Revision of First Year Topics

Luke Collins
https://drmenguin.com

june mmxix

Elementary Algebra

1. Solve the following equations for x.
[Do not use the quadratic formula for quadratics, use either factorisation or completing the square.]

a) $4x - 3 = 7(x-7) + 10$

b) $t = \dfrac{2t - x}{2t + x}$

c) $2x^2 + 12 = 25x$

d) $x^2 = 2x + 5$

e) $x^2 = ax + 5$

f) $x(5x + 13) = 18$

g) $x(5x + 13) = 19$

h) $x = \sqrt{1 - x}$

i) $x(x^4 + 12) = 7x^3$

j) $x = \dfrac{2t - x}{2t + x}$

k) $x^2 + \dfrac{15}{x^2} = 8$

l) $\sqrt{2x - 1} + \sqrt{3x + 1} = 7$

2. The sum $S_n = 1 + \cdots + n$ of the first n positive integers is given by

$$S_n = \tfrac{n}{2}(n + 1).$$

a) Find the sum $1 + \cdots + 100$.

b) If $1 + \cdots + n = 405\,450$, what is n?

c) Show that if $s = 1 + \cdots + n$, then

$$n = \tfrac{1}{2}(-1 + \sqrt{1 + 8s}).$$

3. a) Using surds, find integers a and b such that

$$11\,\frac{a+b\sqrt{3}}{(2\sqrt{3}+1)^2}=\frac{13-4\sqrt{3}}{2\sqrt{3}-1}.$$

b) Use *completing the square* to solve the equation

$$\frac{1}{x}-\frac{1}{\sqrt{2}}=\frac{1}{x+\sqrt{2}},$$

giving your solution(s) in the form $p\sqrt{2}+q\sqrt{10}$ for rational p and q.

4. The quadratic $p = ax^2 + bx + c$ has roots α and β.

a) Prove that $-{}^{b}\!/\!{}_{a} = \alpha + \beta$ and that ${}^{c}\!/\!{}_{a} = \alpha\beta$.

b) Recall that the *discriminant* of a quadratic is $\Delta(p) = b^2 - 4ac$. Explain why, over $\mathbb{R}$, p has distinct roots if $\Delta(p) > 0$, a repeated root if $\Delta(p) = 0$, and no roots if $\Delta(p) < 0$.

c) Define the "dis-k-riminant"

$$\Delta_k(p) = kb^2 - (1+k)^2ac.$$

Show that $\Delta_k(p) = 0$ precisely when one root of p is k times the other. (Notice that Δ_1 is our usual discriminant $\Delta(p)$.)

5. Consider the quadratic $p \in \mathbb{R}[x]$ defined by

$$p(x) = 5x^2 - 2ax + a^2 - 1.$$

a) Determine the range of values of a for which p has two distinct roots in $\mathbb{R}$.

b) For what values of a are the roots repeated?

c) If one root of p is double the other, what are the possible values of a?

d) If the roots of p are α and β, show that the quadratic with roots ${}^{\alpha^2}\!/\!_{\beta}$ and ${}^{\beta^2}\!/\!_{\alpha}$ is

$$q(x) = 25(a^2-1)x^2 + 2a(11a^2-15)x + 5(a^2-1)^2.$$

If for some value of a, the roots of q are $-{}^5\!/\!_6$ and ${}^9\!/\!_{50}$, what are the corresponding roots of p?

6. Consider the quadratic $p = 2x^2 - 5x + a$, where a is an integer. Suppose p can be factorised over the integers (i.e., p has rational roots).

   a) Prove that $(n+5)(n-5)$ is divisible by 8 if and only if n is odd.

   b) Show that a must be of the form $\frac{1}{8}(n+5)(n-5)$ where n is odd.

   c) Obtain a factorisation of p in terms of n.

