$\boxed{\textbf{2017 AUMC Solutions}}$

1. Number plus its digits sum to 2017.

There are exactly two such numbers: $\boxed{\text{1994 and 2012}}$. If the digits are a,b,c,d we have

$$1000a + 100b + 10c + d + a + b + c + d = 2017;$$

i.e., $1001a + 101b + 11c + 2d = 2017$. This tells us that $a = 1$ or 2. Examine first the case $a = 1$. Then $101b + 11c + 2d = 1016$. From the fact that $11c + 2d \leq 99 + 18 = 117$ it follows that $101b \geq 899$. This implies $b = 9$ and $11c + 2d = 107$. As $2d \leq 18$ we have $11c \geq 89$ and $c \geq 9$, and therefore $c = 9$ and $d = 4$. Thus the only solution with $a = 1$ is 1994.

Now examine the case $a = 2$. Then $101b + 11c + 2d = 15$, so $b = 0$, $c \leq 1$, and c is odd, so $c = 1$ and $d = 2$. Thus the only solution with $a = 2$ is 2012.

2. A maximum value.

The maximum value is $\boxed{23/12}$. We have

$$\begin{aligned}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - \left(\frac{1}{x+1} + \frac{1}{y+2} + \frac{1}{z+3}\right) &= \left(\frac{1}{x} - \frac{1}{x+1}\right) + \left(\frac{1}{y} - \frac{1}{y+2}\right) + \left(\frac{1}{z} - \frac{1}{z+3}\right) \\ &= \frac{1}{x(x+1)} + \frac{2}{y(y+2)} + \frac{3}{z(z+3)}\end{aligned}.$$

Each of these last three fractions is a decreasing function of a single variable, and takes its maximum value at the lowest possible value of x, y, or z, respectively, namely at $x = y = z = 1$. Thus the maximum value is

$$\frac{1}{1 \cdot 2} + \frac{2}{1 \cdot 3} + \frac{3}{1 \cdot 4} = \frac{6 + 8 + 9}{12} = \frac{23}{12}.$$

3. Which is older?

$\boxed{\text{Tomasson is 10 years older than Tomas}}$. Here is a proof. Let n be the integer such that Tomas is n years older than Tomasson. Then, as the sum of the ages of Anders, John, Steven, Peter and Tomas is the same as the sum of the ages of Andersson, Johnson, Stevenson, Peterson and Tomasson, when we subtract these sums we get 0:

$$1 + 2 + 3 + 4 + n = 0.$$

Thus $n = -10$; i.e., Tomasson is 10 years older than Tomas.

4. A singular matrix.

We have $(A+B)(A-B) = A^2 + BA - AB - B^2 = (A^2 - B^2) + (BA - AB) = 0 + 0 = 0$. In view of the fact that $A - B \neq 0$, this proves that $A + B$ is singular.

5. How many cards were removed?

There were $\boxed{38}$ cards removed. Let n be the number of cards in the reduced deck. Then the probability of choosing the four aces is

$$\frac{1}{\binom{n}{4}} = \frac{4!(n-4)!}{n!} = \frac{24}{n(n-1)(n-2)(n-3)} = \frac{1}{1001},$$

so

$$n(n-1)(n-2)(n-3) = (24)(1001) = 24 \cdot 7 \cdot 11 \cdot 13 = 12 \cdot 14 \cdot 11 \cdot 13 = 14 \cdot 13 \cdot 12 \cdot 11.$$

Thus $n = 14$. Then the number of cards removed at the beginning was $52 - 14 = 38$.

6. Is this set bounded?

$\boxed{S \text{ is unbounded}}$. Rewrite the equation in the form $m^2 = n^3 + 4n^2 = n^2(n+4)$. If k is any integer and $n = k^2 - 4$, then $n^2(n+4) = n^2k^2$, so all points (m, n) with $n = k^2 - 4$ and $m = kn$ satisfy the equation. This set of points is clearly unbounded.

