NAME: SOLUTIONS	CIRCLE:	Zweck 11:30am	Zweck 2:30pm

1 /12	2 /15	3 /10	4 /12	5 /10	6 /4	7 /12	T /75

MATH 2415 (Fall 2015) Exam II, Nov 6th

No books or notes! You may use a scientific calculator provided it does not allow for access to the internet. Show all work and give **complete explanations**. This is 75 minute exam.

(1) [12 pts]
(a) Calculate the directional derivative of the function $f(x, y) = x^2y^3$ in the direction of the vector $\mathbf{v} = (3, 4)$ at the point $(x, y) = (1, 2)$.

$$\nabla f(x,y) = (2xy^3, 3x^2y^2)$$

$$\nabla f(1,2) = (16, 12)$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{5}(3,4)$$

$$(D_{\vec{u}} f)(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u} = \frac{1}{5}(16, 12) \cdot (3,4) = \frac{96}{5}$$

(b) Let $f(x, y) = y - x^2$. Find the gradient of f at the point $(1, 3)$. Sketch the level curve of f through the point $(1, 3)$, together with the gradient at that point.

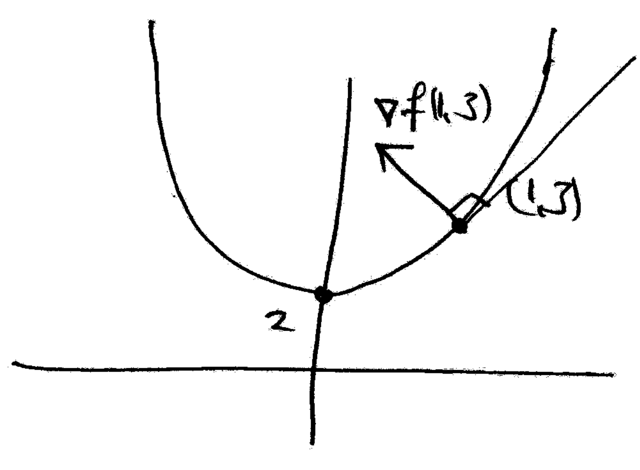
$$\nabla f(x,y) = (-2x, 1) = (-2, 1) \text{ @ } (x,y) = (1,3)$$

LEVEL CURVE

$$y - x^2 = k \text{ where } k = f(1,3) = 2$$

So $y = x^2 + 2$

$\nabla f(1,3)$ IS $\perp$ to LEVEL CURVE AND POINTS IN DIRECTION OF INCREASING f.



(2) [15 pts] Suppose that $z = f(x, y)$ is a function such that

(a,b)	$f(a,b)$	$\nabla f(a,b)$	$f_{xx}(a,b)$	$f_{xy}(a,b)$	$f_{yy}(a,b)$
$(1,2)$	0	(0,0)	5	3	1
$(7,-2)$	0	(0,1)	5	3	1
$(3,4)$	7	(0,0)	-5	-3	-2
$(5,-3)$	68	(0,0)	8	-4	2
$(2,1)$	35	(0,0)	5	3	2

Identify any local maxima, minima, and saddle points of f. Explain the reasons for your answers.

(1,2) CRITICAL POINT ✓

$D = \det\begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix} = 5 \times 1 - 3 \times 3 = -4 < 0$

SADDLE POINT

(7,-2) $\nabla f(7,-2) = (0,1) \neq (0,0)$ NOT CPT.

(3,4) CPT

$D = \det\begin{pmatrix} 5 & -3 \\ -3 & -2 \end{pmatrix} = 10 - 9 = 1 > 0$

$f_{xx} = -5 < 0$ LOCAL MAX

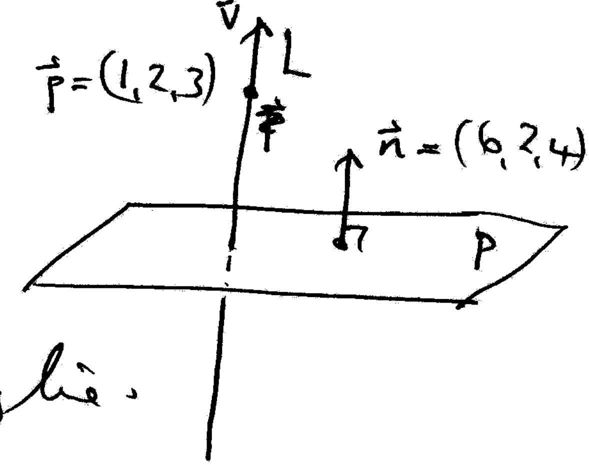
(5,-3) CPT $D = \det\begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} = 16 - 16 = 0$

