The use of quaternions in the computation of 3D rotations

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Contents

1 Introduction

1.1 What are quaternions?

Quaternions are a 4-dimensional number system that extend the complex numbers, in a similar way that the complex numbers extend the real numbers. Denoted $\mathbb{H}$, they are of the form:

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad a, b, c, d \in \mathbb{R}, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1. \tag{1}$$

The first term is referred to as the real part of the quaternion, while the other three components are imaginary. Each of these components (which can be thought of as axes in a 4-dimensional space) are orthogonal (perpendicular), and as such no real transformation on any component can map it onto another.

In the modern day, they are used extensively in representing the rotations of 3-dimensional objects in computer systems, due to the relative computational ease of calculating such rotations, which only involves the multiplication of real numbers, and is not susceptible to problems like gimbal lock, unlike Euler angles, which are perhaps easier to visualise and more widely used outside computational settings. [1]

1.2 The structure of this essay

After this brief introduction, some widely used key terms relating to quaternions will be defined, which will be essential to understanding the following chapters. Next, the additive and (more importantly) multiplicative properties of quaternions will be covered, which in effect govern how they are able to represent 3D rotations, and here the definition - and need for - of the quaternion number system will be derived.

Having covered these areas, the mechanism in which they can be used for rotations will be discussed in depth, including relevant proofs and how to construct rotations with quaternions. The essay will conclude by discussing the applications and desirable properties of quaternions.

The main body of the essay, where most of the explanations are, after the key definitions and properties are stated, starts from Section 2.3, on page 6.

Wherever any information, or inspiration, is taken from a source, it is cited in square brackets, with the numbers corresponding to the bibliography entries. All explanations and diagrams that are not followed by a citation are my original work.

This essay has been written with LaTeX using the VimTeX plugin. Figure 1 has been created with GeoGebra and Asymptote, and Figure 2 and Figure 3 have been made with TikZ.

1.3 Definitions

1.3.1 Pure quaternions

A pure quaternion is a quaternion that has no real component, meaning for a quaternion $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, $a = 0$. [2]

They are also known as vector quaternions, as pure quaternions can be used to represent vectors in quaternion form, where the $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ components of the quaternion represent the x, y, and z components of a vector respectively.

For example, the vector $(2, 3, 1)$ can be represented as the quaternion $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

1.3.2 Unit quaternions

A unit quaternion is a quaternion with norm 1, i.e: $|q| = 1$ for a quaternion, q. [3]

1.3.3 The norm of a quaternion

The norm, $|q|$, of a quaternion, q, is defined as:

$$\begin{gathered} |q| = \sqrt{a^2 + b^2 + c^2 + d^2}, \\ \text{where } q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}. \end{gathered} \tag{2}$$

This could be thought of as similar to the magnitude of a vector, giving the theoretical "length" of the quaternion. [2]

This is referred to as the norm, as it is what each component of a quaternion needs to be divided by to normalise it (to get its corresponding unit quaternion - one with the same theoretical "direction").

For example, to normalise the quaternion

$$q = 2 + 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}, \tag{3}$$

one would first find its norm:

$$|q| = \sqrt{2^2 + 3^2 + (-4)^2 + 5^2} = 3\sqrt{6}, \tag{4}$$

and then divide each of its components by the norm:

$$q_u = \frac{2}{3\sqrt{6}} + \frac{3}{3\sqrt{6}}\mathbf{i} + \frac{4}{3\sqrt{6}}\mathbf{j} + \frac{5}{3\sqrt{6}}\mathbf{k}. \tag{5}$$

It can be proven that this is indeed a unit quaternion, as

$$|q_u| = \sqrt{\left(\frac{2}{3\sqrt{6}}\right)^2 + \left(\frac{3}{3\sqrt{6}}\right)^2 + \left(\frac{4}{3\sqrt{6}}\right)^2 + \left(\frac{5}{3\sqrt{6}}\right)^2} = 1. \tag{6}$$

1.3.4 The conjugate of a quaternion

Every quaternion, q, has a conjugate, $\bar{q}$, which is defined as:

$$\begin{gathered} \bar{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}, \\ \text{where } q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}. \end{gathered} \tag{7}$$

Note that $\bar{q}$ can also be written q^*. [3]

2 Defining Properties of Quaternions

The algebraic field of quaternions is defined by certain core properties, which are explored below.

The additive properties are mostly included for context and completeness, while the multiplicative properties are essential for the rotation calculations that will follow - it is in effect these multiplicative properties that mean quaternions are able to compute 3D rotation.

Throughout this section it is to be assumed that $q_n \in \mathbb{H}$, and $a_n, b_n, c_n, d_n \in \mathbb{R}$.

