LAST NAME:	FIRST NAME:	CIRCLE: Martynova Zweck

1 /10	2 /10	3 /10	4 /10	5 /10	
6 /10	7 /10	8 /10	9 /10	10 /10	T /100

MATH 2415 Final Exam, Spring 2017

No books or notes! **NO CALCULATORS! Show all work and give complete explanations**. This 2 hours 45 mins exam is worth 100 points.

(1) [10 pts] Let $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 4\mathbf{j}$, and $\mathbf{w} = 2\mathbf{i} + a\mathbf{j} + 3\mathbf{k}$, for some scalar, a.

(a) Find a unit vector parallel to $\mathbf{u}$.

$\|u\| = \sqrt{(1)^2+(-1)^2+(2)^2} = \sqrt{6}$

$\therefore$ Unit vector parallel to u is:

$$u/\|u\| = \frac{1}{\sqrt{6}}(i-j+2k) = \boxed{\frac{1}{\sqrt{6}}i - \frac{1}{\sqrt{6}}j + \frac{2}{\sqrt{6}}k}$$

(b) Find a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$

The vector perpendicular to u and v both is given by

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 1 & 4 & 0 \end{vmatrix} = (0-8)\,i - (0-2)\,j + (4+1)\,k$$

$$\boxed{= -8i + 2j + 5k}$$

(c) Find the value of a that makes $\mathbf{w}$ perpendicular to $\mathbf{u}$.

We use the fact that dot product of perpendicular vectors is 0.

$\therefore u \cdot \omega = (1, -1, 2) \cdot (2, a, 3) = 0$

$\Rightarrow (1)(2) + (-1)(a) + (2)(3) = 0$

$\Rightarrow 2 - a + 6 = 0$

$\Rightarrow \boxed{a = 8}$

(2) [10 pts] Find a parametrization of the line that contains the point $(1, 3, -2)$ and is parallel to both the plane $x + 2y + z = 4$ and the plane $2x - y + z = 1$.

The normals of two planes are: $n_1 = (1, 2, 1)$ and $n_2 = (2, -1, 1)$.
Thus, the direction of the line is given by,

$$
\begin{aligned}
l &= n_1 \times n_2 \\
&= \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{vmatrix} \\
&= (2+1)i - (1-2)j + (-1-4)k \\
&= 3i + j - 5k.
\end{aligned}
$$

And since the line passes through the point $(1, 3, -2)$, the parametrization of required line is given by,

$$
\boxed{\begin{aligned} x &= 3t + 1 \\ y &= t + 3 \\ z &= -5t - 2 \end{aligned}}
$$

(3) [10 pts] Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is the vector field $\mathbf{F}(x, y) = y\,\mathbf{i} - x\,\mathbf{j}$ and where C is the circle $(x-2)^2 + (y+5)^2 = 9$.

Let us first write parametric equations of the circle $(x-2)^2+(y+5)^2=9$:

$$x = 2+3\cos t$$
$$y = -5+3\sin t \quad , \quad t \in [0, 2\pi]$$

Therefore, $\vec{r}(t) = x i + y j = (2+3\cos t) i + (-5+3\sin t) j$

$\Rightarrow \vec{r}'(t) = -3\sin t \; i + 3\cos t \; j$

And $F(\vec{r}(t)) = (-5+3\sin t) i - (2+3\cos t) j$

$$\therefore F(\vec{r}(t)) \cdot \vec{r}'(t) = [(-5+3\sin t) i - (2+3\cos t) j][-3\sin t \; i + 3\cos t \; j]$$
$$= (-5+3\sin t)(-3\sin t) - (2+3\cos t)(3\cos t)$$
$$= 15\sin t - 9\sin^2 t - 6\cos t - 9\cos^2 t$$
$$= 15\sin t - 6\cos t - 9(\sin^2 t + \cos^2 t)$$
$$= 15\sin t - 6\cos t - 9$$

$$\therefore \int_C F \cdot dr = \int_0^{2\pi} F(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$
$$= \int_0^{2\pi} [15\sin t - 6\cos t - 9]\,dt$$
$$= 15[-\cos t]_0^{2\pi} - 6[\sin t]_0^{2\pi} - 9[t]_0^{2\pi}$$
$$= 15\{-1+1\} - 6\{0\} - 9\{2\pi - 0\}$$
$$\boxed{= -18\pi}$$

(4) [10 pts] Determine whether or not the vector field $\mathbf{F}(x, y) = (y^2 - 2xy)\mathbf{i} + (2xy - x^2)\mathbf{j}$ is conservative. If it is conservative, find a potential function for $\mathbf{F}$.

