LAST NAME:	FIRST NAME:	CIRCLE:		
SOLUTIONS		Zweck 10:00am	Khafizov 11:30am	Khafizov 2:30pm

1 /10	2 /10	3 /10	4 /10	5 /10	
6 /10	7 /10	8 /10	9 /10	10 /10	T /100

MATH 2415 Final Exam, Spring 2016

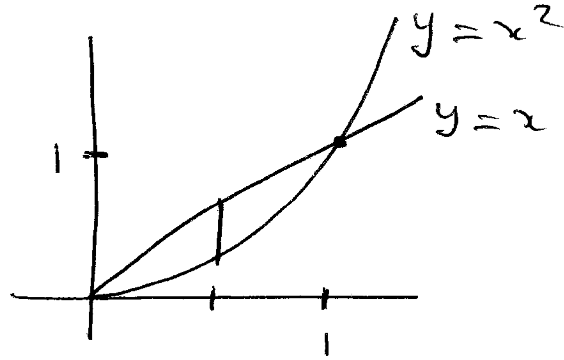
No books or notes! **NO CALCULATORS! Show all work and give complete explanations**. This 2 hours 45 mins exam is worth 100 points.

(1) [10 pts] Evaluate the double integral $\iint_D 2\frac{y}{x}\,dA$, where D is bounded by $y = x$ and $y = x^2$. Write your final answer in the box, and explain the reasons for your answer in the space below.

Final Answer:

D is $0 \le x \le 1$

$x^2 \le y \le x$



$$\iint_D \frac{2y}{x}\,dA = \int_{x=0}^{x=1} \frac{1}{x} \int_{y=x^2}^{y=x} 2y\,dy\,dx$$

$$= \int_{x=0}^{x=1} \frac{1}{x} \left[y^2\right]_{y=x^2}^{y=x} dx$$

$$= \int_{x=0}^{x=1} \frac{x^2 - x^4}{x}\,dx = \int_0^1 x - x^3\,dx$$

$$\cancel{= \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}}$$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

(2) [10 pts] (a) Use the Chain Rule to find $\partial z/\partial t$ if $z = e^{xy}$, $x = \sin t$, $y = t^2$. Write your final answer in the box, and explain the reasons for your answer in the space below.

Final Answer:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

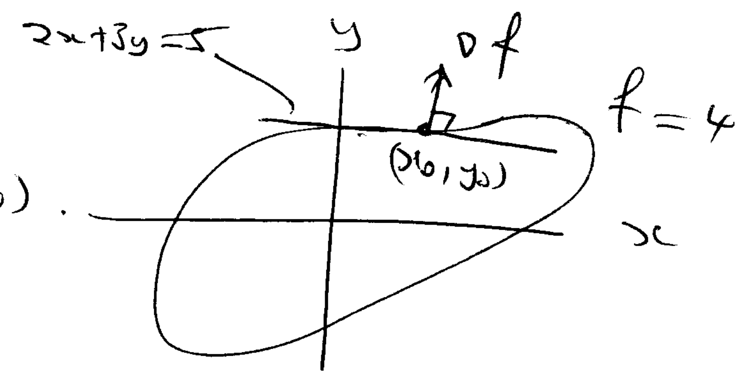
$$= y(t)e^{x(t)y(t)}\cos t + x(t)e^{x(t)y(t)}\,2t$$

$$= t^2 e^{t^2\sin t}\cos t + \sin t\, e^{t^2\sin t}\,2t$$

$$= [t\cos t + 2\sin t]\,t e^{t^2\sin t}$$

(b) Suppose that $2x + 3y = 5$ is the tangent line to a curve $f(x, y) = 4$ at the point $(x_0, y_0) = (1, 1)$. Find the unit vector in the direction of ∇f at the point (x_0, y_0).
Write your final answer in the box, and explain the reasons for your answer in the space below.

Final Answer: OR IN THE OPPOSITE DIRECTION OF



Let $\vec{v}$ be tangent vector to level curve at (x_0, y_0).

We have

$$\vec{v} = (x_1 - x_0,\ y_1 - y_0)$$

where $(x_0, y_0) = (1, 1)$ and $(x_1, y_1) = (2, \frac{1}{3})$ are two points on line $2x + 3y = 5$

So $\vec{v} = (1, 1) - (2, \frac{1}{3}) = (-1, \frac{2}{3})$

Can scale $\vec{v}$ to $\vec{w} = -3\vec{v} = (3, -2)$.

