LAST NAME:	FIRST NAME:	CIRCLE:		
JACOBI	CARL		Dahal 4pm	Li 1pm
		Li 5:30pm	Zweck 11:30am	Zweck 1pm

1 /10	2 /10	3 /10	4 /10	5 /10	
6 /10	7 /10	8 /10	9 /10	10 /10	T /100

MATH 2415 Final Exam, Fall 2019

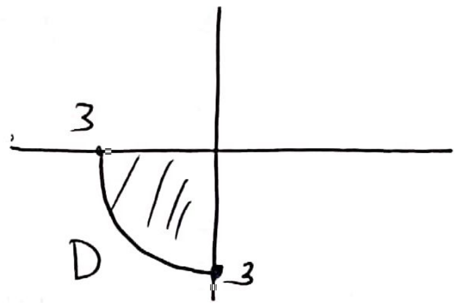
No books or notes! **NO CALCULATORS! Show all work and give complete explanations.** This 2 hours 45 mins exam is worth 100 points.

(1) [10 pts] Let D be the portion of the disc $x^2 + y^2 \leq 9$ that lies in the third quadrant of the xy-plane. Calcuate $\iint_D x\,dA$.

D is

$$0 \leq r \leq 3$$

$$\pi \leq \theta \leq \frac{3\pi}{2}$$



So

$$\iint_D x\,dA = \int_{\theta=\pi}^{3\pi/2} \int_{r=0}^{3} (r\cos\theta)\, r\,dr\,d\theta$$

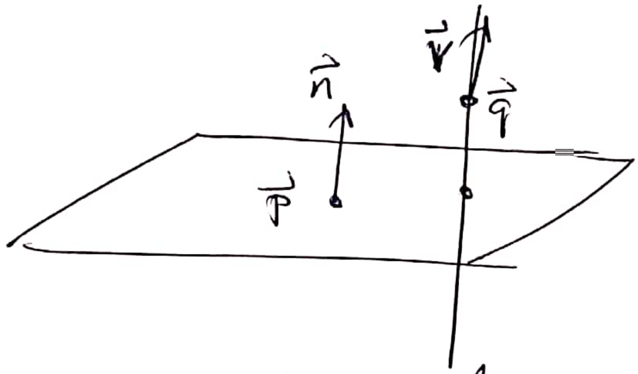
$$= \left[\int_{\pi}^{3\pi/2} \cos\theta\,d\theta\right]\left[\int_0^3 r^2\,dr\right]$$

$$= \left[\sin\theta\right]_{\pi}^{3\pi/2} \left[\frac{r^3}{3}\right]_0^3 = (-1)\frac{3^3}{3} = -9$$

(2) [10 pts]

(a) Find the (level set) equation of the plane through the point $(1, 2, 3)$ that is perpendicular to the line with parametrization $\mathbf{r}(t) = (2 - 3t, 4 + 6t, -1 - t)$.

$\vec{r}(t) = \vec{q} + t\vec{v}$



$\vec{q} = (2, 4, -1)$, $\vec{v} = (-3, 6, -1)$

Plane goes through

$\vec{p} = (1, 2, 3)$ with normal

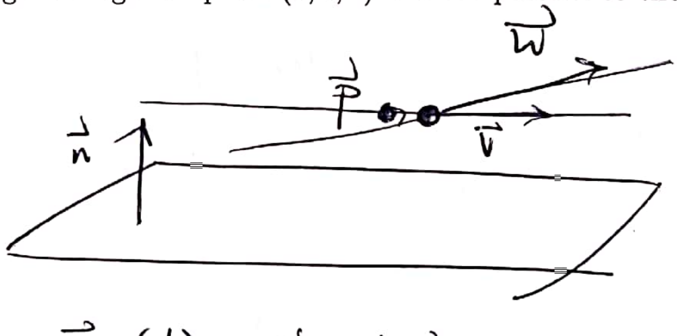
$\vec{n} = \vec{v} = (-3, 6, -1)$. So level set equation is

$0 = (\vec{r} - \vec{p}) \cdot \vec{n} = (x - 1, y - 2, z - 3) \cdot (-3, 6, -1)$

$$\boxed{-3(x-1) + 6(y-2) - (z-3) = 0}$$

(b) Find parametrizations of **two** different lines that go through the point $(1, 0, 2)$ and are parallel to the plane $3x - 2y + 4z = 10$.

