Second Annual Arkansas Undergraduate Mathematics Competition, February 26, 2005

No CALCULATORS, COMPUTERS, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which may simply mean showing all work and reasoning.) Have fun!

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1. Cylindrical volume.

If the radius of a right circular cylinder is decreased by 20% and the height is increased by 40%, find the percentage change in the volume, and whether it is an increase or a decrease.

2. Roots of a quadratic.

Given that the roots of the equation $x^2+px+q=0$ differ by 1, and that p and q are positive, express p as a function of q as simply as you can.

3. Area spanned by roots.

Find the area of the quadrilateral in the complex plane whose vertices are the roots of the equation

$$x^4 - 4x^3 + 5x^2 - 4x + 1 = 0.$$

4. Progressive triples.

Determine all ordered triples (a, b, c) of real numbers satisfying the following three conditions:

(i) In the order a, b, c, the numbers are in arithmetic progression.
(ii) In the order a, c, b, the numbers are in geometric progression.
(iii) The sum of the three numbers is 30.

5. Not a square.

Let n be a positive integer. Prove that

$$n^4 + 2n^3 + 2n^2 + 2n + 1$$

is not the square of an integer.

6. Probability.

Adolph chooses a real number a at random in the interval (0,8) and Bertha independently chooses a real number b at random in the interval (0,4). What is the probability that $a^2 > b^3$? (Choosing "at random" in an interval means that the probability of choosing from any subinterval is proportional to the length of the subinterval.)

7. Find this year's term.

Let $\{x_n\}$ and $\{y_n\}$ be sequences satisfying the recursions $x_{n+1} = x_n^3 - 3x_n$ and $y_{n+1} = y_n^3 - 3y_n$. Let $z_n = x_n^2 - y_n$. If $z_0 = 2$, find z_{2005}.

8. Last four digits.

What are the last four digits of 7^{2005} (in decimal form)?

9. Limit of a sequence.

Let

$$a_n = \frac{1^{\frac{7}{2}} + 2^{\frac{7}{2}} + \cdots + n^{\frac{7}{2}}}{n^{\frac{9}{2}}}.$$

Show that $\lim_{n \to \infty} a_n$ exists and find its value.

10. Maximum value.

Find the maximum (real) value of the function

$$f(x) = \sqrt{7x - x^2} - \sqrt{9x - x^2 - 14}$$

for real x.