Eleventh Annual Arkansas Undergraduate Mathematics Competition, February 15, 2014

No CALCULATORS, COMPUTERS, CELL-PHONES, BOOKS, NOTES or **NON-TEAM-MEMBERS** may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team name and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. Logarithms and exponents.

Find all real values of x satisfying

$$x^{2\log_{10} x} = 10^{50}.$$

2. Base b integers.

The expression of the positive integer n in the positive base b is $n = 1366_b$. The expression of $2n$ in base b is $2n = 2721_b$. Determine both b and n in base 10.

3. Lists of 2014 words.

Each of A, B, C makes a list of 2014 words. If a word appears on more than one list, it is removed from every list on which it appears. In the end, A's list has 1900 words and B's has 1980. What is the smallest number of words that could be left on C's list?

4. It's a right angle.

Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $AB + CD = AD$. Let P be the point on AD such that $AP = AB$. Show that $\angle BPC$ is a right angle.

5. A quadratic with real roots.

Let a be a real number. Show that if the equation $x^2 - ax + a = 0$ has real solutions r and s, then $r^2 + s^2 \geq 2(r + s)$.

6. Floor function equation.

Recall that for real x, $\lfloor x \rfloor$ is defined to be the greatest integer less than or equal to x. Determine, with proof, how many positive integers n with $1 \leq n \leq 2014$ there are which satisfy the equation

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor = \frac{n}{2} + \frac{n}{3} + \frac{n}{4}.$$

7. Integral inequality.

Show that

$$0 < \int_5^{10} \frac{t^3 dt}{t^7 + t^2 - 24} < .0025.$$

8. Coefficient of x^2, with constant term 2014.

The product of two of the four roots of the equation $x^4 - 39x^3 + kx^2 + 1976x - 2014 = 0$ is -53. Determine, with proof, all possible values of k.

9. The square of an integer.

Find all integers n such that $n^2 + 19n + 97$ is the square of an integer.

10. A global minimum.

Find the minimum value of

$$f(x) = x^2 + 2\cos 2x - 4x\cos x$$

for real x.