Linear Transformations Exercises

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6.1 Which of the following transformations are linear transformations?

(a) $T : (x, y)^\mathsf{T} \mapsto (0, y)^\mathsf{T}$

(b) $T : (x, y)^\mathsf{T} \mapsto (x, 5)^\mathsf{T}$

(c) $T : (x, y, z)^\mathsf{T} \mapsto (x, x - y)^\mathsf{T}$

(d) $T : (x, y, z)^\mathsf{T} \mapsto \begin{pmatrix} x + y \\ z \end{pmatrix}$

(e) $T : (x, y)^\mathsf{T} \mapsto (3x + 1, y)^\mathsf{T}$

(f) $T : f(x) \mapsto \dfrac{\mathrm{d}}{\mathrm{d}x} f(x)$

(g) $T : f(x) \mapsto x f(x)$

(h) $T : \mathbb{C}^2 \to \mathbb{C}$ where $T : (x, y)^\mathsf{T} \mapsto x + y$

(i) $T : \mathbb{C}^2 \to \mathbb{C}$ where $T : (x, y)^\mathsf{T} \mapsto xy$

(j) $T : \mathbb{C}^2 \to \mathbb{C}$ where $T : (x, y)^\mathsf{T} \mapsto \bar{y}$

($\bar{x}$ is the complex conjugate of $x = a + bi$ defined by $\bar{x} = a - bi$.)

6.2 A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T : (x, y)^\mathsf{T} \mapsto (-x + 3y, x - 4y)^\mathsf{T}$. Determine the transformation matrix for T and hence calculate $T(2, 5)^\mathsf{T}$.

6.3 A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T : (x, y)^\mathsf{T} \mapsto (x - 2y, 2x + 3y)^\mathsf{T}$. Given $T(\mathbf{u}) = (-1, 5)^\mathsf{T}$ determine $\mathbf{u}$.

6.4 $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$T\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}, \qquad T\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 10 \end{pmatrix}, \qquad T\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}.$$

Find the transformation matrix for T.

6.5 Rotate the position vector $(2, 1)^\mathsf{T} \in \mathbb{R}^2$ by angle $\pi/6$ anti-clockwise about the origin.

6.6 Reflect the position vector $(5, 3)^\mathsf{T} \in \mathbb{R}^2$ about the line that passes through $(0, 0)$ and makes an angle $\pi/3$ with the x-axis.

6.7 A square with side lengths 2 is centred at the co-ordinates $(3, 2)$. It is to be translated so the centre is at the origin, rotated by an angle $\pi/3$ clockwise about the origin and then translated back to its initial position.

(a) Write down a matrix containing the homogeneous co-ordinates for the vertices of the square.

(b) Determine the transformation matrices that perform the three transformations.

(c) Calculate the composite transformation matrix and apply with to the co-ordinate matrix from part (a).