2.1 Additive properties

2.1.1 Closure

Quaternions are considered closed upon addition. This means that:

$$q_1, q_2 \in \mathbb{H} \implies q_1 + q_2 \in \mathbb{H}. \tag{8}$$

In words, the addition of any two quaternions will necessarily produce a quaternion. [4]

2.1.2 Commutativity and associativity

Quaternion addition is both commutative,

$$q_1 + q_2 = q_2 + q_1, \tag{9}$$

and associative [4],

$$(q_1 + q_2) + q_3 = q_1 + (q_2 + q_3). \tag{10}$$

While this may seem obvious, this is not to be assumed for all number systems. In fact, the multiplication of quaternions, though associative, is not commutative. This will be further discussed later.

2.1.3 Calculating addition and subtraction

Summing and finding the difference of two quaternions is relatively straightforward - it simply consists of adding or subtracting each component of the quaternions: [4]

$$\begin{aligned} q_1 + q_2 &= (a_1 + b_1\mathbf{i} + c_1\mathbf{k} + d_1\mathbf{k}) + (a_2 + b_2\mathbf{i} + c_2\mathbf{k} + d_2\mathbf{k}) \\ &= (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\mathbf{k} + (d_1 + d_2)\mathbf{k}, \end{aligned} \tag{11}$$

$$\begin{aligned} q_1 - q_2 &= (a_1 + b_1\mathbf{i} + c_1\mathbf{k} + d_1\mathbf{k}) - (a_2 + b_2\mathbf{i} + c_2\mathbf{k} + d_2\mathbf{k}) \\ &= (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\mathbf{k} + (d_1 - d_2)\mathbf{k}. \end{aligned} \tag{12}$$

2.2 Multiplicative properties

2.2.1 Closure

As with addition, quaternions are considered closed upon multiplication, meaning:

$$q_1, q_2 \in \mathbb{H} \implies q_1 q_2 \in \mathbb{H}. \tag{13}$$

Thus, the product of any two quaternions will necessarily be a quaternion. [4]

2.2.2 Commutativity and associativity

Although quaternion multiplication, like addition, is associative: [4]

$$(q_1 \times q_2) \times q_3 = q_1 \times (q_2 \times q_3), \tag{14}$$

it is crucially not commutative: [2]

$$q_1 \times q_2 \neq q_2 \times q_1. \tag{15}$$

This may seem unintuitive at first, but it explains how multiple different rotations can be encoded within one quaternion even though the order of rotations matters (performing the same rotations in a different order produces a different result). This will be explored in much more depth later.

Note that while the multiplication of two quaternions is not commutative, the multiplication of a quaternion and a real number is:

$$nq_1q_2 = q_1nq_2 = q_1q_2n \neq q_2q_1n, \quad n \in \mathbb{R}. \tag{16}$$

2.2.3 Distributivity

As with both real and complex numbers, quaternion multiplication is considered to be distributive over addition. This means that multiplying a quaternion by the sum of two other quaternions is equal to taking the sum of multiplying the quaternion with each quaternion being added:

$$q_1(q_2 + q_3) = q_1q_2 + q_1q_3. \tag{17}$$

2.2.4 Calculating multiplication

The first step in working out the product of two quaternions is distributing each component:

$$\begin{aligned}
&(a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k})(a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) \\
&= a_1a_2 + a_1b_2\mathbf{i} + a_1c_2\mathbf{j} + a_1d_2\mathbf{k} \\
&\quad + b_1a_2\mathbf{i} + b_1b_2\mathbf{i}^2 + b_1c_2\mathbf{ij} + b_1d_2\mathbf{ik} \\
&\quad + c_1a_2\mathbf{j} + c_1b_2\mathbf{ji} + c_1c_2\mathbf{j}^2 + c_1d_2\mathbf{jk} \\
&\quad + d_1a_2\mathbf{k} + d_1b_2\mathbf{ki} + d_1c_2\mathbf{kj} + d_1d_2\mathbf{k}^2.
\end{aligned} \tag{18}$$

Do note that the order of the quaternion's imaginary components $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ must stay the same, while the order of the real components is not relevant, as discussed in Section 2.2.2.

The result of this distribution, however, is not the final product - it can be further simplified using the defining quaternion multiplication rules, which will be explored in the following subsection.

2.3 The $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components

2.3.1 The defining rules

While the following rules [5] could be considered to be simply multiplicative rules, they almost single-handedly define quaternions as a number system, and so deserve their own section.

They were famously carved into a bridge by William Rowan Hamilton in 1843 [1], after discovering that four-, not three-, dimensional numbers (quaternions) were in fact needed to describe 3D rotations:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \tag{19a}$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \tag{19b}$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \tag{19c}$$

$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{19d}$$

2.3.2 Why four dimensions are needed

These rules could be explained by simply stating they are how they are by definition of the quaternion number system, but by considering why four-, rather than three-, dimensional numbers are needed, the reasons behind each of the above rules will become clear.

It is trivial to consider why complex (two-dimensional) numbers cannot be used for three-dimensional rotations: they simply do not have the scope to even represent a three-dimensional point, let alone transform it. From this point, the natural solution seems to be the use of three-dimensional numbers.

