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An investigation of some theoretical aspects of reversible computing

by

Brett Gordon Giles

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Abstract

The categorical semantics of reversible computing must be a category which combines the concepts of partiality and the ability to, at least partially, reverse a map in the category. Inverse categories, restriction categories where each map is a partial isomorphism provide the necessary machinery for reversible computing. This thesis explores inverse categories and relates them to both quantum computing, by showing that commutative Frobenius algebras form an inverse category and to standard non-reversible computing by showing the equivalence of discrete inverse categories to discrete Cartesian restriction categories.

While restriction categories have products and coproducts, we show the direct application of these to inverse categories is not the right notion to consider. Therefore we introduce corresponding structures in inverse categories, including inverse products, inverse sums, meets and disjoint joins. We show that a commutative Frobenius algebra over a restriction category with a biproduct has both inverse sums and inverse products and disjoint joins.

We provide an equivalence between a discrete inverse categories (an inverse category with an inverse product) and a discrete Cartesian restriction categories. Here, discrete means that the diagonal map $\Delta : A \to A \times A$ has a partial inverse. We also show the construction giving the equivalence translates inverse sums to coproducts in the Cartesian restriction category.

Finally, we relate this directly to standard computing by giving the analogues for Turing objects and partial combinatory algebras in an inverse category and show these correspond directly with a Turing object and PCA in a Cartesian restriction category.

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Dedication

To my wife, Marie “Soliloquy” Gelinas Giles.

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List of Symbols, Abbreviations and Nomenclature

Symbol	Definition
$\overline{f}$	$\overline{f}$ is the restriction of f, see Definition 3.1.1.
$\cup$	Set union.
$\cap$	Set intersection.
$\emptyset$	The empty set.
$\partial_0 f$	The domain object of the map f, see Definition 2.1.1.
$\partial_1 f$	The codomain object of the map f, see Definition 2.1.1.
$\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{D}, \mathbb{R}$	Categories, see Definition 2.1.1.
$\in$	Element of, in sets.
$\to$	Separate domain and range of a map.
$\mathbb{X}(A, B)$	Hom-Set in $\mathbb{X}$.
$\{x \vert \text{condition on } x\}$	Define a set via a condition.
$\exists x.$	This means "there exists an x such that".
$\mathbb{N}$	The set of natural numbers, i.e., $\{0, 1, 2, \ldots\}$.
$[a_{ij}]$	The matrix whose i, j element is a_{ij}.
$\sec f$	A right inverse of f.
$f_\diamond$	A left inverse of f.
$\mathrm{K}_E(\mathbb{B})$	The Karoubi envelope. See Definition 2.2.2.
$\langle f, g\rangle$	The product map of f, g. See Definition 2.6.3.
$A \times B$	The product of A, B. See Definition 2.6.3.
$[f, g]$	The coproduct map of f, g. See Definition 2.6.4.
$A + B$	The coproduct of A, B. See Definition 2.6.4.
$A \boxplus B$	The biproduct of A, B. See Definition 2.6.6.
$\alpha : F \Rightarrow G$	α is a natural transformation.

$F \cong G$	The natural transformation $F \Rightarrow G$ is an isomorphism.
$\lvert S\rvert$	The cardinality of the set S.
$F \vdash G$	F is the left adjoint of G.
$\oplus, \otimes, \odot$	Tensors of categories.
$\uparrow$	Signifies that a function is undefined at some value.
$\leq$	An ordering relation.
$\mathcal{O}(()A)$	Restriction idempotents of A.
$f \smile g$	f is compatible with g. See Definition 3.3.1.
$f \vee g$	The join of f, g. See Definition 3.3.2.
$\setminus$	Minus operation on sets.
$\cap$	Meets of maps, see Definition 3.4.1.
$f^{(-1)}$	Partial inverse of f, see Definition 3.5.1.
$A \triangleleft B$	A is a retract of B.
$\hat{f}$	$\hat{f}$ is the range of f, see Definition 3.6.1.
$\top$	Restriction terminal object. See Definition 3.9.2.
$!$	The unique map to the restriction terminal object.
$\mathcal{M}$	A collection of monic maps.
$\forall X,$	For all X.
$f^{\dagger}$	Apply the functor $\dagger$ to f.
$\implies$	Implies.
$\mathbb{C}$	The field of complex numbers.
$f \xrightarrow{\cong} g$	f is isomorphic to g.
$\widetilde{\mathbb{X}}$	The completion of the discrete inverse category $\mathbb{X}$.
$f \overset{h}{\simeq} g$	f is equivalent to g. See Definition 5.1.2.
(f, C)	Equivalence class of maps $f : A \to B \otimes C$.
$f : A \to B_{\lvert C}$	Alternate way to write $(f, C) : A \to B$.

$\doteq$	Defines the left hand side as the right hand side.
$?$	The unique map from the initial object.
$\perp$	Disjointness relationship. See Definition 6.2.1.
$\underline{\perp}$	Open disjointness. See Definition 6.2.4.
$\sqcup$	Disjoint join. See Definition 6.3.1.
$\uplus$	The disjoint union of sets.
$f \mathbin{\triangledown} g$	Partial operation on f, g. See Definition 7.1.6.
$f \vartriangle g$	Partial operation on f, g. See Definition 7.1.6.
$\iff$	If and only if.

Chapter 1

Introduction

1.1 Summary

A "quantum" setting has a fundamental duality given by the "dagger" of dagger categories [1, 50]. On the other hand, classical computation is fundamentally asymmetric and has no duality. In passing from a quantum setting to a more classical setting, one may want to keep this duality for as long as possible and, thus, consider the intermediate step of passing to "reversible" computation — which has an obvious self-duality given by the ability to reverse the computation. It is reasonable to wonder whether one can then pass from a reversible setting to a classical setting quite independently from the underlying quantum setting. Such an abstract passage would allow a direct translation into the reversible world of the classical notions of computation, for example.

Of course, from a quantum setting, it is already possible to pass directly to a classical setting by taking the homomorphisms between special coalgebras, where "special" means the coalgebra must be the algebra part of a separable Frobenius algebra. That the coalgebra should be special in this manner may be justified by regarding this as a two step process through reversible computation. However, this leaves some gaps: How does on pass, in general, between a quantum setting to a reversible setting and how does one obtain a classical setting from a reversible setting? This thesis answers those questions.

1.2 Background

Reversible computing, at the level of Turing machines, was shown to be equivalent to standard computing by Bennet [6] in 1973. Since approximately 2000, there has been an increased interest in reversible computing, as evidenced by the number of papers found in internet searches. (This may have been engendered by the obvious relationship to Quantum computing (e.g., [43]) where computation is done by a series of unitary transforms, each of which is reversible, followed by an irreversible measurement.)

The semantics of reversible computing has been examined in a variety of ways. Broadly, the research may be broken into two classes. The first class consists of those that introduce a reversible language and then describe the semantics of that language in some way. Examples of this include: Zuliani [53], the quantum Guarded Command Language which has a reversible operational semantics; Mu et.al. [42], the language **Inv** composed of partial injective functions and Sabry and James [31], the language Theseus based on isomorphisms of finite sums and products of types. The second class focuses on some algebraic model and then introduces reversibility to that model. Examples of this include Abramsky [2] with Linear Combinatory Algebras, Di Pierro et.al. [46] with Combinatory Algebras, Danos and Krivine [23] who extend CCS [38] to Reversible CCS (RCCS) and Phillips and Uladowski [45] with an alternate approach to RCCS.

An important aspect of the treatment of reversible computing is the consideration of partiality in programs, i.e., it is possible for programs to not return an answer for certain inputs. The above references consider partiality to a greater or lesser degree, but none of them treat it as a central consideration.

Partiality was shown to have an algebraic treatment, restriction categories, by Cockett and Lack in [17–19]. The algebraic treatment consists of an operator on maps, denoted by a bar over the map, such that for a map $f : A \to B$ in a category, then $\overline{f} : A \to A$ is an idempotent map in the category. Full details are given here in Chapter 3.

In much of the research on reversibility, specific conditions are placed on some aspect of the computational model or reversible language to ensure "programs" in this model are reversible. The variety of models and languages tends to obscure the fundamental commonality of reversibility. We contend that our treatment of the theory of reversibility will provide clarification allowing one to see the relationships between a variety of approaches.

1.2.1 Algebraic models of reversibility

In [2], Abramsky considers linear logic as his computational model. This is done by producing a Linear Combinatory Algebra [4] from the involutive maps over a term algebra and showing these are bi-orthogonal automata. (An automata is considered orthogonal if it is non-ambiguous and left-linear. It is bi-orthogonal when both the automata and its converse are both orthogonal). While the paper does use structurally reversible term rewritings as its basis for computation, the greater emphasis is placed on how this leads to a linear combinatory algebra and the universality of that model.

In [23], Danos and Krivine extend CCS (Calculus for Communicating Systems) [38,39] to produces RCCS, which adds reversible transitions to CCS. This is done by adding a syntax for backtracking, together with a labelling which guides the backtracking. The interesting aspect of this paper is the applicability to multi-processor programs.

Phillips and Uladowski [45] take a different approach to creating a reversible CCS from that of Danos and Krivine. Rather, their stated goal is to use a structural approach, inspired by [2]. The paper is only an initial step in this process, primarily explaining how to turn dynamic rules (such as choice operators) into a series of static rules that keep all the information of the input. For example (from the paper), in standard CCS, we have the rule

$$\frac{X \to X'}{X + Y \to X'}.$$

To preserve information and allow reversibility, this is replaced with

$$\frac{X \to X'}{X + Y \to X' + Y}.$$

1.2.2 Reversible languages

Early examples of reversible languages include Janus [36], an imperative language written as an experiment in producing a language that did not erase information. However, it does not appear that any semantics were developed for this language.

Additionally, there are special purpose reversible languages, such as biXid [33], a language developed explicitly to transform XML [8] from one schema to another. The main novelty of biXid is that a single program targets two schemas and will transform in either direction.

In the realm of languages which possess an explicit semantics, we start with Zuliani [53], who examines logical reversibility via comparing the probabilistic Guarded Command Language ($pGCL$) [40] to the quantum Guarded Command Language ($qGCL$) [47]. Zuliani provides a method for transforming an irreversible $pGCL$ program into a reversible one. This is accomplished via an application of expectation transform semantics to the $pGCL$ program. Interestingly, in this work, partial programs are specifically excluded from the definition of reversible programs. The initial definition of a reversible program is strict, i.e., the program is equivalent to $skip$ which does nothing. To alleviate this and allow us to extract the output, Zuliani follows the example of [6] and modifies the result so that the output is copied before reversing the rest of the program.

In [41] and [42], Mu et.al. introduce the language **Inv**, a language that is composed only of partial injective functions. The language has an operational semantics based on determinate relations and converses. They provide a variety of examples of the language, including translations from XML to HTML and simple functions such as $wrap$, which wraps its argument into a list. They continue by describing how non-injective functions may be converted to injective ones in **Inv** via the addition of logging. In fact, they use this logging to argue the language is equivalent in terms of power to the reversible Turing machine of [6].

The approach of Mu et.al. is a good example of a specific case which is describable by the theory presented in this Thesis. Their approach of insisting on injective functions is a specific case of an inverse category. When given two functions f, g of **Inv**, constructing their union, $f \cup g$ requires that both $\operatorname{dom} f \cap \operatorname{dom} g = \phi$ and $\operatorname{range} f \cap \operatorname{range} g = \phi$. This is an explicit creation of a disjoint join as introduced in Section 6.3.

A more recent entry into the field of reversible languages is that of Theseus, [31], by Sabry and James. Theseus is a functional language which compiles to a graphical language [29,30] for reversible computation, based on isomorphisms of finite sums and products of types. Their chosen isomorphisms include commutativity and associativity for sums and products, units for product and distributivity of product over sums. The basic graphical language is extended with recursive types and looping operators and therefore introduces partiality due to the possibility of non-terminating loops.

The basis of Theseus is reminiscent of what we produce in this thesis, however, our approach differs significantly in focusing first on partiality and the expressiveness this gives us, rather than building a specific language.

Finally, we note there are a number of quantum programming languages which, as noted, included reversible operations. Our first example is LQPL [24], a compiled language based on the semantics of [48]. The language includes a variety of reversible operations (unitary transforms) as primitives, a linear type system and an operational semantics. More recently Quipper [25, 26], which focuses on methods to handle very large circuits, is a language embedded in Haskell [44]. Quipper uses quantum *and* classical circuits as an underlying model. An interesting aspect of reversibility in Quipper is the inclusion of an operator to compute the reverse of a given circuit.

1.3 Objectives

We intend to propose a categorical semantics for reversible computing. Based upon our review of current research as noted in Section 1.2, reversibility lacks a unifying semantic model. Standard computing has Cartesian closed categories [5], while quantum computing has had much success with dagger compact closed categories [1, 49, 50].

In this thesis, we will present a type of restriction category which abstracts partial reversible computations in as general a way as possible. We will show this category admits product-like and sum-like structures. We will also show this type of restriction category is equivalent to a specific type of Cartesian restriction category, showing there is a relation to standard computing models.

Finally, we develop the structure of inverse Turing category and inverse partial combinatory algebras, directly based on Turing category and partial combinatory algebras in restriction categories from [14, 16].

1.4 Contributions

The main contributions of this thesis are:

1. The characterization of inverse categories with restriction products and restriction coproducts. We show in Proposition 4.2.1 and Proposition 6.1.5 that each of products and coproducts impose a trivialization of the structure of the base category.

2. The definition and characterization of the inverse product in an inverse category (known as a discrete inverse category) and showing that it provides meets for the inverse category in Proposition 4.3.6. We show that the inverse sub-category of a discrete Cartesian restriction category is a discrete inverse category in Lemma 4.3.7.

3. The creation of an equivalence relation on a discrete inverse category and show in Theorem 5.1.10 that the category resulting from factoring out the equivalence relation is a discrete Cartesian restriction category.

4. Showing that the category of discrete inverse categories is equivalent to the category of discrete Cartesian restriction categories in Theorem 5.2.6.

5. The development of disjointness relations and disjoint joins in inverse categories and how they arise from suitable tensors on the inverse category in Proposition 7.2.5.

6. Showing that inverse categories with disjoint joins and restriction zeroes give rise to a matrix category and that the original disjoint sum category is equivalent to the matrix category in Proposition 9.2.5.

7. The proof that a distributive inverse category (one with inverse products and disjoint joins where the inverse product distributes over the disjoint join) gives a distributive restriction category when factoring out the equivalence relation from Item 3 above in Corollary 10.3.4.

8. The proofs that an inverse Turing category produces a Turing category, Theorem 11.2.2 and that an inverse partial combinatory algebra results in a partial combinatory algebra in the equivalent discrete Cartesian restriction category, Proposition 11.3.3.

1.5 Outline

We assume a knowledge of basic algebra including definitions and properties of groups, rings, fields, vector spaces and matrices. The reader may consult [35] if further details are needed.

Chapter 2 introduces the various categorical concepts that will be used throughout this thesis.

