Walks and Canonical Double Coverings of Comain Graphs

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Faculty of Science
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S^3 Annual Science Conference
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Outline

1. Introduction
   - Definition of a Graph
   - Representing Graphs as Matrices
   - Graph Isomorphisms
2. Canonical Double Covers
   - Definition of CDC
   - Some Easy Observations about CDC
   - How many such graphs are there?
3. Walks
   - The Walk Matrix of a Graph
   - Main Eigenvalues and Eigenvectors

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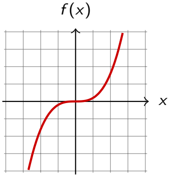

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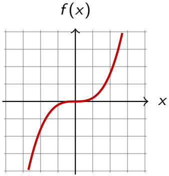

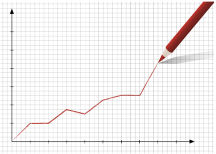

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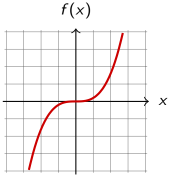

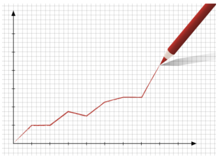

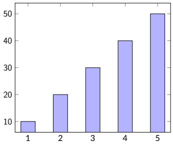

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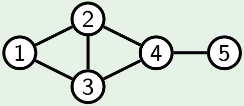

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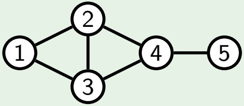

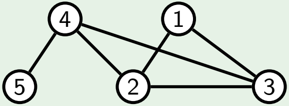

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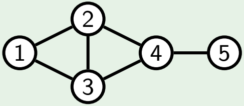

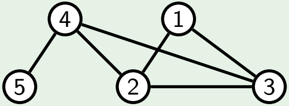

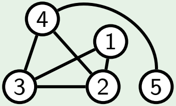

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$$a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j \text{, i.e. } \{v_i, v_j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

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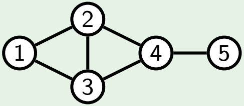

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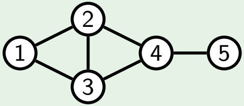

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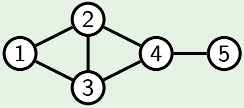

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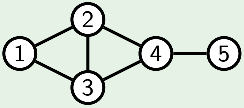

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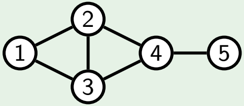

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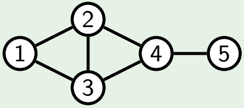

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- Each entry on the diagonal is 0, since we consider simple graphs

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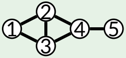
The eigenvalues of 

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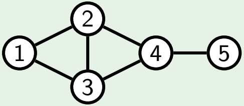

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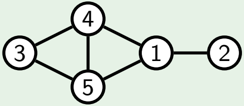

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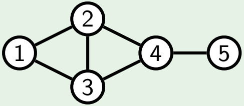

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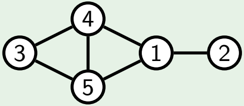

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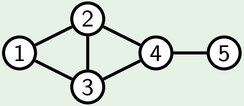

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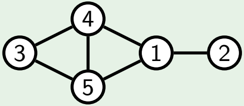

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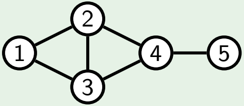

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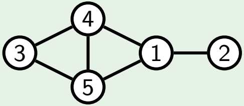

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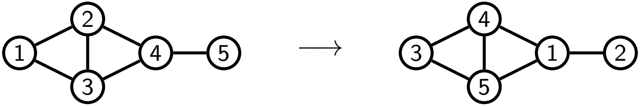

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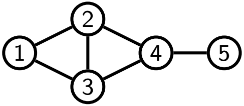

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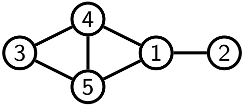

