Homework 1

1. **Estimating** $(1-x)$ **using** $\exp(\cdot)$ **function.** For $x \in [0,1)$, we know that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

   (a) **(5 points)** Prove that $1 - x \leqslant \exp\left(-x - \frac{x^2}{2}\right)$.

   Solution.
   Note that $\frac{x^k}{k} \geqslant 0$ for every $x \in [0,1)$ and for every positive integer k. Therefore, we have

$$-\frac{x^3}{3} - \frac{x^4}{4} - \ldots \leqslant 0,$$

   which implies that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \leqslant -x - \frac{x^2}{2}.$$

   Taking $\exp(\cdot)$ on both sides of the inequality, we get

$$1 - x \leqslant \exp\left(-x - \frac{x^2}{2}\right).$$

(b) **(5 points)** For $x \in [0, 1/2]$, prove that

$$1 - x \geqslant \exp\left(-x - x^2\right).$$

Solution.

The two common approaches provided by students are

i. Use Taylor expansion on $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$. Since $x \in [0, 1/2], x^2 \in [0, 1/4]$, reduce the problem to showing

$$\frac{1}{2} \geqslant \frac{x^2}{2} \geqslant \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Then, use the geometric series formula to compute $\sum_{n=1}^{\infty} \frac{x^n}{n!}$. Finally, take exponential on both sides of the inequality to get the desired bound.

ii. Let $f(x) = \ln(1-x) + x + x^2$. Show that $f(0) = 0$ and $f'(x) > 0$ for $x \in [0, 1/2]$. Conclude that $f(x) \geqslant 0$ for all $x \in [0, 1/2]$.

Here, we provide an alternative proof.

Proof. First we shall show that $\frac{x^k}{k} \leqslant \frac{x^2}{2^{k-1}}$ for every $x \in [0, 1/2]$ and for every positive integer $k \geqslant 2$. It suffices to show that $x^k \cdot 2^{k-1} \leqslant x^2 \cdot k$, which is equivalent to $x^{k-2} \cdot 2^{k-1} = (2x)^{k-2} \cdot 2 \leqslant k$.

Since $x \in [0, 1/2]$, we have $0 \leqslant 2x \leqslant 1$, so $(2x)^{k-2} \leqslant 1$, which implies that $(2x)^{k-2} \cdot 2 \leqslant 2 \leqslant k$. Therefore $\frac{x^k}{k} \leqslant \frac{x^2}{2^{k-1}}$, in other words, $-\frac{x^k}{k} \geqslant \frac{x^2}{2^{k-1}}$. Applying this inequality for $k = 2, 3, \ldots,$ we have

$$\begin{aligned}
\ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \\
&\geqslant -x - \frac{x^2}{2} - \frac{x^2}{2^3} - \frac{x^2}{2^4} - \cdots \\
&\geqslant -x - \frac{x^2}{2 \cdot 2^0} - \frac{x^2}{2 \cdot 2^1} - \frac{x^2}{2 \cdot 2^2} - \cdots \\
&= -x - \frac{x^2}{2}\left(\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} \cdots\right) \\
&= -x - \frac{x^2}{2} \cdot 2 && \text{By simplifying the geometric sum} \\
&= -x - x^2
\end{aligned}$$

Taking exponential on both sides, we get for $x \in [0, 1/2]$,

$$1 - x \geqslant \exp\left(-x - x^2\right).$$

$\square$

2. **Tight Estimations** Provide meaningful upper and lower bounds for the following expressions.

   (a) **(5 points)** $S = \sum_{i=1}^{\infty} i^{-\frac{17}{15}}$.
   Note: Please evaluate/simplify the expression/bound as much as possible.
   Hint: Your upper and lower bounds should be constants.
   Solution.
   First, observe that $S_n = \sum_{i=1}^{\infty} i^{-\frac{17}{15}} = 1+\sum_{i=2}^{\infty} i^{-\frac{17}{15}}$. For $x \in [1,\infty)$, $f(x) = x^{-\frac{17}{15}}$ monotonically decreases. Then,