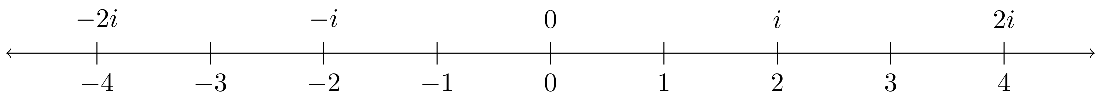
Consider a three-dimensional number, t, where:

$$t = a + b\mathbf{i} + c\mathbf{j}, \quad a, b, c \in \mathbb{R}. \tag{20}$$

It would be possible to represent three-dimensional points in a number like t simply by assigning each of a, b, c to the point's x, y, z co-ordinates. For example, $(2, 3, 4)$ could be represented as $2 + 3\mathbf{i} + 4\mathbf{j}$.

However, where this number system fails is where its multiplicative rules are defined. It is logical to state that $\mathbf{i}^2 = \mathbf{j}^2 = -1$ (this ensures orthogonality to the real component), but the problem comes with defining $\mathbf{ij}$.

It is essential that each component is orthogonal to each other, as this is what allows each component to represent a perpendicular axis. Consider two components not being orthogonal to each other. This would imply that each value in one axis, or of one component can be mapped to a value in the other. For example, if $\mathbf{i} = 2$, doubling any real value would yield its corresponding value in the $\mathbf{i}$ "axis". At this point, it would be illogical to refer to them as seperate axes, as the $\mathbf{i}$ component would simply be equal to a multiplication by 2, and such placing it in the real axis:



As such, the only way to have seperate axes is for no component to be equal to a scalar multiple of another. $\mathbf{i}^2 = -1 \implies \mathbf{i} = \sqrt{-1}$ achieves this, as no real number can equal $\sqrt{-1}$, therefore ensuring the orthogonality of the two axes.

Since $\pm\mathbf{i}$, $\pm\mathbf{j}$, and ± 1 all have a magnitude of 1 (a "distance" of 1 from the origin when thinking of this graphically), the magnitude of $\mathbf{ij}$ must also be 1. This leaves three possibilities for the value of $\mathbf{ij}$: $\pm 1, \pm\mathbf{i}, \pm\mathbf{j}$, since these are the only numbers with a magnitude of 1 in this number system.

Having established that:

$$\mathbf{i}^2 = \mathbf{j}^2 = -1, \tag{21}$$

exploring each case individually shows that none are valid:

$$\mathbf{ij} = \pm\mathbf{i} \implies \mathbf{j} = \pm 1, \tag{22a}$$

$$\mathbf{ij} = \pm\mathbf{j} \implies \mathbf{i} = \pm 1, \tag{22b}$$

$$\mathbf{ij} = \pm\mathbf{1} \implies \mathbf{i} = \mp\mathbf{j} \quad \text{or} \quad \mathbf{j} = \mp\mathbf{i}. \tag{22c}$$

This is because the real, $\mathbf{i}$ and $\mathbf{j}$ components must be orthogonal, and in each of the three cases above one component ends up not being orthogonal to another. For example, if $\mathbf{i} = -\mathbf{j}$, it follows that $\mathbf{i}$ and $\mathbf{j}$ are scalar multiples, and as such no longer orthogonal.

The only solution that can be found to this problem is equating the result of $\mathbf{ij}$ to a new component that is orthogonal to the real, $\mathbf{i}$, and $\mathbf{j}$ "axes". In other words, for the $\mathbf{i}$ and $\mathbf{j}$ components to remain orthogonal, a new component is needed. We can call this new component $\mathbf{k}$, and define it as follows:

$$\mathbf{k}^2 = -1, \quad \mathbf{ij} = \mathbf{k}. \tag{23}$$

We now also need to define the multiplicative rules of $\mathbf{k}$ with $\mathbf{i}$ and $\mathbf{j}$. This is now fairly straightforward: we know that any component multiplied by another must form a seperate component, and that the product of a component must be unique with any other component. Considering what we know about $\mathbf{i}$:

$$\mathbf{i} \times 1 = \mathbf{i}, \tag{24a}$$
$$\mathbf{i} \times \mathbf{i} = -1, \tag{24b}$$
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}. \tag{24c}$$

We can derive the equation for the multiplication of $\mathbf{k}$ and $\mathbf{i}$:

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}. \tag{25}$$

This is because $\mathbf{i}$ multiplies with $1, \mathbf{i}$, and $\mathbf{j}$ to make $\mathbf{i}, -1$, and $\mathbf{k}$ respectively, so the only remaining component that $\mathbf{k}$ can multiply with $\mathbf{i}$ to make is $\mathbf{j}$. Do note that the order of multiplication here is reversed, and this order is significant (in fact $\mathbf{ik} = -\mathbf{j}$) due to the non-commutativity of quaternion multiplication (Section 2.2.2), which is needed to enable them to encode rotations - the exact mechanisms of this will be explained later.

