LAST NAME:	FIRST NAME:	CIRCLE:			
SOLUTIONS		Makhijani 8:30am	Makhijani 11:30am	Makhijani 2:30pm	Zweck 11:30am

1 /12	2 /12	3 /12	4 /12	5 /15	6 /12	T /75

MATH 2415 [Spring 2019] Exam II, Apr 5th

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 90 minute exam is worth 75 points.

(1) [12 pts]
(a) Calculate the (level set) equation of the tangent plane to the graph of $z = f(x, y) = x^2 + 2y^2 + 3x + y$ at $(x, y) = (2, 1)$.

$$z = f(2,1) + \frac{\partial f}{\partial x}(2,1)(x-2) + \frac{\partial f}{\partial y}(2,1)(y-1)$$

$$\frac{\partial f}{\partial x} = 2x+3 = 7 \text{ @ } (2,1)$$

$$\frac{\partial f}{\partial y} = 4y+1 = 5 \text{ @ } (2,1)$$

$$f(2,1) = 4+2+6+1 = 13$$

So $\boxed{z = 13 + 7(x-2) + 5(y-1)}$

(b) Use your answer to (a) to estimate $f(2.1, 0.8)$.

$$f(2.1, 0.8) \simeq 13 + 7(2.1-2) + 5(0.8-1)$$

$$= 13 + 0.7 - 1 = 12.7$$

(2) [12 pts] Let $f(x,y) = e^x - y^2$.

(a) What is the direction of steepest ascent at $(x,y) = (0,1)$?

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (e^x, -2y) = (e^0, -2)$$

$$= (1,-2) \ @ \ (0,1)$$

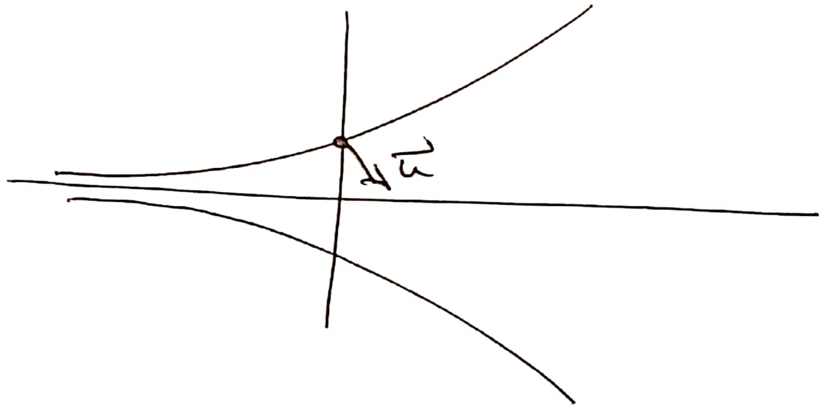
So $\vec{u} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{1}{\sqrt{5}}(1,-2)$

(b) Sketch the level curve $f(x,y) = 0$, together with the direction of steepest ascent of f at $(x,y) = (0,1)$.

$e^x - y^2 = 0$

$y^2 = e^x$

$y = \pm e^{x/2}$



(c) In which directions is that rate of change of f equal to zero at $(x,y) = (0,1)$?

$$0 = D_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u} = (1,-2)\cdot(u_1, u_2)$$

$u_1 - 2u_2 = 0$

$\boxed{u_1 = 2u_2}$ So $\boxed{\vec{u} = \pm(2,1)/\sqrt{5}}$

(d) Let $(x,y) = \mathbf{r}(t)$ be a curve with $\mathbf{r}(2) = (0,1)$ and $\mathbf{r}'(2) = (-2,3)$. Let $z = f(\mathbf{r}(t))$. Find $\frac{dz}{dt}$ at $t = 2$.

$$\frac{dz}{dt} = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$\frac{dz}{dt}(2) = \nabla f(\vec{r}(2)) \cdot \vec{r}\,'(2)$$

$$= \nabla f(0,1) \cdot (-2,3) = (1,-2)\cdot(-2,3) = -8.$$

(3) [12 pts]
(a) Show that

$$(x, y, z) = \mathbf{r}(u, v) = (v, 2\cos u, 3\sin u)$$

is a parametrization of an elliptical cylinder. **Hint:** Find an equation of the form $F(x, y, z) = 0$ for this surface by eliminating u and v from the equations above.

