# Walks and Canonical Double Coverings of Comain Graphs

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DEPARTMENT OF MATHEMATICS
**Faculty of Science**
L-Università ta' Malta

*$S^3$ Annual Science Conference*
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# Outline

1. Introduction
   - Definition of a Graph
   - Representing Graphs as Matrices
   - Graph Isomorphisms
2. Canonical Double Covers
   - Definition of CDC
   - Some Easy Observations about CDC
   - How many such graphs are there?
3. Walks
   - The Walk Matrix of a Graph
   - Main Eigenvalues and Eigenvectors

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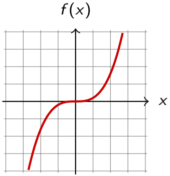

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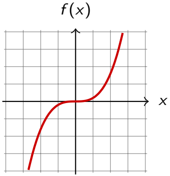


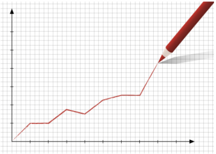

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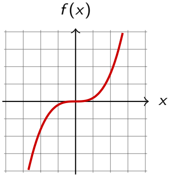


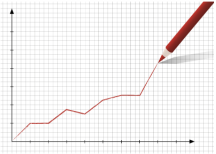

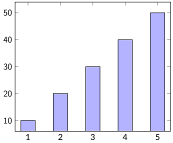

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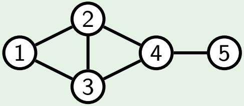

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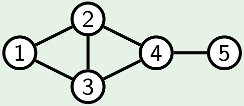


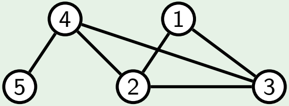

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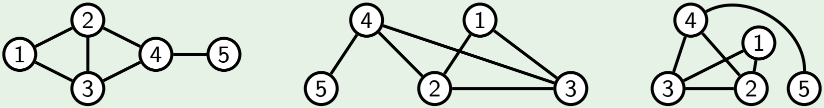

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$$a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j\text{, i.e. } \{v_i, v_j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

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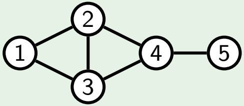

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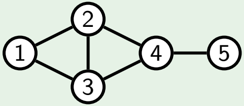


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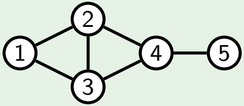


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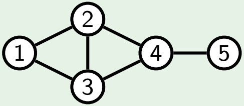


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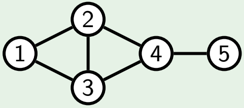


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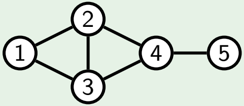


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- Each entry on the diagonal is 0, since we consider simple graphs

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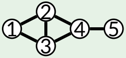
The eigenvalues of 

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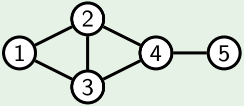


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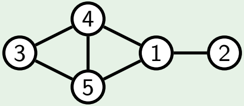


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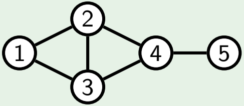


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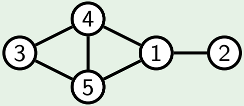


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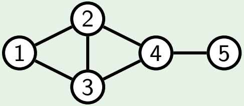


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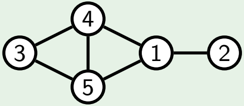


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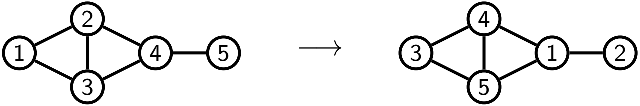

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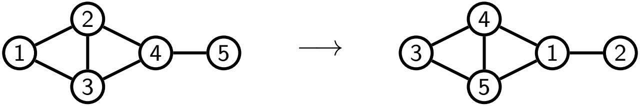


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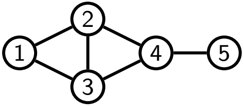


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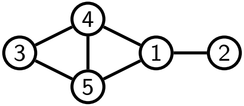