$\vec{v} \perp \nabla f$. So $\dfrac{\nabla f}{|\nabla f|} = \dfrac{(2, -3)}{\sqrt{4+9}}$ or $\dfrac{\nabla f}{|\nabla f|} = \dfrac{-(2, -3)}{\sqrt{4+9}}$

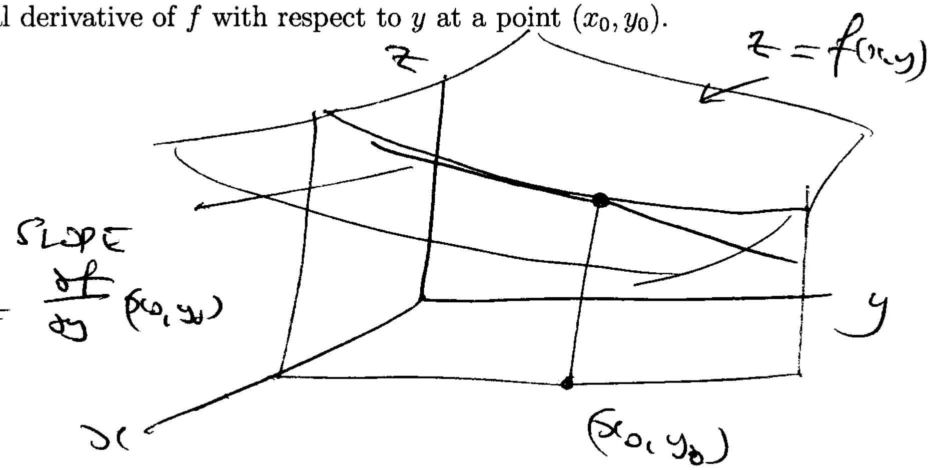
(3) [10 pts]

(a) Let $z = f(x, y)$ be a function of two variables. State the limit definition of the partial derivative of f with respect to y at a point (x_0, y_0).

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

(b) Let $z = f(x, y)$ be a function of two variables. Write a sentence and draw a picture that explains the geometrical meaning of the partial derivative of f with respect to y at a point (x_0, y_0).

$\frac{\partial f}{\partial y}(x_0, y_0)$ IS THE SLOPE AT $y = y_0$ OF SLICE OF SURFACE $z = f(x, y)$ IN PLANE $x = x_0$



(c) Now let $z = f(x, y) = e^{4y}\sin(x^2 + y^2)$. Calculate the partial derivative of f with respect to y at a point $(x_0, y_0) = (-4, 3)$.

$$\frac{\partial f}{\partial y} = 4e^{4y}\sin(x^2+y^2) + e^{4y}\,2y\cos(x^2+y^2)$$

$$\frac{\partial f}{\partial y}(-4, 3) = 4e^{12}\sin(25) + 6e^{12}\cos(25)$$

(4) [10 pts] Find (a) the local maximum values, (b) the local minimum values and (c) saddle point(s) of $f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$, if they exist. Write your final answer in the box, and explain the reasons for your answer in the space below.

Final Answer:

$$\frac{\partial f}{\partial x} = y + - x^{-2} = 0 \quad \text{at} \quad y = \frac{1}{x^2} \quad (1)$$

$$\frac{\partial f}{\partial y} = x - y^{-2} = 0 \quad \text{at} \quad x = \frac{1}{y^2} \quad (2)$$

Plug (2) into (1) to get $y = \frac{1}{\left(\frac{1}{y^2}\right)^2} = y^4$

OR $y(1-y^3) = 0.$

Get $y=0$ or $y=1$

$\boxed{y=0}$ f not defined. So no CPT.

$\boxed{y=1}$ Get $x = 1$ by (2)

CPT at $(1, 1)$

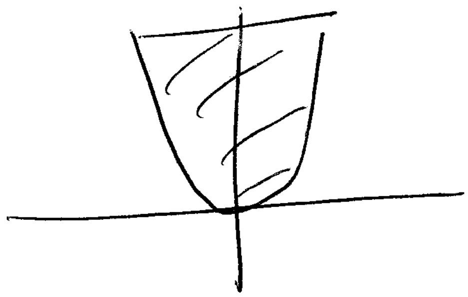
$$D = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \det \begin{bmatrix} 2x^{-3} & 1 \\ 1 & 2y^{-3} \end{bmatrix} = \det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

