Sixteenth Annual Arkansas Undergraduate Mathematics Competition, February 23, 2019

No CALCULATORS, ELECTRONIC DEVICES, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team name and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. Probabiliy of a palindrome.

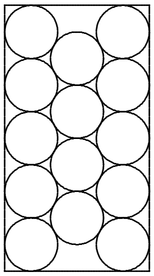
Automobile license plates in a certain state bear a sequence of 3 letters followed by a sequence of 3 decimal digits. E.g., MAT 142. Assuming all 3-letter, 3-digit sequences are equally likely, what is the probability that a randomly chosen plate contains at least one palindrome (a 3-letter or 3-digit sequence that reads the same forward and backward). E.g., MAT 141. Express your answer as a rational fraction in lowest terms.

2. Ratio of the 2019th term to the 2018th.

The first term of a sequence of numbers is 1. For all $n > 1$, the product of the first n terms is n^3. What is the ratio of the 2019th term to the 2018th?

3. Lentgh of a rectangle

In the diagram at the right are 14 congruent circles enclosed in a rectangle. The circles are tangent to one another and to the sides of the rectangle as indicated in the diagram. If the short side of the rectangle has length 1, what is the length of the long side?



4. Minimum value of a function.

For $x > 0$, define

$$f(x) = \frac{\left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2}{\left(x + \frac{1}{x}\right)^3 + \left(x^3 + \frac{1}{x^3}\right)}.$$

Determine whether $f(x)$ has a minimum value for $x > 0$, and if so, find it.

5. A sum of squares.

Find the sum of all positive integers k such that both k and $k+99$ are squares of integers.

6. A floor function value.

Given that the real number t satisfies the equation

$$\left\lfloor t+\frac{11}{85}\right\rfloor+\left\lfloor t+\frac{12}{85}\right\rfloor+\left\lfloor t+\frac{13}{85}\right\rfloor+\cdots+\left\lfloor t+\frac{64}{85}\right\rfloor=256,$$

find $\lfloor 85t\rfloor$. (The symbol $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.)

7. Three numbers adding up to 2019.

The real numbers a, b, c satisfy $a+b+c=2019$ and $1/a+1/b+1/c=5$. Determine the value of

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

8. A fraction greater than or equal to 3/4.

The real numbers x_k satisfy $1\le x_1\le x_2\le x_3\le x_4\le x_5\le x_6\le 2$. Prove that

$$\frac{x_1+x_3+x_5}{x_2+x_4+x_6}\ge\frac{3}{4}.$$

9. A functional equation.

Find all functions f on the real numbers satisfying

$$f(1-x)+2=xf(x).$$

10. A convergent series.

The sequence $\{a_n\}$ is positive, monotone increasing, and bounded. Prove that the series

$$\sum_{n=1}^{\infty}\left(1-\frac{a_n}{a_{n+1}}\right)$$

converges.