

Walks and Canonical Double Coverings of Comain Graphs

Luke Collins Irene Sciriha

DEPARTMENT OF MATHEMATICS
Faculty of Science
L-Università ta' Malta

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L-Università
ta' Malta



Outline

- 1 Introduction
 - Definition of a Graph
 - Representing Graphs as Matrices
 - Graph Isomorphisms
- 2 Canonical Double Covers
 - Definition of CDC
 - Some Easy Observations about CDC
 - How many such graphs are there?
- 3 Walks
 - The Walk Matrix of a Graph
 - Main Eigenvalues and Eigenvectors

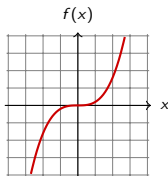
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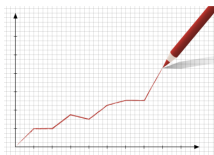
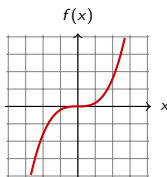
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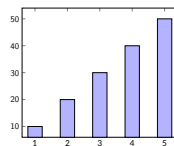
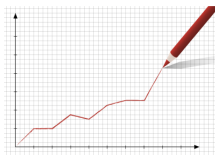
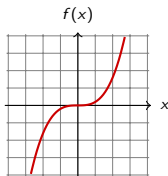
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$V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$ define a graph.

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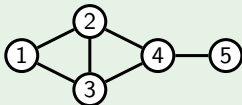
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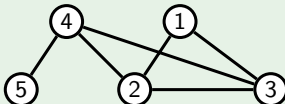
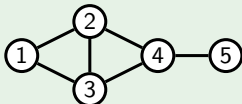
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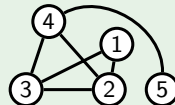
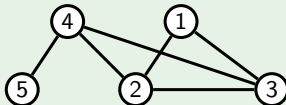
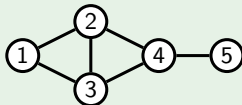
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$$a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j, \text{ i.e. } \{v_i, v_j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

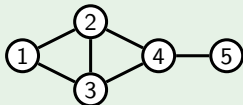
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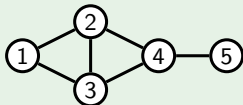
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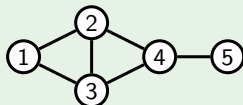


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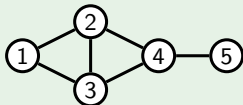
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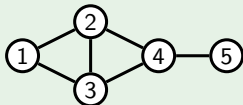
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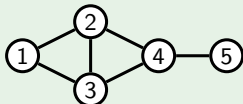
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- The adjacency matrix is symmetric
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- Each entry on the diagonal is 0, since we consider simple graphs

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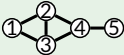
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The eigenvalues of  are those of its adjacency matrix $\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

Graph Isomorphisms

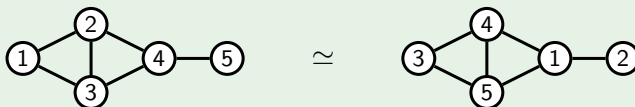
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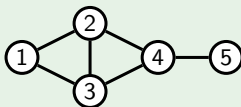
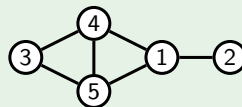
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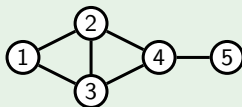
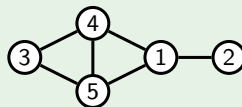
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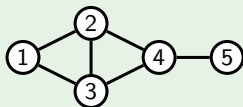
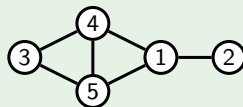
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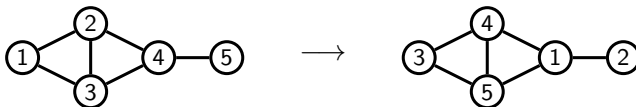
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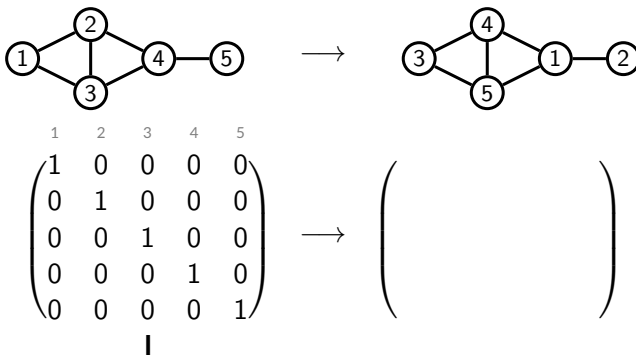