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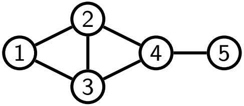

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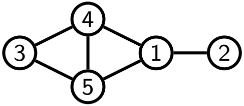

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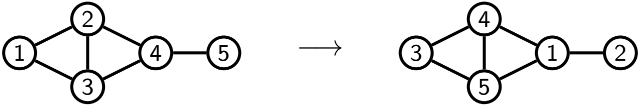


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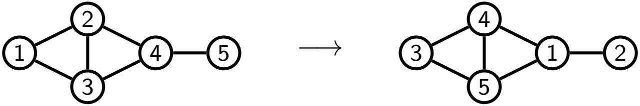


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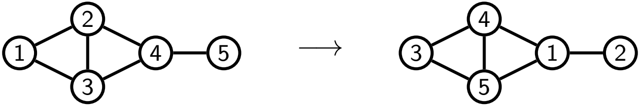


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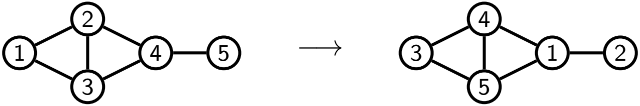


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In most cases, the labelling of the vertices in a graph is not important.

# Definition of $\mathrm{CDC}(G)$

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**Definition (Canonical Double Cover)**

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \dots, n\}$, denoted $\mathrm{CDC}(G)$, is the graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

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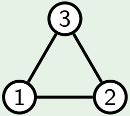

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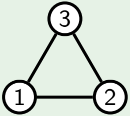


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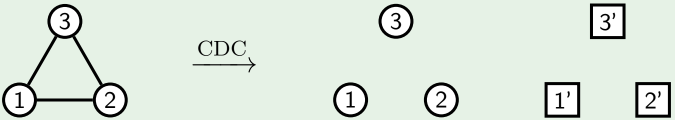

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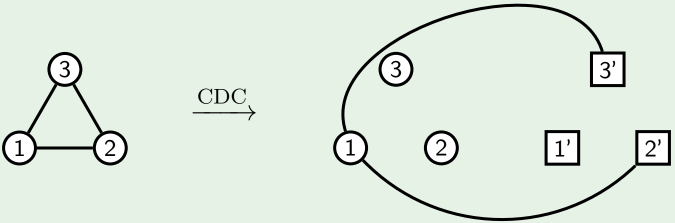

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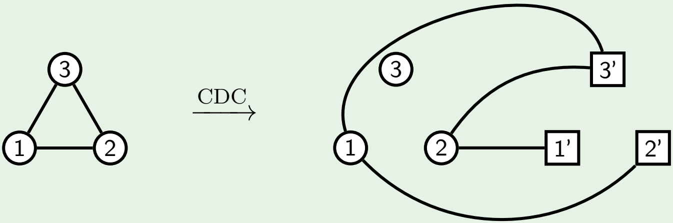

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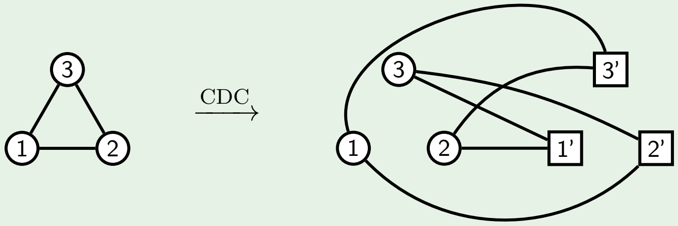

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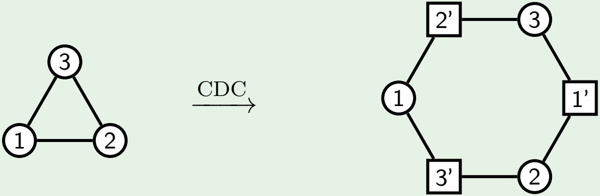

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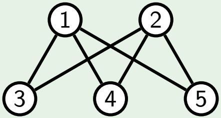

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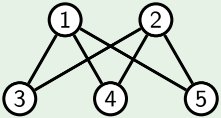


