Referát

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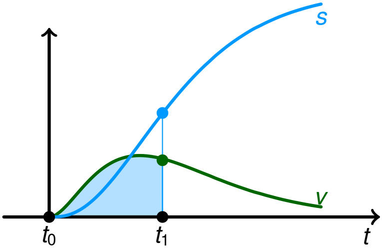
Katedra matematiky a deskriptivní geometrie

Bakalářská matematika II

Pohyb hmotného bodu



- Rychlost v závislosti na čase ... $v(t)$
- Dráha v závislosti na čase ... $s(t)$
- Lze pozorovat jejich závislost?

  Ano! Dráha odpovídá ploše pod grafem rychlosti!
- Lze tuto závislost vyjádřit matematicky?

  Ano! Dvěma ekvivalentními způsoby:

$$\boxed{\big(s(t)\big)' = v(t)} \iff \boxed{\int v(t)\,\mathrm{d}t = s(t) + s(t_0)}$$

„rychlost je derivací dráhy“ „dráha je integrálem rychlosti“

Primitivní funkce a neurčitý integrál

$$v \sim f, \qquad s \sim F, \qquad t \sim x, \qquad s(t_0) \sim c$$

$$\boxed{(F(x))' = f(x)} \quad \Longleftrightarrow \quad \boxed{\int f(x)\,\mathrm{d}x = F(x) + c}$$

„ F je primitivní funkcí k f “

$\int$	$f(x)$	$\mathrm{d}x$	x	$c \in \mathbb{R}$
neurčitý integrál	integrand	diferenciál	integrační prom.	integrační konst.

$$\left(3x^5\right)' = 15x^4 \quad \Longleftrightarrow \quad \int (15x^4)\,\mathrm{d}x = \underline{\underline{3x^5 + c, \ c \in \mathbb{R}}}$$

„ $3x^5$ je primitivní k $15x^4$ “

$$\left(3x^5 + 6\right)' = 15x^4$$

„ $3x^5 + 6$ je primitivní k $15x^4$ “

Příklad 1.1:

$$\int \frac{2^x}{3}\,\mathrm{d}x = \frac{1}{3}\int 2^x\,\mathrm{d}x = \underline{\underline{\frac{1}{3}\frac{2^x}{\ln 2} + c}}$$

pravidla a vzorce

$\boxed{\int cf = c\int f}$ $\boxed{\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + c \quad (a > 0,\ a \neq 1)}$

Příklad 1.2:

$$\int \left(x + \frac{1}{x} + \sqrt{x} + \frac{1}{\sqrt{x}}\right)\mathrm{d}x = \int x\,\mathrm{d}x + \int \frac{1}{x}\,\mathrm{d}x + \int x^{\frac{1}{2}}\,\mathrm{d}x + \int x^{-\frac{1}{2}}\,\mathrm{d}x$$

$$= \frac{x^2}{2} + \ln|x| + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = \underline{\underline{\frac{1}{2}x^2 + \ln|x| + \frac{2}{3}\sqrt{x^3} + 2\sqrt{x} + c}}$$

pravidla a vzorce

$\boxed{\int (f \pm g) = \int f \pm \int g}$ $\boxed{\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c\ (n \neq -1)}$ $\boxed{\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c\ (n = 1)}$

Příklad 1.3:

$$\int (8\cos x - 3\sin x)\,\mathrm{d}x = \underline{\underline{8\sin x + 3\cos x + c}}$$

vzorce

$\boxed{\int \cos x\,\mathrm{d}x = \sin x + c}$ $\boxed{\int \sin x\,\mathrm{d}x = -\cos x + c}$

Příklad 1.4:

$$\int \frac{4}{\sqrt{4-4x^2}}\,\mathrm{d}x = \frac{4}{2}\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \underline{\underline{2\arcsin x + c}}$$

vzorce

$\boxed{\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + c}$

Příklad 1.5:

$$\int \frac{4x-8}{2x^2-8x+7}\,\mathrm{d}x = \underline{\underline{\ln\left|2x^2-8x+7\right|+c}}$$

vzorce

$$\boxed{\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)|+c}$$

Příklad 1.6:

$$\int \operatorname{tg} x\,\mathrm{d}x = -\int \frac{-\sin x}{\cos x}\,\mathrm{d}x = \underline{\underline{-\ln|\cos x|+c}}$$