Repeating this process with $\mathbf{j}$ yields the following:

$$\mathbf{j} \times 1 = \mathbf{j}, \tag{26a}$$
$$\mathbf{j} \times \mathbf{j} = -1, \tag{26b}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \tag{26c}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i}. \tag{26d}$$

And so we have now defined all the multiplications of $\mathbf{k}$, while preserving orthogonality:

$$\mathbf{k} \times 1 = \mathbf{k}, \tag{27a}$$
$$\mathbf{k} \times \mathbf{k} = -1, \tag{27b}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j}, \tag{27c}$$
$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}. \tag{27d}$$

Putting these all together, we get:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \tag{28a}$$
$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \tag{28b}$$
$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \tag{28c}$$
$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{28d}$$

The product of any two seperate components $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ yields the other, and reversing the order flips the sign of the result (e.g. $\mathbf{ij} = \mathbf{k} \implies \mathbf{ji} = -\mathbf{k}$).

Note that these equations (28) are the same as those defined for quaternions (19) on page 7. We have now in effect derived the quaternion number system by establishing that a three-dimensional number system cannot be defined.

2.3.3 Simplifying the multiplication

In Section 2.2.4 (page 6), the distributivity of quaternion multiplication was demonstrated through calculating the product of two sample quaternions, but with the multiplicative rules that we have defined (19), we can now simplify this futher:

$$
\begin{aligned}
&(a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k})(a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) \\
&= a_1a_2 + a_1b_2\mathbf{i} + a_1c_2\mathbf{j} + a_1d_2\mathbf{k} \\
&\quad + b_1a_2\mathbf{i} + b_1b_2\mathbf{i}^2 + b_1c_2\mathbf{ij} + b_1d_2\mathbf{ik} \\
&\quad + c_1a_2\mathbf{j} + c_1b_2\mathbf{ji} + c_1c_2\mathbf{j}^2 + c_1d_2\mathbf{jk} \\
&\quad + d_1a_2\mathbf{k} + d_1b_2\mathbf{ki} + d_1c_2\mathbf{kj} + d_1d_2\mathbf{k}^2 \\
&= a_1a_2 + a_1b_2\mathbf{i} + a_1c_2\mathbf{j} + a_1d_2\mathbf{k} \\
&\quad + b_1a_2\mathbf{i} - b_1b_2 + b_1c_2\mathbf{k} - b_1d_2\mathbf{j} \\
&\quad + c_1a_2\mathbf{j} - c_1b_2\mathbf{k} - c_1c_2 + c_1d_2\mathbf{i} \\
&\quad + d_1a_2\mathbf{k} + d_1b_2\mathbf{j} - d_1c_2\mathbf{i} - d_1d_2 \\
&= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) \\
&\quad + (a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1)\mathbf{i} \\
&\quad + (a_1c_2 - b_1d_2 + a_2c_1 + b_2d_1)\mathbf{j} \\
&\quad + (a_1d_2 + b_1c_2 - b_2c_1 + a_2d_1)\mathbf{k}.
\end{aligned}
\tag{29}
$$

3 Calculating Rotations

3.1 Forming an equation

To calculate a rotation with quaternions, a multiplication of form $qr\bar{q}$ [3] needs to be carried out, where q is the unit quaternion defining the rotation, r is the vector being rotated (as a pure quaternion), and $\bar{q}$ is the conjugate quaternion of q. This multiplication will yield a pure quaternion, like r, that describes the rotated vector (since the real component of r is 0, the product's real component will necessarily also be 0).

This may seem arbitrary at the moment, but it can be vaguely thought of as q describing a rotation on r, while $\bar{q}$ exists as a "balancing force". The true mechanism behind how this works, and how q can be determined to perform the desired rotation, will be explained through this chapter.

3.1.1 The need for unit quaternions

For this rotation to work, q (and by definition also $\bar{q}$), need to be unit quaternions, so that the multiplication only rotates the vector quaternion r, and does not otherwise transform it.

Consider multiplying two quaternions and a vector quaternion, q_1, r, and q_2 (as needs to be done for rotations) such that:

$$r' = q_1 r q_2. \tag{30}$$

The only way to ensure that $|r'| = |r|$ (which is needed to ensure the magnitude of the rotated vector is the same as that of the original one, meaning pure rotation has occurred, without any enlargements or stretches) is if $|q_1| = |q_2| = 1$, as multiplying quaternions also multiplies their norms: $|q_1 q_2| = |q_1||q_2|$.