$$
\begin{aligned}
& \int_{i}^{i+1} x^{-\frac{17}{15}}\,dx \leqslant i^{-\frac{17}{15}} \leqslant \int_{i-1}^{i} x^{-\frac{17}{15}}\,dx \\
\iff & \sum_{i=2}^{\infty}\int_{i}^{i+1} x^{-\frac{17}{15}}\,dx \leqslant \sum_{i=2}^{\infty} i^{-\frac{17}{15}} \leqslant \sum_{i=2}^{\infty}\int_{i-1}^{i} x^{-\frac{17}{15}}\,dx \\
\iff & \int_{2}^{\infty} x^{-\frac{17}{15}}\,dx \leqslant \sum_{i=2}^{\infty} i^{-\frac{17}{15}} \leqslant \int_{1}^{\infty} x^{-\frac{17}{15}}\,dx \\
\iff & -\frac{15}{2}x^{-\frac{2}{15}}\Big|_{x=2}^{\infty} \leqslant \sum_{i=2}^{\infty} i^{-\frac{17}{15}} \leqslant -\frac{15}{2}x^{-\frac{2}{15}}\Big|_{x=1}^{\infty} \\
\iff & \frac{15}{2}\cdot 2^{-\frac{2}{15}} \leqslant \sum_{i=2}^{\infty} i^{-\frac{17}{15}} \leqslant \frac{15}{2} \\
\iff & 1+\frac{15}{2}\cdot 2^{-\frac{2}{15}} \leqslant \sum_{i=1}^{\infty} i^{-\frac{17}{15}} \leqslant 1+\frac{15}{2}
\end{aligned}
$$

   Therefore,

$$1+\frac{15}{2}\cdot 2^{-\frac{2}{15}} \leqslant S_n \leqslant \frac{17}{2}.$$

   The following lower bound is also acceptable but not as tight as the previous lower bound.
   For $x \in [1,\infty)$, $f(x) = x^{-\frac{17}{15}}$ monotonically decreases. Then, for $i \geqslant 1$, the summation $\sum_{i=1}^{\infty} i^{-\frac{17}{15}}$ is an upper bound for the integration $\int_1^{\infty} x^{-\frac{17}{15}}\,dx$,

$$
\begin{aligned}
& \sum_{i=1}^{\infty} i^{-\frac{17}{15}} \geqslant \int_{1}^{\infty} x^{-\frac{17}{15}}\,dx \\
\iff & \sum_{i=1}^{\infty} i^{-\frac{17}{15}} \geqslant -\frac{15}{2}x^{-\frac{2}{15}}\Big|_{x=1}^{\infty} \\
\iff & \sum_{i=1}^{\infty} i^{-\frac{17}{15}} \geqslant \frac{15}{2}
\end{aligned}
$$

Therefore,

$$\frac{15}{2} \leqslant S_n \leqslant \frac{17}{2}.$$

(b) **(10 points)** $A_n = {}_{2n}P_n$ Hint: Note that ${}_{2n}P_n = \frac{(2n)!}{(2n-n)!}$.
Note: Please evaluate/simplify the expression/bound as much as possible.
Hint: You may want to start by upper and lower bounding $S_n = \sum_{i=1}^{n} \ln i$.
Solution. Let us first bound $B_n = n!$.

$$\int_{i-1}^{i} \ln(t)\,\mathrm{d}t \leqslant \ln(i) \leqslant \int_{i}^{i+1} \ln(t)\,\mathrm{d}t.$$

Using the upper bound for values $i = 1, 2, \ldots, n$, we get the following inequality:

$$\begin{aligned} S_n = \sum_{i=1}^{n} \ln(i) &\leqslant \sum_{i=1}^{n} \int_{i}^{i+1} \ln(t)\,\mathrm{d}t \\ &= \int_{1}^{n+1} \ln(t)\,\mathrm{d}t \\ &= [t\ln(t) - t]_{t=1}^{n+1} \\ &= (n+1)\ln(n+1) - n \end{aligned}$$