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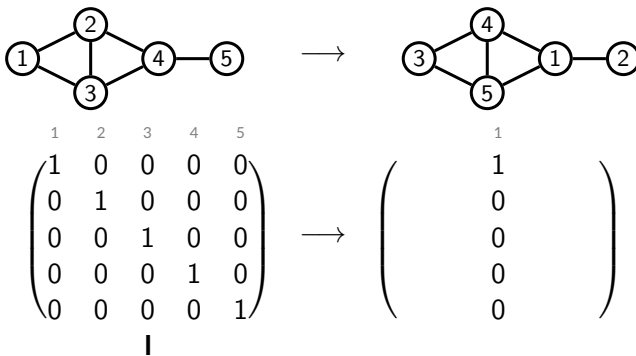


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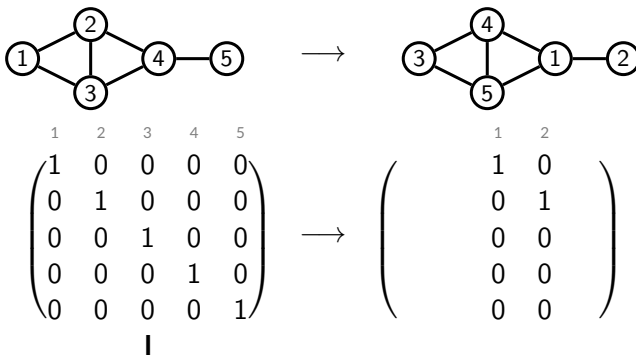


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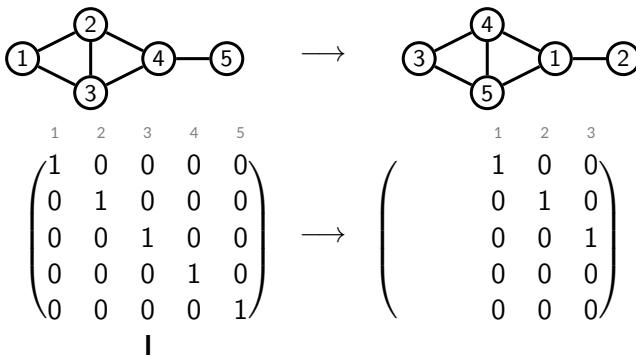


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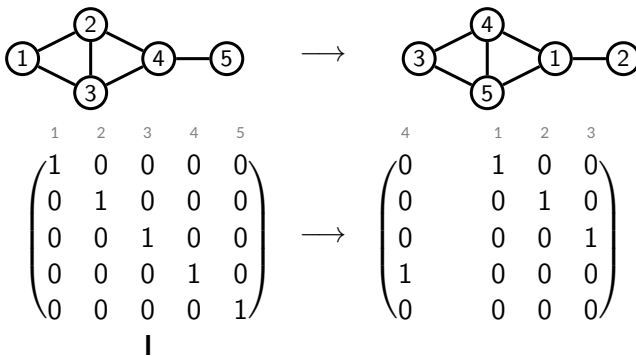


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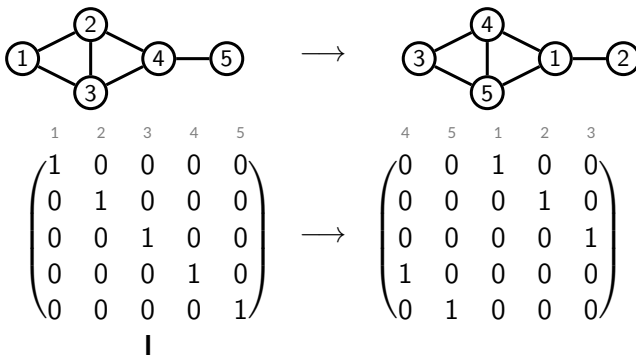