NO CONCLUSION POSSIBLE

(2,1) CPT $D = \det\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = 10 - 9 = 1 > 0$

$f_{xx} = +5 > 0$ LOCAL MIN

(3) [10 pts]
(a) Find a parametrization of the line that contains the point $(1, 2, 3)$ and is perpendicular to the plane $6(x-1) + 2(y+3) + 4(z-8) = 0$.



Since Line is perpendicular to Plane and normal to plane is

$\vec n = (6,2,4)$ we know

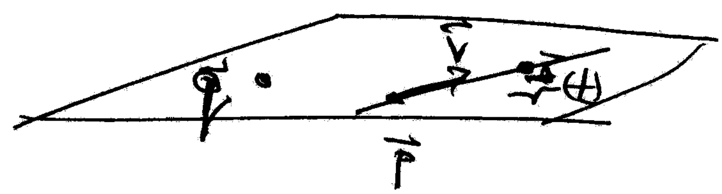
$\vec v = \vec n = (6,2,4)$ is a vector along line.

Given $\vec p = (1,2,3)$ is point on line

So $\vec r(t) = \vec p + t\vec v = (1,2,3) + t(6,2,4)$

(b) Find an equation of the form $Ax + By + Cz = D$ for the plane that contains both the point $(2, 4, 6)$ and line with parametrization $\mathbf{r}(t) = (7 - 3t, 3 + 4t, 5 + 2t)$.

$\vec q = (2,4,6)$ is point in plane



$\vec r(t) = \vec p + t\vec v$

$= (7,3,5) + t(-3,4,2)$

is line in plane.

So two vectors in plane are $\vec v = (-3,4,2)$ and

$\vec w = \vec q - \vec p = (7,3,5) - (2,4,6) = (5,-1,-1)$

So normal to plane is

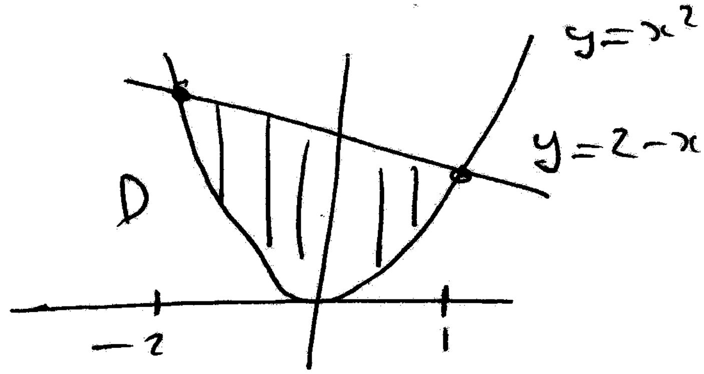
$$\vec n = \vec v \times \vec w = \begin{vmatrix} \vec i & \vec j & \vec k \\ -3 & 4 & 2 \\ 5 & -1 & -1 \end{vmatrix} = (-2, 7, -17)$$

So

$0 = \vec n \cdot (\vec x - \vec p) = -2(x-2) + 7(y-4) - 17(z-6)$

or $2x - 7y + 17z = 78$.

(4) [12 pts] Let D be the domain the the xy-plane that is bounded by the curves $y = x^2$ and $y = 2 - x$. Calculate $\iint_D x\,dA$.



The 2 curves meet at

$$x^2 = 2 - x$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$x = -2 \text{ or } x = +1$$

Type I Region: $\boxed{\begin{matrix} -2 \le x \le 1 \\ x^2 \le y \le 2-x \end{matrix}}$

