$\boxed{\textbf{2015 AUMC Solutions}}$

1. Coin weights.

$\boxed{\text{A silver coin weighs 117/4 grams and a gold coin 143/4 grams}}$. To find these, let g be the weight of a gold coin and s the weight of a silver coin, in grams. Then

$$9g = 11s \quad \text{and} \quad 8g + s = 10s + g - 13.$$

The second equation simplifies to $7g - 9s = -13$, and on solving simultaneously we find $s = 117/4$ and $g = 143/4$.

2. A 2015-term sum.

The sum is $\boxed{1{,}015{,}560}$. For, the sum of the arithmetic progression

$$\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n} = \frac{n(n-1)}{2n} = \frac{n-1}{2}.$$

Thus the given sum simplifies to

$$\frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \cdots + \frac{2015}{2} = \frac{(2015)(2016)}{4} = (2015)(504) = 1,015,560.$$

3. A rational square root.

Let $x = b - c$, $y = c - a$ and $z = a - b$. Then $x + y + z = 0$, $xyz \neq 0$, and we want to show that

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{y^2z^2 + x^2z^2 + x^2y^2}{x^2y^2z^2}$$

is the square of a rational number. For this it suffices to show that the numerator is the square of a rational number. If we write $z^2 = (x + y)^2$, the numerator becomes

$$y^2(x + y)^2 + x^2(x + y)^2 + x^2y^2,$$

and upon multiplying out and collecting terms we obtain

$$x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4,$$

which is $(x^2 + xy + y^2)^2$, as one may directly verify.

Problem 3, second solution.

This time let
$$x = \frac{1}{b-c},\ y = \frac{1}{c-a} \quad \text{and } z = \frac{1}{a-b}.$$
Then
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0,$$
and we want to show that $x^2 + y^2 + z^2$ is the square of a rational number. Now,
$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + xz).$$
But
$$xy + yz + xz = xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0,$$
so $x^2 + y^2 + z^2 = (x + y + z)^2$. ■

4. The same sum.

This is an application of the pigeon-hole principle. There are $\binom{10}{5} = 252$ different five-element subsets of a ten-element set. Since each element is less than or equal to 50, each sum of a 5-element subset is less than 250, so there are fewer than 250 possible different sums. It follows that some two of the 252 five-element subsets have the same sum.

5. A differential equation with only the zero solution.

As f is differentiable it is continuous on $[a, b]$ and therefore has an absolute maximum and an absolute minimum value on $[a, b]$. Let c be a point in $[a, b]$ where $f(x)$ achieves its maximum value. If c is in (a, b) then by standard theorems from elementary calculus, $f'(c) = 0$ and $f''(c) \leq 0$. From the differential equation it follows that $f(c) = f''(c) \leq 0$, and therefore $f(x) \leq f(c) \leq 0$ for all x in $[a, b]$. If c is at a or b we have $f(x) \leq f(c) = 0$ for all x in $[a, b]$, so in all cases $f(x) \leq 0$ for all x in $[a, b]$. Similarly if d is a point where $f(x)$ achieves its minimum value on $[a, b]$, then $f(d) = 0$ and $f''(d) \geq 0$, and it follows that $f(x) \geq 0$ for all x in $[a, b]$. Thus $f(x) = 0$ for all x in $[a, b]$.

6. 2015 integers, mostly composite.

If $k = mr$ where r is an odd integer, $r \geq 3$, then $10^k + 1 = (10^m)^r + 1$ has $10^m + 1$ as a factor:

$$(x^r + 1) = (x + 1)(x^{r-1} - x^{r-2} + \cdots - x + 1).$$

Thus any noncomposite numbers in S must be of the form $10^k + 1$ where k is a power of 2. There are only 10 such elements in S, somewhat less than 1% of the 2015 elements. Thus more than 99% are composite.

7. Limit of a function.

No, the limit need not exist. Here are two quite different counterexamples. The first is an unbounded function, the second a bounded one.

(1) Let $f(x) = 1/(1 - x)$ for $0 \leq x < 1$, and for $n \leq x < n + 1$, let

$$f(x) = \frac{1}{n} f(x - n) = \frac{1}{n}\left(\frac{1}{1 - (x - n)}\right) = \frac{1}{n(n + 1 - x)}.$$

Then for $0 \leq a < 1$,

$$f(a + n) = \frac{1}{n} f(a) \to 0 \quad \text{as } n \to \infty,$$

and for $a \geq 1$, the sequence $\{f(a + n)\}$ is a subsequence of such a sequence $\{f(b + n)\}$ with $0 \leq b < 1$, so every sequence $\{f(a + n)\}$ converges to 0. But in every interval $[n, n + 1)$ there is a number x_n such that $f(x_n) = 1$: For $n \leq x < n + 1$,

$$f(x) = \frac{1}{n(n + 1 - x)} = 1$$

provided that $n + 1 - x = 1/n$; i.e., $x_n = n + 1 - (1/n)$. Thus $\lim_{x\to\infty} f(x)$ does not exist.

