§ 2 Linear Representations.

Let G be a finite group. Let V be a finite-dimensional vector over a field F.

A **linear representation** of G on V is a homomorphism

$$\begin{aligned} \phi = \phi_V : G &\longrightarrow GL(V) \\ g &\longmapsto \phi_g \end{aligned}$$

<u>Remark:</u> ϕ being a homomorphism means that $\forall\, g, h \in G$

$$\phi_{gh} = \phi_g \phi_h$$

We define the **dimension** or **degree** of ρ to be the dimension of V as a vector space over F.

Other perks of ϕ being a homomorphism:

- $\ker \phi \triangleleft G$
  normal subgroup
- $G / \ker \phi \simeq \mathrm{Im}(G) \leq GL(V)$ subgroup

The first homomorphism theorem.

We say that ϕ is faithful if $\ker \phi$ is trivial.

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§ Linear actions

We say that G acts linearly on V if there exists a map

$$* : G \times X \longrightarrow X$$
$$(g, x) \longmapsto gx$$

such that:

- (action)

$$ex = x \qquad \forall x \in X.$$
$$(gh)x = g(hx) \qquad \forall g, h \in G$$

- (linearity)

$$g(x+y) = \underbrace{gx}_{\in V} + \underbrace{gy}_{\in V}$$
$$g(\lambda x) = \lambda(gx)$$

$\forall x, y \in V$, $g \in G$ and $\lambda \in F$.

We say that V is a G-module or a G-space.

Remark:

If G acts linearly on V then the map

$$\phi : G \to GL(V)$$

that sends $g \mapsto \phi_g$

where $\phi_g : x \mapsto gx$ is a representation of G.

Conversely, given a representation $\phi : G \to GL(V)$, we can define a linear action via $gx = \phi_g(x)$. $\forall x \in V$, $g \in G$.

Group Algebra:

Let F be a field.

Let A be an F-vector space and suppose that A also has a ring structure (w/ identity) on F.

(i.e. elements of A can be added subtracted & multiplied together, but not necessarily divided.)

Suppose for all $x, y \in A$ and $k \in F$, we have:

$$k(\underbrace{xy}_{\text{EA.}}) = (\underbrace{kx}_{\text{scalar mult.}})y = x(ky)$$

then A is an F-algebra.

$$\text{group} \longmapsto \text{ring}$$

$$F\text{-vect. space} \longmapsto F \text{ algebra.}$$

Examples: ① Matrices!

$$M_n(F) : \{ n \times n \text{ matrices over } F \}.$$

② $\text{End}(V) = \{ F\text{-linear transformations } T : V \to V \}$

where V is an F-vect. space.

exercise: prove that this is an F algebra.

Proof: $\text{End}(V)$ is a vector space w/ addition:

$$(f+g)v = f(v) + g(v)$$

and multiplication:

$$f \circ g(v) = f(g(v))$$

It is an F algebra under scalar #:

$$c\, f\circ g(v) = c(f(g(v))) = f(g(cv))$$

③ The algebra $F[G]$:

Let G be a finite group. Let $F(G)$ be the set of formal sums:

$$\left\{ \sum_{g\in G} k_g\, g \;\middle|\; k_g \in F \right\}$$

This has an F-vector space structure on it.

Also there is an embedding

$$\begin{aligned} G &\hookrightarrow F[G] \\ g &\longmapsto 1g + \sum_{\substack{h\neq g \\ h\in G}} 0\cdot h \end{aligned}$$

This is a vector space, so to turn it into an algebra, we have to define multiplication.

Let

$$(ag)\cdot(bh) = (ab)(gh) \quad \text{where } g, h, gh \in G \text{ and } a, b, ab \in F.$$

and extend linearly.

Excercise: Check that this is an F-algebra.

Back to representations:

Let V (defined over F) be a G-module. Show that V is an F[G]-module.