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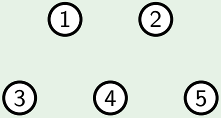

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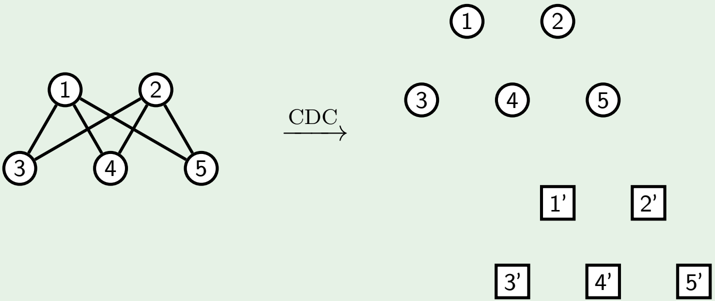

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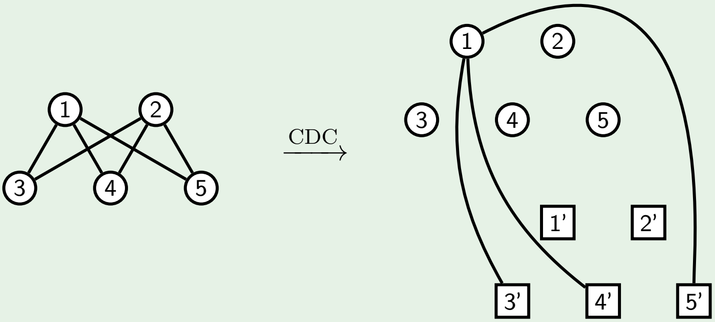

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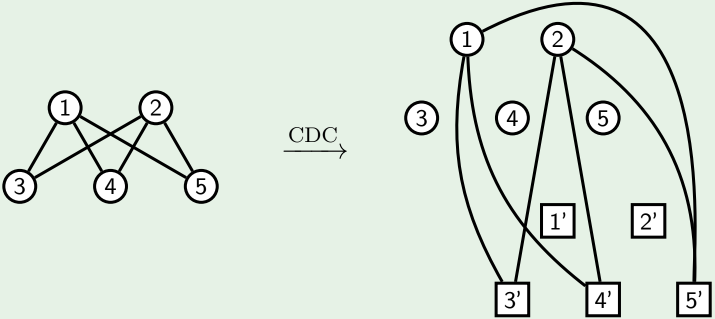

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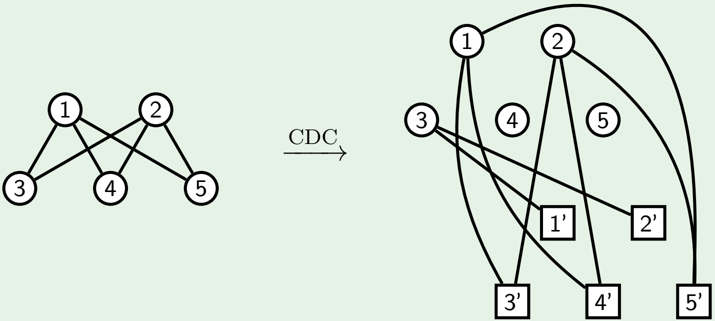

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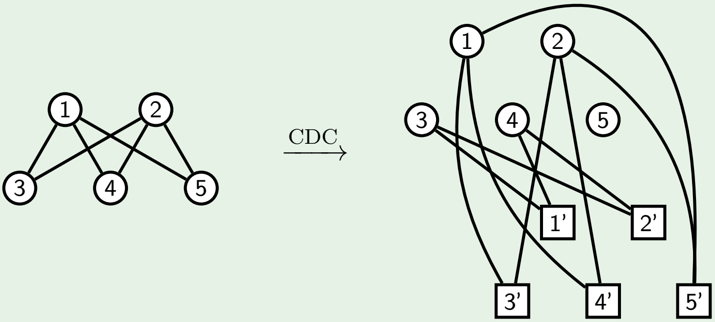