Chapter 3 describes restriction categories, an algebraic way to axiomatize partiality of maps in categories. After introducing restrictions, we discuss joins, meets, ranges and partial map categories. We describe the product in restriction categories, and describe discrete

Cartesian restriction categories, which will be used through the rest of this thesis. Various examples of types of restriction categories are given throughout the chapter.

Chapter 4 introduces inverse categories and provides examples of inverse categories and restriction categories which are not inverse categories. We show that inverse categories with a restriction product collapses to a restriction pre-order, that is, all parallel maps agree wherever they are both defined. Then, Section 4.3 introduces the concept of inverse products and explores the properties of the inverse product. Inverse categories with inverse products are called *discrete inverse categories*.

Chapter 5 then presents the construction of a Cartesian restriction category from a discrete inverse category. Section 5.1.1 presents the details of the equivalence relation for the maps for the Cartesian restriction category, while Section 5.1 and Section 5.1.3 prove that it is a Cartesian restriction category. Section 5.2 culminates in Theorem 5.2.6 giving an equivalence adjunction between discrete inverse categories and discrete Cartesian restriction categories. We conclude the chapter with a series of examples of the construction.

Having explored how to introduce a product-like construction for inverse categories, Chapter 6 begins the exploration of how to add a coproduct-like construction. Paralleling the previous chapter, we show the existence of a restriction coproduct implies that an inverse category must be a pre-order, i.e., that all parallel maps are equal. Section 6.2 defines a disjointness relation in an inverse category. We show that disjointness may be defined on all maps or equivalently only on the restriction idempotents of the inverse category. This allows us to now define the disjoint join in Section 6.3.

Chapter 7 shows how specific conditions on a symmetric monoidal tensor in an inverse category allow us to define both a disjointness relation and a disjoint join from that tensor. We conclude this chapter with Section 8.2 which expands the example of commutative Frobenius algebras, giving a disjointness relation.

A major source of examples of discrete inverse categories is commutative Frobenius algebras. These are discussed in Chapter 8. In the chapter, we first provide a background on dagger categories and frobenius algebras. We then introduce the category of commutative Frobenius algebras and show that it is a discrete inverse category. Next, we show this category has a disjointness relation.

Chapter 9 builds upon the previous chapters to introduce coproduct-like constructions into the inverse category. We explore the relationship between disjoint joins and disjoint sums. This culminates in showing that an inverse category $\mathbb{X}$ with a tensor generating disjoint joins gives rise to a matrix category over $\mathbb{X}$ and that $\mathbb{X}$ is equivalent to its matrix category.

In Chapter 10 we show that the construction introduced in Chapter 5 also lifts an disjoint sum up to a coproduct and in fact will create a distributive restriction category when the inverse product distributes over the disjoint join.

In Chapter 11, we introduce Turing categories and partial combinatory algebras in a Cartesian restriction category. We then discuss the corresponding structures in inverse categories: Inverse Turing categories and inverse partial combinatory algebras, showing the equivalence of these structures to the ones in discrete Cartesian restriction categories.

Our conclusions and thoughts for potentially interesting areas to explore further are then given in Chapter 12.

Chapter 2

Introduction to Categories

This chapter introduces categories and fixes notation for them. More details for category theory can be found from, e.g., [5], [15], [37] and [52].

2.1 Definition of a category

A category may be defined in a variety of equivalent ways. As much of our work will involve the exploration of partial and reversible maps, their domains and codomains, we choose a definition that highlights the algebraic nature of these.

Definition 2.1.1. A *category* $\mathbb{A}$ is a directed graph consisting of objects $\mathbb{A}_o$ and maps $\mathbb{A}_m$. Each $f \in \mathbb{A}_m$ has two associated objects in $\mathbb{A}_o$, called the domain, $\partial_0 f$, and codomain, $\partial_1 f$. When $\partial_0 f$ is the object X and $\partial_1 f$ is the object Y, we will write $f : X \to Y$. For $f, g \in \mathbb{A}_m$, if $f : X \to Y$ and $g : Y \to Z$, there is a map called the *composite* of f and g, written fg,[1] such that $fg : X \to Z$. For any $W \in \mathbb{A}_o$ there is an *identity* map $1_W : W \to W$. Additionally, these two axioms must hold:

[**C.1**] for $f : X \to Y,\ 1_X f = f = f 1_Y$; (Unit laws)

[**C.2**] given $f : X \to Y,\ g : Y \to Z$ and $h : Z \to W$, then $f(gh) = (fg)h$. (Associativity)

Throughout this thesis, we will be working with *small* categories, that is, those categories whose collection of maps and collection of objects is, in fact, a set. We will give categories of "all" sets as an example, and the reader should take that to mean all sets contained within a sufficiently large set.

We give a few examples of categories:

[1] Note that composition is written in diagrammatic order throughout this thesis.

Example 2.1.2. We set $\mathbf{1}$ to be the category consisting of a single object A and the identity arrow $id : A \to A$. This is obviously a category, with $\mathbf{1}_o = A$ and $\partial_0 id = \partial_1 id = A$.

There is also the category $\mathbf{2}$:

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, "f"] & B \arrow[loop right, "id_B"] \end{tikzcd}$$

where f is the only non-identity arrow.

Example 2.1.3 (Preorders are categories)**.** Take any partially ordered set $(P, \leq)$. Define $f : a \to b$ for $a, b \in P$ if and only if $a \leq b$. This is a category as we always have:

(i) $a \leq a$ (Identity);

(ii) $a \leq b$ and $b \leq c$ implies $a \leq c$ (Composition);

Note that we have at most one map between any two objects in P, hence [**C.1**] and [**C.2**] are immediately satisfied

Example 2.1.4 (Dual Category)**.** Given a category $\mathbb{B}$, we may form the *dual* of $\mathbb{B}$, written $\mathbb{B}^{op}$ as the following category:

Objects: The objects of $\mathbb{B}$;

Maps: $f^{op} : B \to A$ in $\mathbb{B}^{op}$ when $f : A \to B$ in $\mathbb{B}$;

Identity: The identity maps of $\mathbb{B}$;

Composition: If $fg = h$ in $\mathbb{B}$, $g^{op} f^{op} = h^{op}$.

Note the format of the previous example, where we list the four basic requirements of a category. This is typically how we will present categories in this thesis. Depending upon the complexity of the definition, we may add further proof that it meets [**C.1**] and [**C.2**].

The previous example is an important one, as we will often speak of *dualizing* a notion, or that concept "x" is the *dual* of concept "y". This means that when "y" holds in a category $\mathbb{B}$, the "x" holds in $\mathbb{B}^{op}$.

2.2 Properties of maps

Many useful properties of maps are generalizations of notions used for sets and functions. We present a few of these in Table 2.1, together with their categorical definition. Throughout Table 2.1, e, f, g are maps in a category C with $e : A \to A$ and $f, g : A \to B$.

Sets	**Categorical Property**	**Definition**
Injective	Monic	f is monic whenever $hf = kf$ means that $h = k$.
Surjective	Epic	The dual notion to monic, g is epic whenever $gh = gk$ means that $h = k$. A map that is both monic and epic is called *bijic*.
Left Inverse	Section	f is a section when there is a map $f^\diamond$ such that $ff^\diamond = 1_A$. f is also referred to as the *left inverse* of $f^\diamond$.
Right Inverse	Retraction	f is a retraction when there is a map $f_\diamond$ such that $f_\diamond f = 1_B$. f is also referred to as the *right inverse* of $f_\diamond$. A map that is both a section and a retraction is called an *isomorphism*.
Idempotent	Idempotent	An endomorphism e is idempotent whenever $ee = e$.

Table 2.1: Properties of Maps In Categories

There are number of basic properties of maps enumerated in this lemma:

Lemma 2.2.1. *In a category* $\mathbb{B}$*,*

(i) If f, g are monic, then fg is monic.

(ii) If fg is monic, then f is monic.

(iii) f being a section means it is monic.

(iv) f, g sections implies that fg is a section.

(v) fg a section means f is a section.

(vi) If $f : A \to B$ is both a section and a retraction, then $f^\diamond = f_\diamond$, where $f^\diamond$ and $f_\diamond$ are as defined in Table 2.1.

(vii) f is an isomorphism if and only if it is an epic section.

Proof.

(i) Suppose $hfg = kfg$. As g is monic, $hf = kf$. As f is monic, this gives us $h = k$ and therefore fg is monic.

(ii) See [5], chapter 2.

(iii) Suppose $hf = kf$. Then $hff^\diamond = kff^\diamond$ giving us $h1 = k1$ and therefore $h = k$ and f is monic.

(iv) We are given $ff^\diamond = 1$ and $gg^\diamond = 1$. But then $fgg^\diamond f^\diamond = ff^\diamond = 1$ and fg is a section.

(v) We are given there is some h such that $(fg)h = 1$. This means $f(gh) = 1$ and f is a section.

(vi) See [15], Lemma 1.2.2.

(vii) See [15], Lemma 1.2.3.

$\square$

Note there are corresponding properties for epics and retractions, obtained by dualizing the statements of Lemma 2.2.1.

Suppose $f : A \to B$ is a retraction with left inverse $f_\diamond : B \to A$. Note that $ff_\diamond$ is idempotent as $ff_\diamond ff_\diamond = f1_B f_\diamond = ff_\diamond$. If we are given an idempotent e, we say e is *split* if there is a retraction f with $e = ff_\diamond$.

In general, not all idempotents in a category will split. The following construction allows us to create a category based on the original one in which all idempotents do split.

Definition 2.2.2. Given a category $\mathbb{B}$ and a set of idempotents E of $\mathbb{B}$, we may create *$Split_E(\mathbb{B})$*. As this is called the Karoubi envelope, this is normally written as $\mathrm{K}_E(\mathbb{B})$, and defined as:

Objects: (A, e), where A is an object of $\mathbb{B}$, $e : A \to A$ and $e \in E$.;

Maps: $f_{d,e} : (A, d) \to (B, e)$ is given by $f : A \to B$ in $\mathbb{B}$, where $f = dfe$.;

Identity: The map $e_{e,e}$ for (A, e).;

Composition: Inherited from $\mathbb{B}$..

When E is the set of all idempotents in $\mathbb{B}$, we write $\mathrm{K}(\mathbb{B})$.

This is the standard idempotent splitting construction, variously known as the Karoubi envelope or Cauchy completion.

2.3 Functors and natural transformations

Definition 2.3.1. A map $F : \mathbb{X} \to \mathbb{Y}$ between categories, as in Definition 2.1.1, is called a *functor*, provided it satisfies the following:

[**F.1**] $F(\partial_0(f)) = \partial_0(F(f))$ and $F(\partial_1(f)) = \partial_1(F(f))$;

[**F.2**] $F(fg) = F(f)F(g)$;

Lemma 2.3.2. *Categories and functors form a category* $\textsc{Cat}$.

Proof.

Objects: Categories.

Maps: Functors.

Identity: The identity functor which takes an object to the same object and a map to the same map.

Composition: Given $F : \mathbb{A} \to \mathbb{B}$, $G : \mathbb{B} \to \mathbb{D}$, define the functor $FG : \mathbb{A} \to \mathbb{D}$ such that $FG(x) = G(F(x))$ which is clearly associative.

Note this would be a *large* category, as its collection of objects would not be a set. $\square$

Definition 2.3.3. A functor $F : \mathbb{X} \to \mathbb{Y}$ is *faithful* when for each pair of objects A, B in $\mathbb{X}$, and map $g : FA \to FB$ in $\mathbb{Y}$, there is a map $f : A \to B$ in $\mathbb{X}$ such that $Ff = g$.

Definition 2.3.4. A functor $F : \mathbb{X} \to \mathbb{Y}$ is *full* when for parallel maps f, f', if $Ff = Ff'$ then $f = f'$.

We may also consider the notion of containment between categories:

Definition 2.3.5. Given the categories $\mathbb{B}$ and $\mathbb{D}$, we say that $\mathbb{B}$ is a *sub-category* of $\mathbb{D}$ when each object of $\mathbb{B}$ is an object of $\mathbb{D}$ and when each map of $\mathbb{B}$ is a map of $\mathbb{D}$.

When $\mathbb{B}$ is a subcategory of $\mathbb{D}$, the functor $J : \mathbb{B} \to \mathbb{D}$ which takes each object to itself in $\mathbb{D}$ and each map to itself in $\mathbb{D}$ is called the inclusion functor. When J is a full functor, we say $\mathbb{B}$ is a full subcategory of $\mathbb{D}$.

We now have the machinery to discus the relationship between a category $\mathbb{B}$ and its splitting $\mathrm{K}(\mathbb{B})$:

Lemma 2.3.6. *Given a category $\mathbb{B}$, then it is a full sub-category of $K(\mathbb{B})$ and all idempotents split in $K(\mathbb{B})$.*

Proof. We identify each object A in $\mathbb{B}$ with the object $(A, 1)$ in $\mathrm{K}(\mathbb{B})$. The only maps between $(A, 1)$ and $(B, 1)$ in $\mathrm{K}(\mathbb{B})$ are the maps between A and B in $\mathbb{B}$, hence we have a full sub-category.

Suppose we have the map $d_{e,e} : (A, e) \to (A, e)$ with $dd = d$, i.e., it is idempotent in C and $\mathrm{K}(\mathbb{B})$. In $\mathrm{K}(\mathbb{B})$, we have the maps $d_{e,d} : (A, e) \to (A, d)$ and $d_{d,e} : (A, d) \to (A, e)$ where $d_{d,e}d_{e,d} = d_{d,d} = 1_{(A,d)}$ and $d_{e,d}d_{d,e} = d_{e,e}$. Hence, it is a splitting of the map $d_{e,e}$. $\square$

Functors with two arguments, e.g., $F : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ which satisfy [**F.1**] and [**F.2**] for each argument independently are called *bi-functors*.

We will often restrict ourselves to specific classes of functors which either *preserve* or *reflect* certain characteristics of the domain category or codomain category. To be more precise, we provide some definitions.

Definition 2.3.7. Given a category $\mathbb{S}$, a *diagram* in a category $\mathbb{B}$ of *shape* $\mathbb{S}$ is a functor $D : \mathbb{S} \to \mathbb{B}$.

Definition 2.3.8. A *property* of a diagram D, written $P(D)$ is a logical relation expressed using the objects and maps of the diagram D.

For example, $P(f : A \to B) = \exists h : B \to A. hf = 1_A$ expresses that f is a retraction.

Definition 2.3.9. A functor F *preserves* the property P over maps f_i and objects A_j when:

$$P(f_1, \ldots, f_n, A_1, \ldots, A_m) \implies P(F(f_1), \ldots, F(f_n), F(A_1), \ldots, F(A_m)).$$

A functor F *reflects* the property P over maps f_i and objects A_j when:

$$P(F(f_1), \ldots, F(f_n), F(A_1), \ldots, F(A_m)) \implies P(f_1, \ldots, f_n, A_1, \ldots, A_m).$$

For example, all functors preserve the properties of being an idempotent or a retraction or section, but in general, not the property of being monic.

