NAME: SOLUTIONS	CIRCLE:	Zweck 11:30am	Zweck 2:30pm

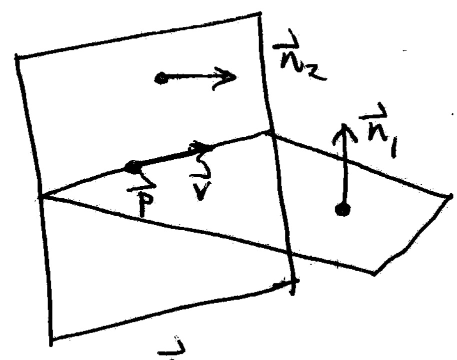
1 /9	2 /12	3 /10	4 /12	5 /12	6 /12	7 /8	T /75

MATH 2415 (Fall ~~2014~~ 2015) Exam I, Oct ~~3rd~~ 9TH

No books or notes! You may use a scientific calculator provided it does not allow for access to the internet. Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 75 minute exam is worth 75 points.

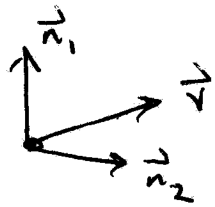
(1) [9 pts] Find a parametrization of the line which is given by the intersection of the planes $x + y + z = 0$ and $x + 2y + 3z = 1$.

$\vec{r}(t) = \vec{p} + t\vec{v}$



where $\vec{p}$ is a point on both planes and $\vec{v}$ is a vector along line.

If $\vec{n}_1, \vec{n}_2$ are normals to the two planes then $\vec{v} \perp \vec{n}_1$ and $\vec{v} \perp \vec{n}_2$ since $\vec{v}$ is in ~~the~~ both planes.



So $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \vec{i} - 2\vec{j} + \vec{k} = (1, -2, 1)$

NEXT ~~to~~ find $\vec{p}$:

$x + y + z = 0$
$x + 2y + 3z = 1$

Set $x = 0$. Solve 2 equations in 2 unknowns: $z = -y$.

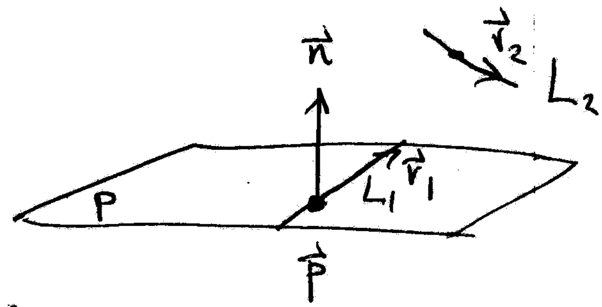
$1 = 2y + 3z = 2y - 3y = -y$

So $\vec{p} = (x, y, z) = (0, -1, 1)$

$\vec{r}(t) = (0, -1, 1) + t\,(1, -2, 1)$

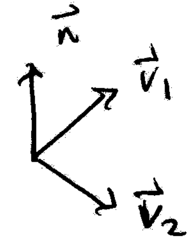
(2) [12 pts] Let L_1 be the line with parametrization $\mathbf{r}_1(t) = (1-2t, 2+3t, 5+4t)$ and let L_2 be the line with parametrization $\mathbf{r}_2(t) = (3+t, 4+t, 2-2t)$.

(a) Find an equation of the form $Ax + By + Cz = D$ for the plane that contains the line L_1 and is parallel to the line L_2.



$\vec{r}_1(t) = (1,2,5) + t(-2,3,4)$
$= \vec{p} + t\vec{v}_1$

$\vec{r}_2(t) = (3,4,2) + t(1,1,-2)$
$= \vec{q} + t\vec{v}_2$

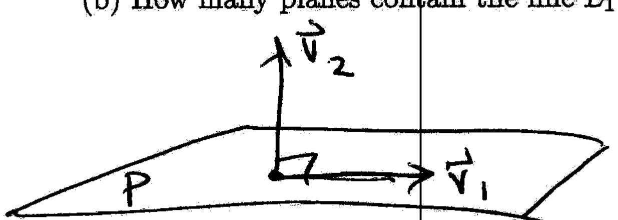


Since our plane contains L_1 the point $\vec{p}$ is in the plane.
and the vector $\vec{v}_1$ is in plane.
Since L_2 is parallel ~~to~~ plane, $\vec{v}_2$ is a vector in plane
So $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 4 \\ 1 & 1 & -2 \end{vmatrix} = (-10, 0, -5)$ is normal to plane

$0 = \vec{n} \cdot (\vec{x} - \vec{p}) = (-10, 0, -5) \cdot (x-1, y-2, z-5)$

$\boxed{0 = -10(x-1) - 5(z-5)}$

(b) How many planes contain the line L_1 and are perpendicular to the plane L_2? Justify your answer.



