1 Problem A

1.1 Analytical Solution

$$cov(X_n, X_{n+1}) = E[X_n X_{n+1}] - EX_n EX_{n+1} \tag{1}$$

$$E[X_n X_{n+1}] = \sum_{i=0}^{\infty} E[X_n X_{n+1} | X_n = i] P(X_n = i) \tag{2}$$

$$= \sum_{i=0}^{\infty} i E[X_{n+1} | X_n = i] P(X_n = i)$$

$$= \sum_{i=0}^{\infty} i(E[X_n | X_n = i] + E[I_n | X_n = i]) P(X_n = i)$$

$$= \sum_{i=0}^{\infty} i(i + E[I_n | X_n = i]) P(X_n = i)$$

$$= \sum_{i=0}^{\infty} i(i + (\alpha_i - \beta_i)) P(X_n = i)$$

$$EX_n = \sum_{i=0}^{\infty} i P(X_n = i) \tag{3}$$

$$EX_{n+1} = EX_n + EI_n \tag{4}$$

$$= EX_n + \sum_{i=0}^{\infty} E[I_n | X_n = i] P(X_n = i)$$

$$= EX_n + \sum_{i=0}^{\infty} (\alpha_i - \beta_i) P(X_n = i)$$

$$\text{From (1), (2)} - \text{(3)} \cdot \text{(4)} \Rightarrow$$

$$\lim_{n\to\infty} cov(X_n, X_{n+1}) = \sum_{i=0}^{\infty} i(i + \alpha_i - \beta_i)\pi_i - \sum_{i=0}^{\infty} i\pi_i \sum_{i=0}^{\infty} (i + \alpha_i - \beta_i)\pi_i \tag{5}$$

1.2 Simulation Results

Define the walk on the positive integers to have a birth rate of 1 for X_1, 0.5 for X_2 and X_3 and 0 for $X_{\geq 4}$. Similarly, the death rate is 0 for X_1, 0.5 for X_2 and X_3, 1 for X_4 and 0 for $X_{>4}$. Running a simulation for $n = 1000000$ generates a covariance of 0.4176591. Using the above equation with the markov chain solver created in homework 1 to generate the pi values, it generates a covariance of 0.4166667, a <1% difference. This result gives a good indication that our math is correct since it is unlikely that the simulation code and the math result would both suffer from two separate errors that generate the same outcome. To run a simulation with our code, run the “RunSimulation” function in “problemA.R”.