Normal to plane is



$\vec{n} = (3, -2, 4)$

Lines are $\vec{r}_1(t) = \vec{p} + t\vec{v}$, $\vec{r}_2(t) = \vec{p} + t\vec{w}$

where $\vec{p} = (1, 0, 2)$ and $\vec{v}, \vec{w}$ are two vectors $\perp \vec{n}$.

<u>CAN CHOOSE</u> $\vec{v} = (-4, 0, 3) \Rightarrow \vec{v} \cdot \vec{n} = 0$

$\vec{w} = (0, 4, 2) \Rightarrow \vec{w} \cdot \vec{n} = 0$

So $\vec{r}_1(t) = (1 - 4t, 0, 2 + 3t)$

$\vec{r}_2(t) = (1, 4t, 2 + 2t)$

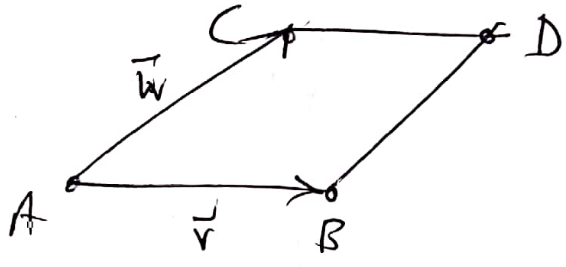
SINCE $\vec{v}$ and $\vec{w}$ are NOT // these 2 lines are different

(3) [10 pts]

(a) Use vectors to find the area of the parallelogram with vertices $A = (1,1)$, $B = (2,6)$, $C = (3,4)$ and $D = (4,9)$.

$\vec{v} = \vec{AB} = (1,5) = \vec{CD}$

$\vec{w} = \vec{AC} = (2,3) = \vec{BD}$



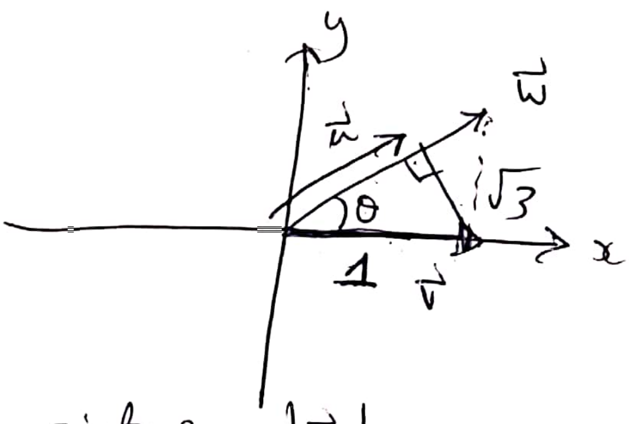
So we have the 4 vertices of ∥gram in correct order.

$A = |\vec{v} \times \vec{w}|$.

$= |-7\vec{k}| = 7$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & 0 \\ 2 & 3 & 0 \end{vmatrix} = -7\vec{k}$$

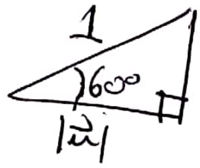
(b) Make a sketch that shows how to project the vector $\mathbf{v} = \mathbf{i}$ onto the vector $\mathbf{w} = \mathbf{i} + \sqrt{3}\mathbf{j}$. Use your sketch to find the component of $\mathbf{v}$ in the direction $\mathbf{w}$.