To find a lower bound, first notice that $\ln(1) = 0$ and so $S_n = \sum_{i=2}^{n} \ln(i)$. Now, by using $\int_{i-1}^{i} \ln(t)\,\mathrm{d}t \leqslant \ln(i)$ for values $i = 2, 3, \ldots, n$, we can find a lower bound for S_n:

$$\begin{aligned} S_n = \sum_{i=2}^{n} \ln(i) &\geqslant \sum_{i=2}^{n} \int_{i-1}^{i} \ln(t)\,\mathrm{d}t \\ &= \sum_{i=1}^{n-1} \int_{i}^{i+1} \ln(t)\,\mathrm{d}t \\ &= \int_{1}^{n} \ln(t)\,\mathrm{d}t \\ &= [t\ln(t) - t]_{1}^{n} \\ &= n\ln(n) - n + 1 \end{aligned}$$

Therefore, we have

$$n\ln(n) - n + 1 \leqslant S_n \leqslant (n+1)\ln(n+1) - n.$$

Taking exponential on both sides, we get:

$$\begin{aligned} & e^{n\ln(n) - n + 1} \leqslant B_n = e^{\ln(n!)} \leqslant e^{(n+1)\ln(n+1) - n} \\ \iff & \frac{n^n}{e^{n-1}} \leqslant B_n \leqslant \frac{(n+1)^{n+1}}{e^n} \end{aligned}$$

Let $B_n = n!$, then we observe that $A_n = \frac{B_{2n}}{B_n}$. According to the above, we have

$$\frac{(2n)^{2n}}{e^{2n-1}} \leqslant B_{2n} \leqslant \frac{(2n+1)^{2n+1}}{e^{2n}}$$

$$\frac{n^n}{e^{n-1}} \leqslant B_n \leqslant \frac{(n+1)^{n+1}}{e^n}$$

Therefore, we have:

$$\frac{1}{\frac{n^n}{e^{n-1}}} \geqslant \frac{1}{B_n} \geqslant \frac{1}{\frac{(n+1)^{n+1}}{e^n}}$$

This implies that:

$$A_n = \frac{B_{2n}}{B_n} \leqslant \frac{B_{2n}}{\frac{n^n}{e^{n-1}}} \leqslant \frac{\frac{(2n+1)^{2n+1}}{e^{2n}}}{\frac{n^n}{e^{n-1}}} = \frac{(2n+1)^{2n+1}}{e^{n+1} \cdot n^n}$$

$$A_n = \frac{B_{2n}}{B_n} \geqslant \frac{B_{2n}}{\frac{(n+1)^{n+1}}{e^n}} \geqslant \frac{\frac{(2n)^{2n}}{e^{2n-1}}}{\frac{(n+1)^{n+1}}{e^n}} = \frac{(2n)^{2n}}{e^{n-1}(n+1)^{n+1}}$$

Therefore, we have

$$\frac{(2n)^{2n}}{e^{n-1}(n+1)^{n+1}} \leqslant A_n \leqslant \frac{(2n+1)^{2n+1}}{e^{n+1} \cdot n^n}.$$

3. **Understanding Joint Distribution.** Ten balls are to be tossed into five bins numbered $\{1, 2, 3, 4, 5\}$. Each ball is thrown into a bin uniformly and independently into the bins. For $i \in \{1, 2, 3, 4, 5\}$, let X_i represent the number of balls that fall into bin i.

   (a) **(5 points)** Find the (marginal) distribution of X_5 and compute its expected value.

   Solution.

   The marginal distribution of X_5 is

$$\mathbb{P}[X_5 = k] = \binom{10}{k} \cdot \left(\frac{1}{5}\right)^k \cdot \left(\frac{4}{5}\right)^{10-k}.$$

   The expected value is

$$\begin{aligned}
\mathbb{E}[X_5] =& \sum_{k=0}^{10} k \cdot \mathbb{P}[X_5 = k] \\
=& \sum_{k=0}^{10} k \cdot \binom{10}{k} \cdot \left(\frac{1}{5}\right)^k \cdot \left(\frac{4}{5}\right)^{10-k} \\
=& \sum_{k=1}^{10} 10 \cdot \binom{10-1}{k-1} \cdot \left(\frac{1}{5}\right)^k \cdot \left(\frac{4}{5}\right)^{10-k} && \left(k \cdot \tbinom{n}{k} = n \cdot \tbinom{n-1}{k-1}\right) \\
=& 10 \cdot \frac{1}{5} \cdot \sum_{k=1}^{10} \binom{10-1}{k-1} \cdot \left(\frac{1}{5}\right)^{k-1} \cdot \left(\frac{4}{5}\right)^{(10-1)-(k-1)} \\
=& 2 \cdot \sum_{j=0}^{9} \binom{9}{j} \cdot \left(\frac{1}{5}\right)^j \cdot \left(\frac{4}{5}\right)^{9-j} \\
=& 2
\end{aligned}$$

   Therefore,

$$\mathbb{E}[X_5] = 2.$$

   (b) **(3 points)** Find the expected value of $X_1 + X_3 + X_5$.