$y = 2\cos u \qquad \cos u = \frac{y}{2}$

$z = 3\sin u \qquad \sin u = \frac{z}{3}$

So $1 = \cos^2 u + \sin^2 u = \left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2$

$F(x,y,z) = \left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 - 1.$

(b) Find a parametrization of the tangent plane to this surface at the point where $(u, v) = (\frac{\pi}{4}, 2)$.

$\vec{l}(s,t) = \vec{r}(\pi/4, 2) + s\frac{\partial \vec{r}}{\partial u}(\pi/4, 2) + t\frac{\partial \vec{r}}{\partial v}(\pi/4, 2)$

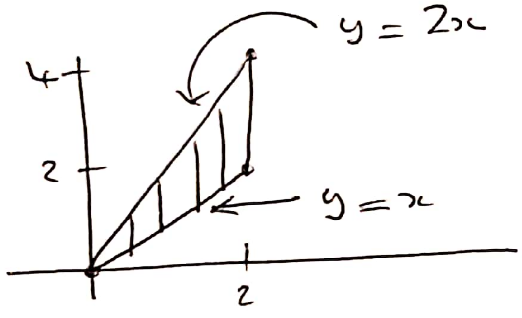
$= \left(2, \frac{2}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) + s\left(0, \frac{-2}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) + t(1, 0, 0)$

as $\frac{\partial \vec{r}}{\partial u} = (0, -2\sin u, 3\cos u) = \left(0, -2/\sqrt{2}, \frac{3}{\sqrt{2}}\right)$

@ $(\pi/4, 2) = (u,v)$

$\frac{\partial \vec{r}}{\partial v} = (1, 0, 0)$

(4) [12 pts] Let D be the triangular domain in the xy-plane with vertices $(0,0)$, $(2,2)$ and $(2,4)$. Calculate $\iint_D (x^2+y^2)\,dA$.



$0 \le x \le 2$

$x \le y \le 2x$

$$\iint_D (x^2+y^2)\,dA = \int_{x=0}^{2}\int_{y=x}^{y=2x} (x^2+y^2)\,dy\,dx$$

$$= \int_{x=0}^{2} \left[x^2 y + y^3/3\right]_{y=x}^{y=2x} dx$$

$$= \int_0^2 \left(x^2 . 2x + \frac{(2x)^3}{3}\right) - \left(x^3 + \frac{x^3}{3}\right) dx$$

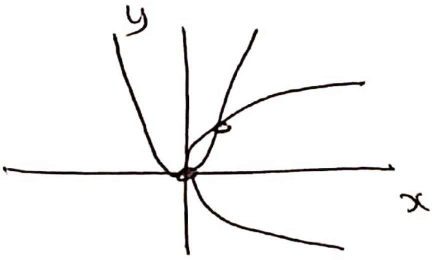
$$= \int_0^2 \left(2 + \frac{8}{3} - 1 - \frac{1}{3}\right) x^3\,dx$$

$$= \frac{10}{3}\left[\frac{x^4}{4}\right]_0^2 = \frac{10}{3}\,\frac{16}{4} = \frac{40}{3}.$$

(5) [15 pts] Find and classify all critical points of $z = f(x, y) = y^3 - 6xy + 8x^3$.

$$0 = \frac{\partial f}{\partial x} = -6y + 24x^2 \Rightarrow y = 4x^2 \quad (1)$$

$$0 = \frac{\partial f}{\partial y} = 3y^2 - 6x \Rightarrow x = \frac{1}{2}y^2 \quad (2)$$



