NAME: SOLUTIONS

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5 /12	6 /12	7 /12	T /100

MATH 2415 (Fall 2012) Exam I, Oct 5

No calculators, books or notes! Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 2 hour exam is worth 100 points.

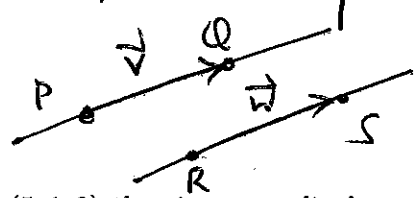
(1) [16 pts]

(a) Is the line through the points $\underset{P}{(-4,-6,1)}$ and $\underset{Q}{(-2,0,-3)}$ parallel to the line through the points $\underset{R}{(10,18,4)}$ and $\underset{S}{(5,3,14)}$?

Let
$$\vec{v} = (-2,0,-3) - (-4,-6,1) = (2,6,-4) = 2(1,3,-2)$$
$$\vec{w} = (5,3,14) - (10,18,4) = (-5,-15,10) = -5(1,3,-2)$$

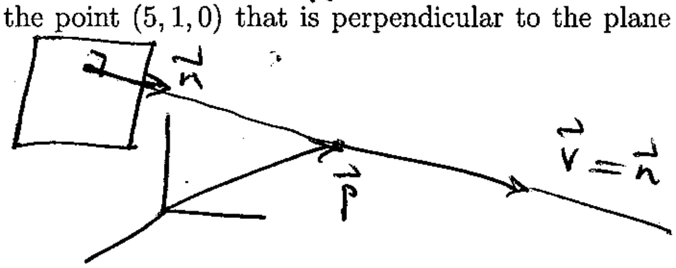
So $\vec{v} = -\frac{2}{5}\vec{w}$. The vectors $\vec{v}, \vec{w}$ are parallel

So the two lines are parallel



(b) Find parametric equations for the line through the point $(5,1,0)$ that is perpendicular to the plane $2x - y + z = 1$.

$$\vec{l}(t) = \vec{p} + t\vec{v}$$



The vector $\vec{v}$ along the line ~~can~~ be chosen to be the normal vector $\vec{n}$ to the plane (since line $\perp$ plane)

So $\vec{v} = \vec{n} = (2,-1,1)$ from eqn of plane

So
$$\vec{l}(t) = (5,1,0) + t(2,-1,1) = (5+2t,\ 1-t,\ t)$$

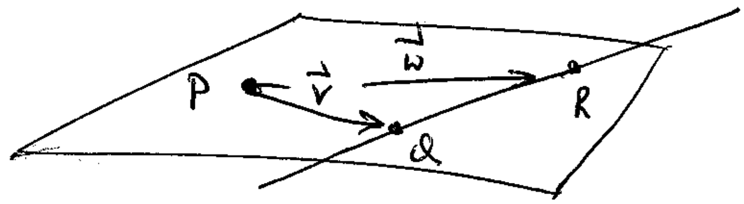
OR
$$x = 5+2t$$
$$y = 1-t$$
$$z = t$$

(2) [16 pts] (For Dr. Zweck's class: When we refer to the *equation* of a plane we mean a *level set equation.*)

(a) Find the equation of the plane that passes through the point $(6, 0, -2)$ and contains the line $x = 4-2t$, $y = 3+5t$, and $z = 7+4t$.

Let $\vec{r}(t) = (4-2t,\ 3+5t,\ 7+4t)$ parametrize line

Here are 3 points in plane



$P = (6, 0, -2)$

$Q = \vec{r}(0) = (4, 3, 7)$

$R = \vec{r}(1) = (2, 8, 11)$

Here are 2 vectors in plane

$\vec{v} = PQ = Q - P = (-2, 3, 9)$, $\vec{w} = PR = R - P = (-4, 8, 13)$

So normal is

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 9 \\ -4 & 8 & 13 \end{vmatrix} = (-33, -10, -4)$$

