

APPENDIX

A. Properties of the Value-at-Risk

Lemma 2. (Properties of the Value-at-Risk). *Given a random variable $X \sim \mathbb{P} = \mathcal{N}(\mu, \sigma^2)$ and a threshold $\epsilon \in [0, 1)$, the following holds.*

- (a) $\text{VaR}_\epsilon^\mathbb{P}[-|X|^2] = -\text{VaR}_\epsilon^\mathbb{P}[|X|^2]$
- (b) For $Y = -|X|$ with a fixed σ^2 ,

$$\frac{d\text{VaR}_\epsilon^\mathbb{P}[Y]}{d\mu} = \frac{\phi\left(\frac{\text{VaR}_\epsilon^\mathbb{P}[Y]-\mu}{\sigma}\right) - \phi\left(\frac{-\text{VaR}_\epsilon^\mathbb{P}[Y]-\mu}{\sigma}\right)}{\phi\left(\frac{\text{VaR}_\epsilon^\mathbb{P}[Y]-\mu}{\sigma}\right) + \phi\left(\frac{-\text{VaR}_\epsilon^\mathbb{P}[Y]-\mu}{\sigma}\right)},$$

where ϕ is the probability density function of the standard normal distribution.

- (c) $\text{VaR}_\epsilon^\mathbb{P}[-|X|] = \sqrt{2}\sigma \cdot \text{erf}^{-1}(\epsilon - 1)$
- (d) $\text{VaR}_\epsilon^\mathbb{P}[X] > \text{VaR}_\epsilon^\mathbb{P}[-|X|]$

Proof: (a)

$$\begin{aligned} P[-|X| \leq \text{VaR}_\epsilon^\mathbb{P}[-|X|]] &= \epsilon, \\ P[-|X|^2 \leq \text{VaR}_\epsilon^\mathbb{P}[-|X|^2]] &= \epsilon \\ \Leftrightarrow P[|X| \geq \sqrt{-\text{VaR}_\epsilon^\mathbb{P}[-|X|^2]}] &= \epsilon \\ \Leftrightarrow P[-|X| \leq -\sqrt{-\text{VaR}_\epsilon^\mathbb{P}[-|X|^2]}] &= \epsilon. \\ \therefore \text{VaR}_\epsilon^\mathbb{P}[-|X|^2] &= -\text{VaR}_\epsilon^\mathbb{P}[|X|^2] \end{aligned}$$

(b) Let $F_Y(y)$ and $f_Y(y)$ be the cumulative probability distribution and the probability density function of a random variable Y , respectively. Let us similarly define $F_X(x)$ and $f_X(x)$ for a random variable X . Also, let Φ be the cumulative distribution function of the standard normal distribution. For simplicity, we denote $\text{Var}_\epsilon^\mathbb{P}[Y]$ by $k(\mu)$, which is also a function of σ but we consider it as a constant.

By definition of VaR, we have

$$\begin{aligned} \epsilon &= F_Y(k(\mu)) = \int_{-\infty}^{k(\mu)} f_Y(y) dy \\ &= \int_{-\infty}^{k(\mu)} (f_X(x) + f_X(-x)) dx \\ &= \int_{-\infty}^{k(\mu)} f_X(x) dx + \int_{-\infty}^{k(\mu)} f_X(-x) dx. \end{aligned}$$

Let $t = -x$. Then,

$$\begin{aligned} F_Y(k(\mu)) &= \int_{-\infty}^{k(\mu)} f_X(x) dx + \int_{\infty}^{-k(\mu)} f_X(t) (-dt) \\ &= \int_{-\infty}^{k(\mu)} f_X(x) dx - \int_{\infty}^{-k(\mu)} f_X(x) dx \\ &= \Phi\left(\frac{k(\mu) - \mu}{\sigma}\right) - \Phi\left(\frac{-k(\mu) - \mu}{\sigma}\right) + 1. \end{aligned}$$

Because $\epsilon = F_Y(k(\mu))$, we have $\frac{dF_Y(k(\mu))}{d\mu} = 0$. This leads to

$$\begin{aligned} \phi\left(\frac{k(\mu) - \mu}{\sigma}\right) \cdot \left(\frac{dk(\mu)}{d\mu} - 1\right) \frac{1}{\sigma} \\ - \phi\left(\frac{-k(\mu) - \mu}{\sigma}\right) \cdot \left(-\frac{dk(\mu)}{d\mu} - 1\right) \frac{1}{\sigma} = 0. \end{aligned}$$

Then

$$\frac{dk(\mu)}{d\mu} = \frac{\phi\left(\frac{k(\mu) - \mu}{\sigma}\right) - \phi\left(\frac{-k(\mu) - \mu}{\sigma}\right)}{\phi\left(\frac{k(\mu) - \mu}{\sigma}\right) + \phi\left(\frac{-k(\mu) - \mu}{\sigma}\right)}.$$

This concludes the proof.

