Performance Modeling and Design of Computer Systems- Ch 10 Exponential Distribution and the Poisson Process

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Overview

Exponential Distribution

- A random variable X is distributed Exponentially with rate λ,

$$X \sim Exp(\lambda)$$

If X has the probability density function:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The cumulative distribution function,

$$F(x) = P\{X \leq x\} = \int_{-\infty}^{x} f(y)dy = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Exponential Distribution
Properties -1

- Mean of Exp Dis.

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{\lambda}$$

- second moment of Exp Dis.

$$E[X^2] = \frac{2}{\lambda^2}$$

- variance of Exp Dis.

$$Var(x) = E[X^2] - (E[X])^2 == \frac{1}{\lambda^2}$$

Exponential Distribution
Properties -2

- Mean of Exp Dis.

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{\lambda}$$

- second moment of Exp Dis.

$$E[X^2] = \frac{2}{\lambda^2}$$

- variance of Exp Dis.

$$Var(x) = E[X^2] - (E[X])^2 == \frac{1}{\lambda^2}$$

- Exp Dist. is memoryless : next state $((t = 1)$th) is independent of current state (t'th)

Exponential Distribution
Properties -3

- Given $X_1 \sim Exp(\lambda_1), X_2 \sim Exp(\lambda_2), X_1 \perp X_2$. then

$$P\{X_1 < X_2\} = P\{X_1 < X_2 | X_2 = x\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

- if $X = min(X_1, X_2)$ then

$$X \sim Exp(\lambda_1 + \lambda_2)$$

$\perp$: independent and identically Distributed

- **Example:** A server can crash bcz of power supply and disk. both one have lifetime with exponential distribution. Find probability that the system failed when it occurs, is caused by the power supply?
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Exponential Distribution

Real Life Examples

- **failure rate function** $r(t)$ **(hazard rate function) :** Let X be a continuous random variable with probability density function $f(t)$ and cumulative distribution function $F(t) = P\{X > t\}$. Then

$$r(t) = \frac{r(t)}{F(t)}$$

- **Decreasing failure rate** : $P\{X > s + t | X > s\}$ goes up as s. $r(t)$ is strictly decreasing in t
    - UNIX job: More CPU a job used up so far, the more CPU likely to use more
- **Increasing failure rate** : $P\{X > s + t | X > s\}$ goes down as s. $r(t)$ is strictly increasing in t
    - A car's lifetime. The older a car is, the less likely that it will survive another, say, t = 6 years.

Relating Exponential to Geometric via δ-Steps

For detailed proof look at page 210

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Definition of the Exponential Distribution

Properties of Exp. Dis.

Relation with other Distribution

Relation with Poisson Process

- **Geometric distribution** can be viewed as the number of flips needed to get a "success"
- **Exponential distribution** is the time until "success."
- **Unification:** imagine each unit of time as divided into n pieces, each of size $\delta = \frac{1}{n}$, and suppose that a trial (flip) occurs every δ time period, rather than at unit times.
- Let

$$X \sim Exp(\lambda)$$

$$Y \sim Geometric(p = \lambda\delta | flip\, every\, \delta - step)$$

- Y denotes the number of flips until success. Let $\tilde{Y}$ is the time until success under Y . $E[\tilde{Y}] =$ (avg. number trials until success) $*$ (time per trial) $= \frac{1}{\lambda\delta} \cdot \delta = \frac{1}{\lambda} \rightarrow$ Implies the distribution of random variable X

Poisson Process

Need to make a slide to express relatin between possion process and exponential distribution

- Poisson process is the most widely used model for arrivals into a system for two reasons
  - Markovian properties of the Poisson process make it analytically tractable.
  - In many cases, it is an excellent model. For example, In communications networks, such as the telephone system, it is a good model for the sequence of times at which telephone calls are originated.
- Poisson process appears often in nature when we are observing the aggregate effect of a large number of individuals or particles operating independently
- **Definition: A Poisson process with rate λ is a sequence of events such that the interarrival times are i.i.d. Exponential random variables with rate λ and $N(0) = 0$.**

Poisson Process
Merging and Splitting

- **Merging :** Given two independent Poisson processes, where process 1 has rate λ_1 and process 2 has rate λ_2 , the merge of process 1 and process 2 is a single Poisson process with rate $(\lambda_1 + \lambda_2)$
- **Splitting :** Given a Poisson process with rate λ , suppose that each event is classified "type A" with probability p and "type B" with probability $1 - p$. Then type A events form a Poisson process with rate $p\lambda$, type B events form a Poisson process with rate $(1 - p)\lambda$, and these two processes are independent.

$$P(AB) = P(A) \cdot P(B)$$