RESEARCH STATEMENT

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My research fits broadly into the framework of moonshine, a branch of mathematics that originated in the late 20th century from a series of numerical coincidences connecting finite groups to modular forms, and has since evolved into a rich theory that uses tools of number theory, algebra, geometry, and physics to shed light on the underlying algebraic structures that the coincidences reflect. I'm very broadly interested in the following questions about various aspects of moonshine.

- $1a$. Given a finite group, what modular forms appear as graded traces for a moonshine module for that group?
- $1b$. In the same vein, if we start with a modular form, is there a group G and a moonshine module for G whose graded dimension is that modular form?
- 2. What sorts of underlying algebraic structures do these modules possess? Current examples include vertex operator algebras, lie algebras, and super vertex operator algebras. We hope to gain a deeper/more theoretical understanding of why moonshine exists in the first place by looking closely at the examples we find.
- 3. How does moonshine interact with other subfields of physics and mathematics? In particular, can we use the theory of moonshine to answer questions in algebra, number theory, physics, etc.?

The first question relates to finding *examples* of moonshine, and the third is concerned with *applications* of moonshine. My graduate research at Emory University has delved into both of these aspects, and I plan on continuing to explore these questions after graduate school.

1. Application to Number Theory.

For a more concrete example of the types of problems I am interested in, consider the following elliptic curve defined over $\mathbb{Q}$.

$$E\colon y^2 = x^3 + 864x - 432 \tag{1.1}$$

It is well-known that the set of rational points $E(\mathbb{Q})$ of an elliptic curve (i.e., the set of points on E whose coordinates are rational numbers) has the structure of a finitely generated abelian group. That is, $E(\mathbb{Q}) = \mathbb{Z}^r \oplus E(\mathbb{Q})_{\mathrm{tor}}$. Here $r \geq 0$ is called the (algebraic) rank of E, and $E(\mathbb{Q})_{\mathrm{tor}}$ is a finite abelian group. Computing the rank of a general elliptic curve is a challenging problem in number theory (cf. [Wiles 2006]).

For $d < 0$ a fundamental discriminant, let E^d denote the d^{th} quadratic twist of E

$$E^d\colon y^2 = x^3 + 864d^2x - 432d^3. \tag{1.2}$$

Let $F(\tau)$ denote the unique (weakly holomorphic) modular form in $M^{+,!}_{\frac{3}{2}}(\Gamma_0(4))$ such that $F(q) = q^{-5} + O(q)$, and let $c(d)$ denote the coefficient of q^{-d} in the q-expansion for $F(\tau)$. Then in [Khaqan 2020], we prove,

Theorem 1.1 ([Khaqan 2020])**.** Let $d < 0$ be a fundamental discriminant which is not a quadratic residue modulo 19. If $c(d) \not\equiv 0 \pmod{19}$, then the Mordell–Weil group $E^d(\mathbb{Q})$ is finite i.e., $\mathrm{rank}(E^d) = 0$.

Note that this theorem gives information about the rank of certain elliptic curves *without any computations on the curves themselves.* This phenomenon is not uncommon in the theory of moonshine. In this document, I will give some background on moonshine and explain how my research fits into the story (cf. Section 2), state a few of my results and describe how to prove them (cf. Section 3), and summarize plans for future work (cf. Section 4).

2. Background

2.1. **Monstrous moonshine.** The complete classification of finite simple groups is one of the greatest achievements of 20th-century mathematics. A culmination of decades of work from hundreds of mathematicians working around the world, it states that almost all finite simple groups fit neatly into three infinite families: the cyclic groups, alternating groups, or finite groups of Lie type. There are 26 exceptions to this rule, the so-called sporadic simple groups. Amongst these 26 sporadic groups, the Monster group $\mathbb{M}$ is distinguished, in part due to its size. The number of elements in the monster is $|\mathbb{M}| = 2^{46}{\cdot}3^{20}{\cdot}5^{9}{\cdot}7^{6}{\cdot}11^{2}{\cdot}13^{3}{\cdot}17{\cdot}19{\cdot}23{\cdot}29{\cdot}31{\cdot}41{\cdot}47{\cdot}59{\cdot}71$ which is $2^5 \cdot 3^7 \cdot 5^3 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 29 \cdot 41 \cdot 59 \cdot 71$ times that of the next largest sporadic simple group, the baby monster.

