

# Revision of First Year Topics

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## Elementary Algebra

1. Solve the following equations for  $x$ .

[Do NOT use the quadratic formula for quadratics, use either factorisation or completing the square.]

a)  $4x - 3 = 7(x - 7) + 10$

b)  $t = \frac{2t - x}{2t + x}$

c)  $2x^2 + 12 = 25x$

d)  $x^2 = 2x + 5$

e)  $x^2 = ax + 5$

f)  $x(5x + 13) = 18$

g)  $x(5x + 13) = 19$

h)  $x = \sqrt{1 - x}$

i)  $x(x^4 + 12) = 7x^3$

j)  $x = \frac{2t - x}{2t + x}$

k)  $x^2 + \frac{15}{x^2} = 8$

l)  $\sqrt{2x - 1} + \sqrt{3x + 1} = 7$

2. The sum  $S_n = 1 + \cdots + n$  of the first  $n$  positive integers is given by

$$S_n = \frac{n}{2}(n + 1).$$

- a) Find the sum  $1 + \cdots + 100$ .

- b) If  $1 + \cdots + n = 405\,450$ , what is  $n$ ?

- c) Show that if  $s = 1 + \cdots + n$ , then

$$n = \frac{1}{2}(-1 + \sqrt{1 + 8s}).$$

3. a) Using surds, find integers  $a$  and  $b$  such that

$$11 \frac{a + b\sqrt{3}}{(2\sqrt{3} + 1)^2} = \frac{13 - 4\sqrt{3}}{2\sqrt{3} - 1}.$$

- b) Use *completing the square* to solve the equation

$$\frac{1}{x} - \frac{1}{\sqrt{2}} = \frac{1}{x + \sqrt{2}},$$

giving your solution(s) in the form  $p\sqrt{2} + q\sqrt{10}$  for rational  $p$  and  $q$ .

4. The quadratic  $p = ax^2 + bx + c$  has roots  $\alpha$  and  $\beta$ .

a) Prove that  $-b/a = \alpha + \beta$  and that  $c/a = \alpha\beta$ .

b) Recall that the *discriminant* of a quadratic is  $\Delta(p) = b^2 - 4ac$ . Explain why, over  $\mathbb{R}$ ,  $p$  has distinct roots if  $\Delta(p) > 0$ , a repeated root if  $\Delta(p) = 0$ , and no roots if  $\Delta(p) < 0$ .

c) Define the “dis- $k$ -riminant”

$$\Delta_k(p) = kb^2 - (1 + k)^2 ac.$$

Show that  $\Delta_k(p) = 0$  precisely when one root of  $p$  is  $k$  times the other. (Notice that  $\Delta_1$  is our usual discriminant  $\Delta(p)$ .)

5. Consider the quadratic  $p \in \mathbb{R}[x]$  defined by

$$p(x) = 5x^2 - 2ax + a^2 - 1.$$

- a) Determine the range of values of  $a$  for which  $p$  has two distinct roots in  $\mathbb{R}$ .
- b) For what values of  $a$  are the roots repeated?
- c) If one root of  $p$  is double the other, what are the possible values of  $a$ ?

- d) If the roots of  $p$  are  $\alpha$  and  $\beta$ , show that the quadratic with roots  $\alpha^2/\beta$  and  $\beta^2/\alpha$  is

$$q(x) = 25(a^2 - 1)x^2 + 2a(11a^2 - 15)x + 5(a^2 - 1)^2.$$

If for some value of  $a$ , the roots of  $q$  are  $-5/6$  and  $9/50$ , what are the corresponding roots of  $p$ ?

6. Consider the quadratic  $p = 2x^2 - 5x + a$ , where  $a$  is an integer. Suppose  $p$  can be factorised over the integers (i.e.,  $p$  has rational roots).
- Prove that  $(n + 5)(n - 5)$  is divisible by 8 if and only if  $n$  is odd.
  - Show that  $a$  must be of the form  $\frac{1}{8}(n + 5)(n - 5)$  where  $n$  is odd.
  - Obtain a factorisation of  $p$  in terms of  $n$ .