This (that $|q_1 q_2| = |q_1||q_2|$) can be proven as follows:

$$\begin{aligned} \text{let } q_1 &= a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}, \\ q_2 &= a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}. \end{aligned} \tag{31}$$

$$\begin{aligned} \implies q_1 q_2 = {} & (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) \qquad (32) \\ & + (a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1)\mathbf{i} \\ & + (a_1c_2 - b_1d_2 + a_2c_1 + b_2d_1)\mathbf{j} \\ & + (a_1d_2 + b_1c_2 - b_2c_1 + a_2d_1)\mathbf{k} \\ & \text{as shown in Section 2.3.3.} \end{aligned}$$

$$\begin{aligned} \implies |q_1q_2|^2 = {} & (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)^2 \qquad (33) \\ & + (a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1)^2 \\ & + (a_1c_2 - b_1d_2 + a_2c_1 + b_2d_1)^2 \\ & + (a_1d_2 + b_1c_2 - b_2c_1 + a_2d_1)^2 \\ = {} & {a_1}^2{a_2}^2 + {a_1}^2{b_2}^2 + {a_1}^2{c_2}^2 + {a_1}^2{d_2}^2 \qquad (34) \\ & + {b_1}^2{a_2}^2 + {b_1}^2{b_2}^2 + {b_1}^2{c_2}^2 + {b_1}^2{d_2}^2 \\ & + {c_1}^2{a_2}^2 + {c_1}^2{b_2}^2 + {c_1}^2{c_2}^2 + {c_1}^2{d_2}^2 \\ & + {d_1}^2{a_2}^2 + {d_1}^2{b_2}^2 + {d_1}^2{c_2}^2 + {d_1}^2{d_2}^2 \\ & \text{after expansion and simplification.} \qquad (35) \end{aligned}$$

$$|q_1|^2 = {a_1}^2 + {b_1}^2 + {c_1}^2 + {d_1}^2, \tag{36}$$

$$|q_2|^2 = {a_2}^2 + {b_2}^2 + {c_2}^2 + {d_2}^2.$$

$$\begin{aligned} \therefore \quad (|q_1||q_2|)^2 &= |q_1|^2|q_2|^2 \\ &= {a_1}^2{a_2}^2 + {a_1}^2{b_2}^2 + {a_1}^2{c_2}^2 + {a_1}^2{d_2}^2 \\ &\quad + {b_1}^2{a_2}^2 + {b_1}^2{b_2}^2 + {b_1}^2{c_2}^2 + {b_1}^2{d_2}^2 \\ &\quad + {c_1}^2{a_2}^2 + {c_1}^2{b_2}^2 + {c_1}^2{c_2}^2 + {c_1}^2{d_2}^2 \\ &\quad + {d_1}^2{a_2}^2 + {d_1}^2{b_2}^2 + {d_1}^2{c_2}^2 + {d_1}^2{d_2}^2. \end{aligned} \tag{37}$$

$$(34) = (37) \implies |q_1q_2|^2 = (|q_1||q_2|)^2 \implies |q_1q_2| = |q_1||q_2|.$$

This shows that if unit quaternions were not used for q and $\bar{q}$ in this multiplication, the norm (magnitude) of r would change, which is not desired, as this represents an enlargement, but if unit quaternions were used (i.e. $|q| = |\bar{q}| = 1$) then $|qr\bar{q}| = |q||r||\bar{q}| = 1 \times |r| \times 1 = |r|$ as desired.

It also shows that $|r|$ does not have to be 1 (i.e. r does not need to represent a unit vector), as the crucial detail is that the magnitude (norm) of r is preserved, which is the case given that $|q| = 1$ as shown above.

3.2 Finding the quaternion of rotation

This section will explain how the values of the components in the formula $r_2 = qr_1\bar{q}$, where r_2 describes the co-ordinates of the point following rotation, can be found.

Since r_1 is simply the position vector of the original point in the form of a pure quaternion (Section 1.3.1), and $\bar{q}$ is the conjugate quaternion of q, all that is left is to define q.

3.2.1 2D rotation with complex numbers

It will help to first consider performing 2-dimensional rotations [6] through complex number multiplication, as quaternion rotation can be thought of as just a "scaled-up" version of this.

Imagine a point (x_1, y_1), that is to be rotated by the angle θ in a 2-dimensional plane. Since we are just rotating this point, without scaling, we know that the distance of the the rotated point, (x_2, y_2), from the origin will be the same as that of the original point. As such, we can consider both points on a circumference of a circle with radius r, where r is the distance of the points from the origin:

$$r = \sqrt{{x_1}^2 + {y_1}^2} = \sqrt{{x_2}^2 + {y_2}^2}. \tag{38}$$

The "aim" of this rotation is, given x_1, y_1, and θ, to be able to calculate x_2 and y_2.

We can start by defining the original and final points in complex number form, as $(x_1 + y_1\mathbf{i})$ and $(x_2 + y_2\mathbf{i})$ respectively. We can also consider two position vectors, u and v, from the origin to each of these points. Note that the x and y components of these vectors are the real and imaginary components of the complex numbers that represent them respectively. The final component to consider is the perpendicular line through u that intersects the point $(x_2 + y_2\mathbf{i})$. This is all illustrated in Figure 1.

The position vector (and thereby the co-ordinates) of the rotated point, $(x_2 + y_2\mathbf{i})$, can therefore be found by adding the component of u up to the point it intersects the perpendicular line through it - this vector will be referred to as a - to the vector, b, from this point to the point $(x_2 + y_2\mathbf{i})$. Therefore, $(x_2 + y_2\mathbf{i}) = a + b$.

