First Annual Arkansas Undergraduate Mathematics Competition, February 28, 2004*

No CALCULATORS, COMPUTERS, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. Have fun!

* Funds for this competition are provided by a grant from the Paul and Virginia Henry Academic Enrichment Program.

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1. The 70 millionth digit.

If the fraction $\frac{1}{70{,}000{,}000}$ is written out in decimal form

$$\frac{1}{70,000,000} = 0.a_1a_2a_3\ldots,$$

what is the digit $a_{70{,}000{,}000}$? (Justify your answer.)

2. The inverse function.

Let $f(x) = x^3 + 3x^2 + 3x$ for x in the set $\boldsymbol{R}$ of real numbers. Prove that f is one-to-one on $\boldsymbol{R}$, and obtain a formula for the inverse function, $f^{-1}(x)$.

3. First quadrant triangle.

Suppose that a and b are positive numbers and that the triangle in the first quadrant bounded by the coordinate axes and the line $ax + by = 12$ has area 18. Find the value of ab, and justify your answer.

4. Vanishing third derivative.

Suppose that the function f has continuous first, second and third derivatives on the closed interval $[-4, 4]$. Given that there are four distinct points x_1, x_2, x_3 and x_4 in $[-4, 4]$ where $f(x)$ vanishes, prove that there is a point c in $(-4, 4)$ such that $f'''(c) = 0$.

5. Second derivative.

Let

$$f(t) = \int_1^t \int_2^x \int_3^y g'(z)\,dz\,dy\,dx,$$

where g is a function with a continuous derivative. Express $f''(t)$ as simply as you can in terms of g.

6. A random divisor.

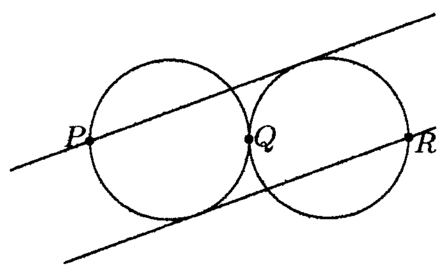
From a list of all the positive divisors of 10^{99}, one is chosen at random. What is the probability that it is a multiple of 10^{88}?

7. Find the second term.

The first term of a sequence is 1, and after the second term, each term is the sum of all preceding terms. If the twelfth term is 2004, what is the second term?

8. Distance between tangents.

Two circles of radius 1 are tangent to each other at point Q. PQ and QR are diameters of the two circles. From P a tangent is drawn to the circle with diameter QR, and from R a parallel tangent is drawn to the circle with diameter PQ. Find the distance d between these two tangent lines.



9. A nonsingular matrix.

Given that A is a square matrix satisfying the equation $A^3 + 3A^2 + 2A + 3I = 0$, where I is the identity matrix and 0 is the zero matrix, show that A is invertible.

10. One of these inequalities holds.

Suppose that $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are positive real numbers. Show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n \qquad \text{or} \qquad \frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \geq n;$$

i.e., at least one of the inequalities holds.