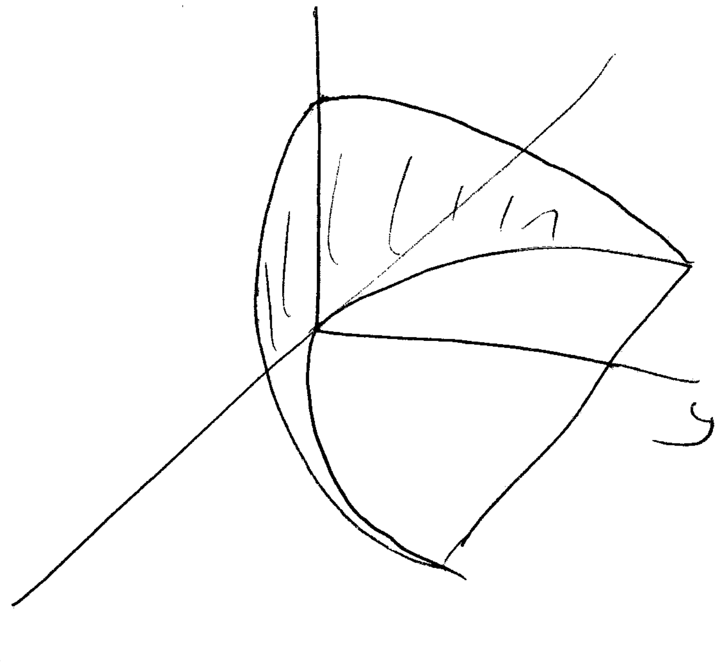
$= 4-1 = 3 > 0$

at $(1,1)$

AND $f_{xx} = 2x^{-3} = 2 > 0$ at $(x,y) = (1,1)$

So LOCAL MIN.





$$\boxed{\begin{array}{l} -1 \le x \le 1 \\ x^2 \le y \le 1 \\ 0 \le z \le 1-y \end{array}}$$

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$$V = \int_{x=-1}^{x=1} \int_{y=x^2}^{y=1} \int_{z=0}^{z=1-y} 1 \, dz \, dy \, dx \quad \boxed{2}$$

$$= \int_{x=-1}^{1} \int_{y=x^2}^{1} (1-y) \, dy \, dx$$

$$\boxed{\begin{array}{l} 0 \le y \le 1 \\ -\sqrt{y} \le x \le \sqrt{y} \\ 0 \le z \le 1-y \end{array}}$$

$$= \int_{x=-1}^{1} \cancel{\int_{y=x^2}^{y=1}} \left[y - \frac{y^2}{2} \right]_{y=x^2}^{y=1} dx$$

$$= 2\int_{x=0}^{1} \left(\frac{1}{2} - x^2 + \frac{x^4}{2} \right) dx \quad \boxed{2}$$

$$\frac{15-10+3}{15} = \frac{8}{15}$$

$$= 2\left[\frac{x}{2} - \frac{x^3}{3} + \frac{x^5}{10} \right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3} + \frac{1}{10} \right]$$

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(6) [10 pts] Let D be the domain in the plane given by $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 \leq 1$. Use the Change of Variables Theorem to calculate $\iint_D 16x^2 + 9y^2\, dxdy$. Hint: Use the change of variables $u = \frac{x}{3}$, $v = \frac{y}{4}$. Write your final answer in the box, and explain the reasons for your answer in the space below.

Final Answer:

$$\iint_D f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$x = 3u$
$y = 4v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$= \det\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = 12$$

If ~~D~~ D is $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 \leq 1$

The S is $u^2 + v^2 \leq 1$.

So

$$\iint_D 16x^2 + 9y^2\,dx\,dy = \iint_S [16(3u)^2 + 9\cdot(4v)^2]\,|12|\,du\,dv$$

$$= 16\times 9\times 12 \iint_S (u^2+v^2)\,du\,dv$$

$$= 16.9.12. \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^2\, r\, d\theta\, dr$$

864π
$\parallel$

$$= 16\times 9\times 12\times 2\pi \quad [r^4/4]_0^1 = 16\times 9\times 12\times 2\pi\cdot\frac{1}{4}.$$

(7) [10 pts] Let S be the surface parametrized by $(x, y, z) = \mathbf{r}(u, v) = (u \cos v, u \sin v, u)$. Find a parametrization for the tangent plane to S at the point where $(u, v) = (2, \frac{\pi}{3})$.

