| LAST NAME: | FIRST NAME: | CIRCLE: | | | |
|---|---|---|---|---|---|
| SOLUTIONS | | Makhijani 8:30am | Makhijani 11:30am | Makhijani 2:30pm | Zweck 11:30am |

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| 1 /10 | 2 /10 | 3 /10 | 4 /10 | 5 /10 | |
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# MATH 2415 Final Exam, Spring 2019

No books or notes! **NO CALCULATORS! Show all work and give complete explanations**. This 2 hours 45 mins exam is worth 100 points.

(1) [10 pts] (a) If $z = f(x, y)$, with $x = e^t$, $y = t^2 + 3t + 2$, $\nabla f = (2xy^2 - y, 2x^2y - x)$ find $z'(t)$ at $t = 0$.

$\vec{r}(0) = (1, 2)$

$r'(t) = (e^t, 2t+3)$

$r'(0) = (1, 3)$

$\cancel{\nabla f(\vec{r}(t)) = \nabla f}$

$z'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$

$z'(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}'(0)$

$= \nabla f(1, 2) \cdot (1, 3)$

$= (2.1.4 - 2, 2.1^2.2 - 1) \cdot (1, 3)$

$= (6, 3) \cdot (1, 3) = \boxed{15}$

(b) Parametrize the surface $(y - 2)^2 + (z - 3)^2 = 4$.

$x = u$ $\quad u \in \mathbb{R}$. $\quad 0 \le v \le 2\pi$

$y = 2\cos v + 2$

$z = 2\sin v + 3$

(2) [10 pts] Let $\mathbf{F}_1(x,y) = (2y - x^2e^{-y})\mathbf{i} + 2xe^{-y}\mathbf{j}$ and $\mathbf{F}_2(x,y) = 2xe^{-y}\mathbf{i} + (2y - x^2e^{-y})\mathbf{j}$

(a) One of these vector fields is conservative. Which one is it and why?

$F_1$// $\quad P = 2y - x^2e^{-y} \qquad Q = 2xe^{-y}$

$\frac{\partial P}{\partial y} = 2 + x^2e^{-y} \qquad \frac{\partial Q}{\partial x} = 2e^{-y} \qquad \frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$ NOT CONS

$F_2$// $\quad P = 2xe^{-y} \qquad Q = 2y - x^2e^{-y}$

$\frac{\partial P}{\partial y} = -2xe^{-y} \qquad \frac{\partial Q}{\partial x} = -2xe^{-y} = \frac{\partial P}{\partial y}$ cons

(b) Find a potential function for the conservative vector field.

$F_2 = \nabla f$ So

$f = \int 2xe^{-y}\,dx = x^2e^{-y} + g(y)$

$f = \int (2y - x^2e^{-y})\,dy = y^2 + x^2e^{-y} + h(x)$

So $\boxed{f(x,y) = y^2 + x^2e^{-y} + C}$

(c) Evaluate $\int_C \mathbf{G} \cdot d\mathbf{r}$ where $C$ is the line segment from $(1,0)$ to $(2,1)$ and $\mathbf{G}$ denotes the conservative vector field you identified in (a).

$\int_C \vec{F}_2 \cdot d\vec{r} = f(2,1) - f(1,0)$ by FTC

$= (1^2 + 2^2e^{-1}) - (0^2 + 1^2e^{-0})$

$= 1 + \frac{4}{e} - 1 = \boxed{\frac{4}{e}}$

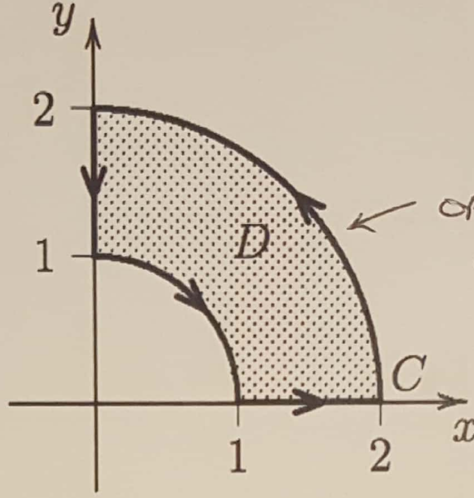
(3) [10 pts] Use Green's theorem to evaluate $\int_C xy^2\,dx - x^2y\,dy$ where $C$ is given in the figure.