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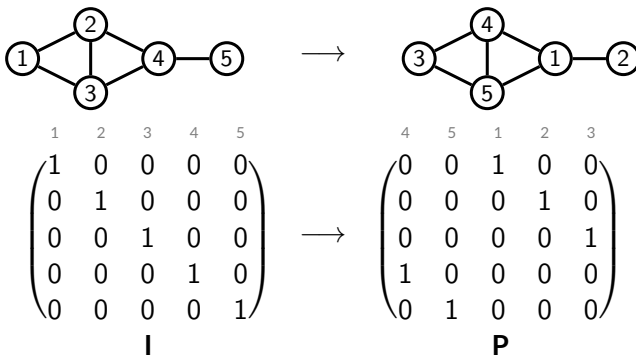


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Moreover, since $\mathbf{P}^T = \mathbf{P}^{-1}$, we are actually doing $\mathbf{A} \longrightarrow \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$. This means that the matrix \mathbf{A} and the resulting new adjacency matrix are *similar*.

Similar matrices have the same:

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In most cases, the labelling of the vertices in a graph is not important.

Definition of $CDC(G)$

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Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\text{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

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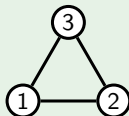
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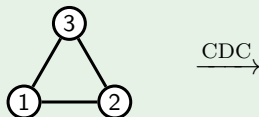


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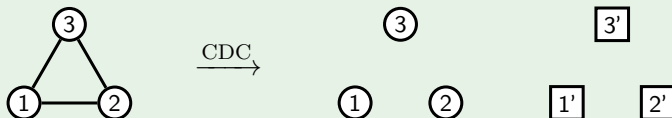


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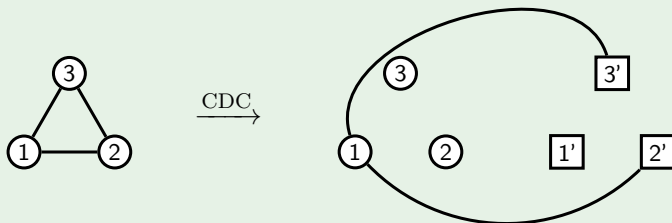


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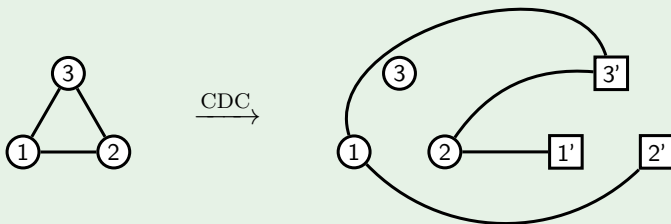


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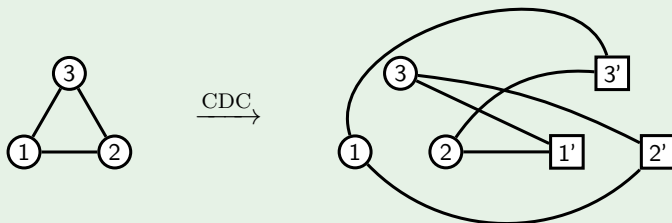


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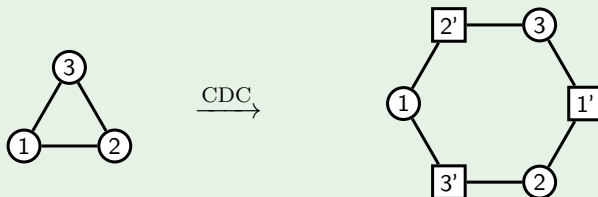


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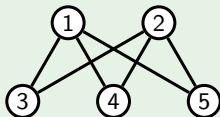