Here, $P = y^2 - 2xy$ and $Q = 2xy - x^2$

$$\Rightarrow \frac{\partial P}{\partial y} = 2y - 2x \qquad \frac{\partial Q}{\partial x} = 2y - 2x$$

Since, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, the vector field $F(x,y)$ is conservative.

$F(x,y) = (y^2 - 2xy)i + (2xy - x^2)j \Rightarrow \dfrac{\partial f}{\partial x} = y^2 - 2xy$ and $\dfrac{\partial f}{\partial y} = 2xy - x^2$

Thus, we have: $\dfrac{\partial f}{\partial x} = y^2 - 2xy$

$$\Rightarrow f = \int (y^2 - 2xy)\,dx$$

$$\boxed{f = xy^2 - x^2y + h(y)} \quad \text{—— (I)}$$

$\cancel{\dfrac{\partial f}{\partial y} = 2xy - x^2}$ $\qquad \Rightarrow \dfrac{\partial f}{\partial y} = 2xy - x^2 + h'(y)$

$\cancel{\Rightarrow f = \int (2xy - x^2)\,dy}$ $\qquad$ But, $\dfrac{\partial f}{\partial y} = 2xy - x^2$

$\cancel{= xy^2 - x^2y + }$ $\qquad \Rightarrow h'(y) = 0$

$\Rightarrow h(y) = c$, constant.

$\therefore$ (I) $\Rightarrow \boxed{f(x,y) = xy^2 - x^2y + c}$ $\leftarrow$ Potential Function for F

(5) [10 pts] Find the absolute maximum and minimum values of the function $f(x, y) = xy - x$ on the triangle with vertices $(-1, 0)$, $(-1, 3)$, and $(2, 0)$.

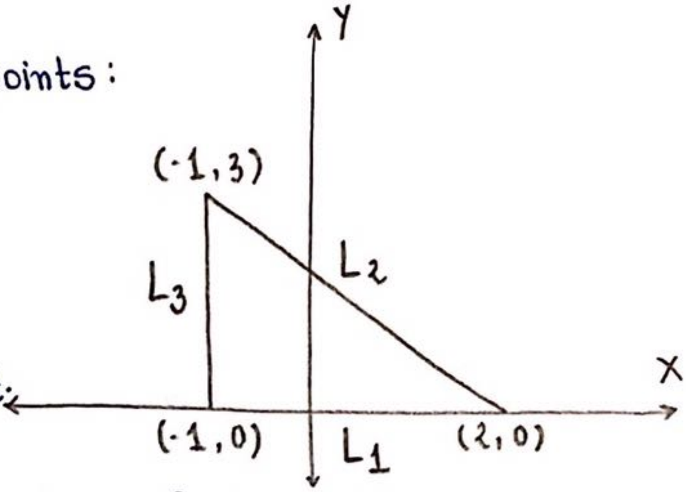
Let us first find the critical points:

$$\frac{\partial f}{\partial x} = y - 1 = 0 \Rightarrow \boxed{y = 1}$$

$$\frac{\partial f}{\partial y} = x = 0 \Rightarrow \boxed{x = 0}$$

Thus, $(0, 1)$ is the only critical point.

~~Now~~ $\boxed{f(0,1) = 0}$



Now, we will find the extreme values of the function on boundary of the triangle.

(I) On line L_1:

$f(x,y) = f(x,0) = -x, \quad -1 \leq x \leq 2$

Thus, the extreme values of the function lies on the end points of the line L_1.

$\therefore \boxed{f(-1,0) = 1}, \quad \boxed{f(2,0) = -2}$

(II) On line L_2:

$L_2 : x + y = 2$

$\therefore f(x,y) = f(x, 2-x) = x(2-x) - x = x - x^2$

$\therefore \frac{df}{dx} = 0 \Rightarrow 1 - 2x = 0 \Rightarrow \boxed{x = \frac{1}{2}} \leftarrow$ critical point.