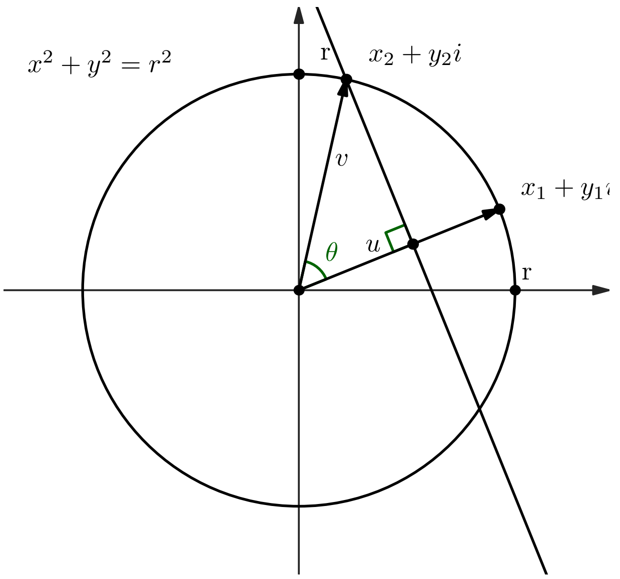


Figure 1: A visual representation of 2-dimensional rotation using complex numbers.

Defining θ as the angle between the vectors v and u:

$$\begin{aligned} \cos\theta = \frac{|a|}{r} &\implies |a| = r\cos\theta = |u|\cos\theta \\ &\implies a = u\cos\theta. \end{aligned} \tag{39}$$

To explain the above, $\cos\theta$ represents the ratio between the magnitude of a, and r, which is also the magnitude of u. Therefore, since a and u are parallel vectors (they are in the same direction), and $\cos\theta$ represents the ratio between them, $a = u\cos\theta$.

A very similar process can be followed to find b, using $\sin\theta$ instead of $\cos\theta$ as b is opposite rather than adjacent to θ, however, while $|b| = |u|\sin\theta$:

$$\sin\theta = \frac{|b|}{r} \implies |b| = r\sin\theta = |u|\sin\theta, \tag{40}$$

it does not follow here that $b = u\sin\theta$, as b is not parallel to u, but perpendicular by definition. Therefore, first we need to rotate u by $\frac{\pi}{2}$ radians to get it parallel with b. This can be done through multiplication with $\mathbf{i}$, as $\mathbf{i}$ is the perpendicular axis to the real numbers. In effect, this multiplication "swaps" the order of the x and y components, by moving the real numbers into the imaginary axis, and the imaginary components into the real axis. In effect:

$$\mathbf{i}^2 = -1 \implies (x + y\mathbf{i}) \times \mathbf{i} = (-y + x\mathbf{i}). \tag{41}$$

By multiplying u by $\mathbf{i}$ to make it parallel to b, it can then be multiplied by $\sin\theta$ to also make its magnitude equal to that of b (as shown above), so:

$$b = u\sin\theta\mathbf{i}. \tag{42}$$

Therefore, as $u = x_1 + y_1\mathbf{i}$:

$$\begin{aligned} v &= x_2 + y_2\mathbf{i} \\ &= a + b \\ &= u\cos\theta + u\sin\theta\mathbf{i} \\ &= (x_1 + y_1\mathbf{i})(\cos\theta + \sin\theta\mathbf{i}). \end{aligned} \tag{43}$$

This shows that a multiplication by $(\cos\theta + \sin\theta\mathbf{i})$ - a complex number with real component $\cos\theta$ and imaginary component $\sin\theta$ - rotates a point in 2-dimensional space by θ radians. It is clear that this is a pure rotation (i.e. the distance of the points from the origin remains unchanged), both by the diagram in Figure 1, and by the identity $\sin^2\theta + \cos^2\theta \equiv 1$:

$$\begin{aligned} &\text{let } c = \cos\theta + \sin\theta\mathbf{i} \\ \implies &|c| = \sqrt{\cos^2\theta + \sin^2\theta} = 1. \end{aligned} \tag{44}$$

Therefore, performing this multiplication cannot change the magnitude of the point, as the magnitude of the complex number performing the rotation is 1. Note that the magnitude of a complex number is defined similarly to that of a quaternion, and multiplying two complex numbers also multiplies their magnitudes, just like with quaternions (Section 3.1.1).

3.2.2 3D rotation with quaternions

Using quaternions to rotate 3-dimensional points is not so different to the process considered above, where complex numbers are used to rotate 2-dimensional points, and most of the principles considered there are also relevant here.

However, there are some key challenges. Firstly, it is not sufficient to simply rotate around the origin, like we did with 2-dimensional points - to rotate a 3-dimensional point, an axis of rotation also has to be defined. Plus, it is much more difficult to diagramatically show this, since we cannot intuitively visualise 4-dimensional space.

