LAST NAME:	FIRST NAME:	CIRCLE:		
SOLUTIONS		Zweck 10:00am	Khafizov 11:30am	Khafizov 2:30pm

1 /12	2 /12	3 /12	4 /15	5 /12	6 /12	T /75

MATH 2415 (Spring 2016) Exam II, Apr 1st

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 90 minute exam is worth 75 points.

(1) [12 pts] Let $z = f(x, y) = x^2 + 4xy - y^3 + x + y$.

(a) Find the direction of steepest ascent and the maximum rate of change of f at $(x, y) = (2, -1)$.

Final Answer:

$$\nabla f = (2x + 4y + 1,\ 4x - 3y^2 + 1)$$

$$\nabla f(2,-1) = (4 - 4 + 1,\ 8 - 3 + 1) = (1, 6)$$

Direction of Steepest Ascent $= \dfrac{\nabla f(2,-1)}{|\nabla f(2,-1)|} = \dfrac{1}{\sqrt{37}}(1, 6)$

Max Rate of Change of f $= |\nabla f(2,-1)| = \sqrt{37}$

(b) Find the directional derivative of f in the direction of the vector $\mathbf{v} = (-3, 4)$ at the point $(x, y) = (2, -1)$.

Final Answer:

$$(D_{\vec{v}/|\vec{v}|} f)(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{v}/|\vec{v}|$$

$$\left(D_{\frac{(-3,4)}{5}} f\right)(2,-1) = \nabla f(2,-1) \cdot \frac{(-3,4)}{5}$$

$$= \frac{1}{5}(1,6) \cdot (-3,4) = \frac{1}{5}(-3+24) = \frac{21}{5}$$

(2) [12 pts] Find the limit if it exists, or show that the limit does not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

Final Answer:

$$= \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^2}$$

$$= \lim_{r\to 0} r\cos^2\theta\sin\theta = 0 \quad \underline{\text{EXIST}}$$

since $-r \le r\cos^2\theta\sin\theta \le r$ as $r\to 0$

$-r \to 0$, $r\cos^2\theta\sin\theta \to 0$, $r \to 0$

by Squeeze Theorem

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2+2y}{\sqrt{x^2+y^2}}$

Final Answer:

$\lim_{(x,y)\to(0,0)}$ Along $y=0$

$$\frac{x^2+2y}{\sqrt{x^2+y^2}} = \lim_{x\to 0}\frac{x^2}{|x|} = \lim_{x\to 0}\frac{|x|^2}{|x|}$$

$$= \lim_{x\to 0}|x| = 0$$

<u>BUT</u>

$\lim_{(x,y)\to(0,0)}$ Along $x=0$ $(y>0)$

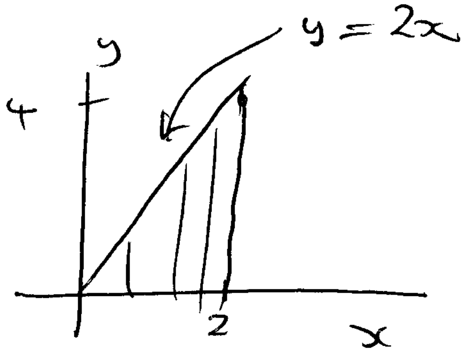
$$\frac{x^2+2y}{\sqrt{x^2+y^2}} = \lim_{y\to 0^+}\frac{2y}{|y|} = \lim_{y\to 0^+} 2\cdot\frac{1}{|1|} = 2$$

Since $0 \neq 2$ LIMIT DNE

(3) [12 pts] Find $\iint_D xy^2\, dA$ where D is the triangular domain with vertices $(0, 0)$, $(2, 0)$ and $(2, 4)$.