So eqn is $(\vec{x} - p)\cdot\vec{n} = 0$ gives

$$\boxed{-33(x-6) - 10(y-0) - 4(z+2) = 0}$$

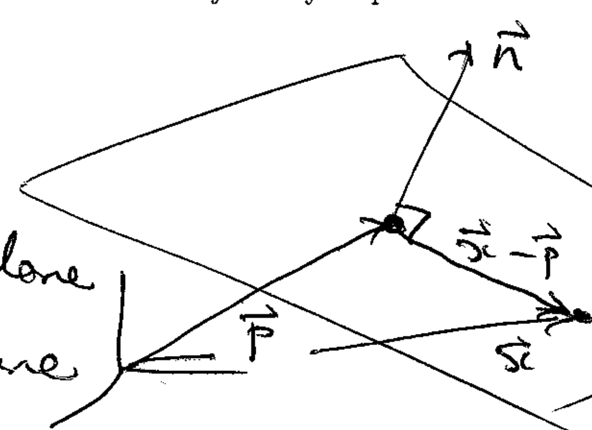
(b) Write down a general equation of a plane that involves the normal vector to the plane. Draw a picture that explains why this equation holds. Be sure to carefully label your picture.

$(\vec{x} - \vec{p})\cdot\vec{n} = 0$

$\vec{p}$ = specific point in plane

$\vec{n}$ = normal vector to plane

$\vec{x}$ = arbitrary pt in plane



$\vec{x} - \vec{p}$ is a vector in plane. So $\vec{x} - \vec{p} \perp \vec{n}$

So $(\vec{x} - \vec{p})\cdot\vec{n} = 0$

(3) [16 pts]
(a) Find the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ where $\mathbf{a} = (3, 6, -2)$ and $\mathbf{b} = (1, 2, 3)$.

$$\mathrm{PROJ}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$$

$$= \frac{(3,6,-2)\cdot(1,2,3)}{|(3,6,-2)|^2}(3,6,-2)$$

$$= \frac{9}{9+36+4}(3,6,-2) = \frac{9}{49}(3,6,-2)$$

(b) Use vector algebra to show that the vector $\mathbf{b} - \mathrm{proj}_{\mathbf{a}}\,\mathbf{b}$ is orthogonal to $\mathbf{a}$. (Here $\mathrm{proj}_{\mathbf{a}}\,\mathbf{b}$ is the projection of $\mathbf{b}$ onto $\mathbf{a}$.) [Note: you need to show this in general not for the specific vectors given in part (a).]

Let $\vec{v} = \vec{b} - \mathrm{PROJ}_{\vec{a}}(\vec{b}) = \vec{b} - \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$,

NTS $\vec{v}\cdot\vec{a} = 0$.

Well $\vec{v}\cdot\vec{a} = \left(\vec{b} - \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}\right)\cdot\vec{a}$

$$= \vec{b}\cdot\vec{a} - \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}\cdot\vec{a}$$

$$= \vec{b}\cdot\vec{a} - \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}|\vec{a}|^2$$

$$= \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{a}$$

$$= 0$$

(4) [16 pts] Make a labelled sketch of the traces of the surface

$$y^2 - x^2 + 4z^2 = 1$$

in the planes $y = 0$, $z = 0$, and $x = k$ for $k = 0, \pm 1, \cancel{\pm 2}$. Then sketch the surface.

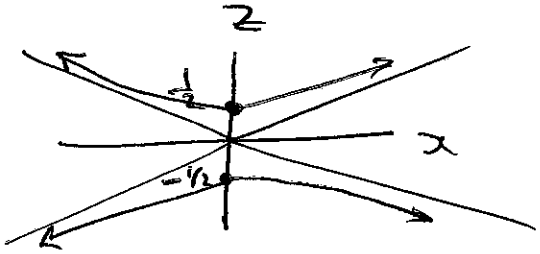
$\boxed{y=0}$ $4z^2 - x^2 = 1$

Asymptotes $4z^2 - x^2 = 0$

$2z = \pm x$

$z = \pm \frac{1}{2} x$

Goes thru $x=0$, $z = \pm \frac{1}{2}$

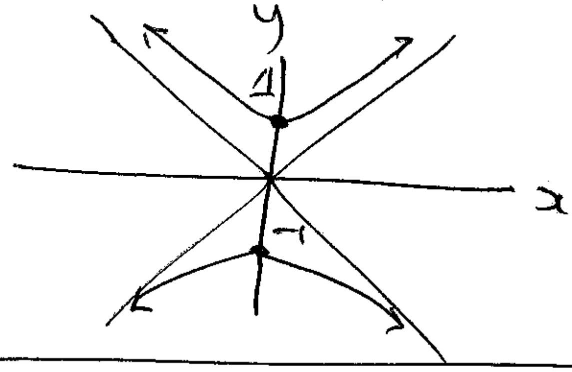


$\boxed{z=0}$ $y^2 - x^2 = 1$

Asymptotes $y^2 - x^2 = 0$

$y = \pm x$

Goes thru $x = 0$, $y = \pm 1$



$\boxed{x=k}$ $y^2 + 4z^2 = 1 + k^2$.

