

Walks and Canonical Double Coverings of Comain Graphs

Luke Collins Irene Sciriha

DEPARTMENT OF MATHEMATICS
Faculty of Science
L-Università ta' Malta

S^3 Annual Science Conference
12th April, 2019



L-Università
ta' Malta



Outline

- 1 Introduction
 - Definition of a Graph
 - Representing Graphs as Matrices
 - Graph Isomorphisms
- 2 Canonical Double Covers
 - Definition of CDC
 - Some Easy Observations about CDC
 - How many such graphs are there?
- 3 Walks
 - The Walk Matrix of a Graph
 - Main Eigenvalues and Eigenvectors

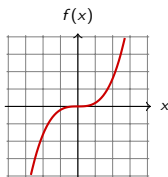
Definition of a Graph

Definition of a Graph

In mathematics, a *graph* is not one of these:

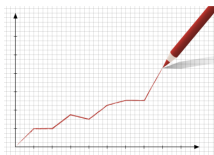
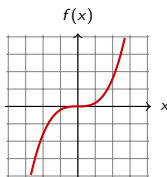
Definition of a Graph

In mathematics, a *graph* is not one of these:



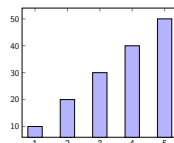
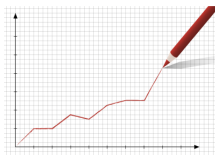
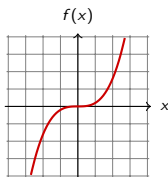
Definition of a Graph

In mathematics, a *graph* is not one of these:



Definition of a Graph

In mathematics, a *graph* is not one of these:



Definition of a Graph

Definition of a Graph

Definition (Graph)

A **graph** G is a pair (V, E) where V is a non-empty finite set, and E is a set of unordered pairs of the elements of V .

Definition of a Graph

Definition (Graph)

A **graph** G is a pair (V, E) where V is a non-empty finite set, and E is a set of unordered pairs of the elements of V .

The elements of the set V are called *vertices*, and the pairs in E are called *edges*.

Definition of a Graph

Definition (Graph)

A **graph** G is a pair (V, E) where V is a non-empty finite set, and E is a set of unordered pairs of the elements of V .

The elements of the set V are called *vertices*, and the pairs in E are called *edges*.

Example

$V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$ define a graph.

Definition of a Graph

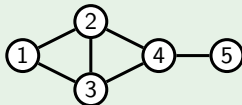
Definition (Graph)

A **graph** G is a pair (V, E) where V is a non-empty finite set, and E is a set of unordered pairs of the elements of V .

The elements of the set V are called *vertices*, and the pairs in E are called *edges*.

Example

$V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$ define a graph.



Definition of a Graph

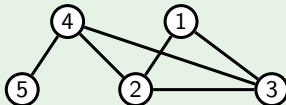
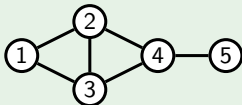
Definition (Graph)

A **graph** G is a pair (V, E) where V is a non-empty finite set, and E is a set of unordered pairs of the elements of V .

The elements of the set V are called *vertices*, and the pairs in E are called *edges*.

Example

$V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$ define a graph.



Definition of a Graph

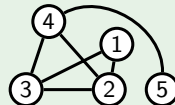
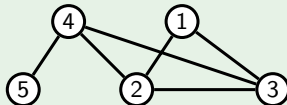
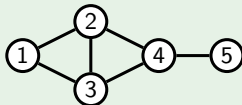
Definition (Graph)

A **graph** G is a pair (V, E) where V is a non-empty finite set, and E is a set of unordered pairs of the elements of V .

The elements of the set V are called *vertices*, and the pairs in E are called *edges*.

Example

$V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$ define a graph.



Representing Graphs as Matrices

Representing Graphs as Matrices

We usually use the letter n for the number of vertices, that is, $n = |V|$.

Representing Graphs as Matrices

We usually use the letter n for the number of vertices, that is, $n = |V|$.

To encode graphs algebraically, we can use an *adjacency matrix*:

Representing Graphs as Matrices

We usually use the letter n for the number of vertices, that is, $n = |V|$.

To encode graphs algebraically, we can use an *adjacency matrix*:

Definition (Adjacency matrix)

The **adjacency matrix** of a graph $G = (V, E)$ is the $n \times n$ matrix (a_{ij}) where

Representing Graphs as Matrices

We usually use the letter n for the number of vertices, that is, $n = |V|$.

To encode graphs algebraically, we can use an *adjacency matrix*:

Definition (Adjacency matrix)

The **adjacency matrix** of a graph $G = (V, E)$ is the $n \times n$ matrix (a_{ij}) where

$$a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j, \text{ i.e. } \{v_i, v_j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

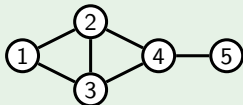
Representing Graphs as Matrices

Example (A simple adjacency matrix)

Representing Graphs as Matrices

Example (A simple adjacency matrix)

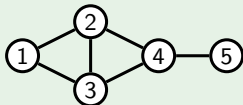
Consider the following graph.



Representing Graphs as Matrices

Example (A simple adjacency matrix)

Consider the following graph. It has the following adjacency matrix.

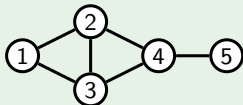


$$\begin{array}{c}
 \begin{matrix} & 1 & 2 & 3 & 4 & 5 \end{matrix} \\
 \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix}
 0 & 1 & 1 & 0 & 0 \\
 1 & 0 & 1 & 1 & 0 \\
 1 & 1 & 0 & 1 & 0 \\
 0 & 1 & 1 & 0 & 1 \\
 0 & 0 & 0 & 1 & 0
 \end{pmatrix}
 \end{array}$$

Representing Graphs as Matrices

Example (A simple adjacency matrix)

Consider the following graph. It has the following adjacency matrix.



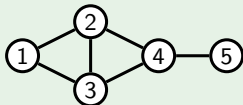
$$\begin{array}{c}
 \begin{matrix} & 1 & 2 & 3 & 4 & 5 \end{matrix} \\
 \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix}
 0 & 1 & 1 & 0 & 0 \\
 1 & 0 & 1 & 1 & 0 \\
 1 & 1 & 0 & 1 & 0 \\
 0 & 1 & 1 & 0 & 1 \\
 0 & 0 & 0 & 1 & 0
 \end{pmatrix}
 \end{array}$$

Note that in general,

Representing Graphs as Matrices

Example (A simple adjacency matrix)

Consider the following graph. It has the following adjacency matrix.



$$\begin{array}{c}
 \begin{matrix} & 1 & 2 & 3 & 4 & 5 \end{matrix} \\
 \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix}
 0 & 1 & 1 & 0 & 0 \\
 1 & 0 & 1 & 1 & 0 \\
 1 & 1 & 0 & 1 & 0 \\
 0 & 1 & 1 & 0 & 1 \\
 0 & 0 & 0 & 1 & 0
 \end{pmatrix}
 \end{array}$$

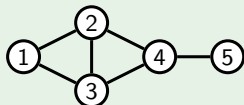
Note that in general,

- The adjacency matrix is symmetric

Representing Graphs as Matrices

Example (A simple adjacency matrix)

Consider the following graph. It has the following adjacency matrix.



$$\begin{array}{c}
 \begin{matrix} & 1 & 2 & 3 & 4 & 5 \end{matrix} \\
 \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix}
 0 & 1 & 1 & 0 & 0 \\
 1 & 0 & 1 & 1 & 0 \\
 1 & 1 & 0 & 1 & 0 \\
 0 & 1 & 1 & 0 & 1 \\
 0 & 0 & 0 & 1 & 0
 \end{pmatrix}
 \end{array}$$

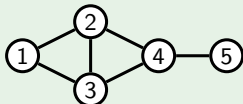
Note that in general,

- The adjacency matrix is symmetric
- Each 1 represents an edge, and each 0 represents a non-edge

Representing Graphs as Matrices

Example (A simple adjacency matrix)

Consider the following graph. It has the following adjacency matrix.



$$\begin{array}{c}
 \begin{matrix} & 1 & 2 & 3 & 4 & 5 \end{matrix} \\
 \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix}
 0 & 1 & 1 & 0 & 0 \\
 1 & 0 & 1 & 1 & 0 \\
 1 & 1 & 0 & 1 & 0 \\
 0 & 1 & 1 & 0 & 1 \\
 0 & 0 & 0 & 1 & 0
 \end{pmatrix}
 \end{array}$$

Note that in general,

- The adjacency matrix is symmetric
- Each 1 represents an edge, and each 0 represents a non-edge
- Each entry on the diagonal is 0, since we consider simple graphs

Representing Graphs as Matrices

Representing Graphs as Matrices

When we use terminology from linear algebra such as

Representing Graphs as Matrices

When we use terminology from linear algebra such as

- *eigenvalues* of a graph,

Representing Graphs as Matrices

When we use terminology from linear algebra such as

- *eigenvalues* of a graph,
- *eigenvectors* of a graph,

Representing Graphs as Matrices

When we use terminology from linear algebra such as

- *eigenvalues* of a graph,
- *eigenvectors* of a graph,
- *eigenspace* of a graph,

Representing Graphs as Matrices

When we use terminology from linear algebra such as

- *eigenvalues* of a graph,
- *eigenvectors* of a graph,
- *eigenspace* of a graph,
- *column space* of a graph,

Representing Graphs as Matrices

When we use terminology from linear algebra such as

- *eigenvalues* of a graph,
- *eigenvectors* of a graph,
- *eigenspace* of a graph,
- *column space* of a graph,

and so on, we are actually referring to the **adjacency matrix** of the graph.


