2009 Arkansas Mathematics Competition Solutions

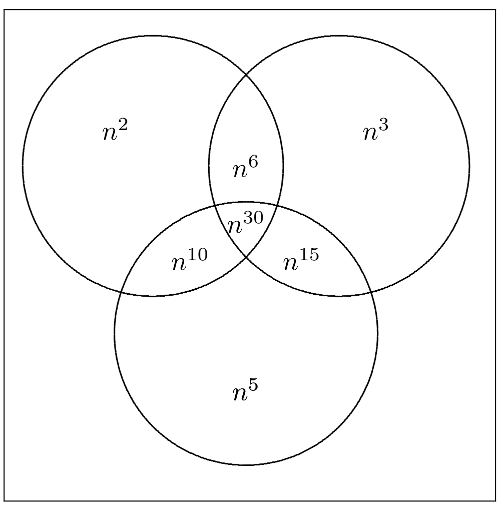
1. The sum is composite.

We have $56a + 56b = 65b + 56b$, so $56(a + b) = 121b$. Thus, in the prime factorization of $56(a+b)$, the factors 11^2 occur. Since 11 does not occur as a factor in 56, 11^2 must be among the factors of $a + b$, which is therefore composite.

2. A probability

The probability is $\boxed{3/16}$. The result described occurs if and only if on the first five tosses there is at most one tail. There are five possible outcomes with exactly one tail and one outcome with no tails, for a total of 6 of the $2^5 = 32$ possible outcomes. Thus the required probability is $6/32 = 3/16$.

3. No squares, cubes or fifth powers.



Let S be the set of integers from 1 to 10^{30}, A, B and C the sets of perfect squares, cubes and fifth powers in S, respectively. Then $|S| = 10^{30}$, $|A| = 10^{15}$, $|B| = 10^{10}$, and $|C| = 10^6$. $A \cap B$, $A \cap C$ and $B \cap C$ are the sets of sixth, tenth and fifteenth powers, respectively, and $A \cap B \cap C$ is the set of 30th powers. We have $|A \cap B| = 10^5$, $|A \cap C| = 10^3$, $|B \cap C| = 10^2$, and $|A \cap B \cap C| = 10$. By the inclusion-exclusion principle,

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= (10^{15} + 10^{10} + 10^6) - (10^5 + 10^3 + 10^2) + 10,\end{aligned}$$

so the number of integers in S which are not squares, cubes or fifth powers is

$$10^{30} - (10^{15} + 10^{10} + 10^6) + (10^5 + 10^3 + 10^2) - 10.$$

4. Solve for x.

The unique solution is $\boxed{x = 6}$. The equation is equivalent to

$$1 + \frac{1}{x-3} + 1 - \frac{1}{x-5} + 1 - \frac{1}{x-7} + 1 + \frac{1}{x-9} = 4,$$

and thus to

$$\frac{1}{x-3} - \frac{1}{x-5} = \frac{1}{x-7} - \frac{1}{x-9}.$$

This, in turn, is equivalent to

$$\frac{1}{(x-3)(x-5)} = \frac{1}{(x-7)(x-9)},$$

and thus to $x^2 - 8x + 15 = x^2 - 16x + 63$, so long as x is not 3,5,7 or 9 (and these values are ruled out by the original equation). This reduces to $8x = 48$, and $x = 6$ is the unique solution.

5. A remainder.

The remainder is $\boxed{25}$. We know that for some positive integer q we have

$$\begin{aligned} n &= 2009q + 1009 \\ &= 7^2 \cdot 41q + 24 \cdot 41 + 25 \\ &= 41(49q + 24) + 25, \end{aligned}$$

so the remainder on division by 41 is 25.

6. Unequal distances from lattice points.

We have

$$d_1^2 = (m - \sqrt{2})^2 + \left(n - \frac{1}{3}\right)^2 \quad \text{and} \quad d_2^2 = (r - \sqrt{2})^2 + \left(s - \frac{1}{3}\right)^2.$$

If $d_1 = d_2$, then on squaring and simplifying we obtain

$$m^2 - 2\sqrt{2}m + n^2 - \frac{2}{3}n = r^2 - 2\sqrt{2}r + s^2 - \frac{2}{3}s,$$

and

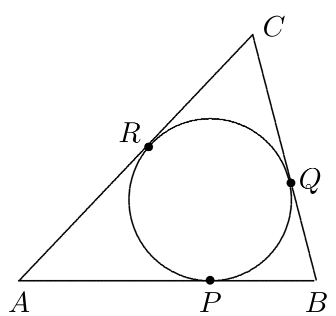
$$m^2 + n^2 - r^2 - s^2 + \frac{2}{3}s - \frac{2}{3}n = 2\sqrt{2}(m - r). \tag{1}$$

If $m - r \neq 0$ we may solve (1) for $\sqrt{2}$ and get $\sqrt{2}$ as a rational number, which it is not. Thus, $m = r$, and (1) simplifies to $n^2 - s^2 + \frac{2}{3}(s - n) = 0$; i.e.,

$$(n - s)\left(n + s - \frac{2}{3}\right) = 0. \tag{2}$$

But n and s are integers, so (2) implies that $n = s$. Thus, when $d_1 = d_2$ we must have $(m, n) = (r, s)$, so $(m, n) \neq (r, s)$ implies $d_1 \neq d_2$.

7. Length of a tangent.



The length of AP is $\boxed{9/2}$. Let Q and R be the points of tangency of sides BC and CA, respectively, to the circle, and $x = AP$. Then $AR = x$, $PB = 7 - x = BQ$, and $CR = 8 - x = CQ$. Thus

$$6 = BC = BQ + QC = (7 - x) + (8 - x) = 15 - 2x,$$

and we have $2x = 9$; $x = 9/2$.

8. It's an integer.

We prove by induction that $r_n := x^n + \frac{1}{x^n}$ is an integer for every n. The case $n = 1$ is given. Suppose that r_k is an integer for every k, $1 \leq k \leq n$. Then

$$\begin{aligned} r_n r_1 &= \left(x^n + \frac{1}{x^n}\right)\left(x + \frac{1}{x}\right) \\ &= x^{n+1} + \frac{1}{x^{n-1}} + x^{n-1} + \frac{1}{x^{n+1}} \\ &= r_{n+1} + r_{n-1}, \end{aligned}$$

so $r_{n+1} = r_n r_1 - r_{n-1}$, which by hypothesis is an integer. ∎

9. An inequality.

For each k we have

$$\frac{1+a_k}{2} \geq \sqrt{a_k}$$

by the AM-GM inequality. (Alternately, from $0 \leq (1-\sqrt{a_k})^2 = 1 - 2\sqrt{a_k} + a_k$ we have $1 + a_k \geq 2\sqrt{a_k}$.) Thus

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n\sqrt{a_1}\sqrt{a_2}\cdots\sqrt{a_n} = 2^n. \blacksquare$$

10. An integral.

We'll show that $\boxed{\int_1^5 (x-1)^2 f''(x)dx = 18}$. Using integration by parts twice, we have

$$\begin{aligned}
\int_1^5 (x-1)^2 f''(x)dx &= (x-1)^2 f'(x)\Big|_1^5 - 2\int_1^5 (x-1)f'(x)dx \\
&= 16f'(5) - 2(x-1)f(x)\Big|_1^5 + 2\int_1^5 f(x)dx \\
&= 16\cdot 2 - 2\cdot 4\cdot f(5) + 2\int_1^5 f(x)dx \\
&= 32 - 8\cdot 3 + 2\cdot 5 = 32 - 24 + 10 = 18.
\end{aligned}$$