NAME: SOLUTIONS	CLASS: 11:30am OR 4pm

1 /14	2 /12	3 /10	4 /12	
5 /15	6 /12	7 /15	8 /10	T /100

MATH 2415 (Fall 2012) Exam II, Nov 9th

No calculators, books or notes! Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 2 hour exam is worth XX points.

(1) [14 pts] Let $z = f(x, y) = 1 + e^x + 3\sin y + x^2y^3$.

(a) Calculate the equation of the tangent plane to the graph of f at $(x, y) = (0, 0)$.

$$\frac{\partial f}{\partial x} = e^x + 2xy^3 = 1 \text{ at } (0,0)$$

$$\frac{\partial f}{\partial y} = 3\cos y + 3x^2y^2 = 3 \text{ at } (0,0)$$

$$f(0,0) = 1 + 1 = 2$$

So $z = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0)$

$$\boxed{z = 2 + x + 3y}$$

(c) What is the maximum rate of change of f at $(0, 0)$ and in which direction does it occur?

$$\nabla f(0,0) = (1,3)$$

$\vec{u} = \frac{\nabla f(0,0)}{|\nabla f(0,0)|} = \frac{(1,3)}{\sqrt{10}}$ is dir

$|\nabla f(0,0)| = \sqrt{10} =$ Max R of C

(2) [12 pts]
(a) Let $z = f(x, y)$ be a function so that

x	1	2	3	2	2
y	5	5	5	4	6
$f(x,y)$	3	4	6	2	7

Estimate $\frac{\partial f}{\partial y}$ at $(x, y) = (2, 5)$.

$$\frac{\partial f}{\partial y}(2,5) \simeq \frac{f(2,5+1) - f(2,5)}{1} = f(2,6) - f(2,5)$$

$$= 7 - 4 = 3$$

OR $$\frac{\partial f}{\partial y}(2,5) \simeq \frac{f(2,6) - f(2,4)}{2} = \frac{7-2}{2} = \frac{5}{2}$$ ETC

(b) Which of the following functions satisfies Laplace's equation $u_{xx} + u_{yy} = 0$?

(i) $u(x, y) = x^3 + 3xy$

$u_x = 3x^2 + 3y$ $\qquad u_y = 3x$

$u_{xx} = 6x$ $\qquad u_{yy} = 0$

$u_{xx} + u_{yy} = 6x \neq 0$ NO

(ii) $u(x, y) = e^{-y} \cos x$

$u_x = -e^{-y} \sin x$ $\qquad u_y = -e^{-y} \cos x$

$u_{xx} = -e^{-y} \cos x$ $\qquad u_{yy} = e^{-y} \cos x$

So $u_{xx} + u_{yy} = -e^{-y} \cos x + e^{-y} \cos x = 0$

YES

(3) [10 pts]

(a) Suppose that $z = f(x, y)$ is a function and $(x, y) = \mathbf{r}(t)$ is a parametrized curve. State the version of the Chain Rule you would use to differentiate the composition $f \circ \mathbf{r}$.

$$(f \circ \vec{r})'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

or if $z = (f \circ \vec{r})(t) = f(\vec{r}(t))$

Then $\dfrac{dz}{dt}(t) = \dfrac{\partial z}{\partial x}(\vec{r}(t))\dfrac{dx}{dt}(t) + \dfrac{\partial z}{\partial y}(\vec{r}(t))\dfrac{dy}{dt}(t)$

(b) Let $z = f(x, y) = x^3y^2 + \ln(x^3)$ and suppose that $(x, y) = \mathbf{r}(t)$ is a parametrized curve so that

t	x	y	$\frac{dx}{dt}$	$\frac{dy}{dt}$
-1	0	0	3	-4
0	1	3	-2	5
1	3	2	5	4

Calculate $\frac{dz}{dt}(0)$.

$$\frac{dz}{dt}(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$

$$= \nabla f(1,3) \cdot (-2, 5)$$

$$\nabla f = \left(3x^2y^2 + \frac{3}{x},\ 2x^3y\right)$$

$$= \left(3 \cdot 1^2 3^2 + \frac{3}{1},\ 2 \cdot 1^3 3\right) = (30, 6)$$

So $z'(0) = (30, 6) \cdot (-2, 5) = -60 + \cancel{45}\,30 = \cancel{-15}\ -30$

(4) [12 pts] Consider the surface that is parametrized by

$$
\begin{aligned}
x &= r\cos\theta,\\
y &= r\sin\theta,\\
z &= r,
\end{aligned}
$$

for $1 \le r \le 3$ and $0 \le \theta \le 2\pi$.

