$\boxed{\textbf{2019 AUMC Solutions}}$

1. Probability of a palindrome.

The probability is $\boxed{7/52}$. The probability of a 3-letter palindrome is $26^2/26^3 = 1/26$ because there are 26 possibilities for the first letter and 26 for the second and just 1 for the third, which must be the same as the first. Similarly the probability of a 3-digit palindrome is $1/10$. The probability that a plate contains both a 3-letter palindrome and a 3-digit palindrome is $(1/26)(1/10) = 1/260$. By the inclusion-exclusion principle, the probability of at least one palindrome is

$$\frac{1}{26} + \frac{1}{10} - \frac{1}{260} = \frac{35}{260} = \frac{7}{52}.$$

2. Ratio of the 2019th term to the 2018th.

It is

$$\frac{2019^3 2017^3}{2018^6}.$$

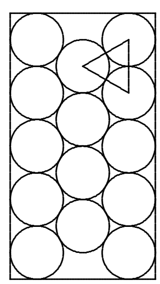
Denote the n-th term by a_n. For $n \geq 2$, $n^3 = a_1 a_2 \cdots a_n = (a_1 a_2 \cdots a_{n-1}) a_n = (n-1)^3 a_n$, so $a_n = n^3/(n-1)^3$. Then

$$\frac{a_{2019}}{a_{2018}} = \frac{2019^3}{2018^3} \cdot \frac{2017^3}{2018^3} = \frac{2019^3 2017^3}{2018^6}.$$

3. Length of a rectangle.

The length of the long side is $\boxed{\frac{5}{1+\sqrt{3}} = \frac{5(\sqrt{3}-1)}{2}}$. If r is the radius of the circle, the long side is evidently $10r$. Consider an equilateral triangle with vertices at the centers of three mutually tangent circles. Its side length is $2r$, so its altitude is $r\sqrt{3}$. Then the short side has length $1 = 2(r + r\sqrt{3})$ so the long side is

$$\frac{10r}{2r(1+\sqrt{3})} = \frac{5}{1+\sqrt{3}} = \frac{5(\sqrt{3}-1)}{2}.$$



4. Minimum value of a function.

The minimum value is $\boxed{f(1) = 6}$. On expanding, multiplying numerator and denominator by x^4 and factoring out $\frac{3}{x}$ we obtain

$$\begin{aligned} f(x) &= \frac{6x^4 + 15x^2 + 18 + 15x^{-2} + 6x^{-4}}{2x^3 + 3x + 3x^{-1} + 2x^{-3}} \\ &= \left(\frac{3}{x}\right)\left(\frac{2x^8 + 5x^6 + 6x^4 + 5x^2 + 2}{2x^6 + 3x^4 + 3x^2 + 2}\right) \\ &= \frac{3}{x}(x^2 + 1) \\ &= 3\left(x + \frac{1}{x}\right). \end{aligned}$$

By the AM-GM inequality, or otherwise, we know $x + \frac{1}{x} \geq 2$, so the minimum value is $f(1) = 6$, as claimed.

5. A sum of squares.

The sum is $\boxed{2627}$. If $k = m^2$ and $k + 99 = n^2$, then $(n - m)(n + m) = n^2 - m^2 = 99 = 1 \cdot 99$ or $3 \cdot 33$ or $9 \cdot 11$. If $n - m = 1$ and $n + m = 99$, then $n = 50$ and $m = 49$ and $k = m^2 = 49^2$. If $n - m = 3$ and $n + m = 33$, then $n = 18$ and $m = 15$ and $k = 15^2$. If $n - m = 9$ and $n + m = 11$, then $n = 10$ and $m = 1$, and $k = 1$. Thus the required sum is $49^2 + 15^2 + 1^2 = 2627$.

6. A floor function value.

We show that $\boxed{\lfloor 85t \rfloor = 400}$. There are 54 terms in the left member of the equation, and each term is either $\lfloor t \rfloor$ or $\lfloor t \rfloor + 1$. From the fact that $54 \cdot 4 < 256 < 54 \cdot 5$ we deduce that $\lfloor t \rfloor = 4$. For some integer k, then, with $0 < k < 54$, the first k terms each have value 4 and the remaining $54 - k$ terms have value 5. Then $4k + 5(54 - k) = 256$; $k = 5 \cdot 54 - 256 = 14$. The 14th term is $\lfloor t + \frac{24}{85} \rfloor$, so

$$\begin{aligned} t + \frac{24}{85} < 5 \leq t + \frac{25}{85}; \\ 5 - \frac{25}{85} \leq t < 5 - \frac{24}{85}. \end{aligned}$$

Then $400 \leq 85t < 401$, and $\lfloor 85t \rfloor = 400$.

7. Three numbers adding up to 2019.

The required sum is $\boxed{10092}$. We have

$$\begin{aligned}(2019)(5) &= (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\\ &= \frac{a}{a}+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{b}{b}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+\frac{c}{c},\\ &= 3+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\end{aligned}$$

so

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c} = (2019)(5)-3 = 10092.$$

(That such real numbers exist is easily verified. For example,, if we put $a=1$ and solve the two equations for b and c we find that b and c are $\frac{2019\pm\sqrt{(2018)(2017)}}{2}$.)

8. A fraction greater than or equal to 3/4.

We have

$$\begin{aligned}\frac{x_1+x_3+x_5}{x_2+x_4+x_6} \geq \frac{1+x_3+x_5}{x_2+x_4+2} &\geq \frac{1+x_2+x_4}{2+x_2+x_4}\\ &= 1-\frac{1}{2+x_2+x_4} \geq 1-\frac{1}{2+1+1} = \frac{3}{4}.\end{aligned}$$

9. A functional equation.

The unique solution is

$$f(x) = \frac{2x-4}{x^2-x+1}.$$

One verifies by substitution that this function satisfies the equation. Conversely, if f satisfies

$$f(1-x)+2 = xf(x) \tag{1}$$

then, replacing x by $1-x$ in (1) we find that

$$f(x)+2 = (1-x)f(1-x). \tag{2}$$

Substituting $f(1-x)$ from (1) into (2) yields

$$f(x)+2 = (1-x)(xf(x)-2),$$

and this may be solved for $f(x)$ to obtain the solution asserted above.

10. A convergent series.

We use the comparison test.

$$1 - \frac{a_n}{a_{n+1}} = \frac{a_{n+1} - a_n}{a_{n+1}} \le \frac{a_{n+1} - a_n}{a_1}.$$

The series

$$\frac{1}{a_1}\sum_{n=1}^{\infty}(a_{n+1} - a_n)$$

telescopes, and the N-th partial sum is $\frac{1}{a_1}(a_N - a_1)$. Because the sequence $\{a_n\}$ is monotone increasing and bounded, it converges, say to L. Then $\dfrac{1}{a_1}\sum_{n=1}^{\infty}(a_{n+1} - a_n)$ converges to $\dfrac{1}{a_1}(L - a_1)$, and by comparison,

$$\sum\left(1 - \frac{a_n}{a_{n+1}}\right)$$

converges.