$\Rightarrow y = 2 - \frac{1}{2} = \frac{3}{2}$

$\therefore f\left(\frac{1}{2}, \frac{3}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) - \frac{1}{2} \boxed{= \frac{1}{4}}$

Also, checking function value at the end points:

$\boxed{f(2,0) = -2}$ and $\boxed{f(-1,3) = -2}$

(III) On line L_3:

$L_3 : x = -1$

$\therefore f(x, y) = f(-1, y) = -y + 1$, $0 \leq y \leq 3$

$\therefore$ Extreme values lie at the end points.

$\boxed{f(-1, 0) = 1}$

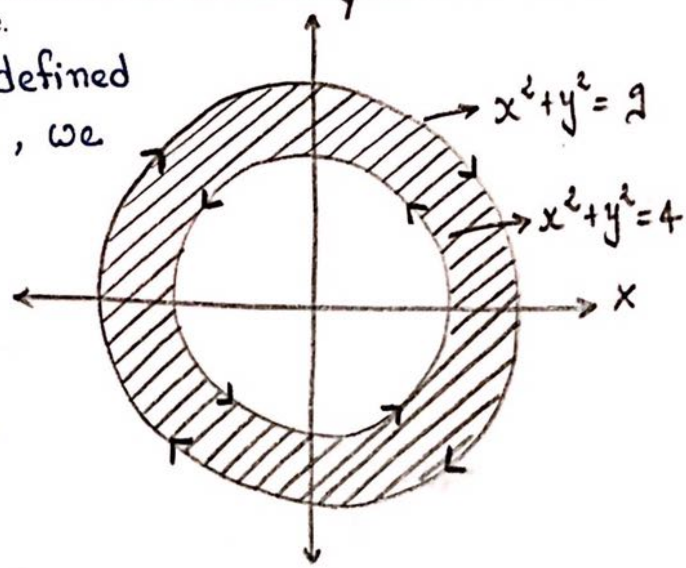
$\boxed{f(-1, 3) = -2}$

→ Combining all these results, we get

$\boxed{\max f(x, y) = 1}$ and $\boxed{\min f(x, y) = -2}$

(6) [10 pts] Use Green's Theorem to evaluate the line integral $\int_C \left(-\frac{1}{3}y^3 + x\right) dx + \left(\frac{1}{3}x^3 - y\right) dy$ where C is the boundary of the annulus, $4 \leq x^2 + y^2 \leq 9$. You should orient C so that the inner circle is traversed counterclockwise and the outer circle is traversed clockwise.

Since, orientation of the annulus defined here is in the negative direction, we have; using Green's Theorem:



$$\int_C \left(-\frac{1}{3}y^3+x\right)dx + \left(\frac{1}{3}x^3-y\right)dy$$

$$= -\iint_D \left[\frac{\partial}{\partial x}\left(\frac{1}{3}x^3-y\right) - \frac{\partial}{\partial y}\left(-\frac{1}{3}y^3+x\right)\right]dA$$

$$= -\iint_D [x^2+y^2]\,dA$$

Converting the integral into polar coordinates, we get:

$$2 \leq r \leq 3$$
$$0 \leq \theta \leq 2\pi$$

$$\therefore \int_C \left(-\frac{1}{3}y^3+x\right)dx + \left(\frac{1}{3}x^3-y\right)dy = -\int_0^{2\pi}\int_2^3 r^2 \cdot r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\left[\frac{r^4}{4}\right]_2^3 dr\,d\theta$$

$$= -\frac{1}{4}\int_0^{2\pi}\{81-16\}\,d\theta$$

$$= -\frac{65}{4}[\theta]_0^{2\pi}$$

$$= \frac{-65(2\pi)}{4}$$

$$\boxed{= -\frac{65\pi}{2}}$$

(7) [10 pts] Use cylindrical coordinates to find the volume of the solid that lies both within the cylinder $x^2 + y^2 = 3$ and the sphere $x^2 + y^2 + z^2 = 4$.

Volume = $\iiint_W 1\,dV$, where W is the described region.

