Fourteenth Annual Arkansas Undergraduate Mathematics Competition, February 25, 2017

No CALCULATORS, ELECTRONIC DEVICES, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team name and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. Number plus its digits sum to 2017.

Find all four-digit decimal integers such that the sum of the integer and its four digits is 2017. It is assumed that the first digit is non-zero.

2. A maximum value.

Determine, with proof, the maximum value of

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-\left(\frac{1}{x+1}+\frac{1}{y+2}+\frac{1}{z+3}\right)$$

for positive integers x, y and z.

3. Which is older?

The first names of five students are Anders, John, Steven, Peter and Tomas. Their last names, in a different order, are Andersson, Johnson, Stevenson, Peterson and Tomasson. Anders is 1 year older than Andersson, John is 2 years older than Johnson, Steven is 3 years older than Stevenson, and Peter is 4 years older than Peterson. Of Tomas and Tomasson, which is older, and by how many years?

4. A singular matrix.

Let A and B be square matrices of the same order which satisfy $A \neq B, AB = BA$ and $A^2 = B^2$. Prove that the matrix $A + B$ is singular.

5. How many cards were removed?

From a standard deck of 52 cards, some are removed but all four aces are among those remaining. If now four cards are chosen at random from the remaining cards, the probability that the cards chosen are the four aces is 1/1001. How many cards were removed from the original deck?

6. Is this set bounded?.

Let S be the set of pairs (m, n) of integers satisfying $m^2 - 4n^2 = n^3$. Determine whether the set S is bounded.

7. Arithmetic mean vs. geometric mean.

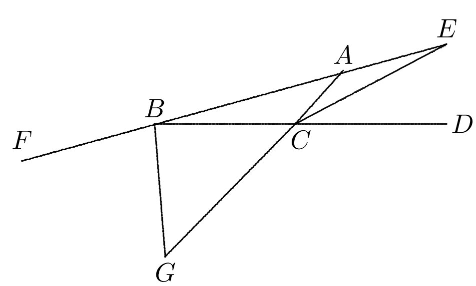
Find all ordered pairs (m, n) of integers with $0 < m < n < 2017$ such that their arithmetic mean exceeds their geometric mean by exactly 800.

8. A geometric series.

The first term of an infinite geometric series is a positive integer, the common ratio is the reciprocal of a negative integer, and the sum of the series is 3. Find, with proof, the third term of the series.

9. Measure of an angle

In the figure at the right D is a point on the extension of side BC of triangle ABC, and F is a point on the extension of side AB. The bisector of angle ACD meets the extension of side BA at E, and the bisector of angle FBC meets the extension of side AC at G. If $CE = BC = BG$, determine the measure of angle ABC.



10. Ten-digit numbers.

How many positive ten-digit decimal integers are there in which every digit is 2 or 3, and no two 3s are adjacent?