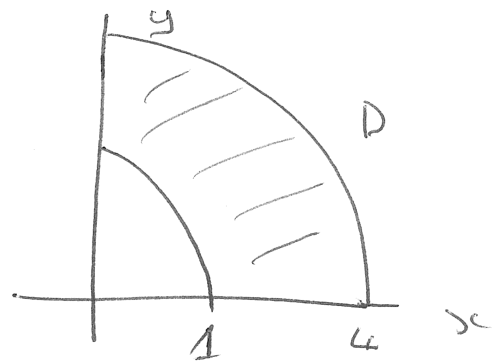
LAST NAME:	FIRST NAME:	CIRCLE:
SOLUTIONS		Li 2:30pm Li 5:30pm Zweck 10am Zweck 1pm

1 /10	2 /10	3 /10	4 /10	5 /10	
6 /10	7 /10	8 /10	9 /10	10 /10	T /100

MATH 2415 Final Exam, Fall 2017

No books or notes! **NO CALCULATORS! Show all work and give complete explanations**. This 2 hours 45 mins exam is worth 100 points.

(1) [10 pts] Evaluate the integral $\iint_D x\,dA$ where D is that quarter of the annulus $1 \le x^2 + y^2 \le 16$ that is in the first quadrant.



$$\iint_D x\,dA = \int_{\theta=0}^{\pi/2}\int_{r=1}^{4} (r\cos\theta)\, r\,dr\,d\theta$$

$$= \left[\int_0^{\pi/2}\cos\theta\,d\theta\right]\left[\int_1^4 r^2\,dr\right]$$

$$= [\sin\theta]_0^{\pi/2}\left[\frac{r^3}{3}\right]_1^4$$

$$= \frac{63}{3} = 21$$

(2) [10 pts] Let $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} + xyz\mathbf{k}$. Calculate

(a) curl($\mathbf{F}$)

$$= \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & 2y & xyz \end{vmatrix}$$

$$= xz\vec{i} - yz\vec{j}$$

(b) div($\mathbf{F}$)

$$= \nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(x\vec{i} + 2y\vec{j} + xyz\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(xyz)$$

$$= 1 + 2 + xy$$

$$= 3 + xy$$

(3) [10 pts] Let $z = f(x, y) = x^2 + 4y^2$, and let C be the level curve $f(x, y) = 4$.
(a) Find a parametrization of the curve C.

$$x^2 + 4y^2 = 4$$

$$x = 2\cos t \qquad 0 \le t \le 2\pi$$

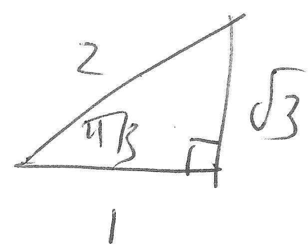
$$y = \sin t$$

As $x^2 + 4y^2 = (2\cos t)^2 + 4(\sin t)^2 = 4$

(b) Use your answer to (a) to find a vector, $\mathbf{v}$, that is tangent to the curve C at the point $(x, y) = (1, \sqrt{3}/2)$.

$$\vec{r}(t) = (2\cos t, \sin t)$$

$$\vec{r}\,'(t) = (-2\sin t, \cos t)$$



$2\cos t = 1$

$\sin t = \sqrt{3}/2 \Rightarrow t = \frac{\pi}{3}$

$$\vec{v} = \vec{r}\,'(\pi/3) = \left(-2 \cdot \frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left(-\sqrt{3}, \frac{1}{2}\right)$$

(c) Find the directional derivative of f in the direction of the vector $\mathbf{v}$ in (b) at the point $(x, y) = (1, \sqrt{3}/2)$.