In POLAR COORDS

D is

$1 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$




ORIENTATION IS CORRECT FOR Green's Thm.

$C = \partial D$

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_C xy^2\,dx - x^2y\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(xy^2)\right] dA$$

$$= \iint_D -2xy - 2xy\; dA$$

$$= -4\iint_D xy\,dA$$

$$= -4\int_{\theta=0}^{\pi/2}\int_{r=1}^{2} r^2\cos\theta\sin\theta\; r\,dr\,d\theta$$

$$= -4\left(\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\right)\left(\int_1^2 r^3\,dr\right)$$

$u = \sin\theta$, $du = \cos\theta\,d\theta$

$$= (-4)\left(\int_{u=0}^{1} u\,du\right) \qquad \left[\frac{r^4}{4}\right]_1^2$$

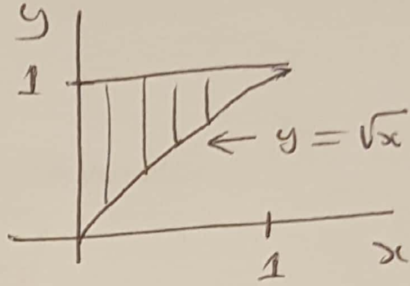
$$= -4\left[u^2/2\right]_0^1 \left(2^4/4 - \frac{1}{4}\right) = -4 \cdot \frac{1}{2} \cdot \frac{15}{4} = \frac{-15}{2}$$

(4) [10 pts] Evaluate $\displaystyle\int_{x=0}^{x=1}\int_{y=\sqrt{x}}^{y=1} \cos(y^3)\, dy dx.$

TYPE I

$0 \le x \le 1$

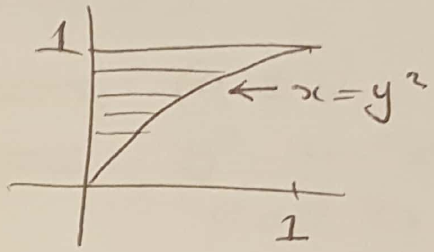
$\sqrt{x} \le y \le 1$




Switch to TYPE II

$0 \le y \le 1$

$0 \le x \le y^2$




So
$$\int_{x=0}^{1}\int_{y=\sqrt{x}}^{y=1} \cos(y^3)\, dy\, dx = \int_{y=0}^{y=1}\int_{x=0}^{x=y^2} \cos(y^3)\, dx\, dy$$

$$= \int_{y=0}^{y=1} \cos(y^3) \left(\int_{x=0}^{x=y^2} 1\, dx\right) dy$$

$$= \int_{y=0}^{y=1} \cos(y^3)\, y^2\, dy \qquad \begin{array}{l} u = y^3 \\ du = 3y^2 dy \end{array}$$

$$= \frac{1}{3}\int_{u=0}^{1} \cos(u)\, du$$

$$= \frac{1}{3}\left[\sin u\right]_0^1 = \frac{1}{3}\sin(1)$$

(5) ~~(3)~~ [10pts] Make a labelled sketch of the traces of the surface

$$y^2 - 4x^2 - z^2 = 1$$

in the planes $x = 0$, $z = 0$, and $y = k$ for $k = 0, \pm 1, \pm 2$. Then sketch the surface.

