LAST NAME: SOLUTIONS	FIRST NAME:	CIRCLE: Martynova 8:30am / Martynova 1pm / Zweck

1 /7	2 /10	3 /12	4 /10	5 /12	6 /12	7 /12	T /75

MATH 2415 (Spring 2017) Exam II, Mar 31st

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 90 minute exam is worth 75 points.

(1) [7 pts] Find the equation of the tangent plane to the surface $z = e^x \cos(xy)$ at $(x, y, z) = (1, \pi/2, 0)$.

$$f(x,y) = e^x \cos(xy)$$

$$\frac{\partial f}{\partial x} = e^x \cos(xy) - e^x y \sin(xy)$$

$$= e^1 \cos(\pi/2) - e^1 \pi/2 \sin \pi/2 = -\frac{e\pi}{2}$$

at $(1, \pi/2)$

$$\frac{\partial f}{\partial y} = -xe^x \sin(xy) = -e \text{ at } (1, \pi/2)$$

Equation of Tangent Plane is

$$z = f(1, \pi/2) + \frac{\partial f}{\partial x}(1, \pi/2)(x-1) + \frac{\partial f}{\partial y}(1, \pi/2)(y - \pi/2)$$

$$\boxed{z = 0 - \frac{e\pi}{2}(x-1) - e(y - \pi/2)}$$

(2) [10 pts] Let $f(x,y) = xy^2$.

(a) Find the direction in which f increases most rapidly at the point $(x,y) = (2,3)$. What is the rate of change of f in this direction?

Direction of most rapid increase is $\vec{v} = \dfrac{\nabla f(2,3)}{|\nabla f(2,3)|}$

NOW

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y^2, 2xy)$$

$$= (9, 12) \quad @ \ (2,3)$$

So $\vec{v} = \dfrac{(9,12)}{\sqrt{9^2+12^2}} = \left(\dfrac{9}{15}, \dfrac{12}{15}\right)$

Rate of change of f in this direction is

$|\nabla f(2,3)| = 15.$

(b) In what directions is the rate of change of f equal to zero at the point $(2,3)$?

We know

$$(D_{\vec{u}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}.$$

We want $\vec{u} = (a,b)$ so that $D_{\vec{u}} f(2,3) = 0.$

So

$$0 = (9, 12) \cdot (a, b)$$

So $(a,b) = \dfrac{(-12, 9)}{15}$ or $\dfrac{(12, -9)}{15}$

(3) [12 pts] (a) Suppose that $z = f(x,y) = \sin(3x^2+4y^2)$ where $x = x(t)$ and $y = y(t)$. If $x(0) = 2$, $y(0) = 1$, $x'(0) = 3$, and $y'(0) = -4$, find $\frac{dz}{dt}$ at $t = 0$.

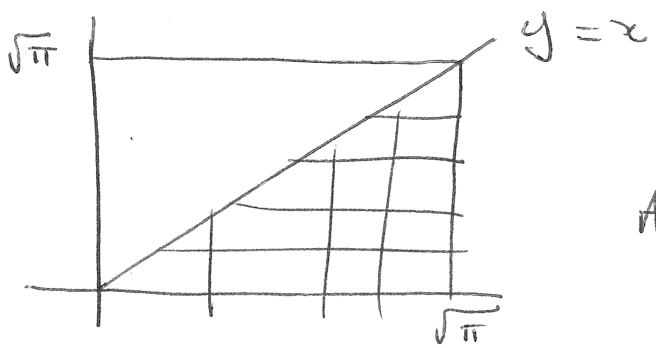
By Chain Rule for Functions on Curves

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad \cancel{\text{then evaluate}}$$

NOW
$$\frac{\partial z}{\partial x} = 6x\cos(3x^2+4y^2) = 6.2\cos(3.4+4.1) \ @(2,1)$$
$$\frac{\partial z}{\partial y} = 8y\cos(3x^2+4y^2) = 8.1.\cos(3.4+4.1) \ @(2,1)$$

So
$$\frac{dz}{dt}(0) = \frac{\partial z}{\partial x}(x(0),y(0))\,x'(0) + \frac{\partial z}{\partial y}(x(0),y(0))\,y'(0)$$
$$= 12\cos(16).3 - 8\cos(16)4 = 4\cos 16$$

(b) Evaluate the iterated integral $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\,dx\,dy$ by reversing the order of integration.



