$\boxed{\textbf{2012 Arkansas Mathematics Competition Solutions}}$

1. Color of the last ball.

The last remaining ball is $\boxed{\text{blue}}$. Originally the number of blue balls is odd, and each performance of the procedure either leaves the number of blues unchanged or reduces it by 2. Thus the number of blue balls in the urn is always odd. When the urn is down to one ball, it must be a blue one.

2. The coefficient of x.

The coefficient of x is $\boxed{495}$. We have $P_1(x) = P_0(x-1)$, $P_2(x) = P_1(x-2) = P_0(x-3)$, and in general,

$$P_n(x) = P_0(x - (1 + 2 + \cdots + n)) = P_0(x - \frac{n(n+1)}{2}),$$

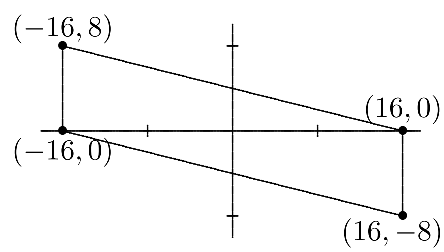
so

$$P_{15}(x) = P_0(x - 120) = (x - 120)^3 + 178(x - 120)^2 + 15(x - 120) - 39,$$

and the coefficient of x in $P_{15}(x)$ is

$$(3)120^2 - (2)(178)(120) + 15 = (360 - 356)120 + 15 = 4 \cdot 120 + 15 = 495.$$

3. Area of the bounded region.



The area of the bounded region is $\boxed{256}$. For $y > 0$ the equation is $x^2 + 4xy + 64y = 256$, which is equivalent to $(x + 2y)^2 = (2y - 16)^2$. Its graph consists of two lines: $x + 2y = 2y - 16$ and $x + 2y = -2y + 16$. These simplify to $x = -16$ and $x + 4y = 16$, and their graphs form the left and upper boundaries of the parallelogram depicted. For $y < 0$ the equation is $x^2 + 4xy - 64y = 256$, equivalent to $(x + 2y)^2 = (2y + 16)^2$. This gives us the lines $x = 16$ and $x + 4y = -16$, the graphs of which form the right and lower boundaries of the parallelogram. The area of the parallelogram is now easily seen to be $(32)(8) = 256$.

4. An inequality.

Because $2+\cos x>0$ for all x, the desired inequality is equivalent to

$$x>\frac{3\sin x}{2+\cos x}.$$

Let $f(x)=x-\dfrac{3\sin x}{2+\cos x}$. Then $f(0)=0$ and we want to show that $f(x)>0$ for all $x>0$. For this it suffices to show that $f'(x)>0$ for $x>0$. Now,

$$\begin{aligned} f'(x) &= 1-\frac{(2+\cos x)(3\cos x)-(3\sin x)(-\sin x)}{(2+\cos x)^2} \\ &= \frac{4+4\cos x+\cos^2 x-6\cos x-3\cos^2 x-3\sin^2 x}{(2+\cos x)^2} \\ &= \frac{\cos^2 x-2\cos x+1}{(2+\cos x)^2} \\ &= \frac{(1-\cos x)^2}{(2+\cos x)^2}\geq 0. \end{aligned}$$

This tells us that the function f is nondecreasing on $(0,\infty)$, so $f(x)\geq f(0)=0$ for all $x>0$. Moreover, $f'(x)>0$ for $0<x<\pi$), so $f(x)>0$ for $0<x<\pi$, and because f is nondecreasing, $f(x)$ remains positive for all $x>0$.

5. A quotient less than 2012.

Because $\dfrac{2n}{2n-1}<\dfrac{2n-1}{2n-2}$,

$$\begin{aligned} \frac{A}{B} &= \left(\frac{2}{1}\cdot\frac{2}{1}\right)\left(\frac{4}{3}\cdot\frac{4}{3}\right)\left(\frac{6}{5}\cdot\frac{6}{5}\right)\cdots\left(\frac{1006}{1005}\cdot\frac{1006}{1005}\right) \\ &< \left(\frac{2}{1}\cdot\frac{2}{1}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\right)\left(\frac{5}{4}\cdot\frac{6}{5}\right)\cdots\left(\frac{1005}{1004}\cdot\frac{1006}{1005}\right), \end{aligned}$$

which telescopes to $2\cdot 1006=2012$.

SECOND SOLUTION

Let A_n be the product of the squares of the first n positive even integers and B_n the product of the squares of the first n positive odd integers. We show by induction that

$$\Pi_n := \frac{A_n}{B_n}<4n \qquad \text{for } n>1.$$

We have $\Pi_1 = 4 = 4 \cdot 1$, and $\Pi_2 = 4 \cdot \dfrac{16}{9} < 4 \cdot 2$. Suppose that $\Pi_k < 4k$. Then

$$\begin{aligned}
\Pi_{k+1} &= \Pi_k \left(\frac{2k+2}{2k+1}\right)^2 < 4k\left(\frac{2k+2}{2k+1}\right)^2 \\
&= 4k\left(1 + \frac{4k+3}{4k^2+4k+1}\right) = 4k + \frac{16k^2+12k}{4k^2+4k+1} \\
&< 4k + \frac{16k^2+16k+4}{4k^2+4k+1} = 4k+4.
\end{aligned}$$

By induction, $\Pi_n < 4n$ for all $n > 1$. The original problem is the case $n = 503$.

