Proof that $(a \leftrightarrow b) \dashv\vdash (a \wedge b) \vee (\neg a \wedge \neg b)$

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1 Deduction Rules

I assume the following deduction rules for predicate logic.

Truth and Falsity

$$\frac{}{\top}\ \top\text{-INT} \qquad\qquad \frac{\bot}{\phi}\ \bot\text{-ELIM}$$

Conjunction

$$\frac{\phi\,,\,\varphi}{\phi \wedge \varphi}\ \wedge\text{-INT} \qquad \frac{\phi \wedge \varphi}{\phi}\ \wedge\text{-ELIM}_1 \qquad \frac{\phi \wedge \varphi}{\varphi}\ \wedge\text{-ELIM}_2$$

Disjunction

$$\frac{\phi}{\phi \vee \varphi}\ \vee\text{-INT}_1 \qquad \frac{\phi}{\varphi \vee \phi}\ \vee\text{-INT}_2 \qquad \frac{\phi \to \psi\,,\,\varphi \to \psi\,,\,\phi \vee \varphi}{\psi}\ \vee\text{-ELIM}$$

Implication

$$\frac{\phi \to \varphi\,,\,\phi}{\varphi}\ \to\text{-ELIM} \qquad\qquad \frac{\mathcal{S}\,,\,\phi \vdash \varphi}{\mathcal{S} \vdash \phi \to \varphi}\ \to\text{-INT}$$

Biconditonal

$$\frac{\phi \to \varphi\,,\,\varphi \to \phi}{\phi \leftrightarrow \varphi}\ \leftrightarrow\text{-INT} \qquad \frac{\phi \leftrightarrow \varphi}{\phi \to \varphi}\ \leftrightarrow\text{-ELIM}_1 \qquad \frac{\phi \leftrightarrow \varphi}{\varphi \to \phi}\ \leftrightarrow\text{-ELIM}_2$$

Negation

$$\frac{\neg\neg\phi}{\phi}\ \neg\text{-ELIM} \qquad\qquad \frac{\phi \to \varphi\,,\,\phi \to \neg\varphi}{\neg\phi}\ \neg\text{-INT}$$

2 The Proof

First, the proof $(a \leftrightarrow b) \vdash (a \wedge b) \vee (\neg a \wedge \neg b)$.

1	$a \leftrightarrow b$	
2	$a \rightarrow b$	($\leftrightarrow$-ELIM$_1$ on line 1)
3	$b \rightarrow a$	($\leftrightarrow$-ELIM$_2$ on line 1)
4	$a \vee \neg a$	(theorem: $\vdash \phi \vee \neg\phi$)
5	$\quad a$	
6	$\quad b$	($\rightarrow$-ELIM on lines 2, 5)
7	$\quad a \wedge b$	($\wedge$-INT on lines 5, 6)
8	$\quad (a \wedge b) \vee (\neg a \wedge \neg b)$	($\vee$-INT$_1$ on line 7)
9	$a \rightarrow (a \wedge b) \vee (\neg a \wedge \neg b)$	($\rightarrow$-INT on lines 5–8)
10	$\quad \neg a$	
11	$\quad b \vee \neg b$	(theorem: $\vdash \phi \vee \neg\phi$)
12	$\quad\quad b$	
13	$\quad\quad a$	($\rightarrow$-ELIM on lines 3, 12)
14	$\quad\quad a \wedge b$	($\wedge$-INT on lines 13, 12)
15	$\quad\quad (a \wedge b) \vee (\neg a \wedge \neg b)$	($\vee$-INT$_1$ on line 14)
16	$\quad b \rightarrow (a \wedge b) \vee (\neg a \wedge \neg b)$	($\rightarrow$-INT on lines 12–15)
17	$\quad\quad \neg b$	
18	$\quad\quad \neg a \wedge \neg b$	($\wedge$-INT on lines 10, 17)
19	$\quad\quad (a \wedge b) \vee (\neg a \wedge \neg b)$	($\vee$-INT$_2$ on line 18)
20	$\quad \neg b \rightarrow (a \wedge b) \vee (\neg a \wedge \neg b)$	($\rightarrow$-INT on lines 17–19)
21	$\quad (a \wedge b) \vee (\neg a \wedge \neg b)$	($\vee$-ELIM on lines 16, 20, 11)
22	$\neg a \rightarrow (a \wedge b) \vee (\neg a \wedge \neg b)$	($\rightarrow$-INT on lines 10–21)
23	$(a \wedge b) \vee (\neg a \wedge \neg b)$	($\vee$-ELIM on lines 9, 22, 4)

Notice that we make use of the law of excluded middle, $\vdash \phi \vee \neg\phi$. The proof is available in Prof Gordon Pace's book on page 44). Next we prove that $(a \leftrightarrow b) \dashv (a \wedge b) \vee (\neg a \wedge \neg b)$, that is, $(a \wedge b) \vee (\neg a \wedge \neg b) \vdash (a \leftrightarrow b)$.

1	$(a \wedge b) \vee (\neg a \wedge \neg b)$	
2	$\quad a$	
3	$\quad\quad a \wedge b$	
4	$\quad\quad b$	($\wedge$-ELIM$_2$ on line 3)
5	$\quad a \wedge b \rightarrow b$	($\rightarrow$-INT on lines 3–4)
6	$\quad\quad \neg a \wedge \neg b$	
7	$\quad\quad \neg a$	($\wedge$-ELIM$_1$ on line 6)
8	$\quad\quad a \wedge \neg a$	($\wedge$-INT on lines 2, 7)
9	$\quad\quad b$	(theorem: $\phi \wedge \neg\phi \vdash \psi$ on line 8)
10	$\quad \neg a \wedge \neg b \rightarrow b$	($\rightarrow$-INT on lines 6–9)
11	$\quad b$	($\vee$-ELIM on lines 5, 10, 1)
12	$a \rightarrow b$	($\rightarrow$-INT on lines 2–11)
13	$\quad b$	
14	$\quad\quad a \wedge b$	
15	$\quad\quad a$	($\wedge$-ELIM$_1$ on line 14)
16	$\quad a \wedge b \rightarrow a$	($\rightarrow$-INT on lines 14–15)
17	$\quad\quad \neg a \wedge \neg b$	
18	$\quad\quad \neg b$	($\wedge$-ELIM$_2$ on line 17)
19	$\quad\quad b \wedge \neg b$	($\wedge$-INT on lines 13, 18)
20	$\quad\quad a$	(theorem: $\phi \wedge \neg\phi \vdash \psi$ on line 19)
21	$\quad \neg a \wedge \neg b \rightarrow a$	($\rightarrow$-INT on lines 17–20)
22	$\quad a$	($\vee$-ELIM on lines 16, 21, 1)
23	$b \rightarrow a$	($\rightarrow$-INT on lines 13–22)
24	$a \leftrightarrow b$	($\leftrightarrow$-INT on lines 12 and 23)

The theorem $\phi \wedge \neg\phi \vdash \psi$ is also proved in Prof Pace's book, on page 43.