SOLUTIONS

LAST NAME:	FIRST NAME:	CIRCLE:	Dahal 4pm	Li 1pm
LAGRANGE	JOSEPH – LOUIS	Li 5:30pm	Zweck 11:30am	Zweck 1pm

MATH 2415 [Fall 2019] Exam II, Nov 1st

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 75 minute exam is worth 75 points. **Your points for each problem will be recorded on the top of the second page.**

(1) [12 pts]

(a) Suppose that $w = f(x, y, z)$, where $x = x(t)$, $y = y(t)$ and $z = z(t)$. Use a tree diagram to write out a formula for $\frac{dw}{dt}$. Use this formula to find $\frac{dw}{dt}$ when $f(x, y, z) = \ln(x^2 + y^2 + z)$, $x = t^3$, $y = \sin t$ and $z = 3t$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \frac{2x(t) \cdot 3t^2 + 2y(t) \cdot \cos t + 3}{x^2(t) + y^2(t) + z(t)}$$

$$= \frac{6t^5 + 2\sin(t)\cos(t) + 3}{t^6 + \sin^2 t + 3t}$$

(b) Find the equation of the tangent plane to the graph of $z = f(x, y) = y^2 e^x$ at $(0, 1)$. Use this tangent plane to approximate $f(0.2, 1.1)$.

$$\frac{\partial f}{\partial x} = y^2 e^x = 1 \ @ \ (0,1)$$

$$\frac{\partial f}{\partial y} = 2y e^x = 2 \ @ \ (0,1)$$

$$f(0,1) = 1$$

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$z = 1 + 1(x - 0) + 2(y - 1) = x + 2y - 1$$

$$f(0.2, 1.1) \simeq 0.2 + 2(1.1) - 1 = 1.4.$$

1 /12	2 /12	3 /12	4 /15	5 /12	6 /12	T /75

(2) [12 pts] Let $f(x,y) = ye^{2x}$.
(a) Find the gradient of f at the point $(0,1)$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2ye^{2x}, e^{2x}) = (2,1) \quad @(0,1)$$

(b) Find the directional derivative of f at the point $(0,1)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

$\vec{x}_0 = (0,1) \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{2}}(1,1)$

$$(D_{\vec{u}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}$$
$$= \nabla f(0,1) \cdot \frac{1}{\sqrt{2}}(1,1) = \frac{1}{\sqrt{2}}(2,1)\cdot(1,1) = \frac{3}{\sqrt{2}}$$

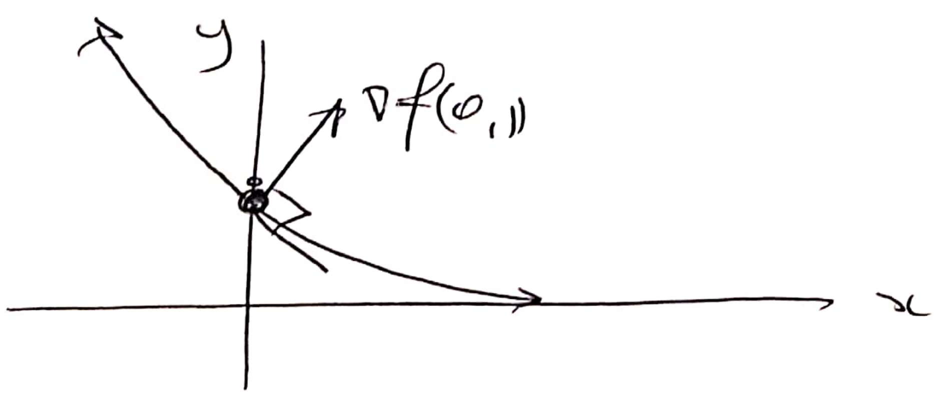
(c) Find the direction of the minimum rate of change in f at (0,1). Also find the minimum rate of change.

$$\vec{u} = \frac{-\nabla f(0,1)}{|\nabla f(0,1)|} = -\frac{1}{\sqrt{5}}(2,1) \text{ as direction}$$

Min Rate $= -|\nabla f(0,1)| = -\sqrt{5}$

(d) Sketch the level curve $f(x,y) = 1$. Add the vector $\nabla f(0,1)$ to your sketch.

