NAME:	CIRCLE: Turi	Zweck 10am	Zweck 4pm

1 /10	2 /12	3 /12	4 /15	5 /16	6 /10	T /75

MATH 2415 (Fall 2014) Exam II, Nov 7th

No books or notes! You may use a scientific calculator provided it does not allow for access to the internet. Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 75 minute exam is worth 75 points.

(1) [10 pts] Let $z = f(x, y) = 3x^2 + 2y^2 - xy$. Suppose that $(x, y) = \mathbf{r}(t)$ is a parametrized curve so that $\mathbf{r}(0) = (2, 3)$ and $\mathbf{r}'(0) = (-3, 4)$. Let $g(t) = f(\mathbf{r}(t))$. Find the slope of g at $t = 0$.

$$g'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$ by Chain Rule

$$g'(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$

$$= \nabla f(2,3) \cdot (-3,4)$$

NOW $$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (6x - y,\ 4y - x)$$

$$\nabla f(2,3) = (12 - 3,\ 12 - 2) = (9, 10)$$

So $$g'(0) = (9,10) \cdot (-3,4) = 40 - 27 = \underline{13}$$

is slope of g at $t = 0$.

(2) [12 pts] Let $z = f(x, y) = 3x^2 + 2y^2 - xy$.
(a) Find the directional derivative of f in the direction of the vector $\mathbf{v} = (3, 4)$ at the point $(x, y) = (1, 2)$.

$$(D_{\vec{u}} f)(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{5}(3,4)$$

$$= \nabla f(1,2) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) \qquad \nabla f = (6x - y, 4y - x)$$

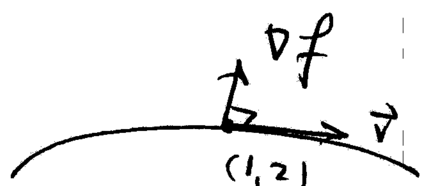
$$= (6-2, 8-1) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \frac{1}{5}(12 + 28) = 8.$$

(b) Find the direction of steepest descent of f at the point $(x, y) = (1, 2)$. What is the rate of change of f in this direction?

$$\vec{u} = -\frac{\nabla f(1,2)}{|\nabla f(1,2)|} = -\frac{(4,7)}{\sqrt{4^2+7^2}}$$

$$\text{R of C} = -|\nabla f(1,2)| = -\sqrt{4^2+7^2}$$

(c) Find a parametrization for the tangent line to the level curve of f that passes through the point $(x, y) = (1, 2)$.



$$\nabla f(1,2) = (4,7)$$

$\vec{v} = (-7, 4)$ so $\perp$ to ∇f

$$\vec{r}(t) = \vec{p} + t\vec{v} = (1,2) + t(-7,4)$$

(3) [12 pts] Let $z = f(x, y) = x^3 + 3x^2 - y^2 + y$. Find all local maxima, minima, and saddle points of f.

$$\frac{\partial f}{\partial x} = 3x^2 + 6x = 3x(x+2) = 0$$

$$\frac{\partial f}{\partial y} = -2y + 1 = 0$$

at $x = 0,\ y = \frac{1}{2}$ CPTS $(0, \frac{1}{2})$

or $x = -2,\ y = \frac{1}{2}$ $(-2, \frac{1}{2})$

$$D = \det \begin{bmatrix} 6x+6 & 0 \\ 0 & -2 \end{bmatrix} = -12(x+1)$$

$\boxed{(0, \frac{1}{2})}$ $D = -12 < 0$ Saddle Point

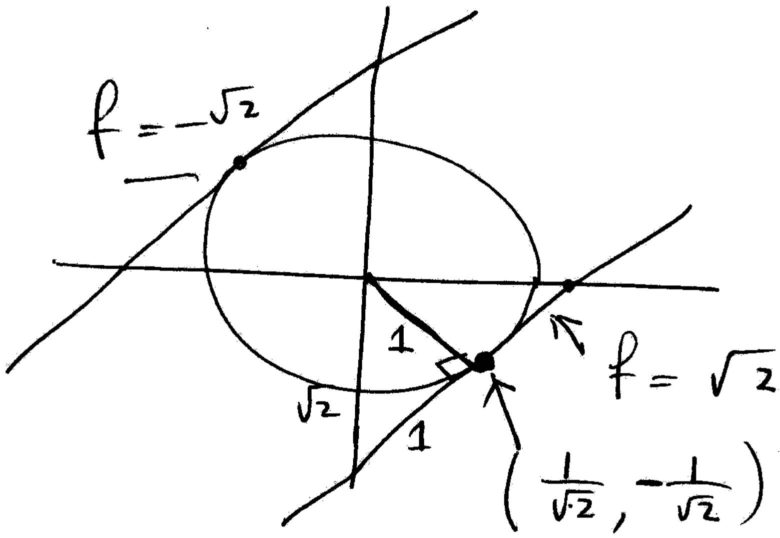
$(-2, \frac{1}{2})$ $D = -12(-2+1) = 12 > 0$

$f_{xx} = 6(x+1) = -6 < 0$ Local MAX

(4) [15 pts] In this problem you will use the **method of Lagrange Multipliers two different ways** to solve the same problem. The problem is to find the absolute maximum and absolute minimum of the function $f(x, y) = x - y$ on the circle $x^2 + y^2 = 1$.

