$\boxed{\textbf{2016 AUMC Solutions}}$

1. A sequence of 2016 terms.

They are $\boxed{\text{(a) } \max(A) = 2017, \text{ (b) } \min(A) = -1, \text{ and (c) element nearest to 0 is } -1/2014}$. Observe first that $f_2(x) = 1 - \frac{1}{x}$ and $f_3(x) = x$. Then $f_4(x) = f_1(x)$ and in general $f_{n+3}(x) = f_n(x)$. Thus

$$f_{3k}(x) = x, \quad f_{3k+1}(x) = \frac{1}{1-x} \quad \text{and } f_{3k+2}(x) = 1 - \frac{1}{x}.$$

From this we see that $a_{3k} = f_{3k}(3k+1) = 3k+1$,

$$a_{3k+1} = f_{3k+1}(3k+2) = \frac{1}{1-(3k+2)} = -\frac{1}{3k+1}, \text{ and}$$

$$a_{3k+2} = f_{3k+2}(3k+3) = 1 - \frac{1}{3k+3} = \frac{3k+2}{3k+3}.$$

Now partition A into three subsets:

$$A_0 = \{a_{3k} : k = 1, 2, 3, \ldots, 672\} = \{4, 7, 10, \ldots, 2017\},$$

$$A_1 = \{a_{3k+1} : k = 0, 1, 2, \ldots, 671\} = \left\{-1, -\frac{1}{4}, -\frac{1}{7}, \ldots, -\frac{1}{2014}\right\},$$

and

$$A_2 = \{a_{3k+2}; k = 0, 1, 2, \ldots, 671\} = \left\{\frac{2}{3}, \frac{5}{6}, \frac{8}{9}, \ldots, \frac{2015}{2016}\right\},$$

and it is clear that the largest and smallest elements are 2017 and -1, respectively, and the element nearest to zero is $-1/2014$.

2. Sum of a series.

The sum is $\boxed{\ln \frac{4}{3}}$. We may write the n-th partial sum as

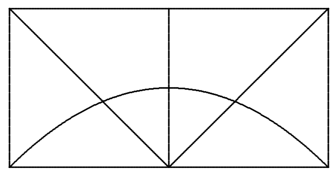
$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \ln \frac{(k+1)(3k+1)}{k(3k+4)} \\
&= \sum_{k=1}^{n} [\ln(k+1) - \ln k + \ln(3k+1) - \ln(3k+4)] \\
&= \qquad (\ln 2 + \ln 3 + \cdots + \ln n + \ln(n+1)) \\
&\quad - (\ln 1 + \ln 2 + \ln 3 + \cdots + \ln n) \\
&\quad + (\ln 4 + \ln 7 + \ln 10 + \cdots + \ln(3n+1)) \\
&\qquad - (\ln 7 + \ln 10 + \cdots + \ln(3n+1) + \ln(3n+4)) \\
&= \ln(n+1) - \ln 1 + \ln 4 - \ln(3n+4) = \ln \frac{4n+4}{3n+4},
\end{aligned}$$

and $\lim_{n\to\infty} S_n = \ln \frac{4}{3}$.

3. Rational values of a function.

The rational values of $f(n)$ are $\boxed{276/6,\ 252 \text{ and } 12324/7}$, occurring when $n = 227,\ 1757$ and 12323, respectively. Let $r = f(n)$. Then $(7r)^2 = n^2 + 49 \cdot 503$, so $(7r - n)(7r + n) = 7 \cdot 7 \cdot 503$, and these factors are prime. Thus $7r - n = 1,\ 7$ or 49, and the corresponding values of $7r + n$ are $49 \cdot 503,\ 7 \cdot 503$ or 503, respectively. If $7r - n = 1$ and $7r + n = 49 \cdot 503 = 24647$, then $14r = 24648$, $r = 12324/7$, and $n = 7r - 1 = 12323$. Similarly, if $7r - n = 7$, then $r = 252$ and $n = 1757$. If $7r - n = 49$, then $r = 276/7$ and $n = 227$.

4. Closer to the center.



The probability is $\boxed{\dfrac{4\sqrt{2}-5}{3}}$. Place the square on a rectangular coordinate system with center at the origen and sides parallel to the axes. Without loss of generality we take the edge of the square to be 2, and by symmetry we can reduce the problem to the same question on the portion of the square where $0 \le x \le y \le 1$. (The figure shows the upper half of the square.) In this region the distance from point (x, y) to the nearest edge is $1 - y$, and the distance to the center is $\sqrt{x^2 + y^2}$. Thus the boundary of the region where the points are nearer to the center has equation $1 - y = \sqrt{x^2 + y^2}$; i.e., $y = \dfrac{1 - x^2}{2}$. This crosses the line $y = x$ at $x = \sqrt{2} - 1$, and the area of this portion of the region is

$$\int_0^{\sqrt{2}-1} \left(\frac{1 - x^2}{2} - x \right) dx = \left[\frac{x}{2} - \frac{x^3}{3} - \frac{x^2}{2} \right]_0^{\sqrt{2}-1} = \frac{4\sqrt{2} - 5}{6}.$$

The area of the part of the square where $0 \le x \le y \le 1$ is $1/2$, so the desired probability is $\dfrac{4\sqrt{2} - 5}{3}$.

