Recall:

We say that $\rho: G \to GL(V)$ is decomposable if there are G-invariant subspaces U, W s.t.

$$V = U \oplus W$$

Recall this means that

$$\rho(U) \subseteq U \quad \text{and} \quad \rho(W) \subseteq W$$

We call ρ a direct sum and denote it by:

$$\rho = \rho_U \oplus \rho_W$$

We say ρ is indecomposable if, you know, ...

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lemma:

$$(\rho_g)_{\mathcal{B}} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}. \qquad \text{etc.}$$

def: direct sum

$$\rho_1 \oplus \rho_2 : G \longrightarrow GL(V_1 \oplus V_2)$$

$$g : v_1 + v_2 \longmapsto \rho_{1(g)}(v_1) + \rho_{2(g)}(v_2).$$

§ Complete reducibility & Maschke's theorem.

def: $\rho : G \to GL(V)$ is **completely reducible** or semisimple iff it is a direct sum of irreps.

simple $\Rightarrow$ semisimple, obviously.

$\not\Leftarrow$ not true.

non example:

$$G = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

$V = \mathbb{C}^2$

G acts on V by matrix mult.

not completely reducible.

(note that G is not finite)

Consider the subspace $W = \{(x,0) \mid x \in F\}$

then $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$

so W is a 1d irrep. Also $V = W \oplus U$ where $U = \{(0,x) \mid x \in F\}$ but U is not a G-space.

First Big Theorem! :)

Maschke's theorem:

G finite, F w/ $ch(F) = 0$.

$\rho : G \longrightarrow GL(V)$

If $W \subseteq V$ is a G-subspace then $\exists$ a G-sub.

space U such that

$$V = W \oplus U$$

Proof:

let W' be any vector space complement of W (this always exists. extend a basis of W to a basis of V, take the other vectors)

then $V = W \oplus W'$ as vec. spaces.

let $q : V \to W$ be the projection of V onto W along W'; i.e. if $v = w + w'$ then

$$q(v) = w.$$

Define $\bar{q} : v \longrightarrow \frac{1}{|G|} \sum_{g \in G} \rho_g\left(q\left(\rho_{g^{-1}} v\right)\right)$

First Q: where does this land?

Recall $\rho_{g^{-1}} v \in V$, $q : V \longrightarrow W$, $\rho_g(W) \subseteq W$ (since W is a G-subspace)

So $\bar{q} : V \to W$.

Claim: $\bar{q}(w) = w$ for $w \in W$.

Let's check: recall that $q(w) = w \ \forall w \in W$ since q is a projection. Also $\rho_{g^{-1}} w \in W$ so

$$q(\rho_{g^{-1}}(w)) = \rho_{g^{-1}}(w)$$

So
$$\bar{q}(w) = \frac{1}{|G|} \sum_{g \in G} \rho_g \rho_{g^{-1}}(w)$$

$$= w.$$

Thus, $\bar{q}$ is a projection of V onto W.

Claim: $\forall h \in G$, $h\bar{q}(v) = \bar{q}(hv) \quad \forall v \in V$.

i.e.
$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d, "\bar{q}"'] & V \arrow[d, "\bar{q}"] \\ W \arrow[r, "h"] & W \end{tikzcd}$$
(W is a G-space).

I'm dropping the ρ's now. So $h\bar{q}$ is $\rho_h \bar{q}$.

<u>Proof:</u>

$$h\bar{q}(v) = \frac{1}{|G|} \sum_{g \in G} hg\, q(g^{-1}h^{-1}hv)$$

$$= \frac{1}{|G|} \sum_{g \in G} (hg)\, q\left((hg)^{-1} h v\right)$$

As you vary $g \in G$, hg also varies over G.
let $hg = g'$ then:

$$= \frac{1}{|G|} \sum_{g' \in G} g' q\, g'^{-1}(hv) = \bar{q}(hv).$$

Finally we claim that

$\ker \bar{q}$ is G-invariant.

<u>Proof:</u> let $v \in \ker(\bar{q})$ & $h \in G$ then

$$h\bar{q}(v) = 0 = \bar{q}(hv) \text{ so } hv \in \ker \bar{q}.$$

thus $V = \operatorname{im}(\bar{q}) \oplus \ker(\bar{q}) = W \oplus \ker \bar{q}$
is a G subspace decomposition.

there is a different proof using inner products over $\mathbb{C}$ (or $\mathbb{R}$).