So

$$\iint_D x\,dA = \int_{x=-2}^{x=+1} \int_{y=x^2}^{y=2-x} x\,dy\,dx$$

$$= \int_{x=-2}^{x=+1} [xy]_{y=x^2}^{y=2-x}\,dx$$

$$= \int_{-2}^{1} (2x - x^2 - x^3)\,dx$$

$$= \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_{-2}^{1}$$

$$= -9/4.$$

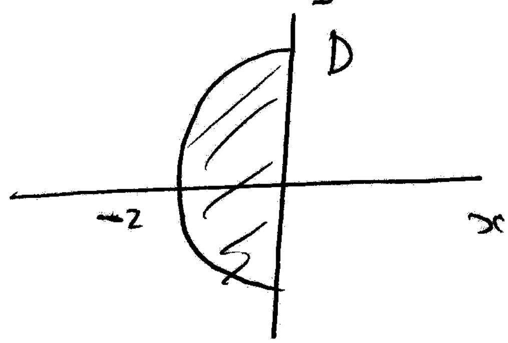
NOTE

Average of $f(x,y) = x$ on D is

$$\frac{\iint_D x\,dA}{\text{Area }(D)} < 0$$

is easy to see from picture.

(5) [10 pts] Let D be the domain the the xy-plane described by the inequalities $x^2 + y^2 \leq 4$ and $x \leq 0$. Calculate $\iint_D x dA$.



In polar coords

$$0 \leq r \leq 2$$

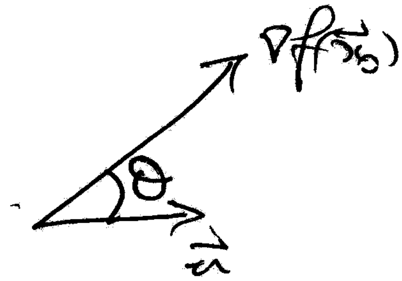
$$\pi/2 \leq \theta \leq \frac{3\pi}{2}$$

$$\iint_D x \, dA = \int_{\theta=\pi/2}^{3\pi/2} \int_{r=0}^{2} (r\cos\theta)\, r \, dr \, d\theta$$

$$= \left[\int_{\pi/2}^{3\pi/2} \cos\theta \, d\theta\right] \left[\int_0^2 r^2 \, dr\right]$$

$$= [\sin\theta]_{\pi/2}^{3\pi/2} \left[\frac{r^3}{3}\right]_0^2 = -2 \cdot \frac{8}{3} = \frac{-16}{3}.$$

(6) [4 pts] Prove that the directional derivative of a function $z = f(x, y)$ at a point (x_0, y_0) is maximized when the derivative is taken in the direction of the gradient of f at (x_0, y_0).