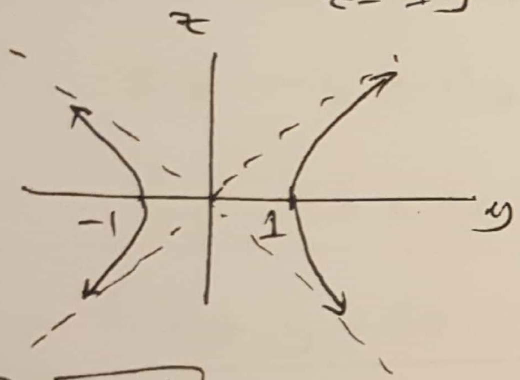
① $\boxed{x=0}$

$y^2 - z^2 = 1$

INTERCEPTS : $(\pm 1, 0)$

ASYMPTOTES : $y^2 - z^2 = 0$

$z = \pm y$



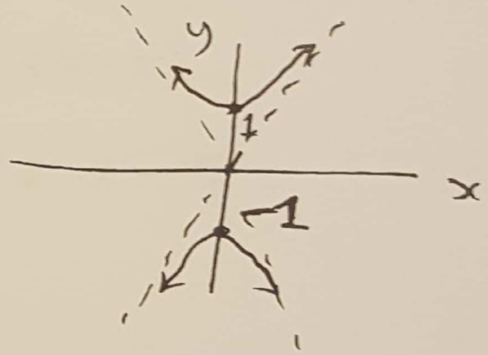

② $\boxed{z=0}$

$y^2 - 4x^2 = 1$

INTERCEPTS : $(0, \pm 1)$

ASYMPTOTES : $y^2 - 4x^2 = 0$

$y = \pm 2x$

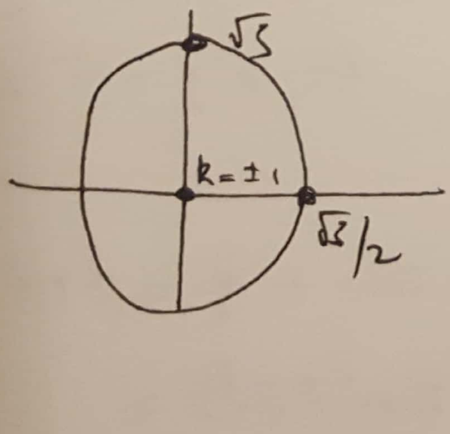


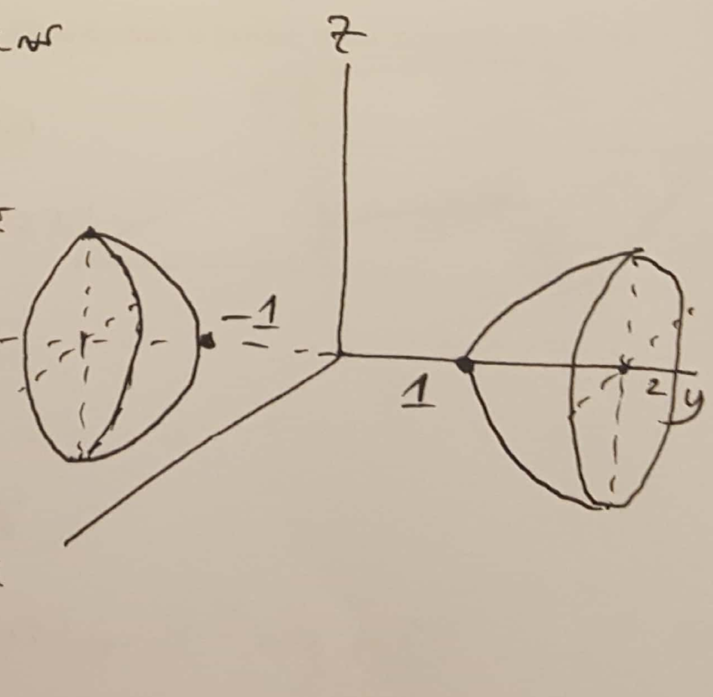

③ $\boxed{y=k}$

$k=0$: $+4x^2 + z^2 = -1$ NO SOLN

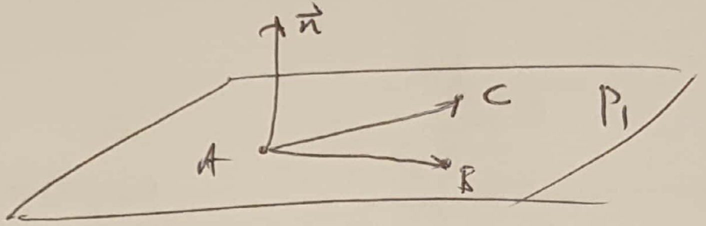
$k=\pm 1$: $4x^2 + z^2 = 0$ ORIGIN

$k=\pm 2$ $4x^2 + z^2 = 3$ ELLIPSE



(6) [10pts] Let $A = (1, 2, 5)$, $B = (2, 2, 7)$, and $C = (3, 5, 8)$ be three points in space.
(a) Let $P_1$ be the plane containing the points $A$, $B$, and $C$. Find an equation of the form $ax + by + cz = d$ for the plane $P_1$.