Fischer and Griess first conjectured the existence of the Monster group in 1973. In 1978, three years before Greiss's explicit construction of the Monster, Fischer, Livingstone, and Thorne computed its character table. They found that the smallest non-trivial irreducible representation of the Monster is 196883-dimensional. The number 196883 rang a bell for John McKay, who along with John Thompson noted that the first few coefficients of the normalized elliptic modular invariant $J(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + O(q^4)$, a central object in the theory of modular forms, can be written as sums involving the first few entries in the first column of the character table (i.e., dimensions of irreducible representations) of the Monster, for example:

$$\begin{aligned} 1 + 196883 &= 196884 \\ 1 + 196883 + 21296876 &= 21493760 \\ 2\cdot 1 + 2\cdot 196883 + 21296876 + 842609326 &= 864299970. \end{aligned} \tag{2.1}$$

This doesn't stop here. If we instead look at the second column (which lists the traces of an element of order two on the same representations), and add the corresponding entries together in a similar fashion,

$$\begin{aligned} 1 + 4371 &= 4372 \\ 1 + 4371 + 91884 &= 96256 \\ 2\cdot 1 + 2\cdot 4371 + 91884 + 1139374 &= 1240002 \end{aligned} \tag{2.2}$$

the numbers we get are Fourier coefficients of another well-known modular form, $T_{2A}(\tau) = q^{-1} + 4372q + 96256q^2 + 1240002q^3 + O(q^4)$.

These observations inspired Thompson's conjecture [Thompson 1979] that there exists an infinite-dimensional $\mathbb{M}$-module V whose graded dimension is $J(\tau)$ and each of whose McKay–Thompson series $T_g(\tau) := \sum_{n\geq -1} \operatorname{tr}(g|V_n)q^n$ is a normalized principal modulus for a genus-zero subgroup Γ_g of $SL_2(\mathbb{R})$. In particular, this means that the phenomenon we described above continues for each of the 194 columns in the character table of the Monster group. Conway and Norton [Conway and Norton 1979] investigated this further, explicitly described the relevant McKay-Thompson series, and christened this phenomenon "monstrous moonshine." In 1988, Frenkel, Lepowsky, and Meurmann [Frenkel et al. 1988] constructed a vertex operator algebra $V^\natural$ whose automorphism group is the Monster, and in 1992, Borcherds [Borcherds 1992] proved that it satisfies the monstrous moonshine conjectures.

Recall that a normalized principal modulus is uniquely determined by its invariance group, so the assignment $g \to \Gamma_g$ determines each of the traces $\operatorname{tr}(g|V_n)$ for $g \in \mathbb{M}$ and $n \in \mathbb{Z}$. In particular, this allows one to compute the structure of V as an $\mathbb{M}$-module *without doing any computations with the Monster itself.*

3. Results

My recent work has focused on a subgroup of the Monster, *the Thompson group,* which is a sporadic simple group of order $2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$. In [Khaqan 2020], we prove the existence of an infinite-dimensional Thompson-module W whose graded dimension is the unique (weakly holomorphic) modular form in $M^{+,!}_{\frac{3}{2}}(\Gamma_0(4))$ with Fourier expansion $6q^{-5} + O(q)$, and whose McKay-Thompson series satisfy certain special properties. In fact, in the paper, we classify all such modules, but here we give a simpler version of the theorem for reference.

Theorem 3.1 ([Khaqan 2020]). There exists an infinite-dimensional graded virtual Th-module

$$W := \bigoplus_{\substack{n>0 \\ n\equiv 0,3 \pmod 4}} W_n \tag{3.1}$$

such that for each rational conjugacy class $[g]$ of Th, the corresponding McKay–Thompson series,

$$\mathcal{F}_g(\tau) = 6q^{-5} + \sum_{\substack{n>0 \\ n\equiv 0,3 \pmod 4}} \operatorname{tr}(g|W_n)\, q^n \tag{3.2}$$

is a specific weakly holomorphic modular form with integer Fourier coefficients that satisfies the following properties:

- It has weight $\frac{3}{2}$, level $4|g|$ with a specific character (dictated by g) and satisfies the Kohnen plus space condition.
- It has a pole of order 5 at the cusp ∞, a pole of order $\frac{5}{4}$ at the cusp $\frac{1}{2|g|}$ if $|g|$ is odd, and vanishes at all other cusps.