Final Answer:

D is Type I



$0 \leq x \leq 2$

$0 \leq y \leq 2x$

So

$$\iint_D xy^2\, dA = \int_{x=0}^{2} \int_{y=0}^{2x} xy^2\, dy\, dx$$

$$= \int_{x=0}^{2} x \left[\frac{y^3}{3}\right]_{y=0}^{y=2x} dx$$

$$= \int_{x=0}^{2} x\, \frac{(2x)^3}{3}\, dx$$

$$= \frac{8}{3} \int_0^2 x^4\, dx$$

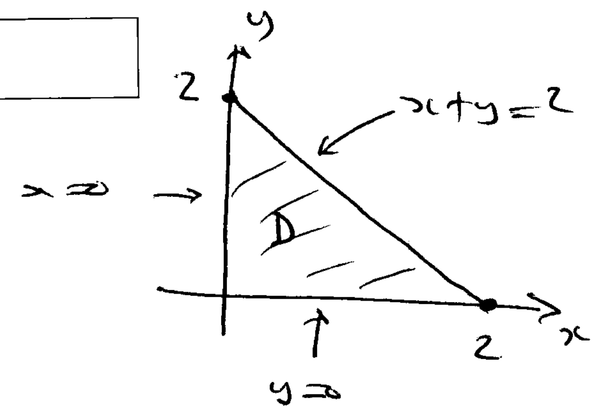
$$= \frac{8}{3} \left[\frac{x^5}{5}\right]_0^2 = \frac{8 \times 32}{15} = \frac{256}{15}$$

(4) [15 pts] Find the absolute maximum and absolute minimum of the following function

$$z = f(x, y) = 2xy - 2x^2 - 5y^2 + 4x + 4y - 4$$

on the triangular domain with vertices $(0, 0)$, $(2, 0)$ and $(0, 2)$.

Final Answer:



① Critical Points of f in D

$$0 = \frac{\partial f}{\partial x} = 2y - 4x + 4 \quad ①$$

$$0 = \frac{\partial f}{\partial y} = 2x - 10y + 4 \quad ②$$

Subtract:

$$\begin{array}{r} 2x - y = 2 \\ 2x - 10y = -4 \\ \hline 9y = 6 \\ y = \frac{2}{3} \end{array}$$

Plug $y = \frac{2}{3}$ into ① to get $x = \frac{y}{2} + 1 = \frac{1}{3} + 1 = \frac{4}{3}$

So CPT at $(x, y) = \left(\frac{4}{3}, \frac{2}{3}\right)$

NOTE SINCE $\frac{4}{3} + \frac{2}{3} = 2$ this critical point lies on the edge $x + y = 2$ of D.

$$f\left(\frac{4}{3}, \frac{2}{3}\right) = 2\frac{8}{9} - 2\frac{16}{9} - 5\frac{4}{9} + \frac{16}{3} + \frac{8}{3} - 4$$

$$= \frac{16 - 32 - 20}{9} + \frac{24 - 12}{3} = -\frac{36}{9} + \frac{12}{3}$$

$$= -4 + 4 = 0.$$

(x,y)	$f(x,y)$
$\left(\frac{4}{3}, \frac{2}{3}\right)$	0 ABS MAX
$\left(0, \frac{2}{5}\right)$	$\frac{-16}{5}$
$(0,0)$	-4
$(0,2)$	-16 ABS MIN
$(1,0)$	-2
$(2,0)$	-4

Use this page if additional space is needed for the solution of Problem 4:

(2) ON $\boxed{x=0, \quad 0 \le y \le 2}$

$$g(y) = f(0,y) = -5y^2 + 4y - 4$$

$$0 = g'(y) = -10y + 4 \Rightarrow y = \frac{2}{5}.$$

So $$f\left(0, \frac{2}{5}\right) = -5\frac{4}{25} + 4\frac{2}{5} - 4 = \frac{-16}{5}$$

ENDPOINTS $f(0,0) = -4$, $f(0,2) = -20 + 8 - 4 = -16$

(3) ON $\boxed{y=0, \quad 0 \le x \le 2}$

$$h(x) = f(x,0) = -2x^2 + 4x - 4$$

$$0 = h'(x) = -4x + 4 \Rightarrow x = 1$$

$$f(1,0) = -2 + 4 - 4 = -2$$

ENDPTS

$f(0,0) = -4$

$f(2,0) = -8 + 8 - 4$
$= -4$

(4) ON $\boxed{x+y=2 \quad 0<x<2}$ $y = 2-x$

$$k(x) = f(x, 2-x) = 2x(2-x) - 2x^2 - 5(2-x)^2 + 4x + 4(2-x) - 4$$
$$= -9x^2 + 24x - 24$$

$0 = k'(x) = -18x + 24 \Rightarrow x = \frac{4}{3}.$ Get $y = 2 - x = \frac{2}{3}.$

So Same CPT as in (1).