$0 \le y \le \sqrt{\pi}$ TYPE II

$y \le x \le \sqrt{\pi}$

As TYPE I:

$0 \le x \le \sqrt{\pi}$

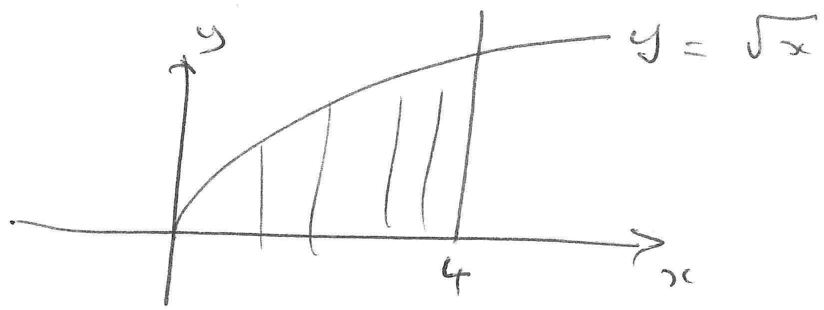
$0 \le y \le x$

So
$$\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}} \cos(x^2)\,dx\,dy = \int_{x=0}^{\sqrt{\pi}}\int_{y=0}^{x} \cos(x^2)\,dy\,dx$$
$$= \int_{x=0}^{\sqrt{\pi}} \cos(x^2)\left(\int_{y=0}^{x} 1\,dy\right)dx = \int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$$
$$\overset{u=x^2}{=} \frac{1}{2}\int_{u=0}^{\pi} \cos(u)\,du = \frac{\sin u}{2}\Big]_0^{\pi} = 0$$

(4) [10 pts] Find the volume of the solid under the surface $z = xy+1$ and above the region in the (x, y)-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 4$.

$$\text{VOL} = \iint_D (xy+1)\, dA$$

where D is domain



$0 \le x \le 4$

$0 \le y \le \sqrt{x}$

So

$$\text{VOL} = \int_{x=0}^{4} \int_{y=0}^{\sqrt{x}} (xy+1)\, dy\, dx$$

$$= \int_{x=0}^{4} \left[\frac{xy^2}{2} + y\right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^4 \frac{x^2}{2} + \sqrt{x}\, dx$$

$$= \left[\frac{x^3}{6} + \frac{2}{3} x^{3/2}\right]_0^4$$

$$= \frac{96}{6} = 16$$

(5) [12 pts] Let S be the surface with parametrization

$$(x, y, z) = \mathbf{r}(u, v) = u \cos v\, \mathbf{i} + 2u \sin v\, \mathbf{j} + u\, \mathbf{k} \qquad \text{for } 0 \le u \le 2 \text{ and } 0 \le v \le \pi/2.$$

(a) Find an equation of the form $F(x, y, z) = 0$ for this surface.

$x = u \cos V \implies \cos V = \frac{x}{u} = \frac{x}{z}$

$y = 2u \sin V \implies \sin V = \frac{y}{2u} = \frac{y}{2z}$

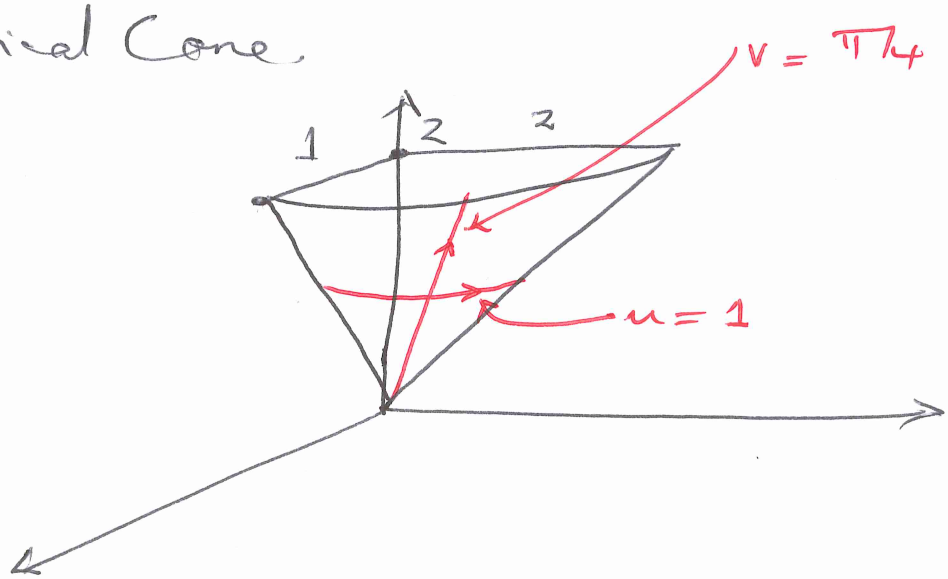
$z = u$.

