

1 propositional logic

a *proposition* is a precise statement that is either **true** (\top) or **false** (\perp), but not both. for example:

- $2 + 2 = 4$ (**true**)
- all dogs have 3 legs (**false**)
- $x^2 < 0$ (**false**)

however, not all statements are propositions. for example:

- eliana is cool
 - *cool* is a subjective term.
- $x^3 < 0$
 - **true** if $x < 0$, **false** otherwise.
- springfield is the capital
 - **true** in illinois, **false** in massachusetts.

1.1 logical connectives

1.1.1 \neg negation

the *negation* (or *not*) of a proposition is **true** *iff* the proposition is **false**.

- $\neg \perp = \top$
- $\neg \top = \perp$

1.1.2 \wedge conjunction

the *conjunction* (or *and*) of two propositions is **true** *iff* **both** propositions are **true**.

- $\top \wedge \top = \top$
- $\top \wedge \perp = \perp$
- $\perp \wedge \top = \perp$
- $\perp \wedge \perp = \perp$

1.1.3 \vee disjunction

the *disjunction* (or *or*) of two propositions is **true** *iff* **at least one** proposition is **true**.

- $\top \vee \top = \top$
- $\top \vee \perp = \top$
- $\perp \vee \top = \top$
- $\perp \vee \perp = \perp$

1.1.4 \oplus exclusive disjunction

the *exclusive disjunction* (or *xor*) of two propositions is **true** iff **exactly one** proposition is **true**.

- $\top \oplus \top = \perp$
- $\top \oplus \perp = \top$
- $\perp \oplus \top = \top$
- $\perp \oplus \perp = \perp$

1.1.5 \rightarrow implication

the *implication* $p \rightarrow q$ is **false** if p is **true**, and q is **false** $p \rightarrow q$ is **true** otherwise. hint: p is called the *hypothesis*, and q is called the *conclusion*.

- $\top \rightarrow \top = \top$
- $\top \rightarrow \perp = \perp$
- $\perp \rightarrow \top = \top$
- $\perp \rightarrow \perp = \top$

1.1.6 \leftrightarrow biconditional

the *biconditional* $p \leftrightarrow q$ is **true** iff p and q assume **the same** truth value.

- $\top \leftrightarrow \top = \top$
- $\top \leftrightarrow \perp = \perp$
- $\perp \leftrightarrow \top = \perp$
- $\perp \leftrightarrow \perp = \top$

1.2 logic operators order

1. \neg negation
 - $\neg q \vee r$ means $(\neg q) \vee r$, not $\neg(q \vee r)$
2. \wedge conjunction
3. \vee disjunction
 - $q \wedge r \vee s$ means $(q \wedge r) \vee s$, not $q \wedge (r \vee s)$
4. \rightarrow implication
 - $q \wedge r \rightarrow s$ means $(q \wedge r) \rightarrow s$, not $q \wedge (r \rightarrow s)$
5. \leftrightarrow biconditional

1.3 other conditional statements

given an implication $p \rightarrow q$:

- $\neg p \rightarrow \neg q$ is its *inverse*
- $q \rightarrow p$ is its *converse*
- $\neg q \rightarrow \neg p$ is its *contrapositive*

1.4 plain words

we can also translate sentences from plain english into propositions. here are some common patterns to memorize:

(remember that p is called the *hypothesis*, and q is called the *conclusion*.)

- q if $p \equiv p \rightarrow q$
- if p , then $q \equiv p \rightarrow q$
- p only if $q \equiv p \rightarrow q$
- p if and only if $q \equiv p \leftrightarrow q$
- neither p nor $q \equiv \neg p \wedge \neg q = \neg(p \vee q)$
- p unless $q \equiv \neg q \rightarrow p$

1.4.1 example 1

proposition: you can see an r-rated movie *only if* you are over 17 *or* you are accompanied by your legal guardian.

let:

- $r \equiv$ you can see an R-rated movie
- $o \equiv$ you are over 17
- $a \equiv$ you are accompanied by your legal guardian

$$\boxed{r \rightarrow (o \vee a)}$$

1.4.2 example 2

proposition: you can have free coffee *if* you are a senior citizen *and* it is a tuesday.

