Homework 1

1. **Estimating** $(1-x)$ **using** $\exp(\cdot)$ **function.** For $x \in [0,1)$, we know that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

   (a) **(5 points)** Prove that $1 - x \leqslant \exp\left(-x - \frac{x^2}{2}\right)$.
   Solution.
   It can be noted that $\frac{x^k}{k}$ is $\geqslant 0$, $\forall x \in [0,\ 1)$ and for every positive integer k.
   This means that $\frac{-x^3}{3} - \frac{x^4}{4} - ... \leqslant 0$
   This further implies the following:
   $ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - ... \leqslant -x - \frac{x^2}{2}$
   Now, taking exponential on both sides of the equation,
   $(1-x) \leqslant exp(-x - \frac{x^2}{2})$.

(b) **(5 points)** For $x \in [0, 1/2]$, prove that

$$1 - x \geqslant \exp\left(-x - x^2\right).$$

Solution.

Firstly, we can note that we can bound the terms in the identity with powers of 2 and above, i.e for $\frac{x^k}{k}$ where $k \geqslant 2$, with $\frac{x^2}{2^{k-1}}$.
Next, we can show that $\frac{x^k}{k} \leqslant \frac{x^2}{2^{k-1}}, \forall x \in [0, \frac{1}{2}]$, for every integer k $\geqslant$ 2.
This is because of the following:
1. $x^k \cdot 2^{k-1} \leqslant k \cdot x^2$
2. $x^{k-2} \cdot 2^{k-1} \leqslant k$
3. $(2x)^{k-2} \cdot 2 \leqslant k$
4. $\forall x \in [0, \frac{1}{2}], 0 \leqslant 2x \leqslant 1$. Therefore, $(2x)^{k-2} \leqslant 1$.
5. This means that $(2x)^{k-2} \cdot 2 \leqslant 2 \leqslant k$. And hence, $\frac{x^k}{k} \leqslant \frac{x^2}{2^{k-1}}$ and therefore $-\frac{x^k}{k} \geqslant -\frac{x^2}{2^{k-1}}$. Finally, we can apply this to k = 2, 3, 4...
6. Using the identity from (a), $ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - ...$
$\geqslant -x - \frac{x^2}{2} - \frac{x^2}{2^2} - \frac{x^2}{2^3} - ...$
$= -x - \frac{x^2}{2}(1 + \frac{1}{2} + \frac{1}{2^2} + ...)$
7. Since $(1 + \frac{1}{2} + \frac{1}{2^2} + ...) = 1 + (\frac{\frac{1}{2}}{1-\frac{1}{2}}) = 2$, by sum of geometric series.
8. Continuing from (6),
$= -x - \frac{x^2}{2} \cdot 2$
$= -x - x^2$
9. Then, taking exponential of both sides of the equation, we arrive at $1 - x \geqslant exp(-x - x^2)$.

2. **Tight Estimations** Provide meaningful upper and lower bounds for the following expressions.

   (a) **(5 points)** $S = \sum_{i=1}^{\infty} i^{-\frac{17}{15}}$.

   Note: Please evaluate/simplify the expression/bound as much as possible.

   Hint: Your upper and lower bounds should be constants.

   Solution.

   Let $f(x) = x^{-\frac{17}{15}}$

   From lecture, we know that we can provide upper and lower bounds using integration techniques. Since this is a decreasing function, the lower bound in this case can be given by $\int_1^{\infty+1} x^{-17/15}$ while the upper bound can be given by $\int_0^{\infty} x^{-17/15}$.

   Since $\int x^{-17/15} = -\frac{15}{2x^{\frac{2}{15}}} + c$,

   The lower bound, can be evaluated as: $\int_1^{\infty+1} x^{-17/15} = [-\frac{15}{2x^{\frac{2}{15}}}]_1^{\infty+1} = \frac{15}{2}$.

   The upper bound, can be evaluated as $\int_0^{\infty} x^{-17/15}$.

   However, for the case where x=0, the integral is undefined. Hence, the upper bound that we will use will be $1 + \int_1^{\infty} x^{-\frac{17}{15}}$ since when $i = 1, i^{-\frac{17}{15}} = 1$. Thus, this evaluates to $1 + \frac{15}{2} = \frac{17}{2}$.