7. a) Solve the equation

$$2^{3x-4} \cdot 5^{x+3} \cdot 7^x = 6^{1-2x}$$

in two different ways, in order to get the solution

$$x = \frac{4\log 2 - 3\log 5 + \log 6}{3\log 2 + \log 5 + 2\log 6 + \log 7}$$

when working one way, and

$$x = \log_{10\,080}\left(\frac{96}{125}\right)$$

when working the other way.

   b) Prove that the two solutions of part (a) are equal to each other.

   c) Solve the equation

$$7^{x+1} - \frac{1}{7^x} = 1$$

writing your answer in the form $\log_7(a + b\sqrt{29})$ for rational a, b.

   d) Solve the set of simultaneous equations

$$\log x + 2\log y = \log(x+2y)$$
$$2^x = 3^y$$

giving your answers in exact form.

8. a) Prove (without using the change of base formula) that for $c \neq 0$,

$$\log_a b = \log_{a^c}(b^c).$$

Deduce that $\log_9 25 = \log_3 5$ and $\log_3 5 = 1/\log_5 3$ by choosing appropriate values of c.

b) In the early stages of the COVID-19 pandemic, the number of known cases in the UK doubled every three days (approximately). If there were 2 000 known cases on the 12th of March 2020, how many days would need to pass in order for the cases to exceed 10 000?

c) Prove that the arithmetic mean of the logarithms of a and b is the logarithm of their geometric mean.

d) Prove that

$$\log 1 + \log 2 + \cdots + \log n = \log(n!).$$

Given that $\log 1 + \log 2 + \cdots + \log n \approx n \log n - n + \frac{1}{2} \log 2\pi n$,[1] deduce the famous Stirling approximation for factorials,

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

e) Use Stirling's formula and base-10 logarithms to obtain that

$$10\,000! \approx 2.8462 \times 10^{35\,659}.$$

How many zeroes does 10 000! end in?

9. Let $p \in \mathbb{R}[x]$ be defined by $p(x) = 3x^4 - 26x^3 + 39x^2 + 4x - 4$.

a) Give the precise statement of the rational roots theorem.

b) Without using the rational roots theorem, prove that any rational root c/d of p must satisfy $c \mid 4$ and $d \mid 3$. Hence determine all possible rational roots of p.

c) Find the remaining roots of p *without* performing long division.

d) What is the remainder when dividing p by $(x - 3)$?

[1] To get an idea as to where this approximation comes from, notice that $\sum_{k=1}^{n} \log k \approx \int_1^n \log t \, dt = n \log n - n + 1$. Obtaining the term in $\log n$ requires more work, but notice that this term is less significant than the main terms. See here for a plot.

10. Solve the following cubic equations.

a) $2x^3 - 15x^2 + 22x + 15 = 0$ b) $3x^3 - 5x^2 + x + 1 = 0$

c) $2x^3 + 7 = x(11x + 9)$ d) $2x^3 - 11x^2 + 19x - 7 = 0$

11. Use the Binomial theorem to expand the following.

a) $(1 + x)^5$ b) $(\sqrt{3} - \sqrt{2})^4$ c) $\left(2x - \frac{y}{x^2}\right)^6$

12. Show that $(x + 1)^6 - (x - 1)^6 = 4x(3x^4 + 10x^2 + 3)$. Deduce that $(x + 1)^6 = (x - 1)^6$ has only $x = 0$ as a solution.

13. a) Determine λ, m and n if

$$(\lambda - x)^m (1 + x)^n = 1 + 7x + 18x^2 + \cdots .$$

b) What is n if $\binom{n}{2} = 5050$?

c) Notice that the formula in question 2 can be written as

$$S_n = \binom{n+1}{2}.$$

Provide a combinatorial argument as to why.

14. Decompose the following rational functions into partial fractions.

a) $\dfrac{x^2 + 1}{(x - 1)(x - 2)(x - 3)}$ b) $\dfrac{1 - 3x}{(2x + 1)(x + 3)}$

c) $\dfrac{8x + 5}{(2x^2 + 1)(x + 3)}$ d) $\dfrac{7x^2 + 40x}{(2x^2 + 1)(x + 3)^2}$

e) $\dfrac{x^3 + 6}{(x - 4)(x + 3)}$ f) $\dfrac{2x^3 - 7x}{(x + 1)^2(x - 4)}$

g) $\dfrac{5x^3 + 3}{(x^2 - 3x + 1)(x^2 + 2)}$ h) $\dfrac{5x^3 + x - 2}{(x^2 - 3x + 1)(x + 2)^2}$

i) $\dfrac{19x^3 + 21x + 14}{(7x^2 + 13x - 2)(4 - x^2)}$ j) $\dfrac{x^4 - 1}{(x^2 - x - 12)(x^2 - x + 12)}$

