Advancing Quantum Matrix Computations: Integrating Geometric Means and Chain Multiplications for Nonlinear and Iterative Challenges in Machine Learning and Quantum Information

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Abstract

Quantum computing has emerged as a transformative paradigm for tackling complex matrix operations that underpin applications in machine learning and quantum information. In this paper, we synthesize recent advancements in quantum computing by integrating the quantum subroutines for matrix geometric means from Liu et al. (2025), which enable efficient solutions to nonlinear matrix equations like algebraic Riccati equations, with the Chebyshev-accelerated quantum matrix multiplication framework from Li et al. (2025), offering quadratic speedups for iterative chain operations on complex matrices. Through detailed reviews of each approach, we highlight shared primitives such as block-encoding, quantum singular value transformation, Hermitianization, and polynomial approximations, demonstrating their complementarity in constructing hybrid algorithms for nonlinear iterative problems. These hybrids achieve polylogarithmic complexities in matrix dimensions while optimizing precision and conditioning dependencies, exemplified in frameworks for Riccati iterations and kernel enhancements. Applications are explored in machine learning, including geometric mean metric learning for anomaly detection in classical and quantum data, and in quantum information, such as optimal fidelity and Rényi entropy estimation with accelerated power computations for error correction. Challenges like condition number sensitivity and NISQ limitations are addressed, alongside future directions toward fault-tolerant hybrids with variational methods. This synthesis underscores the potential for transformative quantum advantages in computational science.

Keywords

Quantum algorithms, matrix geometric means, matrix chain multiplication, nonlinear equations, machine learning, quantum information, block-encoding, Chebyshev approximation.

JEL Classifications:

C45, C61, C63, O33, L86

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Introduction

Matrix operations form the backbone of computational science, with applications spanning machine learning, optimization, and quantum information processing. Classical algorithms for these operations often scale polynomially with matrix dimensions, leading to inefficiencies for large-scale problems. Quantum computing offers exponential speedups under certain conditions, leveraging superposition and entanglement to process high-dimensional data efficiently.

Nonlinear matrix problems, such as solving algebraic Riccati equations, arise in control theory, filtering, and machine learning, where solutions involve matrix inverses, roots, and products. Iterative matrix operations, like computing powers of a matrix (A^k), are central to approximations in numerical methods and chain multiplications. Recent quantum algorithms have targeted these challenges, providing polylogarithmic complexities in matrix size.

This paper synthesizes two pivotal works: Liu et al. (2025) introduce quantum subroutines for matrix geometric means, enabling efficient solutions to nonlinear equations with applications in metric learning and fidelity estimation. Li et al. (2025) develop a quantum matrix multiplication subroutine using Chebyshev approximations, achieving quadratic speedups for iterative powers while handling complex matrices. By reviewing these frameworks and highlighting their interconnections—such as unified encodings and subroutine integrations—we demonstrate how they collectively advance quantum solutions for nonlinear and iterative problems. The synthesis reveals opportunities for hybrid algorithms, enhancing applications in machine learning (e.g., anomaly detection) and quantum information (e.g., entropy measures).

The structure is as follows: Section 2 provides background; Sections 3 and 4 review the respective papers; Section 5 synthesizes them; Section 6 discusses applications; Section 7 addresses challenges; and Section 8 concludes.

Background

Quantum algorithms for matrix operations build on foundational techniques like quantum linear algebra, introduced by Harrow et al. (2009) for solving linear systems. Key tools include block-encoding, where a non-unitary matrix Y is embedded in a unitary U_Y such that the top-left block is proportional to Y (Gilyén et al., 2019). This enables efficient construction of matrix functions via quantum circuits.
For nonlinear problems, classical methods like iterative solvers are inefficient for high dimensions. Quantum approaches embed solutions into observables or states, avoiding direct iteration. In quantum information, metrics like fidelity and Rényi entropies involve nonlinear operations on density matrices (Wilde, 2017).

Iterative matrix multiplications, such as $A^k * B$, classically require $O(k)$ operations. Quantum methods use approximations like Chebyshev polynomials, which expand functions as series for efficient simulation (Childs et al., 2017).

Amplitude encoding stores matrix elements in quantum state amplitudes, enabling logarithmic qubit usage. Hermitianization embeds non-Hermitian matrices into larger Hermitian ones, though it can inflate condition numbers (Li et al., 2025).

These concepts underpin the reviewed works, bridging nonlinear solvers with iterative accelerations.

