# Matrices Exercises

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Semester 1

1.1 (a) Write down the $3 \times 3$ matrix $A$ whose entries are given by $a_{ij} = i + j$.

(b) Write down the $4 \times 4$ matrix $B$ whose entries are given by $b_{ij} = (-1)^{i+j}$.

(c) Write down the $4 \times 4$ matrix $C$ whose entries are given by

$$c_{ij} = \begin{cases} -1, & i > j, \\ 0, & i = j, \\ 1, & i < j. \end{cases}$$

1.2 The Hilbert matrix is the $n \times n$ matrix $H$ where the value of its elements are $h_{ij} = \dfrac{1}{i+j-1}$.

(a) Write down the $4 \times 4$ Hilbert matrix.

(b) Show that an $n \times n$ Hilbert matrix is symmetric.

1.3 Given the matrices

$$A = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 0 \\ -1 & 5 \end{pmatrix},$$

$$C = \begin{pmatrix} 5 \\ 9 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 1 & 3 \\ 4 & -2 & 3 \end{pmatrix},$$

$$E = \begin{pmatrix} 1 & 2 \\ 0 & 6 \\ -2 & 3 \end{pmatrix}, \qquad F = \begin{pmatrix} 1 & -2 & 4 \end{pmatrix},$$

$$G = \begin{pmatrix} 4 & 2 & 3 \\ -2 & 6 & 0 \\ 0 & 7 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 2 & -2 \\ 2 & -3 & 4 \end{pmatrix}.$$

Calculate the following where possible:

(a) $A + B$ (b) $B + C$ (c) $A^\mathsf{T}$ (d) $C^\mathsf{T}$

(e) $3B - A$ (f) $(F^\mathsf{T})^\mathsf{T}$ (g) $A^\mathsf{T} + B^\mathsf{T}$ (h) $(A + B)^\mathsf{T}$

1.4 Using the matrices from exercise 1.3 calculate the following where possible:

(a) $AB$ (b) $BA$ (c) $AC$ (d) $CA$

(e) $C^\mathsf{T}C$ (f) $CC^\mathsf{T}$ (g) $DE$ (h) $GH$

(i) $A(DE)$ (j) $(AD)E$ (k) $A^3$ (l) $G^4$

1.5 Calculate the determinants of the square matrices from exercise 1.3.

1.6 For each non-singular matrix from exercise 1.3 calculate its inverse. Show that your answers are correct.

1.7 Show that $AA^\top$ is a symmetric matrix. Hint: use the properties of matrix transpose.

1.8 Show that $(AB)^{-1} = B^{-1}A^{-1}$. Hint: use the associativity law.

1.9 If $A$ and $B$ are $n \times n$ matrices is the following equation true?

$$(A + B)^2 = A^2 + 2AB + B^2$$

If not, under what conditions would it be true?

1.10 An involutory matrix is a matrix that is its own inverse, i.e., it satisfies the equation $A^2 = I$. Under what conditions is the following matrix an involutory matrix?

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

1.11 Which of the following statements are true? For the false statements, give one counter example where the statement doesn't hold.

(a) If $A = B$ then $AC = BC$
(b) If $AC = BC$ then $A = B$
(c) For $[O]_{ij} = 0$, if $AB = O$ then $A = O$ or $B = O$
(d) If $A + C = B + C$ then $A = B$
(e) If $A^2 = I$ then $A = \pm I$
(f) If $B = A^2$ and if $A$ is an $n \times n$ symmetric matrix then $b_{ii} \geq 0$ for $i = 1, 2, \ldots, n$
(g) If $AB = C$ and if two of the matrices are square then so is the third
(h) If $AB = C$ and if $C$ has a single column then so does $B$
(i) If $A^2 = I$ then $A^n = I$ for all integers $n \geq 2$

1.12 Given the matrices

$$A = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 0 \\ -1 & 5 \end{pmatrix},$$

solve the following equations for $X$.

(a) $5X = A$ (b) $X + A = I$ (c) $2X - B = A$ (d) $XA = I$
(e) $BX = A$ (f) $A^2 = X$ (g) $X^2 = B$ (h) $(X + A)B = I$