Fifteenth Annual Arkansas Undergraduate Mathematics Competition, February 24, 2018

No CALCULATORS, ELECTRONIC DEVICES, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team name and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. A harmonic sum greater than 1.

Prove that for every positive integer n,

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}>1.$$

2. A real number evaluation.

Suppose that a and b are real numbers with $a \neq b$. If $a^2+b^2=5ab$, what is the value of $(a+b)/(a-b)$?

3. A property of integers greater than 2018.

Suppose that n is an integer with $n>2018$. Show that n^4+3n^2+1 is not a perfect square.

4. An arithmetic sum.

Given that $a_1, a_2, a_3, \ldots$ is an arithmetic progression with

$$a_1+a_2+a_3+\cdots+a_{40}=S$$

and

$$a_2+a_4+a_6+\cdots+a_{40}=T,$$

evaluate the sum

$$a_4+a_8+a_{12}+\cdots+a_{40}.$$

5. Sum of tangents equals their product.

Let A, B and C be real numbers satisfying $A + B + C = 0$. Prove that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

6. Minimizing a sum of lengths.

The coordinates of points A, B and C in the Cartesian plane are $(5, 5)$, $(2, 1)$ and $(0, k)$, respectively. For what value of k is the sum of the lengths of AC and BC a minimum? Justify your answer.

7. A 2017-2018 integral equation.

Find all differentiable functions f satisfying the equation

$$(f(x))^{2018} = \int_1^x (f(t))^{2017} dt.$$

8. Triangle construction.

Consider three line segments of lengths a, b and $\sqrt{ab}$, where $a > b$. (i) For what values of a/b can these segments be the sides of a nondegenerate triangle? (ii) For what values of a/b is this a right triangle?

9. The area of a square.

A point P in the interior of the square $ABCD$ is at a distance 13 from A, 5 from C and 8 from D. What is the area of the square?

10. A periodic function.

Let f be a real valued function defined on all of $\boldsymbol{R}$ and satisfying

$$f(x + 1) = \frac{1}{2} + \sqrt{f(x) - f(x)^2}.$$

Prove that f is periodic; i.e., that there exists $c > 0$ such that $f(x + c) = f(x)$ for all x.