(a) First solve the problem **graphically** by sketching the circle and some appropriately chosen level curves, $f(x, y) = k$.

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$



$x - y = \pm\sqrt{2}$

$y = x \mp \sqrt{2}$

(b) Now solve the problem by setting up the appropriate **equations** and solving them algebraically.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases}$$

$1 = \lambda 2x$ $\qquad x = \frac{1}{2\lambda}$

$-1 = \lambda 2y$ $\qquad y = \frac{-1}{2\lambda}$

$x^2 + y^2 = 1$

$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-1}{2\lambda}\right)^2 = 1$ $\qquad x = \pm\frac{1}{2/\sqrt{2}} = \pm\frac{\sqrt{2}}{2} = \pm\frac{1}{\sqrt{2}}$

$\frac{2}{4\lambda^2} = 1$ $\qquad y = \mp\frac{1}{\sqrt{2}}$

$\lambda^2 = \frac{1}{2}$

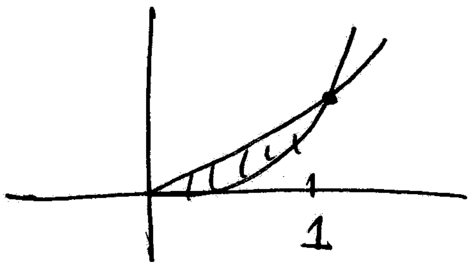
$\boxed{\lambda = \pm\frac{1}{\sqrt{2}}}$

$(x,y) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ $\quad \lambda = \frac{1}{\sqrt{2}}$ $\quad f = \sqrt{2}$

$(x,y) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\quad \lambda = -\frac{1}{\sqrt{2}}$, $\quad f = -\sqrt{2}$

(5) [16 pts]

(a) Let D be the region in the first quadrant (i.e, $x \geq 0$ and $y \geq 0$) of the xy-plane that is bounded by the curves $y = x^3$ and $y = x^4$. Calculate $\iint_D y\,dA$.



$$\int_{x=0}^{x=1} \int_{y=x^4}^{y=x^3} y \, dy \, dx$$

$0 < x < 1$

$x^4 < y < x^3$

$$= \frac{1}{2} \int_{x=0}^{x=1} \left[y^2 \right]_{y=x^4}^{y=x^3} dx$$

$$= \frac{1}{2} \int_0^1 x^6 - x^8 \, dx$$

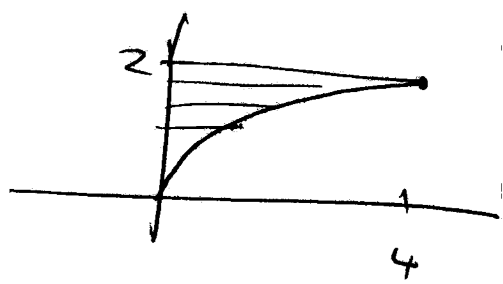
$$= \frac{1}{2} \left(\frac{1}{7} - \frac{1}{9} \right) = \frac{1}{63}$$

(b) Evaluate the integral by reversing the order of integration:

$$I = \int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3 + 1} \, dy dx.$$

$0 < x < 4$

$\sqrt{x} < y < 2$



OR $0 < y < 2$

$0 < x < y^2$

$$I = \int_0^2 \int_{x=0}^{x=y^2} \frac{1}{y^3+1} \, dx dy = \int_0^2 \frac{y^2}{y^3+1} \, dy \qquad u = y^3 + 1, \; du = 3y^2 dy$$

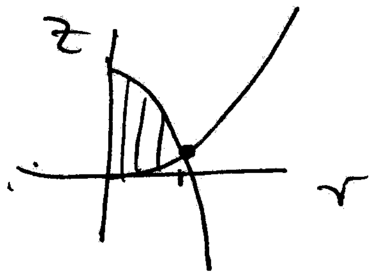
$$= \frac{1}{3} \int_1^9 \frac{du}{u} = \frac{1}{3} \left[\ln |u| \right]_1^9 = \frac{1}{3} \ln 9.$$

(6) [10 pts] Find the volume of the solid bounded by the surfaces $z = 2x^2 + 2y^2$ and $z = 4 - x^2 - y^2$.

$z = 2r^2$

$z = 4 - r^2$

Meet at



$2r^2 = 4 - r^2$

$3r^2 = 4$

$r = \frac{2}{\sqrt{3}}$

$0 < r < 2/\sqrt{3}$

$0 < \theta < 2\pi$

$2r^2 < z < 4 - r^2$

$$\int_{r=0}^{2/\sqrt{3}} \int_{\theta=0}^{2\pi} [(4-r^2) - 2r^2] r \, d\theta \, dr$$

$$= 2\pi \int_0^{2/\sqrt{3}} (4r - 3r^3) \, dr = 2\pi \left[2r^2 - \frac{3}{4} r^4\right]_0^{2/\sqrt{3}}$$

$$\cancel{= 2\pi} \quad = 2\pi \left[2 \frac{4}{3} - \frac{3}{4} \frac{4^2}{3^2}\right] = \boxed{\frac{8\pi}{3}}$$

Please sign the following honor statement:

On my honor, I pledge that I have neither given nor received any aid on this exam.

Signature: ______________________________