NAME: SOLUTIONS				CIRCLE: Turi		Zweck 10am	Zweck 4pm

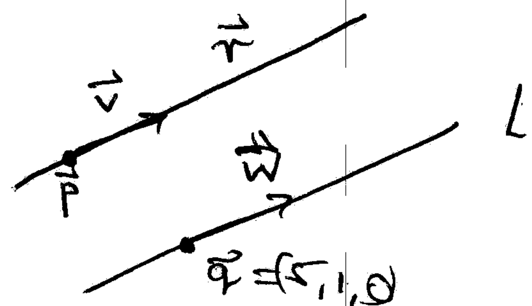
1 /10	2 /12	3 /10	4 /10	5 /12	6 /12	6 /9	T /75

MATH 2415 (Fall 2014) Exam I, Oct 3rd

No books or notes! You may use a scientific calculator provided it does not allow for access to the internet. Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 75 minute exam is worth 75 points.

(1) [10 pts]
(a) Find a parametrization of the line, L, through the point $(5,1,0)$ that is parallel to the line with parametrization $\mathbf{r}(t) = (3+4t, -2+7t, 1-6t)$.

$$\vec{r}(t) = (3,-2,1) + t(4,7,-6)$$
$$= \vec{p} + t\vec{v}$$



AS 2 LINES ARE //

For L:

$$\vec{l}(t) = \vec{q} + t\vec{w} = \vec{q} + t\vec{v}$$
$$= (5,1,0) + t(4,7,-6)$$
$$= (5+4t,\ 1+7t,\ -6t) = (x,y,z)$$

(b) Find the point of intersection of the line L in (a) and the plane $x+y+z=1$.

PLUG

$x = 5+4t$

$y = 1+7t$

$z = -6t$

INTO $1 = x+y+z = (5+4t) + (1+7t) + (-6t)$

$$1 = 6+5t$$
$$t = -1$$

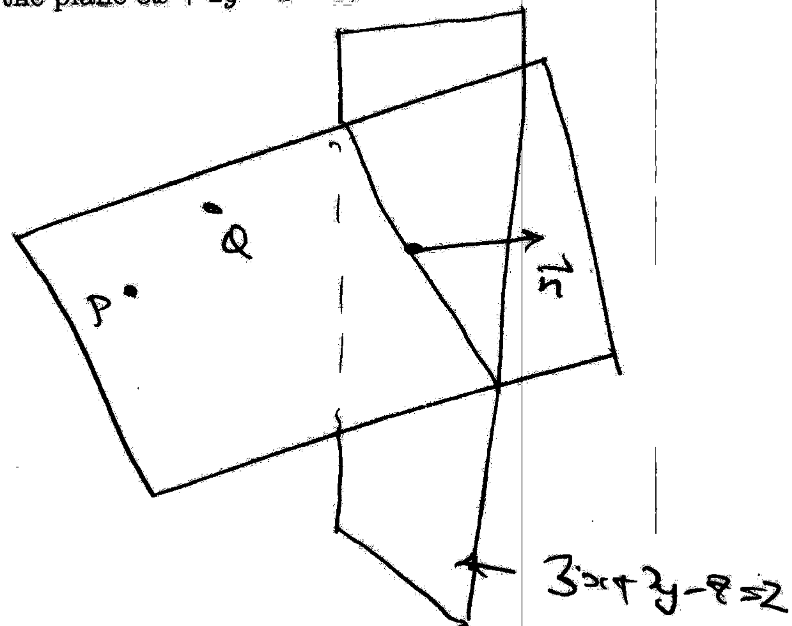
So Point of Intersection is

$$\vec{p} = \vec{r}(-1) = (1,-6,6)$$

(2) [12 pts] Find an equation of the form $z = Ax + By + C$ for the plane that passes through the points $(1, 2, 3)$ and $(-2, 3, 6)$ and which is perpendicular to the plane $3x + 2y - z = 2$.

