Representation Theory for Finite Groups.

We begin by reviewing some group theory:

Def (Group)

A group $(G, \circ)$ is a set G together with a binary operation

$$\circ : G \times G \longrightarrow G$$

that satisfies the following axioms:

- Associativity:

  $\forall g, h, k \in G,$

  $$(g \circ h) \circ k = g \circ (h \circ k)$$

- Identity element:

  $\exists e \in G$ such that:

  $$eg = ge = g \quad \forall g \in G$$

and

- Inverses:

  $\forall g \in G, \exists h \in G$ such that

  $$g \circ h = h \circ g = e.$$

  where e is an identity element.

Examples:

(1) $(\mathbb{Z}, +)$ is a cyclic group of infinite order.
$(\mathbb{Z}/n\mathbb{Z}, +)$ is a cyclic group of order n.

(2) The symmetric group S_n is the set of all permutations of the set $X_n = \{1, 2, 3, \dots, n\}$. $|S_n| = n!$

Define the alternating group A_n for yourself.

(3) Quaternion group

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle$$

- $|Q_8| = 8$.

- Check that in $GL_2(\mathbb{C})$, $x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, & $y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generate a copy of Q_8.

(4) Dihedral group: $D_{2m} = \langle x, y \mid x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle$

- $|D_{2m}| = 2m$.

Def (conjugacy class)

Fix $g \in G$. The conjugacy class of $g \in G$ is

$$Cl_G(g) = \{ xgx^{-1} : x \in G \}.$$

Fact:

$$|Cl_G(g)| = [G : C_G(g)]$$

where $C_G(g) := \{ x \in G \mid xg = gx \}$ is the centralizer of $g \in G$.

[Start here:].

§ Group actions:

Let G be a group and X be a set. Then we say G **acts on X** if there exists a map.

$$* : G \times X \longrightarrow X$$
$$(g, x) \longmapsto gx$$

such that:

① $ex = x \ \forall x \in X$ and $e \in G$ the identity element.

② $(gh)x = g(hx)$ ——— $*$

where $g, h \in G$ and $x \in X$.

example:

The standard example is $G = S_n$ and $X = \{1, 2, \dots, n\}$. Then $g \in S_n$ acts on X in the obvious way. e.g. let $g \in S_3$ be the transposition $(1,2)$, then

$$((1,2), 1) \longmapsto 2$$
$$((1,2), 2) \longmapsto 1$$

and $((1,2),3) \longmapsto 3$ etc.

Inspired by this example, we will now define the permutation representation of G.

Let G be a group that acts on a set X.

let $\theta : G \longrightarrow \mathrm{Sym}(X)$ be the function:

$$g \longmapsto \left(\theta_g : \begin{matrix} X \to X \\ x \longmapsto gx \end{matrix}\right)$$

then θ is a group homomorphism:

$$\theta(g_1 g_2) = \theta_{g_1 g_2} : x \longmapsto (g_1 g_2)x$$

by $\textbf{—}\!\ast$ above:

$$\theta(g_1 g_2) = \theta_{g_1 g_2}\left(: x \longmapsto g_1(g_2 x).\right)$$

$$= \theta_{g_1} \circ \theta_{g_2}\left(: x \longmapsto g_1(g_2 x)\right)$$

$$= \theta_{g_1} \circ \theta_{g_2}$$

Also θ_g is a permutation w/ inverse $\theta_{g^{-1}}$

θ is called the permutation representation of G on X.

§ Linear Algebra:

Let F be a field. Usually $F = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, but sometimes $F = \mathbb{F}_p$ or $\overline{\mathbb{F}_p}$ (modular rep theory).

Let V be a finite dimensional vector space over F.

If $\dim_F V = n < \infty$, we can choose a basis $\{e_1, \dots, e_n\}$ over F and identify V with F^n:

$$\begin{aligned} V &\longrightarrow F^n \\ \sum_{i=1}^{n} a_i e_i &\longmapsto (a_1, a_2, \dots, a_n) \end{aligned}$$

Then, the group:

$$GL(V) := \left\{ \theta : V \longrightarrow V \;\middle|\; \theta \text{ is linear and invertible} \right\}.$$

(under composition) can be identified w/ the group of matrices:

$$GL_n(F) := \left\{ \begin{array}{l} n \times n \text{ invertible matrices w/} \\ \text{entries in } F \end{array} \right\}$$

in the following way:

Let $\theta \in GL(V)$, then θ acts on the basis elements

$\{e_1, \dots, e_n\}$ in the following way:

$$\theta(e_j) = \sum_{i=1}^{n} a_{ij} e_i$$

Consider the map $\varphi : GL(V) \longrightarrow GL_n(F)$

$$\theta \longmapsto A_\theta = \{a_{ij}\}_{i,j}$$

(matrix).

Homework: check that this is a group isomorphism.