$ye^{2x} = 1 \iff y = e^{-2x}$



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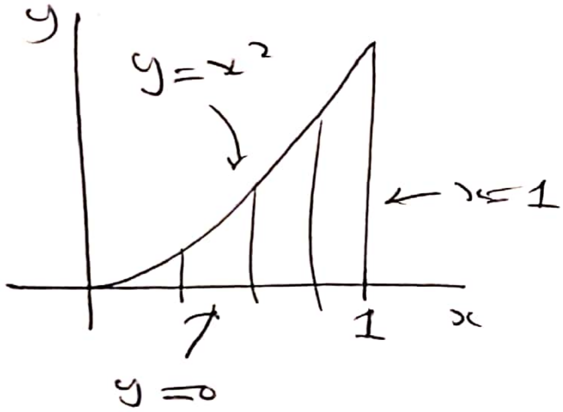
(3) [12 pts] Calculate $\iint_D \cos(x^3+1)\, dA$ where D is the domain in the plane bounded by $y=0$, $x=1$, and $y=x^2$.



$0 \le x \le 1$

$0 \le y \le x^2$

$$\iint_D \cos(x^3+1)\, dA = \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \cos(x^3+1)\, dy\, dx$$

$$= \int_{x=0}^{x=1} \cos(x^3+1) \left[\int_{y=0}^{y=x^2} 1\, dy \right] dx$$

$$= \int_{x=0}^{x=1} \cos(x^3+1)\, x^2\, dx$$

$u = x^3+1$

$du = 3x^2 dx$

$$= \frac{1}{3} \int_{u=0}^{u=2} \cos(u)\, du = \frac{1}{3} \left[\sin u \right]_{u=1}^{u=2}$$

$$= \frac{1}{3} \left[\sin(2) - \sin(1) \right]$$

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(4) [15pts] Find and classify all critical points of the function $f(x, y) = 2x^3 - 3x^2y + 3y^2 + 12x^2$.

$$0 = \frac{\partial f}{\partial x} = 6x^2 - 6xy + 24x$$

$$0 = 6x(x - y + 4) \quad (1)$$

$$0 = \frac{\partial f}{\partial y} = -3x^2 + 6y \quad (2)$$

From (1) $x = 0$ or $y = x + 4$

$\boxed{x = 0}$ By (2) $y = 0.$ $\boxed{(0, 0)}$

$\boxed{y = x + 4}$ By (2) $0 = -3x^2 + 6x + 24$

$= -3(x^2 - 2x - 8)$

$= -3(x - 4)(x + 2)$

So $x = 4 \Rightarrow y = 8$ $\boxed{(4, +8)}$

or $x = -2 \Rightarrow y = 2$ $\boxed{(-2, 2)}$

$$D = \det\begin{bmatrix} 12x - 6y + 24 & -6x \\ -6x & 6 \end{bmatrix}$$

$$D(0, 0) = \det\begin{bmatrix} 24 & 0 \\ 0 & 6 \end{bmatrix} > 0$$

PTO

$$D(4, 8) = \det\begin{bmatrix} 48 - 48 + 24 & -24 \\ -24 & 6 \end{bmatrix} = 6 \times 24 - 24 \times 24 < 0.$$

$$D(-2,2) = \det \begin{bmatrix} -24 -12 + 24 & 12 \\ 12 & 6 \end{bmatrix}$$

$$= -6 \times 12 - 12 \times 12 < 0.$$

	D	f_{xx}	CLASSIFICATION
(0,0)	+	+	LOCAL MIN
(-2,2)	−	*	SADDLE POINT
(4,8)	−	*	SADDLE POINT

(5) [12 pts] Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function $f(x, y) = (x+3)^2 + (y-3)^2$ on the circle $x^2 + y^2 = 8$.

$g = x^2 + y^2$

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \Rightarrow 2(x+3) = \lambda 2x \Rightarrow x+3 = \lambda x \quad (1)$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \Rightarrow 2(y-3) = \lambda 2y \Rightarrow y-3 = \lambda y \quad (2)$$

$$g = k \Rightarrow x^2 + y^2 = 8 \Rightarrow x^2 y^2 = 8 \quad (3)$$

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(1) + (2) : $x + y = \lambda (x+y)$