$$\vec{AB} = (1, 0, 2)$$
$$\vec{AC} = (2, 3, 3)$$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 2 & 3 & 3 \end{vmatrix} = -6\vec{i} + \vec{j} + 3\vec{k}$$

$\vec{P} = A$

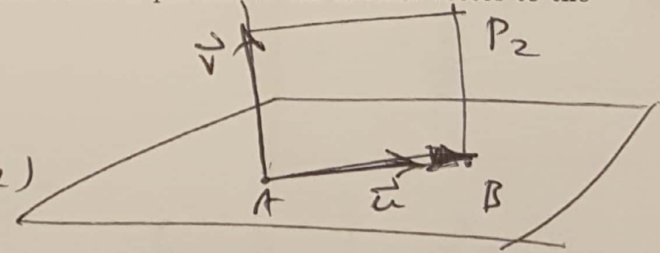
$$0 = (\vec{r} - \vec{P}) \cdot \vec{n} = (x-1, y-2, z-5) \cdot (-6, 1, 3)$$

$$0 = -6(x-1) + y - 2 + 3(z-5)$$

$$= -6x + y + 3z + 6 - 2 - 15$$

$$\boxed{-6x + y + 3z = 11}$$

(b) Let $P_2$ be the plane containing the line segment $AB$ and which is parallel to the normal vector to the plane $P_1$. Find a parametrization for the plane $P_2$.

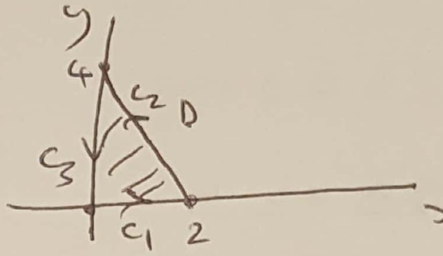



$P_2$ is plane thru $\vec{P} = A = (1, 2, 5)$
containing vectors $\vec{u} = \vec{AB} = (1, 0, 2)$
and $\vec{v} = \vec{n} = (-6, 1, 3)$

PAR

$$\vec{r}(s,t) = \vec{P} + s\vec{u} + t\vec{v}$$
$$= (1, 2, 5) + s(1, 0, 2) + t(-6, 1, 3)$$
$$= (1 + s - 6t, 2 + t, 5 + 2s + 3t)$$

(7) [10 pts] Let $D$ be the closed triangular domain with verticies $(0,0)$, $(2,0)$, and $(0,4)$. Find the absolute maximum and minimum of the function $f(x,y) = xy - x - y$ on $D$.




① CRITICAL PTS IN D

$\nabla f = (y-1, x-1) = (0,0)$

at $(x,y) = (1,1)$ which is in $D$.

$f(1,1) = 1-1-1 = -1$

② $\underline{C_1}$ $g(x) = f(x,0) = -x$ ON $[0,2]$

NO CPTS.

ENDPTS $g(0) = 0$, $g(2) = -2$

③ $\underline{C_3}$ $h(y) = f(0,y) = -y$

NO CPTS

ENDPTS $f(0,0) = 0$, $f(0,4) = -4$

④ $\underline{C_2}$ Eq$^n$ is $y = -2x + 4$

CPTS $k(x) = f(x, -2x+4)$

$= x(-2x+4) - x - (-2x+4)$

$= -2x^2 + 4x - x + 2x - 4$

$k(x) = -2x^2 + 5x - 4$

$0 = k'(x) = -4x + 5 \Rightarrow x = \frac{5}{4}$.

$y = -\frac{5}{\cancel{4}\,2} + 4 = \frac{3}{2}$

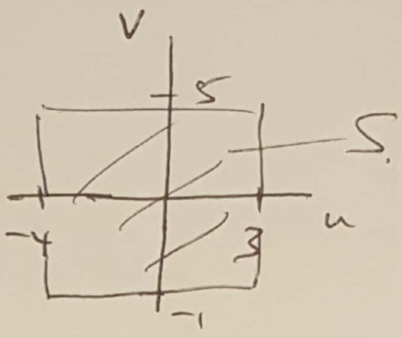
ENDPTS: [illegible]