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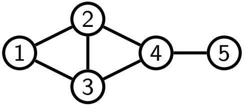

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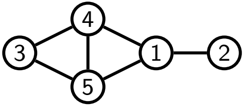

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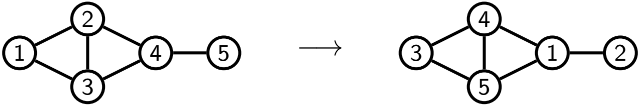

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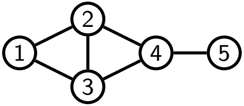
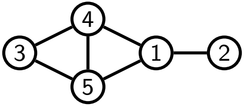
In the example the relabelling was: $1 \to 3,\ \ 2 \to 4,\ \ 3 \to 5,\ \ 4 \to 1,\ \ 5 \to 2$



$\longrightarrow$



$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}$$
$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{I}} \longrightarrow \begin{array}{c} \begin{array}{ccccc} & & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} & & 1 & 0 & 0 \\ & & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \end{pmatrix} \end{array}$$

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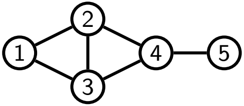

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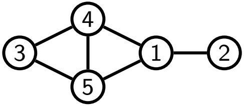

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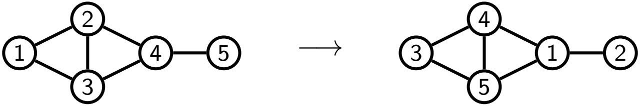

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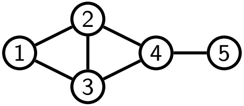

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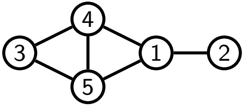

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In most cases, the labelling of the vertices in a graph is not important.

Definition of $\mathrm{CDC}(G)$

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Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\mathrm{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

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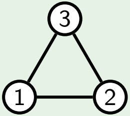

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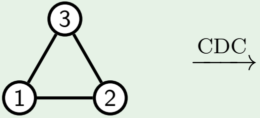

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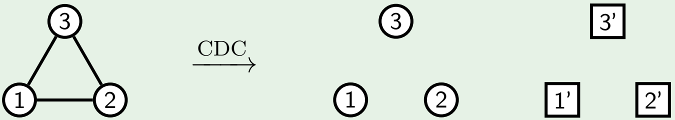

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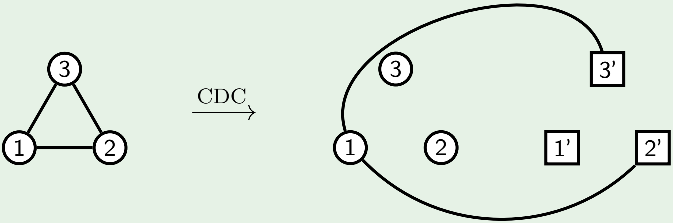

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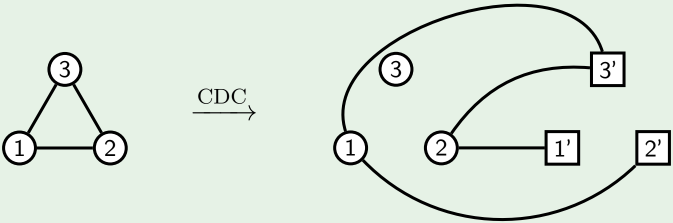

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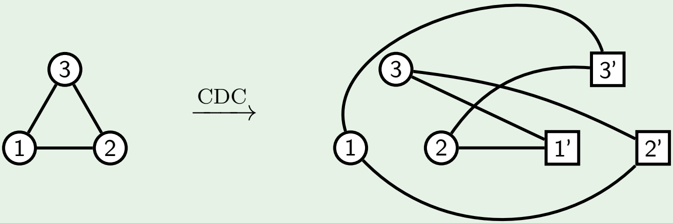

