NAME: Solutions	CIRCLE:	Zweck 11:30am	Zweck 2:30pm

1	/10	2	/12	3	/8	4	/8	5	/12		
6	/10	7	/10	8	/10	9	/12	10	/8	T	/100

MATH 2415 Final Exam, Fall 2015 (Zweck)

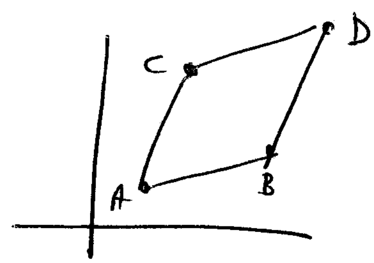
No books or notes! You may use a scientific calculator provided it does not allow for access to the internet. **Show all work and give complete explanations**. This 2 hours 45 mins exam is worth 100 points.

(1) [10 pts]

(a) Find the area of the parallelogram with vertices $\overset{A}{(1,1)}$, $\overset{B}{(3,4)}$, $\overset{C}{(5,6)}$ and $\overset{D}{(7,9)}$.

$$\vec{v} = B - A = (2,3) = D - C$$

$$\vec{w} = C - A = (4,5) = D - B$$



$$A = |\vec{v} \times \vec{w}| = \left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ 4 & 5 & 0 \end{matrix} \right\|$$

$$= |(2\times 5 - 3\times 4)\vec{k}| = |-2| = 2$$

(b) Calculate the vector projection of $\mathbf{u} = (1, 2, -4)$ onto $\mathbf{v} = (3, -2, 1)$.

$$\mathrm{PROJ}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\frac{\vec{v}}{|\vec{v}|} = (1,2,-4)\cdot(3,-2,1)\,\frac{(3,-2,1)}{(\sqrt{9+4+1})^2}$$

$$= \frac{3-4-4}{14}(3,-2,1)$$

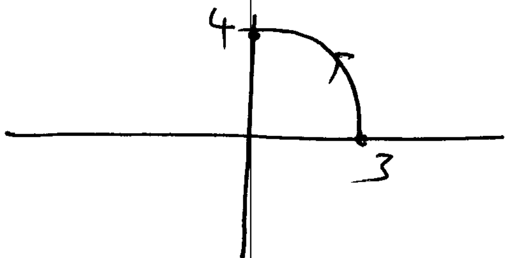
$$= \frac{-5}{14}(3,-2,1)$$

$\begin{array}{r} 64 \\ 27 \\ \hline 37 \\ 24 \\ \hline 148 \end{array}$

$$\frac{4}{7}\left(4^3 - 3^3\right) = 4\,\frac{64-27}{7} = \frac{4}{7}\,37 = \frac{148}{7}$$

(2) [12 pts] Let C be the curve in $\mathbf{R}^2$ parametrized by $(x, y) = \mathbf{r}(t) = (3\cos t, 4\sin t)$ for $0 \le t \le \pi/2$.

(a) Sketch the curve C.



$$\vec{r}\,'(t) = (-3\sin t, 4\cos t)$$

$$|\vec{r}\,'(t)| = \sqrt{3^2 + 7\cos^2 t}$$

(b) Calculate $\int_C f\,ds$ where $f(x, y) = xy$.

$$\int_C f\,ds = \int_0^{\pi/2} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

$$= \int_0^{\pi/2} 3\cos t \cdot 4\sin t \cdot \sqrt{9 + 7\cos^2 t}\;dt$$

$u = 9 + 7\cos^2 t$

$du = -14\cos t\sin t\,dt$

$$= -\frac{12}{14}\int_{16}^{9} \sqrt{u}\,du$$

$$= +\frac{6}{7}\int_9^{16} \sqrt{u}\,du = \frac{6}{7}\,\frac{2}{3}\left[u^{\frac{3}{2}}\right]_9^{16} = \frac{4}{7}\left(16^{3/2} - 9^{3/2}\right)$$

(c) Let $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$. Find a function $g = g(t)$ and numbers a and b so that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b g(t)\,dt$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \left(4\sin t\,\vec{i} + 9\cos^2 t\,\vec{j}\right) \cdot \left(-3\sin t\,\vec{i} + 4\cos t\,\vec{j}\right) dt$$

$$= \int_0^{\pi/2} \underbrace{-12\sin^2 t + 36\cos^3 t}_{g(t)}\;dt$$

(3) [8 pts] Find the limit if it exists, or show that the limit does not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{Along}\\ x=0}} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{Along } y=x}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2} \neq 0 \quad \text{DNE}$$

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

$$= \lim_{r\to 0} \frac{r^3 \cos^2\theta \sin\theta}{r^2}$$

$$= \lim_{r\to 0} r \cos^2\theta \sin\theta$$

$$= 0 \quad \text{by Sandwich Thm}$$

(4) [8 pts] Let $z = f(x, y) = 3x^2 + 4xy + 5y^2$.