   Solution. Note that $\mathbb{E}[X_i] = \mathbb{E}[X_1] = 2$. By linearity of expectation, we have

$$\mathbb{E}[X_1 + X_3 + X_5] = \mathbb{E}[X_1] + \mathbb{E}[X_3] + \mathbb{E}[X_5] = 6.$$

(c) **(7 points)** Find $\mathbb{P}[X_2 = 3 | X_1 + X_3 + X_5 = 6]$.
Solution. We first note that $X_1 + X_3 + X_5 = 6$ is equivalent to $X_2 + X_4 = 4$ By Bayes rule,

$$\begin{aligned}\mathbb{P}[X_2 = 3|X_2 + X_4 = 4] =& \frac{\mathbb{P}[X_2 = 3, X_2 + X_4 = 4]}{\mathbb{P}[X_2 + X_4 = 4]}\\ =& \frac{\mathbb{P}[X_2 = 3, X_4 = 1]}{\mathbb{P}[X_2 + X_4 = 4]}\end{aligned}$$

To compute $\mathbb{P}[X_2 + X_4 = 4]$, we can view bin $2, 4$ as one bin, where each ball will be in bin $2, 4$ with probability $\frac{2}{5}$, and be outside of bin $2, 4$ with probability $1 - \frac{2}{5} = \frac{3}{5}$. This gives us $\mathbb{P}[X_2 + X_4 = 4] = \binom{10}{4} \cdot \left(\frac{2}{5}\right)^4 \cdot \left(\frac{3}{5}\right)^6$.
To compute $\mathbb{P}[X_2 = 3, X_4 = 1]$, we view all the other bins (X_1, X_3, X_5) as one giant bin that each ball has probability $\frac{3}{5}$ of being in. We can now choose the 4 balls that will be in X_2 or X_4, then from those 4 balls choose one that will go in X_4, while the remaining balls fall in the giant bin (X_1, X_3, X_5). This gives us $\mathbb{P}[X_2 = 3, X_4 = 1] = \binom{10}{4} \cdot \binom{4}{1} \cdot \left(\frac{1}{5}\right)^3 \cdot \frac{1}{5} \cdot \left(\frac{3}{5}\right)^6$.
Together we have

$$\begin{aligned}\mathbb{P}[X_2 = 3|X_2 + X_4 = 4] =& \frac{\mathbb{P}[X_2 = 3, X_4 = 1]}{\mathbb{P}[X_2 + X_4 = 4]}\\ =& \frac{\binom{10}{4} \cdot \binom{4}{1} \cdot \left(\frac{1}{5}\right)^3 \cdot \frac{1}{5} \cdot \left(\frac{3}{5}\right)^6}{\binom{10}{4} \cdot \left(\frac{2}{5}\right)^4 \cdot \left(\frac{3}{5}\right)^6}\\ =& \frac{\binom{4}{1} \cdot \left(\frac{1}{5}\right)^3 \cdot \frac{1}{5}}{\left(\frac{2}{5}\right)^4}\\ =& \binom{4}{1} \cdot \left(\frac{\frac{1}{5}}{\frac{2}{5}}\right)^4\\ =& \binom{4}{1} \cdot \left(\frac{1}{2}\right)^4\\ =& \frac{1}{4}\\ =& 0.25\end{aligned}$$

Alternatively, we can view $\mathbb{P}[X_2 = 3|X_2 + X_4 = 4]$ as us being in a world where only bin X_2 and X_4 exist, and we are throwing 4 balls into those two bins. Therefore, $\mathbb{P}[X_2 = 3 | X_2 + X_4 = 4] = \binom{4}{1} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right) = 4 \cdot \frac{1}{2^4} = \frac{1}{4} = 0.25$

4. **Sending one bit.**

   Alice intends to send a bit $b \in \{0,1\}$ to Bob. When Alice sends the bit, it goes through a series of n relays before reaching Bob. Each relay flips the received bit independently with probability p before forwarding that bit to the next relay.