(c) Let $-|X| = Y$. Then the probability density function of Y is

$$f_Y(y) = f_X(x) + f_X(-x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{y^2}{2\sigma^2}\right), y \leq 0.$$

And the cumulative distribution function of Y is

$$F_Y(y) = \int_{-\infty}^y f_Y(t) dt = \int_{-\infty}^y \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt.$$

Meanwhile,

$$F_Y(0) = \int_{-\infty}^0 f_Y(t) dt = 1$$

because Y is defined in $(-\infty, 0]$. Then

$$\begin{aligned} F_Y(0) &= \int_{-\infty}^y f_Y(t) dt + \int_y^0 f_Y(t) dt \\ &= \int_{-\infty}^y f_Y(t) dt - \int_0^y f_Y(t) dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{-\infty}^y f_Y(t) dt &= 1 + \int_0^y f_Y(t) dt \\ &= 1 + \int_0^y \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt. \end{aligned}$$

Let $t/\sqrt{2}\sigma = k$. Then

$$\begin{aligned} F_Y(y) &= 1 + \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \int_0^{y/\sqrt{2}\sigma} \exp(-k^2) \cdot \sqrt{2}\sigma dk \\ &= 1 + \frac{2}{\sqrt{\pi}} \int_0^{y/\sqrt{2}\sigma} \exp(-k^2) dk \\ &= 1 + \text{erf}(y/\sqrt{2}\sigma). \end{aligned}$$

$\text{VaR}_\epsilon^\mathbb{P}[Y]$ is the value of y when $F_Y(y) = \epsilon$. Therefore,

$$\text{VaR}_\epsilon^\mathbb{P}[Y] = \sqrt{2}\sigma \cdot \text{erf}^{-1}(\epsilon - 1).$$

(d) Let $\text{VaR}_\epsilon^\mathbb{P}[X] = k$, $\text{VaR}_\epsilon^\mathbb{P}[-|X|] = k'$. By definition,

$$\int_k^\infty f_X(x) dx = 1 - \epsilon = \int_{k'}^0 f_{-|X|}(x) dx.$$

Also,

$$\begin{aligned} \int_k^\infty f_X(x) dx &= \int_k^0 f_X(x) dx + \int_0^{-k} f_X(x) dx + \int_{-k}^\infty f_X(x) dx \\ &= \int_k^0 f_X(x) dx + \int_k^0 f_X(-x) dx + \int_{-k}^\infty f_X(x) dx \\ &= \int_k^0 f_{-|X|}(x) dx + \int_{-k}^\infty f_X(x) dx. \end{aligned}$$

Therefore,

$$\begin{aligned} & \int_{-k}^{\infty} f_X(x)dx > 0 \\ \implies & \int_{k'}^0 f_{-|X|}(x)dx - \int_k^0 f_{-|X|}(x)dx \\ & = \int_{k'}^k f_{-|X|}(x)dx > 0. \end{aligned}$$

Therefore, $\text{VaR}_\epsilon^\mathbb{P}[X] > \text{VaR}_\epsilon^\mathbb{P}[-|X|]$. ■

B. Properties of the Conditional Value-at-Risk

Lemma 1. (Properties of the Conditional Value-at-Risk). *Given a random variable $X \sim \mathbb{P} = \mathcal{N}(\mu, \sigma^2)$ and a threshold $\epsilon \in [0, 1)$, the following holds.*

- (a) $\text{CVaR}_\epsilon^\mathbb{P}[-|X|^2] \leq -\text{CVaR}_\epsilon^\mathbb{P}[-|X|]^2$
- (b) For a fixed σ^2 , $\text{CVaR}_\epsilon^\mathbb{P}[-|X|]$ is monotonically decreasing with respect to $|\mu|$.
- (c) $\text{CVaR}_\epsilon^\mathbb{P}[-|X|] = \kappa \cdot \sigma$, where $\kappa = \frac{1}{1-\epsilon} \sqrt{2/\pi} [\exp(-[erf^{-1}(\epsilon-1)]^2) - 1]$
- (d) If $\text{VaR}_\epsilon^\mathbb{P}[X] > 0 \wedge \mu > 0$, or $\text{VaR}_\epsilon^\mathbb{P}[X] < 0 \wedge \mu < 0$, then $-|\mu| + \delta \cdot \sigma > \text{CVaR}_\epsilon^\mathbb{P}[-|X|]$.

