Levenberg Marquardt

$$y = y(x; a) \qquad M \text{ parameters } a_0, \dots a_{M-1}$$

$$\chi^2(a) = \sum_{i=0}^{N-1} \left[\frac{y_i - y(x_i; a)}{\sigma_i} \right]^2$$

gradient $$\frac{\partial \chi^2}{\partial a_k} = -2 \underbrace{\sum_{i=0}^{N-1} \left[\frac{[y_i - y(x_i; a)]}{\sigma_i^2} \frac{\partial y(x_i; a)}{\partial a_k} \right]}_{\beta} \quad k = 0, 1, \dots M-1$$

Hessian $$\frac{\partial^2 \chi^2}{\partial a_k \partial a_l} = 2 \underbrace{\sum_{i=0}^{N-1} \frac{1}{\sigma_i^2} \left[\frac{\partial y(x_i; a)}{\partial a_k} \frac{\partial y(\dots)}{\partial a_l} - [y_i - y(\dots)] \frac{\partial^2 y(\dots)}{\partial a_l \partial a_k} \right]}_{\alpha}$$

$$\beta_k = -\frac{1}{2} \frac{\partial \chi^2}{\partial a_k} \qquad \alpha_{kl} = \frac{1}{2} \underbrace{\frac{\partial^2 \chi^2}{\partial a_k \partial a_l}}_{D}$$

$$\frac{\partial \chi^2}{\partial a_k} = -2\beta_k \qquad \frac{\partial^2 \chi^2}{\partial a_k \partial a_l} = 2\alpha_{kl}$$

(Gradient) (Hessian)

$$\alpha_{kl} = \frac{1}{2} D$$

$$a_{min} - a_{cur} = D^{-1} \left[-\nabla \chi^2(a_{cur}) \right]$$

$$\delta a = D^{-1} \left[-\nabla \chi^2(a_{cur}) \right]$$

$$\delta a = 2 \cdot \alpha \cdot \left[+2\beta_k \right]$$

negligible $$y_i - y(\dots) \frac{\partial^2 y}{\partial a_l \partial a_k}$$

$$\sum_{l=0}^{M-1} \alpha_{kl} \, \delta a_l = \beta_k$$

$$\alpha_{kl} = \sum_{i=0}^{N-1} \frac{1}{\sigma^2} \left[\frac{\partial y(x_i; a)}{\partial a_k} \frac{\partial y(x_i; a)}{\partial a_l} \right]$$

$$\beta_k = \sum_{i=0}^{N-1} \left(\frac{[y_i - y(x_i; a)]}{\sigma_i^2} \frac{\partial y(x_i; a)}{\partial a_k} \right)$$

step

$\delta a_\ell = \text{const} \cdot \beta_\ell$ steepest descent

gradient

LM use $\text{const} = \dfrac{1}{\lambda a_{\ell\ell}} \implies \lambda a_{\ell\ell}\,\delta a_\ell = \beta_\ell$

proportion diagonals of hessian

New Hessian matrix $\alpha'_{jj} \equiv \alpha_{jj}(1+\lambda)$ diagonals

$\alpha'_{jk} \equiv \alpha_{jk} \quad (j \neq k)$ off diagonals

$$\sum_{\ell=0}^{M-1} \alpha'_{k\ell}\,\delta a_\ell = \beta_k$$

LM algorithm

1) compute $\chi^2(a_{guess})$
2) Pick small λ (ex $\lambda = 0.001$)
3) solve $\sum_{\ell=0}^{M-1} \alpha'_{k\ell}\,\delta a_\ell = \beta_k$ for δa
4) evaluate $\chi^2(a+\delta a)$
5) if $\chi^2(a+\delta a) \geq \chi^2(a)$ increase λ by say 10x & reevaluate 3→5

if $\chi^2(a+\delta a) < \chi^2(a)$ decrease λ by $\frac{1}{10}$ update $a_{current} = a+\delta a$, and reevaluate 3→5

$$\begin{bmatrix} \alpha'_{11} & \alpha'_{12} & \cdots & \alpha'_{1M} \\ \alpha'_{21} & \alpha'_{22} & \cdots & \alpha'_{2M} \\ \vdots & & \ddots & \vdots \\ \alpha'_{M1} & \alpha'_{M2} & \cdots & \alpha'_{MM} \end{bmatrix} \begin{bmatrix} \delta a_1 \\ \delta a_2 \\ \vdots \\ \delta a_M \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_M \end{bmatrix}$$

$A \cdot d = b \qquad d = A^{-1} b$