Tenth Annual Arkansas Undergraduate Mathematics Competition, February 23, 2013

No CALCULATORS, COMPUTERS, CELL-PHONES, BOOKS, NOTES or **NON-TEAM-MEMBERS** may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. An integral.

Evaluate

$$\int_{1/e}^{e^2} |\ln x| dx.$$

2. An arithmetic progression of 2013 terms.

Given that $a_1, a_2, a_3, \ldots, a_{2013}$ is an arithmetic progression, prove that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{2012}}+\sqrt{a_{2013}}} = \frac{2012}{\sqrt{a_1}+\sqrt{a_{2013}}}.$$

3. Length of a crease.

A rectangular sheet of paper $ABCD$ is folded so that corner B meets the opposite edge CD at B', forming a crease EF where E lies on edge AB and F lies on edge BC. If $|AB| = 21$, $|BC| = 12$ and $|AE| = 6$, find the length $|EF|$.

4. Roots are three consecutive integers.

Find all pairs (p, q) of real numbers such that the roots of the equation

$$x^3 - px^2 + 47x - q = 0$$

are three consecutive integers.

5. A polynomial of degree 2013.

Determine the remainder when the polynomial $x^{2013} + x^{1013} + x^{513} + x^{113} + x^2$ is divided by the polynomial $x^3 - x$.

6. Integer values of a quotient.

(a) How many positive integers n are there such that $(n-1)^2/(n+29)$ is an integer?
(b) What is the largest such n?

7. A quotient of tangents.

Given that

$$\frac{\sin(a+b)}{\sin(a-b)} = \frac{m+n}{m-n},$$

obtain $\frac{\tan a}{\tan b}$ as a function of m and n.

8. $n + m = 2013$.

Does there exist a positive integer a which expressed in decimal form has n digits, and a^3 has m digits, and $n + m = 2013$?

9. Consecutive positive terms.

The sequence $a_1, a_2, a_3, \ldots$ of real numbers has the property that $a_n = a_{n-1} + a_{n+2}$ for all $n \geq 2$. What is the largest number of consecutive terms that can be positive?

10. Rational or irrational?

Recall that the n-th triangular number is $T_n = 1 + 2 + \cdots + n = n(n+1)/2$. Let $N = 0.1360518\ldots$, where the k-th digit is the last (decimal) digit of T_k. Determine whether N is rational.