$\Rightarrow (x+y)(\lambda - 1) = 0$

$\Rightarrow y = -x$ or $\lambda = 1$

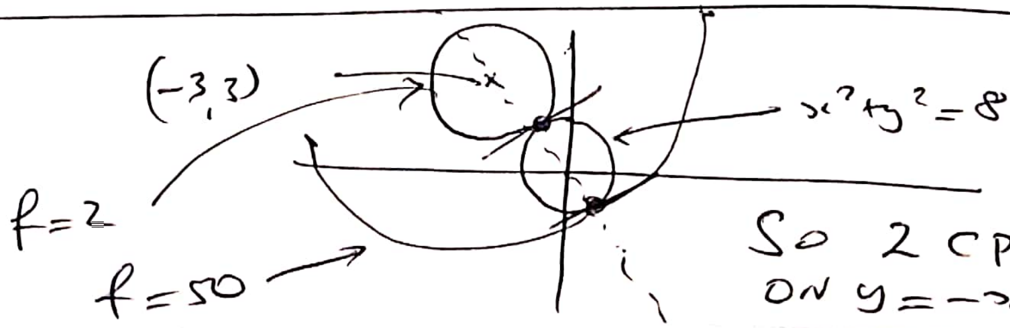
$\boxed{\lambda = 1}$ By (1) $x + 3 = x \Rightarrow 3 = 0$ ✗, Dead end

$\boxed{y = -x}$ By (3) $2x^2 = 8 \Rightarrow x = \pm 2.$

Since $y = -x$ get $y = \mp 2$

$(2, -2)$: $\lambda = \frac{x+3}{x} = \frac{5}{2}$ $\quad f(2,-2) = 50$

$(-2, 2)$: $\lambda = \frac{x+3}{x} = -\frac{1}{2}$ $\quad f(-2,2) = 2$



So 2 CPs

ON $y = -x$

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(6) [12 pts] Let S be the surface with parametrization

$$(x, y, z) = \mathbf{r}(u, v) = (u\cos v, u\sin v, u^2), \qquad \text{for } 0 \le u \le 3 \text{ and } 0 \le v \le 2\pi.$$

(a) Show that S is part of a paraboloid. **Hint:** Find an equation of the form $F(x, y, z) = 0$ for this surface.

$$x^2 + y^2 = (u\cos v)^2 + (u\sin v)^2$$

$$= u^2(\cos^2 v + \sin^2 v) = u^2 = z$$

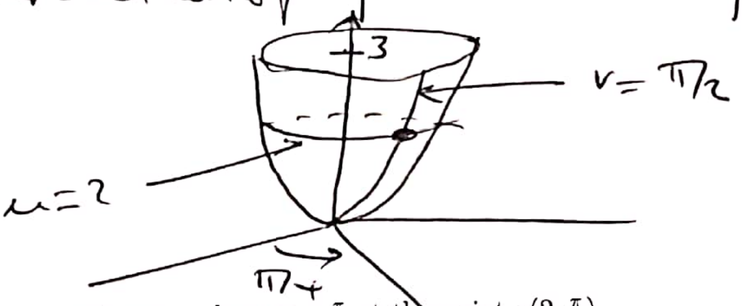
So for all (u,v) $\vec{r}(u,v)$ lies on $z = x^2 + y^2$

which is a paraboloid

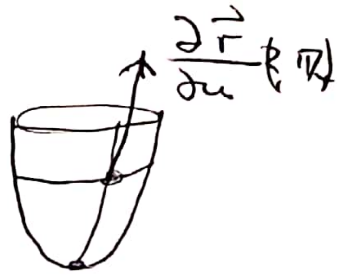
(b) Sketch the surface S, together with the grid curves where (i) $u = 2$ and (ii) $v = \frac{\pi}{4}$. (Label these curves!)

$\underline{u=2}$ is intersection of paraboloid with plane $z=4$.
– A circle.

$\underline{v = \pi/4}$ is intersection of paraboloid with plane at $\theta = \pi/4$



(c) Calculate the tangent vector to the grid curve where $v = \frac{\pi}{4}$ at the point $\mathbf{r}(2, \frac{\pi}{4})$.



$$\vec{v} = \frac{\partial \vec{r}}{\partial u}(2, \pi/4) \qquad (v \text{ is constant})$$

$$\frac{\partial \vec{r}}{\partial u} = (\cos v, \sin v, 2u) = (\cos \pi/4, \sin \pi/4, 4)$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 4\right) \text{ @ } (u,v) = (2, \pi/4)$$