Thirteenth Annual Arkansas Undergraduate Mathematics Competition, February 27, 2016

No CALCULATORS, COMPUTERS, CELL-PHONES, BOOKS, NOTES or **NON-TEAM-MEMBERS** may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team name and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. A sequence of 2016 terms.

Let $D = \boldsymbol{R}\backslash\{0,1\}$ be the domain consisting of the set of real numbers with 0 and 1 deleted. A sequence $\{f_n\}$ of real valued functions is defined recursively on D by

$$f_1(x) = \frac{1}{1-x} \quad \text{and for } n \geq 1,\ f_{n+1}(x) = f_1(f_n(x)).$$

Now define the sequence $a_n = f_n(n+1),\ n = 1, 2, 3, \ldots$, and let

$$A = \{a_1, a_2, \ldots, a_{2016}\}.$$

Find (a) the maximum, (b) the minimum element of A, and (c) the element nearest to 0.

2. Sum of a series.

Evaluate

$$\sum_{k=1}^{\infty} \ln \frac{(k+1)(3k+1)}{k(3k+4)}.$$

Be careful to justify your answer.

3. Rational values of a function .

For positive integers n, find all rational values assumed by the function

$$f(n) = \sqrt{\left(\frac{n}{7}\right)^2 + 503}.$$

4. Closer to the center.

A point is chosen at random (uniform distribution) in a square. What is the probability that it is nearer to the center than to an edge?

5. Which is larger?.

Determine, with proof, which is larger, 2016^{2016} or 2017^{2015}.

6. Doubling the area.

Find all integer pairs (m, n) with $m \leq n$ such that an $m \times n$ rectangle is doubled in area when each of the dimensions in increased by 7. You must show that you have all such pairs.

7. Floor function equation.

Find all real numbers x satisfying the equation

$$\frac{19x + 16}{10} = \left\lfloor \frac{4x + 7}{3} \right\rfloor.$$

As usual, $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u.

8. Real roots.

Given that a, b and c are real numbers, prove that the roots of the equation

$$(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$$

are real.

9. Last two digits.

For integers $n \geq 1$, let $f(n) = 2^3 + 2^{3^2} + 2^{3^3} + \cdots + 2^{3^n}$. Determine, with proof, the last two digits of $f(2016)$.

10. Area of a triangle.

Two sides of triangle ABC have lengths 9 and 15, and the length of the median CD to the third side is 7. Find the area of the triangle.