(We can define the action of F[G] on V by:

$$\left(\sum_{g\in G} k_g \; g\right) v := \sum_{g \in G} k_g (gv)$$

Check that this is a group action)

§ Matrix Representations:

R is a matrix representation of G of degree n if R is a homomorphism

$$R : G \longrightarrow GL_n(F)$$

e.g.

let $G = C_4 = \langle x \mid x^4 = 1 \rangle$.

let $n = 1$, and $F = \mathbb{C}$.
then $R : G \longrightarrow GL_1(\mathbb{C})$ (invertible 1x1 matrices)
$\simeq \mathbb{C}^*$ (since 0 isn't)

is a group homomorphism if

$$R(e) = 1$$

and

$R(x) =$ an element in $\mathbb{C}$ whose 4^{th} power is 1

(right? because
$R(ee) = R(e) = R(e)R(e)$
so $R(e) = R(e)^2$
so it's either 0 or 1.)

$= \pm 1, \pm i.$

so there are 4 matrix representations of degree 1 on C_4.

Exercise: generalize this. How many degree 1 representations does C_n have?

Given a linear rep. $\rho : G \to GL(V)$ with $\dim_F V = n$, we can fix a basis $\mathcal{B}$ of V to get a matrix representation of G:

$$\begin{aligned} R : G &\longrightarrow GL_n(F) \\ g &\longmapsto [\underbrace{\rho(g)}_{\in GL(V)}]_{\mathcal{B}} \end{aligned}$$

$\mathcal{B}$ ↝ written in the basis.

Conversely, if we start w/ a matrix representation, (deg n) we can get a linear representation $\rho : G \to GL(V)$ where $V = F^n$ by assigning $g \mapsto \rho_g$ and

$$\rho_g(v) = \underbrace{R_g}_{\text{matrix}}(v) \qquad \forall v \in F^n.$$

Example: The trivial representation:

$$\begin{aligned} \rho : G &\to GL(F) \\ g &\mapsto (\mathrm{id} : F \to F) \end{aligned}$$

Matrix:

$$\underset{m\times 1}{\begin{pmatrix} \ \end{pmatrix}} \ \underset{1\times n}{\overset{\in F}{(a)}} = \underset{1\times 1}{(b)}$$

$$\downarrow$$

$$(1)\,(a) = (b)$$

e.g.

let $G = C_4 = \langle x \mid x^4 = 1 \rangle$.

let $n=2$, and $F = \mathbb{C}$.
then $R : G \longrightarrow GL_2(\mathbb{C})$ (invertible 2×2 matrices)

is a group homomorphism if

$$R(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$R(x) = X \in GL_2(\mathbb{C}) \text{ s.t. } X^4 = 1.$$

Check that any choice of X is isomorphic to a diagonal matrix w/ entries $\{\pm 1, \pm i\}$

§ G-homomorphisms.

Fix G and F and let V_1, V_2 be F-vector spaces.

Consider $\rho_1 : G \longrightarrow GL(V_1)$

and $\rho_2 : G \longrightarrow GL(V_2)$

two representations of G.

The linear map $\Phi : V_1 \longrightarrow V_2$ is called a G-homomorphism if

$$\Phi \circ \rho_1(g) = \rho_2(g) \circ \Phi$$

$$\begin{tikzcd}
\rho_1(g) : V_1 \arrow[r] \arrow[d, "\Phi"'] & V_1 \arrow[d, "\Phi"] \\
\rho_2(g) : V_2 \arrow[r] & V_2
\end{tikzcd}$$

We say that ϕ intertwines ρ_1 and ρ_2.

The space of all G-homomorphisms is denoted $Hom_G(V, V')$

We say ϕ is a G-isomorphism if it is a bijective G-homomorphism, and we say that ρ_1 & ρ_2 are isomorphic.

Remark: If ρ_1 and ρ_2 are isomorphic, they have the same dimension.

In terms of matrix representations:

$$R_1 : G \longrightarrow GL_n(F)$$

and

$$R_2 : G \longrightarrow GL_n(F)$$

are isomorphic if there exists a non singular matrix X such that

$$X \underbrace{R_1(g)}_{n\times n \text{ matrix}} = R_2(g) X \quad \forall g \in G.$$

In terms of G-actions:

$$\phi(\underbrace{gv}_{\in V_1}) = g(\underbrace{\phi(v)}_{\in V_2}) \quad \forall v \in V_1.$$

§ Subrepresentations:

Let $\rho : G \to GL(V)$ be a G-rep.