A functor $F : \mathbb{B} \to \mathbb{D}$ induces a map between hom-sets in $\mathbb{B}$ and hom-sets in $\mathbb{D}$. For each object A, B in $\mathbb{B}$ we have the map:

$$F_{AB} : \mathbb{B}(A, B) \to \mathbb{D}(F(A), F(B)).$$

Definition 2.3.10. Given functors $F, G : \mathbb{X} \to \mathbb{Y}$, a *natural transformation* $\alpha : F \Rightarrow G$ is a collection of maps in $\mathbb{Y}$, $\alpha_X : F(X) \to G(X)$, indexed by the objects of $\mathbb{X}$ such that for all $f : X_1 \to X_2$ in $\mathbb{X}$ the following diagram in $\mathbb{Y}$ commutes:

$$\begin{tikzcd}
F(X_1) \arrow[r, "F(f)"] \arrow[d, "\alpha_{X_1}"'] & F(X_2) \arrow[d, "\alpha_{X_2}"] \\
G(X_1) \arrow[r, "G(f)"'] & G(X_2)
\end{tikzcd}$$

In the case where a natural transformation is a collection of isomorphisms, we write $\alpha : F \cong G$ or simply $F \cong G$.

2.4 Enrichment of categories

If $\mathbb{X}$ is a category, then the maps from A to B in $\mathbb{X}$ are denoted $\mathbb{X}(A, B)$. If $\mathbb{X}(A, B)$ is a set for all objects in $\mathbb{X}$, we say $\mathbb{X}$ is *enriched* in sets. More generally, categories may be enriched in any monoidal category. For example a category may be enriched in abelian groups, vector spaces, posets, categories or commutative monoids.

Specific types of enrichment may force a structure on a category. Examples:

1. If $\mathbb{X}$ is enriched in sets of cardinality of 0 or 1, then $\mathbb{X}$ is a preorder.
2. If $\mathbb{X}$ is enriched in pointed sets with the monoid of smash product, the $\mathbb{X}$ has zero morphisms.
3. If $\mathbb{X}$ is enriched in abelian groups, then $\mathbb{X}$ is a preadditive category (has zero morphisms and has finite products which are the same as the coproduct).

2.5 Examples of categories

In this section, we will offer a few examples of categories.

Example 2.5.1. A group G may be considered as a one-object category $\mathbb{G}$, with object $\{*\}$. The elements of the group are the maps between $\{*\}$. As G is a group, there is an identity, composition is given by the group multiplication and additionally, each map has an inverse. As $\mathbb{G}(\{*\}, \{*\}) = G$, this category is enriched in groups.

Four categories derived from sets are SETS, PAR, REL and PINJ:

Example 2.5.2 (SETS)**.** In this category, the maps are given by all set functions.

Objects: Sets;

Maps: Set functions;

Identity: The identity function;

Composition: Standard composition of functions.

Example 2.5.3 (PAR)**.** For this example, the maps are all partial set functions.

Objects: Sets;

Maps: Partial set functions;

Identity: The identity function;

Composition: Standard composition of functions.

Example 2.5.4 (REL)**.** REL is often of interest in quantum programming language semantics:

Objects: All sets;

Maps: Relations: $R : X \to Y$;

Identity: $1_X = \{(x, x) | x \in X\}$;

Composition: $RS = \{(x, z) | \, \exists y.(x, y) \in R \text{ and } (y, z) \in S\}$.

Note that REL is enriched in posets, via set inclusion. PAR can be viewed as a subcategory of REL, with the same objects, but only allowing maps which are partial functions, i.e., deterministic relations where if $(x, y), (x, y') \in f$, then $y = y'$. PAR is also enriched in posets, via the same inclusion ordering as in REL.

Example 2.5.5 (PINJ)**.** Our final example based on sets is one that will be used throughout this thesis. The category PINJ consists of the partial injective functions over sets. Similarly to PAR, it may be considered as a subcategory of REL. The maps f, g (relations in REL) in PINJ are defined as follows:

$$(x, y) \in f \text{ and } (x, y') \in f \text{ implies } y = y', \tag{2.1}$$

$$(x, y) \in f \text{ and } (x', y) \in f \text{ implies } x = x'. \tag{2.2}$$

Objects: All sets;

Maps: Relations: $f : X \to Y$ which satisfy Equation (2.1) and Equation (2.2);

Identity: $1_X = \{(x, x) | x \in X\}$;

Composition: $fg = \{(x, z) | \, \exists y.(x, y) \in f \text{ and } (y, z) \in g\}$.

Example 2.5.6 (Top)**.** This is the category of topological spaces.

Objects: Topological spaces;

Maps: Continuous functions;

Identity: The identity function;

Composition: Function composition.

As the composition of continuous maps is also continuous, this is a category.

Our next example shows maps in categories need not always be something normally thought of as a function or relation.

Example 2.5.7 (Matrix category)**.** Given a rig R (i.e., a ri**n**g minus **n**egatives, e.g., the natural numbers), one may form the category Mat(R). For example, the category of matrices over natural numbers is:

Objects: $\mathbb{N}$;

Maps: $[r_{ij}] : n \to m$ where $[r_{ij}]$ is an $n \times m$ matrix over $\mathbb{N}$;

Identity: I_n;

Composition: Matrix multiplication.

Our last example describes a construction on a category

Example 2.5.8 (Slice category)**.** Given a category $\mathbb{B}$ and an object A in $\mathbb{B}$, form the *slice category* $\mathbb{B}/A$:

Objects: Maps of $\mathbb{B}$ whose codomain is A, i.e., $f : B \to A$ for some B.

Maps: Triples $(f_1, g, f_2) : f_1 \to f_2$ such that

$$\begin{tikzcd} B_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & B_2 \arrow[dl, "f_2"] \\ & A & \end{tikzcd}$$

commutes.

Identity: $(f, 1_B, f) : f \to f)$.

Composition: $(f_1, g, f_2)(f_2, h, f_3) = (f_1, gh, f_3)$ which is well defined as $ghf_3 = gf_2 = f_1$.

There is also the concept of the *simple slice* category [7], $\mathbb{B}[A]$ defined as:

Objects: Objects of $\mathbb{B}$

Maps: $f : C \to D$ is given by maps $f : C \times A \to D$.

Identity: projection – π_0

Composition: $C \xrightarrow{f} D \xrightarrow{g} E$ is given by $C \times A \xrightarrow{1\times\Delta} C \times A \times A \xrightarrow{f\times 1} D \times A \xrightarrow{g} E$.

Note the definition of simple slice category relies on products in categories, to be introduced in the next section.

2.6 Limits and colimits in categories

We shall review only a few basic limits/colimits in categories, in order to set up notation and terminology. First we discuss initial and terminal objects.

Definition 2.6.1. An *initial object* in a category $\mathbb{B}$ is an object which has exactly one map to each other object in the category. The dual notion is *terminal object.* Every object in the category has exactly one map to the terminal object.

Lemma 2.6.2. *Suppose I, J are initial objects in $\mathbb{B}$. Then there is a unique isomorphism $i : I \to J$.*

Proof. First, note that by definition there is only one map from I to I — which must be the identity map. As I is initial there is a map $i : I \to J$. As J is initial there is a map $j : J \to I$. But this means $ij : I \to I = 1$ and $ji : J \to J = 1$ and hence i is the unique isomorphism from I to J. $\square$

Dually, we have the corresponding result to Lemma 2.6.2 for terminal objects — they are also unique up to a unique isomorphism.

In categories, following the terminology in sets, we normally designate the initial object by 0 and the terminal object by 1. A map *from* the terminal object to another object in the category is often referred to as an *element* or a *point*.

We now turn to products and co-products.

Definition 2.6.3. Let A, B be objects of the category $\mathbb{B}$. Then the object $A \times B$ is a *product* of A and B when:

- There exist maps π_0, π_1 with $\pi_0 : A \times B \to A$, $\pi_1 : A \times B \to B$;
- Given an object C with maps $f : C \to A$ and $g : C \to B$ there is a unique map $\langle f, g\rangle$ such that the following diagram commutes:

$$\begin{tikzcd} & & A \\ C \arrow[urr, "f"] \arrow[r, dotted, "\langle f,g\rangle"] \arrow[drr, "g"'] & A \times B \arrow[ur, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ & & B \end{tikzcd}$$

A co-product is the dual of a product.

Definition 2.6.4. Let A, B be objects of the category $\mathbb{B}$. Then the object $A + B$ is a *coproduct* of A and B when:

- There exist maps $\amalg_1, \amalg_2$ with $\amalg_1 : A \to A + B$, $\amalg_2 : B \to A + B$;
- Given an object C with maps $h : A \to C$ and $k : B \to C$ there is a unique map $[h, k]$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[dr, "\amalg_1"'] \arrow[drr, "h"] & & \\
& A + B \arrow[r, dotted, "{[h,k]}"] & C \\
B \arrow[ur, "\amalg_2"] \arrow[urr, "k"'] & &
\end{tikzcd}$$

It is possible for an object to both a limit and a co-limit at the same time:

Definition 2.6.5. Given a category $\mathbb{B}$, any object that is both a terminal and initial object is called a *zero object*. This object is labelled $\mathbf{0}$.

Note that any category with a zero object has a special map, $\mathbf{0}_{A,B}$ between any two objects A, B of the category given by: $\mathbf{0}_{A,B} : A \to \mathbf{0} \to B$.

Definition 2.6.6. In a category $\mathbb{B}$, with products and coproducts, where:

$$\amalg_i \pi_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and given any two objects, A, B, $A \times B$ is the same as $A + B$, then $A \times B$ is referred to as the *biproduct* and designated as $A \boxplus B$. A category $\mathbb{D}$ is said to have *finite biproducts* when it has a zero object $\mathbf{0}$ and when each pair of objects A, B have a biproduct $A \boxplus B$.

Note the biproduct is often written as $\oplus$. As we will be using $\oplus$ frequently in this thesis where it is not a biproduct, this alternate notation for $\boxplus$ will be used instead.

Note that a category with finite biproducts is enriched in commutative monoids. If $f, g : A \to B$, define $f + g : A \to B$ as $\langle id_A, id_A \rangle (f \boxplus g) [id_B, id_B]$. The unit for the addition is $\mathbf{0}_{A,B}$. Throughout this thesis, when working in a category with finite biproducts, $\langle id, id \rangle$ will be designated by Δ and $[id, id]$ will be designated by ∇.

2.7 Symmetric Monoidal Categories

Definition 2.7.1. A *symmetric monoidal category* [5, 37] $\mathbb{D}$ is a category equipped with a monoid $\otimes$ (a bi-functor $\otimes : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$) together with four families of natural isomorphisms: $a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, $u^r_A : A \otimes I \to A$, $u^l_A : I \otimes A \to A$ and $c_{A,B} : A \otimes B \to B \otimes A$, which satisfy coherence diagrams and equations shown in Figures 2.1, 2.2, 2.3, 2.4 and 2.5. The isomorphisms are referred to as the *structure isomorphisms* for the symmetric monoidal category. I is the unit of the monoid. A symmetric monoidal category where each of $a_{A,B,C}$, u^r_A, u^l_A and $c_{A,B}$ are identity maps is called a *strict symmetric monoidal category*.

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "a_{A,B,(C\otimes D)}"] \arrow[d, "1\otimes a_{B,C,D}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a_{(A\otimes B),C,D}"] & ((A \otimes B) \otimes C) \otimes D \arrow[d, "a_{A,B,C}\otimes 1"] \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "a_{A,(B\otimes C),D}"'] & & (A \otimes (B \otimes C)) \otimes D
\end{tikzcd}$$

Figure 2.1: Pentagon diagram for associativity in an SMC.

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "a_{A,I,B}"] \arrow[dr, "1\otimes u^l_B"'] & & (A \otimes I) \otimes B \arrow[dl, "u^r_A\otimes 1"] \\
& A \otimes B &
\end{tikzcd}$$

and $u^r_I = u^l_I : I \otimes I \to I$

Figure 2.2: Unit diagram and equation in an SMC.

The essence of the coherence diagrams is that any diagram composed solely of the structure isomorphisms will commute.

$$
\begin{tikzcd}
A \otimes B \arrow[r, "c_{A,B}"] \arrow[dr, equal] & B \otimes A \arrow[d, "c_{B,A}"] \\
 & A \otimes B
\end{tikzcd}
$$

Figure 2.3: Symmetry in an SMC.

$$
\begin{tikzcd}
A \otimes I \arrow[rr, "c_{A,I}"] \arrow[dr, "u^r_A"'] & & I \otimes A \arrow[dl, "u^l_A"] \\
 & A &
\end{tikzcd}
$$

Figure 2.4: Unit symmetry in an SMC.

$$
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "c_{(A\otimes B),C}"] \arrow[d, "a^{-1}_{A,B,C}"'] & C \otimes (A \otimes B) \arrow[d, "a_{C,A,B}"] \\
A \otimes (B \otimes C) \arrow[d, "1\otimes c_{B,C}"'] & (C \otimes A) \otimes B \arrow[d, "c_{C,A}\otimes 1"] \\
A \otimes (C \otimes B) \arrow[r, "a_{A,C,B}"] & C \otimes (A \otimes B)
\end{tikzcd}
,\qquad
\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "c_{A,(B\otimes C)}"] \arrow[d, "a_{A,B,C}"'] & (B \otimes C) \otimes A \arrow[d, "a^{-1}_{B,C,A}"] \\
(A \otimes B) \otimes C \arrow[d, "c_{A,B}\otimes 1"'] & B \otimes (C \otimes A) \arrow[d, "1\otimes c_{C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "a^{-1}_{B,A,C}"] & B \otimes (A \otimes C)
\end{tikzcd}
$$

Figure 2.5: Associativity symmetry in an SMC.

2.8 Adjoint functors and equivalences

Referring to Section 2.3, we consider some relationships between categories. While two categories may be isomorphic (i.e. there is an invertible functor between them), we normally consider two alternate relations between categories: Equivalence and adjointness.

Definition 2.8.1. Given two categories $\mathbb{A}$ and $\mathbb{B}$, we say they are *equivalent* when there exists two functors, $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$ and natural isomorphisms such that $FG \cong I : \mathbb{A} \to \mathbb{A}$ and $GF \cong I : \mathbb{B} \to \mathbb{B}$.

Example 2.8.2 (Skeleton)**.** Consider the category of all finite sets, $\textsc{Sets}_f$. This is equivalent to the category of ordinal sets, $\textsc{Ord}$, $(0 = \phi, 1 = \{0\}, \ldots, n = \{0, 1, \ldots, n-1\}, \ldots)$ [37]. Take $F : \textsc{Ord} \to \textsc{Sets}_f$ as the inclusion functor. Set $G : \textsc{Sets}_f \to \textsc{Ord}$ such that $G : S \mapsto |S|$. We have immediately that $FG = I$ and as $\phi_S : |S| \to S$ in $\textsc{Sets}_f$is an isomorphism, we have $\phi : GF \to I$.

