

# 1 propositional logic

a *proposition* is a precise statement that is either **true** ( $\top$ ) or **false** ( $\perp$ ), but not both. for example:

- $2 + 2 = 4$  (**true**)
- all dogs have 3 legs (**false**)
- $x^2 < 0$  (**false**)

however, not all statements are propositions. for example:

- eliana is cool
  - *cool* is a subjective term.
- $x^3 < 0$ 
  - **true** if  $x < 0$ , **false** otherwise.
- springfield is the capital
  - **true** in illinois, **false** in massachusetts.

## 1.1 logical connectives

### 1.1.1 $\neg$ negation

the *negation* (or *not*) of a proposition is **true** *iff* the proposition is **false**.

- $\neg \perp = \top$
- $\neg \top = \perp$

### 1.1.2 $\wedge$ conjunction

the *conjunction* (or *and*) of two propositions is **true** *iff* **both** propositions are **true**.

- $\top \wedge \top = \top$
- $\top \wedge \perp = \perp$
- $\perp \wedge \top = \perp$
- $\perp \wedge \perp = \perp$

### 1.1.3 $\vee$ disjunction

the *disjunction* (or *or*) of two propositions is **true** *iff* **at least one** proposition is **true**.

- $\top \vee \top = \top$
- $\top \vee \perp = \top$
- $\perp \vee \top = \top$
- $\perp \vee \perp = \perp$

### 1.1.4 $\oplus$ exclusive disjunction

the *exclusive disjunction* (or *xor*) of two propositions is **true** iff **exactly one** proposition is **true**.

- $\top \oplus \top = \perp$
- $\top \oplus \perp = \top$
- $\perp \oplus \top = \top$
- $\perp \oplus \perp = \perp$

### 1.1.5 $\rightarrow$ implication

the *implication*  $p \rightarrow q$  is **false** if  $p$  is **true**, and  $q$  is **false**  $p \rightarrow q$  is **true** otherwise. hint:  $p$  is called the *hypothesis*, and  $q$  is called the *conclusion*.

- $\top \rightarrow \top = \top$
- $\top \rightarrow \perp = \perp$
- $\perp \rightarrow \top = \top$
- $\perp \rightarrow \perp = \top$

### 1.1.6 $\leftrightarrow$ biconditional

the *biconditional*  $p \leftrightarrow q$  is **true** iff  $p$  and  $q$  assume the same truth value.

- $\top \leftrightarrow \top = \top$
- $\top \leftrightarrow \perp = \perp$
- $\perp \leftrightarrow \top = \perp$
- $\perp \leftrightarrow \perp = \top$

## 1.2 logic operators order

1.  $\neg$  negation
  - $\neg q \vee r$  means  $(\neg q) \vee r$ , not  $\neg(q \vee r)$
2.  $\wedge$  conjunction
3.  $\vee$  disjunction
  - $q \wedge r \vee s$  means  $(q \wedge r) \vee s$ , not  $q \wedge (r \vee s)$
4.  $\rightarrow$  implication
  - $q \wedge r \rightarrow s$  means  $(q \wedge r) \rightarrow s$ , not  $q \wedge (r \rightarrow s)$
5.  $\leftrightarrow$  biconditional

## 1.3 other conditional statements

given an implication  $p \rightarrow q$ :

- $\neg p \rightarrow \neg q$  is its *inverse*
- $q \rightarrow p$  is its *converse*
- $\neg q \rightarrow \neg p$  is its *contrapositive*

## 1.4 plain words

we can also translate sentences from plain english into propositions. here are some common patterns to memorize:

(remember that  $p$  is called the *hypothesis*, and  $q$  is called the *conclusion*.)

- $q$  if  $p \equiv p \rightarrow q$
- if  $p$ , then  $q \equiv p \rightarrow q$
- $p$  only if  $q \equiv p \rightarrow q$
- $p$  if and only if  $q \equiv p \leftrightarrow q$
- neither  $p$  nor  $q \equiv \neg p \wedge \neg q = \neg(p \vee q)$
- $p$  unless  $q \equiv \neg q \rightarrow p$

### 1.4.1 example 1

proposition: you can see an r-rated movie *only if* you are over 17 *or* you are accompanied by your legal guardian.

let:

- $r \equiv$  you can see an R-rated movie
- $o \equiv$  you are over 17
- $a \equiv$  you are accompanied by your legal guardian

$$\boxed{r \rightarrow (o \vee a)}$$

### 1.4.2 example 2

proposition: you can have free coffee *if* you are a senior citizen *and* it is a tuesday.

let:

- $c \equiv$  you can have free coffee
- $s \equiv$  you are a senior citizen
- $t \equiv$  it is a tuesday

$$\boxed{(s \wedge t) \rightarrow c}$$

### 1.4.3 example 3

proposition: if you are under 17 and are not accompanied by your legal guardian, then you cannot see the r-rated movie.

we can reuse our definitions from example 1:

- $r \equiv$  you can see an R-rated movie
- $o \equiv$  you are over 17
- $a \equiv$  you are accompanied by your legal guardian

$$\boxed{(\neg o \wedge \neg a) \rightarrow \neg r}$$

this is the *contrapositive* of example 1: we swapped both sides and negated them both.

