Third Annual Arkansas Undergraduate Mathematics Competition, February 25, 2006

No CALCULATORS, COMPUTERS, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which may simply mean showing all work and reasoning.) Have fun!

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1. System of equations.

Find all solutions (x, y, z) of the system

$$\begin{aligned} xy &= 9/8 \\ yz &= 4/7 \\ xz &= 9/14. \end{aligned}$$

2. Minimum element.

Let $S = \{x\colon x^{2008} + 7x^{2004} = 8x^{2006}\}$. Find the minimum element of S.

3. Fourth term of a G.P.

The first term of a geometric progression is $\sqrt{2}$ and the second term is $\sqrt[3]{2}$. Find the fourth term, and express it as simply as you can.

4. Common linear factor.

Find all real values of k for which the following quadratic polynomials have a common linear factor:

$$x^2 + kx + 8 \quad \text{and} \quad 2x^2 + kx - 5$$

5. Trigonometric equation.

Solve the equation

$$\cos\theta = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{3}}}$$

for θ in the interval $0 < \theta < \pi/2$.

6. Coefficient of $(x+1)^2$.

If the polynomial

$$P(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - x^9 + x^{10} - x^{11}$$

is rewritten in the form

$$a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3 + \cdots + a_{11}(x+1)^{11},$$

where the coefficients $a_0, a_1, \ldots, a_{11}$ are constants, what is the value of a_2?

7. An inequality.

If x is a positive number different from 1, and n is a positive integer, prove that

$$\frac{1 - x^{2n+1}}{1 - x} > (2n+1)x^n.$$

8. A mean value?

Let $0 < a < b$ and assume that the function f is continuous on $[a, b]$ and differentiable on (a, b). Show that if

$$\frac{f(a)}{a} = \frac{f(b)}{b},$$

then there is a point c in (a, b) such that

$$f'(c) = \frac{f(c)}{c}.$$

9. Quotient of two series.

Let

$$S = \sum_{n=1}^{\infty} \frac{1}{n^3} \qquad \text{and} \qquad T = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}.$$

Evaluate, with proof, the quotient $\frac{S}{T}$.

10. Area of a triangle.

In the triangle ABC, the altitude from A divides BC into segments of length 4 and 15. If the tangent of the angle at A is $19/4$, find the area of the triangle ABC.