In order for a plane P to contain line L_1 (and hence vector $\vec{v}_1$) and be perpendicular to line L_2 (and hence have normal vector $\vec{v}_2$ to plane) we need $\vec{v}_2 \perp \vec{v}_1$, ie $\vec{v}_2 \cdot \vec{v}_1 = 0$

But $\vec{v}_1 \cdot \vec{v}_2 = (-2,3,4) \cdot (1,1,-2) = -2 + 3 - 8 \neq 0$

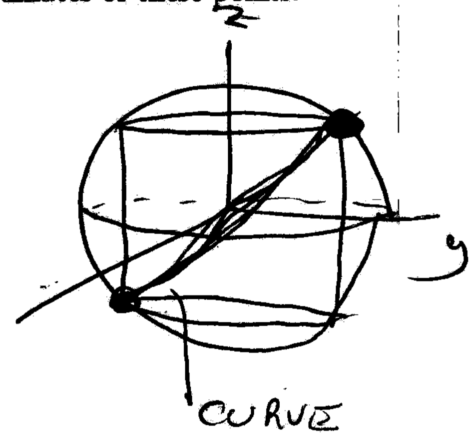
SO THERE ARE NO SUCH PLANES

(3) [10 pts] Let C be the curve with parametrization $\mathbf{r}(t) = (\cos t, \sin t, \frac{2\sqrt{3}}{\pi} t)$ and let S be the sphere of radius 2, centered at the origin.

(a) The curve, C, intersects the surface, S, in two points. Find the coordinates of these points.

$x = \cos t$
$y = \sin t$
$z = \frac{2\sqrt{3}}{\pi} t$

Equation of sphere radius 2 is
$x^2+y^2+z^2 = 2^2 = 4$



Plug curve parametrization into sphere equation + solve for t:

$$4 = \cos^2 t + \sin^2 t + \frac{4\times 3}{\pi^2} t^2 = 1 + \frac{12}{\pi^2} t^2$$

$$\boxed{t = \pm \frac{\pi}{2}} \qquad (x,y,z) = (0, \pm 1, \pm\sqrt{3})$$

(b) Find the arclength of that segment of the curve C that lies within the sphere S.

$$\vec{r}'(t) = \left(-\sin t, \cos t, \frac{2\sqrt{3}}{\pi}\right)$$

$$L = \int_{-\pi/2}^{\pi/2} |\vec{r}'(t)|\, dt$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2 + \left(\frac{2\sqrt{3}}{\pi}\right)^2}\, dt$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \frac{12}{\pi^2}} = \pi\sqrt{1+\frac{12}{\pi^2}}$$

$$= \sqrt{12 + \pi^3}$$

(4) [12 pts]

(a) Calculate the linearization of the function $z = f(x, y) = x^3y^2$ about the point $(x_0, y_0) = (2, -1)$.

$$L(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$(x_0,y_0) = (2,-1)$ $\qquad f(2,-1) = 2^3(-1)^2 = 8$

$$\frac{\partial f}{\partial x} = 3x^2y^2 = 3 \times 4 \times (-1)^2 = 12 \text{ @ } (2,-1)$$

$$\frac{\partial f}{\partial y} = 2x^3y = 2 \cdot 2^3(-1) = -16 \text{ @ } (2,-1)$$

So

$$z = L(x,y) = 8 + 12(x-1) - 16(y+1)$$

(b) Let $z = f(x, y)$ and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Using the table of values below, calculate the partial derivatives $g_u(0, 0)$ and $g_v(0, 0)$.

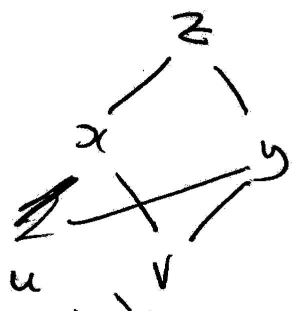
$$\boxed{\frac{\partial x}{\partial u} = e^u \quad \frac{\partial y}{\partial u} = e^u \quad \frac{\partial x}{\partial v} = \cos v \quad \frac{\partial y}{\partial v} = -\sin v}$$

	f	g	f_x	f_y
$(0,0)$	3	6	4	8
$(1,2)$	6	3	2	5

$\vec r(0,0) = (e^0 + \sin 0, e^0 + \cos 0) = (1,2)$

Let $\vec r(u,v) = (e^u + \sin v, e^u + \cos v) = (x,y)$

$z = g(u,v) = f(x(u,v), y(u,v))$



$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial g}{\partial u}(0,0) = \frac{\partial f}{\partial x}(\vec r(0,0))\frac{\partial x}{\partial u}(0,0) + \frac{\partial f}{\partial y}(\vec r(0,0))\frac{\partial y}{\partial u}(0,0)$$