We say that $W \subseteq V$ is a G-subspace if W is a subspace and

$$\rho_g(W) \subseteq W \quad \forall g \in G.$$

e.g. $\{0\}$ or V.

We say ρ is irreducible or simple if there is no proper G-subspace.

e.g. any 1 d rep is irred.

If W is a G-subspace then the corresponding map

$$G \longrightarrow GL(W)$$
$$g \longmapsto \rho_g|_W \quad (w \mapsto gw \in W).$$

if you like the action notation

Lemma: Let $\rho : G \longrightarrow GL(V)$ be a rep. & W be a G-subspace.

Let $\mathcal{B} = \{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$ be a basis of V containing a basis $\{v_1, \dots, v_m\}$ of W.

Then the matrix of ρ_g in this basis is a block matrix:

$$\left(\begin{array}{c|c} \overset{m\times k}{*} & \overset{m\times m}{*} \\ \hline \underset{l\times k}{0} & \underset{l\times m}{*} \end{array}\right).$$

Proof: Recall that $\rho_g(W) \subseteq W$. So

$$\rho_g \begin{pmatrix} \left.\begin{matrix} * \\ * \\ \vdots \\ * \end{matrix}\right\} m \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \left.\begin{matrix} * \\ * \\ \vdots \\ * \end{matrix}\right\} m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

in particular:

$$\rho_g \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

but $\rho_g(e_1)$ is the first column etc.

Examples: ① $C_m = \langle x \mid x^m = 1 \rangle$ has precisely m $\mathbb{C}$-irreps, all of deg 1, given by:

$$\rho_k : x^j \longrightarrow \left(e^{2\pi i/m}\right)^{jk}$$

In fact: all irreps of a fin abelian group are 1d, will prove this later.

②

$$G = D_6 = \langle x, y \mid y^2 = 1, x^3 = 1, yxy^{-1} = x^{-1} \rangle$$

$\simeq S_3$ (the smallest non abelian finite group)

HW problem: G has 2 irreps of degree 1 and 1 of degree 2. Find all of them.

HW problem: (maybe TA sheet).

Find all irreps of D_8.

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§ Decomposable representations.

We say that $\rho: G \longrightarrow GL(V)$ is decomposable if there are G-invariant subspaces U and W such that $V = U \oplus W$. We say ρ is a direct sum $\rho_U \oplus \rho_W$. An indecomposable rep is what you'd think it is.

Lemma: Suppose $\rho: G \to GL(V)$ decomposes as $V = U_1 \oplus U_2$.

If $\mathcal{B}$ is a basis of V compatible w/ U_1 & U_2

i.e. $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1$ & $\mathcal{B}_2$ are bases of U_1 & U_2 resp.

then $[\rho(g)]_{\mathcal{B}} = \left[\begin{array}{c|c} [\rho_{U_1}(g)]_{\mathcal{B}_1} & 0 \\ \hline 0 & [\rho_{U_2}(g)]_{\mathcal{B}_2} \end{array}\right]$

On the other hand,

let

$$\rho_1 : G \longrightarrow GL(V_1) \quad \&$$

$$\rho_2 : G \longrightarrow GL(V_2)$$

be two reps of G. then the direct sum of ρ_1, ρ_2 is the rep:

$$\rho_1 \oplus \rho_2 : G \longrightarrow GL(V_1 \oplus V_2)$$

such that

$$(\rho_1 \oplus \rho_2)_g : (v_1, v_2)$$
$$\downarrow$$
$$\left((\rho_1)_g(v_1) , (\rho_2)_g(v_2) \right)$$

In terms of matrix reps:

$$R_1 : G \to GL_n F , \quad R_2 : G \to GL_m(F),$$

$$R_1 \oplus R_2 : G \longrightarrow GL_{(n+m)} F$$

$$g \mapsto \left[\begin{array}{c|c} R_1(g) & 0 \\ \hline 0 & R_2(g) \end{array}\right]$$

In the next lecture, we will show that all group representations are completely decomposable into irreps. This is a feature of group rep theory & not that common in other types of rep theory.