4 < 0.$$

10. Sums in pairs.

They are $\boxed{3,\ 10,\ 15,\ 18,\ 22}$. To find them, denote them in nondecreasing order by $a \leq b \leq c \leq d \leq e$. The sums, written in nondecreasing order, are 13, 18, 21, 25, 25, 28, 32, 33, 37, 40. Then $a + b = 13$, $a + c = 18$, $d + e = 40$ and $c + e = 37$. Also, the sum of all 10 integers is

$$\begin{aligned}
272 &= 13 + 18 + 21 + 25 + 25 + 28 + 32 + 33 + 37 + 40 \\
&= (a + b) + (a + c) + (a + d) + (a + e) + (b + c) + (b + d) + (b + e) \\
&\qquad + (c + d) + (c + e) + (d + e) \\
&= 4(a + b + c + d + e),
\end{aligned}$$

so $a + b + c + d + e = \frac{1}{4}(272) = 68$. Then $c = 68 - (a + b) - (d + e) = 68 - 13 - 40 = 15$, so $a = 3$, $b = 10$, $e = 22$ and $d = 18$.