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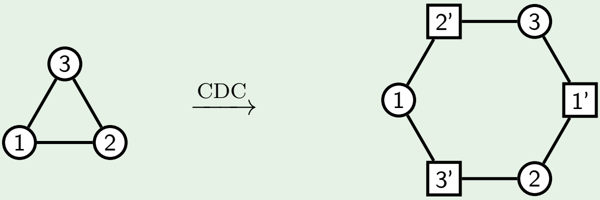

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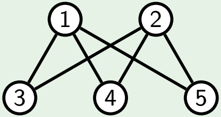

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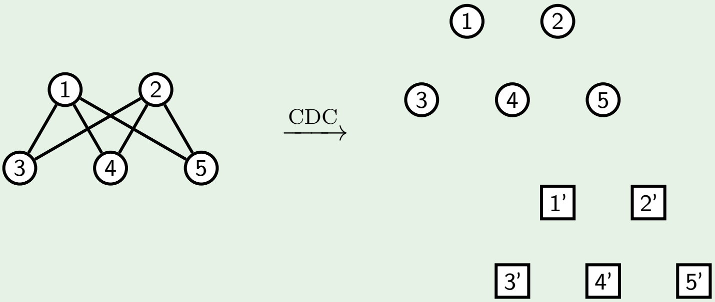

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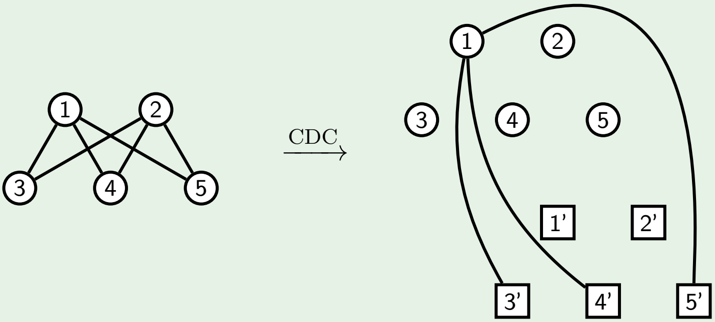

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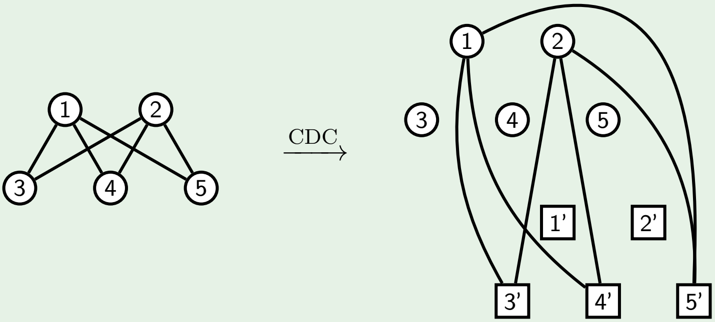

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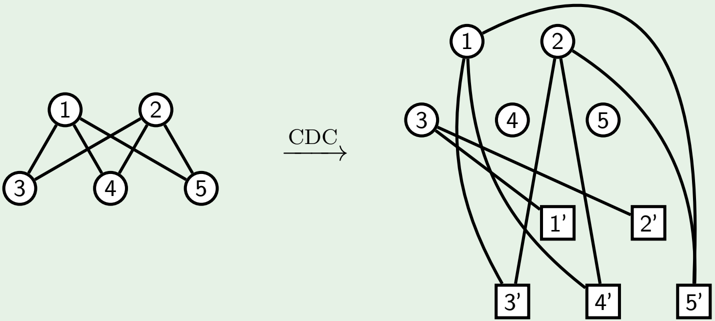