Definition of $\text{CDC}(G)$ – Another example

Example $\text{CDC}(K_{2,3}) = K_{2,3} \dot{\cup} K_{2,3}$

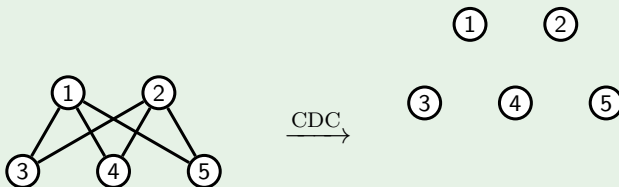
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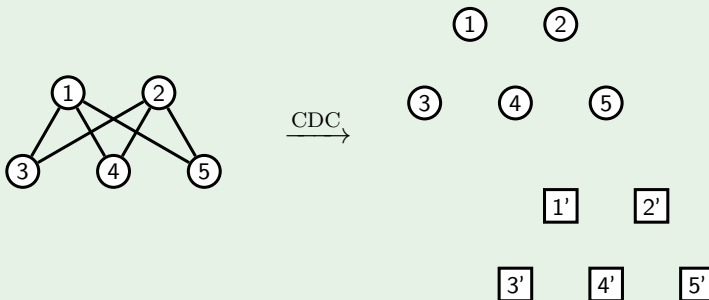
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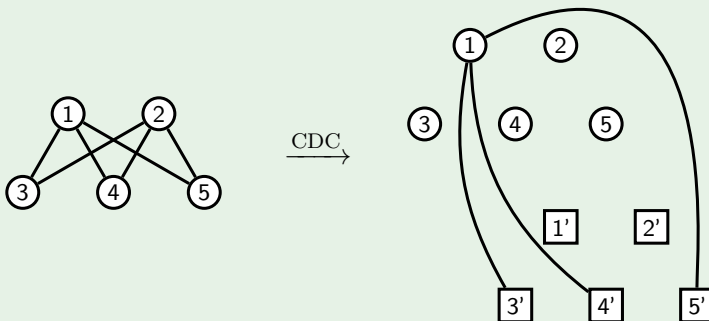
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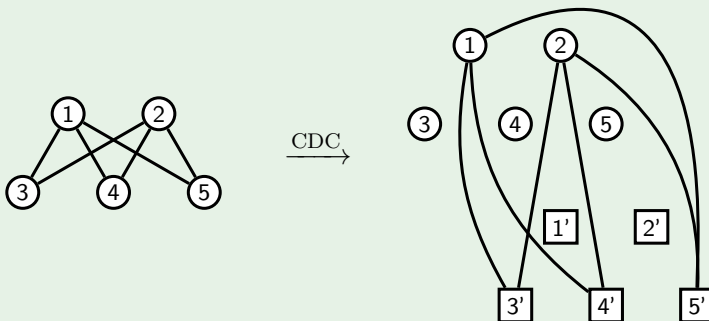
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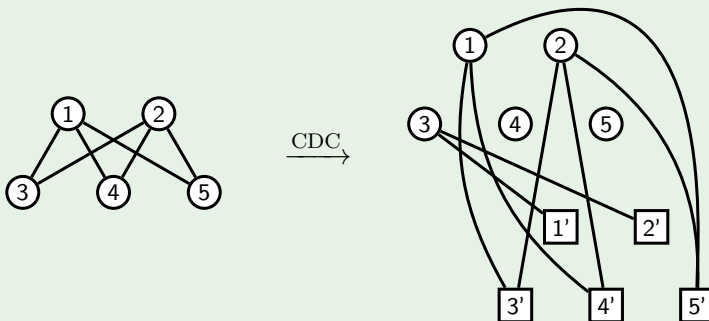
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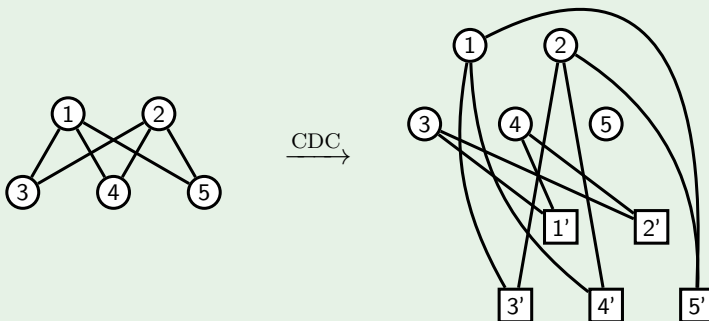
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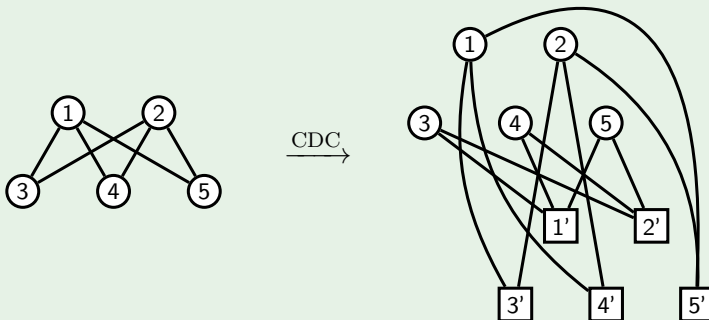
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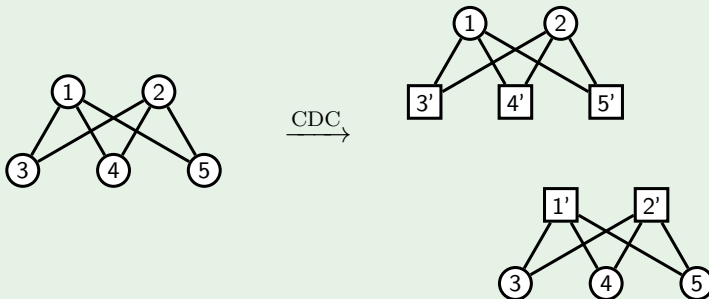
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- If G has adjacency matrix \mathbf{A} , then the adjacency matrix of $\text{CDC}(G)$ is given by