(a) Calculate the equation of the tangent plane to the graph of f at $(x_0, y_0) = (2, -1)$.

$$\frac{\partial f}{\partial x} = 6x + 4y = 12 - 4 = 8 \quad @(2,-1)$$

$$\frac{\partial f}{\partial y} = 4x + 10y = 8 - 10 = -2 \quad @(2,-1)$$

$$f(2,-1) = 3.4 + 4.2 + 5 = 9$$

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$\boxed{z = 9 + 8(x-2) - 2(y+1)} = 9 - 16 - 2 + 8x - 2y$$

$$\boxed{z = -9 + 8x - 2y}$$

(b) Suppose now that an ant is walking on a hot plate in the xy-plane and that the function $z = f(x, y)$ is the temperature of a hot plate at the point (x, y). Suppose that at time $t = 0$ the position of the ant is $\mathbf{x} = (2, -1)$ and the velocity of the ant is $\mathbf{v} = (4, 3)$. What is the rate of change of the temperature of the ant's feet at time $t = 0$?

$$g(t) = f(\vec{r}(t))$$

$$\nabla f(2,-1) = (8, -2)$$

$$g'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

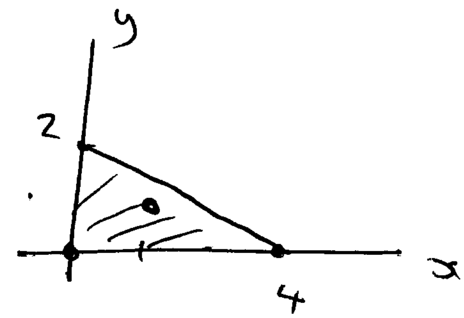
$$g'(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$

$$= \nabla f(2,-1) \cdot (4,3)$$

$$= (8, -2) \cdot (4,3)$$

$$= 32 - 6 = 26$$

(5) [12 pts] Find the absolute maximum and absolute minimum of the function $f(x,y) = x + y - xy$ on the triangle in the xy-plane with vertices $(0,0)$, $(4,0)$, and $(0,2)$.



$\nabla f = (1-y,\ 1-x) = (0,0)$

at $(x,y) = (1,1)$

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$f(1,1) = 1 + 1 - 1 = 1$

$\boxed{x = 0, \quad 0 < y < 2}$

$g(y) = f(0,y) = y$

$g'(y) = 1 \neq 0$

2

$g(0) = 0, \quad g(2) = 2$

$\boxed{y = 0 \quad 0 < x < 4}$

2

$h(x) = x$

(x,y)	$f(x,y)$
$(1,1)$	1
1 $(0,0)$	$\boxed{0 \text{ min}}$
1 $(0,2)$	2
1 $(4,0)$	$\boxed{4 \text{ max}}$
$\left(\frac{3}{2}, \frac{5}{4}\right)$	$\frac{7}{8}$

$\boxed{x + 2y = 4, \quad 0 < x < 4.}$ 2

$y = \frac{4-x}{2} \qquad y = 2 - \frac{x}{2}$

$f\left(\frac{3}{2}, \frac{5}{4}\right) = \frac{3}{2} + \frac{5}{4} - \frac{15}{8} = \frac{12+10-15}{8} = \frac{7}{8}$

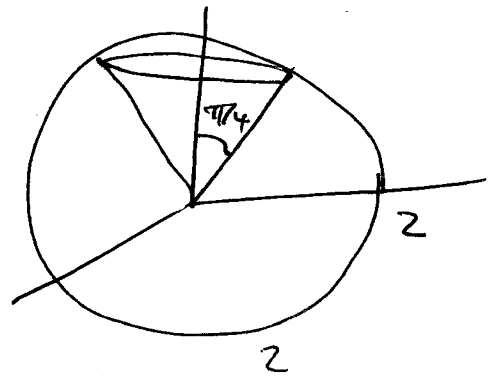
$k(x) = f\left(x, \frac{4-x}{2}\right) = x + \frac{4-x}{2} - \frac{x(4-x)}{2}$