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§ Inner Product:

Recall that for V a $\mathbb{C}$-space, $\langle , \rangle$ is an inner product if

1. $\langle v, w \rangle = \overline{\langle w, v \rangle}$
2. $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$
3. $\langle v, v \rangle > 0$ if $v \neq 0$.

We say $\langle , \rangle$ is G-invariant if

$$\langle gv, gw \rangle = \langle v, w \rangle$$

Claim: Suppose that $W \subseteq V$ is a G-invariant subspace w/ a G-invariant $\langle , \rangle$ on V. Then $W^{\perp}$ is also a G-invariant subspace and

$$V = W \oplus W^{\perp}$$

Proof: WTS: $\forall g \in G, \forall v \in W^\perp, gv \in W^\perp$

Recall that by def $v \in W^\perp \Leftrightarrow \langle v, w \rangle = 0$ $\forall w \in W$. Thus,

$$\langle gv, gw \rangle = 0 \quad \forall g \in G, w \in W. \quad (1)$$

(Since $\langle , \rangle$ is G-invariant)

but note that this means:

$$\langle gv, w' \rangle = 0 \quad \forall w' \in W, \forall g \in G.$$

Since we can take $w = g^{-1} w'$ in eq (1).

Thus, $\forall g \in G$, $\langle gv, w \rangle = 0$ $\forall w \in W$ so

$$gv \in W^\perp$$

Weyl's unitary trick: There is a G-invariant inner product on any complex rep $\rho : G \to GL(V)$.

Proof: There is always an inner product on the $\mathbb{C}$-space V:

let $\{e_1, \dots, e_n\}$ be a basis & define $(e_i, e_j) = \delta_{ij}$.

Now define: $\langle v, w \rangle = \dfrac{1}{|G|} \sum_{g \in G} (gv, gw)$.

Claim: $\langle , \rangle$ is G-invariant:

Proof:

$\forall h \in G$,

$$\langle hv, hw \rangle = \frac{1}{|G|} \sum_{g \in G} (g\,hv, g\,hw)$$

$$= \frac{1}{|G|} \sum_{g' \in G} (g'v, g'w) = \langle v, w \rangle.$$

where $g' = gh$ or $g = g' h^{-1}$.

$\blacksquare$

§ Regular Representation:

Consider the $|G|$-dimensional $\mathbb{F}$-vector space defined by

$$\mathbb{F}G = \operatorname{span}\{e_g \mid g \in G\}$$

This is a vector space over $\mathbb{F}$, and we can define multiplication by setting:

$$e_g e_h = e_{gh}$$

$\mathbb{F}G$ is called the group algebra of G.

Define the action:

$$\rho_{reg} : G \longrightarrow GL(\mathbb{F}G).$$

by:

$$h\left(\sum_{g \in G} a_g e_g\right) = \sum_{g \in G} a_g e_{hg}$$

This is called the regular representation of G.

Claim: The regular representation is faithful.

(HW!) (Hint: consider the kernel $\{h \in G \mid h$ acts as identity on FG i.e.

$$\sum_{g\in G} a_{h^{-1}g}\, e_g = \sum_{g\in G} a_g e_g \Big)$$

Claim: Every irrep of G is isomorphic to a subrep of the regular representation.

Proof: let $\rho : G \to GL(V)$ be an irrep and pick an element $v \neq 0, v \in V$.

Define $\theta : FG \to V$ by.

$$\theta\Big(\sum_{g\in G} a_g e_g\Big) \longmapsto \sum_{g\in G} a_g\, \rho_g(v)$$

Then θ is a G-homomorphism:

$$\begin{tikzcd} FG \arrow[r, "\rho_{reg}(h)"] \arrow[d, "\theta"'] & FG \arrow[d, "\theta"] \\ V \arrow[r, "\rho_h"'] & V \end{tikzcd}$$

since

$$\theta(\rho_{reg}(h)(\sum_{g\in G} a_g e_g))$$

$$= \theta(\sum_{g\in G} a_g e_{hg}) = \sum_{g\in G} a_g \rho_{hg}(v)$$

and

$$\rho_h(\theta(\sum_{g\in G} a_g hg)) = \rho_h(\sum_{g\in G} a_g \rho_g(v))$$

$$= \sum_{g\in G} a_g \rho_{hg}(v).$$

* $\mathrm{Im}(\theta)$

let $w \in \mathrm{Im}(\theta)$ then $w = \theta(\sum_{g\in G} a_g e_g) = \sum_{g\in G} a_g \rho_g(v)$

and

$$\rho_h(w) = \sum_{g\in G} a_g \rho_{hg}(v)$$

$$= \theta(\sum_{g\in G} a_g e_{hg})$$

So $\mathrm{Im}(\theta)$ is a G-subspace & since V is irreducible

$\mathrm{Im}(\theta) = V$.

