$\boxed{\text{2008 Arkansas Mathematics Competition Solutions}}$

1. A function of two integers.

It is $\boxed{3783}$. By an easy induction we have $f(m,n) = f(m+k, n-k)$ for all m, n, and k, so

$$f(2008, 1776) = f(2008 + 1775, 1776 - 1775) = f(3783, 1) = 3783.$$

2. A real product.

They are the complex numbers of the form $\boxed{z = x - (2x+2)i}$; i.e., $\boxed{\{x + yi \colon y = -2x - 2\}}$. We are looking for the complex numbers $z = x + yi$ such that $\Im(z+1)(\overline{z} - 2i) = 0$. We calculate

$$\begin{aligned}\Im(z+1)(\overline{z} - 2i) &= \Im(z\overline{z} + \overline{z} - 2iz - 2i) \\ &= \Im(x^2 + y^2 + x - yi - 2i(x+yi) - 2i) \\ &= -y - 2x - 2,\end{aligned}$$

which is zero if and only if $y = -2x - 2$.

3. Difference of two terms.

$\boxed{a_{2008} - a_{2006} = 1/2}$. Let d be the common difference $a_2 - a_1$. We have

$$S = \frac{1}{2}(a_1 + a_{2008})(2008) = 1004(a_1 + a_{2008})$$

and

$$2T = 1004(a_2 + a_{2008}) = 1004(a_1 + d + a_{2008}) = S + 1004d.$$

Then $251 = 2T - S = 1004d$, so $d = 1/4$, and $a_{2008} - a_{2006} = 2d = 1/2$.

4. A multiple of 7.

As $2008 = 7 \cdot 286 + 6$,

$$2008^{8002} \equiv 6^{8002} \equiv (-1)^{8002} \equiv 1 \pmod 7.$$

Similarly, $8002 = 7 \cdot 1143 + 1$, so

$$8002^{2008} \equiv 1^{2008} \equiv 1 \pmod 7,$$

and therefore

$$2008^{8002} - 8002^{2008} \equiv 0 \pmod 7.$$

5. A global minimum.

The minimum value is $\boxed{\sqrt[3]{49}}$. Because the cube root is a strictly increasing function, $f(x)$ takes its minimum value where $g(x) = f(x)^3$ does. Now,

$$\begin{aligned} g(x) &= (3\sin x - 10)^2 - (4\cos x)^2 \\ &= 9\sin^2 x - 60\sin x + 100 - 16(1 - \sin^2 x) \\ &= 25\sin^2 x - 60\sin x + 84 \\ &= (5\sin x - 6)^2 + 48. \end{aligned}$$

Because $5\sin x \leq 5$, we have $(5\sin x - 6)^2 \geq 1$, so $g(x) \geq 49$, and when $\sin x = 1$, $g(x) = 49$. Thus, the minimum value of $g(x)$ is 49, and the minimum value of $f(x)$ is $\sqrt[3]{49}$.

(The minimum value can also be found using the derivative, but this is a less tidy computation than the above.)

6. Probability of one white and one black.

The probability of one white and one black is $\boxed{1/3}$. More generally, if there are $n+1$ urns numbered 0 to n with the k-th having exactly k black and $n-k$ white marbles, the probability that one black and one white are drawn is 1/3. Let A_k be the event that the urn containing exactly k black balls is drawn and then one black and one white marble are drawn, and let $A = \cup_{k=0}^{n} A_k$ be the event that one black marble and one white are drawn. Then

$$P(A_k) = \frac{1}{n+1}\frac{\binom{k}{1}\binom{n-k}{1}}{\binom{n}{2}} = \frac{2k(n-k)}{(n+1)(n)(n-1)}.$$

The events A_k are mutually exclusive and their union is A, so

$$\begin{aligned} P(A) = \sum_{k=0}^{n} P(A_k) &= \frac{2}{(n+1)(n)(n-1)}\left(n\sum_{k=0}^{n} k - \sum_{k=0}^{n} k^2\right) \\ &= \frac{2}{(n+1)(n)(n-1)}\left(\frac{n^2(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}\right) \\ &= \frac{2}{n-1}\left(\frac{n}{2} - \frac{2n+1}{6}\right) = \frac{2}{n-1}\cdot\frac{n-1}{6} = \frac{1}{3}. \end{aligned}$$

7. Limit of a quotient.

The limit is $\boxed{r - as}$, for

$$\begin{aligned}\lim_{x\to a}\frac{xf(a)-af(x)}{x-a} &= \lim_{x\to a}\frac{xf(a)-af(a)+af(a)-af(x)}{x-a}\\ &= \lim_{x\to a}\frac{(x-a)f(a)-a(f(x)-f(a))}{x-a}\\ &= \lim_{x\to a} f(a) - a\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\ &= f(a)-af'(a) = r-as.\end{aligned}$$

8. Tangent of $x + y$.

We will show that $\boxed{\tan(x+y) = 2008}$. We have

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{1004}{1-\tan x\tan y}.$$

Also,

$$2008 = \cot x + \cot y = \frac{1}{\tan x} + \frac{1}{\tan y} = \frac{\tan x + \tan y}{\tan x \tan y} = \frac{1004}{\tan x\tan y},$$

so $\tan x \tan y = 1/2$. Then

$$\tan(x+y) = \frac{1004}{1-1/2} = 2008.$$

9. Distance from the fourth corner.

The length $\boxed{DP = \sqrt{145}}$. Place the rectangle on a coordinate system with A at (0,0), B at $(b,0)$ and D at $(0,d)$. Let the coordinates of P be (x,y). Then

$$x^2+y^2 = 64, \qquad (b-x)^2+y^2 = 144, \qquad \text{and } (b-x)^2+(d-y)^2 = 225.$$

Note also that $DP^2 = x^2 + (d-y)^2$. Adding the first and third equations above and subtracting the second, we obtain

$$x^2 + (d-y)^2 = 145,$$

so $DP = \sqrt{145}$.

10. No real roots.

Let $P(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x = x(x-1)(x^4 + x^2 + 1)$. If $x \le 0$ or $x \ge 1$ then $P(x) \ge 0$ so $P(x) + \frac{3}{4} > 0$. Note that $x(x-1) = (x - \frac{1}{2})^2 - \frac{1}{4}$, so for $0 < x < 1$ we have $-\frac{1}{4} \le x(x-1) < 0$, and $1 < x^4 + x^2 + 1 < 3$. It follows that

$$-\frac{3}{4} < x(x-1)(x^4 + x^2 + 1) < 0,$$

and therefore that $P(x) + \frac{3}{4} > 0$ again. Thus $P(x) + \frac{3}{4} > 0$ for all real x.