Performance Modeling and Design of Computer Systems- Ch 12 Transition to Continuous-Time Markov Chains

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Overview

Continuous-Time Markov Chain (CTMC)

Definition

- **Continuous-Time Markov Chain (CTMC)**: a continuous-time stochastic process $\{X(t), t \geq 0\}$ s.t., $\forall s, t \geq 0$ and $\forall i, j, x(u)$,

$$P\{X(t+s) = j | X(s) = i, X(u) = x(u), 0 \leq u \leq s\}$$

$$= P\{X(t+s) = j | X(s) = i\} (by M.P.)$$

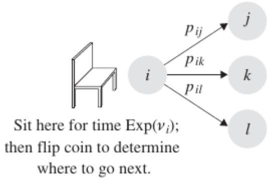
$$= P\{X(t) = j | X(0) = i\} = P_{ij}(t) (stationarity)$$

- We assume throughout that the state space is countable (though continuous)
- τ_i = time until the CTMC leaves state i, given that it is currently in state i.
- τ_i is memoryless and exponentially distributed

Continuous-Time Markov Chain (CTMC)-2

Explanation

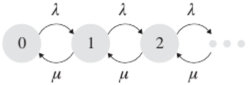
- A CTMC
  - The amount of time the process spends in state i before making a transition is Exponentially distributed with some rate (v_i).
  - When the process leaves state i, it will next enter state j with probability (p_{ij}) independent of the time spent at state i.
- p_{ij} , probability of when we leave i we next go to state j , is and independent of time, t, (by stationarity property).
- p_{ij} is independent of the time spent in state i, τ_i (by the Markovian property).



Continuous-Time Markov Chain (CTMC)

Example

- Suppose we are in state i, where $i \geq 1$. Then the next event is either an arrival or a departure.



- X_A = time to next arraival. $X_A \sim Exp(\lambda)$ regardless of how long we are in the current state.
- X_D = time to the next departure. Then $X_D \sim Exp(\mu)$ regardless of how long we are in current state. X_A & X_D are independent of each other.
- Arrival & departure events happen in parallel. One of these, will occur first. So, τ_i = ime until we leave state i, has the distribution: $\tau_i \sim Exp(\lambda + \mu) \rightarrow v_i = \lambda + \mu)$
- probability of when we leave state i we will next go to state $(i+1)$ is $P\{X_A < X_D\} = \frac{\lambda}{\lambda+\mu}$

Solving CTMCs-1

- **Goal:** Finding $\tau_j = \lim_{t\to\infty} P_{ij}(t)$ = limiting probability of being in state j
- **Solution:** Model CTMC as DTMC and find solution
- We are not discussing full details of derivation
- **Balance Equation:** the rate at which jobs leave state j in the CTMC = the rate at which jobs enter state j in the CTMC, for each state j .

$$\pi_j v_j = \sum_i \pi_i Q_{ij}$$

- **LHS**=total rate of transitions leaving state j= limiting probability of being in state j(π_j) * rate the MC leaves state j given that it is in state $j(v_j)$) .

Solving CTMCs -2

- **RHS**$=$ total rate of transitions entering state j from any state$= \sum \forall i$ (limiting proba- bility of being in state $i(\pi_i)$ * rate the MC leaves state i to go to state j given that it is in state i (q_{ij})
- Balance Equation Become

$$\Rightarrow \pi_j \sum_i \pi_i Q_{ij} = \sum_i \pi_i Q_{ij}$$

- Just solve the balance equation to find the limiting probability

Summary Theorem for CTMCs

- CTMCs can basically be viewed as DTMCs in which the time step goes to zero $(t \to 0)$, $\Rightarrow$ all the ergodicity theory that we developed for DTMCs carries over to CTMCs.
- **Summary Theorem for CTMCs:** Given an irreducible CTMC, suppose $\exists \pi_i$ s.t. $\forall j$,

$$\pi_j v_j = \sum_i \pi_i q_{ij}$$

and

$$\sum_i \pi_i = 1$$

- *In an irreducible Markov Chain, the process can go from any state to any state, whatever be the number of steps it requires.*