$\boxed{\text{2010 Arkansas Mathematics Competition Solutions}}$

1. A probability.

The probability is $\boxed{\frac{100}{649}}$. There are $\binom{60}{6}$ different sets of 6 balls which could be drawn. Of these, $\binom{10}{1}\binom{20}{2}\binom{30}{3}$ include exactly 1 red, 2 white and 3 blue balls, so the desired probability is

$$\frac{\binom{10}{1}\binom{20}{2}\binom{30}{3}}{\binom{60}{6}},$$

which reduces to $\frac{100}{(11)(59)} = \frac{100}{649}$.

2. An inequality.

We have $a^2 + 2ab + b^2 = c^2 + 2cd + d^2$ and $a^2 + b^2 > c^2 + d^2$, so $ab < cd$. Then

$$\begin{aligned} a^3 + b^3 &= (a+b)^3 - 3ab(a+b) \\ &= (c+d)^3 - 3ab(c+d) \\ &> (c+d)^3 - 3cd(c+d) \\ &= c^3 + d^3. \end{aligned}$$

3. The perimeter.

The perimeter is $\boxed{60 - 10\sqrt{6}}$. Let the legs be a and b, and the hypotenuse c. We have $a + b = 20$, $a + c = 30$, and $c^2 = a^2 + b^2$. Then

$$a^2 + b^2 = c^2 = (30 - a)^2 = 900 - 60a + a^2,$$

so $b^2 = 900 - 60a = 900 - 60(20 - b) = -300 + 60b$, and $b^2 - 60b + 300 = 0$. Thus

$$b = \frac{60 \pm \sqrt{2400}}{2} = 30 \pm 10\sqrt{6}.$$

From the fact that $a + b = 20$ we know that $b < 20$, so b cannot be $30 + 10\sqrt{6}$. Thus $b = 30 - 10\sqrt{6}$, and $a + c = 30$, so $a + b + c = 60 - 10\sqrt{6}$.

4. Find $f(-2010)$.

$\boxed{f(-2010) = (-11)(2010) = -22110}$, as we shall show. We have

$$f(2010) = a(2010)^6 + b(2010)^4 + 5(2010) - 9$$

and

$$f(-2010) = a(2010)^6 + b(2010)^4 - 5(2010) - 9.$$

Subtracting, we obtain

$$f(2010) - f(-2010) = (10)(2010),$$

so

$$f(-2010) = f(2010) - (10)(2010) = -2010 - 10(2010) = (-11)(2010) = -22110.$$

5. An integral.

The value is $\boxed{606.5}$.

$$\begin{aligned}\int_1^{2010} \frac{dx}{1 + \lfloor \log_{10} x \rfloor} &= \int_1^{10} 1dx + \int_{10}^{100} \frac{dx}{2} + \int_{100}^{1000} \frac{dx}{3} + \int_{1000}^{2010} \frac{dx}{4} \\ &= 9 + \frac{90}{2} + \frac{900}{3} + \frac{1010}{4} \\ &= 9 + 45 + 300 + 252.5 = 606.5.\end{aligned}$$

6. Power series.

They are the pairs

$$\left\{ (a, r) \colon |r| < 1 \text{ and } a = \frac{(1-r)^2}{1+r} \right\}.$$

With $a \neq 0$ the series $\sum_{n=0}^{\infty} ar^n$ converges if and only if $|r| < 1$, and in this case

$$\sum_{n-0}^{\infty} ar^n = \frac{a}{1-r} \qquad \text{and} \qquad \sum_{n=0}^{\infty} (ar^n)^2 = a^2 \sum_{n=0}^{\infty} r^{2n} = \frac{a^2}{1-r^2}.$$

Then the desired condition becomes

$$\frac{a^2}{1-r^2} = \frac{a^3}{(1-r)^3}.$$

As $a \neq 0$ and $r \neq 1$, we may divide out the common factor $a^2/(1-r)$ to obtain the equivalent equation

$$\frac{1}{1+r} = \frac{a}{(1-r)^2}.$$

Thus the desired equation is satisfied for arbitrary r with $|r| < 1$ and $a = (1-r)^2/(1+r)$.

