Performance Modeling and Design of Computer Systems- Ch 8 Discrete-Time Markov Chains

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Overview

Background & Motivation

- Upto Chapter 7, We have only deduced some approximate bound for performance evaluation of closed systems.
- For Open system we have derived very few useful info
- **Goal**: How we can derive more performance matrix of a system
- **Approach Used in This chapter:** Using **Markov Model**
  - Not all system can be modeled using Markov chain
  - Exponential and Geomoetric distribution shows better suitability to Markov chain
  - In many cases, non Markovian workloads can be approximated by mixtures of Exponential distributions, and hence still lend themselves to Markov chain analysis
- **Question**: How to understand whether a workload is suitable for analysis uinsg Markov chain

Markov Chains

- 2 variations
  - **Discrete-Time Markov Chains** (DTMCs): Every event is broken up into synchronized time steps. An event (arrival or departure) can only occur at the end of a time step.
    $\rightarrow$ Not so Good for modeling Computer Systems
  - **Continuous-Time Markov Chains (CTMCs)**: events can happen at any moment in time.
    $\rightarrow$ Convenient for modeling systems.
  - **A stochastic process** is simply a sequence of random variables.
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Discrete-time Markov chain

- **DTMC (discrete-time Markov chain)** is a stochastic process $\{X_n, n = 0, 1, 2, ...\}$, where X_n denotes the state at (discrete) time step n and such that, $\forall n \geq 0,\ \forall i, j,\ and\ \forall i_0, \cdots i_{n-1}$

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\}$$
$$= P\{X_{n+1} = j | X_n = i\} = P_{ij} (by stationarity)$$

  Here P_{ij} is independent of the time step and of past history.
- **Interpretation**
  - **Markovian Property (1st equality)**: conditional distribution of any future state X_{n+1} , given past states $X_0, X_1, \cdots, X_{n-1}$, and given the present state X_n , is independent of past states and depends only on the present state X_n .
  - **"stationary" property**: second equality indicates that the transition probability is independent of time.

Discrete-time Markov chain

transition probability matrix

- **Transition probability matrix** (P_{ij}): (i, j))entry, P_{ij} , represents the probability of moving to state j on the next transition, given that the current state is i
- P, might have infinite order, if there are infinitely many states.
- $\sum_j P_{ij} = 1, \, \forall i$, because, given that the DTMC is in state i, it must next transition to some state j
- Example Page 132

Powers of P

n-Step Transition Probabilities

- Let $P^n = (p^n)_{ij} = P \cdot P \cdots P$, multiplied n times.
- **Meaning of** P^n **:** For a M -state DTMC, as shown

$$P_{ij}{}^n = \sum_{k=0}^{M-1} P_{ik}{}^{n-1} P_{kj}$$

$=$ Probability of being in state j in n steps,

given we are in state i now.

- **Limiting Probabilities**: For large n

$$\pi_j = \lim_{n \to \infty} P_{ij}{}^n = (\lim_{n \to \infty} P^n)_{ij}$$

$=$ limiting probability of being in state j

infinitely far into the future, given that we started in state i

- in π_j there is no $i \to$ means independent of initial state

- A Markov chain for which **limiting probabilities** exist is said to be **stationary** or in **steady state** if the initial state is chosen according to the stationary probabilities.
- **Finding the Limiting Probabilities in a Finite-State DTMC** : given the limiting distribution $\{\pi_j, j = 0, 1, 2, \cdots, (M-1)\}$ exists, we can obtain it by solving the stationary equations

$$\overrightarrow{\pi} \cdot P = \overrightarrow{\pi} \; and \sum_{i=0}^{M-1} \pi_i = 1$$

Where $\overrightarrow{\pi} = (\pi_0, \pi_1, \cdots, \pi_{M-1})$

- Example and use case : page 138

Infinite-State DTMCs

n-Step Transition Probabilities

- Markov chain with an infinite number of states
- P_{ij} has infinite order
- **limiting probability distribution on the states** $\overrightarrow{\pi} = (\pi_0, \pi_1, \cdots)$ where $\pi_j = \lim_{n \to \infty} {P_{ij}}^n$ and $\sum_{i=0}^{\infty} \pi_j = 1$
- If limiting distrivution exits then $\overrightarrow{\pi}$ is a stationary distribution also (Proof in book)
- How to Solve infinitie number of Equations :

$$\pi_i = (\frac{r}{s})^i \pi_0$$

Look at book page 144

- **I still need to work on understading how to use this formula for using stationary distribution**
- **We are skipping chapter 9 as it is too therorectical discussion**