Pozorování: Je-li F primitivní funkcí k f, potom

$$\left(\int f(x)\,\mathrm{d}x\right)' = \left(F(x)+c\right)' = \left(F(x)\right)' + (c)' = f(x) + 0 = f(x)$$

$$\ldots \quad \boxed{\left(\int f\right)' = f}$$

a obráceně: $$\int (F(x))'\,\mathrm{d}x = \int f(x)\,\mathrm{d}x = F(x) + c$$

$$\ldots \quad \boxed{\int F' = F + c}$$

Důsledek:

$$f\cdot g + c = \int (f\cdot g)' = \int (f'\cdot g + f\cdot g') = \int f'\cdot g + \int f\cdot g'$$

$$\ldots \quad \boxed{\int f'\cdot g = f\cdot g - \int f\cdot g'}$$

(konstanta se „skrývá“ v integrálu)

Příklad 1.7:

$$\int x^2 \sin x \,\mathrm{d}x = \left| \begin{array}{ll} f'(x) = x^2 & g(x) = \sin x \\ f(x) = \dfrac{x^3}{3} & g'(x) = \cos x \end{array} \right| = \frac{x^3}{3}\sin x - \int \frac{x^3}{3}\cos x \,\mathrm{d}x = \textbf{?}$$

. . . jiná volba:

$$\int x^2 \sin x \,\mathrm{d}x = \left| \begin{array}{ll} f'(x) = \sin x & g(x) = x^2 \\ f(x) = -\cos x & g'(x) = 2x \end{array} \right| = -x^2\cos x + \int 2x\cos x \,\mathrm{d}x \qquad \textbf{(!)}$$

$$= \left| \begin{array}{ll} f'(x) = \cos x & g(x) = 2x \\ f(x) = \sin x & g'(x) = 2 \end{array} \right| = -x^2\cos x + \left(2x\sin x - \int 2\sin x \,\mathrm{d}x\right)$$

$$= \underline{\underline{-x^2\cos x + 2x\sin x + 2\cos x + c}}$$

pravidla a vzorce

$\boxed{\int f'g = fg - \int fg'}$ $\boxed{\int \sin x \,\mathrm{d}x = -\cos x + c}$ $\boxed{\int \cos x \,\mathrm{d}x = \sin x + c}$

Příklad 1.8:

$$\int \frac{\ln x}{x^2}\,\mathrm{d}x = \int x^{-2}\ln x\,\mathrm{d}x = \left| \begin{array}{ll} f'(x) = x^{-2} & g(x) = \ln x \\ f(x) = -\frac{1}{x} & g'(x) = \frac{1}{x} \end{array} \right| = -\frac{1}{x}\ln x + \int \frac{1}{x}\cdot\frac{1}{x}\,\mathrm{d}x$$

$$= -\frac{1}{x}\ln x + \int x^{-2}\,\mathrm{d}x = \underline{\underline{-\frac{1}{x}\ln x - \frac{1}{x} + c}}$$

vzorce

$$\boxed{\int f'g = fg - \int fg'} \qquad \boxed{\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c}$$

. . . jiná volba:

$$\int \frac{\ln x}{x^2}\,\mathrm{d}x = \int x^{-2}\ln x\,\mathrm{d}x = \left| \begin{array}{ll} f'(x) = \ln x & g(x) = x^{-2} \\ f(x) = \textbf{?} & g'(x) = -2x^{-3} \end{array} \right|$$

. . . nevhodná

Příklad 1.9:

$$\int \frac{e^{2x}-1}{e^x}\,\mathrm{d}x = \left| \begin{array}{c} t = e^x \\ (t)'\,\mathrm{d}t = (e^x)'\,\mathrm{d}x \\ \mathrm{d}t = e^x\,\mathrm{d}x \\ \frac{1}{t}\,\mathrm{d}t = \mathrm{d}x \end{array} \right| = \int \frac{t^2-1}{t}\cdot\frac{1}{t}\,\mathrm{d}t = \int (1-t^{-2})\,\mathrm{d}t = t - \frac{t^{-1}}{-1} + c$$

$$= \underline{\underline{e^x + e^{-x} + c}}$$

Příklad 1.10:

$$\int \cos(7x-5)\,\mathrm{d}x = \left| \begin{array}{c} t = 7x-5 \\ \mathrm{d}t = 7\,\mathrm{d}x \\ \frac{1}{7}\,\mathrm{d}t = \mathrm{d}x \end{array} \right| = \frac{1}{7}\int \cos t\,\mathrm{d}t = \frac{1}{7}\sin t + c = \underline{\underline{\frac{1}{7}\sin(7x-5)+c}}$$

$$\boxed{F' = f \implies \int f(ax+b)\,\mathrm{d}x = \tfrac{1}{a}F(ax+b)+c}$$

Příklad 1.11:

$$\int \frac{1}{3x+4}\,\mathrm{d}x = \underline{\underline{\frac{1}{3}\ln|3x+4|\,x + c}}$$