Representing Graphs as Matrices

When we use terminology from linear algebra such as

- *eigenvalues* of a graph,
- *eigenvectors* of a graph,
- *eigenspace* of a graph,
- *column space* of a graph,

and so on, we are actually referring to the **adjacency matrix** of the graph.

Example

The eigenvalues of  are those of its adjacency matrix $\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

Graph Isomorphisms

Graph Isomorphisms

Two graphs are *isomorphic* if one can obtain the other by relabelling.

Graph Isomorphisms

Two graphs are *isomorphic* if one can obtain the other by relabelling.

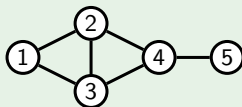
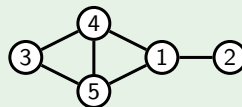
Example



Graph Isomorphisms

Two graphs are *isomorphic* if one can obtain the other by relabelling.

Example


 \cong


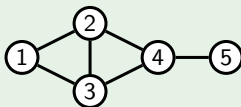
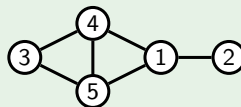
$$\begin{array}{c}
 \begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
 \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
 \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}
 \end{array}
 \end{array}$$

Graph Isomorphisms

Two graphs are *isomorphic* if one can obtain the other by relabelling.

Example


 \approx


$$\begin{array}{c}
 \begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
 \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}
 \end{array}
 \end{array}$$

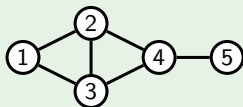
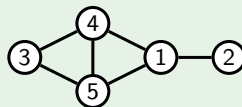
$$\begin{array}{c}
 \begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
 \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}
 \end{array}
 \end{array}$$

Even though we see that they are the same, their adjacency matrices are completely different!

Graph Isomorphisms

Two graphs are *isomorphic* if one can obtain the other by relabelling.

Example


 \approx


$$\begin{array}{c}
 \begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
 \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}
 \end{array}
 \end{array}$$

 $=$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
 \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}
 \end{array}
 \end{array}
 \mathbf{P}$$

Even though we see that they are the same, their adjacency matrices are completely different! ... or are they?

Graph Isomorphisms – Permutation Matrices

A *permutation* matrix \mathbf{P} is a matrix obtained from the identity matrix \mathbf{I} by simply rearranging the columns of \mathbf{I} .

Graph Isomorphisms – Permutation Matrices

A *permutation* matrix \mathbf{P} is a matrix obtained from the identity matrix \mathbf{I} by simply rearranging the columns of \mathbf{I} . Consequently, they are orthogonal:

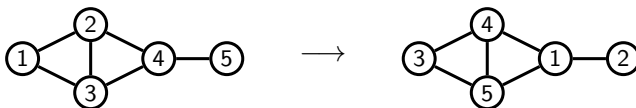
$$\mathbf{P}\mathbf{P}^T = \mathbf{I}.$$

Graph Isomorphisms – Permutation Matrices

A *permutation* matrix \mathbf{P} is a matrix obtained from the identity matrix \mathbf{I} by simply rearranging the columns of \mathbf{I} . Consequently, they are orthogonal:

$$\mathbf{P}\mathbf{P}^T = \mathbf{I}.$$

In the example the relabelling was: $1 \rightarrow 3$, $2 \rightarrow 4$, $3 \rightarrow 5$, $4 \rightarrow 1$, $5 \rightarrow 2$

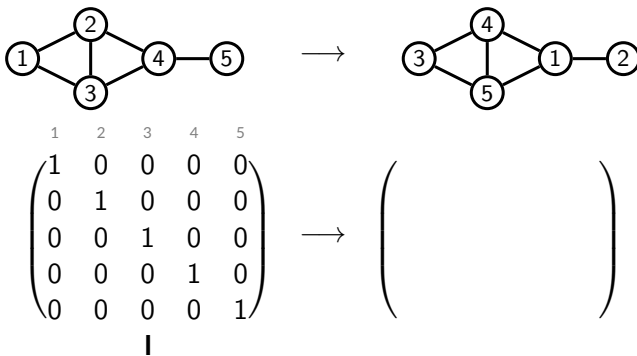


Graph Isomorphisms – Permutation Matrices

A *permutation* matrix \mathbf{P} is a matrix obtained from the identity matrix \mathbf{I} by simply rearranging the columns of \mathbf{I} . Consequently, they are orthogonal:

$$\mathbf{P}\mathbf{P}^T = \mathbf{I}.$$

In the example the relabelling was: $1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 5, 4 \rightarrow 1, 5 \rightarrow 2$

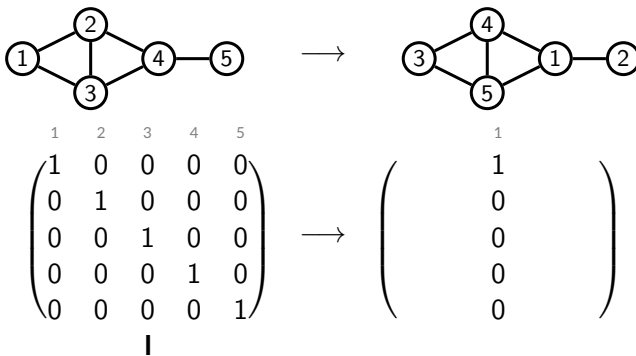


Graph Isomorphisms – Permutation Matrices

A *permutation* matrix \mathbf{P} is a matrix obtained from the identity matrix \mathbf{I} by simply rearranging the columns of \mathbf{I} . Consequently, they are orthogonal:

$$\mathbf{P}\mathbf{P}^T = \mathbf{I}.$$

In the example the relabelling was: $1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 5, 4 \rightarrow 1, 5 \rightarrow 2$

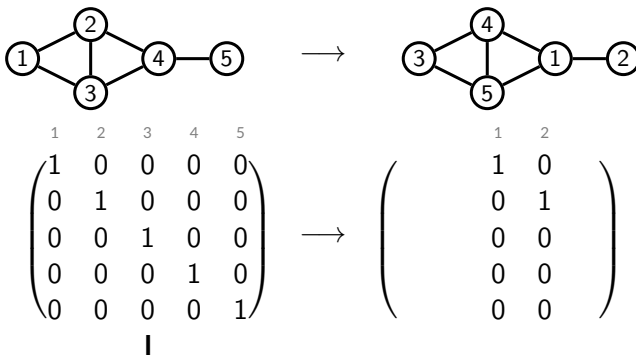


Graph Isomorphisms – Permutation Matrices

A *permutation* matrix \mathbf{P} is a matrix obtained from the identity matrix \mathbf{I} by simply rearranging the columns of \mathbf{I} . Consequently, they are orthogonal:

$$\mathbf{P}\mathbf{P}^T = \mathbf{I}.$$

In the example the relabelling was: $1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 5, 4 \rightarrow 1, 5 \rightarrow 2$

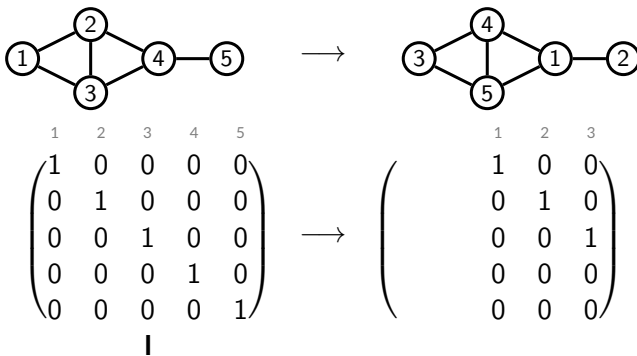


Graph Isomorphisms – Permutation Matrices

A *permutation* matrix \mathbf{P} is a matrix obtained from the identity matrix \mathbf{I} by simply rearranging the columns of \mathbf{I} . Consequently, they are orthogonal:

$$\mathbf{P}\mathbf{P}^T = \mathbf{I}.$$

In the example the relabelling was: $1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 5, 4 \rightarrow 1, 5 \rightarrow 2$

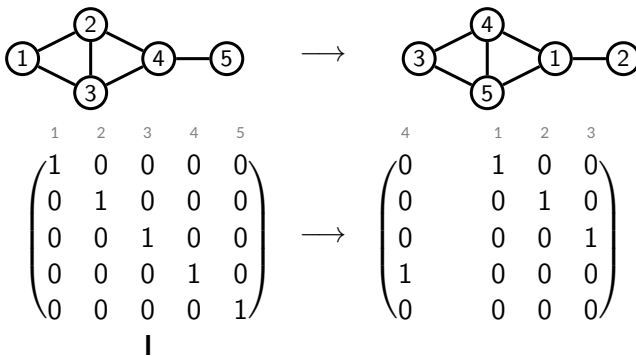


Graph Isomorphisms – Permutation Matrices

A *permutation* matrix \mathbf{P} is a matrix obtained from the identity matrix \mathbf{I} by simply rearranging the columns of \mathbf{I} . Consequently, they are orthogonal:

$$\mathbf{P}\mathbf{P}^T = \mathbf{I}.$$

In the example the relabelling was: $1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 5, 4 \rightarrow 1, 5 \rightarrow 2$

