1 propositional logic

a *proposition* is a precise statement that is either true ($\top$) or false ($\bot$), but not both. for example:
- $2 + 2 = 4$ (true)
- all dogs have 3 legs (false)
- $x^2 < 0$ (false)

however, not all statements are propositions. for example:
- eliana is cool
  - *cool* is a subjective term.
- $x^3 < 0$
  - true if $x < 0$, false otherwise.
- springfield is the capital
  - true in illinois, false in massachusetts.

1.1 logical connectives

1.1.1 $\neg$ negation

the *negation* (or *not*) of a proposition is true *iff* the proposition is false.
- $\neg\bot = \top$
- $\neg\top = \bot$

1.1.2 $\wedge$ conjunction

the *conjunction* (or *and*) of two propositions is true *iff* **both** propositions are true.
- $\top \wedge \top = \top$
- $\top \wedge \bot = \bot$
- $\bot \wedge \top = \bot$
- $\bot \wedge \bot = \bot$

1.1.3 $\vee$ disjunction

the *disjunction* (or *or*) of two propositions is true *iff* **at least one** proposition is true.
- $\top \vee \top = \top$
- $\top \vee \bot = \top$
- $\bot \vee \top = \top$
- $\bot \vee \bot = \bot$

1.1.4 $\oplus$ exclusive disjunction

the *exclusive disjunction* (or *xor*) of two propositions is true *iff* **exactly one** proposition is true.

- $\top \oplus \top = \bot$
- $\top \oplus \bot = \top$
- $\bot \oplus \top = \top$
- $\bot \oplus \bot = \bot$

1.1.5 $\rightarrow$ implication

the *implication* $p \rightarrow q$ is false if p is true, and q is false $p \rightarrow q$ is true otherwise. hint: p is called the *hypothesis*, and q is called the *conclusion*. you can also rewrite it in $\neg p \vee q$.

- $\top \rightarrow \top = \top$
- $\top \rightarrow \bot = \bot$
- $\bot \rightarrow \top = \top$
- $\bot \rightarrow \bot = \top$

1.1.6 $\leftrightarrow$ biconditional

the *biconditional* $p \leftrightarrow q$ is true *iff* p and q **assume the same** truth value.

- $\top \leftrightarrow \top = \top$
- $\top \leftrightarrow \bot = \bot$
- $\bot \leftrightarrow \top = \bot$
- $\bot \leftrightarrow \bot = \top$

1.2 logic operators order

1. $\neg$ negation
   - $\neg q \vee r$ means $(\neg q) \vee r$, not $\neg(q \vee r)$
2. $\wedge$ conjunction
3. $\vee$ disjunction
   - $q \wedge r \vee s$ means $(q \wedge r) \vee s$, not $q \wedge (r \vee s)$
4. $\rightarrow$ implication
   - $q \wedge r \rightarrow s$ means $(q \wedge r) \rightarrow s$, not $q \wedge (r \rightarrow s)$
5. $\leftrightarrow$ biconditional

1.3 other conditional statements

given an implication $p \rightarrow q$:

- $\neg p \rightarrow \neg q$ is its *inverse*
- $q \rightarrow p$ is its *converse*

- $\neg q \to \neg p$ is its *contrapositive*

1.4 plain words

we can also translate sentences from plain english into propositions. here are some common patterns to memorize:

(remember that p is called the *hypothesis*, and q is called the *conclusion*.)

- q if $p \equiv p \to q$
- if p, then $q \equiv p \to q$
- p only if $q \equiv p \to q$
- p if and only if $q \equiv p \leftrightarrow q$
- neither p nor $q \equiv \neg p \land \neg q = \neg(p \lor q)$
- p unless $q \equiv \neg q \to p$

1.4.1 example 1

proposition: you can see an r-rated movie *only if* you are over 17 *or* you are accompanied by your legal guardian.

let:

- $r \equiv$ you can see an R-rated movie
- $o \equiv$ you are over 17
- $a \equiv$ you are accompanied by your legal guardian

$$\boxed{r \to (o \lor a)}$$

1.4.2 example 2

proposition: you can have free coffee *if* you are a senior citizen *and* it is a tuesday.

let:

- $c \equiv$ you can have free coffee
- $s \equiv$ you are a senior citizen
- $t \equiv$ it is a tuesday

$$\boxed{(s \land t) \to c}$$

1.4.3 example 3

proposition: if you are under 17 and are not accompanied by your legal guardian, then you cannot see the r-rated movie.

we can reuse our definitions from example 1:

- $r \equiv$ you can see an R-rated movie
- $o \equiv$ you are over 17
- $a \equiv$ you are accompanied by your legal guardian

$$\boxed{(\neg o \wedge \neg a) \rightarrow \neg r}$$

this is the *contrapositive* of example 1: we swapped both sides and negated them both.