This serves as an example of moonshine for the Thompson's group (cf. Question 1). We prove this theorem using standard techniques in moonshine that were first suggested by Thompson and have been used extensively since. For each rational conjugacy class, we define f_g^{wh} to be the projection onto the Kohnen plus-space of the Rademacher sum of weight $\frac{3}{2}$ and index -5 that transforms under the action of $\Gamma_0(4|g|)$ with a specific character ψ_g.

$$f_g^{wh} = 6R^{[-5],+}_{\frac{3}{2},4|g|,\psi_g}(\tau) = 6q^{-5} + \sum_{\substack{n>0 \\ n\equiv 0,3 \pmod 4}} a_g(n)q^n. \tag{3.3}$$

Rademacher sums, when convergent, define mock modular forms of the specified weight and level. For f_g^{wh}, the space of shadows turns out to be zero, so f_g^{wh} satisfies the properties listed in the theorem for each g (cf. [Cheng and Duncan 2014; Kohnen 1985; Bruinier et al. 2004]). Specifying the behavior at the poles implies that the McKay–Thompson series $\mathcal{F}_g(\tau)$ is uniquely determined up to the addition of cusp forms. Thus, if the theorem is true, we would have:

$$\mathcal{F}_g(\tau) = f_g^{wh}(\tau) + f_g^{cusp}(\tau) \tag{3.4}$$

where f_g^{cusp} lies in $S^+_{\frac{3}{2}}(\Gamma_0(4|g|), \psi_g)$.

In order for the collection $\{\mathcal{F}_g(\tau), g \in Th\}$ to be the McKay–Thompson series of a virtual module (as in Theorem 3.1), they must satisfy congruences modulo certain powers of primes that divide the order of the Thompson group. In simplified terms, the coefficients of $\mathcal{F}_g(\tau)$ need to satisfy all the congruences that the irreducible rational characters of Th do.

We are now ready to connect this back to the elliptic curves. The (weak) *Birch and Swinnerton-Dyer Conjecture* states that the rank of a general elliptic curve equals the order of vanishing of its L-function $L_E(s)$ at $s = 1$. The Birch and Swinnerton-Dyer Conjecture has been proven in special cases, and in particular, it is known that if $L_E(1)$ is non-zero, then $E(\mathbb{Q})$ is finite [Gross and Zagier 1986; Kolyvagin 1989].

Let g be an element of order 19 in the Thompson group. Then $S_g := S^+_{\frac{3}{2}}(\Gamma_0(4|g|), \psi_g)$ has dimension 1, so we let $b_g(n)$ denote the coefficients of the unique normalized cusp form in S_g. From the proof of Theorem 3.1, we get,

$$\mathcal{F}_g(\tau) = 6q^{-5} + \sum_{\substack{n>0 \\ n\equiv 0,3 \pmod 4}} (a_g(n) + 18b_g(n))\, q^n, \tag{3.5}$$

where the 18 is a consequence of the congruences dictated by the irreducible rational characters of Thompson. Furthermore, since W is a virtual module for the Thompson group, we know the following congruence holds for each $p|\#Th$ (and in particular for p = 19) and for all $n > 0$,

$$\dim(W_n) \equiv \operatorname{tr}(g_p|W_n) \pmod p \tag{3.6}$$

where g_p is an element of order p. Thus, for all $n > 0$, we have:

$$\dim(W_n) \equiv a_g(n) + 18b_g(n) \pmod{19} \tag{3.7}$$

Suppose for now that $a_g(n)$ is zero for some n. Then, since the graded dimension of W_n is just $6F(\tau)$, we would get that if $19|c(d)$ as in the statement of the theorem, then $19|b_g(n)$. For $n = d < 0$ a fundamental discriminant such that $\left(\frac{d}{19}\right) = -1$, a lemma of Duncan, Mertens and Ono [Duncan et al. 2017; Lemma 6.5] gives us that $L_{E^d}(1) \not\equiv 0 \pmod{19}$ and thus, in particular, $L_{E^d}(1) \neq 0$.

To prove that $a_g(d)$ is zero for d as above, we let $\mathcal{Q}_D^{(N)}$ be the set of positive definite quadratic forms $Q = [a, b, c] := ax^2 + bxy + cy^2$ of discriminant $-D = b^2 - 4ac < 0$ such that $N|a$. Then, $\Gamma_0(N)$ acts on $\mathcal{Q}_D^N$ with finitely many orbits. For $Q = [a, b, c] \in \mathcal{Q}_D^{(N)}$, we let τ_Q denote the unique root of $Q(x, 1)$ in the upper half-plane $\mathbb{H}$. Then we can use Corollary 1.3 of [Miller and Pixton 2010] to write

$$a_g(n) = \text{const.*} \sum_{Q \in \mathcal{Q}_{5n}^{(19)}/\Gamma_0(19)} \chi_5(Q) \frac{J_{19}^+(\tau_Q)}{\omega^{(19)}(Q)}, \tag{3.8}$$

where J_{19}^+ is the normalized principle modulus for the group $\Gamma_0^+(19)$ (see [Miller and Pixton 2010] for a definition of $\omega^{(N)}(Q)$ and $\chi_D(Q)$). The important thing to note in Equation (3.8) is the following: for $n = d < 0$ a fundamental discriminant such that $\left(\frac{d}{19}\right) = -1$, we have that $5d$ is not a square mod 19. Thus, there are no quadratic forms $[a, b, c]$ of discriminant $b^2 - 4ac = 5d$ such that $19|a$, and so $a_g(n) = 0$. This completes the proof of Theorem 3.1.

