LAST NAME: SOLUTIONS	FIRST NAME:	CIRCLE: Li 2:30pm Li 5:30pm Zweck 10am Zweck 1pm

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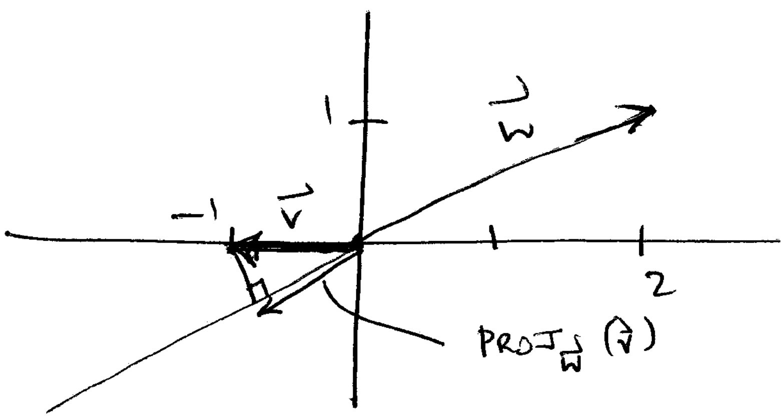
MATH 2415 (Fall 2017) Exam I, Sep 29th

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 90 minute exam is worth 75 points.

(1) [12 pts] Let $\mathbf{v} = (-1, 0)$ and $\mathbf{w} = (2, 1)$.
(a) Make a labelled sketch showing the vector projection of $\mathbf{v}$ onto $\mathbf{w}$.



(b) Calculate the vector projection of $\mathbf{v}$ onto $\mathbf{w}$.

$$\text{PROJ}_{\vec{w}}(\vec{v}) = \frac{\vec{w} \cdot \vec{v}}{|\vec{w}|^2}\vec{w} = -\frac{2}{5}(2,1) = \left(-\frac{4}{5}, -\frac{2}{5}\right)$$

as $\quad \vec{w} \cdot \vec{v} = (-1, 0) \cdot (2, 1) = -2$

$$|\vec{w}|^2 = 2^2 + 1^2 = 5.$$

(2) [12 pts]
(a) Let L be the line through the point $\mathbf{p} = (1, 0, 3)$ that contains the vector $\mathbf{v} = (0, 1, 2)$. Let P be the plane $x + y + z = 7$. The line L and the plane P intersect in a point. Find the coordinates of this point.

$L: \vec{r}(t) = \vec{p} + t\vec{v} = (1, 0, 3) + t(0, 1, 2)$

$= (1, t, 3+2t) = (x, y, z)$

PLUG INTO equation of plane and solve for t:

$7 = 1 + t + 3 + 2t = 4 + 3t \Rightarrow \boxed{t = 1}$

So point is $\vec{q} = \vec{r}(1) = (1, 1, 5)$

(b) Let L_1 and L_2 be the lines parametrized by $\mathbf{r}_1(t) = (1, t, 0)$ and $\mathbf{r}_2(t) = (t, 2t, 3t)$, respectively. Do the lines L_1 and L_2 lie in the same plane? Explain.

$\boxed{\text{METHOD 1}}$ If 2 lines lie in same plane they must either be parallel or intersect.

ⓐ $\vec{r}_1(t) = (1, 0, 0) + t(0, 1, 0)$ contains vector $\vec{v} = (0, 1, 0)$

$\vec{r}_2(t) = (0, 0, 0) + t(1, 2, 3)$ contains vector $\vec{w} = (1, 2, 3)$

$\vec{v}$ and $\vec{w}$ are NOT $\parallel$.