$$\boxed{\begin{aligned} &f\left(\tfrac{5}{4}, \tfrac{3}{2}\right) \\ &= \tfrac{5}{4}\tfrac{3}{2} - \tfrac{5}{4} - \tfrac{3}{2} \\ &= \tfrac{15}{8} - \tfrac{10}{8} - \tfrac{12}{8} \\ &= -\tfrac{7}{8}. \end{aligned}}$$

| $(x,y)$ | $f(x,y)$ | |
|---|---|---|
| $(1,1)$ | $-1$ | |
| $(0,0)$ | $0$ | ABS MAX |
| $(2,0)$ | $-2$ | |
| $(0,4)$ | $-4$. | ABS MIN. |
| $\left(\frac{5}{4}, \frac{3}{2}\right)$ | $-7/8$. | |

(8) [10 pts] Evaluate $\iint_R (x+2y)(y-3x)\,dA$ where $R$ is the parallelogram enclosed by the lines $x+2y=-4$, $x+2y=3$, $y-3x=-1$, $y-3x=5$.

Let $\boxed{\begin{aligned} u &= x+2y \\ v &= y-3x \end{aligned}}$




$-4 \le u \le 3$

$-1 \le v \le 5.$

$3u+v = 3x+6y+y-3x = 7y$

$y = \frac{3}{7}u + \frac{1}{7}v$

$x = u-2y = u - \frac{6}{7}u - \frac{2}{7}v = \frac{1}{7}u - \frac{2}{7}v$

$\boxed{\begin{aligned} x &= \tfrac{1}{7}u - \tfrac{2}{7}v \\ y &= \tfrac{3}{7}u + \tfrac{1}{7}v \end{aligned}}$

$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{vmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{vmatrix} = \frac{1}{49} + \frac{6}{49}$

$= \frac{1}{7}$

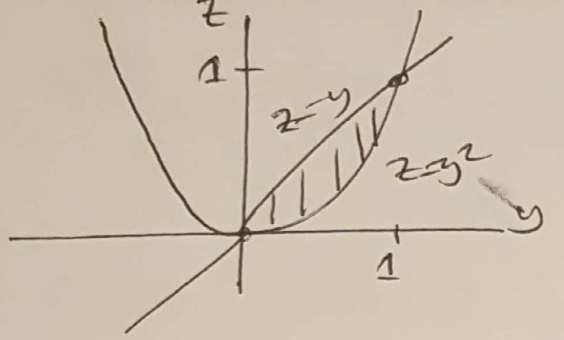
So

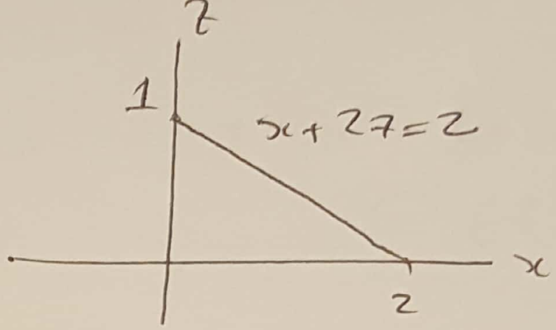
$$\iint_R (x+2y)(y-3x)\,dx\,dy = \iint_S uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

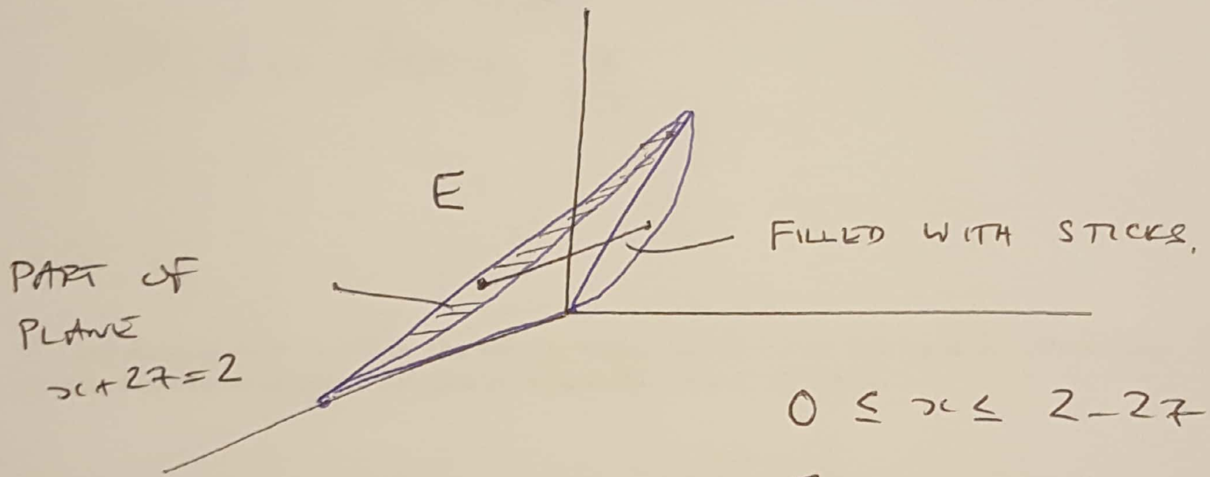
$$= \int_{u=-4}^{3} \int_{v=-1}^{5} uv.\ \frac{1}{7}\,du\,dv = \frac{1}{7}\left[\frac{u^2}{2}\right]_{-4}^{3} \left[\frac{v^2}{2}\right]_{-1}^{5}$$