   Thus, the upper bound $= \frac{17}{2}$, and the lower bound $= \frac{15}{2}$.

(b) **(10 points)** $A_n = {}_{2n}P_n$ Hint: Note that ${}_{2n}P_n = \frac{(2n)!}{(2n-n)!}$.

Note: Please evaluate/simplify the expression/bound as much as possible.

Hint: You may want to start by upper and lower bounding $S_n = \sum_{i=1}^{n} \ln i$.

Solution.

For $S_n = \sum_{i=1}^{n} ln\ i$,

1. $\int_{i}^{i+1} ln(t)\ dt \geqslant ln(i)$
2. $\int_{i-1}^{i} ln(t)\ dt \leqslant ln(i)$

For 1., $\sum_{i=1}^{n} ln(i) \leqslant \sum_{i=1}^{n} \int_{i}^{i+1} ln(t)\ dt = \int_{1}^{n+1} ln(t)\ dt = [tln(t) - t]_{1}^{n+1}$
$= (n+1)ln(n+1) - n$

For 2., we can first note that $\ln(1) = 0$ and hence $S_n = \sum_{i=2}^{n} ln\ i$ instead.
$\sum_{i=2}^{n} ln(i) \leqslant \sum_{i=2}^{n} \int_{i-1}^{i} ln(t)\ dt$
$= \sum_{i=1}^{n-1} \int_{i}^{i+1} ln(t)\ dt = \int_{1}^{n} ln(t)\ dt = [tln(t) - t]_{1}^{n} = nln(n) - n + 1$. Hence, the upper bound for S_n is $(n+1)ln(n+1) - n$ and the lower bound for S_n is given by $nln(n) - n + 1$.

From the above, we can derive the following:
$nln(n) - n + 1 \leqslant ln(n!) = \sum_{i=1}^{n} ln(i) \leqslant (n+1)ln(n+1) - n$
$\Leftrightarrow \frac{n^n}{e^{n-1}} \leqslant n! \leqslant \frac{(n+1)^{n+1}}{e^n}$

Taking exponential on all sides,
$e^{nln(n)-n+1} \leqslant n! = e^{ln(n!)} \leqslant e^{(n+1)ln(n+1)-n}$

Let $B_n = n!$. Hence, $A_n = 2_nP_n = \frac{(2n)!}{(2n-n)!} = \frac{(2n)!}{n!}$. Then, we can observe that $A_n = \frac{B_{2n}}{B_n}$. Then, using the result we obtained above,
$\frac{(2n)^{2n}}{e^{2n-1}} \leqslant B_{2n} \leqslant \frac{(2n+1)^{2n+1}}{e^{2n}}$
$\frac{1}{\frac{n^n}{e^{n-1}}} \geqslant \frac{1}{B_n} \geqslant \frac{1}{\frac{(n+1)^{n+1}}{e^n}}$.

This further implies the following:
$\frac{B_{2n}}{\frac{n^n}{e^{n-1}}} \geqslant A_n = \frac{B_{2n}}{B_n} \geqslant \frac{B_2n}{\frac{(n+1)^{n+1}}{e^n}}$
$\frac{\frac{(2n+1)^{2n+1}}{e^{2n}}}{\frac{n^n}{e^{n-1}}} \geqslant A_n \geqslant \frac{\frac{n^n}{e^{n-1}}}{\frac{(n+1)^{n+1}}{e^n}}$
$\frac{(2n+1)^{2n+1}}{e^{n+1} \cdot n^n} \geqslant A_n \geqslant \frac{n^n \cdot e}{(n+1)^{n+1}}$.

Hence, the upper bound is $\frac{(2n+1)^{2n+1}}{e^{n+1} \cdot n^n}$ while the lower bound is $\frac{n^n \cdot e}{(n+1)^{n+1}}$.

3. **Understanding Joint Distribution.** Ten balls are to be tossed into five bins numbered $\{1, 2, 3, 4, 5\}$. Each ball is thrown into a bin uniformly and independently into the bins. For $i \in \{1, 2, 3, 4, 5\}$, let X_i represent the number of balls that fall into bin i.