$P = (1, 2, 3)$

$Q = (-2, 3, 6)$



The normal vector $\vec{n}$ to $3x+2y-z=2$ is a vector that lies in the plane we need to find.

The vectors

$$\vec{v} = \overrightarrow{PQ} = Q - P = (-2,3,6) - (1,2,3) = (-3,1,3)$$

and $\vec{w} = \vec{n} = (3, 2, -1)$

lie in this plane. So normal to plane is

$$\vec{N} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & 3 \\ 3 & 2 & -1 \end{vmatrix}$$

$$= -7\vec{i} + 6\vec{j} - 9\vec{k}$$

So equation is

$$0 = (\vec{x} - P) \cdot \vec{N} = (x-1, y-2, z-3) \cdot (-7, 6, -9)$$

$$= -7(x-1) + 6(y-2) - 9(z-3)$$

OR $\quad -7x + 6y - 9z = -22 \qquad z = -\frac{1}{9}(7x - 6y - 22)$

(3) [10 pts] Find the equation for the tangent plane to the surface $z = x^2y^3$ at the point $(x, y) = (3, 2)$.

$$Z = f(x,y) = x^2 y^3$$

$$\frac{\partial f}{\partial x} = 2xy^3 = 2.3, 2^3 = 48 \quad @ (3,2)$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 = 108 \quad @ (3,2)$$

$$f(3,2) = 72$$

TP

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$z = 72 + 48(x-3) + 108(y-2)$$

(4) [10 pts] Let C be the curve that is parametrized by $(x, y, z) = \mathbf{r}(t) = (2t, t^2, \frac{1}{3}t^3)$. Find the arc length of C between the points $P = (0, 0, 0)$ and $Q = (2, 1, \frac{1}{3})$.

$$P = \vec{r}(0) \qquad \vec{r}\,'(t) = (2, 2t, t^2)$$

$$Q = \vec{r}(1)$$

$$L = \int_0^1 |\vec{r}\,'(t)|\, dt$$

$$= \int_0^1 \sqrt{4 + 4t^2 + t^4}\, dt$$

$$= \int_0^1 \sqrt{(2+t^2)^2}\, dt = \int_0^1 |2+t^2|\, dt$$

$$= \int_0^1 2+t^2\, dt \quad \text{as} \quad 2+t^2 > 0 \text{ on } [0,1]$$

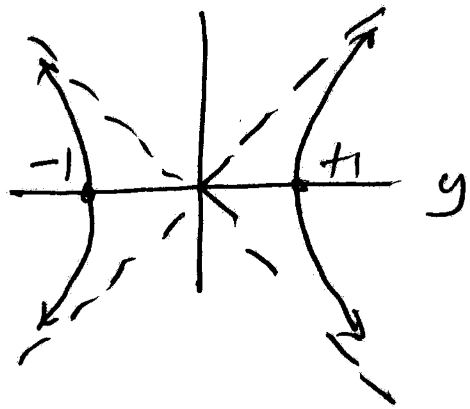
$$= \left[2t + t^3/3\right]_0^1 = \boxed{7/3}$$

(5) [12 pts] Make a labelled sketch of the traces of the surface

$$4x^2 - y^2 + z^2 = -1$$

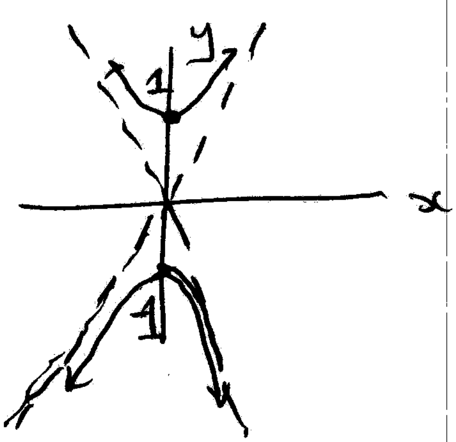
in the planes $x = 0$, $z = 0$, and $y = k$ for $k = 0, \pm 1, \pm 2$. Then sketch the surface.