Příklad 1.12:

$$\int \frac{\mathrm{d}x}{x^2+2x+10} = \left| \begin{array}{l} x^2+2x+10 \\ = \underbrace{x^2+2x+1^2}-1^2+10 \\ = (x+1)^2+9 \\ = (x+1)^2+3^2 \end{array} \right| = \int \frac{\mathrm{d}x}{(x+1)^2+3^2}$$

$$= \underline{\underline{\frac{1}{3}\operatorname{arctg}\frac{x+1}{3}+c}}$$

vzorce

$$\boxed{\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\operatorname{arctg}\frac{x}{a}+c \quad (a\neq 0)}$$

Příklad 1.13:

$$\int \frac{3x}{2x^2+4x+20}\,\mathrm{d}x = \frac{3}{4}\int \frac{2x}{x^2+2x+10}\,\mathrm{d}x$$

$$= \frac{3}{4}\left(\int \frac{2x+2}{x^2+2x+10}\,\mathrm{d}x - 2\int \frac{\mathrm{d}x}{x^2+2x+10}\right)$$

$$\overset{\text{Př 1.12}}{=} \frac{3}{4}\left(\ln\left|x^2+2x+10\right| - 2\frac{1}{3}\operatorname{arctg}\frac{x+1}{3} + c\right)$$

$$= \underline{\underline{\frac{3}{4}\ln(x^2+2x+10) - \frac{1}{2}\operatorname{arctg}\frac{x+1}{3} + c}}$$

Příklad 1.14:

$$\int \frac{5x-14}{x^3-x^2-4x+4}\,\mathrm{d}x \overset{\spadesuit}{=} \int \left(\frac{3}{x-1}-\frac{1}{x-2}-\frac{2}{x+2}\right)\mathrm{d}x$$

$$= \underline{\underline{3\ln|x-1|-\ln|x-2|-2\ln|x+2|+c}}$$

$$\spadesuit \qquad \frac{5x-14}{x^3-x^2-4x+4}=\frac{3}{x-1}-\frac{1}{x-2}-\frac{2}{x+2}$$

výpočet

Příklad 1.15:

$$\int \frac{x^3+1}{x^4-3x^3+3x^2-x}\,\mathrm{d}x \overset{\diamondsuit}{=} \int \left(-\frac{1}{x}+\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{2}{(x-1)^3}\right)\mathrm{d}x$$

$$= -\ln|x|+2\ln|x-1|+\frac{(x-1)^{-1}}{-1}+2\frac{(x-1)^{-2}}{-2}+c$$

$$= \underline{\underline{-\ln|x|+2\ln|x-1|-\frac{1}{x-1}-\frac{1}{(x-1)^2}+c}}$$

$$\diamondsuit \qquad \frac{x^3+1}{x^4-3x^3+3x^2-x}=-\frac{1}{x}+\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{2}{(x-1)^3}$$

výpočet

Příklad 1.16:

$$\int \frac{5x+2}{x^3+2x^2+10x}\,\mathrm{d}x \overset{\spadesuit}{=} \int \left(\frac{1}{5}\cdot\frac{1}{x} - \frac{1}{10}\cdot\frac{2x+2}{x^2+2x+10} + \frac{24}{5}\cdot\frac{1}{x^2+2x+10} \right) \mathrm{d}x$$

$$= \underline{\underline{\frac{1}{5}\ln|x| - \frac{1}{10}\ln(x^2+2x+10) + \frac{24}{5}\cdot\frac{1}{3}\operatorname{arctg}\frac{x+1}{3} + c}}$$

$$\spadesuit \qquad \frac{5x+2}{x^3+2x^2+10x} = \frac{1}{5}\cdot\frac{1}{x} - \frac{1}{10}\cdot\frac{2x+2}{x^2+2x+10} + \frac{24}{5}\cdot\frac{1}{x^2+2x+10}$$

výpočet

Příklad 1.17:

$$\int \frac{x^4+3x^3+12x^2+15x+2}{x^3+2x^2+10x}\,\mathrm{d}x \overset{\diamondsuit}{=} \int \left(x+1+\frac{5x+2}{x^3+2x^2+10x} \right) \mathrm{d}x$$

$$\overset{\text{Př 1.16}}{=} \underline{\underline{\frac{x^2}{2} + x + \frac{1}{5}\ln|x| - \frac{1}{10}\ln(x^2+2x+10) + \frac{8}{5}\operatorname{arctg}\frac{x+1}{3} + c}}$$

$$\diamondsuit \qquad \frac{x^4+3x^3+12x^2+15x+2}{x^3+2x^2+10x} = x+1+\frac{5x+2}{x^3+2x^2+10x}$$

