CS 381 Midterm 1 Cheatsheet

1 Master Theorem

(Easy Form) For $a, b > 1$, and $T(n)$ be defined on nonnegative integers in the form of $T(n) = aT(\frac{n}{b}) + f(n)$, and let $f(n) \in \Theta(n^c)$, the following asymptotic bounds apply:

a. Case 1: $\frac{a}{b^c} < 1$, $T(n) \in \Theta(n^c)$

b. Case 2: $\frac{a}{b^c} = 1$, $T(n) \in \Theta(n^c \log n)$

c. Case 3: $\frac{a}{b^c} > 1$, $T(n) \in \Theta(n^{\log_b a})$

(General Form) $T(n) = aT(\frac{n}{b}) + f(n)$ for some positive function $f(n) \geq 1$. (A) If $f(n) \in \Omega(n^c)$ for some constant $c > \log_b a$ then $T(n) \in \Theta(f(n))$ as long as the regularity condition holds (i.e., $af(n/b) \leq (1-\epsilon)f(n)$ for some constant $0 < \epsilon < 1$). (B) If $f(n) \in \Theta(n^{\log_b a} \log^k n)$ for some constant $k \geq 0$ then $T(n) \in \Theta(n^{\log_b a} \log^{k+1} n)$. (C) If $f(n) \in O(n^c)$ for some constant $c < \log_b a$ then $T(n) \in \Theta(n^{\log_b a})$.

2 Definitions

Big-O: $f(n) \in \mathcal{O}(g(n))$ if there exists $c, n_0 > 0$ such that for all $n > n_0$, $f(n) \leq c \cdot g(n)$.

little-o: $f(n) \in o(g(n))$ if for all $c > 0$, there exists $n_0 > 0$ such that for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$.

Big-Ω: $f(n) \in \Omega(g(n))$ if there exists $c, n_0 > 0$ such that for all $n > n_0$, $c \cdot g(n) \leq f(n)$

little-ω: $f(n) \in \omega(g(n))$ if for all $c > 0$, there exists $n_0 > 0$ such that for all $n \geq n_0$, $c \cdot g(n) \leq f(n)$.

Big-Θ: $f(n) \in \Theta(g(n))$ if there exists $c_1, c_2, n_0 > 0$ such that for all $n > n_0$, $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$.

3 Summations/Bounds

$$\sum_{i=0}^{n} ar^i = \frac{a(1-r^{n+1})}{1-r} \qquad \sum_{i=0}^{\infty} ar^i = \frac{a}{1-r} \quad (r < 1)$$

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n) \qquad \log(n!) = \Theta(n \log n)$$

4 Probability

Inclusion-Exclusion Principle

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Independence

If E and F are independent, then

$$P(E \cap F) = P(E)P(F)$$
$$P(E|F) = P(E)$$

Conditional Independence

If E and F are conditionally independent on X, then

$$P(E \cap F|X) = P(E|X)P(F|X)$$
$$P(E|F \cap X) = P(E|X)$$

Bayes Theorem

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

Law of Total Probability

$$P(X) = \sum_i P(X|y_i)P(y_i)$$

De Morgan's Laws

$$(P(E) \cup P(F))^C = P(E)^C \cap P(F)^C$$
$$(P(E) \cap P(F))^C = P(E)^C \cup P(F)^C$$

Union Bound

$$P\left(\bigcup_{i=1}^{\infty} X_i\right) \leq \sum_{i=1}^{\infty} P(X_i)$$

Markov's Inequality

$$P(X \geq a) \leq \frac{E[X]}{a}$$

5 Other Useful Formulae

Binomial Coefficient:

$$\binom{n}{c} = \frac{n!}{c!(n-c)!}$$

Limit Rules

1. If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$, and $g'(x) \neq 0$, $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$
2. $\lim_{x \to \infty} \frac{f(x)}{g(x)} = c \Rightarrow f \in \Theta(g)$, for a constant c
3. $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \Rightarrow f \in o(g)$
4. $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty \Rightarrow f \in \omega(g)$