Sixth Annual Arkansas Undergraduate Mathematics Competition, February 28, 2009

No CALCULATORS, COMPUTERS, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. The sum is composite.

Given that the integers a and b satisfy $56a = 65b$, prove that the integer $a + b$ is composite.

2. A probability.

A fair coin is tossed repeatedly. What is the probability that the fourth occurence of heads precedes the second occurence of tails?

3. No squares, cubes or fifth powers.

How many of the integers from 1 to 10^{30} inclusive are not perfect squares, perfect cubes, or perfect fifth powers?

4. Solve for x.

Find all real numbers x satisfying

$$\frac{x-2}{x-3} + \frac{x-6}{x-5} + \frac{x-8}{x-7} + \frac{x-8}{x-9} = 4.$$

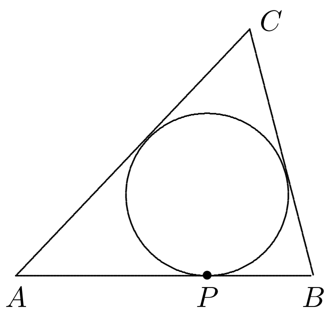
5. A remainder.

When the positive integer n is divided by 2009, the remainder is 1009. What is the remainder when n is divided by 41?

6. Unequal distances from lattice points.

Let (m, n) and (r, s) be two distinct lattice points (i.e., points with integer coordinates) in the plane, and let $d_1,\ d_2$ be their respective distances from $(\sqrt{2}, 1/3)$. Show that $d_1 \neq d_2$.

7. Length of a tangent.



A circle is inscribed in triangle ABC, which has side lengths $BC = 6$, $AB = 7$ and $CA = 8$. Find the length AP, where P is the point of tangency of side AB to the circle.

8. It's an integer.

Suppose that x is a real number such that $x + \frac{1}{x}$ is an integer. Prove that

$$x^{2009} + \frac{1}{x^{2009}}$$

is an integer.

9. An inequality.

Let $a_1,\ a_2, \cdots, a_n$ be positive numbers with $a_1 a_2 \cdots a_n = 1$. Prove that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 2^n.$$

10. An integral.

The function f and its first two derivatives f' and f'' are continuous on the half-line $(0, \infty)$, with $f(5) = 3,\ f'(5) = 2$ and $\int_1^5 f(x)dx = 5$. Find the value of $\int_1^5 (x-1)^2 f''(x)dx$.