5. Which is larger?.

$\boxed{2016^{2016}}$ is the larger. For,

$$\left(\frac{2017}{2016}\right)^{2015} = \left(1 + \frac{1}{2016}\right)^{2015} = \sum_{k=0}^{2015} \binom{2015}{k} \left(\frac{1}{2016}\right)^k$$

$$= \sum_{k=0}^{2015} \frac{2015 \cdot 2014 \cdots (2015 - k + 1)}{1 \ \cdot \ 2 \ \cdots \ k} \cdot \frac{1}{2016^k} = \sum_{k=0}^{2015} \left(\frac{2015}{1 \cdot 2016}\right) \left(\frac{2014}{2 \cdot 2016}\right) \cdots \left(\frac{2016 - k}{k \cdot 2016}\right),$$

which is smaller than 2016 because there are exactly 2016 terms in the sum, and each is smaller than 1. It follows that

$$2017^{2015} < 2016^{2016}. \blacksquare$$

6. Doubling the area.

They are $\boxed{(8{,}105),\ (9{,}56)\text{ and }(14{,}21)}$. The condition is

$$2mn = (m+7)(n+7) = mn + 7m + 7n + 49.$$

This is equivalent to $mn-7m-7n+49 = 98$; i.e., to $(m-7)(n-7) = 98$. As $m-7$ and $n-7$ are integers with $m \le n$, there are 6 possibilities: $(m-7, n-7) = (1, 98), (2, 49), (7, 14), (-98, -1), (-49, -2)$ or $(-14, -7)$. Only the first three of these give positive values for both m and n, and these give us $(m, n) = (8, 105), (9, 56)$ or $(14, 21)$.

7. Floor function equation.

The solutions are $\boxed{-6/19,\ 4/19,\ 14/19\text{ and }24/19}$. Substitution shows that these satisfy the equation. We need to show that these are the only solutions. The number x satisfies the equation if and only if

$$\frac{19x+16}{10}$$

is an integer, and

$$\frac{19x+16}{10} \le \frac{4x+7}{3} < \frac{19x+16}{10} + 1.$$

This inequality is equivalent to

$$-\frac{8}{17} < x \le \frac{22}{17},$$

which in turn is equivalent to

$$\frac{12}{17} < \frac{19x+16}{10} \le \frac{69}{17}.$$

The integers in this range are 1, 2, 3 and 4, and $\dfrac{19x+16}{10}$ takes the values 1, 2, 3, 4 at

$$-\frac{6}{19}, \frac{4}{19}, \frac{14}{19}, \frac{24}{19},$$

respectively.

8. Real roots.

On expansion and simplification the equation becomes

$$3x^2 - 2(a+b+c)x + (ab+ac+bc) = 0.$$

The discriminant $\Delta = 4(a+b+c)^2 - 12(ab+ac+bc)$, so it suffices to show that $(a+b+c)^2 \geq 3(ab+ac+bc)$. This is equivalent to $a^2+b^2+c^2 \geq ab+ac+bc$. But we know that

$$a^2+b^2 \geq 2ab \quad (\text{because } (a-b)^2 \geq 0), \tag{1}$$

$$b^2+c^2 \geq 2bc, \tag{2}$$

$$a^2+c^2 \geq 2ac. \tag{3}$$

Adding (1), (2) and (3) we obtain $2(a^2+b^2+c^2) \geq 2(ab+ac+bc)$, so the inequality $\Delta \geq 0$ follows at once, and the roots are real.

9. Last two digits. The last two digits of $f(2016)$ are $\boxed{00}$. We first look at 2^{3^n} modulo 100. We observe that if $0 \leq m < 100$, then for any positive integer x,

$$(100x+m)^3 = 10^6x^3 + 3\cdot 10^4x^2m + 3\cdot 10^2xm^2 + m^3 \equiv m^3 \pmod{100}.$$

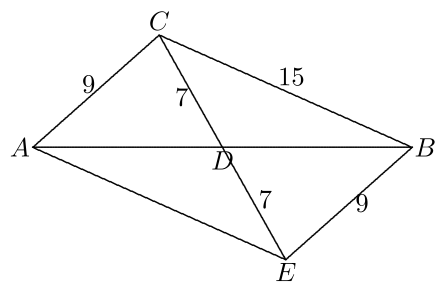
Thus we have $2^3 = 08$, $2^{3^2} = (2^3)^3 = (08)^3 \equiv 12 \pmod{100}$; $2^{3^3} = (2^{3^2})^3 \equiv 12^3 \equiv 28 \pmod{100}$; $2^{3^4} \equiv 28^3 \equiv 52 \pmod{100}$; and $2^{3^5} \equiv 52^3 \equiv 08 \pmod{100}$. We see that the values repeat in a cycle of length 4: 08, 12, 28, 52. The sums then proceed as follows:

$$08, \quad 20, \quad 48, \quad 00,$$

and because the fourth one is 00, the fifth is again 08, and this cycle will be repeated. Thus the last two digits of $f(n)$ are 08, 20, 48 or 00, according as n is 1, 2, 3 or 0 modulo 4. As 2016 is 0 mod 4, the last two digits of $f(2016)$ are 00.

10. Area of a triangle .

The area is $\boxed{10\sqrt{38}}$. Extend the median CD its own length to E, as shown. Then triangles BDE and ADC are congruent, so $BE = AC = 9$. It follows that triangles ABC and BCE have the same area, and BCE has side lengths 9, 14, 15, so by Heron's formula, with semiperimeter $s = 19$ the area is $\sqrt{(19)(10)(5)(4)} = \sqrt{3800}$.



SECOND SOLUTION

Let r be the common lengths of AD and DB. Triangles ADC and BDC have equal areas (common base r and same altitude). Express these areas in terms of r using Heron's formula, solve the resulting equation for r. This gives $r = \sqrt{104}$. Substitute back to get area of triangle $ADC = \sqrt{950}$, double it to get area of triangle ABC.