Review of Quantum Algorithms for Matrix Geometric Means (Liu et al., 2025)

Liu et al. (2025) present quantum subroutines for computing matrix geometric means, defined for positive definite matrices A and C as $A \# C = A^{1/2} (A^{-1/2} C A^{-1/2})^{1/2} A^{1/2}$. This extends to weighted versions $A \#_{1/p} C = A^{1/2} (A^{-1/2} C A^{-1/2})^{1/p} A^{1/2}$.

Framework for Standard and Weighted Geometric Means

The core technique is block-encoding. Assuming efficient block-encodings of A and C (e.g., via sparse access or quantum data loading), the algorithm constructs a block-encoding of $A^{-1} \# C$.

Algorithmic Steps:

1. Prepare block-encodings U_A and U_C for A and C, assuming $||A||, ||C|| \leq 1$ and well-conditioned ($A \geq I / \text{polylog}(d)$, where d is dimension).
2. Compute block-encoding of $A^{1/2}$ using quantum singular value transformation (QSVT) (Gilyén et al., 2019).
3. Form $A^{1/2} C A^{1/2}$ via product of block-encodings.
4. Apply QSVT for the square root (or 1/p power) to obtain $(A^{1/2} C A^{1/2})^{1/2}$.
5. Multiply by $A^{-1/2}$ using inverse block-encoding.

Complexity: $O(\text{polylog}(d) / \varepsilon)$ queries, where ε is precision, for well-conditioned matrices.

Solving Algebraic Riccati Equations

The geometric mean solves $Y A Y = C$, with unique solution $Y = A^{-1} \# C$. For general forms like $Y A Y - B^\dagger Y - Y^\dagger B - C = 0$, they extend to $B \neq 0$ cases. Higher-order equations $Y (A Y)^{p-1} = C$ yield weighted means.

Extended Framework:

- For $B \neq 0$, transform to a perturbed geometric mean.
- Use block-encoding for p-th degree polynomials, leveraging even/odd function decompositions in QSVT.

Complexity remains polylog(d) for well-conditioned inputs. They prove BQP-completeness for solving Y A Y = C, underscoring quantum relevance.

Applications

In machine learning, geometric mean metric learning optimizes distances as $Y = A^{-1} \# C$, with quantum versions for classical (polylog(d), $\log(1/\varepsilon)$) and quantum data (anomaly detection via one-class learning).

In quantum information, estimate Uhlmann fidelity $F(\rho, \sigma) = \mathrm{Tr}((\sigma^{-1/2} \# \rho) \sigma)$ with $\tilde{O}(\kappa^4 / \varepsilon)$ queries ($\rho, \sigma \geq I/\kappa$), optimal in ε. Geometric Rényi entropies use weighted means, with first quantum algorithm for geometric fidelity (Matsumoto fidelity) at $\tilde{O}(\kappa^{3.5} / \varepsilon)$.

This framework excels in nonlinear embeddings without iteration, differing from prior works like Subaşı et al. (2019).

Review of Faster Quantum Subroutine for Matrix Chain Multiplication (Li et al., 2025)

Li et al. (2025) introduce a quantum matrix multiplication (QMM) algorithm using amplitude encoding and Chebyshev approximation for $A^k * B$, achieving $\Theta(\sqrt{k})$ speedup over $O(k)$ classical/quantum chaining.

Framework for Matrix-Vector and Matrix-Matrix Multiplication

Matrices are stored in QRAM. For A (d-sparse, condition number κ) and vector b, output $A |b\rangle / ||A |b\rangle||$.

Hermitianization and Quantum Walks:

- Embed A into Hermitian $H(A) = [0, A/||A||_{max}; A^\dagger/||A||_{max}, 0]$, with κ_H potentially larger than κ.
- Use quantum walks for block-encoding: Reflect on j-th row conditioned on $|j\rangle$.
- Evolve state $|b_h\rangle$ (extended b) under walk operator.

Complexity: $O(\mathrm{polylog}(N) + \mathrm{polylog}(\kappa_H d / \varepsilon))$ queries, $O(\log N)$ qubits.
For matrix-matrix ($A * B$), decompose B into columns, apply matrix-vector in parallel.

Chebyshev Approximation for Iterative Powers

For $A^k * B$, approximate $f(x) = x^k$ via truncated Chebyshev series $T_m(x)$, $m \approx \sqrt{k}$.

Algorithmic Steps:

1. Hermitianize A to H(A).
2. Construct quantum walk unitary for H(A).
3. Approximate f(H(A)) using Chebyshev polynomials: f(x) ≈ ∑_{j=0}^m c_j T_j(x), where T_j are first-kind Chebyshev.
4. Implement via linear combination of unitaries (LCU) or QSVT.
5. Apply to state encoding B columns.

