Two-Graphs and NSSDs: An Algebraic Approach

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1. Introduction
    - Spectrum and Seidel Switching
    - Defining Two-Graphs
2. Regular Two-Graphs
    - The Involution **M**
    - Descendant Form of a Regular Two-Graph
    - Results about Descendants of Regular Two-Graphs
3. Strongly Regular Graphs
    - Definition
    - Structure of Descendants of Regular Two-Graphs
4. Conference Graphs

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The **adjacency matrix** of a graph $G = (V, E)$ is the $n \times n$ matrix (a_{ij}) where

$$a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j\text{, i.e. } \{v_i, v_j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

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Eventually we will work with a variant of the Seidel matrix, which we will introduce later.

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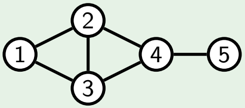

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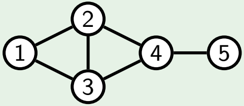

$$\Longleftrightarrow \quad \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 & 1 \\ -1 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{pmatrix} \end{array}$$

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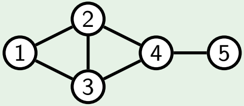

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Note that if $\mathbf{A}$ and $\mathbf{S}$ are the adjacency and Seidel matrices of G respectively,

$$\mathbf{S} = \mathbf{J} - \mathbf{I} - 2\mathbf{A}$$

where $\mathbf{J}$ is the matrix consisting entirely of 1's and $\mathbf{I}$ is the identity matrix.

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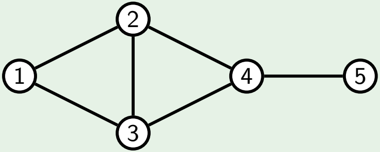

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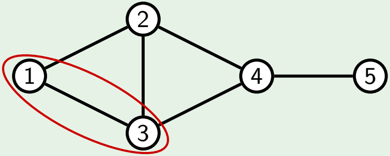

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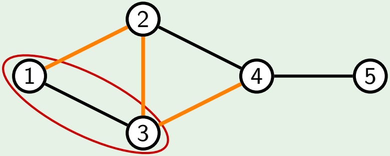

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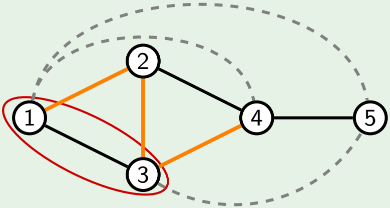

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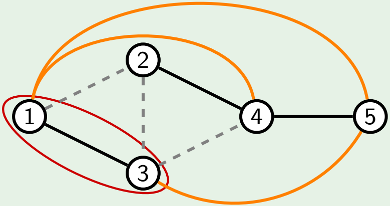

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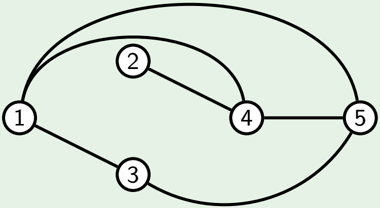

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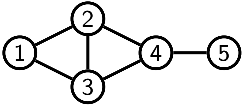

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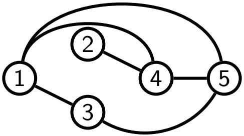

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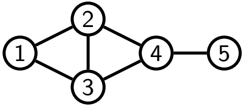

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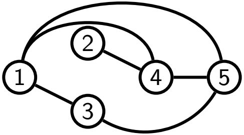

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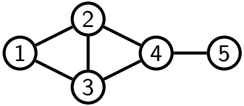

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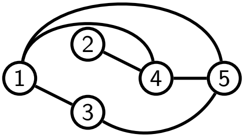

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It follows that $\mathbf{S}$ and $\mathbf{S}_{\mathrm{SS}_U(G)}$ are similar, and therefore G and $\mathrm{SS}_U(G)$ have the same Seidel spectrum.