   (a) **(5 points)** Find the (marginal) distribution of X_5 and compute its expected value.

   Solution.

   Marginal distribution of X_5 is a binomial distribution:
   $P(X_5 = k) = \binom{10}{k}(\frac{1}{5})^k(\frac{4}{5})^{10-k}$
   Expected value $= \sum_{k=0}^{10} k \cdot P(X_5 = k)$
   $= n \cdot p$ by formula of binomial distribution
   $= 10 \cdot \frac{1}{5}$
   $= 2.$

   (b) **(3 points)** Find the expected value of $X_1 + X_3 + X_5$.

   Solution.

   Expected value of $X_1+X_3+X_5 =$ Sum of expectation of 3 binomial distributions with same value of $n, p = 3 * (n \cdot p) = 3 * 2 = 6$

(c) **(7 points)** Find $\mathbb{P}[X_2 = 3 | X_1 + X_3 + X_5 = 6]$.

Solution.

Since the balls are thrown into a bin uniformly and independently,
$\mathbb{P}[X_2 = 3 | X_1 + X_3 + X_5 = 6] = \mathbb{P}[\frac{(X_2=3)\cdot\mathbb{P}(X_1+X_3+X_5=6)}{\mathbb{P}[X_1+X_3+X_5=6]}]$
$= \mathbb{P}(X_2 = 3)$
$= \binom{10}{3}(\frac{1}{5})^3(\frac{4}{5})^{10-3}$
$= 0.20133$ (5s.f.)
$= 0.201$ (3s.f.)

4. **Sending one bit.**

Alice intends to send a bit $b \in \{0,1\}$ to Bob. When Alice sends the bit, it goes through a series of n relays before reaching Bob. Each relay flips the received bit independently with probability p before forwarding that bit to the next relay.

(a) **(5 points)** Show that Bob will receive the correct bit with probability

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} p^{2k} \cdot (1-p)^{n-2k}.$$

Hint: Be careful that Alice could be sending either 0 or 1.

Solution.

Since we are only interested in receiving the correct bit, and we know the fact that given an odd number of relay flips will cause Bob to receive the incorrect bit, while an even number of relay flips will cause Bob to receive the correct bit.
1. Let $k \in \mathbb{N}$ and k denote the number of relay flips. Then, let E_k denote the event where exactly $2k$ relays flips the bit.
2. This means that the probability to receive the correct bit $= (\cup_{k=0}^{\lfloor \frac{n}{2} \rfloor} E_k)$ where n denotes the total number of relays.
3. Now, we can also note that $E_0, E_1, ..., E_{\lfloor \frac{n}{2} \rfloor}$ are all mutually disjoint events. This means that $(\cup_{k=0}^{\lfloor \frac{n}{2} \rfloor} E_k) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} E_k$.
4. Then, the number of possibilities to choose $2k$ relays among n relays is given by: $\binom{n}{2k}$. Since there are only 2 outcomes, either a bit is flipped or a bit is not flipped, the outcome follows a binomial distribution.
5. This means that the probability that 2k specified relays flip the bit given n relays is given by: $p^{2k}(1-p)^{n-2k}$.
6. This means that $P(E_k) = \binom{n}{2k} p^{2k}(1-p)^{n-2k}$. Then, using result from 3., this means that Bob will receive the correct bit with probability $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} p^{2k}(1-p)^{n-2k}$.

(b) **(5 points)** Let us consider an alternative way to calculate this probability. We say that the relay has *bias* q if the probability it flips the bit is $(1-q)/2$. The bias q is a real number between -1 and $+1$. Show that sending a bit through two relays with bias q_1 and q_2 is equivalent to sending a bit through a single relay with bias $q_1 \cdot q_2$.

Solution.

