Ordinary Differential Equations

Pure Mathematics A-Level

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We cover the following first/second order linear ordinary differential equations (ODEs). The *order* of an ODE is determined by the highest derivative. By *linear* we mean that all the derivatives have power 1, and by *ordinary*, we mean that the derivatives are of a function (y) of only one independent variable (x).

First Order ODEs

1. **Separable**

   These are equations which can be brought to the form

   $$f(y)\,\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$

   Integrating both sides gives a solution.

2. **Product-Rule Equations**

   These are equations which are the direct result of applying the product rule. They generally have the form

   $$f(y)g'(x) + f'(y)g(x) = h(x)$$

   which can be transformed to

   $$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = h(x)$$

   Again, integrating both sides gives a solution.

3. **Solvable by Integrating Factors**

   Equations of the form

   $$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = g(x)$$

   can be reduced to exact equations by multiplying throughout by

   $$I(x) = \exp\int f(x)\,\mathrm{d}x$$

   known as the *integrating factor* (Note that $\exp x \equiv e^x$).

Second Order ODEs

1. **Homogeneous with Constant Coefficients**

   A differential equation is *homogeneous* when it equals zero. Here we consider the equation

   $$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

   where $a, b, c \in \mathbb{R}$ are constants. First, we solve the **auxiliary equation**

   $$ak^2 + bk + c = 0$$

   whose solutions are $k = k_1$ and $k = k_2$. The general solution is then given by

   $$y = \begin{cases} c_1e^{k_1x} + c_2e^{k_2x} & \text{if } \; k_1 \neq k_2 \\ e^{kx}(c_1 + c_2x) & \text{if } \; k = k_1 = k_2 \\ e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x) & \text{if } \; k = \alpha \pm \beta i \in \mathbb{C} \end{cases}$$

   where c_1, c_2 are arbitrary constants.

2. **Inhomogeneous with Constant Coefficients**

   A differential equation is *inhomogeneous* if it is not homogeneous. Here we consider the equation

   $$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x) \neq 0$$

   where $a, b, c \in \mathbb{R}$ are constants.

   We solve by following these steps:

   (i) Solve the **homogeneous equation**

   $$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

   to obtain the **complementary function** $\mathrm{cf}(x)$.

   (ii) Obtain a **trial solution** in the form of $f(x)$ which caters for second order differentiation. These can be shown in the table below.

	$f(x)$	Trial Solution, $\mathrm{ts}(x)$
Polynomials	a $ax + b$ $ax^2 + bx + c$ $\vdots$	λ $\lambda x + \mu$ $\lambda x^2 + \mu x + \eta$ $\vdots$
Exponentials*	$ae^{\alpha x}$	$\lambda e^{\alpha x}$ if $k_1 \neq \alpha \neq k_2$ $\lambda x e^{\alpha x}$ if $k_1 = \alpha \neq k_2$ $\lambda x^2 e^{\alpha x}$ if $k_1 = \alpha = k_2$
Trigonometric	$a \cos \alpha x + b \sin \alpha x$	$\lambda \cos \alpha x + \mu \sin \alpha x$

   *Note that k_1, k_2 are the solutions to the auxiliary equation solved in part (i).

Observe that any of the constants in $f(x)$ can be zero, but the trial solution can still have nonzero constants. E.g. if $f(x) = x^2$, then we still take the trial solution to be $\lambda x^2 + \mu x + \eta$. Similarly, the function $f(x) = x^2 + \sin 2x$ has trial solution $\lambda x^2 + \mu x + \eta + \phi \cos 2x + \psi \sin 2x$.

(iii) Determine the trial solution derivatives $\mathrm{ts}'(x)$ and $\mathrm{ts}''(x)$, and substitute them in the original ODE. Compare coefficients to determine correct values for the constants so that the result will equal $f(x)$.

The trial solution with the constant(s) found is called the **particular integral**, $\mathrm{pi}(x)$.

(iv) The general solution is given by $y = \mathrm{cf}(x) + \mathrm{pi}(x)$.

Remark

A remark about this method for second order ODEs. The reason such a method works is due to the fact that differentiation is a *linear operator*, i.e. we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\alpha\, f(x) + \beta\, g(x)) = \alpha \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \beta \frac{\mathrm{d}}{\mathrm{d}x} g(x)$$

So upon substitution of the solution $y = \mathrm{cf}(x) + \mathrm{pi}(x)$ into our ODE, we have the following:

$$\begin{aligned}
\mathcal{LHS} &= a\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\mathrm{cf}(x) + \mathrm{pi}(x)) + b\,\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cf}(x) + \mathrm{pi}(x)) + c\,(cf(x) + pi(x)) \\
&= \underbrace{a\,\mathrm{cf}''(x) + b\,\mathrm{cf}'(x) + c\,\mathrm{cf}(x)}_{=0} + \underbrace{a\,\mathrm{pi}''(x) + b\,\mathrm{pi}'(x) + c\,\mathrm{pi}(x)}_{=f(x)} \\
&= 0 + f(x) \\
&= f(x) = \mathcal{RHS}
\end{aligned}$$

It is very often the case that students naïvely ask:

> *Why do we need the complementary function? Isn't the particular integral alone enough to make the $\mathcal{LHS} = \mathcal{RHS}$, and therefore sufficient as a solution?*

The reason we need the complementary function is that it makes our solution more general (much like adding $+c$ when integrating). We have a function which annihilates the $\mathcal{LHS}$, thus incorporating it into our general solution will give us all possible solutions to the ODE. Without it, we are assuming that only the substitution $y = 0$ can make the $\mathcal{LHS} = 0$, which is not the case. In fact, if initial conditions are given (so we would wish to determine a particular solution) then we would definitely need the complementary function.