7. The local maximum value.

The local maximum value is $\boxed{\frac{2}{5\sqrt{5}}}$. To find it, we first find a and b from the given information. We have $f'(x) = 3x^2 + 2ax + b$, so

$$f'\left(\frac{1}{\sqrt{5}}\right) = \frac{3}{5} + \frac{2a}{\sqrt{5}} + b = 0. \tag{1}$$

Also,

$$f\left(\frac{1}{\sqrt{5}}\right) = \frac{1}{5\sqrt{5}} + \frac{a}{5} + \frac{b}{\sqrt{5}} = \frac{-2}{5\sqrt{5}}. \tag{2}$$

From (1) we deduce

$$2\sqrt{5}a + 5b = -3, \tag{1a}$$

and from (2) we have

$$\sqrt{5}a + 5b = -3. \tag{2a}$$

Solving (1a) and (2a) simultaneously we obtain $a = 0$ and $b = -3/5$. Thus

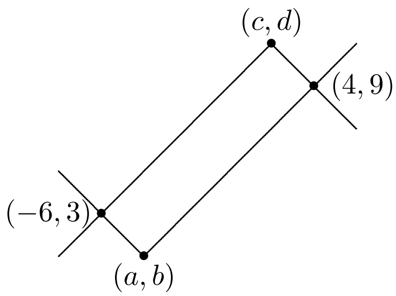
$$f(x) = x^3 - \frac{3x}{5};$$

$$f'(x) = 3x^2 - \frac{3}{5} = 0 \text{ at } x = \pm\frac{1}{\sqrt{5}}.$$

The local minimum is at $x = 1/\sqrt{5}$, as was given, and the local maximum is at $x = -1/\sqrt{5}$. Finally, the local maximum value is

$$f\left(-\frac{1}{\sqrt{5}}\right) = -f\left(\frac{1}{\sqrt{5}}\right) = \frac{2}{5\sqrt{5}}.$$

8. Graph intersections.



We'll show that $\boxed{(a,b,c,d)=(-4,1,2,11)}$. The graph of $y=|x-a|+b$ is V-shaped, with vertex at (a,b) and slopes ± 1. The graph of $y=-|x-c|+d$ is an inverted V-shape with vertex at (c,d) and slopes ± 1. One intersection must lie to the left of both vertices and one to the right of both. The segment from (a,b) to $(-6,3)$ has slope -1 and the segment from (a,b) to $(4,9)$ has slope 1, so $a+6=-b+3$ and $a-4=b-9$. These two equations yield $a=-4,\ b=1$.

Similarly we find $c-4=-d+9$ and $c+6=d-3$, giving us $c=2,\ d=11$.

9. Integer solutions.

The lattice points on the graph are $\boxed{(7,4),(7,-4),(-7,4),\text{ and }(-7,-4)}$. Write the equation in the form $1475=2y^2(x^2-32)+19y^2=(x^2-32)(2y^2+19)+19\cdot 32$. Equivalently, $(x^2-32)(2y^2+19)=1475-608=867=3\cdot 17^2$. Because $2y^2+19>0$, we must have $x^2-32>0$, so x^2-32 is a positive factor of $3\cdot 17^2$; i.e., x^2-32 is one of 1, 3, 17, 51, 289 or 867. The only choice here making x an integer is $x^2-32=17$; $x^2=49$, and $x=\pm 7$. Along with this value of x we need $2y^2+19=51$, so $2y^2=32$ and $y^2=16$. Thus, the only first quadrant solution is $(x,y)=(7,4)$, and by symmetry we obtain the four points given at the outset.

10. A rational value.

We'll show that $\boxed{\csc x+\cot x=11/6}$. If $\sec x+\tan x=r=17/5$, then from the fact that $\sec^2 x-\tan^2 x=1$ we have

$$\sec x-\tan x=\frac{\sec^2 x-\tan^2 x}{\sec x+\tan x}=\frac{1}{r}.$$

This together with $\sec x+\tan x=r$ tells us that

$$2\tan x=r-\frac{1}{r}=\frac{r^2-1}{r}\quad\text{and}\quad 2\sec x=r+\frac{1}{r}=\frac{r^2+1}{r}.$$

Then

$$\sin x=\frac{\tan x}{\sec x}=\frac{r^2-1}{r^2+1}\quad\text{and}\quad \cos x=\frac{\sin x}{\tan x}=\frac{2r}{r^2+1},$$

so

$$\csc x+\cot x=\frac{r^2+1}{r^2-1}+\frac{2r}{r^2-1}=\frac{r+1}{r-1}=\frac{22}{5}\cdot\frac{5}{12}=\frac{11}{6}.$$