This is a specific example of a the *skeleton* of a category being equivalent to the original category. The skeleton $\mathbb{A}$ of a category $\mathbb{B}$ is any full subcategory of $\mathbb{B}$ such that each object $B \in \mathbb{B}$ is isomorphic in $\mathbb{B}$ to exactly one object of $\mathbb{A}$.

We now turn to adjoint functors.

Definition 2.8.3. Given $G : \mathbb{A} \to \mathbb{B}$ is a functor and $A \in \mathbb{A}$, then an object U in $\mathbb{B}$ together with a map $\eta_A : A \to G(U)$ is a *universal pair* for the G at A if whenever there is a map $f : A \to G(Y)$, there is a unique $f^\# : U \to Y$ such that

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & G(U) \arrow[d, dotted, "G(f^\#)"] \\ & G(Y) \end{tikzcd}$$

is a commutative diagram.

Let us consider what happens when we have the situation above, but the U in $\mathbb{B}$ is definable based on the $A \in \mathbb{A}$.

Lemma 2.8.4. *Suppose for $G : \mathbb{B} \to \mathbb{A}$ is a functor such that for each $A \in \mathbb{A}$, there is an operation $F : \mathbb{A} \to \mathbb{B}$ such that $(F(A), \eta_A)$ is a universal pair. Then:*

- *F is a functor with $F(g) = (g\eta)^{\#}$;*
- *$\eta_A : A \to G(F(A))$ is a natural transformation;*
- *$\epsilon_B = 1^{\#}_{G(B)} : F(G(B)) \to B$ is a natural transformation;*
- *The triangle equalities, $\eta_{G(B)}G(\epsilon_B) = 1_B$ and $F(\eta_A)\epsilon_{F(A)} = 1_{F(A)}$ hold.*

Conversely, if we have functors F, G with transformation η and ϵ which satisfy the triangle identities, then each $(F(A), \eta_A)$ is universal for G at A.

Proof. See Proposition 2.2.2 [15]. □

When we have this occurring we say F is *left adjoint* to G (and G is the *right adjoint* of F). The transformation η is referred to as the *unit* and the transformation ϵ is referred to as the *counit*. This is written as:

$$(\eta, \epsilon) : F \vdash G : A \to B.$$

Revisiting Example 2.8.2 above, we see that $(1, \phi) : F \vdash G : \textsc{Ord} \to \textsc{Sets}_f$.

Chapter 3

Restriction categories

Restriction categories were introduced in [17–19] as a way to give an algebraic axiomatization of partial maps. We will introduce restriction categories and provide a variety of results.

3.1 Definitions

Definition 3.1.1. A *restriction category* is a category $\mathbb{X}$ together with a *restriction operator* on maps,

$$\frac{f : A \to B}{\overline{f} : A \to A,}$$

where f is a map of $\mathbb{X}$ and A, B are objects of $\mathbb{X}$, such that the following four *restriction identities* hold, whenever the compositions are defined:

[**R.1**] $\overline{f}f = f$	[**R.2**] $\overline{g}\,\overline{f} = \overline{f}\,\overline{g}$
[**R.3**] $\overline{\overline{f}g} = \overline{f}\,\overline{g}$	[**R.4**] $f\overline{g} = \overline{fg}f$.

Definition 3.1.2. A *restriction functor* is a functor which preserves the restriction. That is, given a functor $F : \mathbb{X} \to \mathbb{Y}$ with $\mathbb{X}$ and $\mathbb{Y}$ restriction categories, F is a restriction functor if:

$$F(\overline{f}) = \overline{F(f)}.$$

Note that any map such that $r = \overline{r}$ is an idempotent, as $rr = \overline{r}r = r$. Such a map is called a *restriction idempotent*.

Here are some basic facts (see e.g., [17] and [20]) for restriction categories.

Lemma 3.1.3. *In a restriction category* $\mathbb{X}$*,* *(i)* $\overline{f}$ *is idempotent;*

(ii) $\overline{fg} = \overline{fg}\,\overline{f}$*;*

(iii) $\overline{fg} = \overline{f\overline{g}}$ *;*

(iv) $\overline{\overline{f}} = \overline{f}$*;*

(v) $\overline{f}\,\overline{g} = \overline{\overline{f}\,\overline{g}}$*;*

(vi) f *monic implies* $\overline{f} = 1$*;*

(vii) $f = \overline{g}f \implies \overline{g}\,\overline{f} = \overline{f}$*.*

Proof.

(i) Using [**R.3**] and then [**R.1**], we see $\overline{f}\,\overline{f} = \overline{\overline{f}f} = \overline{f}$.

(ii) Using [**R.1**], [**R.3**] and then [**R.2**], $\overline{fg} = \overline{\overline{f}fg} = \overline{f}\,\overline{fg} = \overline{fg}\,f$.

(iii) Using (ii), [**R.3**] and then [**R.4**], $\overline{fg} = \overline{fg}\,\overline{f} = \overline{\overline{fg}f} = \overline{f\overline{g}}$.

(iv) By (iii), $\overline{f} = \overline{1f} = \overline{1\overline{f}} = \overline{\overline{f}}$.

(v) Using [**R.3**], $\overline{\overline{f}\,\overline{g}} = \overline{f}\,\overline{\overline{g}} = \overline{f}\,\overline{g}$.

(vi) By [**R.1**] $\overline{f}f = 1f$, hence when f is monic, $\overline{f} = 1$.

(vii) $\overline{g}\,\overline{f} = \overline{\overline{g}f} = \overline{f}$.

□

Note that by Lemma 3.1.3, all maps $\overline{f}$ are restriction idempotents as $\overline{f} = \overline{\overline{f}}$.

Definition 3.1.4. A map $f : A \to B$ in a restriction category is said to be *total* when $\overline{f} = 1_A$.

Lemma 3.1.5. *The total maps in a restriction category form a sub-category* $Total(\mathbb{X}) \subseteq \mathbb{X}$.

Proof. First, as the identity map 1 is monic, by Lemma 3.1.3, we have $\overline{1} = 1$ and therefore the identity map is in $Total(\mathbb{X})$. If f, g are composable maps in $Total(\mathbb{X})$, then $\overline{f\,g} = \overline{f\overline{g}} = \overline{f} = 1$ and hence fg is in $Total(\mathbb{X})$. Therefore, $Total(\mathbb{X})$ is a sub-category of $\mathbb{X}$. □

Example 3.1.6 (PAR)**.** Continuing from Example 2.5.3, PAR is a restriction category. The restriction of $f : A \to B$ is:

$$\overline{f}(x) = \begin{cases} x & \text{if } f(x) \text{ is defined,} \\ \uparrow & \text{if } f(x) \text{ is } \uparrow. \end{cases}$$

In PAR, the total maps correspond precisely to the functions that are defined on all elements of the domain.

Example 3.1.7 (REL)**.** The category REL from Example 2.5.4 is a *not* a restriction category with the candidate restriction of $R = \{(a,b)\}$ being $\overline{R} = \{(a,a)|\exists b.(a,b) \in R\}$. The axiom that fails is [**R.4**], as can be seen by setting $R = \{(1,1),(1,2)\}, S = \{(2,3)\}$. Then we have $\overline{RS} = \{(1,1)\}$ and therefore $\overline{RS}R = R$. However, $R\overline{S} = R\{(2,2)\} = \{(1,2)\}$.

Example 3.1.8 (PINJ)**.** From Example 2.5.5, we see PINJ is a restriction category and in fact is a sub-restriction category of PAR. We will show the four restriction axioms:

[**R.1**] $\overline{f}f = \{(x,z)|\exists x.(x,x) \in \overline{f} \text{ and } (x,z) \in f\} = \{(x,z)|(x,z) \in f\} = f$,

[**R.2**] $\overline{f}\overline{g} = \{(x,z)|\exists y.(x,y) \in \overline{f} \text{ and } (y,z) \in \overline{g}\} = \{(x,x)|(x,x) \in \overline{f} \text{ and } (x,x) \in \overline{g}\} = \overline{g}\overline{f}$,

[**R.3**] $\overline{\overline{f}g} = \overline{\{(x,y)|(x,x) \in \overline{f}, (x,y) \in g\}} = \{(x,x)|(x,x) in \overline{f}, (x,x) \in \overline{g}\} = \overline{f}\overline{g}$,

[**R.4**] $f\overline{g} = \{(x,y)|(x,y) \in f, (y,y) \in \overline{g}\} = \{(x,y)|(x,y) \in f, \exists z.(y,z) \in g\}$

$$= \{(x,y)|(x,y) \in f, \exists z.(x,z) \in fg\} = \{(x,y)|(x,y) \in f, (x,x) \in \overline{fg}\} = \overline{fg}f.$$

Example 3.1.9 (TOP_p)**.** This is the category of topological spaces with partial functions.

Objects: Topological spaces;

Maps: Any partial function f, where f is defined on some open subset of $\partial_0 f$;

Identity: The identity function;

Composition: Function composition;

Restriction: The restriction of $f : A \to B$ is:

$$\overline{f}(x) = \begin{cases} x & \text{if } f(x) \text{ is defined,} \\ \uparrow & \text{if } f(x) \text{ is } \uparrow. \end{cases}$$

3.2 Partial order enrichment

We may use the restriction to define a partial order on the hom-sets of a restriction category. Intuitively, we would think of a map f being less than a map g if f is defined on fewer elements than g *and* they agree where they are defined. This sub-section will bring precision to the above intuition.

Definition 3.2.1. In a restriction category, for any two parallel maps $f, g : A \to B$, define $f \leq g$ iff $\overline{f}g = f$.

Lemma 3.2.2. *For any restriction category $\mathbb{X}$, it is enriched in partial orders under the ordering $\leq$ from Definition 3.2.1 and the following hold:*

(i) $f \leq g \implies \overline{f} \leq \overline{g}$;

(ii) $\overline{fg} \leq \overline{f}$;

(iii) $f \leq g \implies hf \leq hg$;

(iv) $f \leq g \implies fh \leq gh$;

(v) $f \leq g$ *and* $\overline{f} = \overline{g}$ *implies* $f = g$;

(vi) $f \leq 1 \iff f = \overline{f}$;

(vii) $\overline{g}f = f$ *implies* $\overline{f} \leq \overline{g}$.

Proof. First, we show the enrichment by showing $\leq$ is a partial order on $\mathbb{X}(A, B)$. With $f, g, h : A \to B$ parallel maps in $\mathbb{X}$, each of the requirements for a partial order is verified below:

Reflexivity: $\overline{f}f = f$ and therefore, $f \leq f$.

Anti-Symmetry: Given $\overline{f}g = f$ and $\overline{g}f = g$, it follows:

$$f = \overline{f}f = \overline{\overline{f}g}f = \overline{f}\,\overline{g}f = \overline{g}\overline{f}f = \overline{g}f = g.$$

Transitivity: Given $f \leq g$ and $g \leq h$,

$$\overline{f}h = \overline{\overline{f}g}h = \overline{f}\,\overline{g}h = \overline{f}g = f$$

showing that $f \leq h$.

We now show the rest of the claims.

(i) The premise is that $\overline{f}g = f$. From this, $\overline{f}\,\overline{g} = \overline{\overline{f}g} = \overline{f}$, showing $\overline{f} \leq \overline{g}$.

(ii) $\overline{hf}hg = h\overline{f}g = hf$ and therefore $hf \leq hg$.

(iii) $\overline{f}g = f$, this shows $\overline{fh}gh = \overline{\overline{f}gh}gh = \overline{f}\,\overline{gh}gh = \overline{f}gh = fh$ and therefore $fh \leq gh$.

(iv) $g = \overline{g}g = \overline{f}g = f$.

(v) As $f \leq 1$ means precisely $\overline{f}1 = f$.

(vi) Assuming $\overline{g}f = f$, we need to show $\overline{f}\,\overline{g} = \overline{f}$. Using [**R.2**] and then [**R.3**] we have $\overline{f}\,\overline{g} = \overline{g}\,\overline{f} = \overline{\overline{g}f} = \overline{f}$. Hence, $\overline{f} \leq \overline{g}$.

$\square$

Lemma 3.2.2 shows that restriction categories are enriched in partial orders.

In a restriction category $\mathbb{X}$, we will use the notation $\mathcal{O}(A)$ for the restriction idempotents of $A \in \mathrm{Ob}\,\mathbb{X}$. $\mathcal{O}(A) = \{x : A \to A | x = \overline{x}\}$. The notation $\mathcal{O}(A)$ was chosen to be suggestive of open sets, as in $\textsc{Top}_p$, see Example 3.1.9.

Lemma 3.2.3. *In a restriction category $\mathbb{X}$, $\mathcal{O}(A)$ is a meet semi-lattice.*

Proof. The top of the meet semi-lattice is 1_A, under the ordering from Definition 3.2.1. The meet of any two idempotents is given by their composition. $\square$

3.3 Joins

Definition 3.3.1. Two parallel maps $f, g : A \to B$ in a restriction category are *compatible*, written as $f \smile g$, when $\overline{f}g = \overline{g}f$. A restriction category $\mathbb{X}$ is a *restriction pre-order* when all parallel pairs of maps are compatible.

Definition 3.3.2. Given $\mathbb{R}$ is a restriction category with zero maps, then $\mathbb{R}$ is said to have *joins* [27] whenever there is an operator $\vee$ defined between compatible maps such that:

(i) $f \leq f \vee g$ and $g \leq f \vee g$,

(ii) $\overline{f \vee g} = \overline{f} \vee \overline{g}$,

(iii) $f, g \leq h$ implies that $f \vee g \leq h$ and

(iv) $h(f \vee g) = hf \vee hg$.

Theorem 3.3.3 (Cockett-Guo)**.** *Given a restriction category* $\mathbb{R}$ *with joins, then*

$$A \xrightarrow{\amalg_1} C \xleftarrow{\amalg_2} B$$

is a coproduct if and only iff:

(i) $\amalg_1$ *and* $\amalg_2$ *are restriction monics;*

(ii) $\overline{\amalg_1^{(-1)}}\,\overline{\amalg_2^{(-1)}} = 0$ *and*

(iii) $\overline{\amalg_1^{(-1)}} \vee \overline{\amalg_2^{(-1)}} = 1_C$.

Proof. See [11]. $\square$

Example 3.3.4 (Joins in Par)**.** In the restriction category Par, the join is given by:

$$(f \vee g)(x) = \begin{cases} f(x)(= g(x)) & \text{when both } f \text{ and } g \text{ are defined;} \\ f(x) & \text{when only } f \text{ is defined;} \\ g(x) & \text{when only } g \text{ is defined;} \\ \uparrow & \text{when both } f \text{ and } g \text{ are undefined.} \end{cases}$$

We now must show each of the items in Definition 3.3.2 hold:

(i) From its definition, we see the

$$\overline{f}(f \vee g) = \begin{cases} f(x) & \text{when } f \text{ is defined,} \\ \uparrow & \text{otherwise.} \end{cases}$$

giving us $f \leq f \vee g$. Similarly, $g \leq f \vee g$.

(ii) $\overline{f \vee g}(x) = x$ wherever f or g is defined, but that is the same as $\overline{f} \vee \overline{g}$.