$\vec{u} = \mathrm{PROJ}_{\vec{w}} \vec{v}$

$|\vec{w}| = (1^2 + \sqrt{3}^2)^{1/2} = 2$

So $\theta = 60°$



Now from picture $|\vec{u}| = \cos 60° = \frac{1}{2}$

So, and direction of $\vec{u} = \dfrac{\vec{u}}{|\vec{u}|} = \dfrac{\vec{w}}{|\vec{w}|}$

So

$$\vec{u} = |\vec{u}| \frac{\vec{u}}{|\vec{u}|} = |\vec{u}| \frac{\vec{w}}{|\vec{w}|} = \frac{1}{2} \cdot \frac{\vec{i} + \sqrt{3}\vec{j}}{2} = \frac{1}{4}(\vec{i} + \sqrt{3}\vec{j})$$

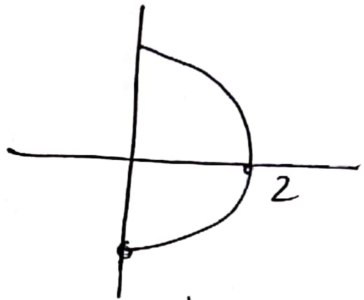
$\vec{r}\,'(t) = (-2\sin t, 2\cos t) \Rightarrow |\vec{r}\,'(t)| = 2$

(4) [10 pts] Let C be the half-circle given by $x^2 + y^2 = 4$ with $x \geq 0$, oriented counter-clockwise.

(a) Calcuate $\int_C x\, ds$.

C is $\vec{r}(t) = (2\cos t, 2\sin t)$

$-\pi/2 \leq t \leq \pi/2$



$$\int_C x\, ds = \int_{-\pi/2}^{\pi/2} (2\cos t)\, |\vec{r}\,'(t)|\, dt$$

~~$\vec{r}\,'(t)$~~

$$= \int_{-\pi/2}^{\pi/2} (2\cos t)\, 2\, dt = 4\left[\sin t\right]_{-\pi/2}^{\pi/2} = 8.$$

(b) Without doing any calculation, find $\int_C y^3\, ds$. Explain your reasoning!

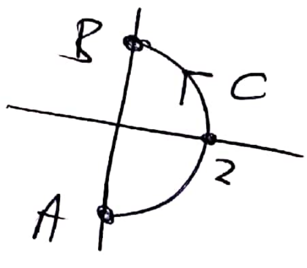
The function $f(x,y) = y^3$ is odd in y

The curve C is symmetric when reflected over the y-axis

Therfore $\int_C y^3 ds = 0$ by Symmetry

(c) Let $f(x, y) = xe^{x^2+y^2}$. Find $\int_C \nabla f \cdot dr$.

By $\boxed{\text{FTC I}}$ (FTC for Line Integrals)



$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

$$= f(0,2) - f(0,-2)$$

$$= 0 - 0 = 0.$$

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(5) [10 pts]
(a) Sketch the surface $x^2 - y^2 + z^2 = 2$ for $0 \leq y \leq \sqrt{2}$. [Hint: Convert into an appropriate cylindrical coordinate system.]