Proof:

(a)

$$\begin{aligned} \text{CVaR}_\epsilon^\mathbb{P}[-|X|^2] &= \mathbb{E}[-|X|^2 : -|X|^2 \geq \text{VaR}_\epsilon^\mathbb{P}[-|X|^2]] \quad (13) \\ &= \mathbb{E}[-|X|^2 : -|X|^2 \geq -\text{VaR}_\epsilon^\mathbb{P}[-|X|]^2] \quad (14) \\ &= \mathbb{E}[-|X|^2 : |X| \leq -\text{VaR}_\epsilon^\mathbb{P}[-|X|]] \quad (15) \\ &= -\mathbb{E}[|X|^2 : |X| \leq -\text{VaR}_\epsilon^\mathbb{P}[-|X|]] \quad (16) \\ &\leq -\mathbb{E}[-|X| : |X| \leq -\text{VaR}_\epsilon^\mathbb{P}[-|X|]]^2 \quad (17) \\ &= -\mathbb{E}[-|X| : -|X| \geq \text{VaR}_\epsilon^\mathbb{P}[-|X|]]^2 \quad (18) \\ &= -\text{CVaR}_\epsilon^\mathbb{P}[-|X|]^2. \quad (19) \end{aligned}$$

Equation (14) holds by Lemma 2 (a), and inequality (17) holds by Jensen's inequality.

(b) With the same notations used in the proof of Lemma 2 (b), we have

$$\begin{aligned} \text{CVaR}_\epsilon^\mathbb{P}[Y] &= \mathbb{E}[Y : Y \geq k(\mu)] = \frac{1}{1-\epsilon} \int_{k(\mu)}^0 y f_Y(y) dy \\ &= \frac{1}{1-\epsilon} \int_{k(\mu)}^0 x (f_X(x) + f_X(-x)) dx \\ &= \frac{1}{1-\epsilon} \left[\int_{k(\mu)}^0 x f_X(x) dx + \int_{-k(\mu)}^0 t f_X(t) dt \right] \\ &= \frac{1}{1-\epsilon} \left[\int_{k(\mu)}^0 x f_X(x) dx - \int_0^{-k(\mu)} x f_X(x) dx \right]. \end{aligned}$$

Let $a(\mu), b(\mu)$ be functions with respect to μ , and standard-

ized random variable $Z = (X - \mu)/\sigma$. Then,

$$\begin{aligned} \int_{a(\mu)}^{b(\mu)} x f_X(x) dx &= \int_{\frac{a(\mu)-\mu}{\sigma}}^{\frac{b(\mu)-\mu}{\sigma}} (\mu + \sigma \cdot z) \frac{1}{\sigma} \phi(z) \sigma dz \\ &= \mu \int_{\frac{a(\mu)-\mu}{\sigma}}^{\frac{b(\mu)-\mu}{\sigma}} \phi(z) dz + \sigma \int_{\frac{a(\mu)-\mu}{\sigma}}^{\frac{b(\mu)-\mu}{\sigma}} z \phi(z) dz. \end{aligned}$$

With $z_a(\mu) = \frac{a(\mu)-\mu}{\sigma}$ and $z_b(\mu) = \frac{b(\mu)-\mu}{\sigma}$, the derivative with respect to μ is

$$\begin{aligned} \frac{d}{d\mu} \int_{a(\mu)}^{b(\mu)} x f_X(x) dx &= \left[\int_{z_a(\mu)}^{z_b(\mu)} \phi(z) dz + \mu \cdot \{ \phi(z_b(\mu)) \cdot z'_b(\mu) - \phi(z_a(\mu)) \cdot z'_a(\mu) \} \right] \\ &\quad + \sigma [z_b(\mu) \phi(z_b(\mu)) z'_b(\mu) - z_a(\mu) \phi(z_a(\mu)) z'_a(\mu)] \\ &= \int_{z_a(\mu)}^{z_b(\mu)} \phi(z) dz + b(\mu) \phi(z_b(\mu)) z'_b(\mu) - a(\mu) \phi(z_a(\mu)) z'_a(\mu). \end{aligned}$$

Using this result and letting $z_k(\mu) = \frac{k(\mu)-\mu}{\sigma}$, $z_{-k}(\mu) = \frac{-k(\mu)-\mu}{\sigma}$ and $z_0(\mu) = \frac{-\mu}{\sigma}$, we can express the derivative of $\text{CVaR}_\epsilon^\mathbb{P}[Y]$ with respect to μ as follows.

