Basic Mathematics
BOOK OUTLINE

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February 2021

1. Introduction

   1.1 Some Revision

   Properties of integers (prime factorisation, hcf, gcd), rational numbers and real numbers (introduced as infinite decimals), solving linear equations, solving simultaneous equations, solving quadratic equations. The trigonometric functions (basic O-level stuff only).

   1.2 Sets

   Expository section to establish notation of the sets, and the ideas of subsets, union, intersection, set difference, Cartesian product, set comprehension, cardinality (informally), the plane being equivalent to $\mathbb{R}^2$, space as $\mathbb{R}^3$, other examles.

   1.3 Surds and Logarithms

   The definition of square roots and indices, how to generalise from the definition of an integer index, to rational indices, to real indices (informally). The definition of a logarithm as the opposite of an index, the laws of logarithms, solving basic logarithmic and indicial equations.

2. Functions, Polynomials and Inequalities

   2.1 Functions

   The abstract idea of a function with examples. Domain and range, the idea of injection, surjection and bijection, how to sketch the graph of a real-valued function by tabulating its values.

   2.2 Polynomials

   Quadratic polynomials and equations, quadratic discriminants, the relationship between roots and coefficients of a quadratic polynomial (Viète's formulæ). The remainder and factor theorems, polynomial division. The rational roots theorem and solving polynomial equations. Partial Fractions.

   2.3 Elementary Functions and Affine Transformations

   The properties and graphs of the functions x, x^2, $\sqrt{x}$, $|x|$, $\frac{1}{x}$, e^x, $\log x$. Affine transformations of graphs of functions: $y = f(x) + k$, $y = f(x + k)$, $y = kf(x)$ and $y = f(kx)$.

2.4 More on functions

Composition of functions, Inverses, addition and multiplication of functions. Domain restriction. Odd and even functions, periodic functions.

2.5 Inequalities

Solving simple inequalities, quadratic inequalities, polynomial inequalities, rational inequalities. Triangle inequality, AM-GM inequality, and Cauchy–Schwarz.

3. Geometry and Matrices

3.1 Geometry in the Plane

Adding an extra ingredient to $\mathbb{R}^2$, namely, a metric. Distance between two points. Straight lines, gradients, midpoint, circles. Angle between two lines, distance between a point and a line.

3.2 Vectors and transformations

Vector operations, dot products, lines and planes. General transformations, linear transformations, matrices for 2D transformations, composition of transformations as matrix multiplication.

3.3 Matrices

More general algebra of matrices (operations), determinant and inverse of 2×2 matrices. Elementary row operations, determinants and inverses of larger matrices. Cross product and applications to planes. Rank of transformations, geometric interpretation.

3.4 Vector Spaces

A few examples of linear transformations on functions.

4. Sequences and Sums

4.1 Sequences

Arithmetic and geometric sequences. Quadratic sequences and difference operators. Convergence of sequences with basic examples.

4.2 Sums

Sigma notation, linearity of Σ. Formulas for $\Sigma 1$, Σn, Σn^2 and Σn^3, proved by perturbation. A factorisation of $x^n - 1$ to find geometric sums. Sums to infinity of geometric sums. Swapping sums.

4.3 Induction and Recursion

Induction and recursion theorems. Proof by induction on sums and matrices, defining functions by recursion. Pascal's triangle and the binomial theorem.

5. Trigonometry

5.1 The Trigonometric Functions

Definition from first principles. Length of arc and area of sector. Values at special angles. The graphs of the functions and their properties. Compound angle identities.

5.2 Trigonometric Equations and Identities

Techniques for solving equations, general solutions. Various identities: double angle identities, triple angle identities. Compound angle transformation. Weierstraß substitution. Sum-to-product and product-to-sum formulæ.

6. Calculus

6.1 Limits, Continuity and Asymptotics.

Intuitive definition of a limit, followed by ϵ-δ definition. Definition of continuity. Proof that linear and quadratic functions are continuous. The notations O, o and $\sim$.

6.2 Differentiation

Definition of derivative as the unique number such that $f(x+h) = f(x)+Ah+o(h)$. Proof that it is equivalent to a limit. Power rule and linearity of differentiation. Rules to differentiate: product rule, quotient rule, chain rule. Critical and stationary points, Fermat's theorem. Finding minima and maxima of functions. The second derivative test. Differentials, the chain rule and Leibniz notation. The inverse function theorem. The implicit function theorem (informally) and parametric and implicit differentiation.

6.3 Integration

Definition as the area under a curve. Riemann sums. The fundamental theorem of calculus. Symbolic integration. Rules of integration, linearity of integration. The power rule, integration by partial fractions, integration by substitution, integration by parts. Trigonometric integrals. Separable differential equations, the mean value of a function. Approximating sums by integrals.

6.4 Taylor's theorem

Extending the definition of derivative to obtain second derivative, third derivative, and so on. Taylor series for various functions. Defining functions using the Taylor series. A note about "holomorphic" functions.

6.5 Other Topics

Intermediate value theorem, numerical approximations, Newton–Raphson, length of arc, volume of solid of revolution, integration by reduction.

7. Combinatorics and Probability

7.1 Basic Combinatorics

Phrasing things in terms of the cardinality of sets. Addition and multiplication rules. Tuples, Permutations, Sets, Multisets. Inclusion-exclusion. Revisiting the binomial theorem. Generating functions.

7.2 Probability

Basic probability, probability trees. Rules for and/or, probability of negated event, conditional probability.

8. Complex Numbers

   8.1 Basic Definitions

   History of cubics, a word about numbers being "real". Argand diagrams, modulus and absolute value. Realising the denominator.

   8.2 The complex exponential

   Defining the complex exponential and logarithms. De Moivre's theorem. Roots of unity. The fundamental theorem of algebra. Trigonometric identities using the complex exponential.

   8.3 Other Topics

   Some basic complex loci, hyperbolic functions, a word about complex analysis.

9. Ordinary Differential Equations

   9.1 First Order ODEs

   Separable, exact and linear (integrating factors).

   9.2 Second Order ODEs

   Linear operators, kernels. Solving second order ODEs with constant coefficients.

10. Curve Sketching & Polar Coordinates

    10.1 Curve Sketching

    Thinking more generally about transformations, sketching $y = |f(x)|$, $y^2 = |f(x)|$, $y = 1/f(x)$. Sketching rational functions up to quadratic.

    10.2 Polar Coordinates

    Common polar curves, areas bounded by polar curves, tangents to polar curves. Swapping between Cartesian and polar coordinates.

Appendices

A. Answers to Exercises

B. Table of Formulæ, Derivatives and Integrals