15. Let a, b and c satisfy $a+b+c=0$. Show that $a^3+b^3+c^3=3abc$. Hence or otherwise, decompose

$$\frac{15x(x^2+11)}{(x^2+x-2)^3-(x^2+1)^3-(x-3)^3}$$

into partial fractions.

16. A fort manned by 45 men has food and water for 100 days.

    d) How long would the same supplies last if the fort were manned by 50 men?

    e) After 25 days, 15 out of the 45 men leave the fort. How many days can the remaining supplies provide for the remaining men in the fort?

17. A cab from Paceville to Tarxien costs €P. A group of friends are thinking of sharing the cab, but 3 of them are not sure if they want to leave yet. It is €1 more expensive per person if the 3 friends do not join the ride.

    Show that $9+12P$ is an odd square number.

18. If x and y are both positive, prove that

$$\left(\frac{1}{x}+\frac{1}{y}\right)(x+y)\geqslant 4.$$

19. Let a/b be a rational number different from 1. Show that a/b plus its reciprocal can never be an integer.

20. A real number α is said to be *transcendental* if there does not exist a polynomial f with rational coefficients such that $f(\alpha)=0$. We know that π and e are both transcendental.

    a) Prove that $\sqrt{2}$ and $\sqrt{3+\sqrt{5}}$ are not transcendental.

    b) Prove that a transcendental number is irrational.

    c) It is not known whether the numbers $\pi+e$ and $\pi\times e$ are rational or irrational. Prove that at least one of them is irrational.

Answers & Hints

1. a) 12
   b) $-\frac{2t(t-1)}{t+1}$
   c) $1/2$, 12
   d) $1 \pm \sqrt{6}$
   e) $\frac{1}{2}(a \pm \sqrt{a^2+20})$
   f) $-18/5$, 1
   g) $-1/10(13 \pm 3\sqrt{61})$
   h) $\frac{1}{2}(\sqrt{5}-1)$
   i) 0, ± 2, $\pm\sqrt{3}$
   j) $-\frac{1}{2}(1+2t \pm \sqrt{4t^2+12t+1})$
   k) $\pm\sqrt{3}$, $\pm\sqrt{5}$
   l) 5
2. a) $\frac{100}{2}(101) = 5050$
   b) 900
   c) Hint: Solve $s = \frac{n}{2}(n+1)$.
3. a) $a = 1, b = 2$
   Hint: Make $a + b\sqrt{3}$ subject of the equation, and simplify using the usual techniques of surds.
   b) $p = \frac{1}{2}$, $q = \pm\frac{1}{2}$.
4. These are standard quadratic theory, refer to the notes. You may assume the result that if p has roots α and β, then we can write it as $a(x-\alpha)(x-\beta)$.

   For part (c), refer to the last example in the notes on Viète's formulæ, the ideas there should be helpful.
5. a) $-\sqrt{5}/2 < a < \sqrt{5}/2$
   b) $a = \pm\sqrt{5}/2$
   c) $a = \pm\frac{3}{37}\sqrt{185}$
   Hint: If the smaller root is α, then $\frac{2a}{5} = \alpha + 2\alpha$ and $\frac{a^2-1}{5} = \alpha(2\alpha)$.
   d) Corresponding roots are $-3/10$ and $1/2$.

   Hint: If the desired roots are α and β, then we are given α^2/β and β^2/α. Without loss of generality, we can assume $\alpha^2/\beta = -5/6$ and $\beta^2/\alpha = 9/50$. Solving these simultaneously gives the required roots.
6. a) If n is odd, we can write it as $2k+1$. Plug this in to $(n+5)(n-5)$ and expand, the result should be clearly divisible by 8 (show this by factorising 8 out of it).