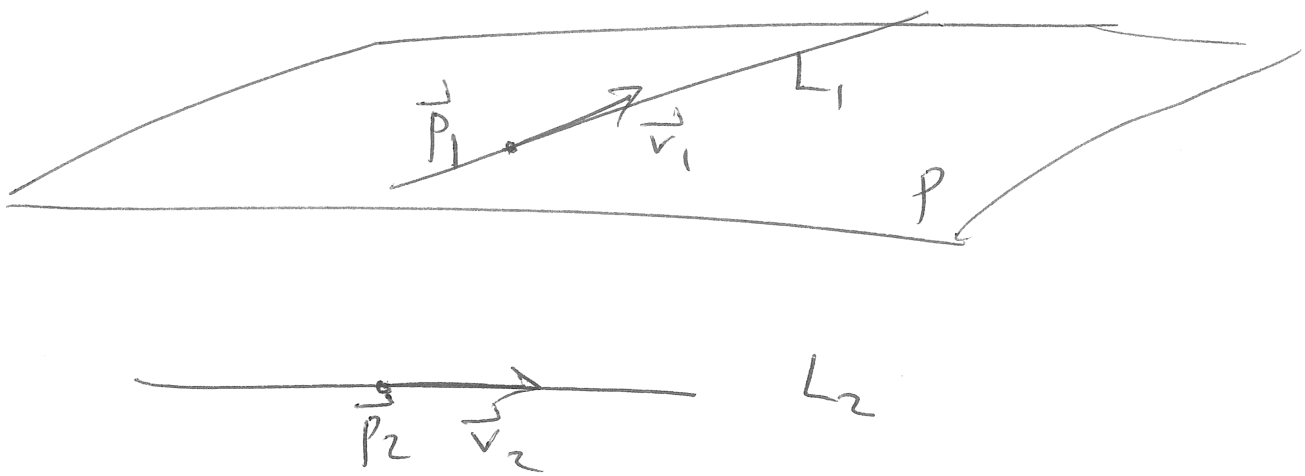
$$(D_{\vec{v}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{v} = 0$$

as $\nabla f(\vec{x}_0)$ is $\perp$ to C while $\vec{v}$ is tangent to C

OR $\nabla f = (2x, 8y) = (2, 4\sqrt{3})$ @ $\vec{x}_0 = (1, \sqrt{3}/2)$

So $(D_{\vec{v}} f)(\vec{x}_0) = (2, 4\sqrt{3}) \cdot (\sqrt{3}, \frac{1}{2}) = 0$

(4) [10 pts] Find an equation of the form $Ax+By+Cz = D$ for the plane that contains the line parametrized by $\mathbf{r}_1(t) = (1+2t, 3+4t, 5-t)$ and that is parallel to the line parametrized by $\mathbf{r}_2(t) = (2+t, 3, -1+4t)$.



$\vec r_1(t) = \vec P_1 + t\vec v_1 \Rightarrow \vec P_1 = (1, 3, 5)$ point in P

$\vec v_1 = (2, 4, -1)$

$\vec r_2(t) = \vec P_2 + t\vec v_2 \Rightarrow \vec v_2 = (1, 0, 4)$

$\vec n = \vec v_1 \times \vec v_2$ is $\perp$ to P

$$= \begin{vmatrix} \vec i & \vec j & \vec k \\ 2 & 4 & -1 \\ 1 & 0 & 4 \end{vmatrix} = (16, -9, -4)$$

So $0 = (\vec r - \vec p)\cdot \vec n = (x-1, y-3, z-5)\cdot(16, -9, -4)$

$\Rightarrow 16x - 9y - 4z = -31$

(5) [10 pts]
(a) Let $u(x,t) = \sin(x+2t)$. Show that u satisfies the wave equation $u_{tt} = 4u_{xx}$.

$u_t = 2\cos(x+2t)$

$u_{tt} = -4\sin(x+2t)$

$u_x = \cos(x+2t)$

$u_{xx} = -\sin(x+2t)$

So $4u_{xx} = -4\sin(x+2t) = u_{tt}$

$u_{tt} = 4u_{xx}$

(b) Let $\mathbf{r}(t) = (t^2, t^3)$ and let $z = f(x,y)$ be a function so that $f(1,1) = 6$, $\frac{\partial f}{\partial x}(1,1) = 4$, and $\frac{\partial f}{\partial y}(1,1) = 5$. Let $g(t) = f(\mathbf{r}(t))$. Calculate $g'(1)$.

$\vec{r}(1) = (1,1)$

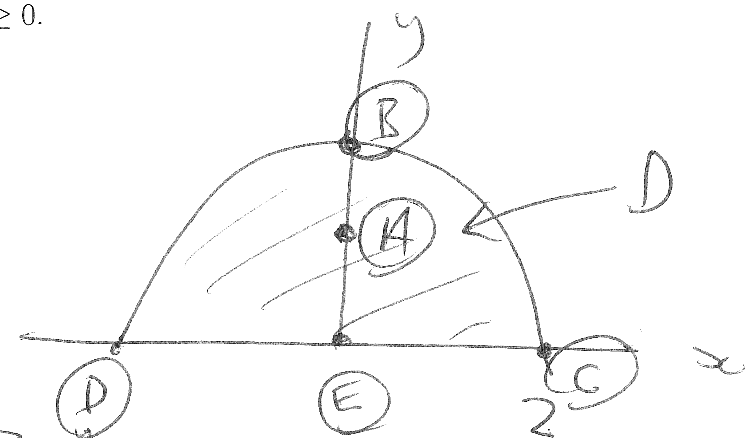
$\vec{r}\,'(t) = (2t, 3t^2)$

$\vec{r}\,'(1) = (2,3)$

$g'(1) = \nabla f(\vec{r}(1)) \cdot \vec{r}\,'(1)$

$= (4,5) \cdot (2,3) = 23$

(6) [10 pts] Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + y^2 - 2y$ on the upper half disk where $x^2 + y^2 \leq 4$ and $y \geq 0$.



