NAME: SOLUTIONS

1 /12	2 /9	3 /16	4 /16	5 /6	6 /16	T /100

MATH 2415 (Spring 2014) Exam I, Feb 28
Dr. Zweck's Class

No calculators, books or notes! Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 75 minute exam is worth 75 points.

(1) [12 pts]

(a) Find a vector parametrization of the line through the points $\overset{\vec{p}}{(0,1,2)}$ and $\overset{\vec{q}}{(2,4,-3)}$.

$$\vec{v} = \vec{q} - \vec{p} = (2,4,-3) - (0,1,2) = (2,3,-5)$$

$$\vec{r}(t) = \vec{p} + t\vec{v} = (0,1,2) + t(2,3,-5)$$

$$\cancel{x = 2t} \quad = (2t, 1+3t, 2-5t)$$

(b) Find two unit vectors that are perpendicular to both of the vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (-1, 1, 0)$.

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{vmatrix} = (-3,-3,3)$$

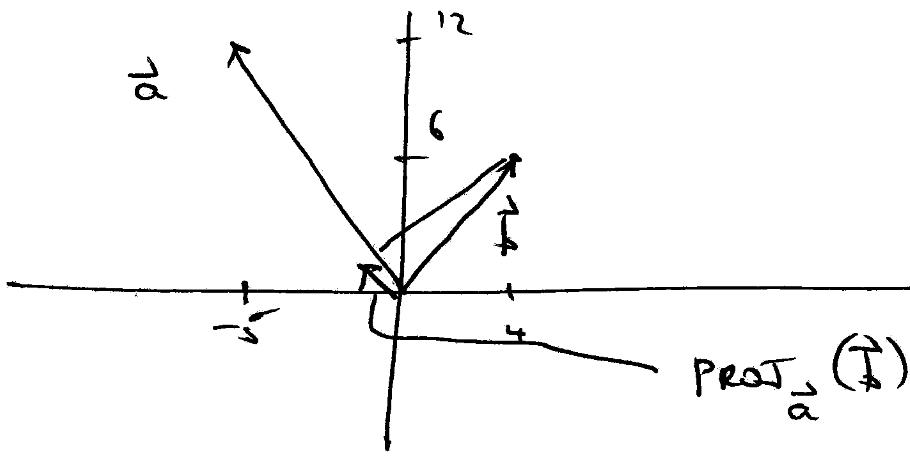
is $\perp$ to $\vec{a}, \vec{b}$

$$\pm \frac{\vec{c}}{|\vec{c}|} = \pm \frac{(-3,-3,3)}{\sqrt{9+9+9}}$$

are our 2 unit vectors.

(2) [9 pts] Let $\mathbf{a} = (-5, 12)$ and $\mathbf{b} = (4, 6)$ be two vectors in the plane.

(a) Draw a picture showing the vectors $\mathbf{a}$ and $\mathbf{b}$ together with the vector projection of $\mathbf{b}$ onto $\mathbf{a}$.



(b) Calculate the vector projection of $\mathbf{b}$ onto $\mathbf{a}$.

$$\mathrm{PROJ}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}$$

$$= \frac{(-5,12)\cdot(4,6)}{\left(\sqrt{5^2+12^2}\right)^2}\,(-5,12)$$

$$= \frac{52}{169}\,(-5,12).$$

$$= \frac{4}{13}\,(-5,12)$$

(3) [16 pts]

(a) Consider the plane whose level set equation is given by $4(x-1)+2(y-5)+6(z-3)=0$. Find a point $\mathbf{p}$ and a pair of vectors $\mathbf{v}$ and $\mathbf{w}$ so that any point $\mathbf{r}$ in this plane can be written in the form $\mathbf{r}=\mathbf{p}+s\mathbf{v}+t\mathbf{w}$ for some scalars s and t.

METHOD 1 Solve level set equation for z.