Let $A \in \{correct, incorrect\}$ be the outcome after we receive the bit after sending it through 2 relays with bias q_1 and q_2, and $B \in \{correct, incorrect\}$ be the outcome after we receive the bit after sending it through 1 relay with bias $q_1 \cdot q_2$.
1. From definition, $P(B = correct) = \frac{(1-q1 \cdot q2)}{2}$
2. Then, for $P(A = correct)$, it can be noted that this event should happen if and only if exactly one of the relays out of the 2 flips the bit. This means that:
$P(A = correct) = \frac{1-q_1}{2} \cdot (1 - \frac{1-q_2}{2}) + \frac{1-q_2}{2} \cdot (1 - \frac{1-q_1}{2})$
$= \frac{1-q_1}{2} - \frac{(1-q_1)(1-q_2)}{4} + \frac{1-q_2}{2} - \frac{(1-q_2)(1-q_1)}{4}$
$= \frac{1}{4}(2(1-q_1) - (1-q_1)(1-q_2) + 2(1-q_2) - (1-q_2)(1-q_1))$
$= \frac{1}{2}((1-q_1) - (1-q_1)(1-q_2) + (1-q_2))$
$= \frac{1}{2}(1 - q_1 - (1 - q_2 - q_1 + q_1q_2) + 1 - q_2)$
$= \frac{1}{2}(1 - q_1 - 1 + q_2 + q_1 - q_1q_2 + 1 - q_2)$
$= \frac{1}{2}(1 - q_1q_2)$
$= \frac{1-q_1 \cdot q_2}{2}$
$= P(B = correct)$
3. Hence, sending a bit through two relays with bias q1 and q2 $= \frac{1-q_1 \cdot q_2}{2}$, which is equivalent to sending a bit through one single relay with bias $q_1 \cdot q_2 = \frac{1-q_1 \cdot q_2}{2}$.

(c) **(5 points)** Prove that the probability you receive the correct bit when it passes through n relays is

$$\frac{1+(1-2p)^n}{2}.$$

Solution.

We can prove the probability that one receives the correct bit when it passes through n relays = $\frac{1+(1-2p)^n}{2}$ using induction $\forall n \in \mathbb{N}^+$.

1. First, the base case. The base case here is when $n = 1$ and the probability that one receives the correct bit would be $(1-p)$ given that p is the probability that a bit is flipped and resulting in the incorrect bit. Since $\frac{1+(1-2p)^1}{2} = \frac{(2-2p)}{2}$ = $(1-p)$. This shows that the theorem applies for the base case.

2. The induction hypothesis(IH) here is that the theorem works and the probability that one receives the correct bit when it passes through n relays = $\frac{1+(1-2p)^n}{2}$. Now, we have to show that it works for $(n+1)$ relays.

3. Let E_1 denote the event that the bit is correct when it has passed through n relays and the $(n+1)$ relay does not flip the bit, while E_2 denote the event that the bit is incorrect when it has passed through n relays and the $(n+1)$ relay flips the bit. In summary, both E_1 and E_2 results in the correct bit after $(n+1)$ relays.

4. However, since E_1 and E_2 are mutually disjoint events, we can further let E denote the event that one receives the correct bit after $(n+1)$ relays. Hence, by property of mutually disjoint events, this means that $E = E_1 \cup E_2$ and hence:

$P(E) = P(E_1) + P(E_2)$
$= (\frac{1+(1-2p)^n}{2} \cdot (1-p)) + ((1 - \frac{1+(1-2p)^n}{2}) \cdot p)$ by the induction hypothesis, IH.
$= \frac{1}{2}((1-p) + (1-p)(1-2p)^n) + \frac{1}{2}(2p - (p + p(1-2p)^n)$
$= \frac{1}{2}(1 - p + (1-p)(1-2p)^n + 2p - p - p(1-2p)^n)$
$= \frac{1}{2}(1 + (1-2p)^n(1-p-p))$
$= \frac{1+(1-2p)^{n+1}}{2}$.

Thus, this proves that the probability that one receives the correct bit when it passes through n relays = $\frac{1+(1-2p)^n}{2}$.

5. **An Useful Estimate.**

For an integers n and t satisfying $0 \leqslant t \leqslant n/2$, define

$$P_n(t) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{t}{n}\right)$$

We will estimate the above expression. (*Remark*: You shall see the usefulness of this estimation in the topic "Birthday Bound" that we shall cover in the forthcoming lectures.)

(a) **(13 points)** Show that

$$\exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{6n^2}\right) \geqslant P_n(t) \geqslant \exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{3n^2}\right).$$

Solution.