In [Khaqan 2020], we also prove a more subtle result for quadratic twists of elliptic curves of conductor 14, which involves using the modules from Theorem 3.1 to detect the non-triviality of the Selmer and Tate–Shafarevich groups of these twists. The common theme is the same: these theorems are examples where one can use moonshine to answer number theoretic questions (cf. Question 3), and they allow one to extract information about elliptic curves *without any computations on the curves themselves*.

4. Future plans

I am actively thinking about two different projects at the moment, and here I will describe them both in some detail.

4.1. **Zagier's Grid.** The function $F(\tau)$ introduced in Theorem 3.1 is part of a bigger story of its own. In [Zagier 2002], Zagier defined a sequence of modular forms of weight $\frac{3}{2}$ in the following way: for each positive integer $D \equiv 0, 1 \pmod 4$, let $g_D(\tau)$ denote the unique form in $M_{\frac{3}{2}}^{+,!}(\Gamma_0(4))$ of the form

$$g_D(\tau) = q^{-D} + \sum_{\substack{d \geq 0 \\ d \equiv 0,3 \pmod 4}} B(D, d) q^d. \tag{4.1}$$

Note that for $D = 5$, this is just $F(\tau)$ from Theorem 3.1, and in particular, $B(5, d) = c(d)$. Furthermore, for each $d > 0$ with $d \equiv 0, 3 \pmod 4$, we define $f_d(\tau)$ to be the unique modular form such that

$$f_d(\tau) = q^{-d} + \sum_{\substack{D > 0 \\ D \equiv 0,1 \pmod 4}} A(D, d) q^D. \tag{4.2}$$

Then, Zagier proved that $A(D, d) = -B(D, d)$ for all D and d (cf. [Zagier 2002; Theorem 4]).

In [Harvey and Rayhaun 2015], Harvey and Reyhaun conjectured the existence of a moonshine module for the Thompson group where the graded dimension is $2f_3(\tau) + 248f_0(\tau)$. Their conjecture was proven in [Griffin and Mertens 2016]. The involvement of both $f_d(\tau)$ and $g_D(\tau)$ in moonshine for the Thompson group begs the question: is there a larger framework that incorporates the entire grid? For instance, for each choice of D, is there a moonshine module for the Thompson group whose graded dimension is $g_D(\tau)$ and whose McKay–Thompson series $\mathcal{F}_g^D(\tau)$ satisfy properties analogous to the ones listed in Theorem 3.1?

This project fits into the broader question: what other modular forms out there can be in a similar relationship with the Thompson group, and can we classify them all? (cf. Question 1)

4.2. **Conway Groups.** Another well-known example of moonshine is one for the Conway groups, Co_0 and Co_1. The largest of the Conway groups, Co_0, is the group of automorphisms of the Leech lattice. It is not a simple group, but the quotient of Co_0 by its center, Co_1 is a sporadic simple group of order $2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$. In [Duncan 2007; Duncan and Mack-Crane 2015], Duncan and Mack-Crane construct super vertex operator algebras $V^{f\natural}$ and $V^{s\natural}$ whose automorphism groups are Co_0 and Co_1 respectively. The McKay–Thompson series in this case also turn out to be normalized principle moduli for genus-zero subgroups of $SL_2(\mathbb{R})$.

I am currently working on computing traces of elements of the Conway group on g-twists of $V^{f\natural}$ and $V^{s\natural}$, for each g in Co_0. This project is aimed towards better understanding the algebraic structures in the moonshine modules that we already have an explicit construction for (cf. Question 3).

4.3. **Long-term plans.** Apart from the two projects mentioned above, one of my long-term research goals is to construct a moonshine module for the Thompson group with an underlying algebraic structure, such as a vertex operator algebra or a lie algebra. Such a construction would provide more insight into the riddles of moonshine, as well as open avenues of future research (cf. Question 3).

Finally, at a future institution, I look forward to collaborating with other researchers in various fields, especially number theory and physics, to further explore some of the connections that moonshine exhibits. I also look forward to including students in my research, both by working with them individually and by teaching courses relevant to my research. For example, I would like to design a topics course consisting of short modules on all the math you would need to learn in order to be able to read a classical paper in moonshine. Some potential topics might include modular forms, representation theory, lie algebras, etc.

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