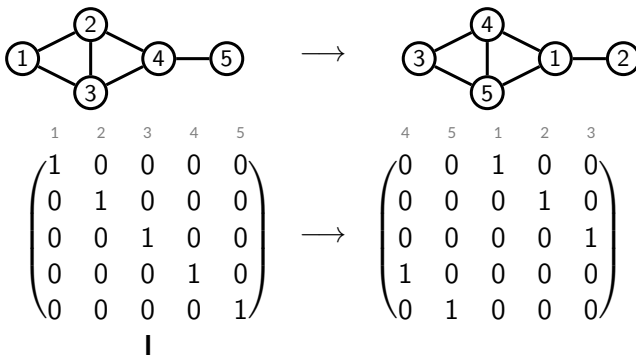


Graph Isomorphisms – Permutation Matrices

A *permutation* matrix \mathbf{P} is a matrix obtained from the identity matrix \mathbf{I} by simply rearranging the columns of \mathbf{I} . Consequently, they are orthogonal:

$$\mathbf{P}\mathbf{P}^T = \mathbf{I}.$$

In the example the relabelling was: $1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 5, 4 \rightarrow 1, 5 \rightarrow 2$

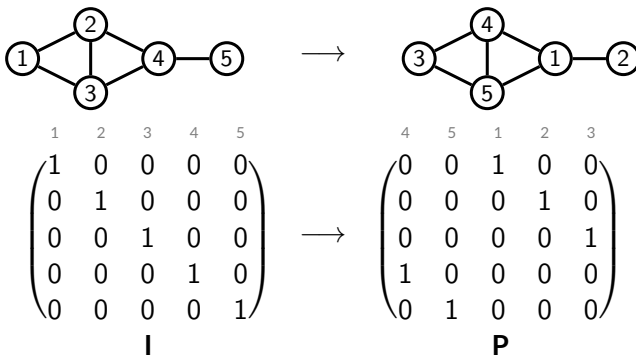


Graph Isomorphisms – Permutation Matrices

A *permutation* matrix \mathbf{P} is a matrix obtained from the identity matrix \mathbf{I} by simply rearranging the columns of \mathbf{I} . Consequently, they are orthogonal:

$$\mathbf{P}\mathbf{P}^T = \mathbf{I}.$$

In the example the relabelling was: $1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 5, 4 \rightarrow 1, 5 \rightarrow 2$



Graph Isomorphisms – Permutation Matrices

Doing $\mathbf{A} \rightarrow \mathbf{P}^T \mathbf{A} \mathbf{P}$ will relabel the vertices of the graph of \mathbf{A} according to the corresponding permutation of \mathbf{P} .

Graph Isomorphisms – Permutation Matrices

Doing $\mathbf{A} \longrightarrow \mathbf{P}^T \mathbf{A} \mathbf{P}$ will relabel the vertices of the graph of \mathbf{A} according to the corresponding permutation of \mathbf{P} .

Moreover, since $\mathbf{P}^T = \mathbf{P}^{-1}$, we are actually doing $\mathbf{A} \longrightarrow \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$. This means that the matrix \mathbf{A} and the resulting new adjacency matrix are *similar*.

Graph Isomorphisms – Permutation Matrices

Doing $\mathbf{A} \rightarrow \mathbf{P}^T \mathbf{A} \mathbf{P}$ will relabel the vertices of the graph of \mathbf{A} according to the corresponding permutation of \mathbf{P} .

Moreover, since $\mathbf{P}^T = \mathbf{P}^{-1}$, we are actually doing $\mathbf{A} \rightarrow \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$. This means that the matrix \mathbf{A} and the resulting new adjacency matrix are *similar*.

Similar matrices have the same:

Graph Isomorphisms – Permutation Matrices

Doing $\mathbf{A} \longrightarrow \mathbf{P}^T \mathbf{A} \mathbf{P}$ will relabel the vertices of the graph of \mathbf{A} according to the corresponding permutation of \mathbf{P} .

Moreover, since $\mathbf{P}^T = \mathbf{P}^{-1}$, we are actually doing $\mathbf{A} \longrightarrow \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$. This means that the matrix \mathbf{A} and the resulting new adjacency matrix are *similar*.

Similar matrices have the same:

- eigenvalues

Graph Isomorphisms – Permutation Matrices

Doing $\mathbf{A} \rightarrow \mathbf{P}^T \mathbf{A} \mathbf{P}$ will relabel the vertices of the graph of \mathbf{A} according to the corresponding permutation of \mathbf{P} .

Moreover, since $\mathbf{P}^T = \mathbf{P}^{-1}$, we are actually doing $\mathbf{A} \rightarrow \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$. This means that the matrix \mathbf{A} and the resulting new adjacency matrix are *similar*.

Similar matrices have the same:

- eigenvalues
- determinant

Graph Isomorphisms – Permutation Matrices

Doing $\mathbf{A} \rightarrow \mathbf{P}^T \mathbf{A} \mathbf{P}$ will relabel the vertices of the graph of \mathbf{A} according to the corresponding permutation of \mathbf{P} .

Moreover, since $\mathbf{P}^T = \mathbf{P}^{-1}$, we are actually doing $\mathbf{A} \rightarrow \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$. This means that the matrix \mathbf{A} and the resulting new adjacency matrix are *similar*.

Similar matrices have the same:

- eigenvalues
- determinant
- rank

Graph Isomorphisms – Permutation Matrices

Doing $\mathbf{A} \rightarrow \mathbf{P}^T \mathbf{A} \mathbf{P}$ will relabel the vertices of the graph of \mathbf{A} according to the corresponding permutation of \mathbf{P} .

Moreover, since $\mathbf{P}^T = \mathbf{P}^{-1}$, we are actually doing $\mathbf{A} \rightarrow \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$. This means that the matrix \mathbf{A} and the resulting new adjacency matrix are *similar*.

Similar matrices have the same:

- eigenvalues
- determinant
- rank
- characteristic polynomial

Graph Isomorphisms – Permutation Matrices

Doing $\mathbf{A} \longrightarrow \mathbf{P}^T \mathbf{A} \mathbf{P}$ will relabel the vertices of the graph of \mathbf{A} according to the corresponding permutation of \mathbf{P} .

Moreover, since $\mathbf{P}^T = \mathbf{P}^{-1}$, we are actually doing $\mathbf{A} \longrightarrow \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$. This means that the matrix \mathbf{A} and the resulting new adjacency matrix are *similar*.

Similar matrices have the same:

- eigenvalues
- determinant
- rank
- characteristic polynomial
- minimum polynomial

Graph Isomorphisms – Permutation Matrices

Doing $\mathbf{A} \rightarrow \mathbf{P}^T \mathbf{A} \mathbf{P}$ will relabel the vertices of the graph of \mathbf{A} according to the corresponding permutation of \mathbf{P} .

Moreover, since $\mathbf{P}^T = \mathbf{P}^{-1}$, we are actually doing $\mathbf{A} \rightarrow \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$. This means that the matrix \mathbf{A} and the resulting new adjacency matrix are *similar*.

Similar matrices have the same:

- eigenvalues
- determinant
- rank
- characteristic polynomial
- minimum polynomial

In most cases, the labelling of the vertices in a graph is not important.

Definition of $\text{CDC}(G)$

Definition of $\text{CDC}(G)$

Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\text{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

Definition of $\text{CDC}(G)$

Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\text{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

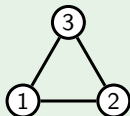
Example $\text{CDC}(K_3) = C_6$

Definition of $\text{CDC}(G)$

Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\text{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

Example $(\text{CDC}(K_3) = C_6)$

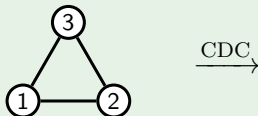


Definition of $\text{CDC}(G)$

Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\text{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

Example ($\text{CDC}(K_3) = C_6$)



Definition of $\text{CDC}(G)$

Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\text{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

Example $(\text{CDC}(K_3) = C_6)$

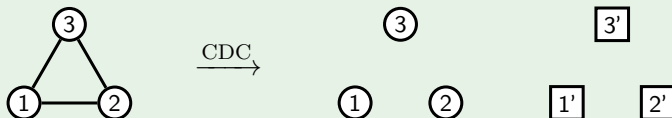


Definition of $\text{CDC}(G)$

Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\text{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

Example ($\text{CDC}(K_3) = C_6$)

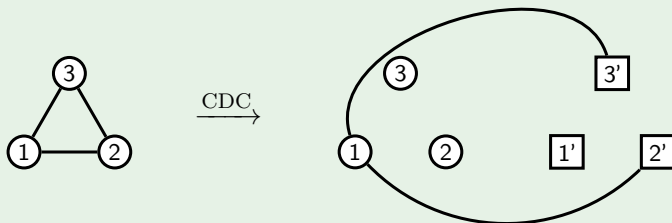


Definition of $\text{CDC}(G)$

Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\text{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

Example ($\text{CDC}(K_3) = C_6$)

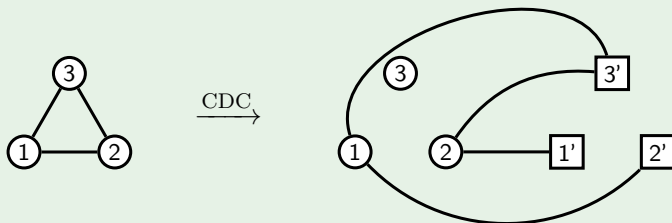


Definition of $\text{CDC}(G)$

Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\text{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

Example ($\text{CDC}(K_3) = C_6$)

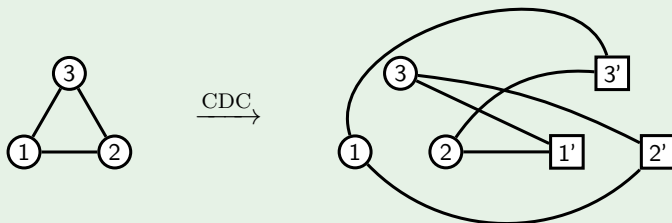


Definition of $\text{CDC}(G)$

Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\text{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

Example ($\text{CDC}(K_3) = C_6$)