Let $r^2 = x^2 + z^2$

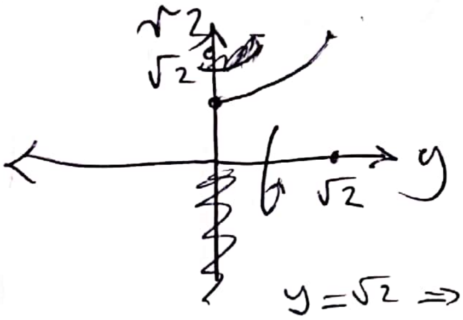
So we used Cyl Coords (y, r, θ) with

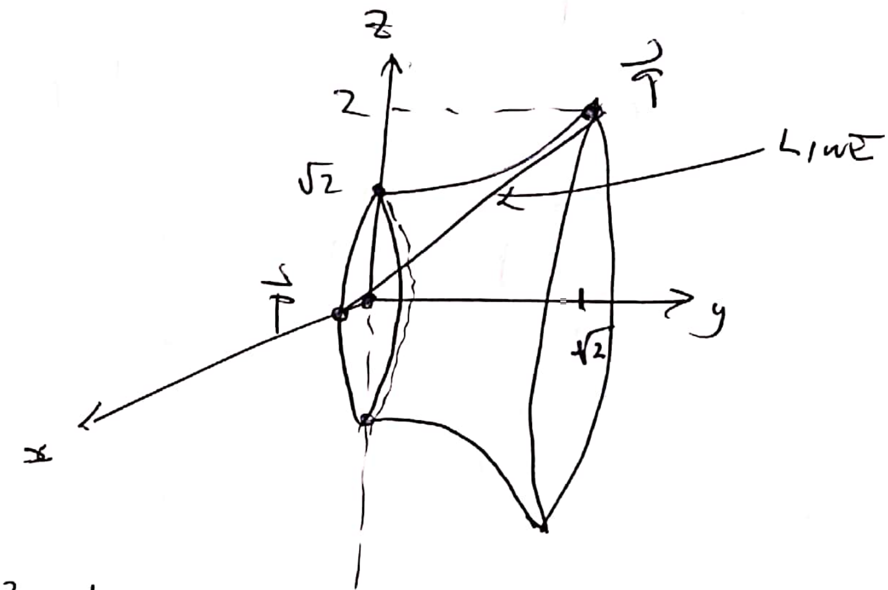
$x = r\cos\theta$

$y = y$

$z = r\sin\theta$

Eqn is $r^2 - y^2 = 2$





$y = \sqrt{2} \Rightarrow r^2 = 4$
$r = 2$

(b) Show that the line through the point $(1, 0, 1)$ in the direction of the vector $(-1, \sqrt{2}, 1)$ lies on the surface in (a). Add this line to your sketch in (a).

$\vec{r}(t) = \vec{p} + t\vec{v} = (1, 0, 1) + t(-1, \sqrt{2}, 1)$

$\vec{r}(t) = (1-t, \sqrt{2}t, 1+t) = (x(t), y(t), z(t))$ (*)

NOTE $\vec{q} = \vec{r}(1) = (0, \sqrt{2}, 2)$

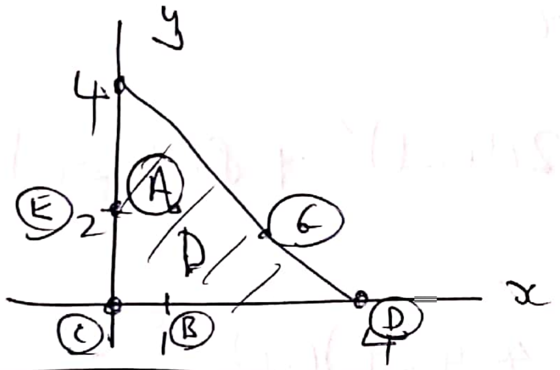
SUBSTITUTE FORMULAE (*) into equation of surface:

$x^2 - y^2 + z^2 = (1-t)^2 + -(\sqrt{2}t)^2 + (1+t)^2$

$= 1 - 2t + t^2 + -2t^2 + 1 + 2t + t^2$

$= 2$ ✓

(6) [10 pts] Find the absolute maximum and minimum of the function $z = f(x,y) = x^2 - 2x - 2y^2 + 8y$ on the triangular domain with vertices $(0,0)$, $(4,0)$, and $(0,4)$.



LABEL	(x,y)	$f(x,y)$
A	$(1,2)$	~~3~~ 7
B	$(1,0)$	-1 $\boxed{\text{MIN}}$
C	$(0,0)$	0
D	$(4,0)$	8
E	$(0,2)$	8
F	$(0,4)$	0
G	$(3,1)$	9 $\boxed{\text{MAX}}$