Par of Plane is

$$L(s,t) = \vec{p} + s\vec{v} + t\vec{w}$$

where $\vec{p}$ is point in plane and $\vec{v}, \vec{w}$ are vectors lying in plane

For Tangent Plane we can choose

$$\vec{p} = \vec{r}(2, \pi/3) = (2\cos \pi/3,\ 2\sin \pi/3,\ 2)$$

$$\vec{v} = \frac{\partial \vec{r}}{\partial u}(2, \pi/3) = (\cos v,\ \sin v,\ 1)\Big|_{(2,\pi/3)}$$

$$= (\cos \pi/3,\ \sin \pi/3,\ 1)$$

$$\vec{w} = \frac{\partial \vec{r}}{\partial v}(2, \pi/3) = (-u\sin v,\ u\cos v,\ 0)\Big|_{(2,\pi/3)}$$

$$= (-2\sin \pi/3,\ 2\cos \pi/3,\ 0)$$

So

$$L(s,t) = (2\cos \pi/3 + s\cos \pi/3 - 2t\sin \pi/3,$$

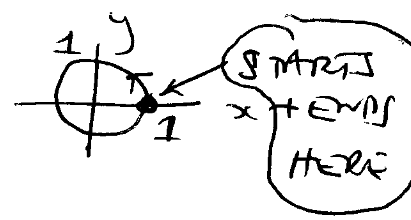
$$2\sin \pi/3 + s\sin \pi/3 + 2t\cos \pi/3,$$

$$2 + s)$$

(8) [10 pts] Make sketches of parametrized curves below. Be sure to label the axes and indicate which sketch goes with which curve.

$$\begin{array}{llllll} C_1 & (x,y,z) & = \mathbf{r}_1(t) & = (\cos t, \sin t, t) & \text{for } 0 \le t \le 2\pi \\ C_2 & (x,y,z) & = \mathbf{r}_2(t) & = (\sin t, \cos t, t) & \text{for } 0 \le t \le 2\pi \end{array}$$

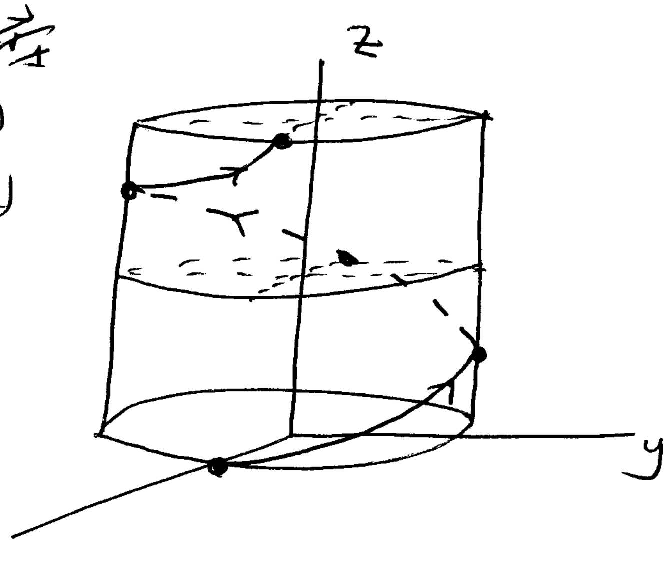
$\boxed{C_1}$ Curve lies on cylinder $x^2+y^2=1$ and z increases linearly with t. Shadow of C_1 onto xy-plane is



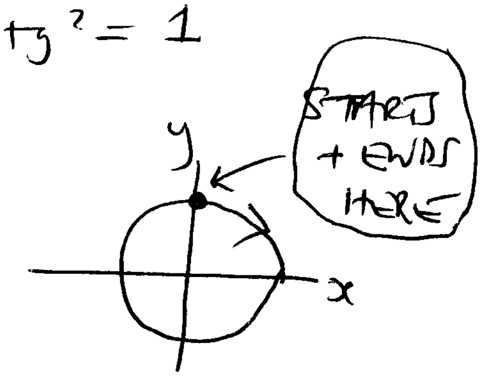
$\vec{r}_1(0) = (1,0,0) \cancel{= \vec{r}_1}$