$$= \frac{\partial f}{\partial x}(1,2)\, e^0 + \frac{\partial f}{\partial y}(1,2)\, e^0$$

$$= 2 \times 1 + 5 \times 1 = 7$$

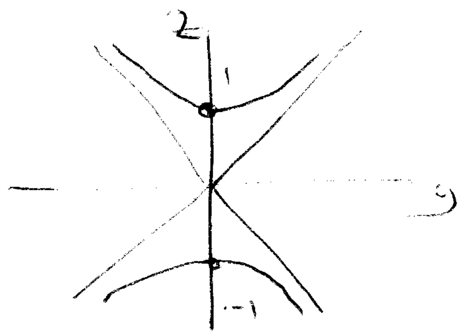
$$\frac{\partial g}{\partial v}(0,0) = f_x(1,2)\cos 0 + f_y(1,2)(-\sin 0) = 2$$

(5) [12 pts] Make a labelled sketch of the traces of the surface

$$4x^2 - 2y^2 + z^2 = 1$$

in the planes $x = 0$, $z = 0$, and $y = k$ for $k = 0, \pm 1, \pm 2$. Then sketch the surface.

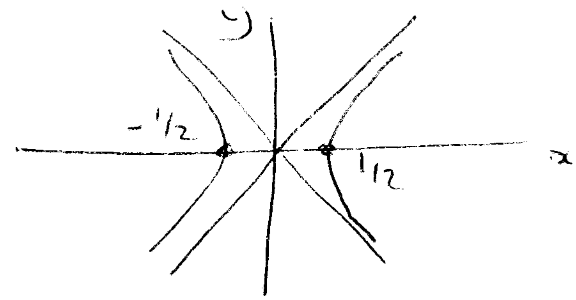
$\boxed{x=0}$ $z^2 - 2y^2 = 1$



ASYMPTOTES $z = \pm\sqrt{2}\,y$

$y=0$, $z = \pm 1$

$\boxed{z=0}$ $4x^2 - 2y^2 = 1$



ASYMPTOTES. $2x = \pm\sqrt{2}\,y$

$y = \pm\sqrt{2}\,x$

$y=0$, $x = \pm\frac{1}{2}$

$\boxed{y=k}$

$4x^2 + z^2 = 1 + 2k^2$

$\underline{k=0}$ $4x^2 + z^2 = 1$

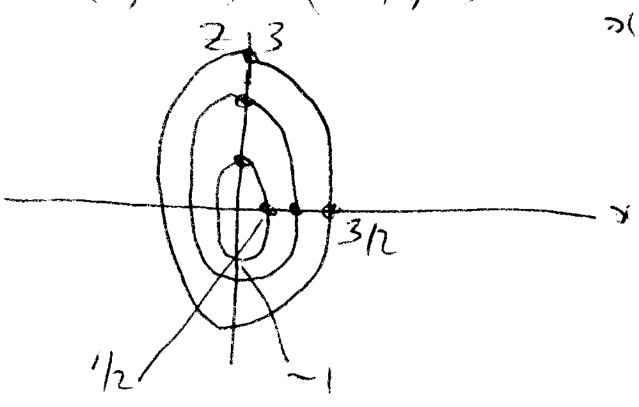
$(0, \pm 1)$ $(\pm\frac{1}{2}, 0)$

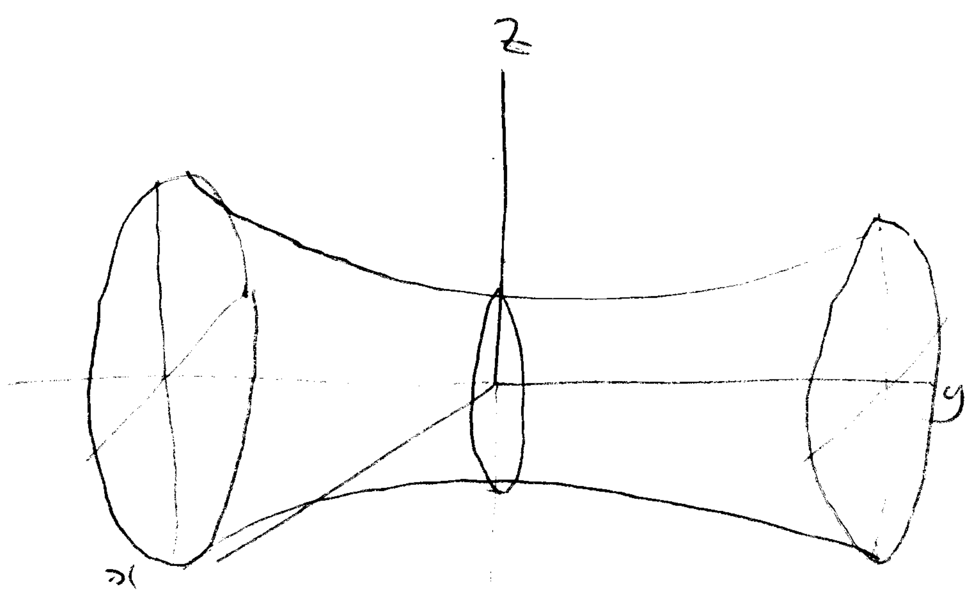
$k = \pm 1$ $4x^2 + z^2 = 3$

$(0, \pm\sqrt{3})$, $(\frac{\pm\sqrt{3}}{2}, 0)$

$k = \pm 2$ $4x^2 + z^2 = 9$

$(0, \pm 3)$ $(\pm 3/2, 0)$





(6) [12 pts] Find the limit if it exists, or show that the limit does not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2}$

Set $x = r\cos\theta$

$y = r\sin\theta$

IDEA ignoring $\cos\theta$, $\sin\theta$, we have $\frac{r^4}{r^2} = r^2 \to 0$ as $r \to 0$

So expect limit to exist.

DETAILS

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{r\cos\theta\, r\sin\theta(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2}$$

$$= \lim_{r\to 0} \frac{r^4}{r^2} \underbrace{\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta)}_{F(\theta)} = \lim_{r\to 0} r^2 F(\theta) = 0$$

by Squeeze Thm as $F(\theta)$ is bounded

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