   For the converse, show that if n is even (i.e., put $n = 2k$) we end up with something not divisible by 8 (it should look like $8t + r$ for some non-zero remainder $1 \leqslant r \leqslant 7$).
   b) By a theorem in the notes, this happens if and only if $\Delta = n^2$ for some n. If we work out Δ, we get $25 - 8a$. If this equals n^2, we get $a = (n+5)(n-5)/8$.
   c) Staring at $2x^2 - 5x + \frac{1}{8}(n+5)(n-5)$ $= \frac{1}{8}(16x^2 - 40x + (n-5)(n+5))$ for long enough, we see it factorises as $\frac{1}{8}(4x-5+n)(4x-5-n)$.
7. a) Hint: For the first way, take log of both sides and apply the laws of logarithms immediately, solving for x. For the second way, use the laws of indices to reduce the equation to the form $a^x = b$ and take $\log_a$ at the end.
   b) Hint: Use the laws of logarithms.
   c) $x = \log_7\left(\frac{1}{14}(1+\sqrt{29})\right)$.
   Hint: Rearrange the given equation to get a quadratic in 7^x. Be careful of the negative branch, $1 - \sqrt{29} < 0$ so $\log_7\left(\frac{1}{14}(1-\sqrt{29})\right)$ does not exist.
   d) $x = \sqrt{(\log_2 3)(2 + \log_2 3)}$
   $y = \sqrt{1 + 2\log_3 2}$
8. a) Hint: If $u = \log_a b$, then $a^u = b$. Apply $(\,\cdot\,)^c$ to both sides and switch back to logarithmic form.

   Take $c = 1/2$ and $c = \log_3 5$ in the examples.
   b) 7 days
   Hint: Solve the indicial equation $2\,000 \times 2^{t/3} = 10\,000$ to get that $t = \lceil 3\log_2 5 \rceil = 7$ days.

c) Hint: The arithmetic mean of two numbers is $\frac{a+b}{2}$, the geometric mean is $(ab)^{1/2}$.

d) For the first part, by the first law of logarithms we see that $\log 1 + \cdots + \log n = \log(1 \cdots n) = \log n!$.

For the second part, equating $\log n!$ and the given expression and doing $e^{(\cdot)}$ both sides, we obtain Stirling's formula.

e) To approximate $10\,000!$, we notice that

$$\begin{aligned}&\sqrt{2\pi\,10\,000}\,\left(\tfrac{10\,000}{e}\right)^{10\,000}\\ &= 100\sqrt{2\pi}\,10^{40\,000}/e^{10\,000}\\ &= 10^{40\,002}\sqrt{2\pi}/e^{10\,000}.\end{aligned}$$

Taking $\log_{10}$ and applying the laws of logarithms, we evaluate $\log_{10}(10\,000!) \approx 35\,659.5$. This means that $10\,000! = a \times 10^{35\,659}$ where $0 < a < 10$.

Now to determine a, we just divide our approximation of $10\,000!$ by $10^{35\,659}$ to get

$$\begin{aligned}&10^{4\,343}\sqrt{2\pi}/e^{10\,000}\\ &= \sqrt{2\pi}\,e^{4\,343\ln 10 - 10\,000}\\ &\approx \sqrt{2\pi}e^{0.12706}\\ &\approx 2.846239,\end{aligned}$$

which agrees with the question.

Finally, to determine the number of zeroes at the end, notice that each zero corresponds to a factor of 10. Thus we need to find the largest k such that $10^k \mid 10\,000!$.