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- Let $G = G_1 \dot{\cup} G_2 \dot{\cup} \dots \dot{\cup} G_k$. Then

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$$\text{CDC}(\text{C}_6) = \text{C}_6 \dot{\cup} \text{C}_6,$$

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- **But** if $\text{CDC}(G) \simeq \text{CDC}(H)$ and G has an isolated vertex, then H must have an isolated vertex as well; i.e.

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This yielded a useful *proof technique*: If we have $\text{CDC}(G) \simeq \text{CDC}(H)$ where G has no isolated vertex, we show that the negation of what we want to prove introduces an isolated vertex in H .

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Here \mathbf{R} and \mathbf{Q} could be any permutation matrices, they don't have to be the inverse (i.e. transpose) of each other.

Theorem

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Theorem

Let G and H be two graphs. Then

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$$\mathbf{P}_{21}^T \mathbf{A}_G \mathbf{P}_{12} + \mathbf{P}_{11}^T \mathbf{A}_G \mathbf{P}_{22} = \mathbf{A}_H \tag{1}$$

$$\mathbf{P}_{21}^T \mathbf{A}_G \mathbf{P}_{11} = \mathbf{P}_{12}^T \mathbf{A}_G \mathbf{P}_{22} = \mathbf{O} \tag{2}$$

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$$\mathbf{P} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$



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Observe that the key to this proof is the contradiction arising from the introduction of an isolated vertex.

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Having the same CDC is very “close” to being isomorphic. On $n \leq 8$ vertices, there are 13 597 non-isomorphic graphs. Taking all

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possible pairs of graphs on at most 8 vertices, it turns out that only 32 pairs are TF-isomorphic.

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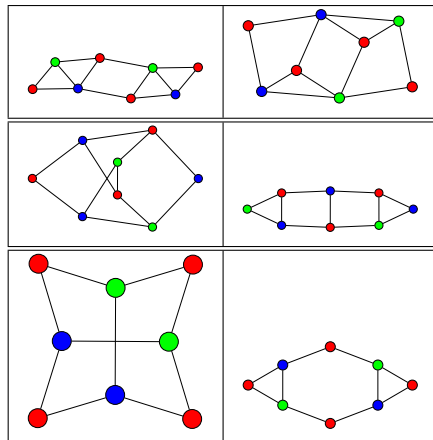
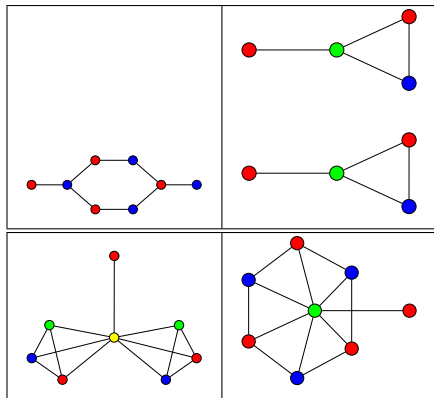
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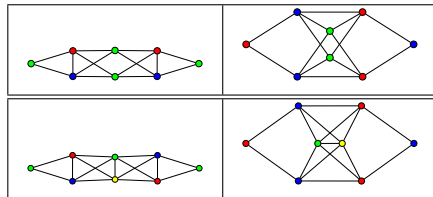
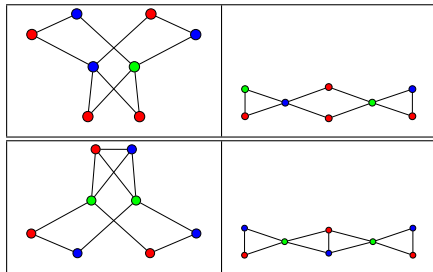
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It is rare for a pair of graphs to be so structurally similar yet not isomorphic.