$$= \frac{1}{28}(9-16)(25-1) = \frac{-7 \cdot 24}{28} = -\frac{24}{4} = \boxed{-6}$$

(9) [10 pts] Let $E$ be the solid region bounded by the surfaces $z = y$, $z = y^2$, $x = 0$, and $x + 2z = 2$. Sketch the region $E$ and calculate $\iiint_E y\, dV$.




$$0 \le x \le 2-2z$$

$$0 \le y \le 1$$

$$y^2 \le z \le y$$

$$\iiint_E y\,dV = \int_{y=0}^{y=1}\int_{z=y^2}^{z=y}\int_{x=0}^{x=(2-2z)} y\, dx\,dz\,dy$$

$$= \int_{y=0}^{y=1} y \int_{z=y^2}^{z=y} (2-2z)\,dz\,dy$$

$$= \int_{y=0}^{1} y \left[2z - z^2\right]_{z=y^2}^{z=y} dy = \int_0^1 y\left(2y - y^2 - 2y^2 + y^4\right) dy$$

$$= \int_{y=0}^{1} y\left(y^4 - 3y^2 + 2y\right) dy = \int_0^1 y^5 - 3y^3 + 2y^2\, dy = \left[\frac{y^6}{6} - \frac{3y^4}{4} + \frac{2y^3}{3}\right]$$

$$= \frac{1}{6} - \frac{3}{4} + \frac{2}{3} = \frac{1}{12}$$

(10) [10 pts]
(a) Let $\mathbf{F}_1(x, y) = x^2\mathbf{i} + y\mathbf{j}$ be the velocity vector field of a fluid flowing in $\mathbb{R}^2$. On average, is the fluid flowing in or out of a small disc centered at the point $(-3, 1)$? Why?

$$\nabla \cdot \vec{F}_1 = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y) = 2x + 1 = -6 + 1 = -5$$

@ $(-3,1)$

Since $\nabla \cdot \vec{F}_1(-3,1) = -5 < 0$

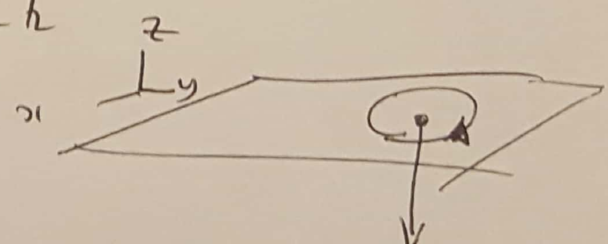
Fluid is flowing $\underline{\underline{\text{IN}}}$

(b) Let $\mathbf{F}_2(x, y) = y^2\mathbf{i} + x^2\mathbf{j}$ be the velocity vector field of a fluid flowing in $\mathbb{R}^2$. On average, is the fluid rotating clockwise or counter-clockwise around the point $(1, 2)$? Why?

$$\nabla \times \vec{F}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + (2x - 2y)\vec{k}$$

$$= 2(x-y)\vec{k}$$

$$(\nabla \times \vec{F}_2)(1,2) = 2(1-2)\vec{k} = -2\vec{k}$$




$\boxed{\text{CLOCKWISE}}$

by RH RULE