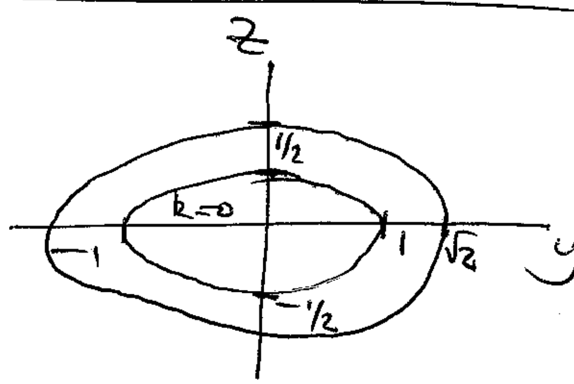
<u>$k=0$</u> $y^2 + 4z^2 = 1$

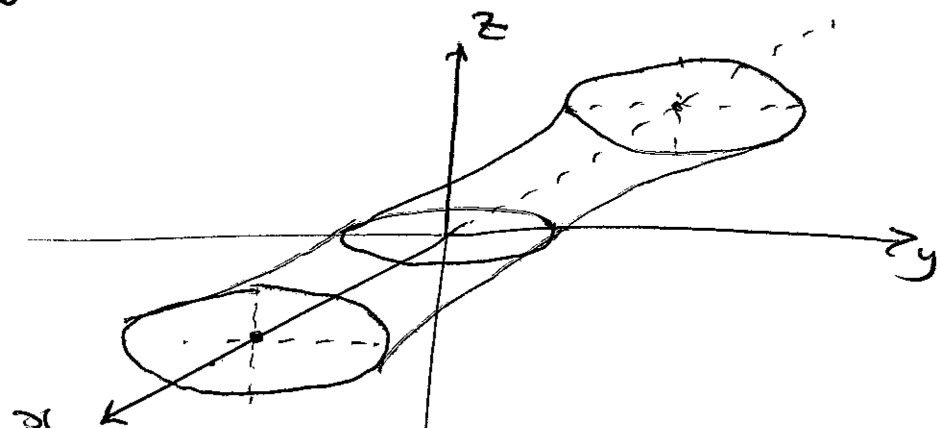
Ellipse Intercepts $(\pm 1, 0)$, $(0, \pm\frac{1}{2})$

<u>$k=\pm 1$</u> $y^2 + 4z^2 = 2$

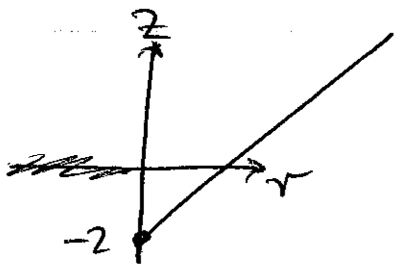
Intercepts $(\pm\sqrt{2}, 0)$, $(0, \pm\frac{\sqrt{2}}{2})$

~~$k = \pm 2$ $y^2 + 4z^2 = 5$~~

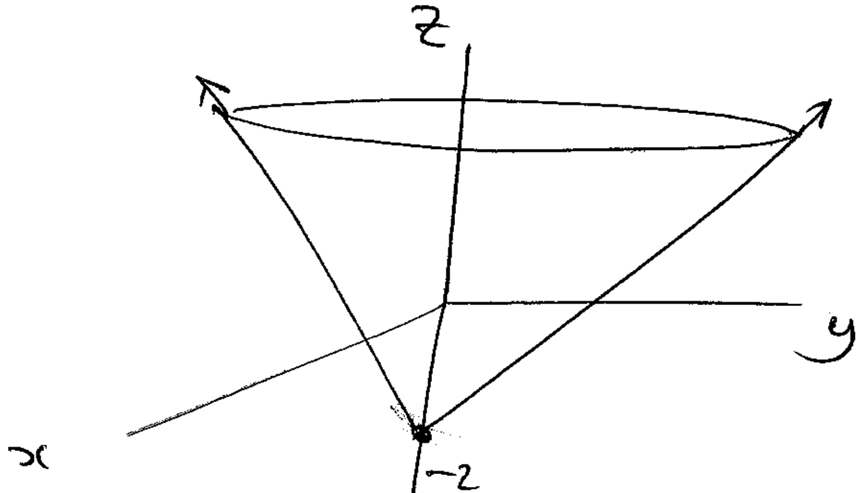




(5) [12 pts]
(a) Sketch the surface whose equation in cylindrical coordinates is $z = r - 2$.



