$\boxed{\textbf{2018 AUMC Solutions}}$

1. Harmonic sum greater than 1.

Here is a proof by induction. Let

$$f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1}.$$

Then

$$f(1) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1.$$

We complete the proof by induction by showing that $f(n+1) > f(n)$.

$$\begin{aligned} f(n+1) - f(n) &= \frac{1}{3n+2} + \frac{1}{3n+3} + \frac{1}{3n+4} - \frac{1}{n+1} \\ &= \frac{1}{3n+2} + \frac{1}{3n+4} - \frac{2}{3n+3} = \frac{6n+6}{(3n+2)(3n+4)} - \frac{2}{3n+3} \\ &= \frac{2(3n+3)^2 - 2(3n+2)(3n+4)}{(3n+2)(3n+3)(3n+4)} = \frac{2}{(3n+2)(3n+3)(3n+4)} > 0. \end{aligned}$$

SECOND SOLUTION

By the AM-HM inequality,

$$\frac{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1}}{2n+1} > \frac{2n+1}{(n+1) + (n+2) + \cdots + (3n+1)},$$

so

$$f(n) > \frac{(2n+1)^2}{(2n+1)(2n+1)} = 1.$$

2. A real number evaluation.

The answer is $\boxed{\sqrt{7/3}}$. We have $(a+b)^2 = a^2 + b^2 + 2ab = 5ab + 2ab = 7ab$, and $(a-b)^2 = a^2 + b^2 - 2ab = 3ab$, so

$$\left(\frac{a+b}{a-b}\right)^2 = \frac{7ab}{3ab} = \frac{7}{3}.$$

Thus $(a+b)/(a-b) = \sqrt{7/3}$.

3. A property of integers greater than 2018.

We note that $(n^2+1)^2 = n^4+2n^2+1 < n^4+3n^2+1 < n^4+4n^2+4 = (n^2+2)^2$. There are no squares which lie between the consecutive squares $(n^2+1)^2$ and $(n^2+2)^2$, so n^4+3n^2+1 is never a perfect square. (It is, of course, true also for integers less than or equal to 2018.)

4. An arithmetic sum.

The required sum is $\boxed{\frac{-S+3T}{2}}$. Let $a = a_1$ and $d = a_2 - a_1$ be the common difference. Then

$$S = 20[a + (a+39d)] = 40a + 780d;$$

$$T = 10[(a+d) + (a+39d)] = 20a + 400d.$$

Solving for a and d we obtain

$$a = \frac{20S - 39T}{20}, \quad d = \frac{2T - S}{20}.$$

Then

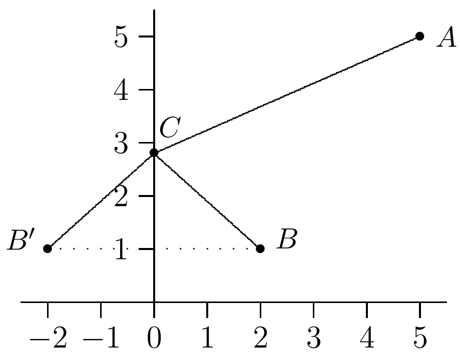
$$\begin{aligned} a_4 + a_8 + a_{12} + \cdots + a_{40} &= 5[(a+3d) + (a+39d)] \\ &= 5(2a + 42d) \\ &= 5\Big(\frac{20S - 39T}{10} + \frac{42T - 21S}{10}\Big) \\ &= \frac{-S+3T}{2}. \end{aligned}$$

5. Sum of tangents equals their product.

Using $C = -(A+B)$, $\tan(-x) = -\tan x$, and the addition formula for the tangent, we have

$$\begin{aligned} \tan A + \tan B + \tan C &= \tan A + \tan B - \tan(A+B) \\ &= \tan A + \tan B - \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= \frac{(\tan A + \tan B)(1 - \tan A \tan B) - (\tan A + \tan B)}{1 - \tan A \tan B} \\ &= \frac{(\tan A + \tan B)(-\tan A \tan B)}{1 - \tan A \tan B} \\ &= -\tan A \tan B \tan(A+B) \\ &= \tan A \tan B \tan C. \end{aligned}$$

6. Minimizing a sum of lengths.



It is with $\boxed{k = \frac{15}{7}}$. Let $B' = (-2, 1)$ be the reflection of B in the y-axis. The lengths of BC and $B'C$ are equal. The sum $AC + B'C$ is minimized when the path from A to B' is a straight line segment. For this it is necessary and sufficient that

$$\frac{k-1}{0+2} = \frac{5-1}{5+2} = \frac{4}{7}.$$

Solve this for k to get $k = \frac{15}{7}$.