(iii) We are given $f, g \leq h$. Calculating

$$\overline{f \vee g}h = \begin{cases} h(x) & \text{when } f \text{ or } g \text{ are defined,} \\ \uparrow & \text{otherwise.} \end{cases}$$

But since $\overline{f}h = f$ and $\overline{g}h = g$, this is the same as $f \vee g$ giving $f \vee g \leq h$.

(iv) Calculating,

$$h(f \vee g) = \begin{cases} hf(x)(= hg(x)) & \text{when } h,\ f \text{ and } g \text{ are defined;} \\ hf(x) & \text{when only } h \text{ and } f \text{ are defined;} \\ hg(x) & \text{when only } h \text{ and } g \text{ are defined;} \\ \uparrow & \text{when } h \text{ or both } f \text{ and } g \text{ are undefined.} \end{cases}$$

This is the same as the definition of $hf \vee hg$.

Example 3.3.5 (Joins in $\textsc{Top}_p$)**.** Recall from Example 3.1.9 that the map $f : A \to B$ is a continuous partial function on some open subset of A. $\overline{f}$ is the identity map on the open subset of A where f is defined, and as such, may be identified with that open subset. As the intersection of open subsets of A is again an open subset of A, given $f, g : A \to B$, define

$$f \vee g(x) = \begin{cases} f(x) & \text{when } x \in \overline{f} \cap \overline{g}, \\ f(x) & \text{when } x \in \overline{f} \setminus \overline{g}, \\ g(x) & \text{when } x \in \overline{g} \setminus \overline{f}, \\ \uparrow & \text{otherwise.} \end{cases}$$

Note this is similar to the definition of $\vee$ in Par and similar reasoning may be used to show it is a join.

3.4 Meets

Definition 3.4.1. A restriction category has *meets* if there is an operation $\cap$ on parallel maps:

$$\frac{A \overset{f}{\underset{g}{\rightrightarrows}} B}{A \xrightarrow{f\cap g} B}$$

such that $f \cap g \leq f, f \cap g \leq g, f \cap f = f, h(f \cap g) = hf \cap hg$.

Meets were introduced in [13]. The following are basic results on meets:

Lemma 3.4.2. *In a restriction category $\mathbb{X}$ with meets, where f, g, h are maps in $\mathbb{X}$, the following are true:*

(i) $f \leq g$ *and* $f \leq h \iff f \leq g \cap h$*;*

(ii) $f \cap g = g \cap f$*;*

(iii) $\overline{f \cap 1} = f \cap 1$*;*

(iv) $(f \cap g) \cap h = f \cap (g \cap h)$*;*

(v) $r(f \cap g) = rf \cap g$ *where* $r = \overline{r}$ *is a restriction idempotent;*

(vi) $(f \cap g)r = fr \cap g$ *where* $r = \overline{r}$ *is a restriction idempotent;*

(vii) $\overline{f \cap g} \leq \overline{f}$ *(and therefore* $\overline{f \cap g} \leq \overline{g}$*);*

(viii) $(f \cap 1)f = f \cap 1$*;*

(ix) $e(e \cap 1) = e$ *where* e *is idempotent.*

Proof.

(i) $f \leq g$ and $f \leq h$ means precisely $f = \overline{f}g$ and $f = \overline{f}h$. Therefore,

$$\overline{f}(g \cap h) = \overline{f}g \cap \overline{f}h = f \cap f = f$$

and so $f \leq g \cap h$. Conversely, given $f \leq g \cap h$, we have $f = \overline{f}(g \cap h) = \overline{f}g \cap \overline{f}h \leq \overline{f}g$. But $f \leq \overline{f}g$ means $f = \overline{f}\,\overline{f}g = \overline{f}g$ and therefore $f \leq g$. Similarly, $f \leq h$.

(ii) From (i), as by definition, $f \cap g \leq g$ and $f \cap g \leq f$.

(iii) $f \cap 1 = \overline{f \cap 1}(f \cap 1) = (\overline{f \cap 1}f) \cap (\overline{f \cap 1}) \leq \overline{f \cap 1}$ from which the result follows.

(iv) By definition and transitivity, $(f \cap g) \cap h \leq f, g, h$ therefore by (i) $(f \cap g) \cap h \leq f \cap (g \cap h)$. Similarly, $f \cap (g \cap h) \leq (f \cap g) \cap h$ giving the equality.

(v) Given $rf \cap g \leq rf$, calculate:

$$rf \cap g = \overline{rf \cap g}rf = \overline{r(rf \cap g)}f = \overline{rrf \cap rg}f = \overline{r(f \cap g)}f = r\overline{f \cap g}f = r(f \cap g).$$

(vi) Using the previous point with the restriction idempotent $\overline{fr}$,

$$\begin{aligned} fr \cap g = f\overline{r} \cap g = \overline{fr}f \cap g = \overline{fr}(f \cap g) &= \overline{fr}\,\overline{f \cap g}f \\ &= \overline{f \cap g}\,\overline{fr}f = \overline{f \cap g}f\overline{r} = (f \cap g)r. \end{aligned}$$

(vii) For the first claim,

$$\overline{f \cap g}\,\overline{f} = \overline{\overline{f}(f \cap g)} = \overline{(\overline{f}f) \cap g} = \overline{f \cap g}.$$

The second claim then follows by (ii).

(viii) Given $f \cap 1 \leq f$:

$$f \cap 1 \leq f \iff \overline{f \cap 1}f = f \cap 1 \iff (f \cap 1)f = f \cap 1$$

where the last step is by item (iii) of this lemma.

(ix) As e is idempotent, $e(e \cap 1) = (ee \cap e) = e$.

$\square$

Additionally, when a restriction category has both meets and joins, we have:

Lemma 3.4.3. *If $\mathbb{R}$ is a meet restriction category with joins, then the meet distributes over the join, i.e.,*

$$h \cap (f \vee g) = (h \cap f) \vee (h \cap g).$$

Proof.

$$\begin{aligned} h \cap (f \vee g) &= \overline{(f \vee g)} h \cap (f \vee g) \\ &= (\overline{f} \vee \overline{g}) h \cap (f \vee g) \\ &= (\overline{f}(h \cap (f \vee g))) \vee (\overline{g}(h \cap (f \vee g))) \\ &= (h \cap \overline{f}(f \vee g)) \vee (h \cap \overline{g}(f \vee g))) \\ &= (h \cap (f \vee g)) \vee (h \cap (f \vee g))). \end{aligned}$$

□

Example 3.4.4 (Meets in Pinj and Par)**.** The restriction category Pinj has meets given by the intersection of the sets defining the maps. First, we note that the hom-set ordering for Pinj is given by set inclusion. We immediately have

$$f \cap g \subseteq f$$

$$f \cap g \subseteq g$$

$$f \cap f = f$$

by the properties of sets and intersections. For the final requirement,

$$\begin{aligned} h(f \cap g) &= \{(x, z) | \exists y.(x, y) \in h, (y, z) \in f \cap g\} \\ &= \{(x, z) | \exists y.(x, y) \in h, (y, z) \in f, (y, z) \in g\} \\ &= \{(x, z) | (x, z) \in hf, (x, z) \in hg\} = hf \cap hg. \end{aligned}$$

Thus, intersection is a meet in Pinj.

Note that the calculations above apply immediately to Par as well and therefore intersection is a meet in Par.

3.5 Partial monics and isomorphisms

Partial isomorphisms play a central role in this thesis. Below we present some of their basic properties.

Definition 3.5.1. For maps f in a restriction category $\mathbb{X}$:

- f is a *partial isomorphism* when there is a *partial inverse*, written $f^{(-1)}$ with $ff^{(-1)} = \overline{f}$ and $f^{(-1)}f = \overline{f^{(-1)}}$;
- f is a *partial monic* if $hf = kf$ implies $h\overline{f} = k\overline{f}$;
- f is a *partial section* if there exists an h such that $fh = \overline{f}$;
- f is a *restriction monic* if it is a section s with a retraction r such that $rs = \overline{rs}$.

Note that restriction monic is a stronger notion than that of monic. Consider two objects A, B in a restriction category where we have $m : A \to B$, $r : B \to A$ with $mr = 1_A$. In this case A is called a *retract* of B, which we will write as $A \triangleleft B$. As m and r need not be unique, we will also write $(m, r)A \triangleleft B$ when the specific section and retraction are to be emphasized. Since m is a section, it is a monic and therefore total. The map rm is idempotent on B as $rmrm = r1m = rm$. A is referred to as a *splitting* of the idempotent rm. Note there is no requirement that $rm = \overline{rm}$ when m is simply monic.

Lemma 3.5.2. *In a restriction category:*

(i) *f, g partial monic implies fg is partial monic;*

(ii) *f a partial section implies f is partial monic;*

(iii) *f, g partial sections implies fg is a partial section;*

(iv) *The partial inverse of f, when it exists, is unique;*

(v) *If f, g have partial inverses and $f\,g$ exists, then $f\,g$ has a partial inverse;*

(vi) *A restriction monic s is a partial isomorphism.*

Proof.

(i) Suppose $hfg = kfg$. As g is partial monic, $hf\overline{g} = kf\overline{g}$. Therefore:

$$h\overline{fg}f = k\overline{fg}f \qquad [\mathbf{R.4}]$$

$$h\overline{fg}\,\overline{f} = k\overline{fg}\,\overline{f} \qquad f\text{partial monic}$$

$$h\overline{fg} = k\overline{fg} \qquad \text{Lemma 3.1.3, (ii).}$$

(ii) Suppose $gf = kf$. Then, $g\overline{f} = gfh = kfh = k\overline{f}$.

(iii) We have $fh = \overline{f}$ and $gh' = \overline{g}$. Therefore,

$$\begin{aligned}
fgh'h &= f\overline{g}h && g \text{ partial section}\\
&= \overline{fg}fh && [\mathbf{R.4}]\\
&= \overline{fg}\,\overline{f} && f \text{ partial section}\\
&= \overline{f}\,\overline{fg} && [\mathbf{R.2}]\\
&= \overline{\overline{f}fg} && [\mathbf{R.3}]\\
&= \overline{fg} && [\mathbf{R.1}].
\end{aligned}$$

(iv) Suppose both $f^{(-1)}$ and $f^{\diamond}$ are partial inverses of f. Then,

$$\begin{aligned}
f^{(-1)} &= \overline{f^{(-1)}}f^{(-1)} = f^{(-1)}ff^{(-1)} = f^{(-1)}\overline{f} = f^{(-1)}ff^{\diamond} = f^{(-1)}f\overline{f^{\diamond}}f^{\diamond}\\
&= \overline{f^{(-1)}}\,\overline{f^{\diamond}}f^{\diamond} = \overline{f^{\diamond}}\,\overline{f^{(-1)}}f^{\diamond} = f^{\diamond}f\overline{f^{(-1)}}f^{\diamond} = f^{\diamond}ff^{(-1)}ff^{\diamond} = f^{\diamond}ff^{\diamond} = f^{\diamond}.
\end{aligned}$$

(v) For $f : A \to B,\ g : B \to C$ with partial inverses $f^{(-1)}$ and $g^{(-1)}$ respectively, the partial inverse of fg is $g^{(-1)}f^{(-1)}$. Calculating $fgg^{(-1)}f^{(-1)}$ using all the restriction identities:

$$fgg^{(-1)}f^{(-1)} = f\overline{g}f^{(-1)} = \overline{fg}ff^{(-1)} = \overline{fg}\,\overline{f} = \overline{f}\,\overline{fg} = \overline{\overline{f}fg} = \overline{fg}.$$

The calculation of $g^{(-1)}f^{(-1)}fg = \overline{g^{(-1)}f^{(-1)}}$ is similar.

(vi) The partial inverse of s is $\overline{rs}\,r$. First, note that $\overline{\overline{rs}\,r} = \overline{rs}\,\overline{r} = \overline{r}\,\overline{rs} = \overline{\overline{r}\,rs} = \overline{rs}$.

Then, it follows that $(\overline{rs}\,r)s = rs = \overline{rs} = \overline{\overline{rs}r}$ and $s(\overline{rs}\,r) = sr\overline{s} = \overline{s}$.

$\square$

3.6 Range categories

Corresponding to Definition 3.1.1 for restriction, which axiomatizes the concept of a domain of definition, we now introduce range categories [12, 13, 27] which algebraically axiomatize the concept of the range for a function.

Definition 3.6.1. A restriction category $\mathbb{X}$ is a *range category* when it has an operator on all maps

$$\frac{f : A \to B}{\hat{f} : B \to B}$$

where the operator satisfies the following:

[RR.1] $\overline{\hat{f}} = \hat{f}$	**[RR.2]** $f\hat{f} = f$
[RR.3] $\widehat{f\overline{g}} = \hat{f}\overline{g}$	**[RR.4]** $\widehat{\hat{f}g} = \widehat{fg}$

whenever the compositions are defined.

Lemma 3.6.2. *In a range category $\mathbb{X}$, the following hold:*

(i) $\hat{g}\hat{f} = \hat{f}\hat{g}$;

(ii) $\overline{f}\hat{g} = \hat{g}\overline{f}$;

(iii) $\widehat{f\hat{g}} = \hat{f}\hat{g}$;

(iv) $\hat{f} = 1$ *when* f *is epic, hence* $\hat{1} = 1$;

(v) $\hat{f}\hat{f} = \hat{f}$;

(vi) $\hat{\hat{f}} = \hat{f}$;

(vii) $\hat{\overline{f}} = \overline{f}$;

(viii) $\hat{g}\widehat{fg} = \widehat{fg}$;

(ix) $\widehat{\hat{f}\hat{g}} = \hat{f}\hat{g}$.

Proof. See, e.g., [27]. $\square$

Lemma 3.6.3. *In a range category:*

(i) $\widehat{hf} \leq \hat{f}$*;* *(ii)* $f' \leq f$ *implies* $\hat{f}' \leq \hat{f}$*.*

Proof.

(i) Noting that $\overline{\widehat{hf}}\hat{f} = \widehat{hf}\hat{f} = \widehat{hf\hat{f}} = \widehat{hf}$, we see $\widehat{hf} \leq \hat{f}$.

(ii) Calculating $\overline{\hat{f}'}\hat{f} = \hat{f}'\hat{f} = \widehat{\overline{f'}f}\hat{f} = \widehat{\overline{f'}f\hat{f}} = \widehat{\overline{f'}f} = \hat{f}'$, we see $\hat{f}' \leq \hat{f}$.

□

Note that unlike restrictions, a range is a *property* of a restriction category. To see this, assume we have two ranges $\widehat{(\_)}$ and $\widetilde{(\_)}$. Then,

$$\hat{f} = \widehat{f\tilde{f}} = \hat{f}\tilde{f} = \tilde{f}\hat{f} = \widetilde{f\hat{f}} = \tilde{f}.$$

Example 3.6.4. In Pinj, $\hat{f} = \{(y,y) | \exists x.(x,y) \in f\}$.

