

## Math 2415

### Problem Section #4

Make sure you do some problems from each section.

#### 12.6: Cylinders and Quadric Surfaces

We recommend that each of you attempts to sketch the curves and surfaces below for yourself. Our goal is not to become great artists (though that would be nice). Rather the sketches are supposed to help you construct an accurate visual representation of the curve or surface in your mind. As you are sketching you can explain to each other what you are doing, how you are doing it, and why it is a good thing to do!

1. In this problem we will make a picture of the surface  $y = x^2$  in  $\mathbb{R}^3$  as follows. The approach below can be used to plot any surface in space whose equation is independent of one of the variables,  $(x, y, z)$ .
  - (a) In  $\mathbb{R}^2$  the equation  $y = x^2$  represents a curve in the plane. Sketch it.
  - (b) However, the set of points  $(x, y, z)$  in  $\mathbb{R}^3$  for which  $y = x^2$  represents a surface in space. To see why, suppose that  $(x_0, y_0)$  is a point on the curve  $y = x^2$  in  $\mathbb{R}^2$ . Explain why for every real number  $z_0$ , the point  $(x_0, y_0, z_0)$  is on the surface  $y = x^2$  in  $\mathbb{R}^3$ .
  - (c) To visualize this surface, translate the curve you sketched in (a) up and down along the  $z$ -axis to form a surface. Use (b) to justify why this works.
  - (d) What would you need to do to make a picture of the surface  $x^2 + 4z^2 = 1$ ?
2. In this problem we will visualize the surface  $z^2 - x^2 - y^2 = 4$  in space. The approach below can be used to plot a surface in space whenever  $x$  and  $y$  always appear in the combination  $x^2 + y^2$  in the equation.
  - (a) First check that when you convert the equation  $z^2 - x^2 - y^2 = 4$  to cylindrical coordinates you get  $z^2 - r^2 = 4$ .
  - (b) Plot the curve  $z^2 - r^2 = 4$  in the  $(r, z)$ -plane.
  - (c) Suppose that  $(r_0, z_0)$  is a point on the curve in (b). Explain why for every  $\theta_0$  between 0 and  $2\pi$ , the point with cylindrical coordinates,  $(r_0, \theta_0, z_0)$ , is on the surface  $z^2 - x^2 - y^2 = 4$ .
  - (d) Rotate the curve in (b) about the  $z$ -axis to obtain a surface in space.
  - (e) Use (c) to explain why the resulting surface is the set of points  $(x, y, z)$  in space so that  $z^2 - x^2 - y^2 = 4$ .
  - (f) What would you need to do to make a picture of the surface  $x^2 + z^2 = y^2$ ?
3. Make a labelled sketch of the traces (slices) of the surface

$$x^2 - y^2 + 4z^2 = 0$$

in the planes  $x = 0$ ,  $z = 0$ , and  $y = k$  for  $k = 0, \pm 1, \pm 2$ . Then make a labelled sketch of the surface. *What do we mean by labelled?* Label the slice you are sketching (e.g.  $x = 0$  or

$y = k$ ), label all axes (e.g.  $x$  or  $z$ ), indicate values of intercepts and other significant points such as vertices, add asymptotes as appropriate, and when graphing multiple curves on the same plot label them with their  $k$ -value. Transfer as much of this information as you can to the sketch of the surface in 3-dimensional space. See the examples in Dr. Zweck's lecture notes: [Lecture 6A \(How to Sketch Quadric Surfaces\)](#) and [Lecture 6B \(Quadric Surfaces\)](#)

**We will take off points for graphing questions like this on the exams if you do not include labeling as requested above!**

4. Use the approach in the previous problem to sketch  $y = 4x^2 + z^2$
5. Make a labelled sketch of the traces of the surface

$$y^2 - 4x^2 - z^2 = 1$$

in the planes  $x = 0$ ,  $z = 0$ , and  $y = k$  for  $k = 0, \pm 1, \pm 2$ . Then sketch the surface.

6. **[Challenge]** Describe and sketch the surface  $z = xy$ . How is it related to the surface  $z = x^2 - y^2$ ?

### 13.1: Parametrized Curves

1. Let  $C$  be the curve in the  $xy$ -plane given by  $x = 2 \sin t$ ,  $y = -\cos t$ . Find an equation of the form  $f(x, y) = 0$  for this curve. Sketch the curve. What is the starting point (where  $t = 0$ )? What direction does the parametrization go along the curve?
2. Describe the motion given by the parametrized curve  $x = \cos t$ ,  $y = -\cos t$  for  $0 \leq t \leq 3\pi$ . You could draw on a white board (make a video if you like!) and provide an accompanying verbal/written explanation.
3. Consider the parametrized curve  $\mathbf{r}(t) = \sin t \mathbf{i} + \sin t \mathbf{j} + \cos t \mathbf{k}$  for  $0 \leq t \leq \pi/2$ .
  - (a) Use algebra to find a cylinder on which this curves lies (there are actually two of them).
  - (b) Use algebra to find a plane on which this curve lies.
  - (c) Use the fact the curve lies on both the plane and cylinder to sketch it. (Note the limited range of  $t$  values!)
  - (d) *Ask your TA for a copy of the blue 3D printed model* and find the curve on the model.
4. Sketch the curve  $x = t \cos t$ ,  $y = t$ ,  $z = 2t \sin t$ , for  $0 \leq t \leq 2\pi$ . To do so, first show that this curve lies on an elliptical cone aligned with the  $y$ -axis. Next explain why shadow of the curve on the  $xz$ -plane is a counter-clockwise spiral. Sketch this spiral in the  $xz$ -plane. Finally try sketching the curve in space.

### 13.1: Curves as Interscetions of Surfaces

1. We can find a parametrization of the curve obtained by intersecting the cone  $z = \sqrt{x^2 + y^2}$  and the plane  $y = 1 + x$  as follows:
  - (a) Parametrize the line  $y = 1 + x$  in the  $xy$ -plane.
  - (b) Now that you know how  $x$  and  $y$  depend on  $t$ , use  $z = \sqrt{x^2 + y^2}$  to get a formula for  $z$  in terms of  $t$ .

(c) In general, what do you need to know about the algebraic forms of the two surfaces in order to use an approach like this? In particular, try a similar approach for the curves obtained by intersecting the pairs of surfaces:

- i.  $z = x^2 + y^2$  and  $x + y = 2$ .
- ii.  $x^2 + 2y^2 + z^2 = 4$  and  $x = z^2$ .
- iii.  $x^2 + y^2 = 4$  and  $z = xy$  **[Hint:** This one is a bit different: Start by parametrizing the circle  $x^2 + y^2 = 4$  in the  $xy$ -plane.]
- iv.  $z = \sqrt{x^2 + y^2}$  and  $z = y + 3$ . **Hint:** This time set the right hand sides of the two equations to be equal. That will give you an equation in  $x$  and  $y$  that represents the shadow of our curve on the  $xy$ -plane. Solve that equation for  $y$  in terms of  $x$ . Then proceed as before.

If you have extra time, try sketching these curves.