(2) Choose a positive irrational number α and let $f(x) = 1$ if $x = n\alpha$ for some integer $n > 0$, and $f(x) = 0$ otherwise. It is clear that $f(x)$ does not have a limit as $x \to \infty$. Also, in any sequence $\{f(a + n)\}$, at most one term is 1 and all others are 0, for if $a + k = n\alpha$ and $a + l = m\alpha$ for some integers k, l, m, n, then $k - l = (n - m)\alpha$. Because α is irrational, this is impossible unless $k - l = 0$. Thus every sequence $\{f(a + n)\}$ converges to 0.

8. An even function.

Let $P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6$. Then

$$0 = P(a) - P(-a) = 2c_1a + 2c_3a^3 + 2c_5a^5 \tag{1}$$

and

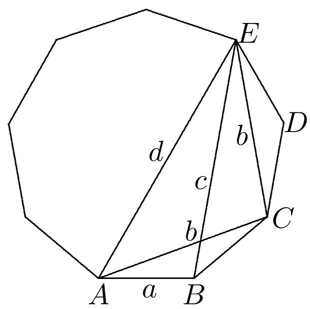
$$0 = P(b) - P(-b) = 2c_1b + 2c_3b^3 + 2c_5b^5. \tag{2}$$

Also, $P'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5$, so $0 = P'(0) = c_1$, and equations (1) and (2) simplify to $2c_3a^3 + 2c_5a^5 = 0$ and $2c_3b^3 + 2c_5b^5 = 0$. Because $a \neq 0$ and $b \neq 0$, these are equivalent to

$$c_3 + c_5a^2 = 0 \qquad \text{and} \qquad c_3 + c_5b^2 = 0.$$

Subtracting the first from the second we obtain $c_5(b^2 - a^2) = 0$, whence $c_5 = 0$ because $b^2 \neq a^2$. It follows that $c_3 = 0$. Thus we have $c_1 = c_3 = c_5 = 0$, so $P(x) = c_0 + c_2x^2 + c^4x^4 + c_6x^6$, and $P(-x) = P(x)$ for all x.

9. Diagonals of a regular nonagon.



The exterior angle of a regular nonagon is 40°, so angle ABC is 140° and each of angles BAC, BCA, DCE and DEC is 20°. Then angle ECA is 100° and each of CEA and CAE is 40°. Thus $\angle EAB = 60°$ and $\angle BCE = 120°$. Let c be the length of the diagonal BE. By the law of cosines in triangle ABE,

$$c^2 = a^2 + d^2 - 2ad\cos 60° = a^2 + d^2 - ad.$$

By the law of cosines in triangle BCE,

$$c^2 = a^2 + b^2 - 2ab\cos 120° = a^2 + b^2 + ab.$$

Equating the two expressions for c^2 and simplifying, we obtain $d^2 - ad = b^2 + ab$, whence $d^2 - b^2 = a(d + b)$, and as $d + b \neq 0$, it follows that $d - b = a$.

10. Distance less than 1/2015?.

$\boxed{\text{Yes, there are such points.}}$ Although the vertical distance between these curves is always at least 1, and gets arbitrarily large as x increases, the horizontal distance becomes arbitrarily small, as we now show. Consider $A = (a, a^3)$ and $B = (b, b^3 + |b| + 1)$, where $a^3 = b^3 + |b| + 1$ and $a > b > 0$. These points are on the horizontal line of height a^3, so the distance between A and B is $a - b$. But

$$a - b = \frac{a^3 - b^3}{a^2 + ab + b^2} = \frac{b + 1}{a^2 + ab + b^2} < \frac{b + 1}{3b^2}.$$

Then for $b > 1$ we have

$$a - b < \frac{2b}{3b^2} < \frac{1}{b}.$$

Thus, if $b = 2015$ and $a = \sqrt[3]{2015^3 + 2015 + 1}$, the distance between $A = (a, a^3)$ and $B = (b, b^3 + |b| + 1)$ is less than $1/2015$.