6. Some five of the 2012 have sum at least 50.

Write the 2012 numbers five times, as in the following array:

$$\begin{array}{ccccccc}
x_1 & x_2 & x_3 & \cdots & x_{2010} & x_{2011} & x_{2012} \\
x_2 & x_3 & x_4 & \cdots & x_{2011} & x_{2012} & x_1 \\
x_3 & x_4 & x_5 & \cdots & x_{2012} & x_1 & x_2 \\
x_4 & x_5 & x_6 & \cdots & x_1 & x_2 & x_3 \\
x_5 & x_6 & x_7 & \cdots & x_2 & x_3 & x_4
\end{array}$$

The sum in each row is 20120, so the sum for the whole array is $5 \cdot 20120 = 100600$. Then at least one of the 2012 column sums is $\geq \frac{100600}{2012} = 50$ because if each column sum were smaller than 50 the array sum would be smaller than 100600.

7. Probabilty that $3^m + 3^n$ is divisible by 5.

The probability is $\boxed{156/625}$. Here is one way to calculate it. Modulo 5, the powers $3^1, 3^2, 3^3, \ldots$ cycle through $3, 4, 2, 1, 3, 4, 2, 1, \ldots$. As the exponents range from 1 to 50, the residues 3 and 4 occur 13 times each, while 2 and 1 occur 12 times each. An equivalent problem then is to put 50 marbles into an urn, with 13 of them bearing the number 3, another 13 the number 4, 12 bearing 2 and 12 bearing 1. We draw two marbles at random with replacement, and ask for the probability that the sum is 5. This will be the case for the pairs $(3,2), (2,3), (4,1)$ and $(1,4)$, of which there are, respectively, $13 \cdot 12,\ 12 \cdot 13,\ 13 \cdot 12,\ 12 \cdot 13$, for a total of $4 \cdot 13 \cdot 12$ pairs. There are 50^2 pairs in all, so the required probability is

$$\frac{4 \cdot 13 \cdot 12}{50^2} = \frac{13 \cdot 12}{25^2} = \frac{156}{625}.$$

8. Three intersection points.

The sums are $\boxed{x_1 + x_2 + x_3 = -1}$ and $\boxed{y_1 + y_2 + y_3 = 20/3}$. The three numbers $x_1,\ x_2,\ x_3$ are the solutions of the equation $x^3+x^2-4x+1 = -2x/3+2$; i.e., of $x^3+x^2-(10/3)x-1 = 0$, and their sum is the negative of the coefficient of x^2, so $x_1 + x_2 + x_3 = -1$. Then

$$\sum_{k=1}^{3} y_k = \sum_{k=1}^{3}\Big(2 - \frac{2}{3}x_k\Big) = 6 - \frac{2}{3}\sum_{k=1}^{3} x_k = 6 + \frac{2}{3} = \frac{20}{3}.$$

9. A sum of 2012 reciprocals.

We show that $\boxed{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_{2012}} = 88 + \dfrac{32}{45}}$. Evaluation of a_n for values of n from 1 to 13 or so suggests that as n increases, a_n changes from k to $k+1$ when n changes from $k(k+1)$ to $k(k+1)+1$. Having guessed this, it is easy to prove.

$$\sqrt{k(k+1)} = \sqrt{k^2+k} < \sqrt{k^2+k+1/4} = k + 1/2 < \sqrt{k^2+k+1}.$$

Thus, the largest n such that $a_n = k$ is $n = k(k+1)$, and the largest n such that $a_n = k-1$ is $(k-1)k$, so the number of values of n with $a_n = k$ is $k(k+1) - (k-1)k = 2k$. Now, $44 \cdot 45 = 1980$, and $45 \cdot 46 = 2070$, so the sum in question is

$$\frac{2}{1} + \frac{4}{2} + \cdots \frac{88}{44} + \frac{2012 - 1980}{45} = 44 \cdot 2 + \frac{32}{45} = 88 + \frac{32}{45}.$$

10. Divergent integrals.

We know that for all $t > 0$, $t+1/t \geq 2$ (this is equivalent to $(t-1)^2 \geq 0$). Thus $g(x)+1/g(x) \geq 2$ for all $x \geq 1$, and therefore for $m > 1$,

$$\int_1^m f(x)g(x)dx + \int_1^m \frac{f(x)}{g(x)}dx = \int_1^m f(x)\Big(g(x) + \frac{1}{g(x)}\Big)dx \geq 2\int_1^m f(x)dx.$$

As the integral on the right diverges as $m \to \infty$, it follows that one of those on the left diverges.