2004 Arkansas Mathematics Competition Solutions

1. The 70 millionth digit.

The 70, 000, 000th digit is 2. Write

$$\frac{1}{70,000,000} = \frac{1}{7} \cdot 10^{-7}.$$

Since $\frac{1}{7} = .\overline{142857}$, the decimal expansion of $\frac{1}{7} \cdot 10^{-7}$ begins with seven zeros followed by the repeated cycle 142857; thus, after the seven initial zeros the cycle always begins at positions congruent to 2 modulo 6. From the fact that $70,000,000 = (11,666,666)(6) + 4$ we see that 70,000,000 is congruent to 4 mod 6, so the 70,000,000th digit is 2.

2. The inverse function.

We have $f'(x) = 3x^2 + 6x + 3 = 3(x+1)^2 > 0$ for all $x \neq -1$, so f is strictly increasing, and therefore one-to-one.

If $y = f(x)$, then $y + 1 = (x+1)^3$, so $x + 1 = \sqrt[3]{y+1}$, and $x = \sqrt[3]{y+1} - 1$. Thus

$$f^{-1}(x) = \sqrt[3]{x+1} - 1.$$

3. First quadrant triangle.

We show that $ab = 4$. The line $ax + by = 12$ intersects the coordinate axes at $(12/a, 0)$ and $(0, 12/b)$, so the triangle formed has area

$$\frac{1}{2}\left(\frac{12}{a}\right)\left(\frac{12}{b}\right) = \frac{72}{ab}.$$

Thus $72/ab = 18$, and $ab = 72/18 = 4$.

4. Vanishing third derivative.

We may, and do, assume without loss of generality, that $x_1 < x_2 < x_3 < x_4$. By the Mean Value Theorem, there exist y_1 in (x_1, x_2), y_2 in (x_2, x_3) and y_3 in (x_3, x_4) such that $f'(y_1) = f'(y_2) = f'(y_3) = 0$. Again by the MVT, there exist z_1 in (y_1, y_2) and z_2 in (y_2, y_3) such that $f''(z_1) = f''(z_2) = 0$. Once more by the MVT, there exists a point c in (z_1, z_2) such that $f'''(c) = 0$.

5. Second derivative.

We'll show that $f''(t) = g(t) - g(3)$.

By the Fundamental Theorem of Calculus we have $\int_3^y g'(z)dz = g(y) - g(3)$, so

$$f(t) = \int_1^t \int_2^x [g(y) - g(3)]dydx = \int_1^t h(x)dx,$$

where $h(x) = \int_2^x [g(y) - g(3)]dy$. By the FTC again,

$$f'(t) = h(t) = \int_2^t [g(y) - g(3)]dy,$$

and

$$f''(t) = g(t) - g(3).$$

6. A random divisor.

The probability is $\frac{9}{625} = 0.0144$. The positive divisors of 10^{99} are

$$\{2^r 5^s : 0 \leq r \leq 99, 0 \leq s \leq 99\}.$$

There are 100^2 of them. Those that are multiples of 10^{88} are

$$\{2^r 5^s : 88 \leq r \leq 99, 88 \leq s \leq 99\}.$$

There are 12^2 of these. Thus, the required probability is

$$\frac{12^2}{100^2} = \frac{9}{625} = .0144.$$

7. Find the second term.

The second term is $\frac{373}{128}$. We have $a_1 = 1$, $a_2 = x$, $a_3 = x + 1$, $a_4 = a_3 + x + 1 = 2(x + 1)$, and in general

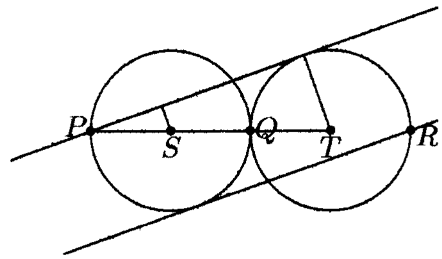
$$a_{n+1} = a_n + (a_{n-1} + a_{n-2} + \cdots + x + 1) = 2a_n.$$

Thus $a_n = 2^{n-3}(x + 1)$, $2004 = a_{12} = 2^9(x + 1)$,

$$x + 1 = \frac{2004}{512} \quad \text{and } a_2 = x = \frac{1492}{512} = \frac{373}{128}.$$

8. Distance between tangents.

The distance is $\frac{4}{3}$. Let S and T be the centers of the circles. The perpendicular distance from T to the upper tangent line is 1, and its distance from the lower tangent line is the same as that from S to the upper tangent line. This distance is $\frac{1}{3}$, by similar triangles, so $d = \frac{4}{3}$.



9. A nonsingular matrix.

We have

$$A(A^2 + 3A + 2I) = (A^2 + 3A + 2I)A = A^3 + 3A^2 + 2A = -3I,$$

so if $U = -(1/3)(A^2 + 3A + 2I)$, then $AU = UA = I$; i.e., $U = A^{-1}$.

10. One of these inequalities holds.

The key fact is that each $\frac{a_i}{b_i} + \frac{b_i}{a_i} \geq 2$. This is because $f(x) = x + \frac{1}{x} \geq 2$ for all $x > 0$, for this inequality is equivalent to $x^2 + 1 \geq 2x$. (Alternatively, one can establish this inequality by examining the derivative, $f'(x)$. We have

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2},$$

which is negative for $0 < x < 1$ and positive for $x > 1$, so $f(x)$ has its global minimum value at $x = 1$, and $f(1) = 2$.)

From the fact that each $\frac{a_i}{b_i} + \frac{b_i}{a_i} \geq 2$ it follows that

$$\left(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}\right) + \left(\frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n}\right) \geq 2n,$$

and therefore at least one of the sums in parentheses is $\geq n$.