   (a) **(5 points)** Show that Bob will receive the correct bit with probability

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} p^{2k} \cdot (1-p)^{n-2k}.$$

   Hint: Be careful that Alice could be sending either 0 or 1.

   Solution.

   Observe that Bob receives the correct bit if and only if there are even number of relays flipping the bits. Define a random variable

$$X_i = \begin{cases} 1 & \text{if the bit is flipped at relay } i, \\ 0 & \text{otherwise} \end{cases}$$

   Define $Y := \sum_{i=1}^{n} X_i$ to be the number of times the bit is flipped. The probability the Bob receivers the correct bit is equivalent as the probability that the bit is flipped even number of times, i.e.

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \mathbb{P}[Y = 2k]$$

   where

$$\mathbb{P}[Y = 2k] = \binom{n}{2k} \cdot p^{k} \cdot (1-p)^{n-2k}.$$

(b) **(5 points)** Let us consider an alternative way to calculate this probability. We say that the relay has *bias* q if the probability it flips the bit is $(1-q)/2$. The bias q is a real number between -1 and $+1$. Show that sending a bit through two relays with bias q_1 and q_2 is equivalent to sending a bit through a single relay with bias $q_1 \cdot q_2$.

Solution.

Consider two relays with bias q_1 and q_2. Then the probability of the first relay flipping the bit is $p_1 = \frac{1-q_1}{2}$ and the second relay flipping the bit is $p_2 = \frac{1-q_1}{2}$. The probability p that the two relays flip the bit is

$$\begin{aligned} p =& p_1 \cdot (1-p_2) + (1-p_1) \cdot p_2 \\ =& \frac{1-q_1}{2} \cdot \frac{1+q_2}{2} + \frac{1+q_1}{2} \cdot \frac{1-q_2}{2} \\ =& \frac{1-q_1+q_2-q_1q_2}{4} + \frac{1+q_1-q_2-q_1q_2}{4} \\ =& \frac{2-2\cdot q_1q_2}{4} \\ =& \frac{1-q_1q_2}{2} \end{aligned}$$

Therefore, it is equivalent to sending a bit through a single relay with bias $q_1 \cdot q_2$.

(c) **(5 points)** Prove that the probability you receive the correct bit when it passes through n relays is

$$\frac{1+(1-2p)^n}{2}.$$

Solution.

By part b, sending a bit through n relays with bias q for each relay is equivalent as sending a bit through a single relay with bias q^n. Then, the probability of receiving the correct bit is $1 - \frac{1-q^n}{2}$ where $q = (1-2p)$. Plugging in we get that the probability you receive the correct bit when it passes through n relays is

$$\frac{1+(1-2p)^n}{2}.$$

5. **An Useful Estimate.**

For an integers n and t satisfying $0 \leqslant t \leqslant n/2$, define

$$P_n(t) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{t}{n}\right)$$

We will estimate the above expression. (*Remark*: You shall see the usefulness of this estimation in the topic "Birthday Bound" that we shall cover in the forthcoming lectures.)

(a) **(13 points)** Show that

$$\exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{6n^2}\right) \geqslant P_n(t) \geqslant \exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{3n^2}\right).$$

Solution.

Upper Bound:

Recall the inequality obtained from problem 1.(a).

$$\ln(1-x) \leqslant -x - \frac{x^2}{2} \Leftrightarrow (1-x) \leqslant \exp\left(-x - \frac{x^2}{2}\right)$$

Then, for all $0 \leqslant t \leqslant n/2$, we have

$$\left(1 - \frac{t}{n}\right) \leqslant \exp\left(-\frac{t}{n} - \frac{t^2}{2n^2}\right).$$