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- A two-graph on n vertices consists of all the n-vertex graphs with the same Seidel spectrum.
- The term ‘two-graph’ originally arose in a combinatorial context, and actually refers to a couple (V, Δ) where $\Delta \subseteq \binom{V}{3}$ is a collection of triples $\{v_1, v_2, v_3\}$ with the property that any 4-subset of V contains an even number of triples of Δ. This is known to be equivalent to our definition.

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- Reverting to the combinatorial definition of 'two-graph', (V, Δ) is said to be regular if every pair of vertices lies in the same number of triples of Δ. This is known to be equivalent to our definition.

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Furthermore, if the spectrum of $\mathbf{M}$ is $1^{(n-k)}(-1)^{(k)}$, then it follows by Cauchy's interlacing inequalities that the spectrum of $\mathbf{B}$ is

$$1^{(n-k-1)}(-1)^{(k-1)}\lambda^{(1)}.$$

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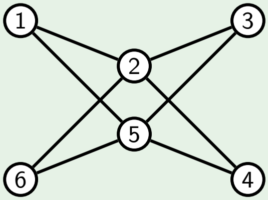

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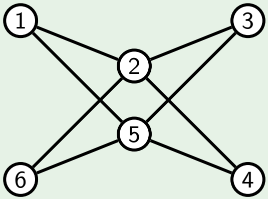

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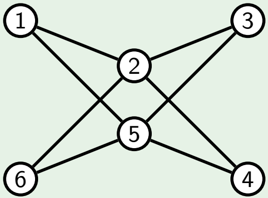

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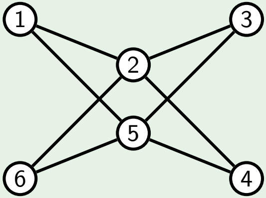

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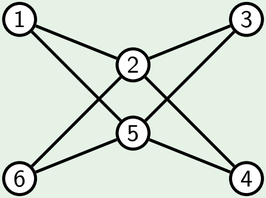

Seidel spectrum: $(-1)^5(5)^1$

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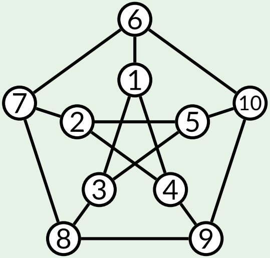

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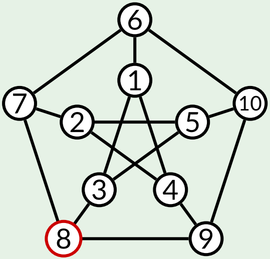

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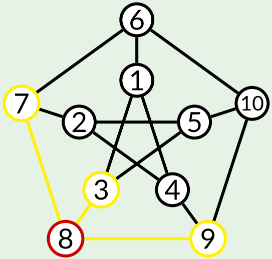

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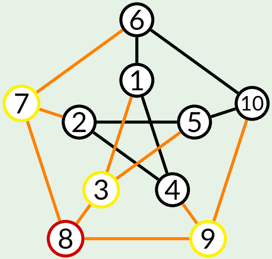

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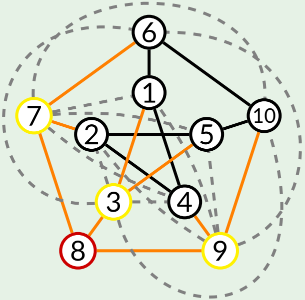

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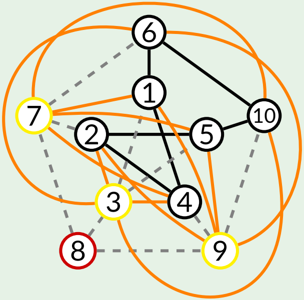

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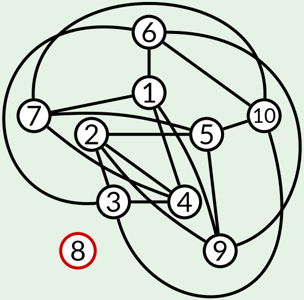

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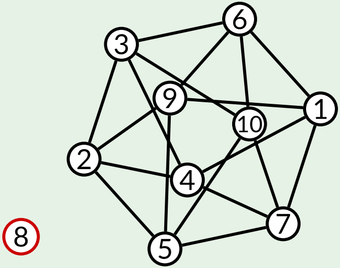

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Results about Descendants of Regular Two-Graphs

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2. Substituting for α and λ, we also get that n and $\mu_1 + \mu_2$ have the same parity (even/odd).