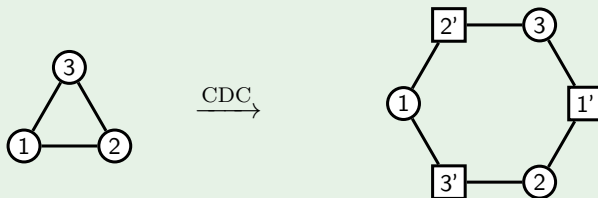


Definition of $\text{CDC}(G)$

Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\text{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

Example ($\text{CDC}(K_3) = C_6$)

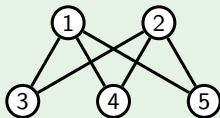


Definition of $\text{CDC}(G)$ – Another example

Example $\text{CDC}(K_{2,3}) = K_{2,3} \dot{\cup} K_{2,3}$

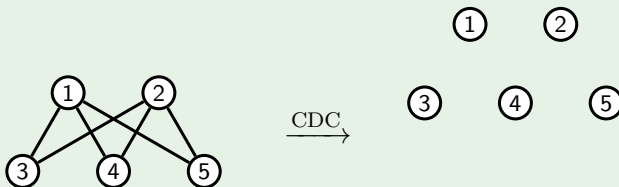
Definition of $\text{CDC}(G)$ – Another example

Example $\text{CDC}(K_{2,3}) = K_{2,3} \dot{\cup} K_{2,3}$



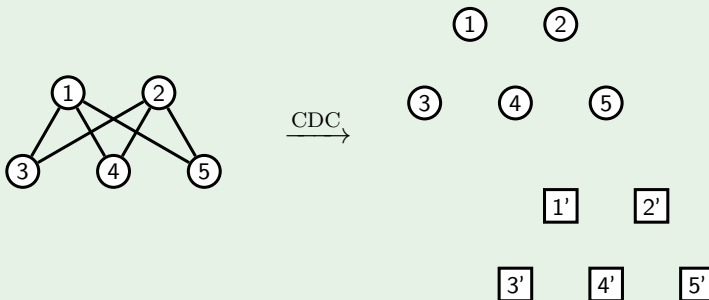
Definition of $\text{CDC}(G)$ – Another example

Example $\text{CDC}(K_{2,3}) = K_{2,3} \dot{\cup} K_{2,3}$



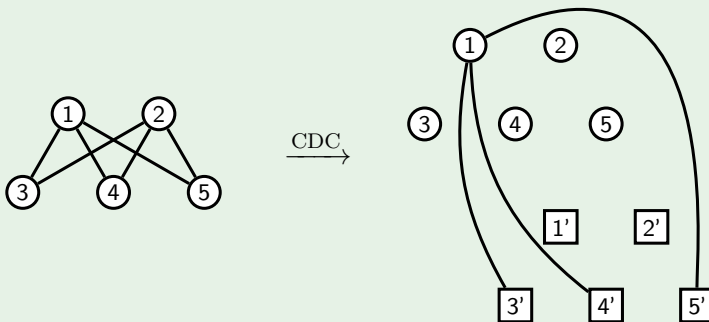
Definition of $\text{CDC}(G)$ – Another example

Example $\text{CDC}(K_{2,3}) = K_{2,3} \dot{\cup} K_{2,3}$



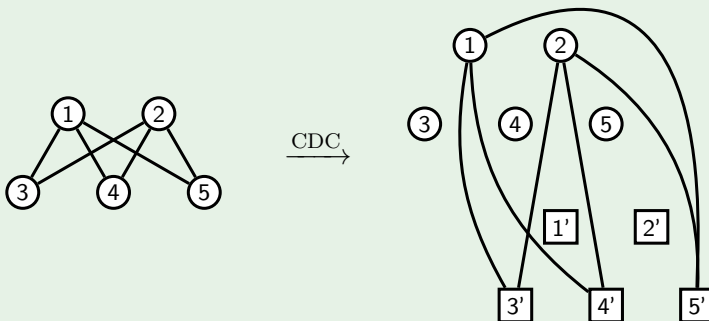
Definition of $\text{CDC}(G)$ – Another example

Example $\text{CDC}(K_{2,3}) = K_{2,3} \dot{\cup} K_{2,3}$



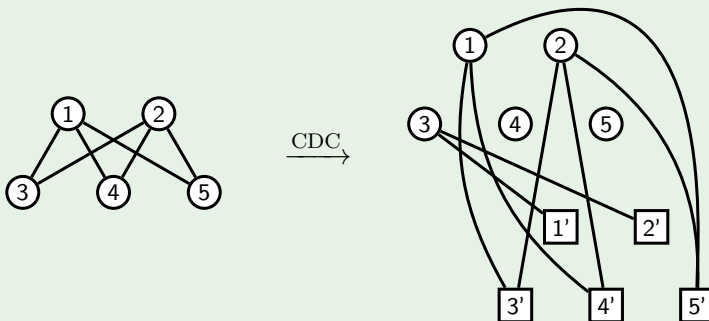
Definition of $\text{CDC}(G)$ – Another example

Example $\text{CDC}(K_{2,3}) = K_{2,3} \dot{\cup} K_{2,3}$



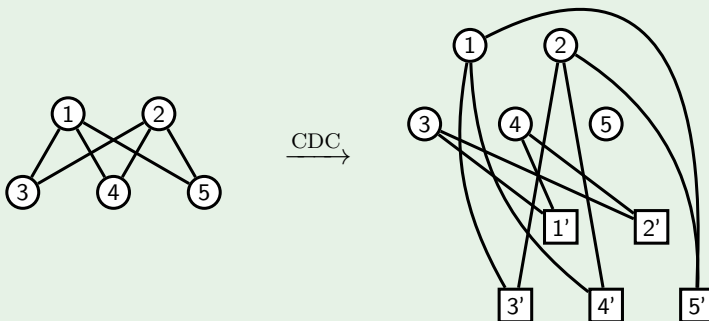
Definition of $\text{CDC}(G)$ – Another example

Example $\text{CDC}(K_{2,3}) = K_{2,3} \dot{\cup} K_{2,3}$



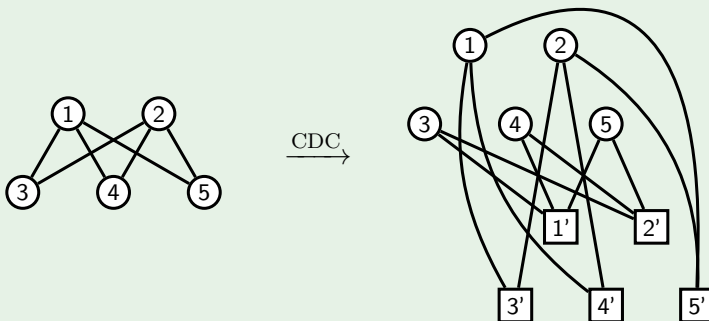
Definition of $\text{CDC}(G)$ – Another example

Example $\text{CDC}(K_{2,3}) = K_{2,3} \dot{\cup} K_{2,3}$



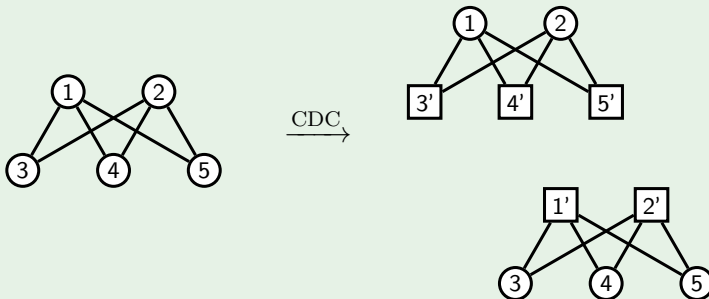
Definition of $\text{CDC}(G)$ – Another example

Example $\text{CDC}(K_{2,3}) = K_{2,3} \dot{\cup} K_{2,3}$



Definition of $CDC(G)$ – Another example

Example $(CDC(K_{2,3}) = K_{2,3} \dot{\cup} K_{2,3})$



Some Easy Observations about CDCs

Some Easy Observations about CDCs

- If G has adjacency matrix \mathbf{A} , then the adjacency matrix of $\text{CDC}(G)$ is given by

$$\begin{matrix} & 1 \dots n & 1' \dots n' \\ \begin{matrix} 1 \\ \vdots \\ n \\ 1' \\ \vdots \\ n' \end{matrix} & \left(\begin{array}{cc} \mathbf{O} & \mathbf{A} \\ \mathbf{A} & \mathbf{O} \end{array} \right) \end{matrix}$$

Some Easy Observations about CDCs

- If G has adjacency matrix \mathbf{A} , then the adjacency matrix of $\text{CDC}(G)$ is given by

$$\begin{matrix} & 1 \dots n & 1' \dots n' \\ \begin{matrix} 1 \\ \vdots \\ n \\ 1' \\ \vdots \\ n' \end{matrix} & \left(\begin{array}{cc} \mathbf{O} & \mathbf{A} \\ \mathbf{A} & \mathbf{O} \end{array} \right) \end{matrix}$$

- Let G be a connected graph. Then

$\text{CDC}(G)$ connected $\iff G$ has an odd cycle,

i.e. if G is not bipartite. Moreover, G bipartite $\implies \text{CDC}(G) = G \dot{\cup} G$.

Some Easy Observations about CDCs

- If G has adjacency matrix \mathbf{A} , then the adjacency matrix of $\text{CDC}(G)$ is given by

$$\begin{matrix} & 1 \dots n & 1' \dots n' \\ \begin{matrix} 1 \\ \vdots \\ n \\ 1' \\ \vdots \\ n' \end{matrix} & \left(\begin{array}{cc} \mathbf{O} & \mathbf{A} \\ \mathbf{A} & \mathbf{O} \end{array} \right) \end{matrix}$$

- Let G be a connected graph. Then

$$\text{CDC}(G) \text{ connected} \iff G \text{ has an odd cycle,}$$

i.e. if G is not bipartite. Moreover, G bipartite $\implies \text{CDC}(G) = G \dot{\cup} G$.