3.7 Split restriction categories

The Karoubi envelope of a restriction category, $\mathrm{K}_E(\mathbb{X})$ as defined in Definition 2.2.2 is a restriction category.

Note that for $f : (A,d) \to (B,e)$, by definition, in $\mathbb{X}$ we have $f = dfe$, giving

$$df = d(dfe) = ddfe = dfe = f \quad \text{and} \quad fe = (dfe)e = dfee = dfe = f.$$

When $\mathbb{X}$ is a restriction category, there is an immediate candidate for a restriction in $\mathrm{K}_E(\mathbb{X})$. If $f \in \mathrm{K}_E(\mathbb{X})$ is $e_1 f e_2$ in $\mathbb{X}$, then define $\overline{f}$ as given by $e_1\overline{f}$ in $\mathbb{X}$. Note that for $f : (A,d) \to (B,e)$, in $\mathbb{X}$ we have:

$$d\overline{f} = \overline{df}d = \overline{f}d.$$

Proposition 3.7.1. *If* $\mathbb{X}$ *is a restriction category and* E *is a set of idempotents, then the restriction as defined above makes* $K_E(\mathbb{X})$ *a restriction category.*

Proof. The restriction takes $f : (A, e_1) \to (B, e_2)$ to an endomorphism of (A, e_1). The restriction is in $\mathrm{K}_E(\mathbb{X})$ as

$$e_1(e_1\overline{f})e_1 = e_1\overline{f}e_1 = \overline{e_1 f}e_1e_1 = \overline{e_1 f}e_1 = e_1\overline{f}.$$

Checking the 4 restriction axioms:

[R.1] $[\![\overline{f}f]\!] = e_1\overline{f}f = e_1 f = [\![f]\!]$.

[R.2] $[\![\overline{g}\overline{f}]\!] = e_1\overline{g}e_1\overline{f} = e_1e_1\overline{g}\overline{f} = e_1e_1\overline{f}\overline{g} = e_1\overline{f}e_1\overline{g} = [\![\overline{f}\overline{g}]\!]$.

[R.3] $[\![\overline{\overline{f}g} \equiv e_1\overline{e_1\overline{f}g} = \overline{e_1e_1\overline{f}ge_1} = \overline{e_1\overline{f}ge_1} = e_1\overline{\overline{f}g} = e_1\overline{f}\overline{g} = e_1e_1\overline{f}\overline{g} = e_1\overline{f}e_1\overline{g} = [\![\overline{f}\,\overline{g}]\!]$.

[R.4] $[\![f\overline{g}]\!] = e_1fe_2\overline{g} = \overline{e_1fe_2g}e_1fe_2 = \overline{e_1e_1fe_2g}e_1e_1fe_2$

$$= e_1\overline{e_1fe_2g}e_1fe_2 = e_1\overline{fg}e_1fe_2 = [\![\overline{fg}f]\!].$$

$\square$

Given this, provided all identity maps are in E, $\mathrm{K}_E(\mathbb{X})$ is a restriction category with $\mathbb{X}$ as a full sub-restriction category, via the embedding defined by taking an object A in $\mathbb{X}$ to the object $(A, 1)$ in $\mathrm{K}_E(\mathbb{X})$.

Proposition 3.7.2. *In a restriction category $\mathbb{X}$, with meets, let R be the set of restriction idempotents. Then, $K(\mathbb{X}) \cong K_R(\mathbb{X})$. That is, splitting over all the idempotents is equivalent to splitting over just the restriction idempotents. Furthermore, $K_R(\mathbb{X})$ has meets.*

Proof. The proof first shows the equivalence of the two categories, then addresses the claim that $\mathrm{K}_R(\mathbb{X})$ has meets.

For equivalence, we require two functors,

$$U : \mathrm{K}_R(\mathbb{X}) \to \mathrm{K}(\mathbb{X}) \text{ and } V : \mathrm{K}(\mathbb{X}) \to \mathrm{K}_R(\mathbb{X}),$$

with:

$$UV \cong I_{\mathrm{K}_R(\mathbb{X})} \tag{3.1}$$

$$VU \cong I_{\mathrm{K}(\mathbb{X})}. \tag{3.2}$$

U is the standard inclusion functor. V will take the object (A, e) to $(A, e \cap 1)$ and the map $f : (A, e_1) \to (B, e_2)$ to $(e_1 \cap 1)f$.

V is a functor as:

Well Defined: If $f : (A, e_1) \to (B, e_2)$, then $(e_1 \cap 1)f$ is a map in $\mathbb{X}$ from A to B and $(e_1 \cap 1)(e_1 \cap 1)f(e_2 \cap 1) = (e_1 \cap 1)(fe_2 \cap f) = (e_1 \cap 1)(f \cap f) = (e_1 \cap 1)f$, therefore, $V(f) : V((A, e_1)) \to V((B, e_2))$.

Identities: $V(e) = (e \cap 1)e = e \cap 1$ by lemma 3.4.2.

Composition: $V(f)V(g) = (e_1 \cap 1)f(e_2 \cap 1)g = (e_1 \cap 1)fe_2(e_2 \cap 1)g = (e_1 \cap 1)f(e_2 \cap e_2)g = (e_1 \cap 1)fe_2g = (e_1 \cap 1)fg = V(fg)$.

Recalling from Lemma 3.4.2, $(e \cap 1)$ is a restriction idempotent. Using this fact, the commutativity of restriction idempotents and the general idempotent identities from 3.4.2, the composite functor UV is the identity on $\mathrm{K}_r(\mathbb{X})$ as when e is a restriction idempotent, $e = e(e \cap 1) = (e \cap 1)e = (e \cap 1)$.

For the other direction, note that for a particular idempotent $e : A \to A$, this gives the maps $e : (A, e) \to (A, e \cap 1)$ and $e \cap 1 : (A, e \cap 1) \to (A, e)$, again by 3.4.2. These maps give the natural isomorphism between I and VU as

$$\begin{tikzcd} (A,e) \arrow[r, "e"] \arrow[dr, "e"'] & (A, e\cap 1) \arrow[d, "e\cap 1"] \\ & (A,e) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} (A,e\cap 1) \arrow[r, "e\cap 1"] \arrow[dr, "e\cap 1"'] & (A, e) \arrow[d, "e"] \\ & (A,e\cap 1) \end{tikzcd}$$

both commute. Therefore, $UV = I$ and $VU \cong I$, giving an equivalence of the categories.

For the rest of this proof, functions in bold type, e.g., $\mathbf{f}$, are in $\mathrm{K}_R(\mathbb{X})$. Functions in normal slanted type, e.g., f are in $\mathbb{X}$.

To show that $\mathrm{K}_R(\mathbb{X})$ has meets, designate the meet in $\mathrm{K}_R(\mathbb{X})$ as $\cap_{\mathrm{K}}$ and define $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$ as the map given by the $\mathbb{X}$ map $f \cap g$, where $\mathbf{f}, \mathbf{g} : (A, d) \to (B, e)$ in $\mathrm{K}_R(\mathbb{X})$ and $f, g : A \to B$ in $\mathbb{X}$. This is a map in $\mathrm{K}_R(\mathbb{X})$ as $d(f \cap g)e = (df \cap dg)e = (f \cap g)e = (fe \cap g) = f \cap g$ where the penultimate equality is by 3.4.2. By definition $\overline{\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}}$ is $d\overline{f \cap g}$.

It is necessary to show $\cap_{\mathrm{K}}$ satisfies the four meet properties.

- $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g} \leq \mathbf{f}$: We need to show $\overline{\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}}\mathbf{f} = \mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$. Calculating now in $\mathbb{X}$:

  $$d\overline{f \cap g}f = \overline{d(f \cap g)}df = \overline{df \cap dg}df = \overline{f \cap g}f = f \cap g$$

  which is the definition of $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$.

- $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g} \leq \mathbf{g}$: Similarly and once again calculating in $\mathbb{X}$,

  $$d\overline{f \cap g}g = \overline{d(f \cap g)}dg = \overline{df \cap dg}dg = \overline{f \cap g}g = f \cap g$$

  which is the definition of $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$.

- $\mathbf{f} \cap_{\mathrm{K}} \mathbf{f} = \mathbf{f}$: From the definition, this is $f \cap f = f$ which is just $\mathbf{f}$.

- $\mathbf{h}(\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}) = \mathbf{hf} \cap_{\mathrm{K}} \mathbf{hg}$: From the definition, this is given in $\mathbb{X}$ by $h(f \cap g) = hf \cap hg$ which in $\mathrm{K}_R(\mathbb{X})$ is $\mathbf{hf} \cap_{\mathrm{K}} \mathbf{hg}$.

$\square$

3.8 Partial map categories

In [17], it is shown that split restriction categories are equivalent to *partial map categories*. The main definitions and results related to partial map categories are given below.

Definition 3.8.1. A collection $\mathcal{M}$ of monics is *a stable system of monics* when:

(i) it includes all isomorphisms;

(ii) it is closed under composition;

(iii) it is pullback stable.

Stable in this definition means that if $m : A \to B$ is in $\mathcal{M}$, then for arbitrary b with codomain B, the pullback

$$\begin{tikzcd}
A' \arrow[r, "a"] \arrow[d, "m'"'] & A \arrow[d, "m"] \\
B' \arrow[r, "b"'] & B
\end{tikzcd}$$

exists and $m' \in \mathcal{M}$. A category that has a stable system of monics is referred to as an $\mathcal{M}$-category.

Lemma 3.8.2. *If $nm \in \mathcal{M}$, a stable system of monics, and m is monic, then $n \in \mathcal{M}$.*

Proof. The commutative square

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "n"'] & A \arrow[d, "nm"] \\ A' \arrow[r, "m"'] & B \end{tikzcd}$$

is a pullback. □

Given a category $\mathbb{B}$ and a stable system of monics, the *partial map category*, $\mathrm{Par}(\mathbb{B}, \mathcal{M})$ is:

Objects: $A \in \mathbb{B}$;

Equivalence Classes of Maps: $(m, f) : A \to B$ with $m : A' \to A$ is in $\mathcal{M}$ and $f : A' \to B$ is a map in $\mathbb{B}$. i.e.,

$$\begin{tikzcd} & A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd};$$

Identity: $1_A, 1_A : A \to A$;

Composition: via a pullback, $(m, f)(m', g) = (m''m, f'g)$ where

$$\begin{tikzcd} & & A'' \arrow[dl, "m''"'] \arrow[dr, "f'"] & & \\ & A' \arrow[dl, "m"'] \arrow[dr, "f"'] & \text{(pb)} & B' \arrow[dl, "m'"] \arrow[dr, "g"] & \\ A & & B & & C \end{tikzcd};$$

Restriction: $\overline{(m, f)} = (m, m)$.

For the maps, $(m, f) \sim (m', f')$ when there is an isomorphism $\gamma : A'' \to A'$ such that $\gamma m' = m$ and $\gamma f' = f$.

In [18], it is shown that:

Theorem 3.8.3 (Cockett-Lack)**.** *Every restriction category is a full sub-category of a partial map category.*

3.9 Restriction products and Cartesian restriction categories

Restriction categories have analogues of products and terminal objects.

Definition 3.9.1. In a restriction category $\mathbb{X}$, a *restriction product* of two objects X, Y is an object $X \times Y$ equipped with *total* projections $\pi_0 : X \times Y \to X, \pi_1 : X \times Y \to Y$ where:

$\forall f : Z \to X, g : Z \to Y, \quad \exists$ a unique $\langle f, g\rangle : Z \to X \times Y$ such that

- $\langle f, g\rangle \pi_0 \leq f$,
- $\langle f, g\rangle \pi_1 \leq g$ and
- $\overline{\langle f, g\rangle} = \overline{f}\,\overline{g} (= \overline{g}\,\overline{f})$.

Definition 3.9.2. In a restriction category $\mathbb{X}$ a *restriction terminal object* is an object $\top$ such that for all objects X, there is a unique total map $!_X : X \to \top$ and the diagram

$$\begin{tikzcd} X \arrow[r, "\overline{f}"] \arrow[d, "f"] & X \arrow[r, "!_X"] & \top \\ Y \arrow[urr, "!_Y"'] & & \end{tikzcd}$$

commutes. That is, $f\,!_Y = \overline{f}\,!_X$. Note this implies that a restriction terminal object is unique up to a unique isomorphism.

Definition 3.9.3. A restriction category $\mathbb{X}$ is *Cartesian* if it has all restriction products and a restriction terminal object.

3.10 Discrete Cartesian restriction categories

Definition 3.10.1. An object A in a Cartesian restriction category is *discrete* when the diagonal map

$$\Delta : A \to A \times A$$

is a partial isomorphism. A Cartesian restriction category where all objects are discrete is called a *discrete* Cartesian restriction category.

Theorem 3.10.2. *A Cartesian restriction category $\mathbb{X}$ is discrete if and only if it has meets.*

Proof. If $\mathbb{X}$ has meets, then

$$\Delta(\pi_0 \cap \pi_1) = \Delta\pi_0 \cap \Delta\pi_1 = 1 \cap 1 = 1.$$

As $\langle \pi_0, \pi_1 \rangle$ is identity,

$$\begin{aligned}
\overline{\pi_0 \cap \pi_1} &= \overline{\pi_0 \cap \pi_1}\langle \pi_0, \pi_1 \rangle \\
&= \langle \overline{\pi_0 \cap \pi_1}\pi_0, \overline{\pi_0 \cap \pi_1}\pi_1 \rangle \\
&= \langle \pi_0 \cap \pi_1, \pi_0 \cap \pi_1 \rangle \\
&= (\pi_0 \cap \pi_1)\Delta
\end{aligned}$$

and therefore, $\pi_0 \cap \pi_1$ is $\Delta^{(-1)}$.

To show the other direction, we set $f \cap g = \langle f, g \rangle \Delta^{(-1)}$. By the definition of the restriction product:

$$f \cap g = \langle f, g \rangle \Delta^{(-1)} = \langle f, g \rangle \Delta^{(-1)} \Delta \pi_0 = \langle f, g \rangle \overline{\Delta^{(-1)}} \pi_0 \leq \langle f, g \rangle \pi_0 \leq f.$$

Then, substituting π_1 for π_0 above, this gives us $f \cap g \leq g$.

For the left distributive law,

$$h(f \cap g) = h\langle f, g \rangle \Delta^{(-1)} = \langle hf, hg \rangle \Delta^{(-1)} = hf \cap hg.$$

The intersection of a map with itself is

$$f \cap f = \langle f, f \rangle \Delta^{(-1)} = (f\Delta)\Delta^{(-1)} = f\overline{\Delta} = f$$

as Δ is total. This shows that $\cap$ as defined above is a meet for the Cartesian restriction category $\mathbb{X}$.

$\square$

Definition 3.10.3. In a Cartesian restriction category, a map $A \xrightarrow{f} B$ is called *graphic* when the maps

$$A \xrightarrow{\langle f,1\rangle} B \times A \qquad \text{and} \qquad A \xrightarrow{\langle \overline{f},1\rangle} A \times A$$

have partial inverses. A Cartesian restriction category is *graphic* when all of its maps are graphic.