(5) [12 pts] Consider the surface that is parametrized by

$$(x, y, z) = \mathbf{r}(\theta, \phi) = (3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi).$$

(a) Find an equation of the form $F(x, y, z) = 0$ for this surface.

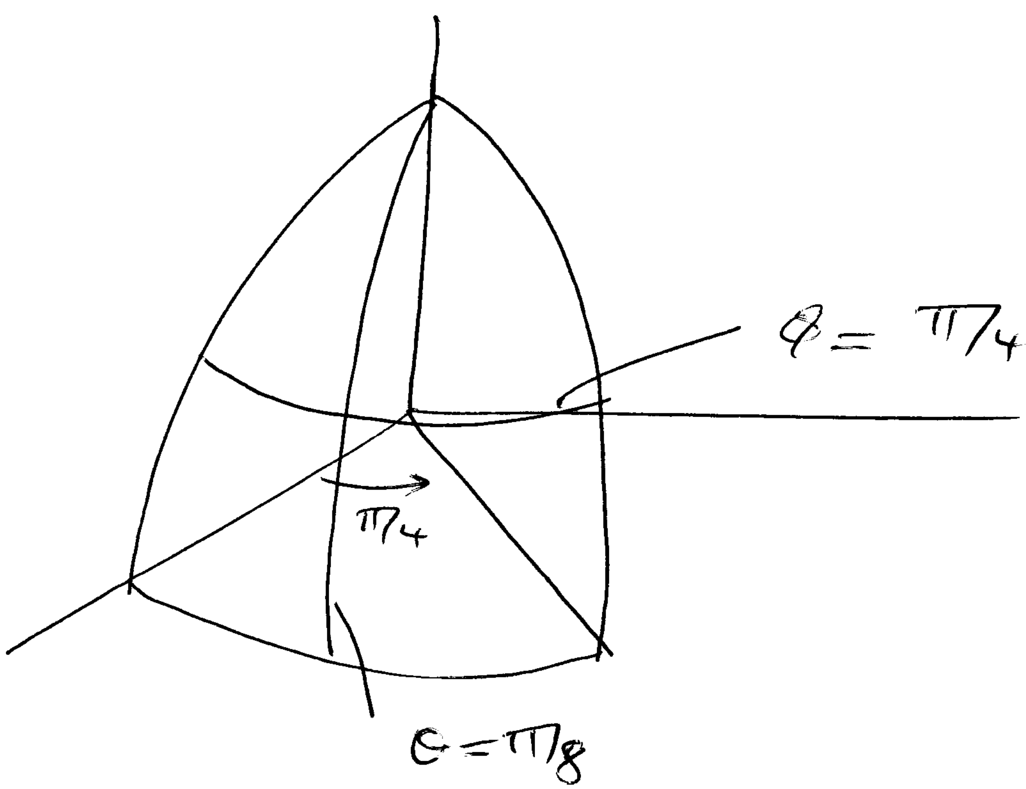
$$x^2+y^2+z^2 = 9(\sin^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta+\cos^2\phi)$$

$$= 9(\sin^2\phi+\cos^2\phi) = 9$$

$$F(x,y,z) = x^2+y^2+z^2-9=0$$

Sphere Radius 3, Center O.

(b) Sketch the graph of that portion of the surface given above for which $0 \leq \theta \leq \frac{\pi}{4}$ and $0 \leq \phi \leq \frac{\pi}{2}$. Also sketch the grid curves $\theta = \frac{\pi}{8}$ and $\phi = \frac{\pi}{4}$ on the surface.



(6) [12 pts] Use the method of Lagrange Multipliers to find the absolute maximum and absolute minimum of the function $z = f(x, y) = x^2y^2$ subject to the constraint $x^2 + 4y^2 = 1$.