$\vec{r}_1(2\pi) = (1,0,2\pi)$

$\vec{r}_1(\pi/2) = (0,1,\pi/2)$



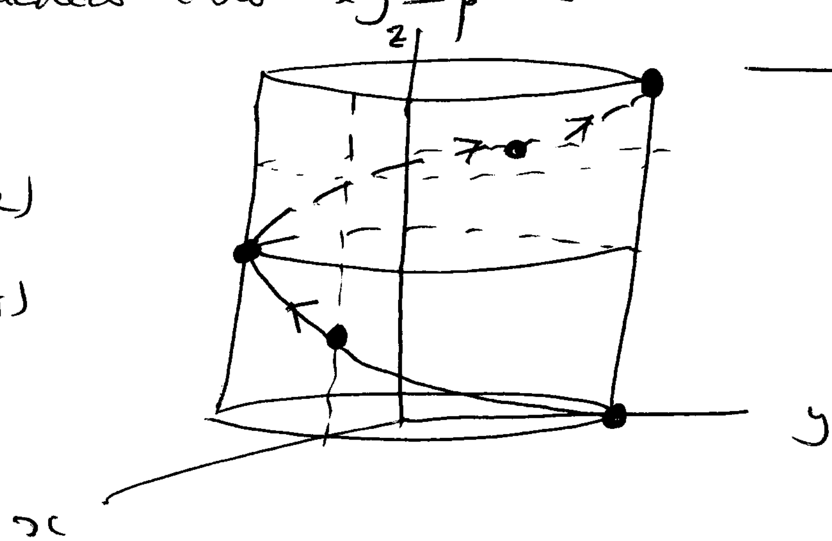
$\boxed{C_2}$ Curve also lies on cylinder $x^2+y^2=1$ and z increases linearly with t. But shadow onto xy-plane is



$\vec{r}_2(0) = (0,1,0)$

$\vec{r}_2(\pi/2) = (1,0,\pi/2)$

$\vec{r}_2(2\pi) = (0,1,2\pi)$



(9) [10 pts]
(a) Let **F** be the vector field given by $\mathbf{F}(x,y,z) = z^2y\,\mathbf{i} + (x^2 - z^2)\,\mathbf{j} + xz\,\mathbf{k}$. Calculate the curl of **F**.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2y & x^2-z^2 & xz \end{vmatrix} = (0 - 2z)\vec{i} - (z - 2zy)\vec{j} + (2x - z^2)\vec{k}$$

$$= (-2z,\ 2zy - z,\ 2x - z^2)$$

(b) Let **F** be the vector field given by $\mathbf{F}(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}$.
Show that $\operatorname{div}\mathbf{F} = 0$ everywhere it is defined.

$$= P\vec{i} + Q\vec{j} + R\vec{k}$$

$$\operatorname{Div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

NOW

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-3/2}\right]$$

$$= (x^2+y^2+z^2)^{-3/2} + x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}\cdot 2x$$

$$= (x^2+y^2+z^2)^{-5/2}\left[x^2+y^2+z^2 - 3x^2\right]$$

So

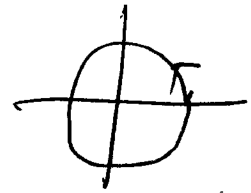
$$\operatorname{Div}(\vec{F}) = (x^2+y^2+z^2)^{-5/2}\Big[\cancel{1-3x^2}\ (x^2+y^2+z^2) - 3x^2 + (x^2+y^2+z^2) - 3y^2 + (x^2+y^2+z^2) - 3z^2\Big]$$

$$= 0.$$

(10) [10 pts] This problem is about the vector field $\mathbf{F}(x,y) = -y\,\mathbf{i} + x\,\mathbf{j}$.

(a) Let C be the unit circle oriented counter clockwise. Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$$\vec{r}(t) = (\cos t, \sin t), \quad 0 \le t \le 2\pi$$



$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)\,dt$$

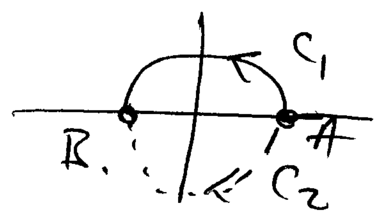
$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} 1\,dt$$

$$= 2\pi$$

(b) Use your answer to (a) to determine whether or not the vector field $\mathbf{F}$ is conservative.

~~Let C_1 = Curve Above~~

~~Let C_2 = Curve that just a~~



$$\int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} = 2\pi$$

So $\int_{C_1} \vec{F}\cdot d\vec{r} \neq \int_{C_2} \vec{F}\cdot d\vec{r}$. Integral depends on path taken from A to B

So $\vec{F}$ NOT CONSERVATIVE

(c) Use a different approach from (b) to determine whether or not $\mathbf{F}$ is conservative.

$$\frac{\partial Q}{\partial x} = 1, \quad \frac{\partial P}{\partial y} = -1 \quad \text{So} \quad \frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$$

So $\vec{F}$ NOT CONSERVATIVE.

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________