$$(D_{\vec{u}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}$$

$$= |\nabla f(\vec{x}_0)| \, |\vec{u}| \cos\theta$$

$$= |\nabla f(\vec{x}_0)| \cos\theta \quad \text{as } |\vec{u}| = 1$$

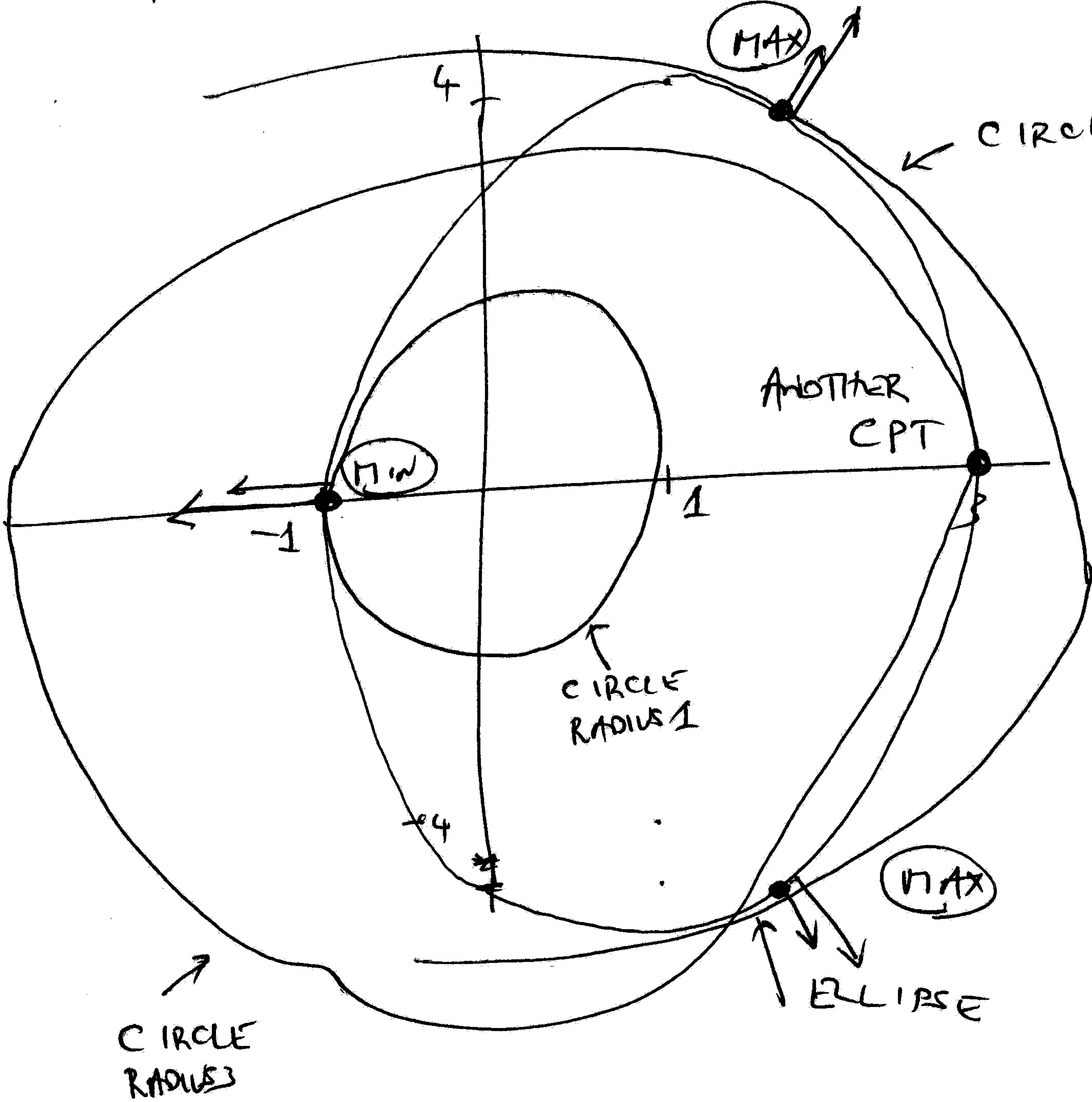
so biggest when $\cos\theta = 1$, $\theta = 0$

So $\vec{u} \parallel \nabla f(\vec{x}_0)$

(7) [12 pts] In this problem you will use the **method of Lagrange Multipliers two different ways** to solve the same problem. The problem is to find the absolute maximum and absolute minimum of the function $f(x, y) = x^2 + y^2$ on the ellipse $4(x-1)^2 + y^2 = 16$. [*Hint: There are four critical points.*]

(a) First, solve the problem **graphically** by sketching the ellipse and some appropriately chosen level curves, $f(x, y) = k$. (This approach will enable you to find the approximate but not necessarily the exact locations and values of the maxima and minima.)

Ellipse has center $(1, 0)$.



(b) Now, solve the problem exactly by setting up the appropriate **equations** and solving them algebraically.

$$\left\{ \begin{array}{l} \nabla f = \lambda \nabla g \\ g = c \end{array} \right\}$$

$$2x = \lambda\, 8(x-1) \quad (1)$$

$$2y = \lambda\, 2y \quad (2)$$

$$4(x-1)^2 + y^2 = 16 \quad (3)$$

By (2) $\quad y(\lambda - 1) = 0$

So $\quad y = 0$ or $\lambda = 1$

$\boxed{y=0}$ By (3)

$$(x-1)^2 = 4$$

$$x - 1 = \pm 2$$

$$x = 3 \text{ or } x = -1$$

$(3,0)$ $\quad f(3,0) = 9$

$(-1,0)$ $\quad f(-1,0) = 1$

GLOBAL MIN.

OR

$\boxed{\lambda = 1}$

By (1) $\quad x = 4x - 4$

So $\quad x = \frac{4}{3}$

By (3) $\quad y^2 = 16 - \frac{4}{9}$

$$y = \pm \frac{\sqrt{140}}{3}$$

So

$$f\left(\frac{4}{3}, \pm\frac{\sqrt{140}}{3}\right) = 17\tfrac{1}{3}$$

TWO GLOBAL MAX

Please sign the following honor statement:

On my honor, I pledge that I have neither given nor received any aid on this exam.

Signature: ______________________________