$\boxed{\text{2011 Arkansas Mathematics Competition Solutions}}$

1. Rectangular parallelepiped of volume 2011.

The required product is $\boxed{2011^2}$. If the edges of the parallelepiped have lengths a, b and c, then the product of the areas of the areas of the three faces meeting at a vertex is

$$(ab)(bc)(ac) = (abc)^2 = (2011)^2.$$

2. The square of an integer.

We have $A = 1 + 10 + 10^2 + \cdots + 10^{2m-1} = (10^{2m} - 1)/9$. Similarly, $B = 4(10^m - 1)/9$, so

$$A + B + 1 = \frac{1}{9}(10^{2m} - 1 + 4 \cdot 10^m - 4 + 9) = \frac{1}{9}(10^{2m} + 4 \cdot 10^m + 4) = \left(\frac{10^m + 2}{3}\right)^2.$$

(Note that $10 \equiv 1 \pmod 3$, so $10^m \equiv 1 \pmod 3$, and therefore $10^m + 2$ is divisible by 3.)

3. Rounding to nearest integer.

$\boxed{\text{No}}$. From the fact that

$$10^{2n} - 10^n < 10^{2n} - 10^n + .25 < 10^{2n} - 10^n + 1$$

it follows that

$$\sqrt{10^{2n} - 10^n} < 10^n - .5 < \sqrt{10^{2n} - 10^n + 1},$$

and therefore the nearest integer to $\sqrt{10^{2n} - 10^n}$ is at most $10^n - 1$, while the nearest integer to $\sqrt{10^{2n} - 10^n + 1}$ is 10^n.

4. Some logs.

We show that $\boxed{\log_4 x = \sqrt{27}/2}$. Let $y = \log_8 x$, $u = \log_2 x$ and $z = \log_2 y$. By the given equation, $z = \log_8 u$. Then $x = 8^y = 2^{3y}$, so $u = 3y$. Further, $y = 2^z$, so $u = 3 \cdot 2^z$. Also, $u = 8^z = 2^{3z}$, so we have $3 = 2^{2z}$, and $2z = \log_2 3$, whence $z = (1/2)\log_2 3$. Then

$$\log_2 x = u = 2^{3z} = 2^{\frac{3}{2}\log_2 3} = 3^{\frac{3}{2}} = \sqrt{27}.$$

Finally, $\log_4 x = \frac{1}{2}\log_2 x = \frac{\sqrt{27}}{2}$.

5. An inequality.

We have

$$0 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx),$$

so that

$$xy + yz + zx = -\frac{1}{2}(x^2 + y^2 + z^2) \leq 0.$$

6. Limit of a sequence.

The limit is $\boxed{\sqrt{2}/2}$. Using the identity

$$\frac{1}{\sqrt{2k-1} + \sqrt{2k+1}} = \frac{\sqrt{2k-1} - \sqrt{2k+1}}{-2}$$

we have

$$a_n = \frac{1}{-2\sqrt{n}}\left(\sqrt{1} - \sqrt{3} + \sqrt{3} - \sqrt{5} + \cdots + \sqrt{2n-1} - \sqrt{2n+1}\right),$$

which telescopes to

$$a_n = \frac{1 - \sqrt{2n+1}}{-2\sqrt{n}} = \frac{\sqrt{2n+1} - 1}{2\sqrt{n}} = \frac{1}{2}\left(\sqrt{2 + \frac{1}{n}} - \sqrt{\frac{1}{n}}\right).$$

From this it is evident that the limit is $\sqrt{2}/2$.

7. The 2011th term.

We show that $\boxed{a_{2011} = \frac{1}{(1006)(2011)}}$ Let $b_n = 1/a_n$. Then $b_1 = 1/a_1$ and

$$b_{n+1} = \frac{1}{a_{n+1}} = \frac{1 + na_n}{a_n} = \frac{1}{a_n} + n = b_n + n.$$

An easy proof by induction then shows that

$$b_n = b_1 + 1 + 2 + \cdots + (n-1) = b_1 + \frac{n(n-1)}{2},$$

so $b_{2011} = 2011 + (2011)(1005) = (1006)(2001)$, and

$$a_{2011} = \frac{1}{(1006)(2011)}.$$

8. An integral root.

We show that $\boxed{a+b+c+d=4n}$. We have

$$(n-a)(n-b)(n-c)(n-d)=4,$$

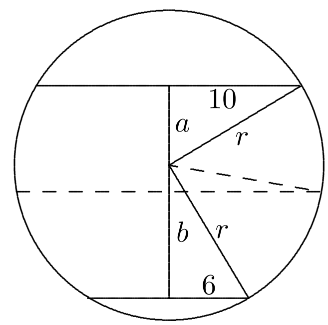
and the four factors on the left are distinct integers. But the only way to express 4 as a product of four distinct integers is as the product (-2)(-1) (1)(2), so $n-a, n-b, n-c$ and $n-d$ are these four integers in some order. Their sum is

$$4n-(a+b+c+d)=-2-1+1+2=0,$$

so $a+b+c+d=4n$.

9. The middle chord.

The length of the chord midway between them is $\boxed{2\sqrt{132}}$, or about 22.98. Let a be the distance from the center of the circle to the chord of length 20, $b=16-a$ the distance from the center to the chord of length 12, and r the radius of the circle. Then



$$a^2+10^2=r^2=b^2+6^2=(16-a)^2+6^2,$$

whence $32a=192$ and $a=6$. Then $r^2=136$, and the distance from the center to the middle chord is 2. If $2L$ is the length of the middle chord then $L^2+2^2=r^2=136$, and $L^2=132$. Thus $2L=2\sqrt{132}=4\sqrt{33}$.

10. Sum of roots.

$\boxed{S \text{ has four elements, and their sum is } 3\pi.}$ The quadratic equation u^2-7u+1 has two distinct positive real roots, call them u_1 and u_2. The tangent function increases from 0 to ∞ on $(0,\pi/2)$. Then there is exactly one pair of real numbers $\{x_1,x_2\}$ in $(0,\pi/2)$ such that $\tan x_1=u_1$ and $\tan x_2=u_2$, and two further numbers $x_3=x_1+\pi$ and $x_4=x_2+\pi$ in $(\pi,3\pi/2)$ such that $\tan x_3=u_1$ and $\tan x_4=u_2$. These are all the members of S (in the other two quadrants the tangent is negative). Because $u_1u_2=1$,

$$\tan x_2=\frac{1}{u_1}=\cot x_1,$$

so $x_1=\frac{\pi}{2}-x_2$; i.e., $x_1+x_2=\frac{\pi}{2}$. Thus

$$x_1+x_2+x_3+x_4=2(x_1+x_2)+2\pi=3\pi.$$