There are many correct answers!

$$z = 3 - \frac{4}{6}(x-1) - \frac{2}{6}(y-5)$$

Set $x = s$
$y = t$
$z = 3 - \frac{2}{3}(s-1) - \frac{1}{3}(t-5)$

So $$\vec{r} = (x,y,z) = \underbrace{\left(0,0,3+\frac{2}{3}+\frac{5}{3}\right)}_{\vec{p}} + s\overbrace{\left(1,0,-\frac{2}{3}\right)}^{\vec{v}} + t\overbrace{\left(0,1,-\frac{1}{3}\right)}^{\vec{w}}$$

METHOD 2 $\boxed{\vec{n} = (4,2,6)}$ Want $\vec{v},\vec{w} \perp \vec{n}$. Guess $\vec{v} = (1,-2,0)$ from $\vec{v}\cdot\vec{n}=0$

$\boxed{\vec{p} = (1,5,3) \text{ FROM } (\vec{x}-\vec{p})\cdot\vec{n}=0}$ Choose $\vec{w} = \vec{v}\times\vec{n} = (-12,-6,10)$

(b) Find the level set equation of the plane through the point $(1,5,2)$ that is perpendicular to the planes $2x+y-2z=2$ and $x+3z=4$. *Hint:* If two planes are perpendicular how are their normal vectors related?

Normal vector $\vec{n}$ to our plane so $\perp$ to normal vectors $\vec{n}_1 = (2,1,-2)$ and $\vec{n}_2 = (1,0,3)$ to the given planes.

So $$\vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix} = (3,-8,-1)$$

Then $(\vec{x}-\vec{x}_0)\cdot\vec{n} = 0$ with $\vec{x}_0 = (1,5,2)$ gives

$$\boxed{3(x-1) - 8(y-5) - 1(z-2) = 0}$$

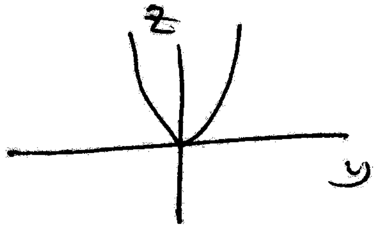
IE $$3x - 8y - z = -39$$

(4) [16 pts] Make a labelled sketch of the traces of the surface

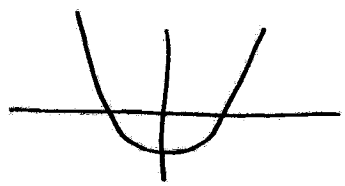
$$z = 4y^2 - x^2$$

in the planes $x = 0$, $x = \pm 1$, $y = 0$, and $z = k$ for $k = 0, \pm 1$. Then sketch the surface.