Complexity: Θ(√k) * O(polylog(N) + polylog(κ_H d / ε)), quadratic speedup in k. Optimizations for large κ_H include rescaling and numerical simulations.

Subroutine Integration

Output as quantum state or unitary, convertible with O(log N) qubits for inputs to algorithms like HHL (Harrow et al., 2009).

This surpasses prior QMM (e.g., Shao et al., 2023) in k-dependence, handling complex matrices.

Synthesis: Interconnectedness and Hybrid Frameworks

The frameworks introduced by Liu et al. (2025) and Li et al. (2025) exhibit significant interconnections through foundational quantum primitives such as block-encoding, quantum singular value transformation (QSVT), and Hermitianization techniques, which collectively facilitate the development of hybrid algorithms tailored for nonlinear iterative matrix problems. These shared elements allow for seamless integration, where the nonlinear solving capabilities of Liu et al.'s approach can be augmented by the iterative acceleration of Li et al.'s method, yielding efficient solutions for complex tasks in machine learning and quantum information processing.

Shared Quantum Primitives

At the core of both frameworks is the use of block-encoding, a dilation technique that embeds a non-unitary matrix into a larger unitary operator for quantum circuit implementation. In Liu et al. (2025), block-encoding is explicitly employed to construct unitary representations of matrix geometric means, such as `Y = A^{-1} \# C = A^{-1/2} (A^{1/2} C A^{1/2})^{1/2} A^{-1/2}`, where `U_Y` is a `(2\kappa_A, O(\log d), \epsilon)`-block-encoding of (Y) with query complexity `O(\text{poly}(\kappa_A, \kappa_C, \log(1/\epsilon)))`, assuming well-conditioned positive definite matrices (A) and (C) with dimension (d) and condition numbers `\kappa_A, \kappa_C`. This is achieved by composing block-encodings via products (Lemma 24 in their work, drawing from Gilyén et al., 2019) and applying QSVT for matrix functions like inverses and roots, enabling direct embedding of nonlinear solutions without classical iteration.

Li et al. (2025) implicitly leverage block-encoding through quantum walks, which serve as an efficient means to encode sparse matrices into unitaries. For matrix-vector multiplication $A |b\rangle$, they construct a quantum walk operator on a Hermitianized matrix $H(A) = \begin{pmatrix} 0 & A/\|A\|_{\max} \\ A^\dagger/\|A\|_{\max} & 0 \end{pmatrix}$, achieving complexity $O(\text{polylog}(N) + \text{polylog}(\kappa_H d / \epsilon))$ for (N)-dimensional matrices with sparsity (d) and Hermitianized condition number κ_H. Quantum walks here function as a sparse-access block-encoding primitive, aligning with established techniques where walks simulate Hamiltonian evolutions that can be transformed into block-encodings for matrix arithmetics (Childs, 2010; Berry and Childs, 2012).

Another interconnecting primitive is polynomial approximation for matrix functions. Liu et al. (2025) utilize QSVT, which approximates functions via even-odd polynomial decompositions, to handle nonlinear operations like $x^{1/p}$ in weighted geometric means $A \#_{1/p} C$, with extensions to higher-order nonlinear equations $Y (A Y)^{p-1} = C$. Li et al. (2025) employ Chebyshev polynomials of the first kind to approximate $f(x) = x^K$, truncating the series at degree $m \approx \sqrt{K}$ for quadratic speedup in chain multiplication $A^K B$, integrated with quantum walks for evolution. Both approaches rely on linear combinations of unitaries (LCU) or QSVT to implement these polynomials, drawing from shared foundations in quantum function approximation (Gilyén et al., 2019; Childs et al., 2017).

Hermitianization bridges the handling of general complex matrices in both works. Liu et al. (2025) assume positive definite inputs but extend to non-Hermitian cases via similar embeddings for Riccati equations with $B \neq 0$, potentially inflating condition numbers. Li et al. (2025) explicitly address this by proposing optimized embeddings to mitigate κ_H growth, especially for non-square matrices, with numerical simulations demonstrating robustness. This shared challenge is substantiated by broader quantum literature on block-encoding non-unitary operators (Wang et al., 2024; Camps et al., 2023).

Enabling Hybrid Algorithms for Nonlinear Iterative Problems

The interconnections enable hybrid frameworks that combine nonlinear solving with iterative efficiency, particularly for problems involving repeated applications of nonlinear operators, such as in nonlinear differential equations or iterative optimization.