In rectangular coordinates, we have

$$x^2+y^2 \leq 3$$

and

$$-\sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2}$$

Converting it to cylindrical coordinates, we get

$$0 \leq r \leq \sqrt{3}$$

$$0 \leq \theta \leq 2\pi$$

$$-\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}$$

$$\therefore \text{Volume} = \int_0^{\sqrt{3}} \left(\int_0^{2\pi} \left(\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1\,dz \right) d\theta \right) r\,dr$$

$$= \int_0^{\sqrt{3}} \int_0^{2\pi} [z]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} \, d\theta \, r\,dr$$

$$= \int_0^{\sqrt{3}} \int_0^{2\pi} 2r\sqrt{4-r^2}\, d\theta\, dr$$

$$= \int_0^{\sqrt{3}} 2r\sqrt{4-r^2}\,[\theta]_0^{2\pi}\, dr$$

$$= 4\pi \int_0^{\sqrt{3}} r\sqrt{4-r^2}\, dr$$

Let $u = \sqrt{4-r^2} \Rightarrow u^2 = 4-r^2$

$\Rightarrow 2u\,du = -2r\,dr$

$\Rightarrow -u\,du = r\,dr$

$$\therefore \text{Volume} = 4\pi \int_2^1 -u^2\,du$$

$$= 4\pi \int_1^2 u^2\,du$$

$$= 4\pi \left[\frac{u^3}{3}\right]_1^2$$

$$= \frac{4\pi}{3}\{8-1\}$$

$$\boxed{= \frac{28\pi}{3}}$$

(8) [10 pts]
Let S be the surface with parametrization

$$(x, y, z) = \mathbf{r}(u, v) = \cos u \sin v\,\mathbf{i} + \sin u \sin v\,\mathbf{j} + 4\cos v\,\mathbf{k} \qquad \text{for } 0 \le u \le 2\pi \text{ and } 0 \le v \le \frac{3\pi}{4}.$$

(a) Find an equation of the form $F(x, y, z) = 0$ for this surface.

Here, $x = \cos u \sin v$, $y = \sin u \sin v$, $z = 4\cos v$

$\therefore \cancel{4(x^2+y^2)}$ $\therefore 16(x^2+y^2) + z^2 = 16\left[\cos^2 u \sin^2 v + \sin^2 u \sin^2 v\right] + 16\cos^2 v$