#### 1.4.4 example for conditional statements

remember in 1.3 where we talked about other conditional statements such as *inverse*, *converse*, and *contrapositive*? we can also translate those into plain words.

take the proposition  $p \rightarrow q$  that:

“if it is raining, then the flowers are watered.”

the *converse* ( $q \rightarrow p$ ) of that would be:

“if the flowers are watered, then it is raining.”

the *inverse* ( $\neg p \rightarrow \neg q$ ) would be:

“if it is not raining, then the flowers are not watered.”

and finally, the *contrapositive* ( $\neg q \rightarrow \neg p$ ) would be:

“if the flowers are not watered, then it is not raining.”

you might notice that the *contrapositive* is the only proposition that is logically equivalent to our original statement  $p \rightarrow q$ . you can find the truth table comparison below in 1.5.1.

## 1.5 activities

### 1.5.1 two propositions

show that an implication  $p \rightarrow q$  and its *contrapositive*  $\neg q \rightarrow \neg p$  always have the same value.

| $p, q$        | $\neg q, \neg p$ | $p \rightarrow q$ | $\neg q \rightarrow \neg p$ | $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow p)$ |
|---------------|------------------|-------------------|-----------------------------|--|
| $\top \top$   | $\perp \perp$    | $\top$            | $\top$                      | $\top$   |
| $\top \perp$  | $\top \perp$     | $\perp$           | $\perp$                     | $\top$   |
| $\perp \top$  | $\perp \top$     | $\top$            | $\top$                      | $\top$   |
| $\perp \perp$ | $\top \top$      | $\top$            | $\top$                      | $\top$   |

### 1.5.2 complex example 1

construct a truth table for  $(p \wedge q) \rightarrow \neg r$ .

| $p, q, r$           | $p \wedge q$ | $\neg r$ | $(p \wedge q) \rightarrow \neg r$ |
|---------------------|--------------|----------|-----------------------------------|
| $\top \top \top$    | $\top$       | $\perp$  | $\perp$                           |
| $\top \top \perp$   | $\top$       | $\top$   | $\top$                            |
| $\top \perp \top$   | $\perp$      | $\perp$  | $\top$                            |
| $\top \perp \perp$  | $\perp$      | $\top$   | $\top$                            |
| $\perp \top \top$   | $\perp$      | $\perp$  | $\top$                            |
| $\perp \top \perp$  | $\perp$      | $\top$   | $\top$                            |
| $\perp \perp \top$  | $\perp$      | $\perp$  | $\top$                            |
| $\perp \perp \perp$ | $\perp$      | $\top$   | $\top$                            |

### 1.6 complex example 2

construct a truth table for  $p \wedge (\neg q \vee r) \rightarrow r$ .

| $p, q, r$          | $\neg q$ | $\neg q \vee r$ | $p \wedge (\neg q \vee r)$ | $p \wedge (\neg q \vee r) \rightarrow r$ |
|--------------------|----------|-----------------|----------------------------|--|
| $\top \top \top$   | $\perp$  | $\top$          | $\top$                     | $\top$                                   |
| $\top \top \perp$  | $\perp$  | $\perp$         | $\perp$                    | $\top$                                   |
| $\top \perp \top$  | $\top$   | $\top$          | $\top$                     | $\top$                                   |
| $\top \perp \perp$ | $\top$   | $\top$          | $\top$                     | $\perp$                                  |
| $\perp \top \top$  | $\perp$  | $\top$          | $\perp$                    | $\top$                                   |

| $p, q, r$           | $\neg q$ | $\neg q \vee r$ | $p \wedge (\neg q \vee r)$ | $p \wedge (\neg q \vee r) \rightarrow r$ |
|---------------------|----------|-----------------|----------------------------|--|
| $\perp \top \perp$  | $\perp$  | $\perp$         | $\perp$                    | $\top$                                   |
| $\perp \perp \top$  | $\top$   | $\top$          | $\perp$                    | $\top$                                   |
| $\perp \perp \perp$ | $\top$   | $\top$          | $\perp$                    | $\top$                                   |

### 1.6.1 plain words activity 1

proposition: you get a free salad only if you order off the extended menu and it is a wednesday.

let:

- $s \equiv$  you can get free salad
- $e \equiv$  you order off the extended menu
- $w \equiv$  it is a wednesday

$$s \rightarrow (e \wedge w)$$

### 1.6.2 plain words activity 2

proposition: if it is raining and today is saturday, then you will either play video games or watch a movie

let:

- $r \equiv$  it is raining
- $s \equiv$  today is saturday
- $g \equiv$  you will play video games
- $m \equiv$  you will watch a movie

$$(r \wedge s) \rightarrow (g \vee m)$$