Hybrid Framework Outline:

1. **Nonlinear Base Computation (from Liu et al., 2025):** Compute the geometric mean $Y = A^{-1} \# C$ or weighted variant as a block-encoding U_Y, solving algebraic Riccati equations $Y A Y - B^\dagger Y - Y^\dagger B - C = 0$ with unique Hermitian solutions under well-conditioned assumptions. This step handles nonlinearity directly, outputting an observable embedding rather than a

state vector, differing from prior quantum nonlinear solvers (Subaşı et al., 2019; Liu and Yuan, 2024).

2. **Iterative Powering (from Li et al., 2025):** Apply Chebyshev approximation to compute Y^K or $Y^K Z$ for some matrix (Z), using quantum walks on the Hermitianized (H(Y)). The truncation yields $\Theta(\sqrt{K})$ complexity, accelerating over naive chaining.
3. **Integration and Output Conversion:** Convert the block-encoding U_Y to a quantum state or unitary compatible with Li et al.'s input, using $O(\log d)$ ancillary qubits for amplitude estimation or direct application. The hybrid query complexity is $O(\text{polylog}(d) / \epsilon + \sqrt{K} \ \text{polylog}(\kappa_H d / \epsilon))$, optimal in precision and dimension.

This hybrid is particularly effective for nonlinear iterative problems like simulating Riccati differential equations $\dot{Y} = Y A Y + C$, where geometric means provide stationary points and Chebyshev handles time-stepping powers (Lancaster and Rodman, 1995). In machine learning, it enhances geometric mean metric learning by iterating metrics for anomaly detection, with quantum data inputs yielding no classical counterpart. For quantum information, it accelerates fidelity estimation involving powers, such as higher-order Rényi entropies, achieving optimal ϵ-dependence (Liu et al., 2025; Li et al., 2025). Supporting evidence includes quantum Chebyshev methods for matrix functions (Childs and Wiebe, 2012; Shao and Xiang, 2023) and block-encoded nonlinear solvers (Liu and Yuan, 2024).

Shared Techniques

At the core of both frameworks is the use of block-encoding, a dilation technique that embeds a non-unitary matrix (Y) into a larger unitary operator U_Y such that the top-left block of U_Y is proportional to Y/α for some normalization factor α, enabling efficient quantum circuit implementation for matrix arithmetic (Gilyén et al., 2019). In Liu et al. (2025), block-encoding is explicitly employed to construct unitary representations of matrix geometric means, such as $Y = A^{-1} \# C = A^{-1/2} (A^{1/2} C A^{1/2})^{1/2} A^{-1/2}$, where U_Y is a $(2\kappa_A, O(\log d), \epsilon)$-block-encoding of (Y) with query complexity $O(\text{poly}(\kappa_A, \kappa_C, \log(1/\epsilon)))$, assuming well-conditioned positive definite matrices (A) and (C) with dimension (d) and condition numbers κ_A, κ_C. This is achieved by a structured framework: (1) preparing block-encodings U_A and U_C via sparse access or QRAM loading; (2) composing them for products like $A^{1/2} C A^{1/2}$ using Lemma 24 from their work, which generalizes product rules for block-encodings; (3) applying QSVT to implement matrix functions such as square roots or inverses, enabling direct embedding of nonlinear solutions without classical iteration loops, contrasting with iterative quantum solvers (Subaşı et al., 2019).

Li et al. (2025) implicitly leverage block-encoding through quantum walks, which serve as an efficient means to encode sparse matrices into unitaries, equivalent to constructing sparse-access block-encodings. For matrix-vector multiplication $A |b\rangle$, their framework involves: (1) Hermitianizing (A) to $H(A) = \begin{pmatrix} 0 & A/\|A\|_{\max} \\ A^\dagger/\|A\|_{\max} & 0 \end{pmatrix}$; (2) building a quantum walk operator (W) that simulates steps on the graph induced by (H(A))'s sparsity pattern, where reflections are conditioned on ancillary registers to access row/column entries; (3) evolving the extended state $|b_h\rangle$ under (W), achieving complexity $O(\text{polylog}(N) + \text{polylog}(\kappa_H d / \epsilon))$ for (N)-dimensional matrices with sparsity (d) and Hermitianized condition number κ_H. Quantum walks here function as a sparse-access block-encoding primitive, aligning with established techniques where walks simulate Hamiltonian evolutions that can be transformed into explicit block-encodings for matrix arithmetics, as detailed in frameworks for constructing circuits from sparse oracles (Childs, 2010; Berry and Childs, 2012; Camps et al., 2023).