We can attempt to visualise a unit quaternion in a 4-dimensional space as in Figure 2, where the shaded 2-dimensional space represents the 3-dimensional imaginary $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ components of the quaternions, while the vertical component of the sphere represents the real component.

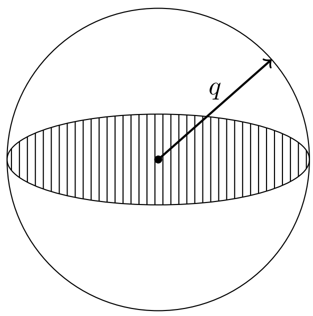


Figure 2: A 3-dimensional representation of quaternions. [7]

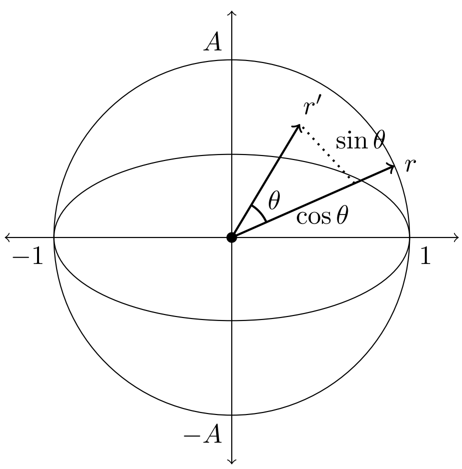


Figure 3: 4-dimensional rotations viewed on a unit circle.

In this representation, the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ space is thought of as one component, perpendicular to the real numbers. Considering the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ space as defining the axis we are rotating around (in the same way that pure quaternions represent vectors - Section 1.3.1), that axis will also be perpendicular to the real numbers, by definition of the orthogonality of the four quaternion components.

We can therefore think of a new diagram, Figure 3, similar to Figure 1, but with the axis perpendicular to the real numbers being the chosen axis to rotate around, instead of $\mathbf{i}$. This axis, A, can be represented as a pure, unit quaternion, where:

$$A = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \qquad \sqrt{x^2 + y^2 + z^2} = 1, \qquad x, y, z \in \mathbb{R}. \tag{45}$$

Figure 3 shows diagramatically how the same method can be used for quaternion rotations as with 2-dimensional complex number rotations, but with a rotation around the axis A instead of around the origin. While in this diagram $|r| = 1$, as mentioned in Section 3.1, this is not necessary.

To rotate r to r', just like with complex numbers, r' can be considered to be the sum of two "vectors" (technically quaternions in this 4-dimensional space): the proportion of r in the direction of (parallel to) r, and the proportion of r perpendicular to r. As shown by the right-angled triangle formed in the diagram (and explained more thoroughly in Section 3.2.1 with complex numbers), the vector parallel to r is $r\cos\theta$, and the vector perpendicular to it is $Ar\sin\theta$. Here, A is used instead of $\mathbf{i}$, as it is being thought of as the perpendicular axis to the real numbers. Therefore, we can write r' (the rotated point) as:

$$\begin{aligned} r' &= r\cos\theta + Ar\sin\theta \\ &= (\cos\theta + A\sin\theta)r \\ &= (\cos\theta + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\sin\theta)r \\ &= qr.^* \\ \implies q &= \cos\theta + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\sin\theta,^* \end{aligned} \tag{46}$$

where q is the quaternion that r (a point being rotataed in pure quaternion form) is multiplied by for a rotation of θ radians around the axis A.

Since A is a (pure) unit quaternion (which implies $x^2 + y^2 + z^2 = 1$), and $\sin^2\theta + \cos^2\theta \equiv 1$, this will guarantee that q will be a unit quaternion too, as desired:

$$\begin{aligned} |q| &= \sqrt{\cos^2\theta + (x\sin\theta)^2 + (y\sin\theta)^2 + (x\sin\theta)^2} \\ &= \sqrt{\cos^2\theta + \sin^2\theta(x^2 + y^2 + z^2)} \\ &= \sqrt{\cos^2\theta + \sin^2\theta} \\ &= 1. \end{aligned} \tag{47}$$

The use of sine for the imaginary axis of rotation and cosine for the real component can also be shown by looking at the sine and cosine functions themselves. For an angle of 0 (or any multiple of 2π - any full rotation), sine and cosine output 0 and 1 respectively. When the angle of rotation, θ, is 0, the desired effect is that qr yields r. If sine and cosine are used as given:

$$r' = qr^* = (\cos\theta + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\sin\theta)r = 1r = r, \tag{48}$$

as desired. However, if the roles of sine and cosine were reversed, when $\theta = 0$:

$$r' = qr^* = (\sin\theta + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\cos\theta)r = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})r. \tag{49}$$

*This is not fully accurate, but should be considered so for the time being. It will be explained by in Section 3.3.