$\boxed{x=0}$ $y^2 - z^2 = +1$



$\boxed{z=0}$ $4x^2 - y^2 = -1$

ASYMPTOTES $y = \pm 2x$

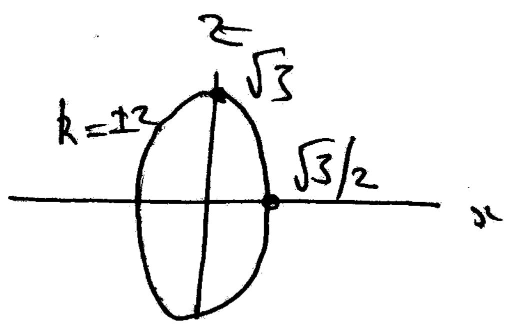


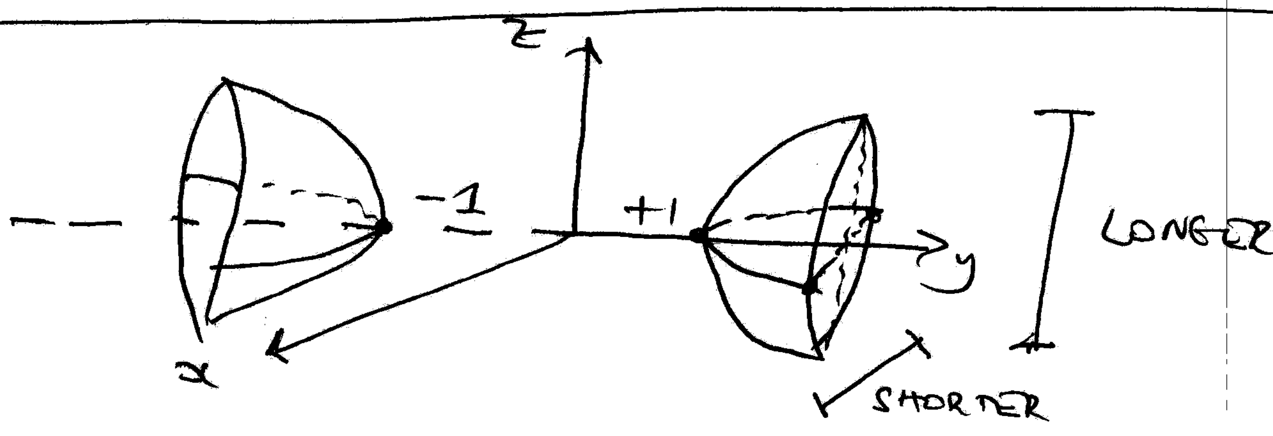
$\boxed{y=k}$ $4x^2 + z^2 = k^2 - 1$

$k=0$ EMPTY SET

$k=\pm 1$ ORIGIN

$k = \pm 2$ ELLIPSE





(6) [12 pts] Find the limit if it exists, or show that the limit does not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{y^4-9x^2}{y^2-3x}$

$$\lim_{(x,y)\to(0,0)} \frac{y^4-9x^2}{y^2-3x}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(y^2-3x)(y^2+3x)}{y^2-3x} \quad \text{FACTOR}$$

$$= \lim_{(x,y)\to(0,0)} y^2+3x \quad \text{CANCEL}$$

$$= 0+0=0$$ AS POLYNOMIALS ARE CONTINUOUS

(b) $\lim_{(x,y)\to(0,0)} \frac{y^4-9x^2}{y^2-3x^2}$

APPROACH $(0,0)$ ALONG $x=0$ GET $\lim_{y\to 0} \frac{y^4}{y^2} = \lim_{y\to 0} y^2 = 0$