7. Arithmetic mean vs. geometric mean.

They are the pairs $\boxed{(1^2, 41^2), (2^2, 42^2), (3^2, 43^2), (4^2, 44^2)}$. In general if $m = k^2$ and $n = (k+40)^2$ then

$$\frac{m+n}{2} - \sqrt{mn} = \frac{k^2 + (k+40)^2}{2} - k(k+40) = \frac{k^2 + k^2 + 80k + 1600 - 2k^2 - 80k}{2} = 800,$$

and for $k = 1, 2, 3, 4$ we have $k^2 < (k+40)^2 < 2017$. We must show these are the only solutions. The requirement is that

$$\frac{m+n}{2} - \sqrt{mn} = 800,$$

i.e., that

$$(\sqrt{n} - \sqrt{m})^2 = n - 2\sqrt{mn} + m = 1600.$$

As $n > m$, this implies that $\sqrt{n} - \sqrt{m} = 40$. Then $\sqrt{n} = \sqrt{m} + 40$, and $n = m + 80\sqrt{m} + 1600$. Thus $\sqrt{m}$ is rational, and is therefore an integer, say $\sqrt{m} = k$. Then $\sqrt{n} = k + 40$, so we have $(m, n) = (k^2, (k+40)^2)$. If $k > 4$, then $(k+40)^2 \geq 45^2 > 2017$. Thus there are just the four solutions given at the outset.

8. A geometric series.

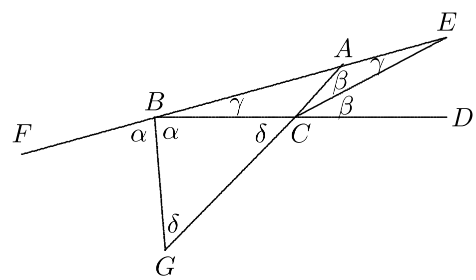
The third term is $\boxed{4/9}$. Let n be the first term and $1/m$ be the common ratio. Then the sum of the series is

$$3 = \frac{n}{1 - 1/m} = \frac{mn}{m-1}.$$

As $m-1$ and m are relatively prime, $(m-1)|n$; say, $n = k(m-1)$. Then $mk = 3$, where m and k are integers. We are given that m is a negative integer, so either $m = -3$ and $k = -1$, or $m = -1$ and $k = -3$. If $m = -1$, the common ratio is -1 and the series is divergent. Therefore $m = -3$, $k = -1$ and the first term is $n = k(m-1) = (-1)(-4) = 4$. The common ratio is $-1/3$, and the third term is $(4)(-1/3)^2 = 4/9$.

9. Measure of an angle.

$\boxed{\text{Angle } ABC = 12°}$. Let α be the common value of angles FBG and GBC. Because $BG = BC$, triangle GBC is isosceles and angles BGC and BCG are equal. Call this common value δ. Then angle BCA is $180° - \delta$. Because $BC = CE$, triangle BCE is isosceles, and angles ABC and AEC are equal. Call this common value γ. Then from triangle ABC we have angle $BAC = \delta - \gamma$. Let β be the common value of angles ACE and ECD. We then have



$$2\alpha + \gamma = 180°, \tag{1}$$

because FBA is on a line. From triangle BCG we have

$$\alpha + 2\delta = 180°. \tag{2}$$

From angles BCG and ACD we have

$$\delta = 2\beta. \tag{3}$$

Summing the angles of triangle BCE we get $2\gamma + 180° - \delta + \beta = 180°$, so

$$\beta + 2\gamma - \delta = 0. \tag{4}$$

Substitute $\delta = 2\beta$ from (3) into (2) and (4):

$$\alpha + 4\beta = 180°, \qquad (2'),$$

$$-\beta + 2\gamma = 0. \qquad (4')$$

Substitute $\beta = 2\gamma$ from (4′) into (2′) to get

$$\alpha + 8\gamma = 180° \qquad (2''.)$$

Now eliminate α between (1) and (2″) to get $15\gamma = 180°$, and we have $\gamma = 12°$.

10. Ten-digit numbers.

There are $\boxed{144}$ such ten-digit numbers. Such a number either begins with a 2, followed by a nine-digit string with the same property, or it begins with a 3 which must then be followed by a 2 and then an eight-digit string with the same property. Let $f(n)$ be the number of n-digit numbers of the desired type. Then the recursion is $f(n) = f(n-1) + f(n-2)$. The initial conditions are $f(1) = 2$ and $f(2) = 3$. The next eight terms in the sequence then are 5, 8, 13, 21, 34, 55, 89, 144.