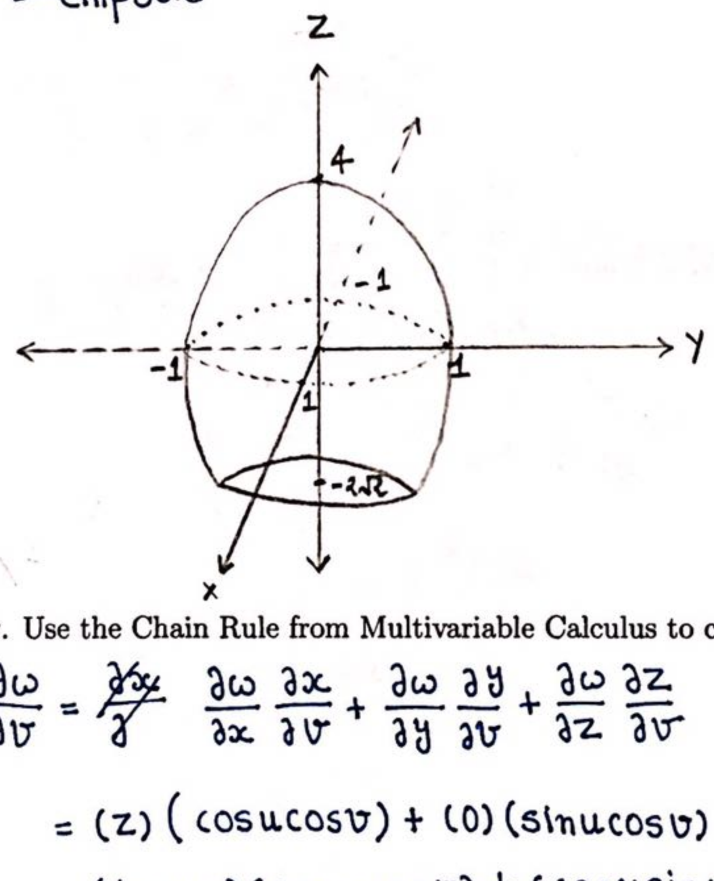
$= 16\sin^2 v + 16\cos^2 v$

$= 16$

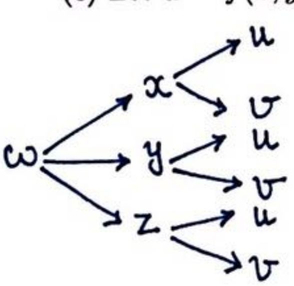
$\Rightarrow \boxed{x^2 + y^2 + \dfrac{z^2}{16} = 1}$, where $-1 \le x \le 1$, $-1 \le y \le 1$ and $-2\sqrt{2} \le z \le 4$

ellipsoid

(b) Sketch the surface, S.



(c) Let $w = f(x, y, z) = xz$. Use the Chain Rule from Multivariable Calculus to calculate $\frac{\partial w}{\partial v}$.



$$\frac{\partial \omega}{\partial v} = \cancel{\frac{\partial x}{\partial}} \; \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial \omega}{\partial z}\frac{\partial z}{\partial v}$$

$= (z)(\cos u \cos v) + (0)(\sin u \cos v) + (x)(-4\sin v)$

$= (4\cos v)(\cos u \cos v) + (\cos u \sin v)(-4\sin v)$

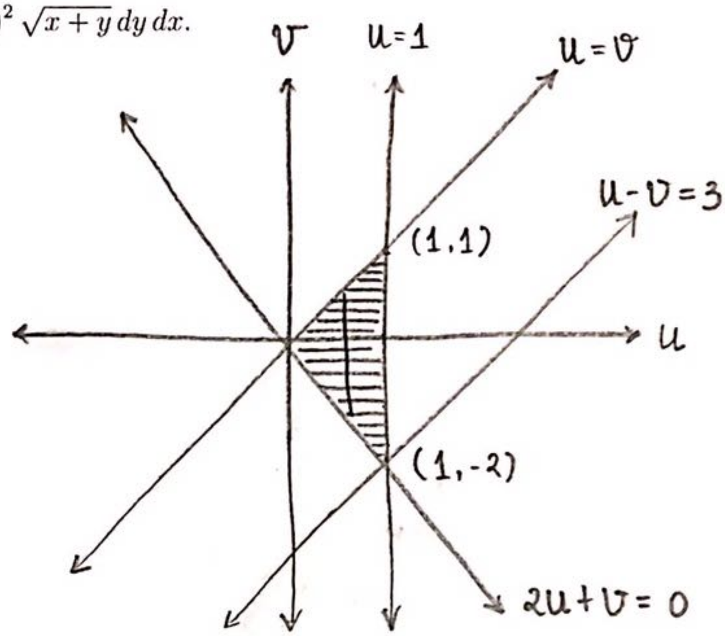
$= 4\cos u \cos^2 v - 4\cos u \sin^2 v$

$= 4\cos u(\cos^2 v - \sin^2 v)$

$\boxed{= 4\cos u \cdot \cos(2v)}$

(9) [10 pts] Use the change of variables $u = x + y$, $v = y - 2x$ to evaluate

$$\int_0^1 \int_0^{1-x} (y-2x)^2 \sqrt{x+y}\, dy\, dx.$$



$u = x+y$, $v = y - 2x$

$\Rightarrow x = \dfrac{u-v}{3}$, $y = \dfrac{2u+v}{3}$

$\therefore y = 0 \Rightarrow \boxed{2u+v=0}$

$y = 1-x \Rightarrow \boxed{u=1}$

$x = 0 \Rightarrow \boxed{u=v}$

$x = 1 \Rightarrow \boxed{u-v=3}$

$f(x,y) = (y-2x)^2\sqrt{x+y}$

$\Rightarrow f(u,v) = \left(\dfrac{2u+v}{3} - \dfrac{2(u-v)}{3}\right)^2 \sqrt{\dfrac{u-v}{3} + \dfrac{2u+v}{3}} = v^2\sqrt{u}$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{vmatrix} = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$$