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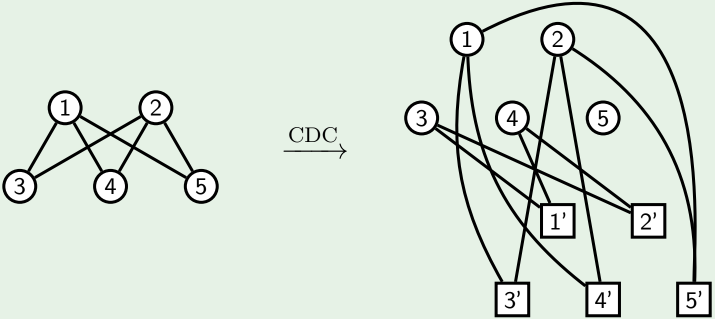

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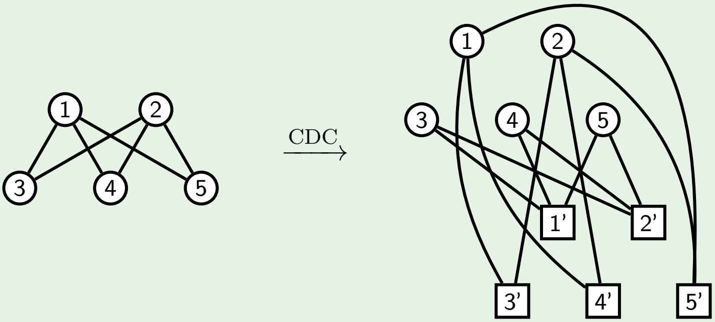

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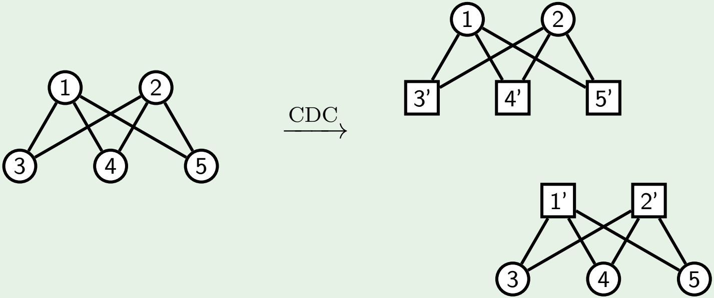

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  This yielded a useful *proof technique*: If we have $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$ where G has no isolated vertex, we show that the negation of what we want to prove introduces an isolated vertex in H.

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Definition (Two-Fold Isomorphism)

Let G and H be two graphs with adjacency matrices $\mathbf{A}_G$ and $\mathbf{A}_H$. We say that G is *two-fold isomorphic* or *TF-isomorphic* to H if

$$\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$$

for some permutation matrices $\mathbf{R}$, $\mathbf{Q}$.

TF-Isomorphisms (Scapellato et al.)

Let us illustrate this proof technique with an important result.
Recall that two graphs G and H with adjacency matrices $\mathbf{A}_G$ and $\mathbf{A}_H$ are isomorphic if and only if there exists a permutation matrix such that

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Here $\mathbf{R}$ and $\mathbf{Q}$ could be any permutation matrices, they don't have to be the inverse (i.e. transpose) of each other.

Theorem

Let **G** *and* **H** *be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic}.$$

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$$\mathbf{P}_{21}^\mathsf{T}\mathbf{A}_G\mathbf{P}_{12} + \mathbf{P}_{11}^\mathsf{T}\mathbf{A}_G\mathbf{P}_{22} = \mathbf{A}_H \tag{1}$$

$$\mathbf{P}_{21}^\mathsf{T}\mathbf{A}_G\mathbf{P}_{11} = \mathbf{P}_{12}^\mathsf{T}\mathbf{A}_G\mathbf{P}_{22} = \mathbf{O} \tag{2}$$

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^{\mathsf{T}}$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$.

$\square$

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$$\mathbf{P} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

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Observe that the key to this proof is the contradiction arising from the introduction of an isolated vertex.

How many different graphs have $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$?