* $\ker\theta$.

Similarly, $\ker\theta = \left\{ \sum_{g\in G} a_g e_g \;\middle|\; \sum_{g\in G} a_g \rho_g(v) = 0 \right\}$ is a G-subspace:

$$\rho^{reg}(h)\left(\sum_{g\in G} a_g e_g\right) = \sum_{g\in G} a_g e_{hg} \in FG$$

and
$$\theta\left(\sum_{g\in G} e_g e_{hg}\right) = \sum_{g\in G} e_g \rho_{hg}(v) = \sum_{g\in G} e_g \rho_h(\rho_g(v))$$
$$= \rho_h\left(\sum_{g\in G} e_g \rho_g(v)\right)$$
$$= \rho_h(0) = 0.$$

So now by Mashke's theorem, $\exists$ G subspace $W \subseteq FG$ s.t.

$$FG = W \oplus \ker\theta$$

and

$$FG/\ker\theta \cong W$$

is

$$\mathrm{Im}\,\theta \cong V$$

(by First fund. theorem.)

this map is a G-isomorphism

$$\begin{tikzcd} FG/\ker\theta \arrow[r, "\rho^{reg}_h"] \arrow[d] & FG/\ker\theta \arrow[d] \\ \mathrm{Im}(\theta) \arrow[r, "\rho_h"'] & \mathrm{Im}(\theta) \end{tikzcd}$$

Thus, V is isomorphic to $W \subseteq FG$. $\boxtimes$

Def: let F be a field & let G act on a set X. let

$$FX := \text{span}\{e_x : x \in X\}$$ be a

vector space of degree $= |X|$ over F, and consider the group action:

$$g\left(\sum_{x\in X} a_x e_x\right) = \sum_{x \in X} a_x e_{gx}$$

The representation $G \longrightarrow GL(FX)$ is called **the corresponding permutation representation.**

e.g. let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{e, \tau, \sigma, \tau\sigma\}$.

Then G acts on the set $X = \{e, \tau, \sigma, \tau\sigma\}$ by left multiplication.

let $FX = \{b_e, b_\tau, b_\sigma, b_{\tau\sigma}\}$ be a 4-dim vector space over a field F. Then the permutation rep. is given by:

$$\tau\left(\alpha_1 b_e + \alpha_2 b_\tau + \alpha_3 b_\sigma + \alpha_4 b_{\tau\sigma}\right)$$

$$= \alpha_1 b_\tau + \alpha_2 b_e + \alpha_3 b_{\tau\sigma} + \alpha_4 b_\sigma \quad \text{etc.}$$

HW: write down the action of all other elements of G.

This is faithful:

$$\ker \rho_{reg} = \{ h \mid h(\sum_{g\in G} a_g e_g) = \sum_{g\in G} \underbrace{a_g e_g}_{\forall g} \}.$$

so $e_{hg} = e_g \ \forall g$ or

$$a_{h^{-1}g} = a_g \ \forall g.$$

every rep is subrep?

$\rho : G \to GL(V)$ irrep. let $0 \neq v \in V$.

let $\theta : FG \to V$

$$\theta\left(\sum_g a_g e_g\right) = \sum a_g \rho_g(v)$$

V is irrep. im θ is a G-subspace?

let $w \in \operatorname{Im}\theta$ then $w = \sum a_g \rho_g(v)$

for some $\sum a_g e_g$.

and $\rho_h(w) = \sum_g a_g \rho_h(\rho_g(w))$

$= \theta\left(\sum a_g e_{hg}\right)$.

So $\operatorname{im}\theta = V$.

Similarly, $\ker\phi$ is a G-subspace of FG.

let $\sum_g a_g e_g \in \ker\phi$. then $\phi\left(\sum_g a_g e_g\right) =$

$\sum_g a_g \rho_g(v) = 0$.

$\rho_h\left(\sum\ \right) = a_g \rho_{hg}(v) = 0$

$\phi\left(a_g e_{hg}(v)\right) = 0$

$\cap$
$\ker\phi$.

Pick, by Maschke's, a G-complement W such that

$$FG \cong \ker\phi \oplus W \text{ \& } W \text{ is a } G\text{-subspace}$$

so
$$FG/\ker\phi \cong \operatorname{im}\phi \cong V$$

$\cong$
W

subrep of P_{reg}.

Irrep.