7. A 2017-2018 integral equation.

They are $\boxed{(1)\ f(x) \equiv 0,\ \text{and } (2)\ f(x) = \dfrac{x-1}{2018}.}$ To prove this, suppose that f is a differentiable function satisfying

$$f(x)^{2018} = \int_1^x f(t)^{2017} dt. \tag{1}$$

Differentiate both members with respect to x to get

$$2018 f(x)^{2017} f'(x) = f(x)^{2017}. \tag{2}$$

One solution of (2) is obviously $f(x) \equiv 0$. Another is any function f with $f'(x) \equiv 1/2018$, and thus $f(x) = \dfrac{x}{2018} + c$ for any constant c. To evaluate c we substitute into (1) and get

$$\begin{aligned}\left(\frac{x}{2018} + c\right)^{2018} &= f(x)^{2018} = \int_1^x f(t)^{2017} dt \\ &= \int_1^x \left(\frac{t}{2018} + c\right)^{2017} dt = \left(\frac{t}{2018} + c\right)^{2018}\Bigg|_1^x \\ &= \left(\frac{x}{2018} + c\right)^{2018} - \left(\frac{1}{2018} + c\right)^{2018}.\end{aligned}$$

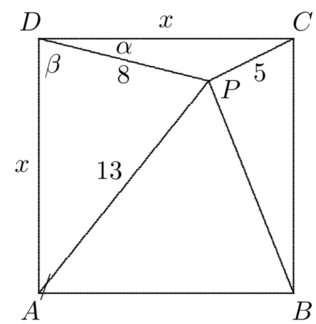
Thus $c = -1/2018$, and $f(x) = \dfrac{x-1}{2018}$.

The question arises whether there can be a function f where $f(x)$ is 0 on part of its domain and $\dfrac{x-1}{2018}$ on the remainder. However such a function cannot be everywhere differentiable. Thus we have just the two solutions given at the outset.

8. Triangle construction.

The answers are: (i) for $\boxed{1 < a/b < (3+\sqrt{5})/2}$, and (ii) for $\boxed{a/b = (1+\sqrt{5})/2}$. We have $a > \sqrt{ab} > b$, and these are the sides of a nondegenerate triangle if and only if $a < \sqrt{ab}+b$. If $u = \sqrt{a/b}$ the condition becomes $u^2 < u+1$; for positive u this is equivalent to $u < (1+\sqrt{5})/2$, and thus $1 < a/b = u^2 < (3+\sqrt{5})/2$. The triangle is a right triangle if and only if $a^2 = ab+b^2$; i.e., $(a/b)^2 = a/b+1$. The unique positive solution of this equation is $a/b = (1+\sqrt{5})/2$.

9. The area of a square.



We show that the area is $\boxed{153}$. Let x be the side length of the square, and α and β be the angles CDP and ADP, respectively. Applying the law of cosines to the triangles CDP and ADP we get $25 = 64+x^2-16x\cos\alpha$ and $169 = 64+x^2-16x\cos\beta$. Since $\cos\beta = \sin\alpha$, these equations are equivalent to

$$\cos\alpha = \frac{x^2+39}{16x} \quad \text{and} \quad \sin\alpha = \frac{x^2-105}{16x}.$$

Then

$$1 = \cos^2\alpha + \sin^2\alpha = \frac{x^4+78x^2+39^2+x^4-210x^2+105^4}{256x^2}.$$

Clear of fractions and simplify to get $x^4 - 194x^2 + 6273 = 0$. The quadratic formula then gives $x^2 = 153$ or 41. The value $x^2 = 41$ is impossible because a square of side $\sqrt{41}$ has no interior point at a distance 13 from a vertex. Therefore the area of the square must be $x^2 = 153$.

10. A periodic function.

Transpose the 1/2 and square both members to get

$$f(x+1)^2 - f(x+1) + \frac{1}{4} = f(x) - f(x)^2.$$

If $g(x) = f(x) - f(x)^2$ we have that $g(x) = 1/4 - g(x+1)$ for all x, and thus that $g(x+1) = 1/4 - g(x+2)$. From these two equations it follows that $g(x+2) = g(x)$ for all x. Then $f(x+1) = 1/2 + \sqrt{g(x)}$ implies that

$$f(x+3) = \frac{1}{2} + \sqrt{g(x+2)} = \frac{1}{2}\sqrt{g(x)} = f(x+1),$$

so f is periodic with period 2.