APPROACH $(0,0)$ ALONG $y=0$ GET $\lim_{x\to 0} \frac{-9x^2}{-3x^2}$

$$= \lim_{x\to 0} 3 = 3$$

Since these 2 limits are not equal

$\lim_{(x,y)\to(0,0)} \frac{y^4-9x^2}{y^2-3x^2}$ DNE

(7) [9 pts] Let $\mathbf{r}(t) = \mathbf{p} + t\mathbf{v}$ be a parametrization of the line, L, through the point $\mathbf{p}$ in the direction of the vector $\mathbf{v}$ and let $\mathbf{q}$ be a point that is not on the line L.

(a) Show that the distance between the point $\mathbf{q}$ and a point $\mathbf{r}(t)$ on the line is given by

$$D(t) = \sqrt{|\mathbf{p}-\mathbf{q}|^2 + 2t(\mathbf{p}-\mathbf{q})\cdot\mathbf{v} + t^2|\mathbf{v}|^2}. \tag{1}$$

$$D(t) = |\vec{r}(t) - \vec{q}| = \sqrt{(\vec{r}(t)-\vec{q})\cdot(\vec{r}(t)-\vec{q})}$$

$$= \sqrt{(\vec{p}-\vec{q}+t\vec{v})\cdot(\vec{p}-\vec{q}+t\vec{v})}$$

$$= \sqrt{|\vec{p}-\vec{q}|^2 + 2t(\vec{p}-\vec{q})\cdot\vec{v} + t^2|\vec{v}|^2}$$

as $\vec{w}\cdot\vec{w} = |\vec{w}|^2$

(b) Use Equation (1) above to show that the point on the line L that is closest to $\mathbf{q}$ is given by

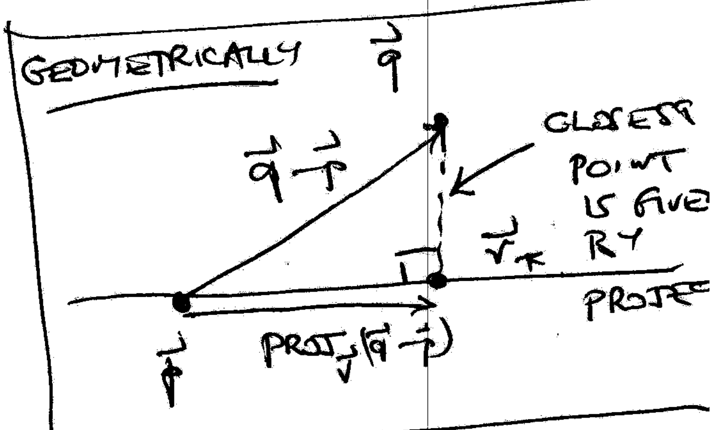
$$\mathbf{r}_* = \mathbf{p} + \mathrm{Proj}_{\mathbf{v}}(\mathbf{q}-\mathbf{p})$$

where $\mathrm{Proj}_{\mathbf{b}}(\mathbf{a})$ is the vector projection of the vector $\mathbf{a}$ onto the vector $\mathbf{b}$.

FIND t SO THAT $D^2(t)$ IS MINIMUM

$$0 = (D^2(t))' = \frac{d}{dt}\left[|\vec{p}-\vec{q}|^2 + 2t(\vec{p}-\vec{q})\cdot\vec{v} + t^2|\vec{v}|^2\right]$$

$$= 2(\vec{p}-\vec{q})\cdot\vec{v} + 2t|\vec{v}|^2$$



GIVES

$$t_* = \frac{(\vec{q}-\vec{p})\cdot\vec{v}}{|\vec{v}|^2}$$

So

$$\vec{r}_* = \vec{p} + t_*\vec{v} = \vec{p} + \frac{(\vec{q}-\vec{p})\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \vec{p} + \mathrm{PROJ}_{\vec{v}}(\vec{q}-\vec{p})$$