Příklad 1.18:

$$\int \cos^5 x \,\mathrm{d}x = \int (\cos^2 x)^2 \cos x \,\mathrm{d}x = \int (1-\sin^2 x)^2 \cos x \,\mathrm{d}x = \left| \begin{array}{c} t = \sin x \\ \mathrm{d}t = \cos x \,\mathrm{d}x \end{array} \right|$$

$$= \int (1-t^2)^2 \,\mathrm{d}t = \int (1-2t^2+t^4)\,\mathrm{d}t = t - \frac{2}{3}t^3 + \frac{1}{5}t^5 + c$$

$$= \underline{\underline{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c}}$$

vzorce

$\boxed{\sin^2 x + \cos^2 x = 1}$ $\quad t = \operatorname{tg}\frac{x}{2} \implies \boxed{\frac{2}{t^2+1}\,\mathrm{d}t = \mathrm{d}x}$ $\boxed{\frac{2t}{t^2+1} = \sin x}$ $\boxed{\frac{1-t^2}{t^2+1} = \cos x}$

Příklad 1.19:

$$\int \frac{1+\sin x}{1-\sin x}\,\mathrm{d}x = \left| t = \operatorname{tg}\frac{x}{2} \right| = \int \frac{1+\frac{2t}{t^2+1}}{1-\frac{2t}{t^2+1}} \cdot \frac{2}{t^2+1}\,\mathrm{d}t = 2\int \frac{\frac{t^2+2t+1}{t^2+1}}{\frac{t^2-2t+1}{t^2+1}} \cdot \frac{1}{t^2+1}\,\mathrm{d}t$$

$$= 2\int \frac{t^2+2t+1}{t^2-2t+1} \cdot \frac{1}{t^2+1}\,\mathrm{d}t = 2\int \frac{t^2+2t+1}{t^4-2t^3+2t^2-2t+1}\,\mathrm{d}t$$

$$= \cdots = 2\left(-\frac{2}{t-1} - \operatorname{arctg} t\right) + c = \underline{\underline{\frac{4}{1-\operatorname{tg}\frac{x}{2}} - x + c}}$$

Příklad 1.20:

$$\int \frac{\mathrm{d}x}{(2+x)\sqrt{1+x}} = \left| \begin{aligned} t^2 &= 1+x \\ 2t\,\mathrm{d}t &= \mathrm{d}x \end{aligned} \right| = \int \frac{2t}{(t^2+1)t}\,\mathrm{d}t = 2\int \frac{\mathrm{d}t}{t^2+1} = 2\operatorname{arctg} t + c$$
$$= \underline{\underline{2\operatorname{arctg}\sqrt{1+x} + c}}$$

Příklad 1.21:

$$\int \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[4]{x}} = \left| \begin{aligned} t^4 &= x \\ 4t^3\,\mathrm{d}t &= \mathrm{d}x \end{aligned} \right| = \int \frac{4t^3}{t^2+t}\,\mathrm{d}t = 4\int \frac{t^2}{t+1}\,\mathrm{d}t$$
$$= 4\int \left(t - 1 + \frac{1}{t+1}\right)\mathrm{d}t = 2t^2 - 4t + \ln(t+1) + c$$
$$= \underline{\underline{2\sqrt{x} - 4\sqrt[4]{x} + \ln(\sqrt[4]{x}+1) + c}}$$

