# 2006 Arkansas Mathematics Competition Solutions

## 1. System of equations.

The solutions are $\boxed{\left(\frac{9}{8}, 1, \frac{4}{7}\right) \text{ and } \left(-\frac{9}{8}, -1, -\frac{4}{7}\right)}$. We have

$$(xyz)^2 = (xy)(yz)(xz) = \left(\frac{9}{8}\right)\left(\frac{4}{7}\right)\left(\frac{9}{14}\right) = \left(\frac{9}{14}\right)^2,$$

so $xyz = 9/14$ or $-9/14$. If $xyz = 9/14$, then $x = (9/14)(7/4) = 9/8,\ y = 1$ and $z = 4/7$. If $xyz = -9/14$, then $x = -9/8,\ y = -1$ and $z = -4/7$.

## 2. Minimum element.

It is $\boxed{-\sqrt{7}}$. The condition on $x$ is equivalent to $x^{2004}(x^4 - 8x^2 + 7) = 0$; i.e.,

$$x^{2004}(x^2 - 7)(x^2 - 1) = 0.$$

The roots of this equation are $0, \pm\sqrt{7}, \pm 1$, and the minimum is $-\sqrt{7}$.

## 3. Fourth term of a G.P.

The fourth term is $\boxed{1}$. The ratio is

$$r = \frac{2^{\frac{1}{3}}}{2^{\frac{1}{2}}} = 2^{-\frac{1}{6}}.$$

The fourth term is the second term multiplied by $r^2$:

$$a_4 = 2^{\frac{1}{3}}(2^{-\frac{1}{6}})^2 = 2^{\frac{1}{3}}2^{-\frac{1}{3}} = 1.$$

**4. Common linear factor.**

They are $\boxed{k = \pm\frac{21}{\sqrt{13}}}$.

Let the common factor be $x - a$ and write the two quadratics as

$$x^2 + kx + 8 = (x-a)(x-b);$$
$$2x^2 + kx - 5 = (x-a)(2x-c).$$

Then $a + b = -k = 2a + c$, $ab = 8$ and $ac = -5$. Thus

$$a + \frac{8}{a} = 2a - \frac{5}{a},$$

giving $a^2 = 13$ and $a = \pm\sqrt{13}$, $b = \frac{8}{a} = \frac{8}{\pm\sqrt{13}}$, and

$$-k = a + b = \pm\left(\sqrt{13} + \frac{8}{\sqrt{13}}\right)$$

so

$$k = \pm\frac{21}{\sqrt{13}}.$$

**5. Trigonometric equation.**

The unique solution is $\boxed{\theta = \pi/24}$. We find it as follows. Multiply the given equation by 2, square both sides and subtract 2 to obtain the equation

$$\sqrt{2 + \sqrt{3}} = 4\cos^2\theta - 2 = 2\cos 2\theta. \tag{1}$$

Now square again and subtract 2 to get

$$\sqrt{3} = 4\cos^2 2\theta - 2 = 2\cos 4\theta.$$

Thus $\cos 4\theta = \sqrt{3}/2$. With $0 < \theta < \pi/2$ we have $0 < 4\theta < 2\pi$, so

$$4\theta = \pi/6 \quad \text{or} \quad 4\theta = 2\pi - \pi/6 = 11\pi/6.$$

But $4\theta = 11\pi/6$ is extraneous because then $2\theta = 11\pi/12$ and $\cos 2\theta$ is negative, contrary to (1). Thus the only candidate solution is $\theta = \pi/24$.

Conversely, if $\theta = \pi/24$, then we may reverse the above steps, taking only positive square roots, and find that $\cos\theta = \sqrt{2 + \sqrt{2 + \sqrt{3}}}$.

**6. Coefficient of $(x+1)^2$.**

We show that $\boxed{a_2 = 220}$. Summing the geometric progression and then writing $y$ for $x+1$, we have

$$\begin{aligned}
P(x) &= \frac{1-x^{12}}{1+x} \\
&= \frac{-1}{y}((y-1)^{12}-1) \\
&= \frac{-1}{y}\left[y^{12} - \binom{12}{1}y^{11} + \binom{12}{2}y^{10} - \cdots - \binom{12}{9}y^3 + \binom{12}{10}y^2 - \binom{12}{11}y + 1 - 1\right] \\
&= -y^{11} + \binom{12}{1}y^{10} - \binom{12}{2}y^9 + \cdots + \binom{12}{9}y^2 - \binom{12}{10}y + \binom{12}{11},
\end{aligned}$$