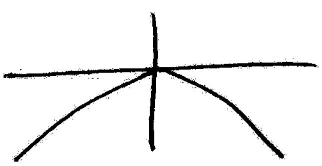
$\boxed{x = 0}$ $z = 4y^2$



$\boxed{x = \pm 1}$ $z = 4y^2 - 1$



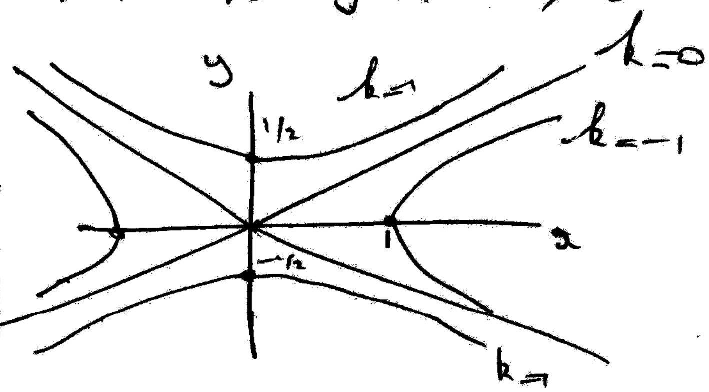
$\boxed{y = 0}$ $z = -x^2$

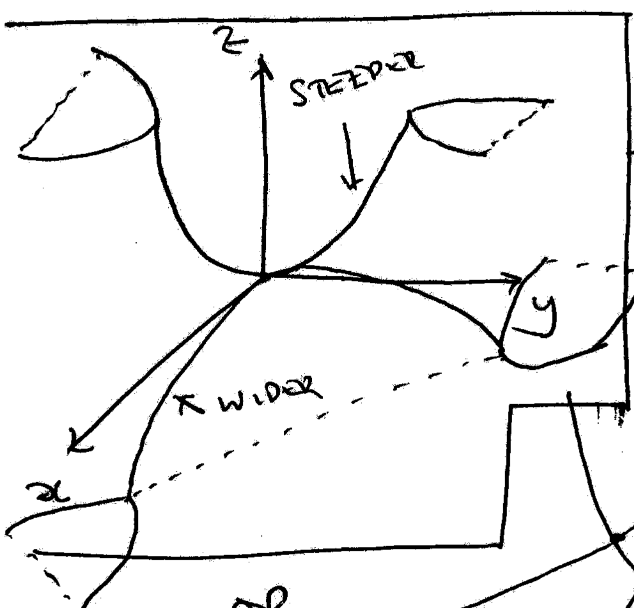


$\boxed{z = k}$

$k = 0$: $y = \pm \frac{x}{2}$

$k = 1$ $1 = 4y^2 - x^2$: $x = 0$, $y = \pm\frac{1}{2}$





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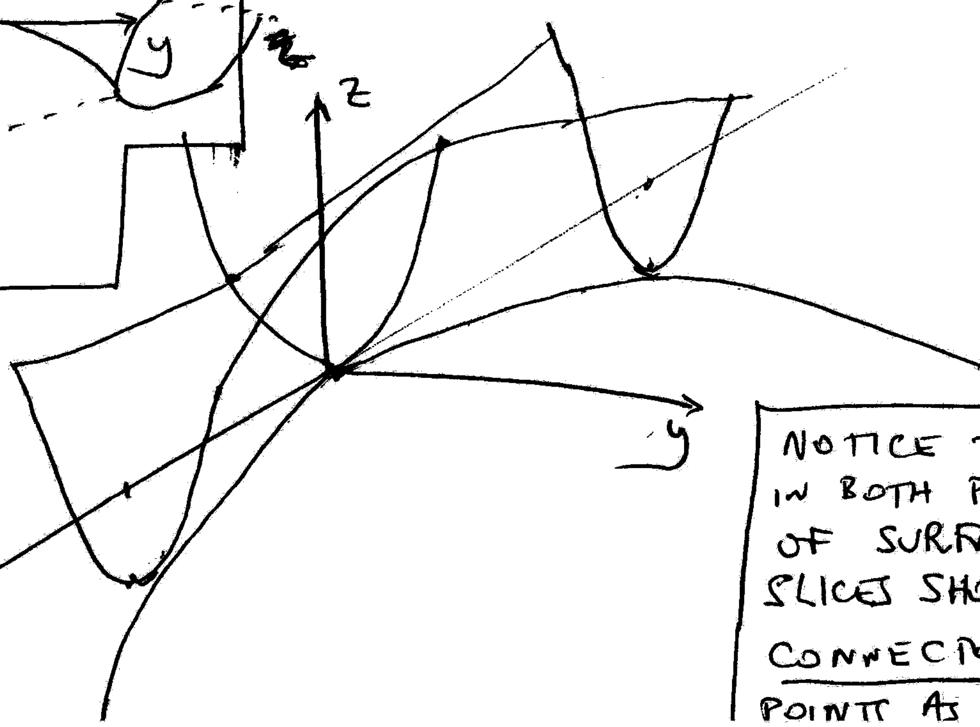


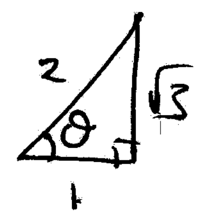
NOTICE THAT IN BOTH PICTURES OF SURFACE THE SLICES SHOWN ARE CONNECTED AT POINTS AS WE DID IN

(5) [6 pts] If $\mathbf{a}\cdot\mathbf{b} = \sqrt{3}$ and $\mathbf{a}\times\mathbf{b} = (1,2,2)$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\sqrt{3} = \vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

$$3 = |\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$$

So dividing these eqns gives



$$\tan\theta = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

$$\theta = 60^\circ.$$

Note $\cos 60^\circ > 0$, $\sin 60^\circ > 0$ as required.

