# Seventh Annual Arkansas Undergraduate Mathematics Competition, February 27, 2010

**No CALCULATORS, COMPUTERS, BOOKS, NOTES or NON-TEAM-MEMBERS** may be consulted.

**PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER.** Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

**1. A probability.**

An urn contains 10 red, 20 white and 30 blue balls. If 6 balls are drawn at random without replacement, what is the probability that 1 red, 2 white and 3 blue balls are drawn? Express your answer in exact form as a fraction in lowest terms.

**2. An inequality.**

Suppose that $a, b, c, d$ are positive numbers such that $a + b = c + d$ and $a^2 + b^2 > c^2 + d^2$. Prove that $a^3 + b^3 > c^3 + d^3$.

**3. The perimeter.**

In a right triangle the sum of the legs is 20, and the sum of the hypotenuse and one of the legs is 30. Find (all possible values of) the perimeter of the triangle.

**4. Find $f(-2010)$.**

Let $f(x) = ax^6 + bx^4 + 5x - 9$, where $a$ and $b$ are certain real numbers whose values are being kept secret. If $f(2010) = -2010$, find, with proof, the value of $f(-2010)$.

**5. An integral.**

Evaluate
$$\int_1^{2010} \frac{dx}{1 + \lfloor \log_{10} x \rfloor},$$
where, for a real number $u$, $\lfloor u \rfloor$ denotes the greatest integer less than or equal to $u$.

**6. Power series.**

Find all pairs $(a, r)$ of non-zero real numbers for which the series $\sum_{n=0}^{\infty} ar^n$ converges and
$$\sum_{n=0}^{\infty} (ar^n)^2 = \Big(\sum_{n=0}^{\infty} ar^n\Big)^3.$$

**7. The local maximum value.**

The function $f(x) = x^3 + ax^2 + bx$ has a local minimum value of $-\frac{2}{5\sqrt{5}}$ at $x = \frac{1}{\sqrt{5}}$. Find the local maximum value, and justify your answer.

**8. Graph intersections.**

Given that $a, b, c, d$ are real numbers such that the graphs of $y = |x-a|+b$ and $y = -|x-c|+d$ intersect at $(-6, 3)$ and $(4, 9)$, determine $a$, $b$, $c$ and $d$.

**9. Integer solutions.**

Find all lattice points $(x, y)$ (i.e., points with integer coordinates) lying on the graph of the equation
$$19x^2 + 2x^2y^2 = 64y^2 + 1475,$$
and show that you have them all.

**10. A rational value.**

Given that $\sec x + \tan x = 17/5$, show that $\csc x + \cot x$ is a rational number, and express it as a fraction in lowest terms.