CRITICAL PTS IN D

$\nabla f = (2x, 2y-2) = (0,0)$ @

$(x,y) = (0,1)$ (A)

$f(0,1) = 1-2 = -1$ MIN

UPPER SEMICIRCLE

$\vec{r}(t) = (2\cos t, 2\sin t)$ $\quad 0 < t < \pi$

$g(t) = f(\vec{r}(t)) = 4 - 4\sin t$

~~MAX @~~ $\quad g'(t) = -4\cos t = 0$ @ $t = \pi/2$

So

(B) $(x,y) = (0,2)$ $\quad f(0,2) = 4 - 4 = 0$

ALSO (C) $f(2,0) = 4$ MAX $\quad$ (D) $f(-2,0) = 4$ MAX

FINALLY ON

$y = 0$

$-2 < x < 2$

Have

$h(x) = f(x,0)$

$= x^2$

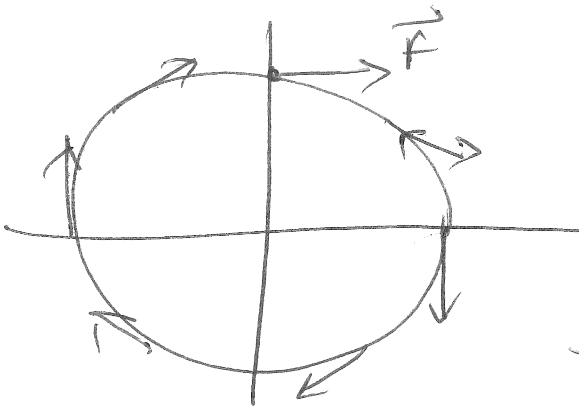
CPT @ $x = 0$

$f(0,0) = 0$

(7) [10 pts] Let $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$ and let C be the unit circle oriented counter clockwise.

(a) Use a sketch to determine whether $\int_C \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.

$\vec{T}$ = UNIT TANGENT VECTOR TO C



$\vec{F} = -\vec{T}$

So $\vec{F} \cdot \vec{T} = -\vec{T} \cdot \vec{T}$

$= -1 < 0$

So $\int_C \vec{F} \cdot d\vec{r}$

$= \int_C (\vec{F} \cdot \vec{T})\, ds < 0$

(b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the definition of the line integral.

$\vec{r}(t) = (\cos t, \sin t) \quad 0 < t < 2\pi$

$\vec{r}'(t) = (-\sin t, \cos t)$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\sin t, -\cos t) \cdot (-\sin t, \cos t)\, dt$

$= -\int_0^{2\pi} 1\, dt = -2\pi$

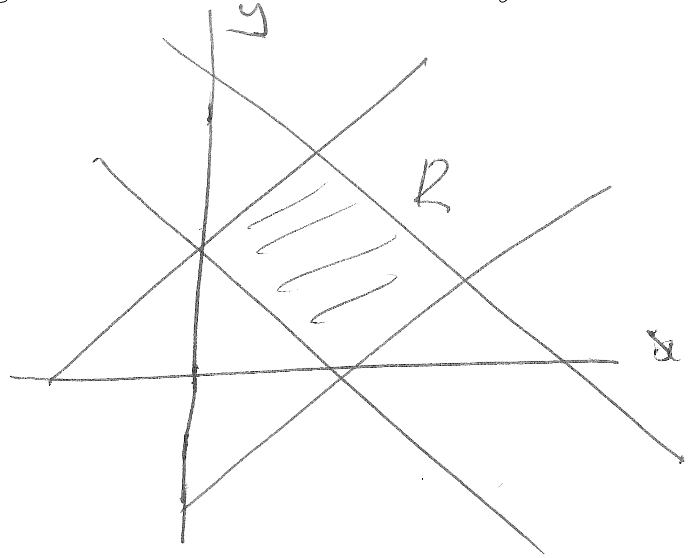
(c) Is $\mathbf{F}$ conservative? Justify your answer.

