§ Schur's lemma.

1. Let $\rho_1 : G \to GL(V)$ and $\rho_2 : G \to GL(W)$ be irreps of G. Then any G-homomorphism $\phi : V \to W$ is either the zero map ($\phi(v) = 0 \ \forall v$) or an isomorphism.

2. Suppose F is algebraically closed and $\rho : G \to GL(V)$ is an irrep. Then any G-homomorphism $\phi : V \to V$ is a scalar multiple of the identity map.

Proof: 1. Recall that $\ker \phi$ is a G-subspace of V. Since V is irreducible, $\ker\phi = 0$ or $\ker \phi = V$.

Similarly, $\operatorname{im} \theta$ is a G-subspace of W, & since W is irreducible, $\operatorname{im}\theta = W$ or $\operatorname{im}\theta = 0$.

Thus, $\theta = 0$ or θ is injective & surjective, so θ is an isomorphism.

2 Since F is algebraically closed, θ has atleast one eigenvalue $\lambda \in F$.

Thus, $\theta - \lambda \mathrm{Id}$ is a singular G-endomorphism on V, thus by 1,

$$\theta - \lambda \mathrm{Id} = 0$$

i.e. $\theta = \lambda \mathrm{Id}$.

The F-space of all G-homomorphisms $V \to W$ is denoted by

$$\mathrm{Hom}_G(V, W)$$

We write $\mathrm{End}_G(V)$ for $\mathrm{Hom}_G(V, V)$

Corollary: If V and W are G-irreps over $\mathbb{C}$, then

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong_G W \\ 0 & \text{otherwise} \end{cases}$$

Proof: If V & W are not isomorphic, then by Schur's, the only G-homomorphism $V \to W$ is 0.

So assume $V \simeq_G W$ and let $\theta_1, \theta_2 \in \mathrm{Hom}_G(V, W)$ both non-zero.

Then, θ_1 is invertible by Schur's and

$$\theta_1^{-1}\theta_2 \in \mathrm{Hom}_G(V, V) \text{ so}$$

$$\theta_1^{-1}\theta_2 = \lambda \, Id \text{ for some } \lambda \in \mathbb{C}. \text{ i.e.}$$

$$\theta_1 = \lambda \theta_2$$

Cor: If G has a faithful complex irrep then the center of G $Z(G)$ is cyclic.

Proof: let $\rho : G \to GL(V)$ be a faithful complex irrep and let $z \in Z(G)$ i.e. $zg = gz$ $\forall g \in G$.

Consider the map $\rho_z : v \longmapsto zv$ for $v \in V$.

This is a G-endomorphism on V:

$$\begin{tikzcd} V \arrow[r, "\rho_g"] \arrow[d, "\rho_z"'] & V \arrow[d, "\rho_z"] \\ V \arrow[r, "\rho_g"'] & V \end{tikzcd}$$

So by Schur, $\rho_z(v) = \lambda_z v$ where $\lambda_z \in \mathbb{C}$ is some scalar. Thus the map

$$\begin{aligned} \Phi : Z(G) &\longrightarrow \mathbb{C}^\times \simeq GL(\mathbb{C}) \\ z &\longmapsto \lambda_z \end{aligned}$$

is a 1-dimensional representation of $Z(G)$.

Claim: Φ is faithful.

Consider
$$\begin{aligned} \ker \Phi &= \{ z \in Z(G) \mid \lambda_z = 1 \} \\ &= \{ z \in Z(G) \mid \rho_z(v) = v \} \\ &= \{ z \in Z(G) \mid \rho(z) = \mathrm{Id} \}. \end{aligned}$$

$= \{e\}$ since ρ is faithful.

Thus $Z(G) \hookrightarrow \mathbb{C}^\times$, and $Z(G)$ is finite, so it is cyclic.

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Corollary: The complex irreps of a finite abelian group are all 1-dimensional.

Proof: Let V be a complex irrep. For $g \in G$, the map $f_g : v \mapsto gv$ is a G-endomorphism of V and since V is irreducible,

$$f_g = \lambda_g \, \mathrm{Id} \text{ for some } \lambda_g \in \mathbb{C}.$$

example: $G = C_4 = \{1, x, x^2, x^3\}$.

	1	x	x^2	x^3
V_1	1	1	1	1
V_2	1	-1	1	-1
V_3	1	i	-1	$-i$?
V_4	1	$-i$	-1	i

x has to act as something that 4^{th} powers up to 1, so the options are $\{\pm 1, \pm i\}$

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hw: show that over $\mathbb{R}$, C_3 has 2 irreps, of dim 1 & 2 respectively.

Lemma: A finite abelian group G has precisely $|G|$ complex irreps.

Proof:

Recall that each finite abelian group is a product of cyclic groups.

Write $G \cong \langle x_1 \rangle \times \dots \times \langle x_k \rangle$

where $o(x_j) = n_j$ and $\prod n_i = |G|$.

let ρ be an irrep. then ρ is 1-dim.

so $\rho : G \to GL(\mathbb{C}) \cong \mathbb{C}^{\times}$ so.

$\rho(1, \dots, x_j, \dots, 1) = \lambda_j \in \mathbb{C}$

where $o(x_j) = n_j$ so $\lambda_j^{n_j} = 1$ so

λ_j is an n_j^{th} root of unity.

If for each j, we fix a λ_j, a

j^{th} root of unity. then,

$$\rho(x_1^{m_1}, x_2^{m_2}, \dots, x_k^{m_k}) =$$

$$\rho(x_1^{m_1}, 1, \dots, 1) \cdot \rho(1, x_2^{m_2}, 1, \dots, 1) \dots \text{ etc}$$

$$= \lambda_1^{m_1} \cdots \lambda_k^{m_k}.$$

So ρ is determined by λ_j's

Since there are precisely n_j n^{th} roots of unity, we have that the # of irreps $=$ $n_1 \cdots n_r = |G|$

example: $G = V_4 = C_2 \times C_2$:

the 4 irreps are:

	1	x_1	x_2	$x_1 x_2$
ρ_1	1	1	1	1
ρ_2	1	1	-1	-1
ρ_3	1	-1	$+1$	-1
ρ_4	1	-1	-1	1