A factor of 10 corresponds to the occurrence of a $2 \cdot 5$ in the prime factorisation of $10\,000!$. Since half of the integers between 1 and $10\,000$ are even, we know the power of 2 in its prime factorisation is at least $10\,000/2 = 5\,000$. Similarly, one fifth of the numbers between 1 and $10\,000$ are divisible by 5, so the power of 5 is at least $10\,000/5 = 2\,000$. But some numbers contribute more than one multiple of 5. In fact, one in every 25 numbers contributes two multiples of five, so in addition to those already counted, we need to add an extra multiple of 5 for each number of this kind, so that's another $10\,000/25 = 400$. Similarly, one in every 125 contributes three multiples of 5. We've already counted 2 contributions for each number of this kind, we need to add an additional multiple for each of them, which gives us $10\,000/125 = 80$. Continuing this way, we see that, in total, the amount of fives in the prime factorisation is therefore $\lfloor \frac{10\,000}{5} + \frac{10\,000}{25} + \frac{10\,000}{125} + \frac{10\,000}{625} + \frac{10\,000}{3\,125} \rfloor = 2\,499$.

For each of these 5's, there is a 2 we can pair them up with (there are many more 2's in fact), but this is the precise number of 5's, so we have that $10\,000!$ ends in $2\,499$ zeroes.

9. a) Let $p = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients ($p \in \mathbb{Z}[x]$). If $x = {}^{c}\!/\!_{d}$ is a root of p (where $\mathrm{hcf}(c,d) = 1$), then $c \mid a_0$ and $d \mid a_n$.

Refer to the notes or look at https://en.wikipedia.org/wiki/Rational_root_theorem if this fact doesn't sound familiar.

b) Hint: for the first part, just reproduce the proof of the rational roots theorem in the notes with $a_0 = -4$ and $a_n = 3$.

$2, -{}^{1}\!/\!_{3}$

Hint: Try putting $x = {}^{s}\!/\!_{t}$ with $s = \pm 1, \pm 2, \pm 4$ and $t = \pm 1, \pm 3$.

c) $\frac{1}{2}(7 \pm \sqrt{41})$

Hint: Let the remaining two roots be α, β. By Viète's formulæ, $2 - \frac{1}{3} + \alpha + \beta = \frac{26}{3}$ and $2(\frac{1}{3})\alpha\beta =$

$\frac{-4}{3}$. Solving simultaneously yields α and β.

d) -100

Hint: Evaluate $f(3)$ and apply the remainder theorem.

10. a) $-1/2$, 3, 5

b) $-1/3$, 1 (repeated)

c) $1/2$, $1/2(5 \pm \sqrt{53})$

d) $1/2$

11. a) $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

b) $49 - 20\sqrt{6}$

c) $64x^6 - 192x^3y + 240y^2 - 160\frac{y^3}{x^3} + 60\frac{y^4}{x^6} - 12\frac{y^5}{x^9} + \frac{y^6}{x^{12}}$.

12. Hint: For the identity, expand using the binomial theorem.

The right-hand side of the identity can be factorised further as $4x(x^2 + 3)(3x^2 + 1)$, which is clearly 0 only if $x = 0$.

13. a) $\lambda = 1$, $m = 3$, $n = 10$

Hint: Use the binomial theorem and explicit expressions for $\binom{n}{k}$, $\binom{m}{k}$ to determine the coefficients of $1, x, x^2$ in terms of λ, m and n, and compare coefficients.

b) 101

Hint: $\binom{n}{2} = \frac{n(n-1)}{2}$.

c) The number of ways of choosing subsets of size 2 from $\{1, \ldots, n+1\}$ can be computed as follows. There are n of the form $\{1, k\}$. Then there are $(n-1)$ of the form $\{2, k\}$ (we are excluding $\{2, 1\}$ since it was already counted as $\{1, 2\}$ before). Similarly there are $(n-2)$ of the form $\{3, k\}$ which weren't already counted. Continuing this way till $\{n, k\}$, which there is only one of (namely $\{n, n+1\}$), we get that $\binom{n+1}{2} = n + \cdots + 1 = S_n$.

