# The Newton-Raphson(?) Method For Computing Square Roots

The following is an attempt to define and validate what I believe is called the Newton-Raphson approximation method for extracting the positive square root of a positive number. (For the rest of this paper, "square root" will be taken to mean a number's positive square root.)

**Definition.** *Let $a$ be a positive real number. A* **Newton-Raphson sequence on $a$** *is a sequence of real numbers $\{x_0,\ x_1,\ldots\}$ such that*

$$\text{(1)} \qquad x_0 \text{ is positive;}$$

$$\text{(2)} \qquad x_{n+1} = \left(x_n + \frac{a}{x_n}\right)/2, \qquad n = 0, 1, \ldots$$

**Theorem.** *Let $a$ be positive, $\{x_n\}$ a Newton-Raphson sequence on a. Then $\lim_{n\to\infty} x_n$ exists and*

$$\lim_{n\to\infty} x_n = \sqrt{a}.$$

**Lemma.** *Let $a$ be positive, $\{x_n\}$ a Newton-Raphson sequence on a. Unless $x_0 = \sqrt{a}$, then for all positive integers n, $x_n > \sqrt{a}$.*

**Proof of Lemma.** We consider the function

$$y(x) = \left(x + \frac{a}{x}\right)/2, \qquad x > 0.$$

Then $y$ is differentiable and

$$y'(x) = \left(1 - \frac{a}{x^2}\right)/2.$$

Moreover, $y'$ is differentiable and

$$y''(x) = \frac{a}{x^3}.$$

We see that $y'(x) = 0$ iff $x = \sqrt{a}$; since $y''(\sqrt{a})$ is positive, $y$ is minimal at $\sqrt{a}$. Since $y(\sqrt{a}) = \sqrt{a}$, for all (positive) $x \neq \sqrt{a}, y(x) > \sqrt{a}$. Then $y(y(x))$ is defined and $y(y(x)) > \sqrt{a}$; similarly for $y(y(y(x)))$, and so on. But $x_1 = y(x_0),\ x_2 = y(x_1) = y(y(x_0)), \ldots$ Hence $x_0 \neq \sqrt{a} \Rightarrow x_1, x_2, x_3 \ldots > \sqrt{a}$.

**Proof of Theorem.** If $x_0 = \sqrt{a}$ the proof is trivial, since all of the $x_n$ equal $\sqrt{a}$. Otherwise, we must show that $|x_n - \sqrt{a}| \to 0$. We note that for $i = 0, 1, \ldots$

$$\begin{aligned} |x_{i+1} - \sqrt{a}| &= \left|\left(x_i + \frac{a}{x_i}\right)/2 - \sqrt{a}\right| \\ &= \frac{|{x_i}^2 - 2x_i\sqrt{a} + a|}{2x_i} \\ &= \frac{(x_i - \sqrt{a})^2}{2x_i}; \end{aligned}$$

therefore

$$\left|\frac{x_{i+1} - \sqrt{a}}{x_i - \sqrt{a}}\right| = \frac{|x_i - \sqrt{a}|}{2x_i}.$$

From the Lemma we know that $x_i - \sqrt{a} > 0$ as long as $i > 0$, so

$$\begin{aligned}
\left|\frac{x_{i+1} - \sqrt{a}}{x_i - \sqrt{a}}\right| &= \frac{x_i - \sqrt{a}}{2x_i} \\
&< \frac{x_i}{2x_i} = 1/2.
\end{aligned}$$

Hence for any integer $m > 0$,

$$\begin{aligned}
\left|\frac{x_{i+m} - \sqrt{a}}{x_i - \sqrt{a}}\right| &= \prod_{j=1}^{m} \left|\frac{x_{i+j} - \sqrt{a}}{x_{i+j-1} - \sqrt{a}}\right| \\
&< \prod_{j=1}^{m} \left(\frac{1}{2}\right) \\
&= \left(\frac{1}{2}\right)^m.
\end{aligned}$$

Setting $i = 1$ gives

$$|x_{1+m} - \sqrt{a}| < \frac{|x_1 - \sqrt{a}|}{2^m};$$

Finally, let $n = m + 1$ (and $n > 1$); then

$$|x_n - \sqrt{a}| < \frac{|x_1 - \sqrt{a}|}{2^{n-1}}$$

and the theorem is proved.