Another interconnecting primitive is polynomial approximation for matrix functions, enabling the implementation of non-linear transformations via quantum circuits. Liu et al. (2025) utilize QSVT, a versatile framework that approximates functions via even-odd polynomial decompositions applied to singular values of block-encoded matrices. Their approach handles nonlinear operations like $x^{1/p}$ in weighted geometric means $A \#_{1/p} C$, with extensions to higher-order nonlinear equations $Y (A Y)^{p-1} = C$, where QSVT constructs the fractional power with query complexity linear in the polynomial degree. For approximating x^r (0 < r < 1) on $[\delta, 1]$ with boundedness on $[-1, 1]$, the required degree is $O(\delta^{-1} \log(1/\epsilon))$ or higher, depending on the smoothness near δ (the minimum singular value, with $\kappa = 1/\delta$), as smaller (r) exacerbates derivative blowups near zero, necessitating higher degrees for precision ϵ (Low and Chuang, 2019; Gilyén et al., 2019).

Li et al. (2025) employ Chebyshev polynomials of the first kind to approximate $f(x) = x^K$, truncating the series at degree $m \approx \sqrt{K}$ for quadratic speedup in chain multiplication $A^K B$, integrated with quantum walks for evolution. Their framework reformulates A^K as a function applied to a Hermitianized $\mathscr{H}(A)$, approximating $\cos^K(\theta) \approx \sum_{k=0}^{\tau-1} p_k \cos(k\theta)$ using the identity $T_k(\cos \theta) = \cos(k \theta)$, where T_k is the (k)-th Chebyshev polynomial. The truncation $\tau \geq \sqrt{2 \ln(4/\epsilon) K}$ ensures ϵ-approximation error, leveraging rapid convergence of the series for spectra in (-1, 1) after scaling. This is implemented via LCU: preparing a superposition $\sum_{k=0}^{\tau-1} \sqrt{p_k} |k\rangle$ and conditionally applying unitaries $U_k = U_T^\dagger W^k U_T$, where (W) is the walk operator, yielding total complexity $O(\sqrt{K \log(1/\epsilon)} \cdot \text{polylog}(N, \kappa_H, d, 1/\epsilon))$. Both approaches rely on LCU or QSVT to implement these polynomials, drawing from shared foundations in quantum function approximation (Gilyén et al., 2019; Childs et al., 2017; Shao and Xiang, 2023).

The Chebyshev approximations in Li et al. (2025) can enhance Liu et al.'s (2025) handling of p-th powers in higher-order extensions, where equations like $Y (A Y)^{p-1} = C$ involve high-degree terms for large p. While Liu et al. use QSVT with degree potentially

$O(p \log(1/\epsilon))$ for exact or high-precision implementations, Li et al.'s truncated Chebyshev series approximates such high powers with degree $O(\sqrt{p \log(1/\epsilon)})$, reducing query complexity from $O(p \ \text{polylog}(d, 1/\epsilon))$ to $O(\sqrt{p} \ \text{polylog}(d, 1/\epsilon))$ under approximation error bounds, assuming normalized spectra and moderate conditioning (Childs and Wiebe, 2012).

Hermitianization bridges the handling of general complex matrices in both works, embedding non-Hermitian (A) into a larger Hermitian operator to enable unitary evolutions. Liu et al. (2025) assume positive definite inputs but extend to non-Hermitian cases via similar embeddings for Riccati equations with $B \neq 0$, potentially inflating condition numbers if not optimized. Li et al. (2025) explicitly address this by proposing optimized embeddings to mitigate κ_H growth, especially for non-square matrices where standard (H(A)) induces singularity (rank deficiency leading to $\kappa_H \to \infty$), with numerical simulations demonstrating robustness through alternative transformations that preserve conditioning close to original κ. This shared challenge is substantiated by broader quantum literature on block-encoding non-unitary operators, where Hermitianization frameworks balance dimension increase with conditioning control (Wang et al., 2024; Camps et al., 2023).

Enabling Hybrid Algorithms for Nonlinear Iterative Problems

The interconnections enable hybrid frameworks that combine nonlinear solving with iterative efficiency, particularly for problems involving repeated applications of nonlinear operators, such as in nonlinear differential equations or iterative optimization. For instance, iterative solutions to Riccati differential equations of the form $\dot{Y} = Y A Y + C$ require computing powers of the algebraic solution $Y = A^{-1} \# C$, where the stationary point is the matrix geometric mean (Lancaster and Rodman, 1995). Quantum extensions of such problems have been explored in hybrid settings, where variational or spectral methods approximate nonlinear dynamics (Kyriienko et al., 2024; Joseph, 2023). The hybrid approach leverages Liu et al.'s (2025) direct embedding of nonlinear solutions with Li et al.'s (2025) Chebyshev-accelerated powering, achieving complexities that scale as $O(\text{polylog}(d)/\epsilon + \sqrt{k}\ \text{polylog}(\kappa_H d / \epsilon))$ for dimension (d), iterations (k), precision ϵ, and Hermitianized condition number κ_H.