Pravidla pro integrování

$$\int cf = c\int f$$
$$\int (f \pm g) = f \pm g$$
$$\int f'g = fg - \int fg'$$

zpět

Vzorce pro integrování

$$\int 0 \, \mathrm{d}x = c$$
$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$
$$\int \frac{1}{x} \, \mathrm{d}x = \ln|x| + c$$
$$\int e^x \, \mathrm{d}x = e^x + c$$
$$\int a^x \, \mathrm{d}x = \frac{a^x}{\ln a} + c, \quad a > 0, \ a \neq 1$$
$$\int \sin x \, \mathrm{d}x = \cos x + c$$
$$\int \cos x \, \mathrm{d}x = \sin x + c$$
$$\int \frac{1}{\sin^2 x} \, \mathrm{d}x = -\operatorname{cotg} x + c$$
$$\int \frac{1}{\cos^2 x} \, \mathrm{d}x = \operatorname{tg} x + c$$
$$\int \frac{1}{1+x^2} \, \mathrm{d}x = \operatorname{arctg} x + c$$
$$\int \frac{1}{a^2+x^2} \, \mathrm{d}x = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + c, \quad a \neq 0$$
$$\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \arcsin x + c$$
$$\int \frac{1}{\sqrt{a^2-x^2}} \, \mathrm{d}x = \arcsin \frac{x}{a} + c, \quad a \neq 0$$
$$\int \frac{1}{\sqrt{a+x^2}} \, \mathrm{d}x = \ln\left|x + \sqrt{x^2+a}\right| + c, \ a \neq 0$$

$$\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \ln|f(x)| + c$$
$$\int f(ax+b) \, \mathrm{d}x = \frac{1}{a} F(ax+b) + c$$

zpět

$$\frac{5x-14}{x^3-x^2-4x+4} \overset{\heartsuit}{=} \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x+2} \overset{\clubsuit}{=} \underline{\underline{\frac{3}{x-1}-\frac{1}{x-2}-\frac{2}{x+2}}}$$

zpět

$\heartsuit$

$$\begin{aligned} x^3-x^2-4x+4 &= x^2(x-1)-4(x-1)=(x-1)(x^2-4) \\ &= \underline{(x-1)(x-2)(x+2)} \end{aligned}$$

$\clubsuit$

$$\frac{5x-14}{(x-1)(x-2)(x+2)} = \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x+2} \qquad \Big|\cdot(x-1)(x-2)(x+2)$$

$$5x-14 = A(x-2)(x+2)+B(x-1)(x+2)+C(x-1)(x-2)$$

$\boxed{x=1} \implies$

$$\begin{aligned} 5\cdot 1-14 &= A\cdot(1-2)(1+2)+B\cdot 0+C\cdot 0 \\ -9 &= -3A \\ \underline{A} &\underline{= 3} \end{aligned}$$

$\boxed{x=2} \implies$

$$\begin{aligned} 5\cdot 2-14 &= A\cdot 0+B\cdot(2-1)(2+2)+C\cdot 0 \\ -4 &= 4B \\ \underline{B} &\underline{= -1} \end{aligned}$$

$\boxed{x=-2} \implies$

$$\begin{aligned} 5\cdot(-2)-14 &= A\cdot 0+B\cdot 0+C\cdot(-2-1)(-2-2) \\ -24 &= 12A \\ \underline{C} &\underline{= -2} \end{aligned}$$

$$\frac{x^3+1}{x^4-3x^3+3x^2-x} \overset{\heartsuit}{=} \frac{A}{x} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} + \frac{B_3}{(x-1)^3} \overset{\clubsuit}{=} \underline{\underline{\frac{-1}{x} + \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3}}}$$

zpět

$\heartsuit$

$$\begin{aligned} x^4-3x^3+3x^2-x &= x^4-x-3x^3+3x^2 = x(x^3-1)-3x^2(x-1) = x(x-1)(x^2+x+1)-3x^2(x-1) \\ &= x(x-1)[x^2+x+1-3x] = x(x-1)[x^2-2x+1] = x(x-1)(x-1)^2 = \underline{x(x-1)^3} \end{aligned}$$

$\clubsuit$

$$\frac{x^3+1}{x(x-1)^3} = \frac{A}{x} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} + \frac{B_3}{(x-1)^3} \qquad \Big| \cdot x(x-1)^3$$

$$x^3+1 = A(x-1)^3 + B_1 \cdot x(x-1)^2 + B_2 \cdot x(x-1) + B_3 \cdot x$$

$\boxed{x=0} \implies$

$$\begin{aligned} 0^3+1 &= A\cdot(0-1)^3 + B_1\cdot 0 + B_2 \cdot 0 + B_3 \cdot 0 \\ 1 &= -A \iff \underline{A=-1} \end{aligned}$$

$\boxed{x=1} \implies$

$$1^3+1 = A\cdot 0 + B_1 \cdot 0 + B_2 \cdot 0 + B_3 \cdot 1 \iff \underline{B_3 = 2}$$