7. a) Solve the equation

$$2^{3x-4} \cdot 5^{x+3} \cdot 7^x = 6^{1-2x}$$

in two different ways, in order to get the solution

$$x = \frac{4 \log 2 - 3 \log 5 + \log 6}{3 \log 2 + \log 5 + 2 \log 6 + \log 7}$$

when working one way (here  $\log$  denotes the natural logarithm  $\log_e$ ), and

$$x = \log_{10\,080} \left( \frac{125}{96} \right)$$

when working the other way.

- Prove that the two solutions of part (a) are equal to each other.
- Solve the equation

$$7^{x+1} - \frac{1}{7^x} = 1$$

writing your answer in the form  $\log_7(a + b\sqrt{29})$  for rational  $a, b$ .

- Solve the set of simultaneous equations

$$\begin{aligned} \log x + 2 \log y &= \log(x + 2y) \\ 2^x &= 3^y \end{aligned}$$

giving your answers in exact form.

8. a) Prove (without using the change of base formula) that for any  $c \neq 0$ ,

$$\log_a b = \log_{ac}(b^c).$$

Deduce that  $\log_9 25 = \log_3 5$  and  $\log_3 5 = 1/\log_5 3$  by choosing appropriate values of  $c$ .

- b) In the early stages of the COVID-19 pandemic, the number of known cases in the UK doubled every three days (approximately). If there were 2 000 known cases on the 12th of March 2020, how many days would need to pass in order for the cases to exceed 10 000?
- c) Prove that the arithmetic mean of the logarithms of  $a$  and  $b$  is the logarithm of their geometric mean.
- d) Prove that

$$\log 1 + \log 2 + \cdots + \log n = \log(n!).$$

Given that  $\log 1 + \log 2 + \cdots + \log n \approx n \log n - n + \frac{1}{2} \log 2\pi n$ ,<sup>1</sup> deduce the famous Stirling approximation for factorials,

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

- e) Use Stirling's formula and base-10 logarithms to obtain that

$$10\,000! \approx 2.8462 \times 10^{35\,659}.$$

9. Let  $p \in \mathbb{R}[x]$  be defined by

$$p(x) = 3x^4 - 26x^3 + 39x^2 + 4x - 4.$$

- a) Give the precise statement of the rational roots theorem.
- b) Without using the rational roots theorem, prove that any rational root  $c/d$  of  $p$  must satisfy  $c \mid 4$  and  $d \mid 3$ . Hence determine all possible rational roots of  $p$ .
- c) Find the remaining roots of  $p$  *without* performing long division.
- d) What is the remainder when dividing  $p$  by  $(x - 3)$ ?

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<sup>1</sup>To get an idea as to where this approximation comes from, notice that  $\sum_{k=1}^n \log k \approx \int_1^n \log t \, dt = n \log n - n + 1$ . Obtaining the term in  $\log n$  requires more work, but notice that this term is less significant than the main terms. See [here for a plot](#).

10. Solve the following cubic equations.

a)  $2x^3 - 15x^2 + 22x + 15 = 0$       b)  $3x^3 - 5x^2 + x + 1 = 0$   
c)  $2x^3 + 7 = x(11x + 9)$       d)  $2x^3 - 11x^2 + 19x - 7 = 0$

11. Use the Binomial theorem to expand the following.

a)  $(1+x)^5$       b)  $(\sqrt{3} - \sqrt{2})^4$       c)  $(2x - \frac{y}{x^2})^6$

12. Show that  $(x+1)^6 - (x-1)^6 = 4x(3x^4 + 10x^2 + 3)$ . Deduce that  $(x+1)^6 = (x-1)^6$  has only  $x=0$  as a solution.

13. (a) Determine  $\lambda$ ,  $m$  and  $n$  if

$$(\lambda - x)^m(1+x)^n = 1 + 7x + 18x^2 + \dots$$

(b) What is  $n$  if  $\binom{n}{2} = 5050$ ?

(c) Notice that the formula in question 2 can be written as

$$S_n = \binom{n+1}{2}.$$

Provide a combinatorial argument as to why.