This yields the product of r and axis of rotation, instead of just r as desired. Generalising, it is clear that (in the first quadrant) increasing θ increases $\sin\theta$ and decreases $\cos\theta$, and the "aim" when rotating is that increasing theta (in the first quadrant) should decrease the parallel (unchanged) component of r and increase the its perpendicular (rotated) component as a proportion of r'. This is also clear when looking at Figure 3. Therefore, it is logical that cosine should be used for the real component, and sine for the imaginary rotation axis.

However, there is a problem with the definition of a quaternion rotation above. In fact, $r' \neq qr, r' = qr\bar{q}$, as stated in Section 3.1. The following section will explain the reason for this, and how the formula above should be changed to reflect this.

3.3 The purpose of $\bar{q}$

While it is true, as shown above, that a mutliplication by qr rotates r by θ radians around the axis, A, defined by q, it also performs a second, undesired, rotation, parallel to A. This is best demonstrated through a stereographic projection [5] [7].

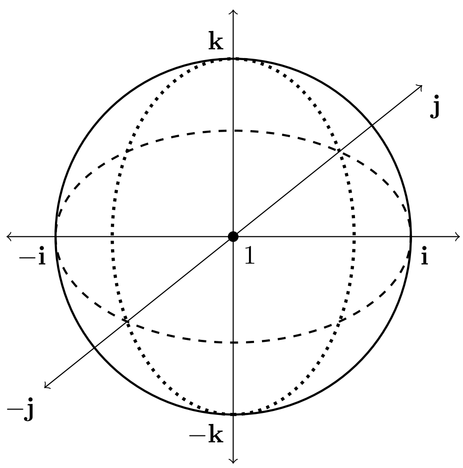


Figure 4: A stereographic projection of unit quaternions in 3-dimensional space. [7]

This stereographic projection (shown in Figure 4) maps every unit quaternion onto a point within 3-dimensional space. Note that the origin reprents 1: the unit quaternion with real component 1 and therefore no imaginary component. The edge of this sphere that has been drawn represents every unit quaternion with a real componenent 0 (i.e. all the pure unit quaternions). The space inside the sphere represents quaternions with real component betweeen 1 and 0, and the space outside the sphere represents the quaternions with negative real components. As a quaternion's real component tends towards -1, its representation in this diagram tends towards being infinitely displaced from the centre.

We are looking to rotate 3-dimensional points, that are represented as pure quaternions, and as such these points will all lie on the edge of the sphere, so we can place our focus on that part of the diagram. Although the quaternion being rotated does not need to be a unit quaternion, this diagram doees not represent quaternions with any other magnitude, so the explanations will assume the point being rotated has magnitude 1. However, the explanations will still be relevant for any other pure quaternion, and the diagram can be modified to show quaternions of any magnitude by multiplying each component in the diagram by the desired magnitude. For example, for a magnitude of 2, on the diagram: $1 \rightarrow 2, -\mathbf{i} \rightarrow -2\mathbf{i}$, etc.

We can now imagine multiplications of quaternions using this diagram. If we consider all the points on the sphere to have been left multiplied by $\mathbf{i}$ (note the importance of the order of multiplication due to the non-commutativity of quaternion multiplication) theta/2 w sterographic projection.

Visualise the rotation... double transformative effects... q bar cancelling out unwanted parallel to axis rotation

3.4 Desirable properties?

combining rotations => why multiplication is not commutative PUT IN DEFINITION SECTION AND REMOVE ALL REFS TO 'WILL BE EXPLAINED LATER'

from euler angles (only once trig for interpolation too)

interpolation (no gimbal lock)

two quaternions (q and -q) for each rotation

References

[1] Wikipedia contributers. *Quaternion*. 30/01/2025. URL: https://en.wikipedia.org/wiki/Quaternion (visited on 21/02/2025).

[2] Justin Wyss-Gallifent. “MATH431: Quaternions”. In: *University of Maryland* (December, 2021). URL: https://math.umd.edu/~immortal/MATH431/book/ch_quaternions.pdf.

[3] D. Rose. “Rotation Quaternions, and How to Use Them”. In: *danceswithcode.net* (May, 2015). URL: https://danceswithcode.net/engineeringnotes/quaternions/quaternions.html.

[4] Angela Bennett and Volodymyr Kindratenko. *Introduction to Quaternionial Algebra*. September 2000. URL: https://users.ncsa.illinois.edu/kindr/emtc/quaternions (visited on 22/02/2025).

[5] Ben Eater and Grant Sanderson. *Visualizing quaternions*. While perhaps used less as a source, this helped immensely with gaining an intuitive understanding for quaternions. October, 2018. URL: https://eater.net/quaternions (visited on 17/02/2025).

[6] Mathimaginatio. *Why Complex Multiplication Produces Rotation*. August 2022. URL: https://www.youtube.com/watch?v=HozImwsr2ho (visited on 23/03/2025).

[7] PenguinMaths. *How quaternions produce 3D rotation*. Inspiration for the diagram. July 2020. URL: https://www.youtube.com/watch?v=jTgdKoQv738&t=565s (visited on 25/03/2025).