$\boxed{x=2} \implies$

$$\begin{aligned} 2^3+1 &= A\cdot(2-1)^3 + B_1\cdot 2(2-1)^2 + B_2\cdot 2(2-1) + B_3 \cdot 2 \\ 9 &= -1\cdot 1 + B_1 \cdot 2 \cdot 1 + B_2 \cdot 2 \cdot 1 + 2 \cdot 2 \\ 6 &= 2B_1 + 2B_2 \\ 3 &= B_1 + B_2 \end{aligned}$$

$\boxed{x=-1} \implies$

$$\begin{aligned} 3^3+1 &= A\cdot(3-1)^3 + B_1\cdot 3(3-1)^2 + B_2\cdot 3(3-1) + B_3 \cdot 3 \\ 28 &= -1\cdot 8 + B_1 \cdot 3 \cdot 4 + B_2 \cdot 3 \cdot 2 + 2 \cdot 3 \\ 30 &= 12B_1 + 6B_2 \\ 5 &= 2B_1 + B_2 \end{aligned}$$

$$\begin{aligned} 3 &= B_1 + B_2 \implies B_2 = 3 - B_1 \\ 5 &= 2B_1 + B_2 \qquad \underline{B_2 = 1} \\ \hline 5 &= 2B_1 + 3 - B_1 \\ \underline{B_1} &\underline{= 2} \end{aligned}$$

$$\frac{5x+2}{x^3+2x^2+10x} \overset{\heartsuit}{=} \frac{A}{x} + \frac{B_1(2x+2)}{x^2+2x+10} + \frac{B_2}{x^2+2x+10} \overset{\clubsuit}{=} \underline{\underline{\frac{\frac{1}{5}}{x} - \frac{\frac{1}{10}(2x+2)}{x^2+2x+10} + \frac{\frac{24}{5}}{x^2+2x+10}}}$$

zpět

$\heartsuit$ $$x^3+2x^2+10x = x(x^2+2x+10)$$

$\clubsuit$

$$\frac{5x+2}{x(x^2+2x+10)} = \frac{A}{x} + \frac{B_1(2x+2)}{x^2+2x+10} + \frac{B_2}{x^2+2x+10} \quad \Big| \cdot x(x^2+2x+10)$$

$$5x+2 = A\cdot(x^2+2x+10) + B_1\cdot x(2x+2) + B_2\cdot x$$

$\boxed{x=0} \implies$

$$2 = A\cdot 10 + B_1\cdot 0 + B_2\cdot 0$$
$$A = \frac{1}{5}$$

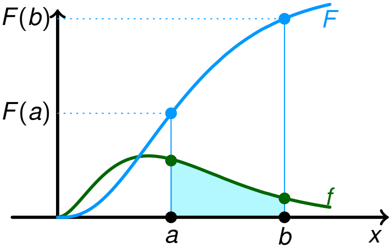
$\boxed{x=1} \implies$

$$7 = A\cdot 13 + B_1\cdot 4 + B_2\cdot 1$$
$$7 = \frac{1}{5}\cdot 13 + 4B_1 + B_2$$
$$\frac{22}{5} = 4B_1 + B_2$$

$\boxed{x=-1} \implies$

$$-3 = A\cdot 9 + B_1\cdot 0 + B_2\cdot(-1)$$
$$-3 = \frac{1}{5}\cdot 9 - B_2$$
$$B_2 = \frac{24}{5}$$

$$\frac{22}{5} = 4B_1 + \frac{24}{5}$$
$$-\frac{2}{5} = 4B_1$$
$$B_1 = -\frac{1}{10}$$



$$\boxed{\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)}$$

Příklad 3.1:

$$\int_1^2 (x^3 - 9x^2 + 18x)\,\mathrm{d}x = \left[\frac{x^4}{4} - 9\frac{x^3}{3} + 18\frac{x^2}{2}\right]_1^2$$

$$= \left(\frac{16}{4} - 9\cdot\frac{8}{3} + 18\cdot\frac{4}{2}\right) - \left(\frac{1}{4} - \frac{9}{3} + \frac{18}{2}\right) = 16 - \frac{47}{4} = \underline{\underline{\frac{17}{4}}}$$