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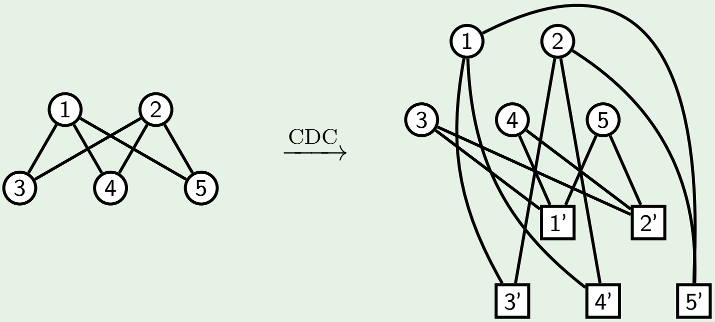

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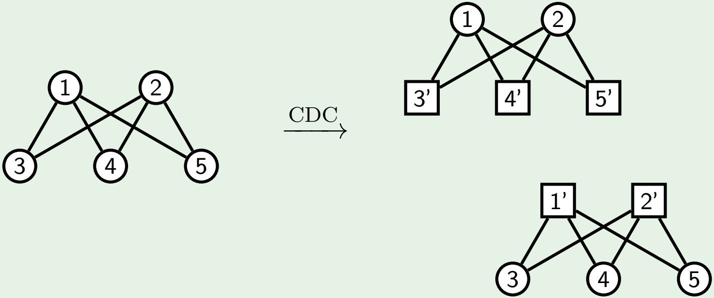

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  This yielded a useful *proof technique*: If we have $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$ where $G$ has no isolated vertex, we show that the negation of what we want to prove introduces an isolated vertex in $H$.

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If we weaken this relationship, we get the following:

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$$\mathbf{A}_G = \mathbf{P}^\top \mathbf{A}_H \mathbf{P}.$$

If we weaken this relationship, we get the following:

## Definition (Two-Fold Isomorphism)

Let $G$ and $H$ be two graphs with adjacency matrices $\mathbf{A}_G$ and $\mathbf{A}_H$. We say that $G$ is *two-fold isomorphic* or *TF-isomorphic* to $H$ if

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for some permutation matrices $\mathbf{R}$, $\mathbf{Q}$.

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for some permutation matrices $\mathbf{R}$, $\mathbf{Q}$.

Here $\mathbf{R}$ and $\mathbf{Q}$ could be any permutation matrices, they don't have to be the inverse (i.e. transpose) of each other.

**Theorem**

*Let* **G** *and* **H** *be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

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so $\mathrm{CDC}(H) \simeq \mathrm{CDC}(G)$.

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$$\implies \begin{pmatrix} \mathbf{P}_{11}^\mathsf{T} & \mathbf{P}_{21}^\mathsf{T} \\ \mathbf{P}_{12}^\mathsf{T} & \mathbf{P}_{22}^\mathsf{T} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}$$

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Multiplying out and comparing entries, we get that

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Multiplying out and comparing entries, we get that

$$\mathbf{P}_{21}^\top \mathbf{A}_G \mathbf{P}_{12} + \mathbf{P}_{11}^\top \mathbf{A}_G \mathbf{P}_{22} = \mathbf{A}_H \tag{1}$$

$$\mathbf{P}_{21}^\top \mathbf{A}_G \mathbf{P}_{11} = \mathbf{P}_{12}^\top \mathbf{A}_G \mathbf{P}_{22} = \mathbf{O} \tag{2}$$

**Proof (continued).**

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^{\mathsf{T}}$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$.

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Observe that the key to this proof is the contradiction arising from the introduction of an isolated vertex.