$$\begin{aligned} (1-\epsilon) \frac{d}{d\mu} \text{CVaR}_\epsilon^\mathbb{P}[Y] &= \frac{d}{d\mu} \left[\int_{k(\mu)}^0 x f_X(x) dx - \int_0^{-k(\mu)} x f_X(x) dx \right] \\ &= \int_{z_k(\mu)}^{z_0(\mu)} \phi(z) dz - k(\mu) \cdot \phi(z_k(\mu)) \cdot z'_k(\mu) \\ &\quad - \int_{z_0(\mu)}^{z_{-k}(\mu)} \phi(z) dz - [(-k(\mu)) \cdot \phi(z_{-k}(\mu)) \cdot z'_{-k}(\mu)] \\ &= \int_{z_k(\mu)}^{z_0(\mu)} \phi(z) dz - \int_{z_0(\mu)}^{z_{-k}(\mu)} \phi(z) dz \\ &\quad - k(\mu) [\phi(z_k(\mu)) \cdot z'_k(\mu) - \phi(z_{-k}(\mu)) \cdot z'_{-k}(\mu)]. \end{aligned}$$

Because

$$z'_k(\mu) = \frac{k'(\mu) - 1}{\sigma}, \quad z'_{-k}(\mu) = \frac{-k'(\mu) - 1}{\sigma},$$

and by Lemma 2 (b),

$$k'(\mu) = \frac{dk(\mu)}{d\mu} = \frac{\phi(z_k(\mu)) - \phi(z_{-k}(\mu))}{\phi(z_k(\mu)) + \phi(z_{-k}(\mu))},$$

we have $\phi(z_k(\mu)) \cdot z'_k(\mu) - \phi(z_{-k}(\mu)) \cdot z'_{-k}(\mu) = 0$. Therefore,

$$\frac{d}{d\mu} \text{CVaR}_\epsilon^\mathbb{P}[Y] = \frac{1}{1-\epsilon} \left[\int_{\frac{k(\mu)-\mu}{\sigma}}^{\frac{-\mu}{\sigma}} \phi(z) dz - \int_{\frac{-\mu}{\sigma}}^{\frac{-k(\mu)-\mu}{\sigma}} \phi(z) dz \right].$$

Note that for any $b \leq 0$, $\int_{a+b}^a \phi(z) dz > \int_a^{a-b} \phi(z) dz$ if $a > 0$ and $\int_{a+b}^a \phi(z) dz < \int_a^{a-b} \phi(z) dz$ if $a < 0$. Therefore,

$$\frac{d}{d\mu} \text{CVaR}_\epsilon^\mathbb{P}[Y] < 0 \text{ for } \mu > 0.$$

and

$$\frac{d}{d\mu} \text{CVaR}_\epsilon^\mathbb{P}[Y] > 0 \text{ for } \mu < 0.$$

That is, $\text{CVaR}_\epsilon^{\mathbb{P}}[Y]$ is monotonically decreasing with respect to $|\mu|$.

(c) Let $-|X| = Y$ and $\text{VaR}_\epsilon[Y] = k$. Then

$$\begin{aligned}\text{CVaR}_\epsilon^{\mathbb{P}}[Y] &= \mathbb{E}[Y : Y \geq k] \\ &= \frac{1}{1-\epsilon} \int_k^0 y f_Y(y) dy \\ &= \frac{1}{1-\epsilon} \int_k^0 y \cdot \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy.\end{aligned}$$

Let $y/\sqrt{2}\sigma = u$, $y = \sqrt{2}\sigma u$, $dy = \sqrt{2}\sigma du$. Then

$$\begin{aligned}&= \frac{1}{1-\epsilon} \int_{k/\sqrt{2}\sigma}^0 \sqrt{2}\sigma u \cdot \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \exp(-u^2) \sqrt{2}\sigma du \\ &= \frac{1}{1-\epsilon} \cdot \frac{2\sqrt{2}}{\sqrt{\pi}} \sigma \int_{k/\sqrt{2}\sigma}^0 u \cdot \exp(-u^2) du \\ &= \frac{1}{1-\epsilon} \cdot \frac{2\sqrt{2}}{\sqrt{\pi}} \sigma \left[-\frac{1}{2} \exp(-u^2) \right]_{k/\sqrt{2}\sigma}^0 \\ &= \frac{1}{1-\epsilon} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \sigma \left[\exp\left(-\left[k/\sqrt{2}\sigma\right]^2\right) - 1 \right] \\ &= \frac{1}{1-\epsilon} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \sigma \left[\exp\left(-\left[\frac{\sqrt{2}\sigma \cdot \text{erf}^{-1}(\epsilon-1)}{\sqrt{2}\sigma}\right]^2\right) - 1 \right] \\ &= \frac{1}{1-\epsilon} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \sigma \left[\exp(-[\text{erf}^{-1}(\epsilon-1)]^2) - 1 \right] \\ &= \kappa \cdot \sigma,\end{aligned}$$

where $k = \sqrt{2}\sigma \cdot \text{erf}^{-1}(\epsilon-1)$ by Lemma 2 (c) and $\kappa = \frac{1}{1-\epsilon} \sqrt{2/\pi} [\exp(-[\text{erf}^{-1}(\epsilon-1)]^2) - 1]$.

