2005 Arkansas Mathematics Competition Solutions

1. Cylindrical volume.

The volume decreases by 10.4%. For, if the original volume is $V_0 = \pi r^2 h$, then the revised volume is

$$V_1 = \pi(.8r)^2(1.4h) = (.64)(1.4)V_0 = .896V_0,$$

which is $V_0 - (.104)V_0$.

2. Roots of a quadratic.

The answer is $p = \sqrt{4q+1}$. Let the roots be r and s, with $r \geq s > 0$. That

$$r - s = 1 \tag{1}$$

is given, and that

$$r + s = -p \tag{2}$$

is known from the fact that $x^2 + px + q = (x-r)(x-s)$. Solving (1) and (2) for r and s we obtain

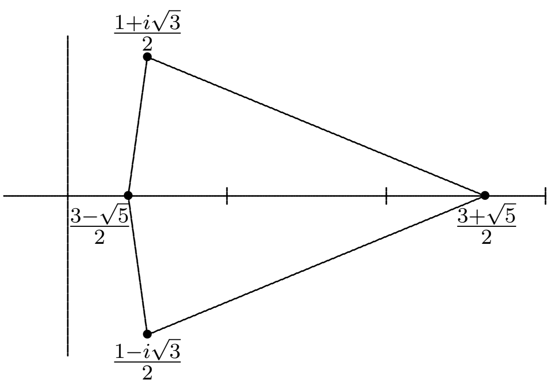
$$r = \frac{1-p}{2} \quad \text{and} \quad s = \frac{-1-p}{2}.$$

Then

$$q = rs = \frac{p^2-1}{4},$$

so $p^2 = 4q+1$, and p is the positive square root: $p = \sqrt{4q+1}$.

3. Area spanned by roots.



The area is $\sqrt{15}/2$. Factor the equation into

$$(x^2 - x + 1)(x^2 - 3x + 1) = 0$$

to find the four roots $\frac{1\pm i\sqrt{3}}{2}$ and $\frac{3\pm\sqrt{5}}{2}$. The x-axis divides the quadrilateral into two congruent triangles, each having base length $\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{3-\sqrt{5}}{2}\right) = \sqrt{5}$ and altitude $\frac{\sqrt{3}}{2}$, so each has area $\frac{\sqrt{15}}{4}$. Thus the total area is $\frac{\sqrt{15}}{2}$.

4. Progressive triples.

The solutions are $(10, 10, 10)$ and $(40, 10, -20)$. Write $a = b - d$ and $c = b + d$, where d is the common difference in the A.P. Then $30 = a + b + c = 3b$, so $b = 10$. From (ii) we have $c^2 = ab = 10a$, so $(10 + d)^2 = 10(10 - d)$; i.e., $d^2 + 30d = 0$. Thus $d = 0$ or $d = -30$. These lead to the solutions $(10, 10, 10)$ and $(40, 10, -20)$, respectively.

5. Not a square.

Let $m = n^4 + 2n^3 + 2n^2 + 2n + 1$. We show that m lies between the consecutive squares $(n^2 + n)^2$ and $(n^2 + n + 1)^2$, so cannot be itself a square integer. We have

$$(n^2 + n)^2 = n^4 + 2n^3 + n^2 < m$$

and

$$(n^2 + n + 1)^2 = n^4 + 2n^3 + 3n^2 + 2n + 1 > m,$$

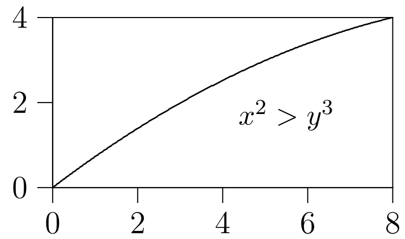
and the proof is complete.

6. Probability.

It is $\frac{3}{5}$. Because of the independence, the "experiment" is equivalent to choosing a point (x, y) at random in the rectangle $(0, 8) \times (0, 4)$, and the probability that $x^2 > y^3$ is the fraction of the total area of this rectangle lying to the right of the curve $y = x^{2/3}$. The area of this portion is

$$\int_0^8 x^{\frac{2}{3}}\,dx = \frac{3}{5}\left[x^{\frac{5}{3}}\right]_0^8 = \frac{3}{5} \cdot 32,$$

which is $\frac{3}{5}$ of the area of the rectangle.