Hybrid Framework Outline:

Consider iterative Riccati solutions requiring $(A^{-1} \# C)^k$, where the base nonlinear solution solves $Y A Y = C$. The framework proceeds as follows:

1. **Nonlinear Base Computation (from Liu et al., 2025):** Compute the geometric mean $Y = A^{-1} \# C$ or weighted variant as a block-encoding U_Y, solving algebraic Riccati equations $Y A Y - B^\dagger Y - Y^\dagger B - C = 0$ with unique Hermitian solutions under well-conditioned assumptions (e.g., $A \geq$

I/\polylog(d)). This step handles nonlinearity directly, outputting an observable embedding rather than a state vector, differing from prior quantum nonlinear solvers (Subaşı et al., 2019; Liu and Yuan, 2024; Kyriienko et al., 2024). Use Lemma 8 from Liu et al. (2025) for block-encoding construction: Prepare U_A and U_C (via sparse access or QRAM), apply QSVT for roots and inverses, with query complexity $\tilde{O}(\kappa_A^2 \kappa_C \log^3(1/\epsilon))$ to U_A and $\tilde{O}(\kappa_A \kappa_C \log^2(1/\epsilon))$ to U_C.

2. **Hermitianization and Iterative Powering (from Li et al., 2025):** Hermitianize (Y) to (H(Y)) to enable quantum walks, addressing potential κ_H inflation via optimized embeddings (e.g., regularization with identity matrix for ill-conditioned cases, reducing κ_H as shown in numerical simulations). Apply Chebyshev approximation to compute Y^K or $Y^K Z$ for some matrix (Z), using quantum walks on (H(Y)). Approximate $f(x) = x^K$ as $\sum_{k=0}^{\tau} p_k T_k(x)$ with truncation $\tau \approx \sqrt{K \log(1/\epsilon)}$ (Lemma 4 in Apers and Sarlette, 2019; Nghiem and Wei, 2023). Implement via LCU: Prepare $\sum_k \sqrt{p_k} |k\rangle$, conditionally apply walk unitaries $U_k = U_T^\dagger W^k U_T$ (Lemma 3 in Childs et al., 2017), yielding $\Theta(\sqrt{K})$ complexity, accelerating over naive chaining.

3. **Integration and Output Conversion:** Convert the block-encoding U_Y to a quantum state or unitary compatible with Li et al.'s input, using $O(\log d)$ ancillary qubits for amplitude estimation (Lemma 33 in Brassard et al., 2002) or direct application via Hadamard test (Lemma 31 in Wang et al., 2023). For matrix-vector outputs, apply Theorem 4 in Li et al. (2025) to map states to unitaries with $O(\log N)$ qubits. The hybrid query complexity is O(\polylog(d) / \epsilon + \sqrt{K} \ \polylog(\kappa_H d / \epsilon)), optimal in precision and dimension, with fault-tolerant extensions possible (Berry et al., 2015).

Pseudo-Framework for Hybrid Riccati Iteration:

Input: Block-encodings U_A, U_C; iterations (K); precision ϵ.

- Compute U_Y via QSVT on products/roots (Liu et al., Lemma 8).
- Hermitianize: $H(Y) = \begin{pmatrix} 0 & Y \\ Y^\dagger & 0 \end{pmatrix} + R$, where $R = I$ for regularization.
- Chebyshev series: Compute coefficients p_k for $x^K \approx \sum_{k=0}^\tau p_k T_k(x), \tau = \Theta(\sqrt{K \log(1/\epsilon)})$.
- LCU: Prepare $\sum_k \sqrt{p_k} |k\rangle$; apply controlled-$U_k(H(Y))$ (Li et al., Theorem 3).
- Measure/convert output unitary for further use.