(6) [16 pts] This problem concerns the parametrized curve $\mathbf{r}(t) = (t\cos t, t\sin t, t)$ for $0 \le t \le 2\pi$.

(a) Calculate the velocity vector of the curve at $t = \pi/2$.

$$\vec{r}\,'(t) = (\cos t + - t\sin t,\ \sin t + t\cos t,\ 1)$$

$$\vec{r}\,'(\pi/2) = (-\pi/2,\ 1,\ 1)$$

(b) Find a formula for the speed of the curve as a function of time.

$$s(t) = |\vec{r}\,'(t)| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1}$$

$$= \sqrt{\cos^2 t + t^2\sin^2 t + \sin^2 t + t^2\cos^2 t + 1}$$

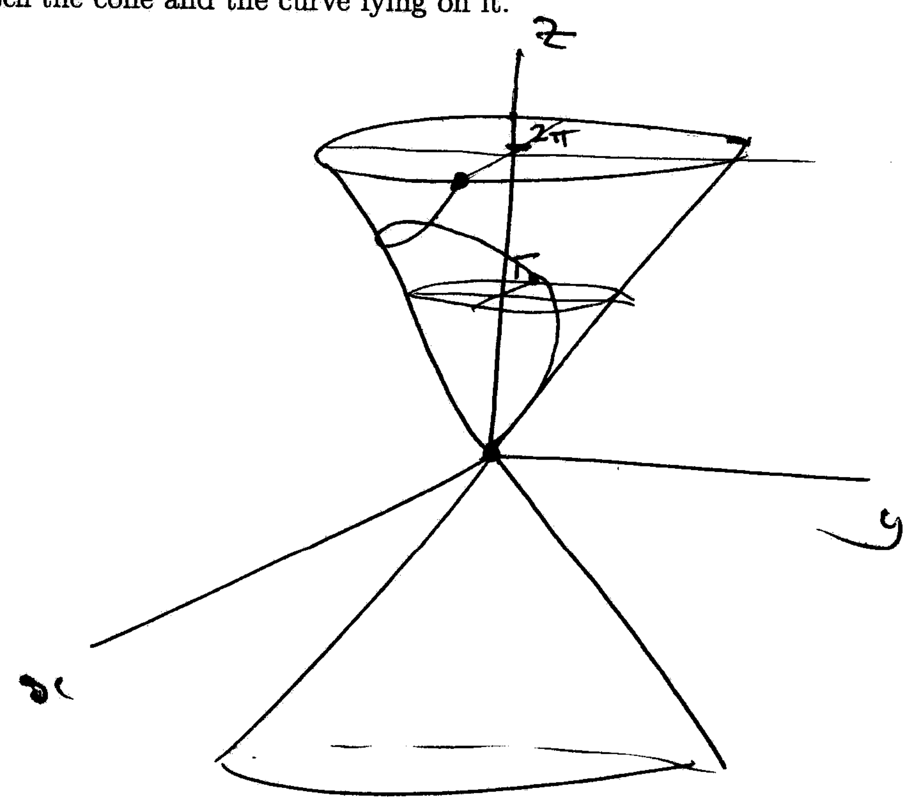
$$= \sqrt{2 + t^2}$$

This problem concerns the parametrized curve $\mathbf{r}(t) = (t\cos t, t\sin t, t)$ for $0 \le t \le 2\pi$.

(c) Show that the curve lies on the cone $z^2 = x^2 + y^2$.

$$x^2+y^2 = (t\cos t)^2 + (t\sin t)^2$$
$$= t^2 = z^2$$

(d) Sketch the cone and the curve lying on it.



Please sign the following honor statement:

On my honor, I pledge that I have neither given nor received any aid on this exam.

Signature: ______________________________