Příklad 3.2:

$$\int_{-1}^{1}(3x^2-4)\ln(x+2)\,\mathrm{d}x = \left| \begin{array}{ll} u' = 3x^2-4 & v = \ln(x+2) \\ u = x^3-4x & v' = \dfrac{1}{x+2} \end{array} \right|$$

$$= \Big[(x^3-4x)\ln(x+2)\Big]_{-1}^{1} - \int_{-1}^{1}\frac{x^3-4x}{x+2}\,\mathrm{d}x = -3\ln 3 - 3\ln 1 - \int_{-1}^{1} x(x-2)\,\mathrm{d}x$$

$$= -3\ln 3 - \left[\frac{x^3}{3} - x^2\right]_{-1}^{1} = -3\ln 3 - \left(\left(\frac{1}{3}-1\right) - \left(-\frac{1}{3}-1\right)\right) = \underline{\underline{-3\ln 3 - \frac{2}{3}}}$$

vzorce

$$\boxed{\int_a^b u'v = [uv]_a^b - \int_a^b uv'}$$

Příklad 3.3: Urči obsah rovinného obrazce ohraničeného křivkami

$$x = -2, \quad x = 3, \quad y = 0, \quad y = x + 3$$

$$x = -2, x = 3, y = 0, y = x + 3 \quad \dots f(x) = x + 3, \quad a = -2, b = 3$$

$$S = \int_{-2}^{3} |x + 3| \, \mathrm{d}x = \int_{-2}^{3} (x + 3) \, \mathrm{d}x = \left[\frac{x^2}{2} + 3x \right]_{-2}^{3} = \left(\frac{9}{2} + 9 \right) - \left(\frac{4}{2} - 6 \right) = \underline{\underline{\frac{35}{2}}}$$

vzorce

$$\boxed{S = \int_a^b |f(x)| \, \mathrm{d}x}$$

Příklad 3.4:

$$y^2 = 2x, x^2 = 2y \quad \ldots f(x) = \sqrt{2x} \quad g(x) = \frac{x^2}{2}$$

$$\begin{aligned} f(x) &= g(x) \\ \sqrt{2x} &= \frac{x^2}{2} \\ x^4 - 8x &= 0 \\ x(x-2)(x^2+2x+4) &= 0 \\ a = 0,\ b &= 2 \end{aligned}$$

$$S = \int\limits_0^2 \left|\sqrt{2x} - \frac{x^2}{2}\right| \,\mathrm{d}x = \int\limits_0^2 \left(\sqrt{2x} - \frac{x^2}{2}\right) \mathrm{d}x = \left[\sqrt{2}\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{6}\right]_0^2 = \left(\frac{8}{3} - \frac{8}{6}\right) = \underline{\underline{\frac{4}{3}}}$$

vzorce

$$\boxed{S = \int\limits_a^b |f(x) - g(x)| \,\mathrm{d}x}$$

Příklad 3.5:

$$y = \ln \sin x, \quad x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle \qquad \ldots f(x) = \ln \sin x$$

$$f'(x) = \frac{\cos x}{\sin x}$$

$$a = \tfrac{\pi}{3},\ b = \tfrac{2\pi}{3}$$

$$d = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\,\mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\,\mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{1-\cos^2 x}\,\mathrm{d}x = \left| \begin{array}{c} t = \cos x \\ \mathrm{d}t = -\sin x\,\mathrm{d}x \end{array} \right|$$

$$= \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{\mathrm{d}t}{t^2-1} = \frac{1}{2}\int_{\frac{1}{2}}^{-\frac{1}{2}} \left(\frac{1}{t-1} - \frac{1}{t+1} \right) \mathrm{d}t = \frac{1}{2}\left[\ln|t-1| - \ln|t+1| \right]_{\frac{1}{2}}^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(\left(\ln\frac{3}{2} - \ln\frac{1}{2} \right) - \left(\ln\frac{1}{2} - \ln\frac{3}{2} \right) \right) = \underline{\underline{\ln 3}}$$

vzorce

$$\boxed{d = \int_a^b \sqrt{1 + f'^2(x)}\,\mathrm{d}x}$$

Příklad 3.6:

$$\left.\begin{aligned} x(t) &= \cos t + t\sin t \\ y(t) &= \sin t - t\cos t \end{aligned}\right\} \; t \in \langle 0, 2\pi \rangle \qquad \ldots x'(t) = t\cos t$$

$$y'(t) = t\sin t$$

$$\alpha = 0,\ \beta = 2\pi$$

$$d = \int_0^{2\pi} \sqrt{(t\cos t)^2 + (t\sin t)^2}\,\mathrm{d}t = \int_0^{2\pi} t\,\mathrm{d}t = \left[\frac{t^2}{2}\right]_0^{2\pi} = \underline{\underline{2\pi^2}}$$

vzorce

$$\boxed{d = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t}$$

Příklad 3.7:

$$x = 1, x = 4, y = 0, xy = 4 \qquad \dots f(x) = \frac{4}{x}$$

$$a = 1,\ b = 4$$

$$V = \pi \int_1^4 \left(\frac{4}{x}\right)^2 \mathrm{d}x = 16\pi \int_1^4 x^{-2}\,\mathrm{d}x = 16\pi \left[-\frac{1}{x}\right]_1^4 = 16\pi\left(-\frac{1}{4}+1\right) = \underline{\underline{12\pi}}$$

vzorce

$$\boxed{V = \pi \int_a^b f^2(x)\,\mathrm{d}x}$$

Příklad 3.8:

$$y^2 = 2x, x^2 = 2y \quad \dots f(x) = \sqrt{2x} \quad g(x) = \frac{x^2}{2}$$

$$f(x) = g(x)$$
$$\sqrt{2x} = \frac{x^2}{2}$$
$$x^4 - 8x = 0$$
$$x(x-2)(x^2+2x+4) = 0$$
$$a = 0, b = 2$$
$$x \in \langle 0, 2 \rangle \implies 2x - \frac{x^4}{4} \geq 0$$

$$V = \pi \int_0^2 \left| 2x - \frac{x^4}{4} \right| \mathrm{d}x = \pi \int_0^2 \left(2x - \frac{x^4}{4} \right) \mathrm{d}x = \pi \left[x^2 - \frac{x^5}{20} \right]_0^2 = \pi \left(4 - \frac{32}{20} \right) = \underline{\underline{\frac{12}{5}\pi}}$$

vzorce

$$\boxed{V = \pi \int_a^b \left| f^2(x) - g^2(x) \right| \mathrm{d}x}$$

Příklad 3.9:

$$y = \frac{x^3}{3}, \quad x \in \langle 0, 2 \rangle \qquad \ldots f(x) = \frac{x^3}{3}$$

$$f'(x) = x^2$$

$$a = 0,\ b = 2$$

$$x \in \langle 0, 2 \rangle \implies \frac{x^3}{3} \geq 0$$

$$P = 2\pi \int_0^2 \left| \frac{x^3}{3} \right| \sqrt{1 + x^4}\,\mathrm{d}x = 2\pi \int_0^2 \frac{x^3}{3} \sqrt{1 + x^4}\,\mathrm{d}x = \left| \begin{array}{c} t = 1 + x^4 \\ \mathrm{d}t = 4x^3\,\mathrm{d}x \end{array} \right| = \frac{2}{12}\pi \int_1^{17} \sqrt{t}\,\mathrm{d}t$$

$$= \frac{2}{12}\pi \left[\frac{2}{3} t^{\frac{3}{2}} \right]_1^{17} = \frac{1}{9}\pi(\sqrt{17^3} - 1) = \underline{\underline{\frac{\pi}{9}(17\sqrt{17} - 1)}}$$

vzorce

$$\boxed{P = 2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2}\,\mathrm{d}x}$$

Příklad 3.10:

$$\left.\begin{aligned} x(t) &= t^2 \\ y(t) &= \frac{t}{3}(t^2-3) \end{aligned}\right\} t \in \left\langle 0, \sqrt{3} \right\rangle \quad \ldots x'(t) = 2t \qquad y(t) = 0$$

$$y'(t) = t^2 - 1 \qquad \frac{t}{3}(t-\sqrt{3})(t+\sqrt{3}) = 0$$

$$\alpha = 0,\ \beta = \sqrt{3}$$

$$P = 2\pi \int_0^{\sqrt{3}} \left|t^2\right| \sqrt{4t^2 + (t^2-1)^2}\,\mathrm{d}t = 2\pi \int_0^{\sqrt{3}} t^2 \sqrt{t^4 + 2t^2 + 1}\,\mathrm{d}t = 2\pi \int_0^{\sqrt{3}} t^2(t^2+1)\,\mathrm{d}t$$

$$= 2\pi \int_0^{\sqrt{3}} (t^4 + t^2)\,\mathrm{d}t = 2\pi \left[\frac{t^5}{5} + \frac{t^3}{3}\right]_0^{\sqrt{3}} = 2\pi\left(\frac{(\sqrt{3})^5}{5} + \frac{(\sqrt{3})^3}{3}\right) = \underline{\underline{\frac{28}{5}\sqrt{3}\pi}}$$

vzorce

$$\boxed{P = 2\pi \int_\alpha^\beta |x(t)| \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t}$$

Konec

(Referát)