14. Decompose the following rational functions into partial fractions.

a)  $\frac{x^2 + 1}{(x-1)(x-2)(x-3)}$       b)  $\frac{1-3x}{(2x+1)(x+3)}$   
c)  $\frac{8x+5}{(2x^2+1)(x+3)}$       d)  $\frac{7x^2+40x}{(2x^2+1)(x+3)^2}$   
e)  $\frac{x^3+6}{(x-4)(x+3)}$       f)  $\frac{2x^3-7x}{(x+1)^2(x-4)}$   
g)  $\frac{5x^3+3}{(x^2-3x+1)(x^2+2)}$       h)  $\frac{5x^3+x-2}{(x^2-3x+1)(x+2)^2}$   
i)  $\frac{19x^3+21x+14}{(7x^2+13x-2)(4-x^2)}$       j)  $\frac{x^4-1}{(x^2-x-12)(x^2-x+12)}$

15. Let  $a$ ,  $b$  and  $c$  satisfy  $a + b + c = 0$ . Show that  $a^3 + b^3 + c^3 = 3abc$ .  
Hence or otherwise, decompose

$$\frac{15x(x^2 + 11)}{(x^2 + x - 2)^3 - (x^2 + 1)^3 - (x - 3)^3}$$

into partial fractions.

16. A fort manned by 45 men has food and water for 100 days.
- d) How long would the same supplies last if the fort were manned by 50 men?
  - e) After 25 days, 15 out of the 45 men leave the fort. How many days can the remaining supplies provide for the remaining men in the fort?
17. A cab from Paceville to Tarxien costs € $P$ . A group of friends are thinking of sharing the cab, but 3 of them are not sure if they want to leave yet. It is €1 more expensive per person if the 3 friends do not join the ride.

Show that  $9 + 12P$  is an odd square number.

18. If  $x$  and  $y$  are both positive, prove that

$$\left(\frac{1}{x} + \frac{1}{y}\right)(x + y) \geq 4.$$

19. Let  $a/b$  be a rational number different from 1. Show that  $a/b$  plus its reciprocal can never be an integer.
20. A real number  $\alpha$  is said to be *transcendental* if there does not exist a polynomial  $f$  with rational coefficients such that  $f(\alpha) = 0$ . We know that  $\pi$  and  $e$  are both transcendental.
- a) Prove that  $\sqrt{2}$  and  $\sqrt{3 + \sqrt{5}}$  are not transcendental.
  - b) Prove that a transcendental number is irrational.
  - c) It is not known whether the numbers  $\pi + e$  and  $\pi \times e$  are rational or irrational. Prove that at least one of them is.