$k'(x) = 1 + -\frac{1}{2} - 2 + x$

$0 = -3/2 + x \qquad x = 3/2 \qquad y = \frac{4 - 3/2}{2} = \frac{5}{4}$

(6) [10 pts] Use spherical coordinates to calculate the triple integral $\iiint_E z\,dV$, where E is the solid region inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$.

$$\boxed{\begin{aligned} &0 < \phi < \pi/4 \\ &0 < \theta < 2\pi \\ &0 < \rho < 2 \end{aligned}}$$



$$\iiint_E z\,dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{2} \boxed{\rho\cos\phi}\,\boxed{\rho^2 \sin\phi}\, d\rho\, d\phi\, d\theta$$

$$= 2\pi \left(\int_0^{\pi/4} \cos\phi \sin\phi\, d\phi\right)\left(\int_0^2 \rho^3\, d\rho\right)$$

$u = \sin\phi$

$$= 2\pi \int_0^{1/\sqrt{2}} u\, du \qquad \left[\frac{\rho^4}{4}\right]_0^2$$

$$= 2\pi \left[\frac{u^2}{2}\right]_0^{\frac{1}{\sqrt{2}}} \quad \frac{2^4}{4}$$

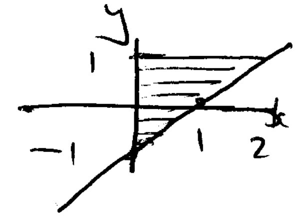
$$= 2\pi \cdot \frac{1}{4} \quad 4 = \underline{\underline{2\pi}}$$

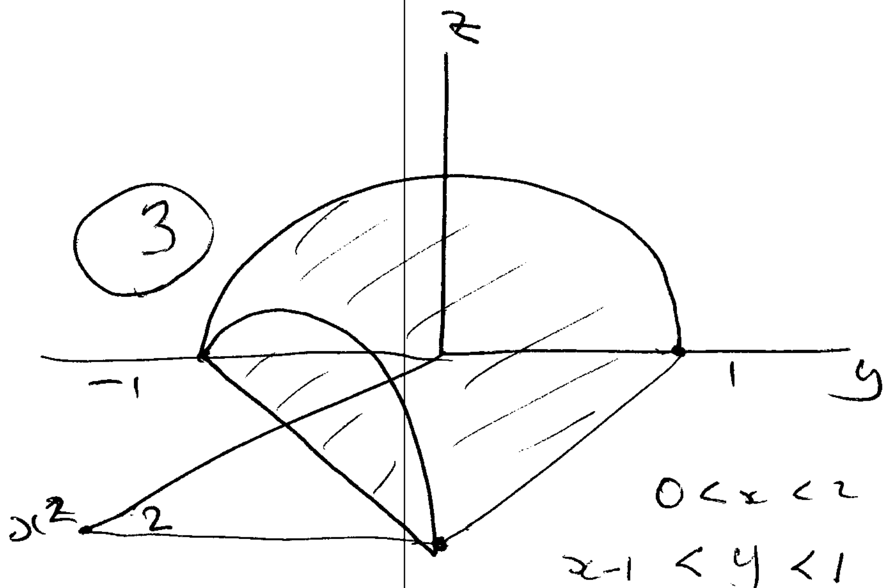
2

(7) [10 pts]

Let E be the solid region in $\mathbf{R}^3$ bounded by the surfaces $z = 1 - y^2$, $y = x - 1$, $x = 0$, and $z = 0$.

(a) Sketch E. Is $\iiint_E y\, dV$ positive or negative? Why?





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$+ve$ 2

AS MOST OF VOLUME IS IN $+y$ REGION

$0 < x < 2$

$x-1 < y < 1$

(b) Calculate $\iiint_E y\, dV$.

$$\boxed{\begin{aligned} -1 &< y < 1 \\ 0 &< z < 1-y^2 \\ 0 &< x < y+1 \end{aligned}}$$

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$$\iiint_E y\, dV = \int_{y=-1}^{1} \int_{z=0}^{1-y^2} \int_{x=0}^{1+y} y\, dx\, dz\, dy$$

$$= \int_{y=-1}^{1} y(1-y^2)(1+y)\, dy$$

$$= \int_{-1}^{1} y(1-y^2)\cancel{dy}(1+y)\, dy$$

$$= \int_{-1}^{1} y + y^2 - y^3 - y^4\, dy$$

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$$= 2\int_0^1 y^2 - y^4\, dy = 2\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = 2\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{4}{15} > 0$$

$0 < v < *$ -2

(8) [10 pts] Use the Change of Variables Theorem to evaluate the integral $\iint_R x\,dA$, where R is the quadrilateral region bounded by the lines $x+2y=2$, $x+2y=4$, $x=0$, and $y=0$. **Hint:** Let $u=x+2y$ and $v=y$.