So $1 = \cos^2 V + \sin^2 V = \left(\frac{x}{z}\right)^2 + \left(\frac{y}{2z}\right)^2$

OR $z^2 = x^2 + \left(\frac{y}{2}\right)^2$, $0 = F(x,y,z) = z^2 - x^2 - \left(\frac{y}{2}\right)^2$

(b) Sketch the surface, S, together with the "grid" curves on S where (i) $u = 1$ and (ii) $v = \pi/4$.

1/4 Elliptical Cone.



(c) Calculate the tangent vector to the grid curve $u = 1$ at the point where $(u, v) = (1, \pi/4)$.

Grid Curve $u = 1$ has parametrization

$\vec{r}(t) = \vec{r}(1, t) = (\cos t, 2\sin t, 1)$

So tangent vector is

$\vec{V} = \vec{r}\,'(\pi/4) = (-\sin \pi/4, 2\cos \pi/4, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 0\right)$

(6) [12 pts] Find and classify all critical points of $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

$$0 = \frac{\partial f}{\partial x} = 6xy - 6x = 6x(y-1) \quad (1)$$

$$0 = \frac{\partial f}{\partial y} = 3x^2 + 3y^2 - 6y \quad (2)$$

By (1) $x = 0$ or $y = 1$

$\boxed{x=0}$ By (2) $0 = 3y^2 - 6y = 3y(y-2)$

So $y = 0$ or $y = 2$

CPTS : $(x,y) = (0,0)$, $(0,2)$

$\boxed{y=1}$ By (2) $3x^2 + 3 - 6 = 0$, $3x^2 = 3$, $x = \pm 1$

CPTS $(1,1)$, $(-1,1)$

$$D = \det\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \det\begin{bmatrix} 6(y-1) & 6x \\ 6x & 6(y-1) \end{bmatrix} = 36[(y-1)^2 - x^2]$$

$f_{xx} = 6(y-1)$

CPT	SIGN(D)	SIGN (f_{xx})	CLASSIFICATION
(0,0)	+	−	MAX
(0,2)	+	+	MIN
(1,1)	−		SADDLE
(−1,1)	−		SADDLE

(7) [12 pts] Use the method of Lagrange multipliers to find the absolute maximum and absolute minimum of the function $f(x, y) = x^2 + y^2$ on the ellipse $x^2 + 2(y+1)^2 = 8$.

$g(x,y) = x^2 + 2(y+1)^2$

$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \implies 2x = \lambda 2x$ ①

$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial x} \implies 2y = \lambda(4y+4)$ ②

$g = k \implies x^2 + 2(y+1)^2 = 8$ ③

By ① $0 = x - \lambda x = x(1-\lambda)$ So $x = 0$ or $\lambda = 1$

$\boxed{\text{CASE } x=0}$ By ③ $(y+1)^2 = 4$

$y + 1 = \pm 2$

$y = 1$ or $y = -3$.

So $(x,y) = (0,1), (0,-3)$

$\boxed{\text{CASE } \lambda = 1}$ By ② $2y = 4y + 4 \implies y = -2$

By ③ $x^2 = 8 - 2 = 6 \implies x = \pm\sqrt{6}$

So $(x,y) = (+\sqrt{6}, -2), (-\sqrt{6}, -2)$

CPT	$f(x,y)$	CLASSIFICATION
$(0,1)$	1	ABS MIN
$(0,-3)$	9	—
$(\sqrt{6}, -2)$	10	ABS MAX
$(-\sqrt{6}, -2)$	10	ABS MAX