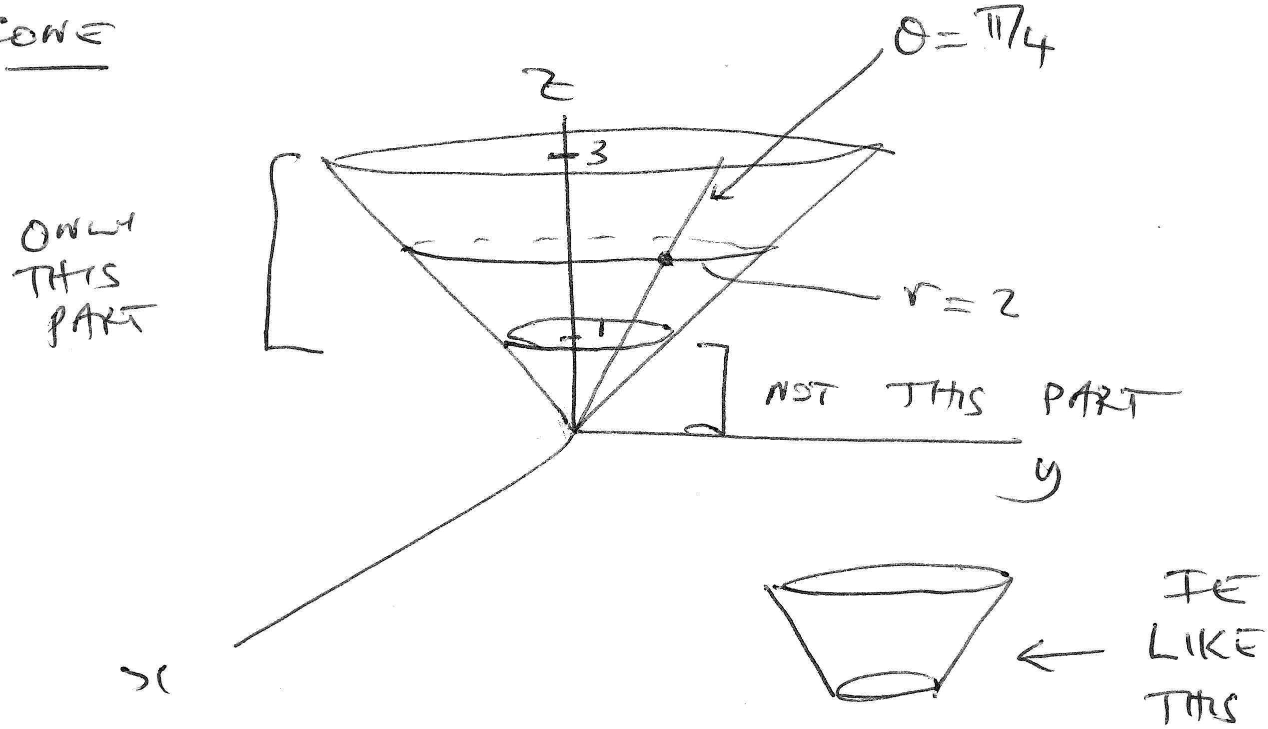
(a) Find an equation of the form $F(x, y, z) = 0$ for this surface.

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 = z^2$$

So $F(x,y,z) = z^2 - x^2 - y^2 = 0.$

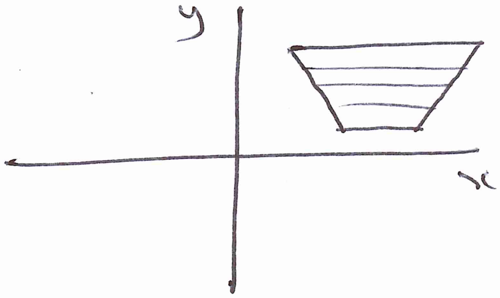
(b) Sketch the graph of the surface. Also sketch the grid curves $\theta = \frac{\pi}{4}$ and $r = 2$ on the surface.

CONE

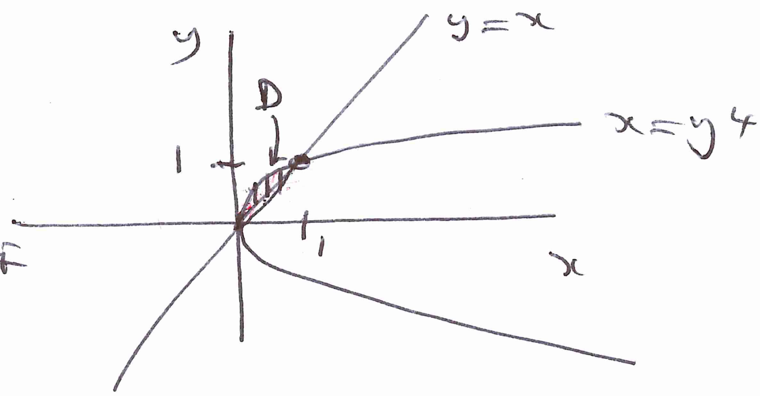


(5) [15 pts]

(a) Draw an example of a region that is Type II but not Type I.



(b) Calculate $\iint_D x^2y^3\,dA$, where D is the domain bounded by the curves $y = x$ and $x = y^4$.