CRITICAL POINTS IN D

$0 = f_x = 2x-2 \Rightarrow x = 1$

$0 = f_y = -4y+8 \Rightarrow y = 2$

GIVES $(x,y) = (1,2)$ A

BOTTOM EDGE

$x = t, \quad y = 0, \quad 0 \le t \le 4$

$g(t) = f(t,0) = t^2 - 2t$

$0 = g'(t) = 2t - 2 \Rightarrow t = 1$

Gives $(x,y) = (1,0)$ B

Endpts ~~$g(0)$~~ $(x,y) = (0,0)$ C

$(x,y) = (4,0)$ D

LEFT EDGE

$x = 0, \ y = t, \ 0 \le t \le 4$

$h(t) = -2t^2 + 8t$

$0 = h'(t) = -4t + 8 \Rightarrow t = 2$

Gives $(x,y) = (0,2)$ E

Endpts $(x,y) = (0,0)$ C

$(x,y) = (0,4)$ F

DIAGONAL EDGE

PTO

DIAGONAL EDGE $\quad x+y=4$

So $x=t, \quad y=4-t$

$$k(t) = t^2 - 2t - 2(4-t)^2 + 8(4-t)$$

$$0 = k'(t) = 2t - 2 - 4(4-t)(-1) - 8$$

$$= 2t - 4t - 2 + 16 - 8$$

$$= -2t + 6$$

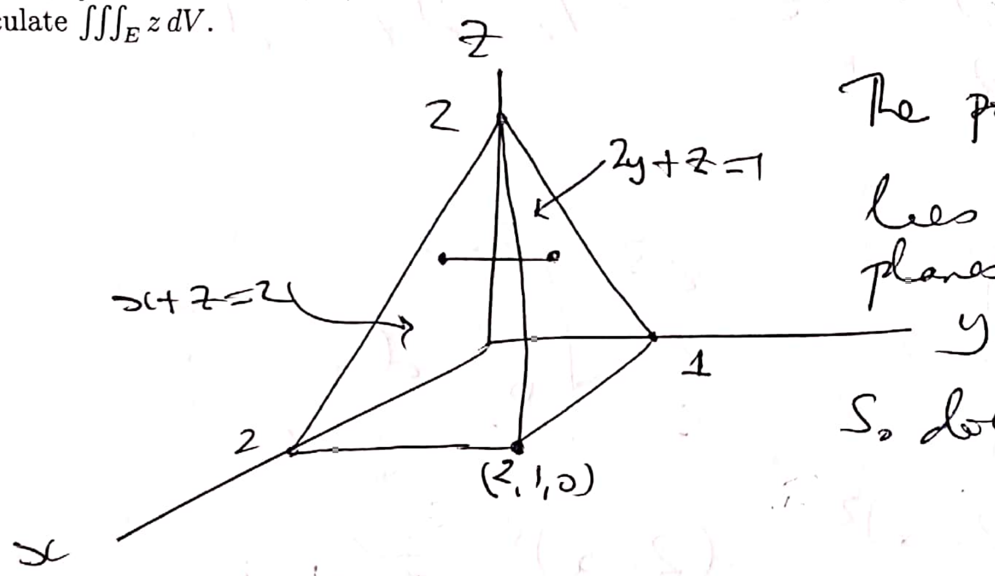
$$\Rightarrow t = 3$$

$$\Rightarrow (x,y) = (3,1) \quad 6$$

$$f(3,1) = 3^2 - 6 - 2.1^2 + 8.1$$

$$= 9$$

(7) [10 pts] Sketch the solid, E, in the first octant that is bounded by the planes $x + z = 2$ and $2y + z = 2$. Calculate $\iiint_E z\, dV$.

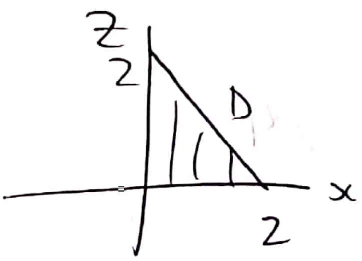


The point $(2,1,0)$ lies on both planes.

So does $(0,0,2)$

E is ~~a~~ Pyramid with ~~square~~ rectangular base and vertex above origin. (off-center)

Fill E with french fries Left → Right



E is Therefore

$$0 \le x \le 2$$

$$0 \le z \le 2 - x$$

$$0 \le y \le \frac{2-z}{2} = 1 - \frac{z}{2}$$

$$\iiint_E z\,dV = \int_{x=0}^{2} \int_{z=0}^{2-x} \int_{y=0}^{1-z/2} z \, dy\, dz\, dx$$

$$= \int_{x=0}^{2} \int_{z=0}^{2-x} z\left(1 - z/2\right) dz\, dx$$

PTO

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$$= \int_{x=0}^{2} \int_{z=0}^{2-x} z - z^2/2 \, dz \, dx$$