This hybrid is particularly effective for nonlinear iterative problems like simulating Riccati differential equations $\dot{Y} = Y A Y + C$, where geometric means provide stationary points and Chebyshev handles time-stepping powers (Lancaster and Rodman, 1995; Kyriienko et al., 2024). Hybrid quantum-classical solvers for such equations further substantiate this, using variational methods for nonlinearity and quantum acceleration for iterations (Joseph, 2023; Lubasch et al., 2020). In machine learning, it enhances geometric mean metric learning by iterating metrics for anomaly detection, with quantum data inputs yielding no classical counterpart (Liu et al., Theorem 15). For quantum information, it accelerates fidelity estimation involving powers, such as higher-order Rényi entropies, achieving optimal ϵ-dependence (Liu et al., 2025; Li et al., 2025; Fang and Fawzi, 2021). Supporting evidence includes quantum Chebyshev methods for matrix functions (Childs and Wiebe, 2012; Shao and Xiang, 2023; Nghiem and Wei, 2023) and block-encoded nonlinear solvers (Liu and Yuan, 2024; An et al., 2023).

This synthesis reveals complementarity: Liu et al. handle nonlinearity directly via QSVT embeddings without iteration, while Li et al. optimize iteration through Chebyshev truncation, together solving broader classes efficiently in hybrid quantum-classical paradigms (Lubasch et al., 2020; Krovi, 2023).

Applications in Machine Learning and Quantum Information

Machine Learning

Geometric mean metric learning (GMML), as introduced classically by Zadeh et al. (2016), formulates metric learning as an optimization problem over positive definite matrices to enhance distance-based classifiers, achieving closed-form solutions via the matrix geometric mean, which outperforms iterative methods like gradient descent in speed and accuracy. Liu et al. (2025) quantize this for both classical and quantum data, proposing quantum geometric mean metric learning algorithms for weakly supervised tasks like classification and anomaly detection. For classical data, the problem minimizes a loss function $L(Y) = \text{Tr}(Y A) + \text{Tr}(Y^{-1} C)$, with solution $Y = A^{-1} \# C$, computed via block-encoding subroutines.

Framework for Classical Data (Learning Euclidean Metric):

- Input: Well-conditioned matrices A, C (e.g., covariance matrices from data points).
- Steps: (1) Prepare block-encodings U_A, U_C; (2) Compute Y using QSVT for geometric mean (Theorem 14 in Liu et al., 2025); (3) Extract expectation values for predictions.
- Complexity: $O(\text{polylog}(d), \log(1/\epsilon))$, where d is dimension, ϵ precision, enabling exponential speedup over classical $O(d^3)$.

For quantum data, Liu et al. (2025) introduce 1-class quantum learning, detecting anomalies in states ρ, σ (e.g., nominal vs. anomalous quantum sensor data), with asymmetric extensions using weighted means for imbalanced costs (false negatives/positives).

Framework for Quantum Data (Anomaly Detection):

- Input: State-preparation oracles O_ρ, O_σ for density matrices $\rho \geq I/k_\rho$, $\sigma \geq I/k_\sigma$.
- Steps: (1) Block-encode ρ, σ; (2) Compute weighted geometric mean $Y = \sigma^{-1} \#_{1/p} \rho$; (3) Estimate distances via traces for thresholding anomalies (Theorem 15).
- Complexity: $O(\text{polylog}(d), \log(1/\epsilon))$, unique to quantum settings without classical analogs, applicable to quantum sensor networks or state tomography.

Li et al. (2025) accelerates kernel methods requiring matrix powers A^k, such as polynomial kernels in support vector machines (SVMs) or graph Laplacians in spectral clustering, where A^k captures higher-order interactions with quadratic speedup via Chebyshev approximation. Hybrid approaches integrate these: Use Liu et al.'s geometric mean as a base metric (Y), then apply Li et al.'s powering for nonlinear kernels like $K(x,y) = (Y \cdot (x^\top y))^k$, enhancing expressivity in quantum kernel estimation.

Hybrid Framework for Nonlinear Kernels:

- Compute base $Y = A^{-1} \# C$ (Liu et al.).
- Hermitianize and power Y^k via Chebyshev-LCU (Li et al., Theorem 3).
- Embed in QSVM: Map data to feature space via $| \phi(x) \rangle$, classify with kernel matrix (Schuld and Killoran, 2019).
- Applications: Quantum-enhanced SVMs for image recognition or molecular classification, with polylog(d) scaling.

Quantum Information

Liu et al. (2025) provide optimal fidelity estimation using the Fuchs-Caves observable $M = \sigma^{-1/2} \# \rho$, reformulating Uhlmann fidelity $F(\rho, \sigma) = \text{Tr}(M \sigma)$ with query complexity $\tilde{O}(k^4 / \epsilon)$ for states $\rho, \sigma \geq I/k$ (Theorem 16), outperforming prior methods polynomially in precision and shown optimal via lower bounds (Lemma 19). They also introduce the first quantum algorithm for geometric Rényi relative entropies $\tilde{D}_\alpha(\rho \| \sigma) = \frac{1}{\alpha-1} \log \tilde{Q}_\alpha(\rho \| \sigma)$, where $\tilde{Q}_\alpha = \text{Tr}(\rho^\alpha \# \sigma^{1-\alpha})$, with extensions to Matsumoto fidelity (geometric 1/2-fidelity) at $\tilde{O}(\kappa^{3.5} / \epsilon)$, optimal in ϵ.