D is Type II and Type I

(I) $0 \le x \le 1$

$\boxed{\cancel{x \le y \le x}}$

SORT OF YUK

$x \le y \le x^{1/4}$

(II) $0 \le y \le 1$

$y^4 \le x \le y$ Nicer

So

$$\iint_D x^2y^3\,dA = \int_{y=0}^{1}\int_{x=y^4}^{x=y} x^2y^3\,dx\,dy$$

$$= \int_{y=0}^{1} y^3\left[\int_{x=y^4}^{x=y} x^2\,dx\right]dy$$

$$= \int_{y=0}^{1} y^3\left[\frac{x^3}{3}\right]_{x=y^4}^{x=y} dy = \frac{1}{3}\int_0^1 y^3(y^3 - y^{12})\,dy$$

$$= \frac{1}{3}\left[y^7/7 - y^{16}/16\right]_0^1 = \frac{1}{3}\left[\frac{1}{7} - \frac{1}{16}\right]$$

$\frac{\cancel{16}\,3}{48}$
$\overline{144}$

(6) [12 pts] Find the local maxima, minima, and saddle points of the function $z = f(x, y) = y^3 - 12xy + 8x^3$.

$\nabla f = (-12y + 24x^2,\ 3y^2 - 12x) = (0, 0)$ at

① $y = 2x^2$

② $4x = y^2$

② into ①: $y = 2\cdot\left(\frac{y^2}{4}\right)^2 = \frac{1}{8}y^4$

$8y - y^4 = 0$

$y(8 - y^3) = 0$

$y = 0$ or $y = 2$

$\boxed{y=0}$ $x = 0$ by ②

$\boxed{(0,0) \text{ CPT.}}$

$D = 144(0-1) = -144 < 0$

$\boxed{\text{SADDLE PT}}$.

$\boxed{y=2}$ $x = \frac{y^2}{4} = \cancel{8}\,1$.

$\boxed{(1,2) \text{ CPT}}$

$D = 144(2\times1\times2 - 1) = 3\times144 > 0.$

$f_{xx} = 48\times1 = 48 > 0$ $\boxed{\text{LOCAL MIN}}$

$$\boxed{\begin{aligned} D &= \det\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y} \\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \\ &= \begin{bmatrix} 48x & -12 \\ -12 & 6y \end{bmatrix} \\ &= 6\cdot 48xy - 144 \\ &= 12(24xy - 12) \\ &= 144(2xy - 1) \end{aligned}}$$

(7) [15 pts] Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function $z = f(x, y) = x^2y$ on the circle $x^2 + y^2 = 1$.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases} \qquad \begin{cases} (2xy, x^2) = \lambda(2x, 2y) \\ x^2+y^2 = 1 \end{cases}$$

① $2xy = \lambda 2x$; $x(\lambda - y) = 0$. (1')

② $x^2 = 2y\lambda$

③ $x^2+y^2=1$

By (1') $x=0$ OR $\lambda = y$.

$\boxed{x=0}$: By ③ $y = \pm 1$.

By ② $\lambda = 0$.

So $(x, y, \lambda) = (0, \pm 1, 0)$ $f(0, \pm 1) = 0$.

$\boxed{\lambda = y}$ By ② $x^2 = 2y^2$, ④

By ③ $3y^2 = 1$, $y = \pm \frac{1}{\sqrt{3}}$

By ④ $x = \pm\sqrt{2}\, y$

So get $(x, y, \lambda) = \left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ OR $\left(\pm\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$f\left(\pm\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3} \cdot \frac{1}{\sqrt{3}}$ $\qquad f\left(\pm\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$

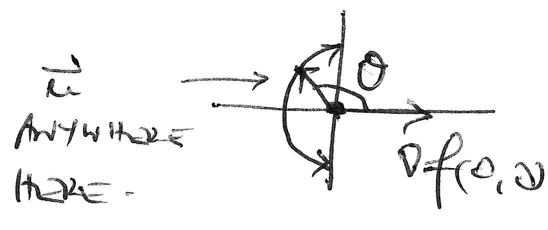
ABS MAX

(8) [10 pts]
Let $z = f(x, y)$ be a function so that $\nabla f(0, 0) = 3\mathbf{i}$.

(a) Let $\mathbf{u} = (\cos\theta, \sin\theta)$ be a unit length vector in direction given by an angle $\theta \in [0, 2\pi]$. For which values of θ is the directional derivative $D_{\mathbf{u}} f(0, 0) < 0$?

$D_{\vec{u}} f(0,0) = \nabla f(0,0) \cdot \vec{u} = 3\vec{i} \cdot \vec{u} = 3\cos\theta < 0$

when $\pi/2 < \theta < 3\pi/2$



(b) What is the equation of the tangent line to the level curve of f at the point $(0, 0)$?

Tangent Line to Level Curve of f is $\perp$ to ∇f.

So T.L. must be y axis as $\nabla f(0,0) = 3\vec{i}$.

So eqn is $\boxed{x = 0}$ (Vertical Line, Goes thru $(0,0)$)

Please sign the following honor statement:

On my honor, I pledge that I have neither given nor received any aid on this exam.

Signature: ______________________________