Find two curves that go to (0,0) along which limits are different. So limit DNE

$x=0$
$$\lim_{(0,y)\to(0,0)} \frac{0^2 y}{0^4+y^2} = \lim_{x\to 0} \frac{0}{y^2} = \lim_{x\to 0} 0 = 0$$

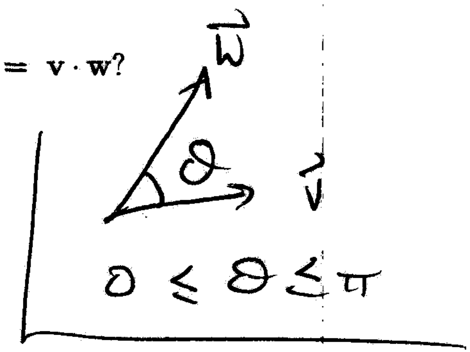
$y=x^3$
$$\lim_{(x,x^2)\to(0,0)} \frac{x^2 \cdot x^2}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{x^4+x^4}$$

$$= \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

SINCE $0 \neq \frac{1}{2}$ LIMIT DNE

(7) [8 pts]

(a) Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors in $\mathbf{R}^3$. Under what conditions is $|\mathbf{v} \times \mathbf{w}| = \mathbf{v} \cdot \mathbf{w}$?



$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$$

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$$

So $|\vec{v} \times \vec{w}| = \vec{v} \cdot \vec{w}$ gives

$\sin\theta = \cos\theta$ (as $|\vec{v}| \neq 0$, $|\vec{w}| \neq 0$)

$\tan\theta = 1$

$\boxed{\theta = \pi/4}$ as $0 \leq \theta \leq \pi$

(b) Suppose that $(x, y, z) = \mathbf{r}(t)$ is a parametrized curve whose speed is constant. Show that the acceleration vector of the curve is always perpendicular to the velocity vector of the curve, *i.e.*, that $\mathbf{r}'(t) \perp \mathbf{r}''(t)$.

CONSTANT SPEED : $|\vec{r}'(t)| = c$

So $c^2 = |\vec{r}'(t)|^2 = \vec{r}'(t) \cdot \vec{r}'(t)$

Take $\frac{d}{dt}$:

$$0 = \frac{d}{dt}(\vec{r}' \cdot \vec{r}') = \vec{r}'' \cdot \vec{r}' + \vec{r}' \cdot \vec{r}''$$

$$= 2\vec{r}'' \cdot \vec{r}'$$

So $\vec{r}'' \cdot \vec{r}' = 0$, $\Rightarrow \vec{r}'' \perp \vec{r}'$. ✓

Please sign the following honor statement:

On my honor, I pledge that I have neither given nor received any aid on this exam.

Signature: ______________________