$P = y \qquad Q = -x$

$\frac{\partial P}{\partial y} = 1 \neq -1 = \frac{\partial Q}{\partial x}$

$\vec{F}$ NOT CONS

(8) [10 pts] Evaluate $\iint_R (x-y)^2 e^{x+y}\,dxdy$ where R is the parallelogram bounded by $x+y=1$, $x+y=3$, $x-y=-2$ and $x-y=1$. **Hint:** Use the Change of Variables Theorem with $u=x+y$ and $v=x-y$.



$u = x+y$

$v = x-y$

$\Rightarrow$

$u+v = 2x$

$x = \frac{u+v}{2}$

$y = u - x = u - \frac{u+v}{2} = \frac{u-v}{2}$

$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$

$= -\frac{\cancel{2}}{4} = -\frac{1}{2}$

$I = \int_{u=1}^{3} \int_{v=-2}^{1} v^2 e^u \frac{1}{2}\, dv\,du$

$= \frac{1}{2}(e^3 - e)\left[\frac{v^3}{3}\right]_{-2}^{1} = \frac{e^3-e}{2}\ \frac{1+8}{3}$

(9) [10 pts] Let S be the surface parametrized by

$$(x, y, z) \;=\; \mathbf{r}(\theta, \phi) \;=\; (2\cos\theta\sin\phi, 3\sin\theta\sin\phi, 4\cos\phi).$$

Calculate a parametrization of the tangent plane to the surface S at the point where $(\theta, \phi) = (\pi/4, \pi/2)$.

$\vec{L}(s,t) = \vec{p} + s\vec{v} + t\vec{w}$ with

$\vec{p} = \vec{r}(\pi/4, \pi/2) = \left(\sqrt{2}, \frac{3}{\sqrt{2}}, 0\right)$

$\vec{v} = \frac{\partial \vec{r}}{\partial \theta}(\pi/4, \pi/2)$

$= \left(-2\sin\theta\sin\phi, 3\cos\theta\sin\phi, 0\right)\Big|_{(\pi/4, \pi/2)}$

$= \left(-\sqrt{2}, \frac{3}{\sqrt{2}}, 0\right)$

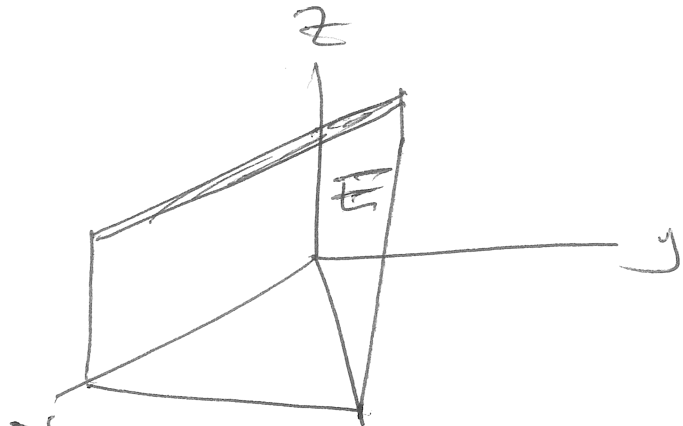
$\vec{w} = \frac{\partial \vec{r}}{\partial \phi}(\pi/4, \pi/2)$

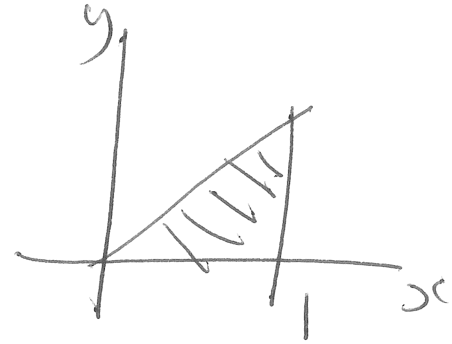
$= \left(2\cos\theta\cos\phi, 3\sin\theta\cos\phi, -4\sin\phi\right)\Big|_{\left(\frac{\pi}{4}, \frac{\pi}{2}\right)}$

$= (0, 0, -4)$

So $\vec{L}(s,t) = \left(\sqrt{2}(1-s), \frac{3}{\sqrt{2}}(1+s), -4t\right)$

(10) [10 pts] Let E be the solid region bounded by the planes $y = x$, $x = 1$, $y = 0$, $z = 1 + y$, and $z = 0$. Calculate $\iiint_E x\, dV$.





$0 < x < 1$

$0 < y < x$

$0 < z < 1+y$

$$\iiint_E x\, dV = \int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{1+y} x \, dz\, dy\, dx$$

$$= \int_{x=0}^{1} x \int_{y=0}^{x} (1+y)\, dy\, dx$$

$$= \int_0^1 x\left(x + x^2/2\right) dx$$

$$= \frac{11}{24}$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________