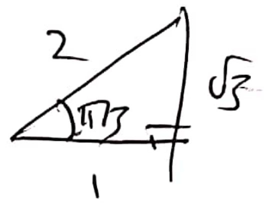
$$= \int_{x=0}^{2} \left[\frac{z^2}{2} - \frac{z^3}{6} \right]_0^{2-x} dx$$

$$= \int_0^2 \frac{(2-x)^2}{2} - \frac{(2-x)^3}{6} \, dx$$

$$= \left[-\frac{(2-x)^3}{6} + \frac{(2-x)^4}{24} \right]_0^2$$

$$= \frac{2^3}{6} - \frac{2^4}{24} = \frac{2^3 \cdot 4 - 2^4}{24}$$

$$= \frac{16}{24} = \frac{2}{3}$$



(8) [10pts]

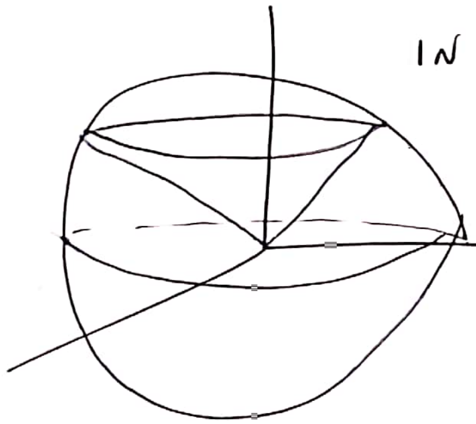
(a) Sketch the curve obtained by intersecting the surfaces whose equations in spherical coordinates are given by $\rho = 2$ and $\phi = \pi/3$. Write down a parameterization of the form $(x, y, z) = \mathbf{r}(t)$ for this curve.

Set $\rho = 2$, $\phi = \pi/3$, $\theta = t$

IN SPH $\to$ RECT COORD CHANGE

$\rho = 2$ = SPHERE

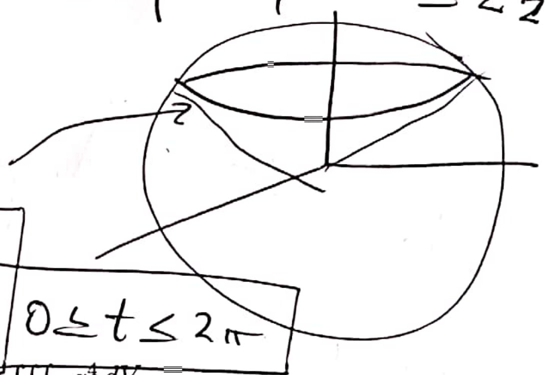
$\phi = \pi/3$ = CONE



$x = \rho \sin\phi \cos\theta = 2\frac{\sqrt{3}}{2}\cos t$

$y = \rho \sin\phi \sin\theta = 2\frac{\sqrt{3}}{2}\sin t$

$z = \rho\cos\phi \quad = 2\frac{1}{2} = 1$



CURVE = CIRCLE ON SPHERE

$$\boxed{(x, y, z) = (\sqrt{3}\cos t, \sqrt{3}\sin t, 1) \quad 0 \le t \le 2\pi}$$

(b) Let E be the solid region $x^2 + y^2 + z^2 \le 16$. Calculate $\iiint_E z^4 \, dV$.

E is

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$0 \le \rho \le 4$

$$\iiint_E z^4 dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{4} (\rho\cos\phi)^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_0^{\pi} \cos^4\phi \sin\phi \, d\phi \int_0^4 \rho^6 d\rho$$

$$= 2\pi \left[\frac{u^5}{5}\right]_{-1}^{1} \left[\frac{\rho^7}{7}\right]_0^4 = \frac{4\pi}{5} \cdot \frac{4^7}{7}$$