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Having the same CDC is very "close" to being isomorphic. On $n \leqslant 8$ vertices, there are 13 597 non-isomorphic graphs. Taking all

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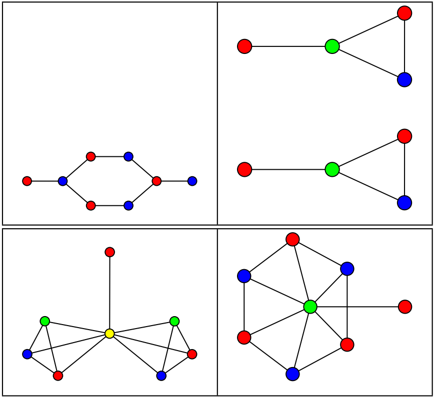
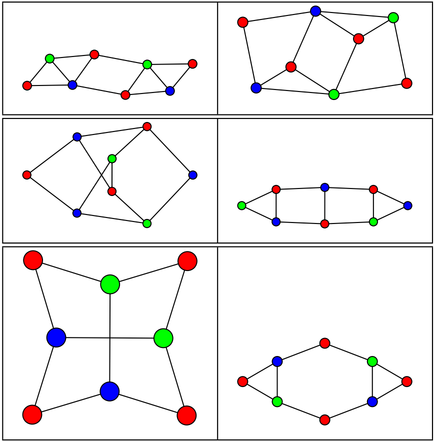
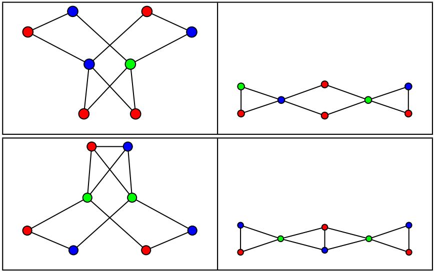
It is rare for a pair of graphs to be so structurally similar yet not isomorphic.

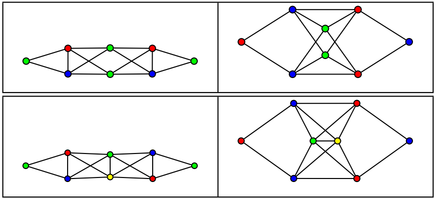
# Here are some of them:





# and some more:





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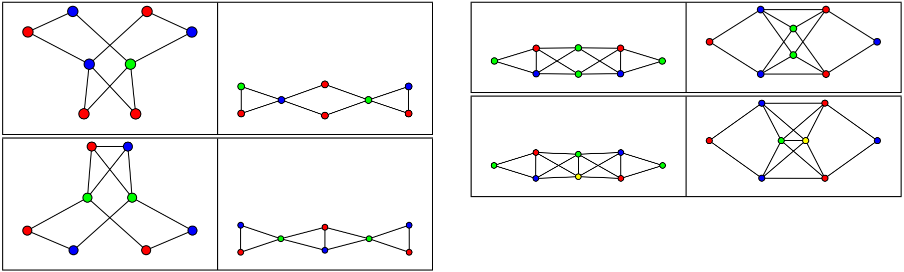

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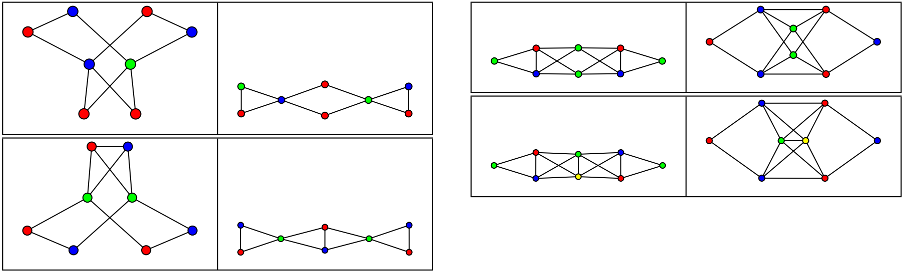

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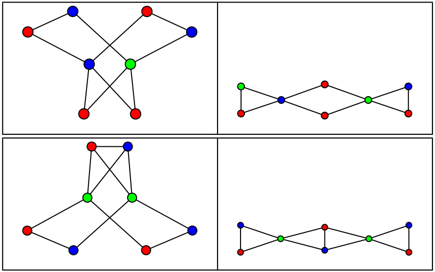