let:

- $c \equiv$ you can have free coffee
- $s \equiv$ you are a senior citizen
- $t \equiv$ it is a tuesday

$$\boxed{(s \wedge t) \rightarrow c}$$

1.4.3 example 3

proposition: if you are under 17 and are not accompanied by your legal guardian, then you cannot see the r-rated movie.

we can reuse our definitions from example 1:

- $r \equiv$ you can see an R-rated movie
- $o \equiv$ you are over 17
- $a \equiv$ you are accompanied by your legal guardian

$$\boxed{(\neg o \wedge \neg a) \rightarrow \neg r}$$

this is the *contrapositive* of example 1: we swapped both sides and negated them both.

1.4.4 example for conditional statements

remember in 1.3 where we talked about other conditional statements such as *inverse*, *converse*, and *contrapositive*? we can also translate those into plain words.

take the proposition $p \rightarrow q$ that:

“if it is raining, then the flowers are watered.”

the *converse* ($q \rightarrow p$) of that would be:

“if the flowers are watered, then it is raining.”

the *inverse* ($\neg p \rightarrow \neg q$) would be:

“if it is not raining, then the flowers are not watered.”

and finally, the *contrapositive* ($\neg q \rightarrow \neg p$) would be:

“if the flowers are not watered, then it is not raining.”

you might notice that the *contrapositive* is the only proposition that is logically equivalent to our original statement $p \rightarrow q$. you can find the truth table comparison below in 1.5.1.

1.5 activities

1.5.1 two propositions

show that an implication $p \rightarrow q$ and its *contrapositive* $\neg q \rightarrow \neg p$ always have the same value.

p, q	$\neg q, \neg p$	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow p)$
$\top \top$	$\perp \perp$	\top	\top	\top
$\top \perp$	$\top \perp$	\perp	\perp	\top
$\perp \top$	$\perp \top$	\top	\top	\top
$\perp \perp$	$\top \top$	\top	\top	\top

1.5.2 complex example 1

construct a truth table for $(p \wedge q) \rightarrow \neg r$.

p, q, r	$p \wedge q$	$\neg r$	$(p \wedge q) \rightarrow \neg r$
$\top \top \top$	\top	\perp	\perp
$\top \top \perp$	\top	\top	\top
$\top \perp \top$	\perp	\perp	\top
$\top \perp \perp$	\perp	\top	\top
$\perp \top \top$	\perp	\perp	\top
$\perp \top \perp$	\perp	\top	\top
$\perp \perp \top$	\perp	\perp	\top
$\perp \perp \perp$	\perp	\top	\top

1.6 complex example 2

construct a truth table for $p \wedge (\neg q \vee r) \rightarrow r$.

p, q, r	$\neg q$	$\neg q \vee r$	$p \wedge (\neg q \vee r)$	$p \wedge (\neg q \vee r) \rightarrow r$
$\top \top \top$	\perp	\top	\top	\top
$\top \top \perp$	\perp	\perp	\perp	\top
$\top \perp \top$	\top	\top	\top	\top
$\top \perp \perp$	\top	\top	\top	\perp
$\perp \top \top$	\perp	\top	\perp	\top

p, q, r	$\neg q$	$\neg q \vee r$	$p \wedge (\neg q \vee r)$	$p \wedge (\neg q \vee r) \rightarrow r$
$\perp \top \perp$	\perp	\perp	\perp	\top
$\perp \perp \top$	\top	\top	\perp	\top
$\perp \perp \perp$	\top	\top	\perp	\top

1.6.1 plain words activity 1

proposition: you get a free salad only if you order off the extended menu and it is a wednesday.

let:

- $s \equiv$ you can get free salad
- $e \equiv$ you order off the extended menu
- $w \equiv$ it is a wednesday

$$s \rightarrow (e \wedge w)$$

1.6.2 plain words activity 2

proposition: if it is raining and today is saturday, then you will either play video games or watch a movie

let:

- $r \equiv$ it is raining
- $s \equiv$ today is saturday
- $g \equiv$ you will play video games
- $m \equiv$ you will watch a movie

$$(r \wedge s) \rightarrow (g \vee m)$$