## Answers & Hints

1. a) 12  
 b)  $-\frac{2t(t-1)}{t+1}$   
 c)  $1/2, 12$   
 d)  $1 \pm \sqrt{6}$   
 e)  $\frac{1}{2}(a \pm \sqrt{a^2 + 20})$   
 f)  $-18/5, 1$   
 g)  $-1/10(13 \pm 3\sqrt{61})$   
 h)  $\frac{1}{2}(\sqrt{5} - 1)$   
 i)  $0, \pm 2, \pm \sqrt{3}$   
 j)  $-\frac{1}{2}(1 + 2t \pm \sqrt{4t^2 + 12t + 1})$   
 k)  $\pm\sqrt{3}, \pm\sqrt{5}$   
 l) 5
2. a)  $\frac{100}{2}(101) = 5050$   
 b) 900  
 c) Hint: Solve  $s = \frac{n}{2}(n+1)$ .
3. a)  $a = 1, b = 2$   
 Hint: Make  $a + b\sqrt{3}$  subject of the equation, and simplify using the usual techniques of surds.  
 b)  $p = \frac{1}{2}, q = \pm\frac{1}{2}$ .
4. These are standard quadratic theory, refer to the notes. You may assume the result that if  $p$  has roots  $\alpha$  and  $\beta$ , then we can write it as  $a(x - \alpha)(x - \beta)$ .  
 For part (c), refer to the last example in the notes on Viète's formulæ, the ideas there should be helpful.
5. a)  $-\sqrt{5}/2 < a < \sqrt{5}/2$   
 b)  $a = \pm\sqrt{5}/2$   
 c)  $a = \pm\frac{3}{37}\sqrt{185}$   
 Hint: If the smaller root is  $\alpha$ , then  $\frac{2a}{5} = \alpha + 2\alpha$  and  $\frac{a^2-1}{5} = \alpha(2\alpha)$ .  
 d) Corresponding roots are  $-3/10$  and  $1/2$ .  
 Hint: If the desired roots are  $\alpha$  and  $\beta$ , then we are given  $\alpha^2/\beta$  and  $\beta^2/\alpha$ . Without loss of generality, we can assume  $\alpha^2/\beta = -5/6$  and  $\beta^2/\alpha = 9/50$ . Solving these simultaneously gives the required roots.
6. a) If  $n$  is odd, we can write it as  $2k+1$ . Plug this in to  $(n+5)(n-5)$  and expand, the result should be clearly divisible by 8 (show this by factorising 8 out of it).  
 For the converse, show that if  $n$  is even (i.e., put  $n = 2k$ ) we end up with something not divisible by 8 (it should look like  $8t+r$  for some non-zero remainder  $1 \leq r \leq 7$ ).  
 b) By a theorem in the notes, this happens if and only if  $\Delta = n^2$  for some  $n$ . If we work out  $\Delta$ , we get  $25 - 8a$ . If this equals  $n^2$ , we get  $a = (n+5)(n-5)/8$ .  
 c) Staring at the quadratic  $2x^2 - 5x + \frac{1}{8}(n+5)(n-5) = \frac{1}{8}(16x^2 - 40x + (n-5)(n+5))$  for long enough, we deduce that it can be factorised as  $\frac{1}{8}(4x-5+n)(4x-5-n)$ .
7. a) Hint: For the first way, take log of both sides and apply the laws of logarithms immediately, solving for  $x$ . For the second way, use the laws of indices to reduce the equation to the form  $a^x = b$  and take  $\log_a$  at the end.  
 b) Hint: Use the laws of logarithms.  
 c)  $x = \log_7\left(\frac{1}{14}(1 + \sqrt{29})\right)$ .  
 Hint: Rearrange the given equation to get a quadratic in  $7^x$ . Be careful of the negative branch,  $1 - \sqrt{29} < 0$  so  $\log_7\left(\frac{1}{14}(1 - \sqrt{29})\right)$  does not exist.  
 d)  $x = \sqrt{(\log_2 3)(2 + \log_2 3)}$   
 $y = \sqrt{1 + 2\log_3 2}$
8. a) Hint: If  $u = \log_a b$ , then  $a^u = b$ . Apply  $(\cdot)^c$  to both sides and switch back to logarithmic form.  
 Take  $c = 1/2$  and  $c = \log_3 5$  in the examples.  
 b) Hint: Solve the equation  $2000 \times 2^{t/3} = 10000$  to get  $t = \lceil 3\log_2 5 \rceil = 7$  days.

c) Hint: The arithmetic mean of two numbers is  $\frac{a+b}{2}$ , the geometric mean is  $(ab)^{1/2}$ .

d) For the first part, by the first law of logarithms we see that  $\log 1 + \dots + \log n = \log(1 \cdots n) = \log n!$ .

For the second part, equating  $\log n!$  and the given expression and doing  $e^{(\cdot)}$  both sides, we obtain Stirling's formula.

e) To approximate  $10\,000!$ , we notice that

$$\begin{aligned} & \sqrt{2\pi \cdot 10\,000} \left(\frac{10\,000}{e}\right)^{10\,000} \\ &= 100\sqrt{2\pi} 10^{40\,000}/e^{10\,000} \\ &= 10^{40\,002}\sqrt{2\pi}/e^{10\,000}. \end{aligned}$$

Taking  $\log_{10}$  and applying the laws of logarithms, we evaluate  $\log_{10}(10\,000!) \approx 35\,659.5$ . This means that  $10\,000! = a \times 10^{35\,659}$  where  $0 < a < 10$ .