- Let $G = G_1 \dot{\cup} G_2 \dot{\cup} \dots \dot{\cup} G_k$. Then

$$\text{CDC}(G) = \text{CDC}(G_1) \dot{\cup} \text{CDC}(G_2) \dot{\cup} \dots \dot{\cup} \text{CDC}(G_k).$$

Some Easy Observations about CDCs

- If $\text{CDC}(G) \simeq \text{CDC}(H)$ and G is connected, is H necessarily connected?

Some Easy Observations about CDCs

- If $\text{CDC}(G) \simeq \text{CDC}(H)$ and G is connected, is H necessarily connected? Answer: **No** since

$$\text{CDC}(\text{C}_6) = \text{C}_6 \dot{\cup} \text{C}_6,$$

and

$$\text{CDC}(\text{C}_3 \dot{\cup} \text{C}_3) = \text{CDC}(\text{C}_3) \dot{\cup} \text{CDC}(\text{C}_3) = \text{C}_6 \dot{\cup} \text{C}_6$$

by the previous result about CDCs of graph unions.

Some Easy Observations about CDCs

- If $\text{CDC}(G) \simeq \text{CDC}(H)$ and G is connected, is H necessarily connected? Answer: **No** since

$$\text{CDC}(C_6) = C_6 \dot{\cup} C_6,$$

and

$$\text{CDC}(C_3 \dot{\cup} C_3) = \text{CDC}(C_3) \dot{\cup} \text{CDC}(C_3) = C_6 \dot{\cup} C_6$$

by the previous result about CDCs of graph unions.

- **But** if $\text{CDC}(G) \simeq \text{CDC}(H)$ and G has an isolated vertex, then H must have an isolated vertex as well; i.e.

$$\text{CDC}(G) \simeq \text{CDC}(H) \iff \text{CDC}(G \dot{\cup} \circ) \simeq \text{CDC}(H \dot{\cup} \circ)$$

Some Easy Observations about CDCs

- If $\text{CDC}(G) \simeq \text{CDC}(H)$ and G is connected, is H necessarily connected? Answer: **No** since

$$\text{CDC}(C_6) = C_6 \dot{\cup} C_6,$$

and

$$\text{CDC}(C_3 \dot{\cup} C_3) = \text{CDC}(C_3) \dot{\cup} \text{CDC}(C_3) = C_6 \dot{\cup} C_6$$

by the previous result about CDCs of graph unions.

- **But** if $\text{CDC}(G) \simeq \text{CDC}(H)$ and G has an isolated vertex, then H must have an isolated vertex as well; i.e.

$$\text{CDC}(G) \simeq \text{CDC}(H) \iff \text{CDC}(G \dot{\cup} \circ) \simeq \text{CDC}(H \dot{\cup} \circ)$$

This yielded a useful *proof technique*: If we have $\text{CDC}(G) \simeq \text{CDC}(H)$ where G has no isolated vertex, we show that the negation of what we want to prove introduces an isolated vertex in H .

TF-Isomorphisms

TF-Isomorphisms

Let us illustrate this proof technique with an important result.

TF-Isomorphisms

Let us illustrate this proof technique with an important result.

Recall that two graphs G and H with adjacency matrices \mathbf{A}_G and \mathbf{A}_H are isomorphic if and only if there exists a permutation matrix such that

$$\mathbf{A}_G = \mathbf{P}^T \mathbf{A}_H \mathbf{P}.$$

TF-Isomorphisms

Let us illustrate this proof technique with an important result.

Recall that two graphs G and H with adjacency matrices \mathbf{A}_G and \mathbf{A}_H are isomorphic if and only if there exists a permutation matrix such that

$$\mathbf{A}_G = \mathbf{P}^T \mathbf{A}_H \mathbf{P}.$$

If we weaken this relationship, we get the following:

TF-Isomorphisms

Let us illustrate this proof technique with an important result.

Recall that two graphs G and H with adjacency matrices \mathbf{A}_G and \mathbf{A}_H are isomorphic if and only if there exists a permutation matrix such that

$$\mathbf{A}_G = \mathbf{P}^T \mathbf{A}_H \mathbf{P}.$$

If we weaken this relationship, we get the following:

Definition (Two-Fold Isomorphism)

Let G and H be two graphs with adjacency matrices \mathbf{A}_G and \mathbf{A}_H . We say that G is *two-fold isomorphic* or *TF-isomorphic* to H if

$$\mathbf{A}_G = \mathbf{R} \mathbf{A}_H \mathbf{Q}$$

for some permutation matrices \mathbf{R}, \mathbf{Q} .

TF-Isomorphisms

Let us illustrate this proof technique with an important result.

Recall that two graphs G and H with adjacency matrices \mathbf{A}_G and \mathbf{A}_H are isomorphic if and only if there exists a permutation matrix such that

$$\mathbf{A}_G = \mathbf{P}^T \mathbf{A}_H \mathbf{P}.$$

If we weaken this relationship, we get the following:

Definition (Two-Fold Isomorphism)

Let G and H be two graphs with adjacency matrices \mathbf{A}_G and \mathbf{A}_H . We say that G is *two-fold isomorphic* or *TF-isomorphic* to H if

$$\mathbf{A}_G = \mathbf{R} \mathbf{A}_H \mathbf{Q}$$

for some permutation matrices \mathbf{R}, \mathbf{Q} .

Here \mathbf{R} and \mathbf{Q} could be any permutation matrices, they don't have to be the inverse (i.e. transpose) of each other.

Theorem

Let **G** and **H** be two graphs. Then

$$\text{CDC}(G) \simeq \text{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Theorem

Let **G** and **H** be two graphs. Then

$$\text{CDC}(G) \simeq \text{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow)

Theorem

Let **G** and **H** be two graphs. Then

$$\text{CDC}(G) \simeq \text{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If G and H are TF-isomorphic, then by definition there are permutation matrices **R**, **Q** such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$.

Theorem

Let **G** and **H** be two graphs. Then

$$\text{CDC}(G) \simeq \text{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If G and H are TF-isomorphic, then by definition there are permutation matrices **R**, **Q** such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

Theorem

Let \mathbf{G} and \mathbf{H} be two graphs. Then

$$\text{CDC}(\mathbf{G}) \simeq \text{CDC}(\mathbf{H}) \iff \mathbf{G} \text{ and } \mathbf{H} \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If \mathbf{G} and \mathbf{H} are TF-isomorphic, then by definition there are permutation matrices \mathbf{R}, \mathbf{Q} such that $\mathbf{A}_{\mathbf{G}} = \mathbf{R}\mathbf{A}_{\mathbf{H}}\mathbf{Q}$. Then

$$\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^T & \mathbf{O} \end{pmatrix}$$

Theorem

Let \mathbf{G} and \mathbf{H} be two graphs. Then

$$\text{CDC}(\mathbf{G}) \simeq \text{CDC}(\mathbf{H}) \iff \mathbf{G} \text{ and } \mathbf{H} \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If \mathbf{G} and \mathbf{H} are TF-isomorphic, then by definition there are permutation matrices \mathbf{R}, \mathbf{Q} such that $\mathbf{A}_{\mathbf{G}} = \mathbf{R}\mathbf{A}_{\mathbf{H}}\mathbf{Q}$. Then

$$\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^T & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{A}_{\mathbf{H}} \\ \mathbf{A}_{\mathbf{H}} & \mathbf{O} \end{pmatrix}$$

Theorem

Let **G** and **H** be two graphs. Then

$$\text{CDC}(G) \simeq \text{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If G and H are TF-isomorphic, then by definition there are permutation matrices **R**, **Q** such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

$$\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^T & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^T & \mathbf{O} \end{pmatrix}$$

Theorem

Let \mathbf{G} and \mathbf{H} be two graphs. Then

$$\text{CDC}(\mathbf{G}) \simeq \text{CDC}(\mathbf{H}) \iff \mathbf{G} \text{ and } \mathbf{H} \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If \mathbf{G} and \mathbf{H} are TF-isomorphic, then by definition there are permutation matrices \mathbf{R}, \mathbf{Q} such that $\mathbf{A}_{\mathbf{G}} = \mathbf{R}\mathbf{A}_{\mathbf{H}}\mathbf{Q}$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^T & \mathbf{O} \end{pmatrix}}_{:= \mathbf{P}} \begin{pmatrix} \mathbf{O} & \mathbf{A}_{\mathbf{H}} \\ \mathbf{A}_{\mathbf{H}} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^T & \mathbf{O} \end{pmatrix}$$

Theorem

Let \mathbf{G} and \mathbf{H} be two graphs. Then

$$\text{CDC}(\mathbf{G}) \simeq \text{CDC}(\mathbf{H}) \iff \mathbf{G} \text{ and } \mathbf{H} \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If \mathbf{G} and \mathbf{H} are TF-isomorphic, then by definition there are permutation matrices \mathbf{R}, \mathbf{Q} such that $\mathbf{A}_{\mathbf{G}} = \mathbf{R}\mathbf{A}_{\mathbf{H}}\mathbf{Q}$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^T & \mathbf{O} \end{pmatrix}}_{:= \mathbf{P}} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{A}_{\mathbf{H}} \\ \mathbf{A}_{\mathbf{H}} & \mathbf{O} \end{pmatrix}}_{\text{CDC}(\mathbf{H})} \begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^T & \mathbf{O} \end{pmatrix}$$