$f(x,y) = x^2y^2$

$g(x,y) = x^2+4y^2 = 1$

$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \Rightarrow 2xy^2 = \lambda 2x$ ① $\Rightarrow x(\cancel{y}^2 - \lambda) = 0 \Rightarrow x = 0$ OR $y = \pm\sqrt{\lambda}$

$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \Rightarrow 2x^2y = \lambda 8y$ ②

$g(x,y) = 1 \Rightarrow x^2 + 4y^2 = 1$ ③

From ①, as above we get $x = 0$ OR $y = \pm\sqrt{\lambda}$.

$\boxed{x=0}$ By ③ $4y^2 = 1$, $y = \pm\frac{1}{2}$

By ② $0 = \pm\lambda 4 \Rightarrow \lambda = 0$

$(x, y, \lambda) = (0, \pm\frac{1}{2}, 0)$ $\quad f(0, \pm\frac{1}{2}) = 0$

$\boxed{y = \pm\sqrt{\lambda}}$ By ② $(x^2 - 4\lambda)y = 0$ So $y = 0$ OR $x^2 = 4\lambda$

With $\boxed{y=0}$ Get by ③ $x = \pm 1$ and $\lambda = y^2 = 0$

$(x, y, \lambda) = (\pm 1, 0, 0)$ $\quad f(\pm 1, 0) = 0$.

With $\boxed{x^2 = 4\lambda}$ By ③ $4\lambda + 4\lambda = 1$, $\lambda = \frac{1}{8}$.

So $x^2 = 4\frac{1}{8} = \frac{1}{2}$, $x = \pm\frac{1}{\sqrt{2}}$, $y^2 = \frac{1}{8}$, $y = \pm\frac{1}{\sqrt{8}}$

Please sign the following honor statement:

On my honor, I pledge that I have neither given nor received any aid on this exam.

Signature:

Get $(x, y, \lambda) = \left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{8}}, \frac{1}{8}\right)$, $f\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{8}}\right) = \frac{1}{16}$

and $(x, y, \lambda) = \left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{8}}, \frac{1}{8}\right)$, $f\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{8}}\right) = \frac{1}{16}$

Y.C.V ("You can check") that all the 8 critical points we have found satisfy ①, ②, ③.

<u>ABS MAX</u> is $\frac{1}{16}$ at $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{8}}\right)$

and $\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{8}}\right)$

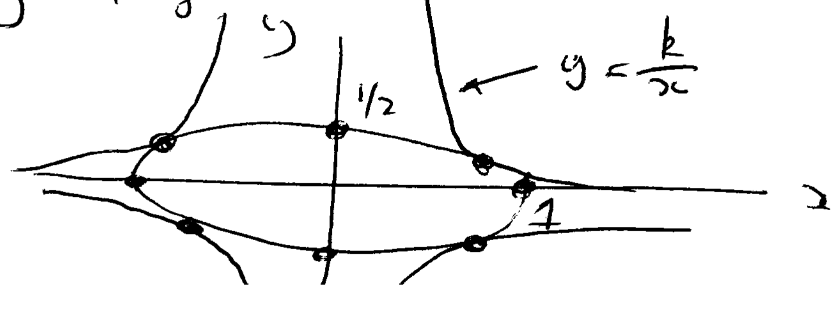
<u>ABS MIN</u> ~~at~~ is 0 at $(\pm 1, 0)$ and $\left(0, \pm\frac{1}{2}\right)$

<u>NOTE</u> AT $(\pm 1, 0)$ and $\left(0, \pm\frac{1}{2}\right)$ we have $\lambda = 0$

So $\nabla f = \lambda \nabla g = 0$. These are critical pts of f in $\mathbb{R}^2$, not just on curve $x^2 + 4y^2 = 1$.

So don't have ~~tangent~~ level curve of f and level curve of g tangent at these 4 points.

$\boxed{\text{See 8 CPs!}}$



$f = k^2$

is $x^2y^2 = k^2$

$y = \pm\frac{k}{x}$