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Having the same CDC is very "close" to being isomorphic. On $n \leqslant 8$ vertices, there are 13 597 non-isomorphic graphs. Taking all

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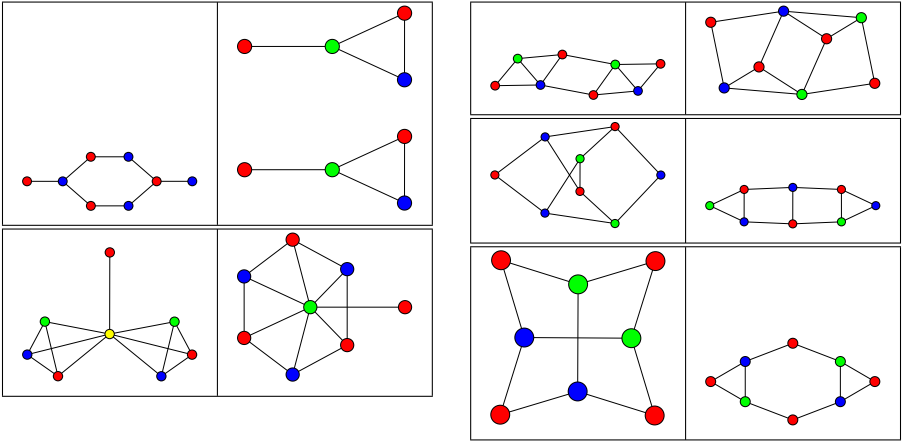
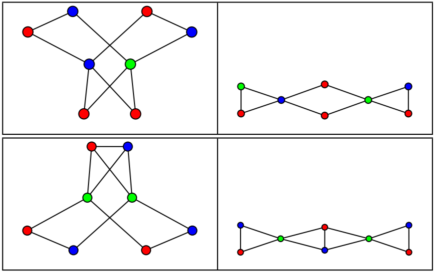
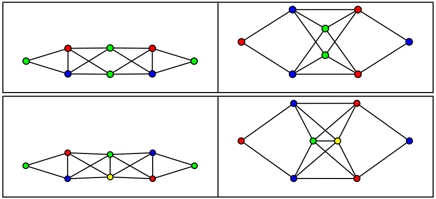
It is rare for a pair of graphs to be so structurally similar yet not isomorphic.

Here are some of them:



and some more:





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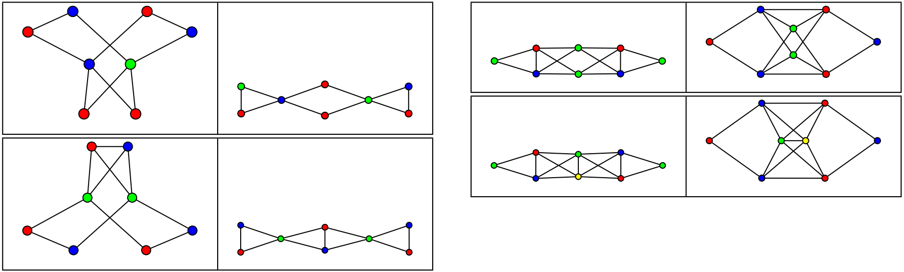

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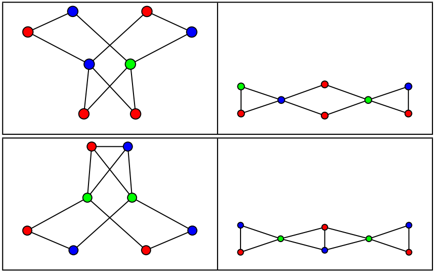