7. Find this year's term.

We'll show that $x_{2005}^2 - y_{2005} = 2$. We have $x_0^2 - y_0 = 2$. For a clue about how $x_n^2 - y_n$ behaves, let's look at $x_1^2 - y_1$.

$$\begin{aligned} x_1^2 - y_1 &= (x_0^3 - 3x_0)^2 - (y_0^3 - 3y_0) \\ &= x_0^6 - 6x_0^4 + 9x_0^2 - (x_0^2 - 2)^3 + 3(x_0^2 - 2) \\ &= x_0^6 - 6x_0^4 + 9x_0^2 - x_0^6 + 6x_0^4 - 12x_0^2 + 8 + 3x_0^2 - 6 \\ &= 2. \end{aligned}$$

Hmm... Suppose that $x_k^2 - y_k = 2$. Then

$$\begin{aligned} x_{k+1}^2 - y_{k+1} &= (x_k^3 - 3x_k)^2 - (y_k^3 - 3y_k) \\ &= x_k^6 - 6x_k^4 + 9x_k^2 - (x_k^2 - 2)^3 + 3(x_k^2 - 2) \\ &= x_k^6 - 6x_k^4 + 9x_k^2 - x_k^6 + 6x_k^4 - 12x_k^2 + 8 + 3x_k^2 - 6 \\ &= 2. \end{aligned}$$

By the induction principle, $x_n^2 - y_n = 2$ for all n, so $x_{2005}^2 - y_{2005} = 2$.

8. Last four digits.

We show that the last four digits are 6807. We find $7^2 = 49$, $7^3 = 343$, and $7^4 = 2401$. Then

$$\begin{aligned} 7^{2005} &= 7 \cdot 7^{2004} \\ &= 7(7^4)^{501} \\ &= 7(1 + 2400)^{501}. \end{aligned}$$

By the binomial theorem,

$$(1 + 2400)^{501} = 1 + 501 \cdot 2400 + k \cdot 2400^2$$

for some integer k, and $k \cdot 2400^2 \equiv 0 \pmod{10000}$. Thus

$$\begin{aligned} 7^{2005} &\equiv 7(1 + (500 + 1)2400) \pmod{10000} \\ &\equiv 7(1 + 2400) \pmod{10000} \\ &\equiv (7)(2401) \pmod{10000} \\ &\equiv 6807 \pmod{10000}. \end{aligned}$$

Thus the last four digits are 6807.

9. Limit of a sequence.

The limit is $\frac{2}{9}$. Write a_n in the form

$$a_n = \frac{1}{n}\left[\left(\frac{1}{n}\right)^{\frac{7}{2}} + \left(\frac{2}{n}\right)^{\frac{7}{2}} + \cdots + \left(\frac{n}{n}\right)^{\frac{7}{2}}\right],$$

and recognize this as a Riemann sum for the integral

$$\int_0^1 x^{\frac{7}{2}}dx = \frac{2}{9}x^{\frac{9}{2}}\bigg|_0^1 = \frac{2}{9}.$$

10. Maximum value.

The maximum value is $\sqrt{10} = f(2)$. Write

$$\begin{aligned} f(x) &= \sqrt{x(7-x)} - \sqrt{(x-2)(7-x)} \\ &= (\sqrt{x} - \sqrt{x-2})\sqrt{7-x}, \end{aligned}$$

and note that $f(x)$ is real if and only if $2 \leq x \leq 7$. Multiplying and dividing by $(\sqrt{x}+\sqrt{x-2})$ we obtain

$$f(x) = \frac{2\sqrt{7-x}}{\sqrt{x}+\sqrt{x-2}},$$

where both numerator and denominator are positive. As x increases from 2 to 7, the denominator increases and the numerator decreases, so $f(x)$ decreases. Thus the maximum value is at $x = 2$, and $f(2) = \sqrt{10}$.