Now to determine  $a$ , we just divide our approximation of  $10\,000!$  by  $10^{35\,659}$  to get

$$\begin{aligned} & 10^{4\,343}\sqrt{2\pi}/e^{10\,000} \\ &= \sqrt{2\pi} e^{4\,343 \ln 10 - 10\,000} \\ &\approx \sqrt{2\pi} e^{0.12706} \\ &\approx 2.846239. \end{aligned}$$

9. a) Let  $p = a_n x^n + \dots + a_0$  be a polynomial with integer coefficients ( $p \in \mathbb{Z}[x]$ ). If  $x = c/d$  is a root of  $p$  (where  $\text{hcf}(c, d) = 1$ ), then  $c \mid a_0$  and  $d \mid a_n$ .

Refer to the notes or look at [https://en.wikipedia.org/wiki/Rational\\_root\\_theorem](https://en.wikipedia.org/wiki/Rational_root_theorem) if this fact doesn't sound familiar.

b) Hint: for the first part, just reproduce the proof of the rational roots theorem in the notes with  $a_0 = -4$  and  $a_n = 3$ .

$$2, -1/3$$

Hint: Try putting  $x = s/t$  with  $s = \pm 1, \pm 2, \pm 4$  and  $t = \pm 1, \pm 3$ .

$$\text{c) } \frac{1}{2}(7 \pm \sqrt{41})$$

Hint: Let the remaining two roots be  $\alpha, \beta$ . By Viète's formulæ,  $2 - \frac{1}{3} + \alpha + \beta = \frac{26}{3}$  and  $2(\frac{1}{3})\alpha\beta = \frac{-4}{3}$ . Solving simultaneously yields  $\alpha$  and  $\beta$ .

$$\text{d) } -100$$

Hint: Evaluate  $f(3)$  and apply the remainder theorem.

$$10. \text{ a) } -1/2, 3, 5$$

$$\text{b) } -1/3, 1 \text{ (repeated)}$$

$$\text{c) } 1/2, 1/2(5 \pm \sqrt{53})$$

$$\text{d) } 1/2$$

$$11. \text{ a) } x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

$$\text{b) } 49 - 20\sqrt{6}$$

$$\text{c) } 64x^6 - 192x^3y + 240y^2 - 160\frac{y^3}{x^3} + 60\frac{y^4}{x^6} - 12\frac{y^5}{x^9} + \frac{y^6}{x^{12}}.$$

12. Hint: For the identity, expand using the binomial theorem.

The right-hand side of the identity can be factorised further as  $4x(x^2 + 3)(3x^2 + 1)$ , which is clearly 0 only if  $x = 0$ .

$$13. \text{ a) } \lambda = 1, m = 3, n = 10$$

Hint: Use the binomial theorem and explicit expressions for  $\binom{n}{k}$ ,  $\binom{m}{k}$  to determine the coefficients of  $1, x, x^2$  in terms of  $\lambda, m$  and  $n$ , and compare coefficients.

$$\text{b) } 101$$

$$\text{Hint: } \binom{n}{2} = \frac{n(n-1)}{2}.$$

c) The number of ways of choosing subsets of size 2 from  $\{1, \dots, n+1\}$  can be computed as follows. There are  $n$  of the form  $\{1, k\}$ . Then there are  $(n-1)$  of the form  $\{2, k\}$  (we are excluding  $\{2, 1\}$  since it was already counted as  $\{1, 2\}$  before). Similarly there are  $(n-2)$  of the form  $\{3, k\}$  which weren't already counted. Continuing this way till  $\{n, k\}$ , which there is only one of (namely  $\{n, n+1\}$ ), we get that  $\binom{n+1}{2} = n + \dots + 1 = S_n$ .

$$14. \text{ a) } \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}$$



- b)  $\frac{1}{2x+1} - \frac{2}{x+3}$   
 c)  $\frac{2(x+1)}{2x^2+1} - \frac{1}{x+3}$   
 d)  $\frac{4x+1}{2x^2+1} - \frac{2}{x+3} - \frac{3}{(x+3)^2}$   
 e)  $x + 1 + \frac{3}{x+3} + \frac{10}{x-4}$   
 f)  $2 - \frac{1}{(x+1)^2} + \frac{4}{x-4}$   
 g)  $\frac{4x}{x^2-3x+1} + \frac{x+3}{x^2+2}$   
 h)  $\frac{2x-1}{x^2-3x+1} + \frac{3}{x+2} - \frac{4}{(x+2)^2}$   
 i)  $\frac{2}{7x-1} + \frac{3}{(x+2)^2} - \frac{2}{x+2} - \frac{1}{x-2}$   
 j)  $1 + \frac{85}{56(x-4)} - \frac{10}{21(x+3)} + \frac{23x-131}{24(x^2-x+12)}$