Lemma 3.10.4. *In a Cartesian restriction category:*

(i) Graphic maps are closed under composition;

(ii) Graphic maps are closed under the restriction;

(iii) An object is discrete if and only if its identity map is graphic.

Proof.

(i) To show closure, it is necessary to show that $\langle fg, 1\rangle$ has a partial inverse. By Lemma 3.5.2, the uniqueness of the partial inverse gives

$$(\langle f, 1\rangle; \langle g, 1\rangle \times 1)^{(-1)} = \langle g, 1\rangle^{(-1)} \times 1; \langle f, 1\rangle^{(-1)}.$$

By the definition of the restriction product, we have $\overline{\langle fg, 1\rangle} = \overline{fg}$. Additionally, a straightforward calculation shows that $\overline{\langle f, 1\rangle; \langle g, 1\rangle \times 1} = \overline{\langle f\langle g, 1\rangle, 1\rangle} = \overline{f; \langle g, 1\rangle} = \overline{\langle f; g, f\rangle} = \overline{fg}\,\overline{f} = \overline{fg}$ where the last equality is by [**R.2**], [**R.3**] and finally [**R.1**].

Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "\langle f,1\rangle"] \arrow[drr, "\langle fg,1\rangle"'] & B \times A \arrow[r, "\langle g,1\rangle\times 1"] & C \times B \times A \\
& & C \times A. \arrow[u, "1\times\langle f,1\rangle"']
\end{tikzcd}$$

Thus,

$$
\begin{aligned}
\langle fg,1\rangle(1\times\langle f,1\rangle)(\langle g,1\rangle^{(-1)}\times 1)\langle f,1\rangle^{(-1)} & \\
&= \langle f,1\rangle(\langle g,1\rangle\times 1)(\langle g,1\rangle^{(-1)}\times 1)\langle f,1\rangle^{(-1)} \\
&= \langle f,1\rangle(\overline{g\times 1})\langle f,1\rangle^{(-1)} \\
&= \overline{\langle f,1\rangle(g\times 1)}\langle f,1\rangle\langle f,1\rangle^{(-1)} \\
&= \overline{\langle f,1\rangle(g\times 1)}\,\overline{\langle f,1\rangle} \\
&= \overline{\langle f,1\rangle}\,\overline{\langle f,1\rangle(g\times 1)} \\
&= \overline{\langle f,1\rangle(g\times 1)} \\
&= \overline{\langle fg,1\rangle}(=\overline{fg})
\end{aligned}
$$

showing that $1\times\langle f,1\rangle(\langle g,1\rangle^{(-1)}\times 1)\langle f,1\rangle^{(-1)}$ is a right inverse for $\langle fg,1\rangle$. For the other direction, note that in general $(hk)^{(-1)} = k^{(-1)}h^{(-1)}$ and that we have $\langle fg,1\rangle = \langle f,1\rangle(\langle g,1\rangle\times 1)(1\times\langle f,1\rangle^{(-1)})$, thus $(1\times\langle f,1\rangle)(\langle g,1\rangle^{(-1)}\times 1)\langle f,1\rangle^{(-1)}$ will also be a left inverse and $\langle fg,1\rangle$ is a restriction isomorphism.

(ii) This follows from the definition of graphic and that $\overline{\langle f,1\rangle} = \overline{f} = \overline{\overline{f}}$.

(iii) Given a discrete object A, the identity on A is graphic as $\langle 1,1\rangle = \Delta$ and therefore $\langle 1,1\rangle^{(-1)} = \Delta^{(-1)}$. Conversely, if $\langle 1,1\rangle = \Delta$ has an inverse, A is discrete by definition.

$\square$

Lemma 3.10.5. *A discrete restriction category is precisely a graphic Cartesian restriction category.*

Proof. The requirement is that $\langle f,1\rangle$ (and $\langle\overline{f},1\rangle$) each have partial inverses. For $\langle f,1\rangle$, the inverse is $\overline{(1\times f)\Delta^{(-1)}}\pi_1$.

To show this, calculate the two compositions. First,

$$\langle f,1\rangle\overline{1\times f\Delta^{(-1)}}\pi_1 = \overline{\langle f,f\rangle\Delta^{(-1)}}\langle f,1\rangle\pi_1 = \overline{f\Delta\Delta^{(-1)}}\langle f,1\rangle\pi_1 = \overline{f}\langle f,1\rangle\pi_1 = \overline{f}.$$

The other direction is:

$$\begin{aligned}
\overline{(1\times f)\Delta^{(-1)}}\pi_1\langle f,1\rangle &= \langle\overline{(1\times f)\Delta^{(-1)}}\pi_1 f, \overline{(1\times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle\overline{(1\times f)\Delta^{(-1)}}(1\times f)\pi_1, \overline{(1\times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle(1\times f)\overline{\Delta^{(-1)}}\pi_1, \overline{(1\times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle(1\times f)\overline{\Delta^{(-1)}}\pi_0, \overline{(1\times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle\overline{(1\times f)\Delta^{(-1)}}(1\times f)\pi_0, \overline{(1\times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle\overline{(1\times f)\Delta^{(-1)}}\pi_0, \overline{(1\times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \overline{(1\times f)\Delta^{(-1)}}\langle\pi_0,\pi_1\rangle \\
&= \overline{(1\times f)\Delta^{(-1)}}
\end{aligned}$$

The above follows in a discrete restriction category, as we have

$$\overline{\Delta^{(-1)}}\pi_1 = \Delta^{(-1)}\Delta\pi_1 = \Delta^{(-1)} = \Delta^{(-1)}\Delta\pi_0 = \overline{\Delta^{(-1)}}\pi_0.$$

For $\langle\overline{f},1\rangle$, the inverse is $\overline{(1\times\overline{f})\Delta^{(-1)}}\pi_1$. Similarly to above,

$$\langle\overline{f},1\rangle\overline{1\times\overline{f}\Delta^{(-1)}}\pi_1 = \overline{\langle\overline{f},\overline{f}\rangle\Delta^{(-1)}}\langle\overline{f},1\rangle\pi_1 = \overline{\overline{f}\Delta\Delta^{(-1)}}\langle\overline{f},1\rangle\pi_1 = \overline{\overline{f}}\langle\overline{f},1\rangle\pi_1 = \overline{f}.$$

The other direction follows the same pattern as for $\langle f,1\rangle$. $\square$

To conclude this section, we give a few examples of Cartesian restriction categories, of both the discrete and non-discrete variety.

Example 3.10.6 (Par is discrete)**.** In Par,

$$\Delta : x \mapsto (x,x) \text{ and } \Delta^{(-1)} : (x,y) \mapsto \begin{cases} x & x = y, \\ \uparrow & x \neq y. \end{cases}$$

Example 3.10.7 (Terminal object is discrete)**.** In any Cartesian restriction category, the terminal object, 1, is discrete as $1 \times 1 \cong 1$.

Example 3.10.8 (Semi-lattice is discrete)**.** As the product is the meet of the semi-lattice and $A \wedge A = A$, we have $\Delta = 1$ and therefore is always invertible. Note that a total Cartesian restriction category must equal a semi-lattice. Also, we see that any Cartesian restriction category which is a restriction preorder will also be discrete.

However, there are also numerous examples of non-discrete Cartesian restriction categories.

Example 3.10.9 (Non-discrete Cartesian restriction categories)**.** We list a few categories which are not discrete:

(i) $\textsc{Top}_p$, the partial maps on open sets, is not discrete.

(ii) Any total non-trivial (i.e., not a semi-lattice) Cartesian category is not discrete.

(iii) $\mathrm{Par}(\mathbb{X}, \mathcal{M})$ is not discrete unless $\Delta : X \to X \times X$ is in $\mathcal{M}$.

(iv) $StabMeetSemiLat^{op}$ is not discrete.

Chapter 4

Inverse categories and products

This chapter will introduce inverse categories. We first give a few results about inverse categories and then proceed to show an inverse category which has restriction products is a restriction pre-order.

Given this, the chapter then turns to focus on adding "products" to an inverse category, by which we mean a structure that behaves in a product-like manner in the inverse category, which we call *inverse products*. These will be defined below in Sub-Section 4.3.1: Inverse products are given by a natural structure on a tensor product which includes a diagonal but lacks projections. The diagonal map is required to give a natural Frobenius structure on each object.

4.1 Inverse categories

Definition 4.1.1. A restriction category in which every map is a partial isomorphism is called an *inverse category*.

Lemma 4.1.2. *In an inverse category, all idempotents are restriction idempotents.*

Proof. Given an idempotent e,

$$\overline{e} = ee^{(-1)} = eee^{(-1)} = e\overline{e} = \overline{e}\overline{e}e = \overline{e}e = e.$$

□

Lemma 4.1.3. *An inverse category $\mathbb{X}$ is a range category, where $\hat{f} = f^{(-1)}f = \overline{f^{(-1)}}$.*

Proof.

[RR.1] $\overline{\hat{f}} = \overline{\overline{f^{(-1)}}} = \overline{f^{(-1)}} = \hat{f}$;

[RR.2] $f\hat{f} = f\overline{f^{(-1)}} = ff^{(-1)}f = \overline{f}f = f$;

[RR.3] $\widehat{f\overline{g}} = \overline{(f\overline{g})^{(-1)}} = \overline{\overline{g}^{(-1)}f^{(-1)}} = \overline{\overline{g}f^{(-1)}} = \overline{g}\,\overline{f^{(-1)}} = \overline{f^{(-1)}}\overline{g} = \hat{f}\overline{g}$;

[RR.4] $\widehat{\hat{f}g} = \overline{\left(\overline{f^{(-1)}}g\right)^{(-1)}} = \overline{g^{(-1)}\overline{f^{(-1)}}^{(-1)}} = \overline{g^{(-1)}\overline{\overline{f^{(-1)}}}} = \overline{g^{(-1)}f^{(-1)}} = \overline{(fg)^{(-1)}} = \widehat{fg}$

□

The property of being an inverse category is preserved by splitting.

Lemma 4.1.4. *When $\mathbb{X}$ is an inverse category, $K_E(X)$ is an inverse category.*

Proof. The inverse of $f : (A, e_1) \to (B, e_2)$ in $\mathrm{K}_E(\mathbb{X})$ is $e_2f^{(-1)}e_1$ as

$$[\![ff^{(-1)}]\!] = e_1fe_2e_2f^{(-1)}e_1 = e_1e_1fe_2f^{(-1)}e_1 = e_1ff^{(-1)}e_1 = e_1e_1\overline{f}e_1 = e_1\overline{f} = [\![\overline{f}]\!]$$

and

$$\begin{aligned}[\![f^{(-1)}f]\!] = e_2f^{(-1)}e_1e_1fe_2 = e_2f^{(-1)}e_1fe_2e_2 = e_2f^{(-1)}fe_2 \\ = e_2e_2\overline{f^{(-1)}}e_2 = e_2\overline{f^{(-1)}} = [\![\overline{f^{(-1)}}]\!].\end{aligned}$$

□

Example 4.1.5 (Pinj is an inverse category)**.** For any map f, $f^{(-1)} = \{(y, x)|(x, y) \in f\}$. Note that $f^{(-1)}$ is a map in Pinj due to the two dual conditions on maps as given in Example 2.5.5.

Example 4.1.6 (Par is not an inverse category)**.** Par, while it is a restriction category, is not an inverse category. For example, let $A = \{1, 2\}$, $B = \{1\}$ and $f = \{(1, 1), (2, 1)\}$ in Par. The restriction of f is $\overline{f} = \{(1, 1), (2, 2)\} = 1_A$. There is no partial function $g : B \to A$ such that $fg = 1_A$.

Example 4.1.7. Generally, let $\mathbb{R}$ be a restriction category, and $\mathbb{X}$ the sub-category of $\mathbb{R}$ having the same objects as $\mathbb{R}$ and only the partial isomorphisms as maps. Then, $\mathbb{X}$ is an inverse category.

Example 4.1.8. A groupoid, which is a category in which every map is an isomorphism, is an inverse category. Note that all maps in the groupoid are total, hence the partial isomorphisms are all isomorphisms.

Example 4.1.9. Given a category $\mathbb{P}$, create a partial map category as in Section 3.8, where the stable system of monics, $\mathcal{M}$, contains of all isomorphisms in $\mathbb{P}$. Then the partial isomorphisms are the maps of the form $\begin{tikzcd} & A' \arrow[dl, "m"'] \arrow[dr, "m'"] & \\ A & & B \end{tikzcd}$, where $m' \in \mathcal{M}$. Its inverse is $\begin{tikzcd} & A' \arrow[dl, "m'"'] \arrow[dr, "m"] & \\ B & & A \end{tikzcd}$. The composition is $\begin{tikzcd} & A \arrow[dl, "m^{-1}m"'] \arrow[dr, "m^{-1}m"] & \\ A & & A \end{tikzcd}$ which is the identity when m is an isomorphism. The partial isomorphisms will have either m or m' be in $\mathcal{M}$, but not be an isomorphism. Taking just the partial isomorphisms gives us an inverse category.

4.2 Inverse categories with restriction products

We start by showing that an inverse category with restriction products is a restriction pre-order and thus, is a very restrictive notion.

Proposition 4.2.1. *Given an inverse category $\mathbb{X}$, if it has restriction products, it is a restriction pre-order as in Definition 3.3.1. That is,*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd} \implies f \smile g.$$

Proof. Notice that $\pi_1{}^{(-1)} = \Delta\pi_1\pi_1{}^{(-1)} = \Delta\overline{\pi_1} = \Delta$. This gives $\overline{\pi_1{}^{(-1)}} = 1$ and therefore π_1 (and similarly, π_0) is an isomorphism.

Starting with the product map $\langle f, g\rangle$,

$$\cfrac{\cfrac{\cfrac{\cfrac{\langle f, g\rangle = \langle f, g\rangle}{\langle f, g\rangle\pi_1\pi_1{}^{(-1)} = \langle f, g\rangle\pi_0\pi_0{}^{(-1)}}}{\overline{f}g\pi_1{}^{(-1)} = \overline{g}f\pi_0{}^{(-1)}}}{\overline{f}g\Delta = \overline{g}f\Delta}}{\overline{f}g = \overline{g}f}$$

which shows that f and g are compatible. $\square$

Corollary 4.2.2. $\mathbb{X}$ *is a inverse category with restriction products if and only if Total($K_r(\mathbb{X})$) is a meet pre-order.*

Proof. Total($\mathbb{X}$), the subcategory of total maps on $\mathbb{X}$, has products and therefore every pair of parallel maps is compatible. As total compatible maps are equal, there is at most one map between any two objects. Hence, Total($\mathbb{X}$) is a pre-order with the meet being the product.

Similarly, from [17] and [19], Total($\mathrm{K}_r(\mathbb{X})$) is an inverse category and has products and is therefore also a meet pre-order. This shows the "only if" side of the corollary.