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Proof.

Let the Seidel eigenvalues of G be μ_1 and μ_2, where G is in descendant form. Using the values of α and λ, the first and last rows of the involution $\mathbf{M}$ are of the form

$$\begin{array}{l} \text{Row } 1 \\ \\ \text{Row } n \end{array} \begin{pmatrix} -\lambda & \pm\alpha & \pm\alpha & \cdots & \pm\alpha & \alpha \\ & & \vdots & & & \\ \alpha & \alpha & \alpha & \cdots & \alpha & -\lambda \end{pmatrix}$$

where we are assuming that the last row corresponds to the isolated vertex. The number of $-\alpha$'s in row 1 is the degree of vertex 1. Since $\mathbf{M}^2 = \mathbf{I}$, the inner product $\langle \text{Row } 1, \text{Row } n \rangle = 0$.

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Proof. *(continued...)*

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$$\langle \text{Row } 1, \text{Row } n \rangle = 0 \implies -\alpha\lambda + (n-2)\alpha^2 - 2\rho_1\alpha^2 - \alpha\lambda = 0$$

where ρ_1 denotes the degree of vertex 1.
Note that ρ_1 is independent of the vertex label 1, since $\langle \text{Row } 1, \text{Row } i \rangle = 0$ for all $1 \leq i \leq n-1$. Thus D is ρ-regular. $\square$

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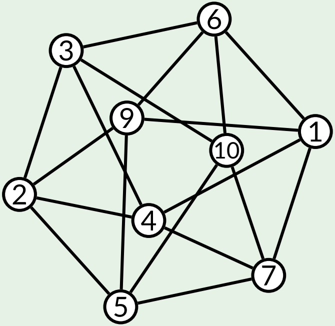
The descendant from the last example is an srg$(9, 4, 1, 2)$.



Structure of Descendants of Regular Two-Graphs

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Consider a descendant form $D \dot\cup K_1$ of a regular two-graph and the following notations for pairs of vertices.

	# of common neighbours	# of common non-neighbours
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$$\tilde{e} + \tilde{\tilde{e}} = \tfrac{1}{2}(n-2) - \frac{\lambda}{\alpha} \qquad \tilde{f} + \tilde{\tilde{f}} = \tfrac{1}{2}(n-2) + \frac{\lambda}{\alpha}$$

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From these it follows that $\tilde{e}$, $\tilde{\tilde{e}}$, $\tilde{f}$ and $\tilde{\tilde{f}}$ are invariant for any pair of adjacent/non-adjacent vertices.

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- n must be even.
- $\frac{\lambda}{\alpha}$ is an integer.

An Application: Conference Graphs

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These have applications in telephone networks. A necessary condition for setting up a conference with n telephone ports and ideal signal loss is the existence of an $n \times n$ conference matrix.

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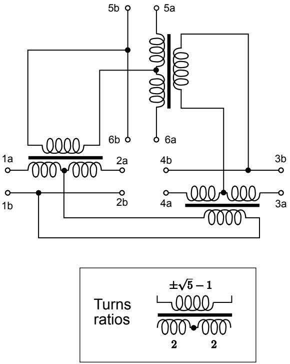

Figure: Implementation of 6-port conference matrix, corresponds to the smallest existing conference graph on $n = 6$ vertices with Seidel eigenvalues $\pm\sqrt{5}$.

Thank you!

DEPARTMENT OF MATHEMATICS
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