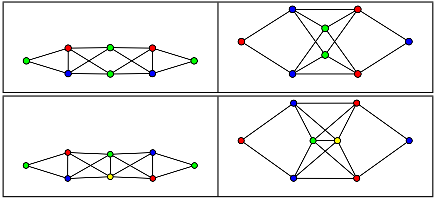

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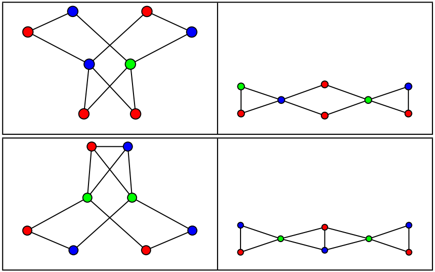

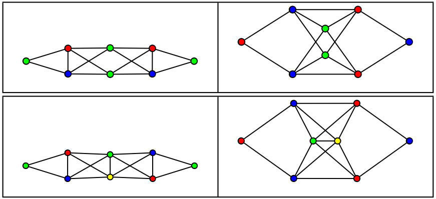

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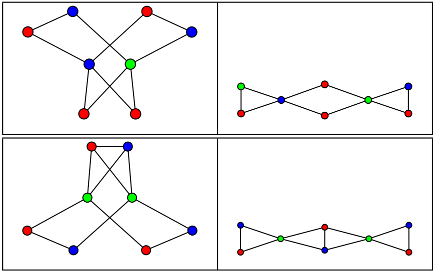
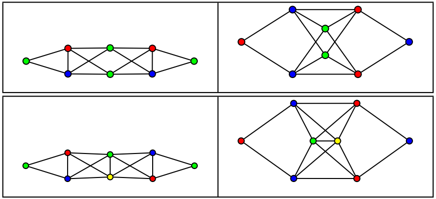
- Main eigenvectors

# and some more:





Graphs with the same CDC have the same:

- Main eigenvalues
- Main eigenspaces
- Main eigenvectors
- Number of walks of any length…

# Walks

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## Definition (Walk)

Let $G$ be a graph. A **walk** in $G$ is a sequence of vertices

$$v_1, v_2, \ldots, v_k$$

such that $\{v_i, v_{i+1}\}$ is an edge for $i = 1, \ldots, k-1$. The **length** of a walk is the number $k$ of vertices.

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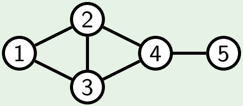
## Example

In our usual example graph, 1234 and 12324 are walks, but 1235 is not.



# Walk Matrix

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$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

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In general, $\mathbf{A}^k\mathbf{j}$ is the vector

$$\begin{pmatrix} \text{\# of walks of length } k \text{ starting at } v_1 \\ \text{\# of walks of length } k \text{ starting at } v_2 \\ \vdots \\ \text{\# of walks of length } k \text{ starting at } v_n \end{pmatrix}.$$

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### Definition (Walk Matrix)

The matrix $\mathbf{W}_k(G)$ is the $n \times k$ matrix whose columns are the first $k$ such vectors, i.e.

$$\mathbf{W}_k(G) = \begin{pmatrix} | & | & | & & | \\ \mathbf{j} & \mathbf{A}\mathbf{j} & \mathbf{A}^2\mathbf{j} & \cdots & \mathbf{A}^{k-1}\mathbf{j} \\ | & | & | & & | \end{pmatrix}.$$

An eigenvector $\mathbf{x}$ of $\mathbf{A}$ is said to be **main** if it is *not* orthogonal to $\mathbf{j}$, i.e. if $\langle \mathbf{x}, \mathbf{j} \rangle \neq 0$.

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### Theorem

*Let $G$ and $H$ be two graphs with the same main eigenvalues and eigenvectors. Then for any $k$, $\mathbf{W}_k(G) = \mathbf{W}_k(H)$.*

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$$\mathbf{j} = \sum_{i=1}^{p} \beta_i \mathbf{x}_i,$$

where the coefficients $\beta_i$ are unique.

Proof (continued).

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Now the $\ell$th column of $\mathbf{W}_k(G)$ is the vector $\mathbf{A}_G^{\ell-1}\mathbf{j}$, so

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## Corollary

Any two graphs with the same CDC have the same walk matrix.

*Thank you!*

DEPARTMENT OF MATHEMATICS
**Faculty of Science**
L-Università ta' Malta