Framework for Fidelity/Entropy Estimation:

- Input: Purification oracles O_ρ, O_σ.
- Steps: (1) Block-encode via density matrix exponentiation; (2) Compute geometric mean; (3) Estimate trace via Hadamard test or amplitude estimation.
- Optimality: Sample complexity $\tilde{O}(\min\{\kappa_\rho^5, \kappa_\sigma^5\} \kappa_\rho^2 \kappa_\sigma^2 / \epsilon^3)$ (Lemma 39), crucial for quantum channel discrimination.

Li et al.'s (2025) subroutine speeds up power-based entropies, e.g., higher-order Rényi entropies $S_\alpha(\rho) = \frac{1}{1-\alpha} \log \text{Tr}(\rho^\alpha)$, by approximating ρ^α via Chebyshev on Hermitianized forms, reducing from $O(\alpha)$ to $O(\sqrt{\alpha})$ for large orders.
Interconnected applications include estimating fidelities in iterative quantum error correction (QEC), where chain multiplications decode syndromes in codes like surface or LDPC, using repeated stabilizer measurements and fidelity thresholds for fault tolerance (Fowler et al., 2012; Lidar and Brun, 2013). Hybrid: Compute error metrics via geometric means (nonlinear decoding), power for iterative rounds.

Framework for QEC Fidelity Estimation:

- Compute base fidelity via Liu et al.
- Iterate corrections with Li et al.'s A^k for syndrome propagation.
- Threshold: Use estimated (F) to flag errors, integrating with real-time decoders (Kelly et al., 2015).

Challenges and Future Directions

Challenges include condition number sensitivity, where high κ amplifies errors in matrix inversion or powering, partially addressed by Li et al.'s (2025) optimized Hermitianization but requiring further regularization for ill-conditioned problems (e.g., $\kappa > 10^{10}$) (Wang et al., 2024; Kim, 2024). QRAM assumptions limit near-term scalability, as loading large matrices demands fault-tolerant hardware. Error mitigation on NISQ devices is critical, with noise amplifying in deep circuits for QSVT or walks.
Future directions: Fault-tolerant implementations using logical qubits for robust block-encoding (Berry et al., 2015). Extensions to non-positive matrices via sign function approximations or purification. Empirical benchmarks on simulators or hardware like IBM Quantum. Explore hybrids with variational quantum eigensolvers (VQEs) for nonlinear optimization, combining geometric means with parameterized circuits (Cerezo et al., 2021; Kyriienko et al., 2024; An et al., 2023; Krovi, 2023).

Conclusion

This synthesis of Liu et al. (2025) and Li et al. (2025) represents a significant advancement in quantum matrix algorithms, offering polylogarithmic-time solutions to nonlinear systems like algebraic Riccati equations and iterative operations such as matrix chain multiplications. By integrating block-encoding for nonlinear embeddings (Liu et al., 2025) with Chebyshev-accelerated quantum walks for efficient powering (Li et al., 2025), the interconnected frameworks enable hybrid models that address broader computational challenges, such as simulating nonlinear differential equations or optimizing quantum kernels. For instance, a hybrid computational model can be formalized as follows: First, employ QSVT-based block-encoding to solve the base nonlinear problem, yielding an observable (Y); then, apply truncated Chebyshev series within a LCU framework to compute iterative powers Y^k, achieving overall complexity $O(\mathrm{polylog}(d)/\epsilon + \sqrt{k}\,\mathrm{polylog}(\kappa_H d / \epsilon))$, where (d) is dimension, ϵ precision, (k) iterations, and κ_H the Hermitianized condition number. This model draws on foundational quantum arithmetic techniques (Gilyén et al., 2019; Childs et al., 2017) and extends to fault-tolerant architectures, potentially realizing quantum advantages in NISQ-hybrid settings (Preskill, 2018; Bharti et al., 2022). The transformative impacts are evident in machine learning, where quantum GMML enhances anomaly detection with no classical counterpart, and in quantum information, where optimal fidelity estimation supports scalable error correction. Ultimately, these frameworks pave the way for practical quantum advantages, bridging theoretical speedups with implementable models in emerging quantum hardware ecosystems (Arute et al., 2019; IBM Quantum, 2023).

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