14. a) $\frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}$

b) $\frac{1}{2x+1} - \frac{2}{x+3}$

c) $\frac{2(x+1)}{2x^2+1} - \frac{1}{x+3}$

d) $\frac{4x+1}{2x^2+1} - \frac{2}{x+3} - \frac{3}{(x+3)^2}$

e) $x + 1 + \frac{3}{x+3} + \frac{10}{x-4}$

f) $2 - \frac{1}{(x+1)^2} + \frac{4}{x-4}$

g) $\frac{4x}{x^2-3x+1} + \frac{x+3}{x^2+2}$

h) $\frac{2x-1}{x^2-3x+1} + \frac{3}{x+2} - \frac{4}{(x+2)^2}$

i) $\frac{2}{7x-1} + \frac{3}{(x+2)^2} - \frac{2}{x+2} - \frac{1}{x-2}$

j) $1 + \frac{85}{56(x-4)} - \frac{10}{21(x+3)} + \frac{23x-131}{24\left(x^2-x+12\right)}$

15. Hint: For the identity, since $a + b + c = 0$, then $c = -a - b$, so we have $a^3 + b^3 + c^3 = a^3 + b^3 + (-a-b)^3$. Expand this to get $3ab(-a-b) = 3abc$.

Partial fractions:

$\frac{4}{x+2} - \frac{5}{x-1} - \frac{3x+4}{x^2+1} + \frac{9}{x-3}$

Hint: Notice that if we sum the brackets in the denominator, we get $(x^2+x-2)-(x^2+1)-(x-3) = 0$, and therefore applying the identity with these as a, b and c, we get

$$\begin{aligned}
&(x^2 + x - 2)^3 - (x^2 + 1)^3 - (x - 3)^3 \\
&= (x^2 + x - 2)^3 + (-(x^2 + 1))^3 \\
&\qquad + (-(x - 3))^3 \\
&= 3(x^2 + x - 2)(x^2 + 1)(x - 3) \\
&= 3(x + 2)(x - 1)(x^2 + 1)(x - 3).
\end{aligned}$$

16. a) We have enough food for a total of 4 500 man-days. (analogous to "man-hours"). This would last 50 men $4500/50 = 90$ days.

b) After 25 days, there is 75 days' worth of food left for 45 men, i.e., it will last $45 \times 75 = 3\,375$ man days. But now we divide this among 30 men, so the result is $3\,375/30 = 112\frac{1}{2}$ days.

17. We have that $\frac{P}{n-3} - \frac{P}{n} = 1$, which means that

$$n = \tfrac{1}{2}(3 + \sqrt{9 + 12P}).$$

Clearly n is a whole number, so $9 + 12P$ must be square, and moreover, odd, so that $3 + \sqrt{9 + 12P}$ is divisible by 2.

18. Hint: The left hand side can be rewritten as $\frac{1}{xy}((x-y)^2 + 4xy)$.

19. If $\frac{a}{b} + \frac{b}{a} = \frac{a^2+b^2}{ab} = k$, then $a^2 - kab + b^2 = 0$. This is a quadratic in a, with discriminant $\Delta = b^2(k^2-4)$. Clearly we want Δ to be a square since a must be an integer (hence a rational root). Δ is a square if and only if $k^2 - 4$ is a square.

    Say $k^2 - 4 = \ell^2$. This rearranges to $(k-\ell)(k+\ell) = 4$. This can only happen if $k - \ell = 1$ and $k + \ell = 4$, or $k - \ell = 4$ and $k + \ell = 1$, or else $k - \ell = k + \ell = 2$. The first two cannot happen since k and ℓ are to be integers. The last one implies that $k = 2$, in other words, that the quadratic only has rational solutions for a if it is $a^2 - 2ab + b^2 = 0$. But this means $(a-b)^2 = 0$, i.e., $a = b$, and the question states that $a/b \neq 1$.

20. a) They are not transcendental since $\sqrt{2}$ is a root of the quadratic $x^2 - 2$ and $\sqrt{3+\sqrt{5}}$ is a root of the quartic $(x^2-3)^2 - 5 = x^4 - 6x^2 - 4$, and both of these have rational coefficients.

    b) Any rational number a/b is the root of the polynomial $ax - b$ which has rational coefficients, so it cannot be transcendental.

    Thus any transcendental number must be irrational.

    c) e and π are both transcendental, which means that $(x-e)(x-\pi) = x^2 - (e+\pi)x + e\pi$ cannot have coefficients which are all rational. Thus at least one of $e + \pi$ and $e\pi$ is irrational.