(d) If $\mu < 0$ and $\text{VaR}_\epsilon^{\mathbb{P}}[X] = k < 0$, then

$$\begin{aligned}\text{CVaR}_\epsilon^{\mathbb{P}}[X] &= \mathbb{E}[X | X \geq k] \\ &= \frac{1}{1-\epsilon} \int_k^\infty x f_X(x) dx \\ &> \frac{1}{1-\epsilon} \int_k^{-k} x f_X(x) dx \\ &= \frac{1}{1-\epsilon} \left[\int_k^0 x f_X(x) dx + \int_0^{-k} x f_X(x) dx \right] \\ &= \frac{1}{1-\epsilon} \left[\int_k^0 x f_X(x) dx + \int_0^k x f_X(-x) dx \right] \\ &= \frac{1}{1-\epsilon} \left[\int_k^0 x f_X(x) dt - \int_k^0 x f_X(-x) dx \right] \\ &\geq \frac{1}{1-\epsilon} \left[\int_k^0 x f_X(x) dt + \int_k^0 x f_X(-x) dx \right] \\ &= \frac{1}{1-\epsilon} \int_k^0 x (f_X(-x) + f_X(x)) dx \\ &= \frac{1}{1-\epsilon} \int_k^0 x f_{-|X|}(x) dx \\ &\geq \frac{1}{1-\epsilon} \int_{\text{VaR}_\epsilon^{\mathbb{P}}[-|X|]}^0 x f_{-|X|}(x) dx \\ &= \mathbb{E}[-|X| : -|X| \geq \text{VaR}_\epsilon^{\mathbb{P}}[-|X|]] \\ &= \text{CVaR}_\epsilon^{\mathbb{P}}[-|X|].\end{aligned}$$

The last inequality is because $k > \text{VaR}_\epsilon^{\mathbb{P}}[-|X|]$ by Lemma 2 (d). Therefore, $\text{CVaR}_\epsilon^{\mathbb{P}}[X] = \mu + \delta \cdot \sigma > \text{CVaR}_\epsilon^{\mathbb{P}}[-|X|]$.

Similarly, if $\mu > 0$ and $\text{VaR}_\epsilon^{\mathbb{P}}[X] > 0$, the mean of $-X$ is smaller than zero and $\text{VaR}_\epsilon^{\mathbb{P}}[-X] < 0$. Thus, we can use the inequality that we derived in the preceding paragraph, that is,

$$\text{CVaR}_\epsilon^{\mathbb{P}}[-X] > \text{CVaR}_\epsilon^{\mathbb{P}}[-|X|].$$

Because X has a symmetric distribution about the mean μ , we have $-\text{VaR}_\epsilon^{\mathbb{P}}[-X] = \text{VaR}_{1-\epsilon}^{\mathbb{P}}[X]$. Also, $\text{CVaR}_{1-\epsilon}^{\mathbb{P}}[X] = \mathbb{E}[X : X \geq \text{VaR}_\epsilon^{\mathbb{P}}[X]] = \mu - \delta \cdot \sigma$ by [30]. Thus,

$$\begin{aligned}\text{CVaR}_\epsilon^{\mathbb{P}}[-X] &= \mathbb{E}[-X : -X \geq \text{VaR}_\epsilon^{\mathbb{P}}[-X]] \\ &= -\mathbb{E}[X : X \leq -\text{VaR}_\epsilon^{\mathbb{P}}[-X]] \\ &= -\mathbb{E}[X : X \leq \text{VaR}_{1-\epsilon}^{\mathbb{P}}[X]] \\ &= -(\mu - \delta \cdot \sigma).\end{aligned}$$

Therefore, if $\text{VaR}_\epsilon^{\mathbb{P}}[X] > 0 \wedge \mu > 0$ or $\text{VaR}_\epsilon^{\mathbb{P}}[X] < 0 \wedge \mu < 0$, then $-|\mu| + \delta \cdot \sigma > \text{CVaR}_\epsilon^{\mathbb{P}}[-|X|]$. ■