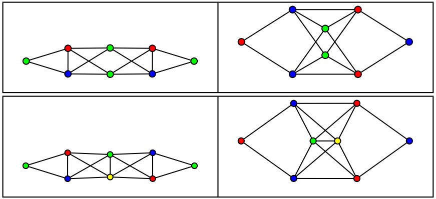

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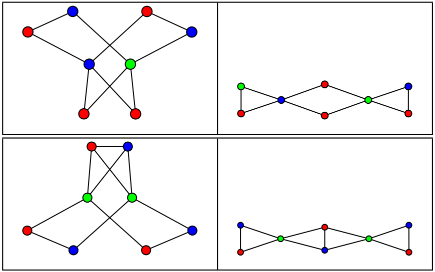

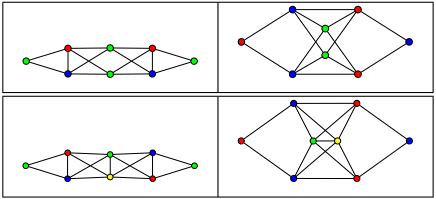

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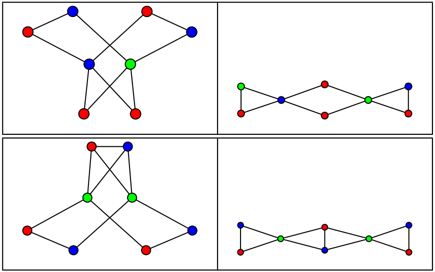

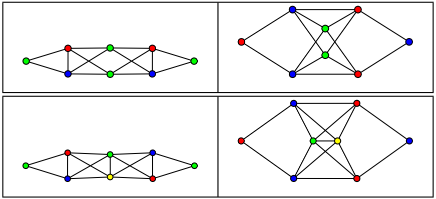

Graphs with the same CDC have the same:

- Main eigenvalues
- Main eigenspaces
- Main eigenvectors

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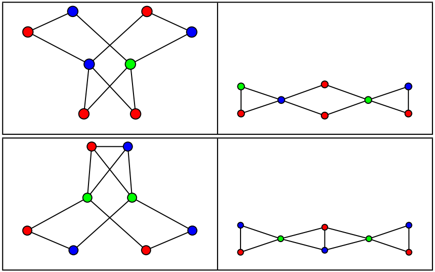

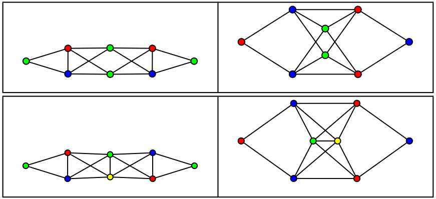

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- Main eigenvalues
- Main eigenspaces
- Main eigenvectors
- Number of walks of any length...

Walks

Walks

Definition (Walk)

Let G be a graph. A **walk** in G is a sequence of vertices

$$v_1, v_2, \ldots, v_k$$

such that $\{v_i, v_{i+1}\}$ is an edge for $i = 1, \ldots, k-1$. The **length** of a walk is the number k of vertices.

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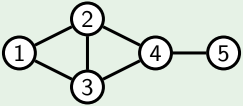
Example

In our usual example graph, 1234 and 12324 are walks, but 1235 is not.



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In general, $\mathbf{A}^k\mathbf{j}$ is the vector

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Definition (Walk Matrix)

The matrix $\mathbf{W}_k(G)$ is the $n \times k$ matrix whose columns are the first k such vectors, i.e.

$$\mathbf{W}_k(G) = \begin{pmatrix} | & | & | & & | \\ \mathbf{j} & \mathbf{A}\mathbf{j} & \mathbf{A}^2\mathbf{j} & \cdots & \mathbf{A}^{k-1}\mathbf{j} \\ | & | & | & & | \end{pmatrix}.$$

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$$\mathbf{j} = \sum_{i=1}^{p} \beta_i \mathbf{x}_i,$$

where the coefficients β_i are unique.

Proof (continued).

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Now the ℓth column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

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Corollary

Any two graphs with the same CDC have the same walk matrix.

Thank you!

DEPARTMENT OF MATHEMATICS
Faculty of Science
L-Università ta' Malta