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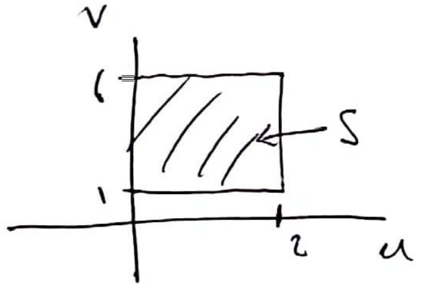
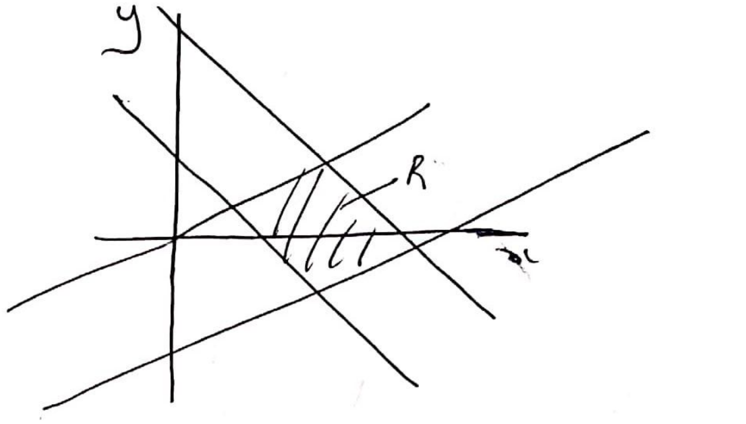
$$= \frac{1}{25}\int_{u=0}^{2}\left(-5u + \frac{3\times 35}{2}\right)du = \frac{1}{25}\left[-\frac{5}{2}u^2 + \frac{105}{2}u\right]_0^2$$

$$= \frac{1}{25}\left(-\frac{5}{2}4 + \frac{105}{2}2\right) = \frac{1}{25}(-10+105) = \frac{95}{25} = \frac{19}{5}$$

(9) [10 pts] Use an appropriate change of variables to evaluate $\iint_R x\,dA$, where R is the parallelogram bounded by the lines $x-3y=0$, $x-3y=2$, $2x-y=1$, and $2x-y=6$.

Let $\boxed{\begin{array}{l} u = x-3y \\ v = 2x-y \end{array}}$ ① ②





SOLVE FOR (x,y) IN TERMS OF (u,v):

$$2u+v = 2x-6y+2x-y = -7y \Rightarrow y = -\frac{1}{7}(2u+v)$$

$$u-3v = x-3y-6x+3y = -5x \Rightarrow x = -\frac{1}{5}(u-3v)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{3}{5} \\ -\frac{2}{7} & -\frac{1}{7} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \\ -2/7 & -1/7 \end{bmatrix}$$

$$= \frac{1}{35} + \frac{6}{35} = \frac{1}{5}$$

So

$$\iint_R x\,dx\,dy = \iint_S x(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_{u=0}^{2}\int_{v=1}^{6}\left(-\frac{u}{5}+\frac{3v}{5}\right)\frac{1}{5}\,du\,dv = \frac{1}{25}\int_{u=0}^{2}\int_{v=1}^{6}(-u+3v)\,dv\,du$$

$$= \frac{1}{25}\int_{u=0}^{2}\left[-uv+\frac{3}{2}v^2\right]_{v=1}^{v=6}du = \frac{1}{25}\int_{u=0}^{2}\left(-6u+\frac{3}{2}36\right)-\left(-u+\frac{3}{2}\right)du$$

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(10) [10 pts] Let $\mathbf{F}(x, y) = x^3\mathbf{i} + y^3\mathbf{j}$ be the velocity vector field of a fluid flowing in $\mathbb{R}^2$.

(a) Calculate $\nabla \cdot \mathbf{F}$.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) = 3x^2 + 3y^2$$

(b) Calculate $\nabla \times \mathbf{F}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}$$

(c) On average, is the fluid rotating clockwise, counter-clockwise, or not rotating at all about the point $(1, 2)$? Why?

NOT ROTATING AT ALL as $\nabla \times \vec{F} = \vec{0}$.

(d) On average, is the fluid flowing in, out, or neither in or out, of a small disc centered at $(1, 2)$? Why?

$$(\nabla \cdot \vec{F})(1,2) = 3(1^2 + 2^2) = 15 > 0$$

So OUT on average