1.4.4 example for conditional statements

remember in 1.3 where we talked about other conditional statements such as *inverse*, *converse*, and *contrapositive*? we can also translate those into plain words.

take the proposition $p \rightarrow q$ that:

"if it is raining, then the flowers are watered."

the *converse* ($q \rightarrow p$) of that would be:

"if the flowers are watered, then it is raining."

the *inverse* ($\neg p \rightarrow \neg q$) would be:

"if it is not raining, then the flowers are not watered."

and finally, the *contrapositive* ($\neg q \rightarrow \neg p$) would be:

"if the flowers are not watered, then it is not raining."

you might notice that the *contrapositive* is the only proposition that is logically equivalent to our original statement $p \rightarrow q$. you can find the truth table comparison below in 1.5.1.

1.5 activities

1.5.1 two propositions

show that an implication $p \to q$ and its *contrapositive* $\neg q \to \neg p$ always have the same value.

p, q	$\neg q, \neg p$	$p \to q$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to p)$
$\top$ $\top$	$\bot$ $\bot$	$\top$	$\top$	$\top$
$\top$ $\bot$	$\top$ $\bot$	$\bot$	$\bot$	$\top$
$\bot$ $\top$	$\bot$ $\top$	$\top$	$\top$	$\top$
$\bot$ $\bot$	$\top$ $\top$	$\top$	$\top$	$\top$

1.5.2 complex example 1

construct a truth table for $(p \wedge q) \to \neg r$.

p, q, r	$p \wedge q$	$\neg r$	$(p \wedge q) \to \neg r$
$\top$ $\top$ $\top$	$\top$	$\bot$	$\bot$
$\top$ $\top$ $\bot$	$\top$	$\top$	$\top$
$\top$ $\bot$ $\top$	$\bot$	$\bot$	$\top$
$\top$ $\bot$ $\bot$	$\bot$	$\top$	$\top$
$\bot$ $\top$ $\top$	$\bot$	$\bot$	$\top$
$\bot$ $\top$ $\bot$	$\bot$	$\top$	$\top$
$\bot$ $\bot$ $\top$	$\bot$	$\bot$	$\top$
$\bot$ $\bot$ $\bot$	$\bot$	$\top$	$\top$

1.6 complex example 2

construct a truth table for $p \wedge (\neg q \vee r) \to r$.

p, q, r	$\neg q$	$\neg q \vee r$	$p \wedge (\neg q \vee r)$	$p \wedge (\neg q \vee r) \to r$
$\top$ $\top$ $\top$	$\bot$	$\top$	$\top$	$\top$
$\top$ $\top$ $\bot$	$\bot$	$\bot$	$\bot$	$\top$
$\top$ $\bot$ $\top$	$\top$	$\top$	$\top$	$\top$
$\top$ $\bot$ $\bot$	$\top$	$\top$	$\top$	$\bot$
$\bot$ $\top$ $\top$	$\bot$	$\top$	$\bot$	$\top$

p, q, r	$\neg q$	$\neg q \vee r$	$p \wedge (\neg q \vee r)$	$p \wedge (\neg q \vee r) \to r$
$\bot$ $\top$ $\bot$	$\bot$	$\bot$	$\bot$	$\top$
$\bot$ $\bot$ $\top$	$\top$	$\top$	$\bot$	$\top$
$\bot$ $\bot$ $\bot$	$\top$	$\top$	$\bot$	$\top$

1.6.1 plain words activity 1

proposition: you get a free salad only if you order off the extended menu and it is a wednesday.

let:

- $s \equiv$ you can get free salad
- $e \equiv$ you order off the extended menu
- $w \equiv$ it is a wednesday

$$\boxed{s \to (e \wedge w)}$$

1.6.2 plain words activity 2

proposition: if it is raining and today is saturday, then you will either play video games or watch a movie

let:

- $r \equiv$ it is raining
- $s \equiv$ today is saturday
- $g \equiv$ you will play video games
- $m \equiv$ you will watch a movie

$$\boxed{(r \wedge s) \to (g \vee m)}$$