ⓑ These ~~planes~~ lines do not intersect: Try to find s, t so that

$\vec{r}_1(s) = \vec{r}_2(t)$

$(1, s, 0) = (t, 2t, 3t)$

So $t = 1$, $s = 2t$, $3t = 0$.

t cannot be both 1 and 0. So don't intersect

(2b) METHOD 2

Pick ~~2~~ points ~~L~~ L_1, 2 points on L_2

~~Show~~

$\vec{P}_1 = \vec{r}_1(0) = (1, 0, 0)$

$\vec{P}_2 = \vec{r}_1(1) = (1, 1, 0)$

$\vec{P}_3 = \vec{r}_2(0) = (0, 0, 0)$

$\vec{P}_4 = \vec{r}_2(1) = (1, 2, 3)$

Let P be plane containing $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$

The normal to this plane is

$$\vec{n} = (\vec{P}_1 - \vec{P}_3) \times (\vec{P}_2 - \vec{P}_3) = +\vec{P}_1 \times \vec{P}_2$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = +\vec{k}$$

So 2 lines don't intersect

If $\vec{P}_4$ is also in this plane then the vector $\vec{v} = \vec{P}_4 - \vec{P}_3$ should be in the plane and so $\vec{v} \cdot \vec{n} = 0$ should hold.

But $(1,2,3) \cdot (0,0,1) = 3 \neq 0$ So 4 points do not lie in plane.

(3) [15 pts] Make a labelled sketch of the traces of the surface

$$y^2 - 4x^2 - z^2 = 1$$

in the planes $x = 0$, $z = 0$, and $y = k$ for $k = 0$, ± 1, ± 2. Then sketch the surface.