and the coefficient of $y^2$ is

$$\binom{12}{9} = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} = 220.$$

**SECOND SOLUTION**

By Taylor's Theorem,

$$P(x) = P(-1) + P'(-1)(x+1) + \frac{P''(-1)}{2!}(x+1)^2 + \cdots + \frac{P^{(11)}(-1)}{11!}(x+1)^{11},$$

and $a_2 = \frac{P''(-1)}{2}$. Now

$$P'(x) = -1 + 2x - 3x^2 + \cdots + 10x^9 - 11x^{10},$$

$$P''(x) = 2 \cdot 1 - 3 \cdot 2x + 4 \cdot 3x^2 - \cdots + 10 \cdot 9x^8 - 11 \cdot 10x^9, \quad \text{and}$$

$$P''(-1) = 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4 + 6 \cdot 5 + 7 \cdot 6 + 8 \cdot 7 + 9 \cdot 8 + 10 \cdot 9 + 11 \cdot 10 = 440,$$

so $a_2 = \frac{440}{2} = 220$.

**7. An inequality.**

$$\begin{aligned}\frac{1-x^{2n+1}}{1-x} &= 1+x+\cdots+x^{n-1}+x^n+x^{n+1}+\cdots+x^{2n}\\ &= x^n+(x^{n-1}+x^{n+1})+(x^{n-2}+x^{n+2})+\cdots+(1+x^{2n}).\end{aligned}$$

For each $k$, $1 \le k \le n$, we have

$$x^{n-k}+x^{n+k}=x^n\left(\frac{1}{x^k}+x^k\right).$$

But for any $y>0$ and $y \neq 1$, $\frac{1}{y}+y>2$. (This follows from $(y-1)^2>0$, which implies $y^2+1>2y$.) Thus, for each $k$,

$$x^n\left(\frac{1}{x^k}+x^k\right)>2x^n,$$

and therefore

$$x^n+\sum_{k=1}^{n}(x^{n-k}+x^{n+k})>x^n+2nx^n=(2n+1)x^n.$$

**8. A mean value?**

Let $g(x)=f(x)/x$ for $a \le x \le b$. Then $g$ is continuous on $[a,b]$ and differentiable on $(a,b)$, and $g(a)=g(b)$, so by Rolle's Theorem there is a point $c$ in $(a,b)$ such that $g'(c)=0$. But

$$g'(x)=\frac{xf'(x)-f(x)}{x^2},$$

so we have

$$0=g'(c)=\frac{cf'(c)-f(c)}{c^2},$$

and hence $cf'(c)-f(c)=0$, and $f'(c)=f(c)/c$.

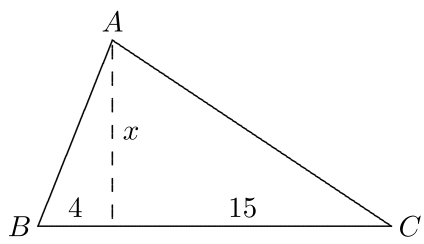
**9. Quotient of two series.**

$\boxed{\frac{S}{T} = \frac{4}{3}}$. Because the series are absolutely convergent, we are justified in writing

$$\begin{aligned} T &= 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \cdots \\ &= 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \cdots \\ &\quad - \frac{2}{2^3} \qquad - \frac{2}{4^3} \qquad - \frac{2}{6^3} - \cdots \\ &= S - \frac{2}{2^3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\right) \\ &= S - \frac{1}{4}S = \frac{3}{4}S, \end{aligned}$$

so $\frac{S}{T} = \frac{4}{3}$.

**10. Area of a triangle.**




The area is $\boxed{95}$. Let $x$ be the length of the altitude from $A$. Then

$$\tan^{-1}\left(\frac{19}{4}\right) = \tan^{-1}\left(\frac{4}{x}\right) + \tan^{-1}\left(\frac{15}{x}\right),$$

so

$$\frac{19}{4} = \tan\left(\tan^{-1}\frac{4}{x} + \tan^{-1}\frac{15}{x}\right) = \frac{4/x + 15/x}{1 - (4/x)(15/x)} = \frac{19x}{x^2 - 60}.$$

Clearing of fractions we obtain the quadratic equation $x^2 - 4x - 60 = (x - 10)(x + 6) = 0$. The only positive root is $x = 10$, so the area of triangle $ABC$ is $(1/2)(19)(10) = 95$.