Therefore,

$$\begin{aligned}
P_n(t) &\leqslant \prod_{i=1}^{t} \exp\left(-\frac{i}{n} - \frac{i^2}{2n^2}\right) \\
&= \exp\left(\sum_{i=1}^{t}\left(-\frac{i}{n} - \frac{i^2}{2n^2}\right)\right) \\
&= \exp\left(-\left(\sum_{i=1}^{t}\frac{i}{n}\right) - \frac{1}{2}\cdot\left(\sum_{i=1}^{t}\frac{i^2}{n^2}\right)\right) \\
&= \exp\left(-\left(\frac{t(t+1)}{2n}\right) - \frac{1}{2}\cdot\left(\frac{t(t+1)(2t+1)}{6n^2}\right)\right) \\
&= \exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{6n^2}\right)
\end{aligned}$$

Lower Bound:
Similarly, recall the inequality obtained from problem 1.(b).

$$\ln(1-x) \geqslant -x - x^2 \Leftrightarrow (1-x) \geqslant \exp\left(-x - x^2\right)$$

Then, for all $0 \leqslant t \leqslant n/2$, we have

$$\left(1 - \frac{t}{n}\right) \geqslant \exp\left(-\frac{t}{n} - \frac{t^2}{n^2}\right).$$

Therefore,

$$\begin{aligned}
P_n(t) &\geqslant \prod_{i=1}^{t} \exp\left(-\frac{i}{n} - \frac{i^2}{n^2}\right) \\
&= \exp\left(\sum_{i=1}^{t}\left(-\frac{i}{n} - \frac{i^2}{n^2}\right)\right) \\
&= \exp\left(-\left(\sum_{i=1}^{t}\frac{i}{n}\right) - \left(\sum_{i=1}^{t}\frac{i^2}{n^2}\right)\right) \\
&= \exp\left(-\left(\frac{t(t+1)}{2n}\right) - \left(\frac{t(t+1)(2t+1)}{6n^2}\right)\right) \\
&= \exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{3n^2}\right)
\end{aligned}$$

(b) **(2 points)** When $t = \sqrt{2cn}$, where c is a positive constant, the expression above is

$$P_n(t) = \exp\Big(-c - \ \Theta\left(1/\sqrt{n}\right) \Big).$$

Solution.
From part (a), we get

$$\exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{6n^2}\right) \geqslant P_n(t) \geqslant \exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{3n^2}\right).$$

Plugging in $t = \sqrt{2cn}$, the upper bound becomes

$$\begin{aligned} P_n(t) &\leqslant \exp\left(-\frac{(\sqrt{2cn})^2}{2n} - \frac{\sqrt{2cn}}{2n} - \frac{\Theta\left((\sqrt{2cn})^3\right)}{6n^2}\right) \\ &= \exp\left(-\frac{2cn}{2n} - \frac{\sqrt{c}}{\sqrt{2n}} - \Theta\left(\frac{2cn\sqrt{2cn}}{6n^2}\right)\right) \\ &= \exp\left(-c - \sqrt{\frac{c}{2}} \cdot \frac{1}{\sqrt{n}} - \Theta\left(\frac{c\sqrt{2c}}{3} \cdot \frac{1}{\sqrt{n}}\right)\right) \\ &= \exp\left(-c - \Theta\left(\frac{1}{\sqrt{n}}\right)\right) \end{aligned}$$

and the lower bound becomes

$$\begin{aligned} P_n(t) &\geqslant \exp\left(-\frac{(\sqrt{2cn})^2}{2n} - \frac{\sqrt{2cn}}{2n} - \frac{\Theta\left((\sqrt{2cn})^3\right)}{3n^2}\right) \\ &= \exp\left(-\frac{2cn}{2n} - \frac{\sqrt{c}}{\sqrt{2n}} - \Theta\left(\frac{2cn\sqrt{2cn}}{3n^2}\right)\right) \\ &= \exp\left(-c - \sqrt{\frac{c}{2}} \cdot \frac{1}{\sqrt{n}} - \Theta\left(\frac{2c\sqrt{2c}}{3} \cdot \frac{1}{\sqrt{n}}\right)\right) \\ &= \exp\left(-c - \Theta\left(\frac{1}{\sqrt{n}}\right)\right) \end{aligned}$$

Therefore,

$$P_n(t) = \exp\left(-c - \Theta\left(1/\sqrt{n}\right)\right).$$