Theorem

Let \mathbf{G} and \mathbf{H} be two graphs. Then

$$\text{CDC}(\mathbf{G}) \simeq \text{CDC}(\mathbf{H}) \iff \mathbf{G} \text{ and } \mathbf{H} \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If \mathbf{G} and \mathbf{H} are TF-isomorphic, then by definition there are permutation matrices \mathbf{R}, \mathbf{Q} such that $\mathbf{A}_{\mathbf{G}} = \mathbf{R}\mathbf{A}_{\mathbf{H}}\mathbf{Q}$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^T & \mathbf{O} \end{pmatrix}}_{:= \mathbf{P}} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}}_{\text{CDC}(\mathbf{H})} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^T & \mathbf{O} \end{pmatrix}}_{\mathbf{P}^T}$$

Theorem

Let \mathbf{G} and \mathbf{H} be two graphs. Then

$$\text{CDC}(\mathbf{G}) \simeq \text{CDC}(\mathbf{H}) \iff \mathbf{G} \text{ and } \mathbf{H} \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If \mathbf{G} and \mathbf{H} are TF-isomorphic, then by definition there are permutation matrices \mathbf{R}, \mathbf{Q} such that $\mathbf{A}_{\mathbf{G}} = \mathbf{R}\mathbf{A}_{\mathbf{H}}\mathbf{Q}$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^T & \mathbf{O} \end{pmatrix}}_{:= \mathbf{P}} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}}_{\text{CDC}(\mathbf{H})} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^T & \mathbf{O} \end{pmatrix}}_{\mathbf{P}^T} = \begin{pmatrix} \mathbf{O} & \mathbf{R}\mathbf{A}_H\mathbf{Q} \\ (\mathbf{R}\mathbf{A}_H\mathbf{Q})^T & \mathbf{O} \end{pmatrix}$$

Theorem

Let G and H be two graphs. Then

$$\text{CDC}(G) \simeq \text{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If G and H are TF-isomorphic, then by definition there are permutation matrices R, Q such that $A_G = RA_HQ$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & R \\ Q^T & \mathbf{O} \end{pmatrix}}_{:=P} \underbrace{\begin{pmatrix} \mathbf{O} & A_H \\ A_H & \mathbf{O} \end{pmatrix}}_{\text{CDC}(H)} \underbrace{\begin{pmatrix} \mathbf{O} & Q \\ R^T & \mathbf{O} \end{pmatrix}}_{P^T} = \begin{pmatrix} \mathbf{O} & RA_HQ \\ (RA_HQ)^T & \mathbf{O} \end{pmatrix} \\ = \begin{pmatrix} \mathbf{O} & A_G \\ A_G & \mathbf{O} \end{pmatrix}$$

Theorem

Let G and H be two graphs. Then

$$\text{CDC}(G) \simeq \text{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If G and H are TF-isomorphic, then by definition there are permutation matrices R, Q such that $A_G = RA_HQ$. Then

$$\underbrace{\begin{pmatrix} O & R \\ Q^T & O \end{pmatrix}}_{:=P} \underbrace{\begin{pmatrix} O & A_H \\ A_H & O \end{pmatrix}}_{\text{CDC}(H)} \underbrace{\begin{pmatrix} O & Q \\ R^T & O \end{pmatrix}}_{P^T} = \begin{pmatrix} O & RA_HQ \\ (RA_HQ)^T & O \end{pmatrix} \\ = \underbrace{\begin{pmatrix} O & A_G \\ A_G & O \end{pmatrix}}_{\text{CDC}(G)}$$

Theorem

Let G and H be two graphs. Then

$$\text{CDC}(G) \simeq \text{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Proof.

(\Leftarrow) If G and H are TF-isomorphic, then by definition there are permutation matrices R, Q such that $A_G = RA_HQ$. Then

$$\underbrace{\begin{pmatrix} O & R \\ Q^T & O \end{pmatrix}}_{:= P} \underbrace{\begin{pmatrix} O & A_H \\ A_H & O \end{pmatrix}}_{\text{CDC}(H)} \underbrace{\begin{pmatrix} O & Q \\ R^T & O \end{pmatrix}}_{P^T} = \begin{pmatrix} O & RA_HQ \\ (RA_HQ)^T & O \end{pmatrix} \\ = \underbrace{\begin{pmatrix} O & A_G \\ A_G & O \end{pmatrix}}_{\text{CDC}(G)}$$

so $\text{CDC}(H) \simeq \text{CDC}(G)$.

Proof (continued).

(\Rightarrow)

Proof (continued).

(\implies) Suppose $\text{CDC}(G) \simeq \text{CDC}(H)$.

Proof (continued).

(\implies) Suppose $\text{CDC}(G) \simeq \text{CDC}(H)$. We can assume that both G and H have no isolated vertices, because if they do, we can pair them off.

Proof (continued).

(\implies) Suppose $\text{CDC}(G) \simeq \text{CDC}(H)$. We can assume that both G and H have no isolated vertices, because if they do, we can pair them off. Now since $\text{CDC}(G) \simeq \text{CDC}(H)$, there exists a permutation matrix \mathbf{P} such that

$$\mathbf{P}^T \begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix} \mathbf{P} = \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}$$

Proof (continued).

(\Rightarrow) Suppose $\text{CDC}(G) \simeq \text{CDC}(H)$. We can assume that both G and H have no isolated vertices, because if they do, we can pair them off. Now since $\text{CDC}(G) \simeq \text{CDC}(H)$, there exists a permutation matrix \mathbf{P} such that

$$\begin{aligned} \mathbf{P}^T \begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix} \mathbf{P} &= \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix} \\ \Rightarrow \begin{pmatrix} \mathbf{P}_{11}^T & \mathbf{P}_{21}^T \\ \mathbf{P}_{12}^T & \mathbf{P}_{22}^T \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} &= \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix} \end{aligned}$$

Proof (continued).

(\implies) Suppose $\text{CDC}(G) \simeq \text{CDC}(H)$. We can assume that both G and H have no isolated vertices, because if they do, we can pair them off. Now since $\text{CDC}(G) \simeq \text{CDC}(H)$, there exists a permutation matrix \mathbf{P} such that

$$\begin{aligned} \mathbf{P}^T \begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix} \mathbf{P} &= \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix} \\ \implies \begin{pmatrix} \mathbf{P}_{11}^T & \mathbf{P}_{21}^T \\ \mathbf{P}_{12}^T & \mathbf{P}_{22}^T \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} &= \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix} \end{aligned}$$

Multiplying out and comparing entries, we get that

Proof (continued).

(\implies) Suppose $\text{CDC}(G) \simeq \text{CDC}(H)$. We can assume that both G and H have no isolated vertices, because if they do, we can pair them off. Now since $\text{CDC}(G) \simeq \text{CDC}(H)$, there exists a permutation matrix \mathbf{P} such that

$$\begin{aligned} \mathbf{P}^T \begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix} \mathbf{P} &= \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix} \\ \implies \begin{pmatrix} \mathbf{P}_{11}^T & \mathbf{P}_{21}^T \\ \mathbf{P}_{12}^T & \mathbf{P}_{22}^T \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} &= \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix} \end{aligned}$$

Multiplying out and comparing entries, we get that

$$\mathbf{P}_{21}^T \mathbf{A}_G \mathbf{P}_{12} + \mathbf{P}_{11}^T \mathbf{A}_G \mathbf{P}_{22} = \mathbf{A}_H \quad (1)$$

$$\mathbf{P}_{21}^T \mathbf{A}_G \mathbf{P}_{11} = \mathbf{P}_{12}^T \mathbf{A}_G \mathbf{P}_{22} = \mathbf{O} \quad (2)$$

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^T$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$.

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^T$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$. Using the obtained equations (1) and (2), we can expand $\mathbf{QA}_G\mathbf{R}$ to get

$$\mathbf{QA}_G\mathbf{R} = \mathbf{A}_H. \quad (3)$$

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^T$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$. Using the obtained equations (1) and (2), we can expand $\mathbf{QA}_G\mathbf{R}$ to get

$$\mathbf{QA}_G\mathbf{R} = \mathbf{A}_H. \quad (3)$$

But are \mathbf{P} and \mathbf{R} permutation matrices?

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^T$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$. Using the obtained equations (1) and (2), we can expand $\mathbf{QA}_G\mathbf{R}$ to get

$$\mathbf{QA}_G\mathbf{R} = \mathbf{A}_H. \quad (3)$$

But are \mathbf{P} and \mathbf{R} permutation matrices? Suppose not.

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^T$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$. Using the obtained equations (1) and (2), we can expand $\mathbf{QA}_G\mathbf{R}$ to get

$$\mathbf{QA}_G\mathbf{R} = \mathbf{A}_H. \quad (3)$$

But are \mathbf{P} and \mathbf{R} permutation matrices? Suppose not. Being the sum of two submatrices of \mathbf{P} , this can only happen if a row (and column) are zero.

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^T$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$. Using the obtained equations (1) and (2), we can expand $\mathbf{QA}_G\mathbf{R}$ to get

$$\mathbf{QA}_G\mathbf{R} = \mathbf{A}_H. \quad (3)$$

But are \mathbf{P} and \mathbf{R} permutation matrices? Suppose not. Being the sum of two submatrices of \mathbf{P} , this can only happen if a row (and column) are zero. But by (3) above, \mathbf{A}_H will have a row of zeros.