$$\therefore \int_0^1 \int_0^{1-x} (y-2x)^2\sqrt{x+y}\, dy\, dx = \int_0^1 \int_{-2u}^{u} v^2\sqrt{u} \cdot |J|\, dv\, du$$

$$= \int_0^1 \int_{-2u}^{u} \frac{4}{9} v^2 \sqrt{u}\, dv\, du$$

$$= \frac{4}{9} \int_0^1 \sqrt{u} \left[\frac{v^3}{3}\right]_{-2u}^{u} du$$

$$= \frac{4}{27} \int_0^1 \sqrt{u}\,(9u^3)\, du$$

$$\therefore \int_0^1 \int_0^{1-x} (y-2x)^2 \sqrt{x+y}\, dy\, dx = \frac{4}{3} \int_0^1 u^{7/2}\, du$$

$$= \frac{4}{3} \left[\frac{u^{9/2}}{9/2} \right]_0^1$$

$$= \frac{4}{3} \times \frac{2}{9}$$

$$\boxed{= \frac{8}{27}}$$

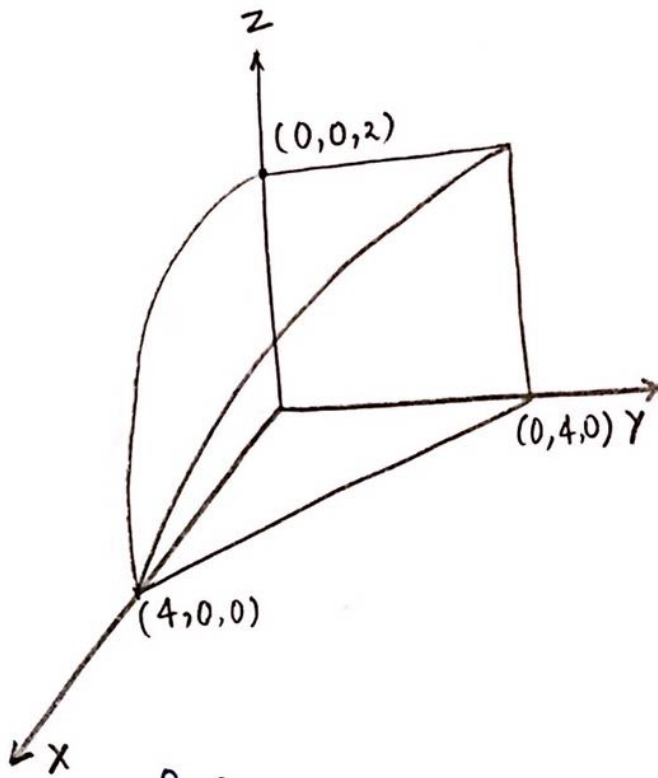
(10) [10 pts] Calculate the volume of the the solid region in the first octant that is bounded by the surfaces $x + y = 4$ and $x = 4 - z^2$.

Here, we have

$$0 \leq z \leq \sqrt{4-x}$$

$$0 \leq y \leq 4-x$$

$$0 \leq x \leq 4$$



$$\text{Volume} = \int_0^4 \int_0^{4-x} \int_0^{\sqrt{4-x}} dz\,dy\,dx$$

$$= \int_0^4 \int_0^{4-x} [z]_0^{\sqrt{4-x}}\, dy\,dx$$

$$= \int_0^4 \int_0^{4-x} \sqrt{4-x}\, dy\,dx$$

$$= \int_0^4 \sqrt{4-x}\,[y]_0^{4-x}\, dx$$

$$= \int_0^4 \sqrt{4-x}\,(4-x)\, dx$$

$$= \int_0^4 (4-x)^{3/2}\, dx$$

Let $4-x = u \Rightarrow dx = -du$

$x = 0 \Rightarrow u = 4$

$x = 4 \Rightarrow u = 0$

$$\text{Volume} = -\int_4^0 u^{3/2}\, du$$

$$= \int_0^4 u^{3/2}\, du$$

$$= \left[\frac{u^{5/2}}{5/2}\right]_0^4$$

$$= \frac{2}{5}\left\{(4)^{5/2} - 0\right\}$$

$$= \frac{2}{5}(2)^5$$

$$\boxed{= \frac{64}{5}}$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________