1. First, we take the natural logarithm of $P_n(t)$.
$ln(P_n(t)) = ln = [(1 - \frac{1}{n})(1 - \frac{2}{n})...(1 - \frac{t}{n})]$
$= ln(1 - \frac{1}{n}) + ln(1 - \frac{2}{n}) + ... + ln(1 - \frac{t}{n})$
$= \sum_{i=0}^{t} ln(1 - \frac{t}{n})$
2. From Q1, we know that the corresponding bounds for $ln(1-x)$ are $-x - \frac{x^2}{2}$ and $-x - x^2$.
3. Then, substituting $\frac{t}{n}$ with x, we get the following bounds:
$(-\frac{t}{n} - \frac{t^2}{2n^2})$ and $(-\frac{t}{n} - \frac{t^2}{n^2})$.
4. Now, we can use integration to estimate the bounds for $P_n(t)$.
5. Integrating both the bounds, we can obtain the following:
$(-\frac{t}{2n} - \frac{t^2}{2n} - \frac{t^3}{6n^2})$ and $(-\frac{t}{2n} - \frac{t^2}{2n} - \frac{t^3}{3n^2})$.
Hence, we can denote the following:
$(-\frac{t}{2n} - \frac{t^2}{2n} - \frac{t^3}{6n^2}) \geqslant ln(P_n(t)) \geqslant (-\frac{t}{2n} - \frac{t^2}{2n} - \frac{t^3}{3n^2})$
However, since the answer from Q1 is an estimate already, a tighter estimate would be to bound the terms after t^3. Hence, a tighter estimate that we can obtain would be:
$(-\frac{t}{2n} - \frac{t^2}{2n} - \frac{\Theta(t^3)}{6n^2}) \geqslant ln(P_n(t)) \geqslant (-\frac{t}{2n} - \frac{t^2}{2n} - \frac{\Theta(t^3)}{3n^2})$
Now, taking exponential on all sides of the equation:
$exp((-\frac{t}{2n} - \frac{t^2}{2n} - \frac{\Theta(t^3)}{6n^2})) \geqslant P_n(t) \geqslant exp(-\frac{t}{2n} - \frac{t^2}{2n} - \frac{\Theta(t^3)}{3n^2})$.

(b) **(2 points)** Show that when $t = \sqrt{2cn}$, where c is a positive constant, the expression above is

$$P_n(t) = \exp\left(-c - \ \Theta\left(1/\sqrt{n}\right) \right).$$

Solution.

Using both of the bounds obtained above,
$exp((-\frac{t}{2n} - \frac{t^2}{2n} - \frac{\Theta(t^3)}{6n^2})) \geqslant P_n(t) \geqslant exp(-\frac{t}{2n} - \frac{t^2}{2n} - \frac{\Theta(t^3)}{3n^2})$
1. When $t = \sqrt{2cn}$, the following result can be obtained:
$\exp\left(-\frac{\sqrt{2cn}}{2n} - \frac{2cn}{2n} - \frac{\Theta((2cn)^{3/2})}{6n^2}\right) \geqslant P_n(t) \geqslant \exp\left(-\frac{\sqrt{2cn}}{2n} - \frac{2cn}{2n} - \frac{\Theta((2cn)^{3/2})}{3n^2}\right)$
$= \exp\left(-\frac{\sqrt{c}}{\sqrt{2n}} - c - \frac{\Theta(c^{3/2}n^{1/2})}{6n^2}\right) \geqslant P_n(t) \geqslant \exp\left(-\frac{\sqrt{c}}{\sqrt{2n}} - c - \frac{\Theta(c^{3/2}n^{1/2})}{3n^2}\right)$
Then, we can note that as n tends to a large number, the terms bounded by Θ can be simplified further to $\Theta(\frac{1}{\sqrt{n}})$:
$\exp(-c - \Theta(1/\sqrt{n})) \geqslant P_n(t) \geqslant \exp(-c - \Theta(1/\sqrt{n}))$.
Thus, $P_n(t) = \exp(-c - \Theta(1/\sqrt{n}))$.

Collaborators for this homework assignment: Josh Tseng, Rohan Purandare, Adam Nasr, Nate Johnson