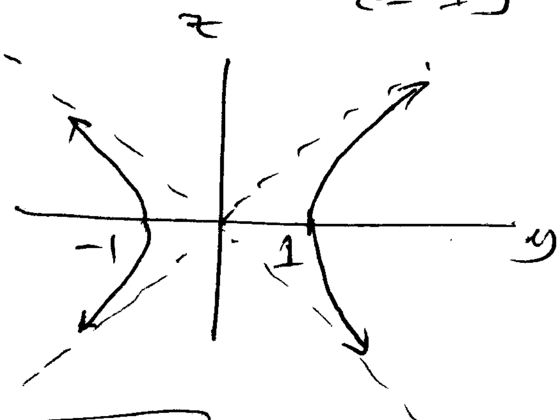
1 $\boxed{x = 0}$

$$y^2 - z^2 = 1$$

INTERCEPTS: $(\pm 1, 0)$

ASYMPTOTES: $y^2 - z^2 = 0$

$z = \pm y$



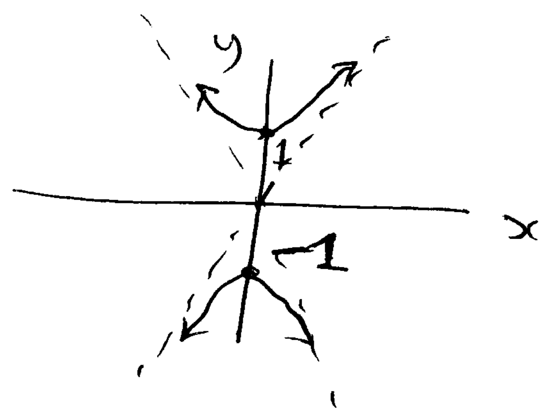
2 $\boxed{z = 0}$

$$y^2 - 4x^2 = 1$$

INTERCEPTS: $(0, \pm 1)$

ASYMPTOTES: $y^2 - 4x^2 = 0$

$y = \pm 2x$

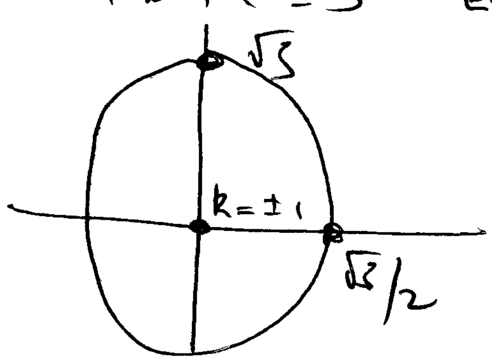


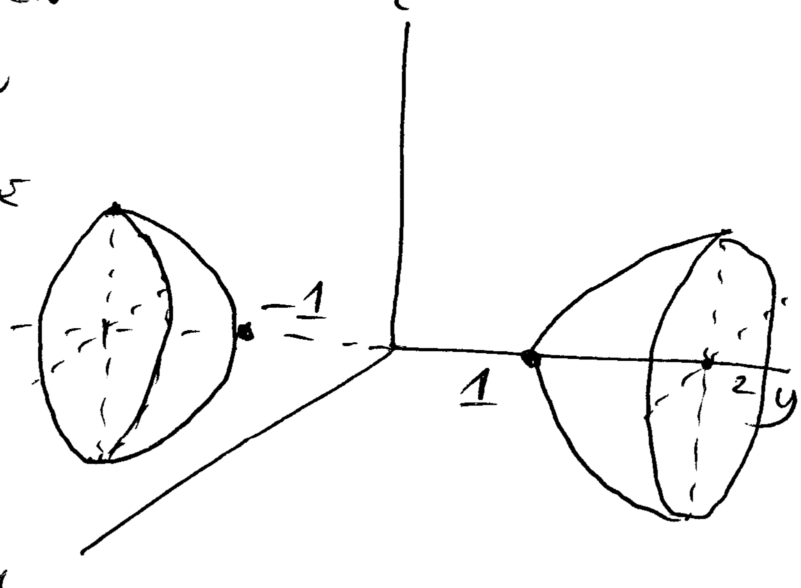
3 $\boxed{y = k}$

$k = 0$: $+4x^2 + z^2 = -1$ NO SOLN

$k = \pm 1$: $4x^2 + z^2 = 0$ ORIGIN

$k = \pm 2$ $4x^2 + z^2 = 3$ ELLIPSE





(4) [12 pts] Let C be the parametrized curve $\mathbf{r}(t) = (3\cos 2t, 4\sin 2t, 5t)$.
(a) Show that the curve C lies on an elliptical cylinder.

$$x = 3\cos 2t \quad \Rightarrow \quad \cos 2t = \frac{x}{3}$$

$$y = 4\sin 2t \quad \Rightarrow \quad \sin 2t = \frac{y}{4}$$

$$1 = \cos^2 2t + \sin^2 2t = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2$$

$$1 = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2$$

ELLIPTICAL CYLINDER

(b) Find a parametrization of the tangent line to the curve C at $t = \pi/8$.

$$\vec{l}(s) = \vec{r}(\pi/8) + (s - \pi/8)\,\vec{r}\,'(\pi/8)$$

$$= \left(\frac{3\sqrt{2}}{2}, 2\sqrt{2}, \frac{5\pi}{8}\right) + (s - \pi/8)\left(-3\sqrt{2}, 4\sqrt{2}, 5\right)$$

as $\vec{r}\,'(t) = (-6\sin 2t, 8\cos 2t, 5)$

$\boxed{\text{METHOD 2}}$ $x = t$

$y = \pm\sqrt{4-x^2} = \pm\sqrt{4-t^2}$

$z = x^2 - 3y^2 = t^2 - 3(4-t^2) = 4t^2 - 12.$

(5) [12 pts]
(a) Parametrize the curve that is given by the intersection of the surfaces $x^2 + y^2 = 4$ and $z = x^2 - 3y^2$.

$\boxed{\text{METHOD I}}$ $\begin{cases} x^2+y^2=4 & \cdots ① \\ z = x^2-3y^2 & \cdots ② \end{cases} \Rightarrow \begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = 4\cos^2 t - 12\sin^2 t \end{cases}$

let $x = 2\cos t$, $y = 2\sin t$

so $x^2+y^2 = 4$ satisfies ①

$z = x^2 - 3y^2 = 4\cos^2 t - 3\cdot 4\sin^2 t$

(b) Let $z = f(x,y) = xe^{-y}$. Make a labelled sketch showing the contours of $f(x,y) = k$ for $k = 0$ $k = \pm 1$, and $k = \pm 2$.