Proof: Consider $A_{\theta_1\theta_2}(e_j) = \sum_{i=1}^{n} c_{ij} e_i$. ✓

$$A_{\theta_1}\left[A_{\theta_2}(e_j)\right] = A_{\theta_1}\left[\sum_{i=1}^{n} b_{ij} e_i\right]$$

$$= \sum_{i=1}^{n} b_{ij} \left(A_{\theta_1} e_i\right)$$

$$= \sum_{i=1}^{n} b_{ij} \left[\sum_{k=1}^{n} a_{ki} e_k\right].$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ki} b_{ij} e_k. \checkmark$$

etc.

The a_{ij} obviously depend on the choice of basis, so the 'isomorphism' depends on the basis as well.

Fact: Matrices A_1, A_2 represent the same element of $GL(V)$ with respect to different bases if and only if they are conjugate ("similar") i.e.

if and only if $\exists X \in GL_n(F)$ such that

$$A_2 = X A_1 X^{-1}$$

Recall the trace of A,

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

where $A = (a_{ij}) \in F_{n \times n}$.

Since $tr(XAX^{-1}) = tr(A)$ we can define $tr(\theta) = tr(A)$ for $\theta \in GL(V)$ independent of basis.

Fact: (Facts are theorems that we won't prove in this class). OVER $\mathbb{C}$

Let V be a finite-dimensional vector space over $\mathbb{C}$.
Let $\theta \in GL(V)$ be idempotent. i.e. $\theta^m = id$ for some m.
Then

θ is diagonalizable.

(i.e. there is some basis in which A_θ is a diagonal matrix).

Def: We call $\mathrm{End}(V) := \{\text{linear maps} : V \to V\}$ the endomorphism algebra of V. It is an F-algebra under the natural addition of linear maps & composition as multiplication.

Fact: Let V be a finite dim vector space over $\mathbb{C}$, and let $\theta \in \mathrm{End}(V)$. Then,

θ is diagonalizable $\iff$ $\exists$ a polynomial f w/ distinct linear factors such that $f(\theta) = 0$.

Proof: If θ is diagonalizable then $\exists$ a basis in which the matrix representing θ is $D_\theta = (\lambda_1, \dots, \lambda_n)$.
Let $(\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_k})$ be the distinct eigenvalues of D_θ and consider the polynomial:

$$f(x) = \prod_{j=1}^{k} (x - \lambda_{i_j})$$

then $f(D_\theta) = 0$ and, $f(\theta) = 0$ since for any polynomial

$p(x)$ and invertible matrix X,

$$p(XDX^{-1}) = X\,p(D)\,X^{-1}$$

(Check this!).

($\Leftarrow$) Now suppose $\exists\, f(x)$ w/ distinct linear factors such that $f(\theta) = 0$.

Claim: $f(\theta)$ is the minimal polynomial of θ. (Check)

let $f(x) = (x-\lambda_1)\cdots(x-\lambda_n)$, and consider the polynomials:

$$f_i(x) = \frac{f(x)}{(x-\lambda_i)} = \prod_{\substack{j=1 \\ j\neq i}}^{n} (x-\lambda_j).$$

Since the λ_i's are distinct, the polynomials $f_i(x)$ are relatively prime so some linear combination of them equals 1:

$$\sum_{i=1}^{n} g_i(x)\,f_i(x) = 1.$$

let $h_i(x) = g_i(x)\,f_i(x)$, and note that:

$$h_1(x) + h_2(x) + \cdots + h_n(x) = 1.$$

This means that for all $v \in V$:

$$v = h_1(\theta)v + h_2(\theta)v + \dots + h_n(\theta)v.$$

and

$$(\theta - \lambda_i)\, h_i(\theta)v = (\theta-\lambda_i)g_i(\theta)f_i(\theta)v$$
$$= g_i(\theta)f(\theta)v = 0.$$

Thus, $h_i(\theta)v$ is an eigenvector w/ eigenvalue λ_i for all $v \in V$ and each $v \in V$ can be written as a sum of these eigenvectors so

$$V = E_{\lambda_1} + E_{\lambda_2} + \dots + E_{\lambda_n}$$

where $E_\lambda = \{v \in V \mid \theta v = \lambda v\}$. Thus θ is diagonalizable.

A useful reference for this section is Keith Conrad's expository note on minimal polynomials.

Finally, fact: let $\{\theta_1, \theta_2, \dots, \theta_n\}$ be a finite set of diagonalizable automorphisms of $V/\mathbb{C}$, then if θ_i commute, they are simultaneously diagonalizable.

§ Linear Representations

Let G be a finite group and V be a finite dimensional vector space over field F.

A **linear representation** of G on V is a homomorphism $\phi : G \to GL(V)$.
This map sends $g \longmapsto \phi(g) := \phi_g$.
So for each g,

$$\phi_g \in GL(V) \text{ i.e. } \phi_g : V \to V.$$

and $\phi_{g_1 g_2} = \phi_{g_1} \phi_{g_2}$

The **dimension or degree** of ϕ is just defined to be the $\dim_F V$

Since $\phi : G \to GL(V)$ is a homomorphism, all the usual fun stuff applies. e.g. $\ker \phi$ is a subgroup of G and $G/\ker\phi \simeq \operatorname{im}(\phi)$.

We say that ϕ is **faithful** if $\ker \phi$ is trivial.