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^T$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$. Using the obtained equations (1) and (2), we can expand $\mathbf{QA}_G\mathbf{R}$ to get

$$\mathbf{QA}_G\mathbf{R} = \mathbf{A}_H. \quad (3)$$

But are \mathbf{P} and \mathbf{R} permutation matrices? Suppose not. Being the sum of two submatrices of \mathbf{P} , this can only happen if a row (and column) are zero. But by (3) above, \mathbf{A}_H will have a row of zeros. This corresponds to an isolated vertex in H

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^T$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$. Using the obtained equations (1) and (2), we can expand $\mathbf{QA}_G\mathbf{R}$ to get

$$\mathbf{QA}_G\mathbf{R} = \mathbf{A}_H. \quad (3)$$

But are \mathbf{P} and \mathbf{R} permutation matrices? Suppose not. Being the sum of two submatrices of \mathbf{P} , this can only happen if a row (and column) are zero. But by (3) above, \mathbf{A}_H will have a row of zeros. This corresponds to an isolated vertex in H — a contradiction. \square

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^T$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$. Using the obtained equations (1) and (2), we can expand $\mathbf{QA}_G\mathbf{R}$ to get

$$\mathbf{QA}_G\mathbf{R} = \mathbf{A}_H. \quad (3)$$

But are \mathbf{P} and \mathbf{R} permutation matrices? Suppose not. Being the sum of two submatrices of \mathbf{P} , this can only happen if a row (and column) are zero. But by (3) above, \mathbf{A}_H will have a row of zeros. This corresponds to an isolated vertex in H — a contradiction. \square

Observe that the key to this proof is the contradiction arising from the introduction of an isolated vertex.

How many different graphs have $\text{CDC}(G) = \text{CDC}(H)$?

How many different graphs have $\text{CDC}(G) = \text{CDC}(H)$?

We have seen that two graphs being TF-isomorphic and having isomorphic CDCs are equivalent.

How many different graphs have $\text{CDC}(G) = \text{CDC}(H)$?

We have seen that two graphs being TF-isomorphic and having isomorphic CDCs are equivalent.

Question: How many non-isomorphic graphs have the same CDCs?

How many different graphs have $\text{CDC}(G) = \text{CDC}(H)$?

We have seen that two graphs being TF-isomorphic and having isomorphic CDCs are equivalent.

Question: How many non-isomorphic graphs have the same CDCs?

Having the same CDC is very “close” to being isomorphic. On $n \leq 8$ vertices, there are 13 597 non-isomorphic graphs. Taking all

$$\binom{13\,597}{2} = 92\,432\,406$$

possible pairs of graphs on at most 8 vertices, it turns out that only 32 pairs are TF-isomorphic.

How many different graphs have $\text{CDC}(G) = \text{CDC}(H)$?

We have seen that two graphs being TF-isomorphic and having isomorphic CDCs are equivalent.

Question: How many non-isomorphic graphs have the same CDCs?

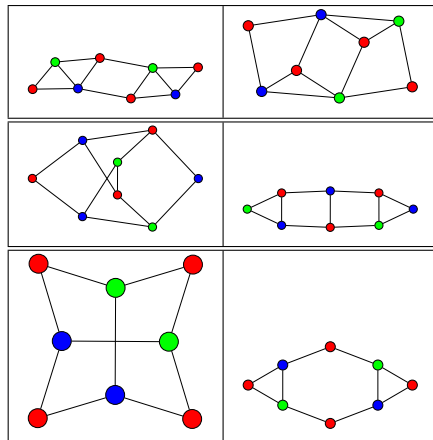
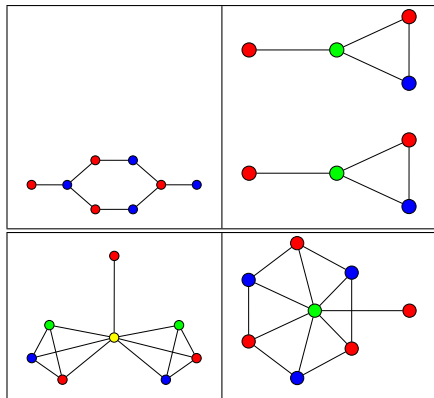
Having the same CDC is very “close” to being isomorphic. On $n \leq 8$ vertices, there are 13 597 non-isomorphic graphs. Taking all

$$\binom{13\,597}{2} = 92\,432\,406$$

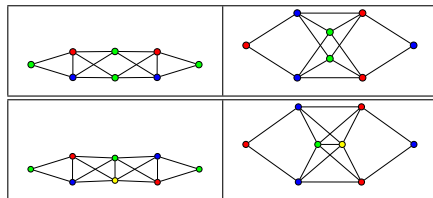
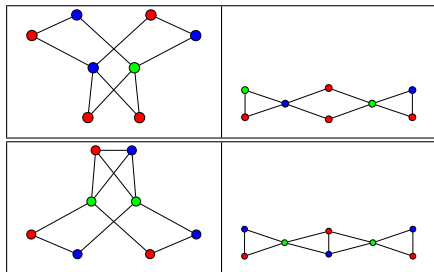
possible pairs of graphs on at most 8 vertices, it turns out that only 32 pairs are TF-isomorphic.

It is rare for a pair of graphs to be so structurally similar yet not isomorphic.

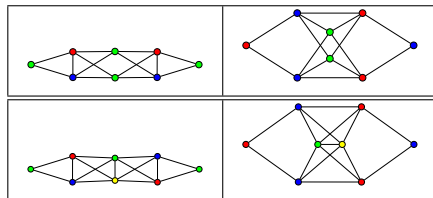
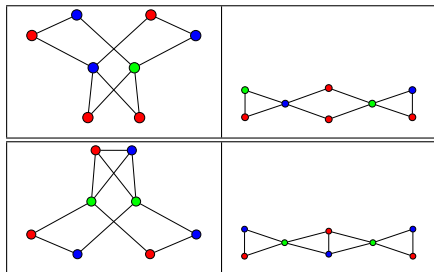
Here are some of them:



and some more:

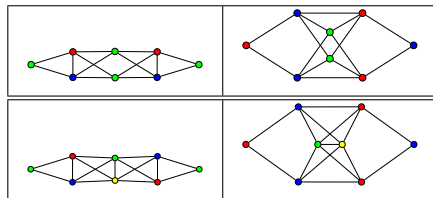
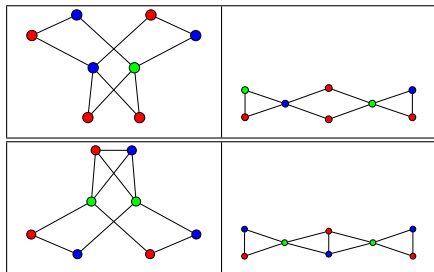


and some more:



Graphs with the same CDC have the same:

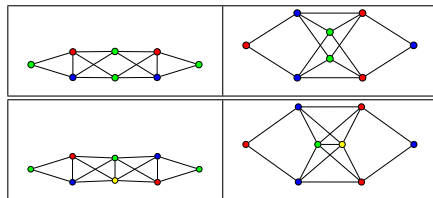
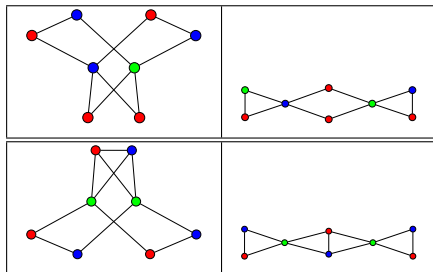
and some more:



Graphs with the same CDC have the same:

- Main eigenvalues

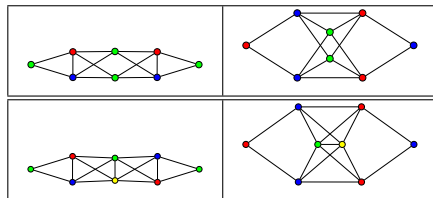
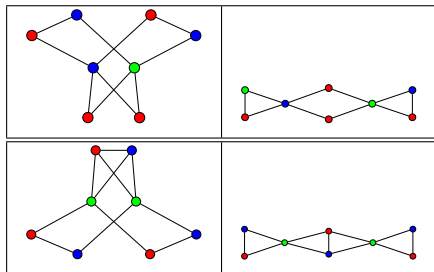
and some more:



Graphs with the same CDC have the same:

- Main eigenvalues
- Main eigenspaces

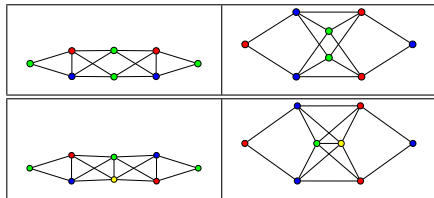
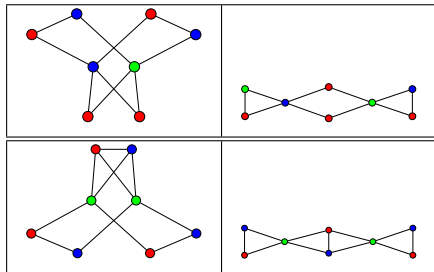
and some more:



Graphs with the same CDC have the same:

- Main eigenvalues
- Main eigenspaces
- Main eigenvectors

and some more:



Graphs with the same CDC have the same:

- Main eigenvalues
- Main eigenspaces
- Main eigenvectors
- Number of walks of any length...

Walks

Walks

Definition (Walk)

Let G be a graph. A **walk** in G is a sequence of vertices

$$v_1, v_2, \dots, v_k$$

such that $\{v_i, v_{i+1}\}$ is an edge for $i = 1, \dots, k - 1$. The **length** of a walk is the number k of vertices.

Walks

Definition (Walk)

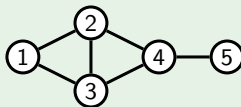
Let G be a graph. A **walk** in G is a sequence of vertices

$$v_1, v_2, \dots, v_k$$

such that $\{v_i, v_{i+1}\}$ is an edge for $i = 1, \dots, k - 1$. The **length** of a walk is the number k of vertices.

Example

In our usual example graph, 1234 and 12324 are walks, but 1235 is not.