Rotate r axis about z axis to get cone vertex $(0, 0, -2)$



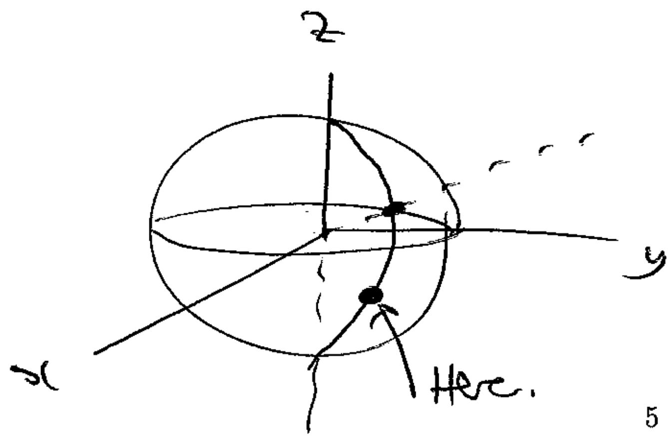
(c) What are the rectangular coordinates of the point whose spherical coordinates are $(\rho, \theta, \phi) = (2, \pi, 3\pi/4)$?

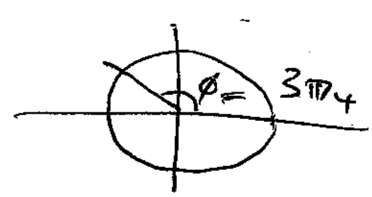
$$x = \rho \sin\phi \cos\theta = 2 \sin\frac{3\pi}{4} \cos\pi = 2\, \frac{1}{\sqrt{2}}(-1) = \frac{-2}{\sqrt{2}}$$

$$y = \rho \sin\phi \sin\theta = 2 \sin\frac{3\pi}{4} \sin\pi = 0.$$

$$z = \rho \cos\phi = 2 \cos 3\pi/4 = 2 \cdot \left(-\frac{1}{\sqrt{2}}\right) = \frac{-2}{\sqrt{2}}$$

$$(-\sqrt{2},\ 0,\ -\sqrt{2})$$

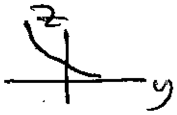




(6) [12 pts] Match each of the functions below with both a contour plot and a graph. *To receive credit you must provide reasons for your answers.*

(a) $f(x,y) = e^{-y}\cos x$ $\boxed{\text{III}, 4}$ Fix $y=0$ get $f(x,0) = \cos x$

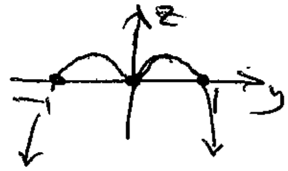
So oscillates $\pm$ along x axis. Fix $x=0$ get $f(0,y) = e^{-y}$



Grows exponentially. But fix $x = \pi/2$ get $f(0,y) = 0$.

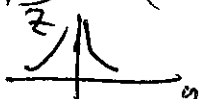
(b) $f(x,y) = y^2 - y^4 - x^2$ $\boxed{\text{IV}, 2}$ $\underline{y=0}$ Get $z = -x^2$.

Fix $x=0$, $z = y^2 - y^4 = y^2(1-y^2)$



(c) $f(x,y) = \frac{1}{4x^2+y^2}$ $\boxed{\text{I}, 3}$ At $(x,y) = (0,0)$, $z = +\infty$.

IF $x=0$ get $z = \frac{1}{y^2}$.

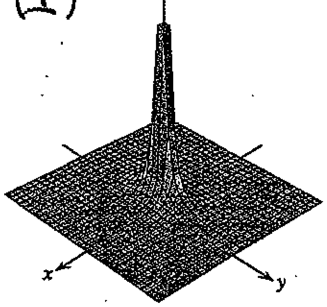


Steeper

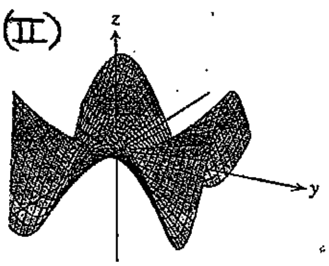
IF $y=0$ get $z = \frac{1}{4x^2}$

Less steep

(I)