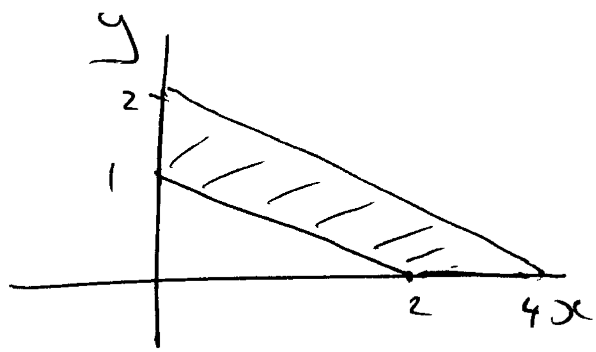
y

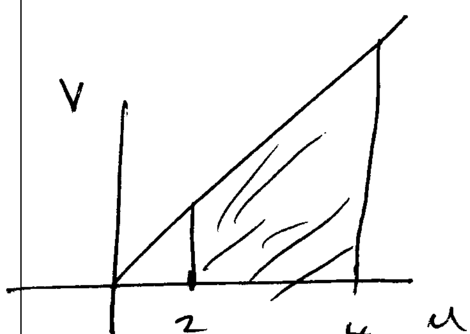
$u = 2$

$u = 4$

$v = 0$

$u = 2v$





$$\boxed{\begin{aligned} 2 &< u < 4 \\ 0 &< v < \tfrac{u}{2} \end{aligned}} \quad \boxed{3}$$

$$\boxed{\begin{aligned} x &= u - 2v \\ y &= v \end{aligned}} \quad \boxed{2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 \quad \boxed{2}$$

$$\iint_R y\,dA = \int_{u=2}^{u=4} \int_{v=0}^{\frac{u}{2}} \boxed{v\,|1|}\,dv\,du \quad \boxed{1}$$

$$= \int_2^4 \left[\frac{v^2}{2}\right]_0^{\frac{u}{2}} du$$

$$= \int_2^4 \frac{u^2}{8}\,du = \frac{u^3}{24}\Big|_2^4$$

$$= \frac{4^3 - 2^3}{24} \quad \boxed{1}$$

$$= \frac{56}{24} = \frac{7}{3}$$

$$\iint_R x\,dA = \int_{u=2}^{4} \int_{v=0}^{\frac{u}{2}} (u-2v).dv\,du$$

$$= \int_{u=2}^{4} \left[uv - v^2\right]_{v=0}^{\frac{u}{2}} du = \int_2^4 \frac{u^2}{2} - \frac{u^4}{4}\,du$$

$$= \left[\frac{u^3}{6} - \frac{u^5}{20}\right]_2^4 = \frac{4^3-2^3}{6} - \frac{4^5-2^5}{20}.$$

(9) [12 pts] Let $\mathbf{F}$ be the vector field in the plane given by $\mathbf{F}(x, y) = x^2y\mathbf{i} + (x^2 - y^2)\mathbf{j}$.

(a) Calculate the divergence of $\mathbf{F}$.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(x^2 - y^2)$$
$$= \boxed{2xy - 2y}$$

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(b) Calculate the curl of $\mathbf{F}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \cancel{\partial} \\ x^2 y & x^2 - y^2 & 0 \end{vmatrix} = \boxed{(2x - x^2)\vec{k}}$$

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(c) Is $\mathbf{F}$ conservative? Why? NO $\nabla \times \vec{F} \neq \vec{0}$

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(d) Suppose that the vector field $\mathbf{F}$ given above is the velocity vector field of a fluid flowing in the plane. On average is the fluid flowing in or out of a small disk centered at the point $(-1, 2)$? Why?

$$\nabla \cdot \vec{F}(-1, 2) = 2(-1)2 - 2 \cdot 2 = -8 < 0$$

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(10) [8 pts]

(a) Define what it means for a vector field to be conservative.

$$\vec{F} = \nabla f$$

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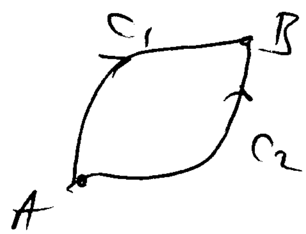
(b) Define what it means for the integral of a vector field to be independent of path.

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$



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(c) Prove that if $\mathbf{F}$ is a conservative vector field then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r} = f(B) - f(A) \quad \forall C$$

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Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________