15. Hint: For the identity, since  $a + b + c = 0$ , then  $c = -a - b$ , so we have  $a^3 + b^3 + c^3 = a^3 + b^3 + (-a-b)^3$ . Expand this to get  $3ab(-a-b) = 3abc$ .  
 Partial fractions:

$$\frac{4}{x+2} - \frac{5}{x-1} - \frac{3x+4}{x^2+1} + \frac{9}{x-3}$$

Hint: Notice that  $(x^2+x-2)-(x^2+1)-(x-3)=0$ , and therefore applying the identity with these as  $a, b$  and  $c$ , we get

$$\begin{aligned} & (x^2+x-2)^3 - (x^2+1)^3 - (x-3)^3 \\ &= (x^2+x-2)^3 + (-(x^2+1))^3 \\ & \quad + (-(x-3))^3 \\ &= 3(x^2+x-2)(x^2+1)(x-3) \\ &= 3(x+2)(x-1)(x^2+1)(x-3). \end{aligned}$$

16. a) We have enough food for a total of 4500 man-days. (analogous to "man-hours"). This would last 50 men  $4500/50 = 90$  days.  
 b) After 25 days, there is 75 days' worth of food left for 45 men, i.e., it will last  $45 \times 75 = 3375$  man days'. But now we divide this among 30 men, so the result is  $3375/30 = 112\frac{1}{2}$  days.  
 17. We have that  $\frac{P}{n-3} - \frac{P}{n} = 1$ , which means that

$$n = \frac{1}{2}(3 + \sqrt{9 + 12P}).$$

Clearly  $n$  is a whole number, so  $9 + 12P$  must be square, and moreover, odd, so that  $3 + \sqrt{9 + 12P}$  is divisible by 2.

18. Hint: The left hand side can be rewritten as  $\frac{1}{xy}((x-y)^2 + 4xy)$ .  
 19. If  $\frac{a}{b} + \frac{b}{a} = \frac{a^2+b^2}{ab} = k$ , then  $a^2 - kab + b^2 = 0$ . This is a quadratic in  $a$ , with discriminant  $\Delta = b^2(k^2 - 4)$ . Clearly we want  $\Delta$  to be a square since  $a$  must be an integer (hence a rational root).  $\Delta$  is a square if and only if  $k^2 - 4$  is a square.

Say  $k^2 - 4 = \ell^2$ . This rearranges to  $(k - \ell)(k + \ell) = 4$ . This can only happen if  $k - \ell = 1$  and  $k + \ell = 4$ , or  $k - \ell = 4$  and  $k + \ell = 1$ , or else  $k - \ell = k + \ell = 2$ . The first two cannot happen since  $k$  and  $\ell$  are to be integers. The last one implies that  $k = 2$ , in other words, that the quadratic only has rational solutions for  $a$  if it is  $a^2 - 2ab + b^2 = 0$ . But this means  $(a - b)^2 = 0$ , i.e.,  $a = b$ , and the question states that  $a/b \neq 1$ .

20. a) They are not transcendental since  $\sqrt{2}$  is a root of the quadratic  $x^2 - 2$  and  $\sqrt{3 + \sqrt{5}}$  is a root of the quartic  $(x^2 - 3)^2 - 5 = x^4 - 6x^2 - 4$ , and both of these have rational coefficients.  
 b) Any rational number  $a/b$  is the root of the polynomial  $ax - b$  which has rational coefficients, so it cannot be transcendental.  
 Thus any transcendental number must be irrational.  
 c)  $e$  and  $\pi$  are both transcendental, which means that  $(x - e)(x - \pi) = x^2 - (e + \pi)x + e\pi$  cannot have coefficients which are all rational. Thus at least one of  $e + \pi$  and  $e\pi$  is irrational.