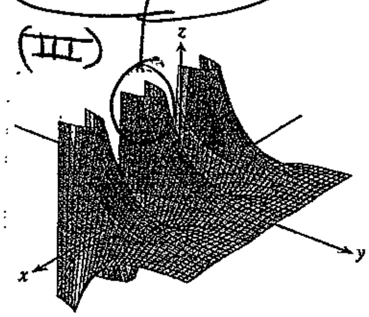


(II)

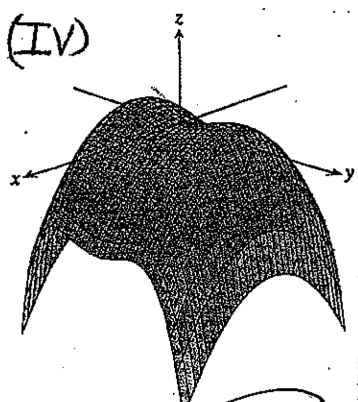


GET STEEPER AS $y \downarrow$

(III)



(IV)



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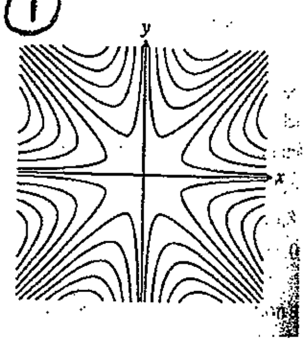

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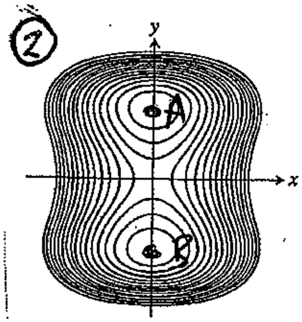
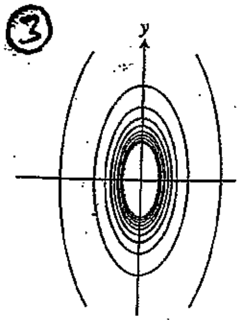


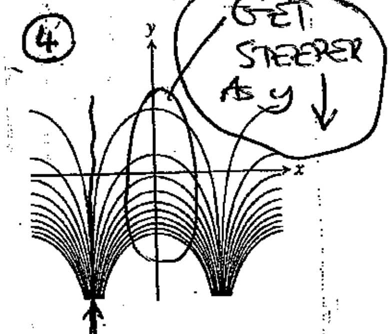
A, B ARE TOP OF HILL

③



LESS STEEP ON $x=0$

④



$f=0$ on this line.

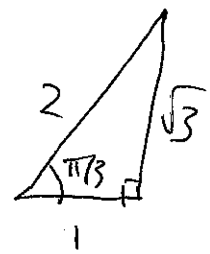
(7) [12 pts]

(a) Calculate a parametrization of the tangent line to the curve $\mathbf{r}(t) = (2\sin t, 3\cos t)$ at $t = \pi/3$

$$\vec{\ell}(s) = \vec{r}(\pi/3) + s\,\vec{r}\,'(\pi/3)$$

$$= (2\sin \pi/3, 3\cos \pi/3)$$

$$+ s(2\cos \pi/3, -3\sin \pi/3)$$



$$= \left(2.\frac{\sqrt{3}}{2}, 3.\frac{1}{2}\right) + s\left(2.\frac{1}{2}, -3.\frac{2}{\sqrt{3}}\right)$$

$$= \left(\sqrt{3} + s, \frac{3}{2} - 2\sqrt{3}s\right)$$

(b) Suppose that $\mathbf{r}$ is a parametrized curve in space with $\mathbf{r}'(0) = (1, 2, 3)$ and $\mathbf{r}''(0) = (4, 5, 6)$. Could the parametization $\mathbf{r}$ have constant speed? Explain!

If $\vec{r}$ has constant speed then from Lecture VELOCITY $\perp$ ACCELERATION for all t

But $\underset{\text{VEL}}{\vec{r}\,'(0)} \cdot \underset{\text{ACC}}{\vec{r}\,''(0)} = (1\ 2\ 3) \cdot (4\ 5\ 6) = 4 + 10 + 18 \neq 0$

So VEL $\not\perp$ ACC AT $t=0$. So can't be constant speed

Please sign the following honor statement:

On my honor, I pledge that I have neither given nor received any aid on this exam.

Signature: ______________________________