Walk Matrix

Walk Matrix

Let $\mathbf{j} = (1, 1, \dots, 1)$ be a vector consisting entirely of ones.

Walk Matrix

Let $\mathbf{j} = (1, 1, \dots, 1)$ be a vector consisting entirely of ones.

Question: What is $\mathbf{A}\mathbf{j}$ for an adjacency matrix \mathbf{A} ?

Walk Matrix

Let $\mathbf{j} = (1, 1, \dots, 1)$ be a vector consisting entirely of ones.

Question: What is $\mathbf{A}\mathbf{j}$ for an adjacency matrix \mathbf{A} ?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Walk Matrix

Let $\mathbf{j} = (1, 1, \dots, 1)$ be a vector consisting entirely of ones.

Question: What is $\mathbf{A}\mathbf{j}$ for an adjacency matrix \mathbf{A} ?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \deg v_1 \\ \deg v_2 \\ \deg v_3 \\ \deg v_4 \\ \deg v_5 \end{pmatrix}$$

Walk Matrix

Let $\mathbf{j} = (1, 1, \dots, 1)$ be a vector consisting entirely of ones.

Question: What is $\mathbf{A}\mathbf{j}$ for an adjacency matrix \mathbf{A} ?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \deg v_1 \\ \deg v_2 \\ \deg v_3 \\ \deg v_4 \\ \deg v_5 \end{pmatrix}$$

What about $\mathbf{A}^2\mathbf{j}$?

Walk Matrix

Let $\mathbf{j} = (1, 1, \dots, 1)$ be a vector consisting entirely of ones.

Question: What is $\mathbf{A}\mathbf{j}$ for an adjacency matrix \mathbf{A} ?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \deg v_1 \\ \deg v_2 \\ \deg v_3 \\ \deg v_4 \\ \deg v_5 \end{pmatrix}$$

What about $\mathbf{A}^2\mathbf{j}$?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \deg v_1 \\ \deg v_2 \\ \deg v_3 \\ \deg v_4 \\ \deg v_5 \end{pmatrix}$$

Walk Matrix

Let $\mathbf{j} = (1, 1, \dots, 1)$ be a vector consisting entirely of ones.

Question: What is $\mathbf{A}\mathbf{j}$ for an adjacency matrix \mathbf{A} ?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \deg v_1 \\ \deg v_2 \\ \deg v_3 \\ \deg v_4 \\ \deg v_5 \end{pmatrix}$$

What about $\mathbf{A}^2\mathbf{j}$?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \deg v_1 \\ \deg v_2 \\ \deg v_3 \\ \deg v_4 \\ \deg v_5 \end{pmatrix} = \begin{pmatrix} \deg v_2 + \deg v_3 \\ \deg v_1 + \deg v_3 + \deg v_4 \\ \deg v_1 + \deg v_2 + \deg v_4 \\ \deg v_2 + \deg v_3 + \deg v_5 \\ \deg v_4 \end{pmatrix}$$

Walk Matrix

In general, $\mathbf{A}^k \mathbf{j}$ is the vector

$$\begin{pmatrix} \# \text{ of walks of length } k \text{ starting at } v_1 \\ \# \text{ of walks of length } k \text{ starting at } v_2 \\ \vdots \\ \# \text{ of walks of length } k \text{ starting at } v_n \end{pmatrix}.$$

Walk Matrix

In general, $\mathbf{A}^k \mathbf{j}$ is the vector

$$\begin{pmatrix} \# \text{ of walks of length } k \text{ starting at } v_1 \\ \# \text{ of walks of length } k \text{ starting at } v_2 \\ \vdots \\ \# \text{ of walks of length } k \text{ starting at } v_n \end{pmatrix}.$$

Definition (Walk Matrix)

The matrix $\mathbf{W}_k(G)$ is the $n \times k$ matrix whose columns are the first k such vectors, i.e.

$$\mathbf{W}_k(G) = \begin{pmatrix} | & | & | & \dots & | \\ \mathbf{j} & \mathbf{A}\mathbf{j} & \mathbf{A}^2\mathbf{j} & \dots & \mathbf{A}^{k-1}\mathbf{j} \\ | & | & | & & | \end{pmatrix}.$$

An eigenvector \mathbf{x} of \mathbf{A} is said to be **main** if it is *not* orthogonal to \mathbf{j} , i.e. if $\langle \mathbf{x}, \mathbf{j} \rangle \neq 0$.

An eigenvector \mathbf{x} of \mathbf{A} is said to be **main** if it is *not* orthogonal to \mathbf{j} , i.e. if $\langle \mathbf{x}, \mathbf{j} \rangle \neq 0$. An eigenvalue of \mathbf{A} is **main** if at least one of its corresponding eigenvectors is **main**.

An eigenvector \mathbf{x} of \mathbf{A} is said to be **main** if it is *not* orthogonal to \mathbf{j} , i.e. if $\langle \mathbf{x}, \mathbf{j} \rangle \neq 0$. An eigenvalue of \mathbf{A} is **main** if at least one of its corresponding eigenvectors is **main**.

Theorem

Let G and H be two graphs with the same main eigenvalues and eigenvectors. Then for any k , $\mathbf{W}_k(G) = \mathbf{W}_k(H)$.

An eigenvector \mathbf{x} of \mathbf{A} is said to be **main** if it is *not* orthogonal to \mathbf{j} , i.e. if $\langle \mathbf{x}, \mathbf{j} \rangle \neq 0$. An eigenvalue of \mathbf{A} is **main** if at least one of its corresponding eigenvectors is **main**.

Theorem

Let G and H be two graphs with the same main eigenvalues and eigenvectors. Then for any k , $\mathbf{W}_k(G) = \mathbf{W}_k(H)$.

Proof.

Suppose G and H have main eigenvalues μ_i , $i = 1 \dots p$ and corresponding main eigenvectors \mathbf{x}_i .

An eigenvector \mathbf{x} of \mathbf{A} is said to be **main** if it is *not* orthogonal to \mathbf{j} , i.e. if $\langle \mathbf{x}, \mathbf{j} \rangle \neq 0$. An eigenvalue of \mathbf{A} is **main** if at least one of its corresponding eigenvectors is **main**.

Theorem

Let G and H be two graphs with the same main eigenvalues and eigenvectors. Then for any k , $\mathbf{W}_k(G) = \mathbf{W}_k(H)$.

Proof.

Suppose G and H have main eigenvalues μ_i , $i = 1 \dots p$ and corresponding main eigenvectors \mathbf{x}_i . Since \mathbf{A}_G and \mathbf{A}_H are real symmetric, then the set of \mathbf{x}_i 's forms a basis for $\text{span}(\{\mathbf{j}\})$.

An eigenvector \mathbf{x} of \mathbf{A} is said to be **main** if it is *not* orthogonal to \mathbf{j} , i.e. if $\langle \mathbf{x}, \mathbf{j} \rangle \neq 0$. An eigenvalue of \mathbf{A} is **main** if at least one of its corresponding eigenvectors is **main**.

Theorem

Let G and H be two graphs with the same main eigenvalues and eigenvectors. Then for any k , $\mathbf{W}_k(G) = \mathbf{W}_k(H)$.

Proof.

Suppose G and H have main eigenvalues μ_i , $i = 1 \dots p$ and corresponding main eigenvectors \mathbf{x}_i . Since \mathbf{A}_G and \mathbf{A}_H are real symmetric, then the set of \mathbf{x}_i 's forms a basis for $\text{span}(\{\mathbf{j}\})$. In particular, we may express \mathbf{j} as

$$\mathbf{j} = \sum_{i=1}^p \beta_i \mathbf{x}_i,$$

where the coefficients β_i are unique.

Proof (continued).

Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\mathbf{A}_G^{\ell-1}\mathbf{j}$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\mathbf{A}_G^{\ell-1}\mathbf{j} = \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\mathbf{A}_G^{\ell-1}\mathbf{j} = \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\mathbf{A}_G^{\ell-1}\mathbf{j} = \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\mathbf{A}_G^{\ell-1}\mathbf{j} = \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_H^{\ell-1} \mathbf{x}_i$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\begin{aligned}\mathbf{A}_G^{\ell-1}\mathbf{j} &= \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_H^{\ell-1} \mathbf{x}_i \\ &= \mathbf{A}_H^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i\end{aligned}$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\begin{aligned}\mathbf{A}_G^{\ell-1}\mathbf{j} &= \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_H^{\ell-1} \mathbf{x}_i \\ &= \mathbf{A}_H^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \mathbf{A}_H^{\ell-1} \mathbf{j},\end{aligned}$$



Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\begin{aligned}\mathbf{A}_G^{\ell-1}\mathbf{j} &= \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_H^{\ell-1} \mathbf{x}_i \\ &= \mathbf{A}_H^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \mathbf{A}_H^{\ell-1} \mathbf{j},\end{aligned}$$

the ℓ th column of $\mathbf{W}_k(H)$. □

Proof (continued).

Now the ℓ th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

$$\begin{aligned}\mathbf{A}_G^{\ell-1}\mathbf{j} &= \mathbf{A}_G^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_G^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mu_i^{\ell-1} \mathbf{x}_i = \sum_{i=1}^p \beta_i \mathbf{A}_H^{\ell-1} \mathbf{x}_i \\ &= \mathbf{A}_H^{\ell-1} \sum_{i=1}^p \beta_i \mathbf{x}_i = \mathbf{A}_H^{\ell-1} \mathbf{j},\end{aligned}$$

the ℓ th column of $\mathbf{W}_k(H)$. □

Corollary

Any two graphs with the same CDC have the same walk matrix.

Thank you!

DEPARTMENT OF MATHEMATICS
Faculty of Science
L-Università ta' Malta



L-Università
ta' Malta