① $k=0 \Rightarrow xe^{-y} = 0$

since $e^{-y} > 0$, so $x=0$ is the solution

② $k = 1 \Rightarrow xe^{-y} = 1 \Rightarrow x = e^{y}$

$e^{-y} = \frac{1}{x} \Rightarrow y = \ln x$

$-y = \ln\frac{1}{x} = -\ln x$

$\Rightarrow y = \ln x$

$k = -1 \Rightarrow xe^{-y} = -1$

$e^{-y} = -\frac{1}{x}$

$-y = \ln(-\frac{1}{x})$, $x<0$

$-y = -\ln(-x)$

$y = \ln(-x)$

③ $k=2 \Rightarrow xe^{-y} = 2$

$e^{-y} = \frac{2}{x}$

$-y = \ln\frac{2}{x} = \ln 2 - \ln x$

$y = \ln x - \ln 2$

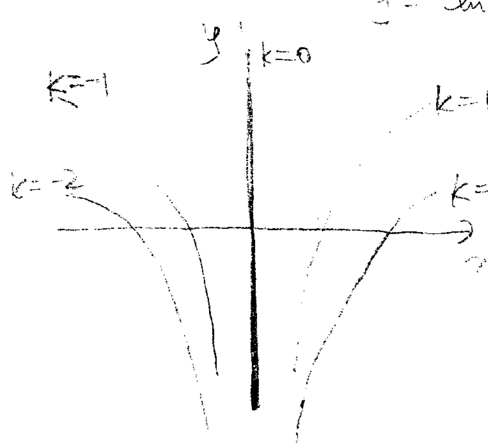
$k=-2 \Rightarrow xe^{-y} = -2$

$e^{-y} = -\frac{2}{x}$

$-y = \ln(-\frac{2}{x})$, $x<0$

$-y = \ln 2 - \ln(-x)$

$y = \ln(-x) - \ln 2$



(6) [12 pts]
(a) Let P be the point with cylindrical coordinates $(r, \theta, z) = (\sqrt{3}, \frac{\pi}{4}, -1)$. Find the spherical coordinates of P.

Cylindrical

$$\begin{cases} x = r\cos\theta = \sqrt{3}\cdot\cos\frac{\pi}{4} = \frac{\sqrt{6}}{2} \\ y = r\sin\theta = \sqrt{3}\cdot\sin\frac{\pi}{4} = \frac{\sqrt{6}}{2} \\ z = z = -1 \end{cases}$$

$\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, -1\right)$ in the rectangular coor.

Spherical

$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{6}{4}+\frac{6}{4}+1} = 2 \\ \theta = \arctan\frac{y}{x} = \arctan 1 = \frac{\pi}{4} \\ \phi = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} = \arccos\frac{-1}{2} = \frac{2\pi}{3} \end{cases}$$

NOTE SINCE $z<0$ we must have $\pi/2 < \phi \leq \pi$.

(b) Convert the equation $z = -\sqrt{x^2+y^2}$ into ~~(a) cylindrical and (b)~~ spherical coordinates.

$$\boxed{\begin{aligned} &\text{a) } \cancel{z = -\sqrt{(r\cos\theta)^2+(r\sin\theta)^2} = -r} \\ &\cancel{\Rightarrow z = r \text{ in cylindrical}} \end{aligned}}$$

b) $\rho\cos\varphi = -\sqrt{(\rho\sin\varphi\cdot\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2}$

$= -\sqrt{\rho^2\cdot\sin^2\varphi}$

$= -\rho\cdot\sin\varphi$

$\tan\phi = -1$ or $\phi = \frac{3\pi}{4}$