PLUG (1) into (2)

$$x = \frac{1}{2}(4x^2)^2 = 8x^4$$

$$x(1 - 8x^3) = 0$$

$$x = 0 \text{ or } x = \frac{1}{2}.$$

$\boxed{x=0}$ $y = 0$ $\boxed{(0,0)}$

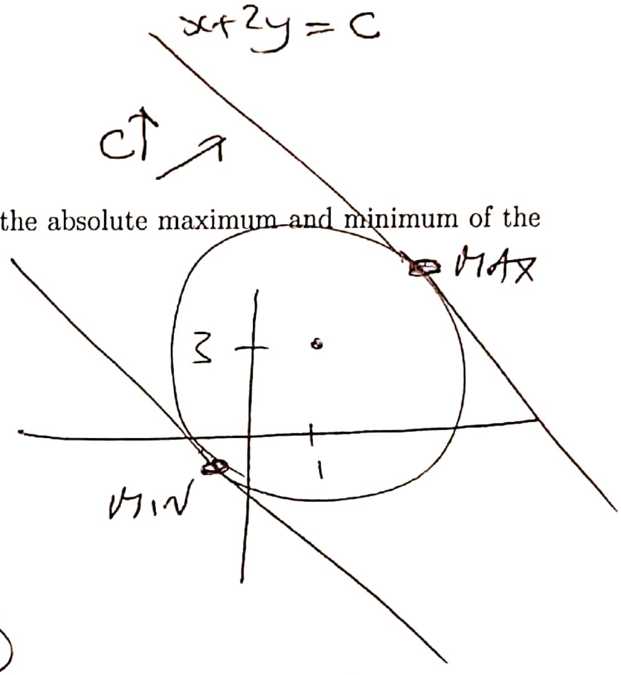
$\boxed{x = 1/2}$ $y = 4\left(\frac{1}{2}\right)^2 = 1$ $\boxed{\left(\frac{1}{2}, 1\right)}$

$$D = \det\begin{bmatrix} 48x & -6 \\ -6 & 6y \end{bmatrix} = 8 \cdot 6^2 xy - 6^2 = 6^2(8xy - 1)$$

$f_{xx} = 48x$

	D	f_{xx}	CLASSIFICATION
$(0,0)$	$-36 < 0$	*	SADDLE POINT
$\left(\frac{1}{2}, 1\right)$	$6^2 \cdot (4-1) > 0$	$24 > 0$	LOCAL MIN

(6) [12 pts] Use the Method of Lagrange Multipliers to find the absolute maximum and minimum of the function $z = x + 2y$ on the circle $(x-1)^2 + (y-3)^2 = 5$.



$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \Rightarrow 1 = 2\lambda(x-1) \quad ①$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \Rightarrow 2 = 2\lambda(y-3) \quad ②$$

$$g = c \Rightarrow (x-1)^2 + (y-3)^2 = 5 \quad ③$$

By ① $x - 1 = \frac{1}{2\lambda}$ $(\lambda \neq 0)$ ④

By ② $y - 3 = \frac{1}{\lambda}$ (") ⑤

Plug into ③ : $5 = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 = \frac{5}{4\lambda^2}$

So $\lambda^2 = \frac{1}{4}$ $\boxed{\lambda = \pm\frac{1}{2}}$

So by ④ $x = 1 \pm 1 = 2$ or 0

by ⑤ $y = 3 \pm 2 = 5$ or 1

So $(x, y, \lambda) = (2, 5, \frac{1}{2})$ $f = 12$ MAX

$(x, y, \lambda) = (0, 1, -\frac{1}{2})$ $f = 2$ MIN.

ALTERNATELY

$\frac{①}{②}$ GIVES

$\frac{1}{2} = \frac{x-1}{y-3}$

SOLVE FOR y TO GET

$y = 2x + 1$.

PLUG INTO ③ TO GET quadratic in x with solutions

$x = 0$, $x = 2$

ETC