Introduction

Strong and Weak Laws of Large Numbers

Concept of Average

Simulation

Performance Modeling and Design of Computer Systems- Ch 5 Sample Paths, Convergence, and Averages

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Overview

1. Introduction
2. Strong and Weak Laws of Large Numbers
3. Concept of Average
4. Simulation

Convergence

- **Definitiion** A sequence $\{a_n : n = 1, 2, .\because.\}$ converges to b as $n \to \infty$ over a sample path , mathmetically

$$\lim_{n \to \infty} a_n = b$$

- Convergence for random variable (2 erquivalent def)
  - The sequence of random variables $\{Y_n : n = 1, 2, .\because.\}$ converges almost surely to μ, written $Y_n \to \mu, as: \; n \to \infty$
  - The sequence converges with probability 1, written $Y_n \to, \mu: \; as: \; n \to \infty$ with probability $p = 1$ if, $\forall k > 0, P\{\lim_{n \to \infty}(Y_n - \mu) > k\} = 0$

Convergence

- **Sample path** of a stochastic process is a particular realisation of the process, i.e. a particular set of values $Y(t)$ for all t
  Example: A coin is tossed n times. the probability of head in each toss forms a path. This is a sample path $\frac{1}{2}$,0, $\ldots, \frac{1}{2}$
- Convergence is considered over a sample path

Weak Law of large numbers

- Let $X_1, X_2, X_3, \ldots$ be i.i.d. (independent and identically distributed) random variables with mean $E[X]$. Let

$$S_n = \sum_{i=1}^{n} X_i$$

and

$$Y_n = \frac{S_n}{n}$$

Then Y_n **converges in probability to** $E[X]$, mathematically $Y_n \to E[X]$, as $n \to \infty$
Equivalently, $\forall k > 0, \lim_{n\to\infty} P\{|Y_n - E[X]| > k)\} = 0$

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Strong Law of large numbers

- Let $X_1, X_2, X_3, \ldots$ be i.i.d. (independent and identically distributed) random variables with mean $E[X]$. Let

  $$S_n = \sum_{i=1}^{n} X_i$$

  and

  $$Y_n = \frac{S_n}{n}$$

  Then Y_n **converges almost surely to** $E[X]$, mathematically $Y_n \rightarrow E[X]$, as $n \rightarrow \infty$
  Equivalently, $\forall k > 0, \lim_{n\rightarrow\infty} P\{|Y_n - E[X]| \geq k)\} = 0$

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Example of Strong & Weak Law of large numbers

- Let X_i are all $0/1$ random variables (coin toss) with mean $\frac{1}{2}$
- Each sample path is an infinite sequence of coin flips
- **Strong Law** : for **"almost every"** sample path, if we average the coin flips far out enough (large n) along the path, we will get convergence to $\frac{1}{2}$
- **Weak Law** : Says convergence is not almost sure. Convergence will happen with a probability.

Time Average versus Ensemble Average

- Scenario: a single FCFS queue, at every second a new job is added to the queue with probability p and at every second the job in service (if there is) completed with probability q , where $q > p$. Let $N(v) =$ number of jobs in the system at time v .
- What is the avg. number of jobs in the queue
- **Time Average**:

$$\overline{N}^{TimeAvg} = \lim_{t\to\infty} \frac{\int_0^t N(v)dv}{t}$$

- **Ensemble Average**:

$$\overline{N}^{Ensemble} = \lim_{t\to\infty} E[N(t)] = \sum_{i=0}^{\infty} ip_i$$

Where,

$$p_i = \lim_{t\to\infty} P\{N(t) = i\}$$

Time Average versus Ensemble Average

Interpretation: Time Average

- Consider the FCFS scenario again
- **Time Average**: Consider a sample path over a long period t, mointor $\#$ of jobs in system , then take avg
  Example: queue start empty. at time 1, an arrival , no departure N $(1) = 1$.
  at time 2, an arrival , no departure N $(2) = 2)$.
  At time 3, an arrival , no departure N $(3) = 3$
  at time 4, no arrival and a departure N $(4) = 2$, etc.
  The average number of jobs in the system by time 4 for this process is $(0 + 1 + 2 + 3 + 2)/5 = 8/5$.
- **Problem of Time Average**: It takes only a single path. If this is one of the **rare** bad path then avg may be not reprsentitive

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Simulation

- Consider the FCFS scenario again
- **Time Average**: at time 1 there is some probability that the system is still empty and there is some probability that the system contains one job. Same for time 2,3,4....
- Calculate the $E[N(i)] = ip(i) =$ expectation that the system have i jobs
- then apply

$$p_i = \lim_{t\to\infty} P\{N(t) = i\}$$

How to Simulate

- For ergodic system (positive recurrent, aperiodic, & irreducible) both are equal
  - **irreducibility**: a processcan reach from any state to any other state (think of the state as the number of jobs in the system). This is important for ensuring that the choice of initial state does not matter.
  - **positive recurrent :** if for any state i, the state is revisited infinitely often, and the mean time between visits to state i (renewals) is finite. Furthermore, every time that we visit state i the system will probabilistically restart itself
- **Time Avg :** sampling a single process over a very long period of time and averaging those samples
- **Ensemble Avg :** generating many independent processes and taking their Ensemble average at some far-out time t
- Ensemble Avg is better :
  - ensemble average can be obtained in parallel, by running simulations on different cores or different machines.
  - independent data points allow us to generate confidence

Average Time in System

- two versions of the average time in system. Assume:
  T_i is the time in system of the ith arrival
  $A(t)$ is the number of arrivals by time t
- **For Time Avg**

$$\overline{T}^{TimeAvg.} = \lim_{t\to\infty} \frac{\sum_{i=1}^{A(t)T_i}}{a(t)}$$

- **For Ensemble Avg.**

$$\overline{T}^{Ensemble} = \lim_{i\to\infty} E[T_i]$$