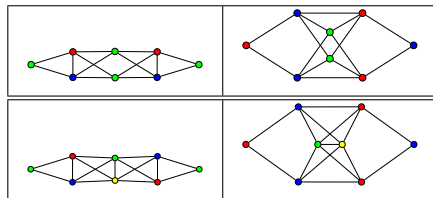
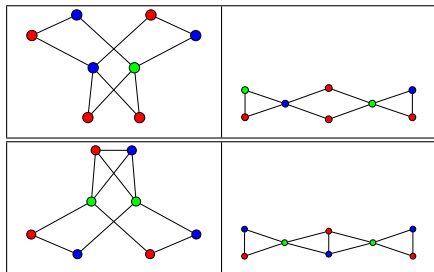
Here are some of them:



and some more:

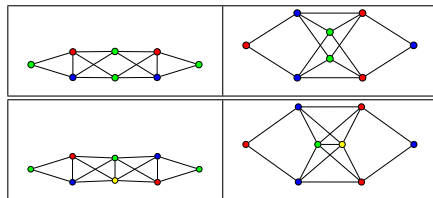
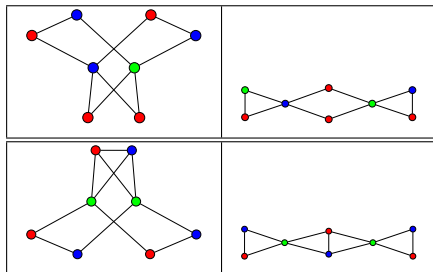


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Graphs with the same CDC have the same:

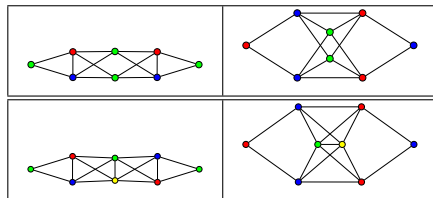
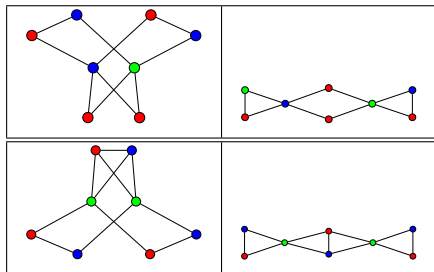
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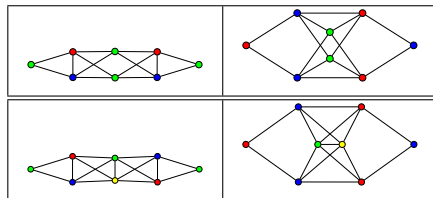
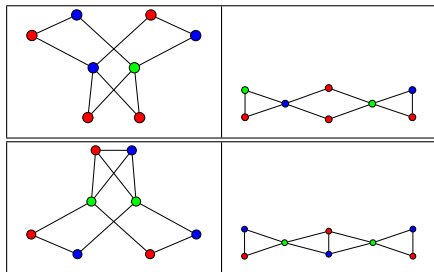
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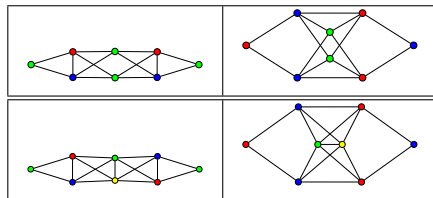
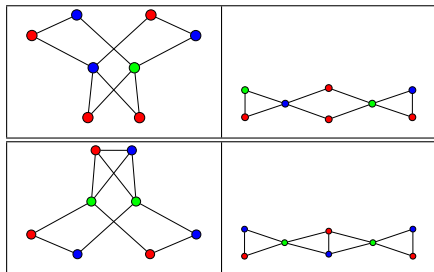
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Graphs with the same CDC have the same:

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- Number of walks of any length...

Walks

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Definition (Walk)

Let G be a graph. A **walk** in G is a sequence of vertices

$$v_1, v_2, \dots, v_k$$

such that $\{v_i, v_{i+1}\}$ is an edge for $i = 1, \dots, k - 1$. The **length** of a walk is the number k of vertices.

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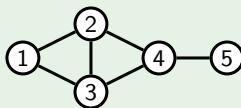
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Example

In our usual example graph, 1234 and 12324 are walks, but 1235 is not.



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Definition (Walk Matrix)

The matrix $\mathbf{W}_k(G)$ is the $n \times k$ matrix whose columns are the first k such vectors, i.e.

$$\mathbf{W}_k(G) = \begin{pmatrix} | & | & | & \dots & | \\ \mathbf{j} & \mathbf{A}\mathbf{j} & \mathbf{A}^2\mathbf{j} & \dots & \mathbf{A}^{k-1}\mathbf{j} \\ | & | & | & & | \end{pmatrix}.$$

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Let G and H be two graphs with the same main eigenvalues and eigenvectors. Then for any k , $\mathbf{W}_k(G) = \mathbf{W}_k(H)$.

Proof.

Suppose G and H have main eigenvalues μ_i , $i = 1 \dots p$ and corresponding main eigenvectors \mathbf{x}_i . Since \mathbf{A}_G and \mathbf{A}_H are real symmetric, then the set of \mathbf{x}_i 's forms a basis for $\text{span}(\mathbb{R}^n \setminus \{\mathbf{j}\}^\perp)$. In particular, we may express \mathbf{j} as

$$\mathbf{j} = \sum_{i=1}^p \beta_i \mathbf{x}_i,$$

where the coefficients β_i are unique.

Proof (continued).

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Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\mathbf{A}_G^{\ell-1}\mathbf{j}$$



Proof (continued).

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$$\mathbf{A}_G^{\ell-1}\mathbf{j} = \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\mathbf{A}_G^{\ell-1}\mathbf{j} = \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\mathbf{A}_G^{\ell-1}\mathbf{j} = \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\mathbf{A}_G^{\ell-1}\mathbf{j} = \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_H^{\ell-1} \mathbf{x}_i$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\begin{aligned}\mathbf{A}_G^{\ell-1}\mathbf{j} &= \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_H^{\ell-1} \mathbf{x}_i \\ &= \mathbf{A}_H^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i\end{aligned}$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\begin{aligned}\mathbf{A}_G^{\ell-1}\mathbf{j} &= \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_H^{\ell-1} \mathbf{x}_i \\ &= \mathbf{A}_H^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \mathbf{A}_H^{\ell-1} \mathbf{j},\end{aligned}$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\begin{aligned}\mathbf{A}_G^{\ell-1}\mathbf{j} &= \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_H^{\ell-1} \mathbf{x}_i \\ &= \mathbf{A}_H^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \mathbf{A}_H^{\ell-1} \mathbf{j},\end{aligned}$$

the ℓ th column of $\mathbf{W}_k(H)$. □

Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\begin{aligned}\mathbf{A}_G^{\ell-1}\mathbf{j} &= \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_H^{\ell-1} \mathbf{x}_i \\ &= \mathbf{A}_H^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \mathbf{A}_H^{\ell-1} \mathbf{j},\end{aligned}$$

the ℓ th column of $\mathbf{W}_k(H)$. □

Corollary

Any two graphs with the same CDC have the same walk matrix.

Thank you!

DEPARTMENT OF MATHEMATICS
Faculty of Science
L-Università ta' Malta



L-Università
ta' Malta

