Eighth Annual Arkansas Undergraduate Mathematics Competition, February 26, 2011

No CALCULATORS, COMPUTERS, CELL-PHONES, BOOKS, NOTES or **NON-TEAM-MEMBERS** may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. Rectangular parallelepiped of volume 2011.

Three faces of a rectangular parallelepiped meet at a common vertex. The volume of the parallelepiped is 2011. Find the product of the areas of these three faces.

2. The square of an integer.

Let m be a positive integer, and A be the integer which in decimal form has $2m$ digits, each equal to 1. Let B be the integer of m digits, each equal to 4. For example, with $m = 2$, $A = 1111$ and $B = 44$. Show that $A + B + 1$ is the square of an integer.

3. Rounding to nearest integer.

Let n be a positive integer. Is it possible to get equal results if

$$\sqrt{10^{2n} - 10^n} \quad \text{and} \quad \sqrt{10^{2n} - 10^n + 1}$$

are rounded to the nearest integer?

4. Some logs.

Suppose that $\log_2(\log_8 x) = \log_8(\log_2 x)$. Find $\log_4 x$, and justify your answer.

5. An inequality.

Given that x, y and z are real numbers with $x + y + z = 0$, prove that $xy + yz + zx \le 0$.

6. Limit of a sequence.

Let
$$a_n = \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \cdots + \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}\right).$$
Find, with proof, $\lim_{n\to\infty} a_n$.

7. The 2011th term.

The sequence $\{a_n\}$ is defined recursively by $a_1 = 1/2011$ and
$$a_{n+1} = \frac{a_n}{1 + na_n} \quad \text{for } n \ge 1.$$
Find a_{2011} in closed form, and justify your answer.

8. An integral root.

Given that a, b, c and d are four distinct integers and that n is an integral root of the equation
$$(x-a)(x-b)(x-c)(x-d) - 4 = 0,$$
evaluate $a + b + c + d$ in terms of n.

9. The middle chord.

Two parallel chords in a circle have lengths 20 and 12. They lie on opposite sides of the center of the circle, and the distance between them is 16. Find the length of the chord parallel to these two and midway between them.

10. Sum of roots.

Let
$$S = \{x : \tan^2 x - 7\tan x + 1 = 0,\ 0 < x < 2\pi\}.$$
(a) How many elements has S? (Explain.)

(b) Find the sum of the elements of S.