For the other direction, if Total($\mathrm{K}_r(\mathbb{X})$) is a meet pre-order, define the product as the meet of the maps and the terminal object as the supremum of all maps. □

Corollary 4.2.3. *Every inverse category with restriction products is a full subcategory of a partial map category of a meet semi-lattice.*

4.3 Inverse products

4.3.1 Inverse product definition

Definition 4.3.1. An *inverse product* on an inverse category $\mathbb{X}$ is given by a symmetric tensor product, based on a restriction bi-functor, $\_ \otimes \_ : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$. The tensor makes $\mathbb{X}$ a symmetric monoidal category and there is a natural "semi-Frobenius" diagonal map, Δ which is canonical. If an inverse category has inverse products, it is called a *discrete inverse category*.

The diagonal map $\Delta_A : A \to A \otimes A$ must be total and create a co-semigroup. It must satisfy the following diagrams:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & A \otimes A \arrow[d, "c_\otimes"] \\ & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "\Delta^{(-1)}"] \arrow[d, "c_\otimes"'] & A \\ A \otimes A \arrow[ur, "\Delta^{(-1)}"'] & \end{tikzcd}$$

Cocommutative and Commutative;

$$
\begin{tikzcd}
A \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] & & A \otimes A \arrow[d, "1\otimes\Delta"] \\
A \otimes A \arrow[dr, "\Delta\otimes 1"'] & & A \otimes (A \otimes A) \\
& (A \otimes A) \otimes A \arrow[ur, "a_{\otimes}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes (A \otimes A) \arrow[rr, "1\otimes\Delta^{(-1)}"] \arrow[d, "{a_{\otimes}}^{(-1)}"'] & & A \otimes A \arrow[d, "\Delta^{(-1)}"] \\
(A \otimes A) \otimes A \arrow[dr, "\Delta^{(-1)}\otimes 1"'] & & A \\
& A \otimes A \arrow[ur, "\Delta^{(-1)}"'] &
\end{tikzcd}
$$

Coassociative and Associative;

$$
\begin{tikzcd}
A \otimes A \arrow[rr, "(\Delta\otimes 1)a_{\otimes}"] \arrow[dr, "\Delta^{(-1)}"] \arrow[dd, "(1\otimes\Delta){a_{\otimes}}^{(-1)}"'] & & A \otimes (A \otimes A) \arrow[dd, "1\otimes\Delta^{(-1)}"] \\
& A \arrow[dr, "\Delta"] & \\
(A \otimes A) \otimes A \arrow[rr, "\Delta^{(-1)}\otimes 1"'] & & A \otimes A
\end{tikzcd}
$$

Δ is Frobenius.

If we define the map:

$ex_{\otimes} = a_{\otimes}(1 \otimes {a_{\otimes}}^{(-1)})(1 \otimes (c_{\otimes} \otimes 1))(1 \otimes a_{\otimes}){a_{\otimes}}^{(-1)}) : (A \otimes B) \otimes (C \otimes D) \to (A \otimes C) \otimes (B \otimes D)$

then we have:

$$
\begin{tikzcd}
A \otimes B \arrow[rr, "\Delta\otimes\Delta"] \arrow[dr, "\Delta"'] & & (A \otimes A) \otimes (B \otimes B) \arrow[dl, "ex_{\otimes}"] \\
& (A \otimes B) \otimes (A \otimes B) &
\end{tikzcd}
$$

Δ is canonical.

Thus, Δ is a cocommutative, coassociative map which together with $\Delta^{(-1)}$ forms a special semi-Frobenius algebra. We use the prefix "semi-" as we are not requiring the unit laws of a Frobenius algebra as will be discussed later in Definition 8.1.10.

Note also, cocommutativity implies commutativity, i.e., that $c_\otimes \Delta^{(-1)} = \Delta^{(-1)}$. One can see this as:

$$\Delta(c_\otimes \Delta^{(-1)}) = (\Delta c_\otimes)\Delta^{(-1)} = \Delta\Delta^{(-1)} = \overline{\Delta} \text{ and}$$
$$(c_\otimes \Delta^{(-1)})\Delta = (c_\otimes \Delta^{(-1)})(\Delta c_\otimes) = \overline{c_\otimes \Delta^{(-1)}}.$$

This means that both $\Delta^{(-1)}$ and $c_\otimes \Delta^{(-1)}$ are partial inverses for Δ and are therefore equal.

Similarly, coassociativity implies associativity as one can show that $(1 \otimes \Delta^{(-1)})\Delta^{(-1)} = {a_\otimes}^{(-1)}(\Delta^{(-1)} \otimes 1)\Delta^{(-1)}$ as

$$\begin{aligned}\Delta(1 \otimes \Delta){a_\otimes}^{(-1)}(\Delta^{(-1)} \otimes 1)\Delta^{(-1)} &= \Delta(\Delta \otimes 1)a_\otimes {a_\otimes}^{(-1)}(\Delta^{(-1)} \otimes 1)\Delta^{(-1)} \\ &= \Delta(\Delta \otimes 1)(\Delta^{(-1)} \otimes 1)\Delta^{(-1)} \\ &= \Delta 1 \Delta^{(-1)} = 1.\end{aligned}$$

Example 4.3.2 (Pinj is a discrete inverse category)**.** In the inverse category Pinj (see Example 4.1.5), suppose we add the tensor given by the Cartesian product of sets. In detail, this means:

$$A \otimes B = \{(a, b) | a \in A, b \in B\}$$
$$f \otimes g = \{((a, c), (b, d)) | (a, b) \in f, (c, d) \in g\}$$
$$1 = \{*\}, \text{ a single element set.}$$

The symmetric monoid isomorphisms are:

$$u^l_\otimes : \{(*, a)\} \mapsto \{a\} \qquad u^r_\otimes : \{(a, *)\} \mapsto \{a\}$$
$$a_\otimes : \{((a, b), c)\} \mapsto \{(a, (b, c))\} \qquad c_\otimes : \{(a, b)\} \mapsto \{(b, a)\}$$

Define $\Delta_A = \{(a, (a, a)) | a \in A\}$. Then Pinj is a discrete inverse category with the inverse product of $\otimes$. The required properties of cocommutativity, coassociativity and exchange are immediately obvious. To show the Frobenius rule for Δ, first note that $\Delta^{(-1)}$ is defined only on the elements of $A \otimes A$ which agree in the first and second co-ordinate. We show the upper triangle of the Frobenius diagram in detail. Equation (4.1) shows the result of applying $\Delta^{(-1)}$ followed by Δ.

$$\Delta(\Delta^{(-1)}(A \otimes A)) = \Delta(\{a|(a,a) \in A \otimes A\}) = \{(a,a)|(a,a) \in A \otimes A\}. \tag{4.1}$$

Applying $(\Delta \otimes 1)a_{\otimes}$ to $A \otimes A$ is shown in Equation (4.2).

$$a_{\otimes}(\Delta \otimes 1(A \otimes A)) = a_{\otimes}(\{((a,a),a')|(a,a') \in A \otimes A\} = \{(a,(a,a'))|(a,a') \in A \otimes A\}. \tag{4.2}$$

Finally, applying $1 \otimes \Delta^{(-1)}$ to the result of Equation (4.2) gives us Equation (4.3).

$$(1 \otimes \Delta^{(-1)})(\{(a,(a,a'))|(a,a') \in A \otimes A\} = \{(a,a)|(a,a) \in A \otimes A\}. \tag{4.3}$$

Thus, we have $\Delta^{(-1)}\Delta = (\Delta \otimes 1)a_{\otimes}(1 \otimes \Delta^{(-1)})$ and the Frobenius condition is satisfied.

Example 4.3.3 (Top$_p$ does not give a discrete inverse category)**.** Recalling Top$_p$ from Example 3.1.9, we know that the partial isomorphisms of Top$_p$ form an inverse category — **INV**(Top$_p$). Additionally, Top$_p$ has a product, given by the standard Cartesian product. This product does work as a tensor in **INV**(Top$_p$), but Δ is not a map in **INV**(Top) and hence it is not a discrete inverse category.

Inverse products are extra structure on an inverse category, rather than a property. An example to demonstrate this is given next.

Example 4.3.4 (Inverse products are additional structure)**.**

Any discrete category (i.e., a category with only the identity arrows) is a trivial inverse category. To create an inverse product on a discrete category, add a commutative, associative, idempotent multiplication, with a unit.

Let $\mathbb{D}$ be the discrete category with four objects a, b, c and d. Then, define two different inverse product tensors, $\otimes$ and $\odot$, with d the unit of each as shown in Table 4.1.

As $\mathbb{D}$ is discrete, Δ is forced to be the identity. One can check easily that each of the conditions for being an inverse product are satisfied by $\otimes$ and by $\odot$ with the trivial diagonal.

$\otimes$	a	b	c	d
a	a	a	a	a
b	a	b	**b**	b
c	a	**b**	c	c
d	a	b	c	d

$\odot$	a	b	c	d
a	a	a	a	a
b	a	b	**a**	b
c	a	**a**	c	c
d	a	b	c	d

Table 4.1: Two different inverse products on the same category.

4.3.2 Diagrammatic Language

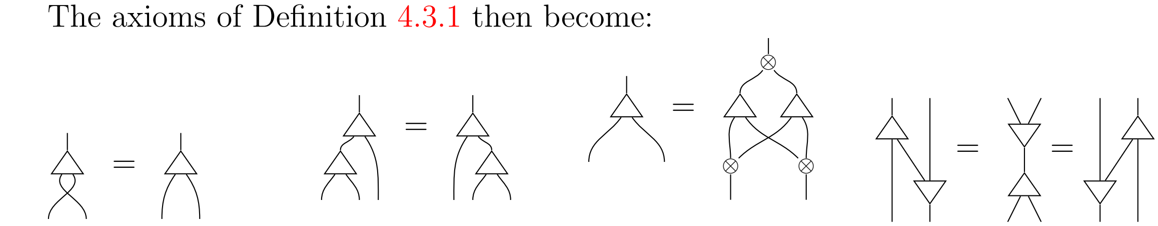
While it is certainly possible to prove results about inverse products using direct algebraic manipulation, it is much more understandable to use circuits or string diagrams. See [51] for a comparison of various graphical languages for monoidal categories. As shown in [32], diagrammatic reasoning is equivalent to reasoning algebraically for symmetric monoidal categories.

In the diagrams, we will use the following representations:

- Δ will be represented by an upward pointing triangle: $\triangle$.
- $\Delta^{(-1)}$ by a downward triangle: $\bigtriangledown$.
- Maps by a rectangle with the name of the map inside: $\boxed{f}$.
- Use of the tensor: $\otimes$.
- Unit introduction (often referred to as an η map): $\circ$.
- Unit removal (often referred to as an ϵ map): $\bullet$.

String diagrams in this thesis are to be read from top to bottom.

The axioms of Definition 4.3.1 then become:



Cocommutativity   Coassociativity   Exchange   Frobenius

The diagram for commutativity is obtained by flipping the diagram of cocommutativity vertically. Similarly, the diagram for associativity is obtained by flipping the diagram for coassociativity vertically.

4.3.3 Properties of discrete inverse categories

We now present some properties of discrete inverse categories.

Lemma 4.3.5. *In a discrete inverse category $\mathbb{X}$ with the inverse product $\otimes$ and Δ, where $e = \overline{e}$ is a restriction idempotent and f, g, h are arrows in $\mathbb{X}$, the following are true:*

(i) $e = \Delta(e \otimes 1)\Delta^{(-1)}$.

(ii) $e\Delta(f \otimes g) = \Delta(ef \otimes g)$ *(and* $= \Delta(f \otimes eg)$ *and* $= \Delta(ef \otimes eg)$.*)*

(iii) $(f \otimes ge)\Delta^{(-1)} = (f \otimes g)\Delta^{(-1)}e$ *(and* $= (fe \otimes g)\Delta^{(-1)}$ *and* $= (fe \otimes ge)\Delta^{(-1)}$.*)*

(iv) $\overline{\Delta(f \otimes g)\Delta^{(-1)}} = \Delta(1 \otimes gf^{(-1)})\Delta^{(-1)}$ *when* $f, g : A \to A$.

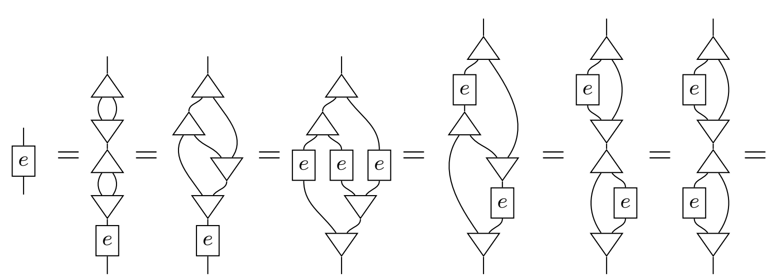
(v) If $\Delta(h \otimes g)\Delta^{(-1)} = \overline{\Delta(h \otimes g)\Delta^{(-1)}}$ *then* $(\Delta(h \otimes g)\Delta^{(-1)})h = \Delta(h \otimes g)\Delta^{(-1)}$.

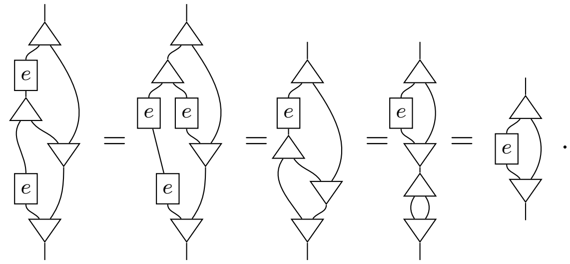
(vi) $\Delta(f \otimes 1) = \Delta(g \otimes 1) \implies f = g$.

(vii) $(f \otimes 1) = (g \otimes 1) \implies f = g$.

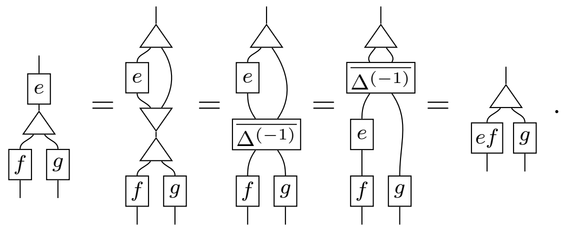
Proof.

(i)



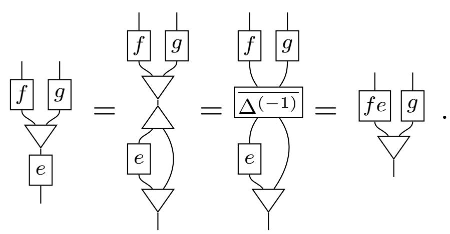


(ii) This equality uses the previous equality, the commutativity of restriction idempotents ([**R.2**]) and the identity $\Delta\overline{\Delta^{(-1)}} = \Delta$.



The second equality $(e\Delta(f \otimes g) = \Delta(f \otimes eg))$ follows by cocommutativity. The third equality, $(e\Delta(f \otimes g) = \Delta(ef \otimes eg))$ follows by naturality of Δ.

(iii) As in (ii), details are only given for the first equality. This proof is obtained by reversing the diagrams of (ii).

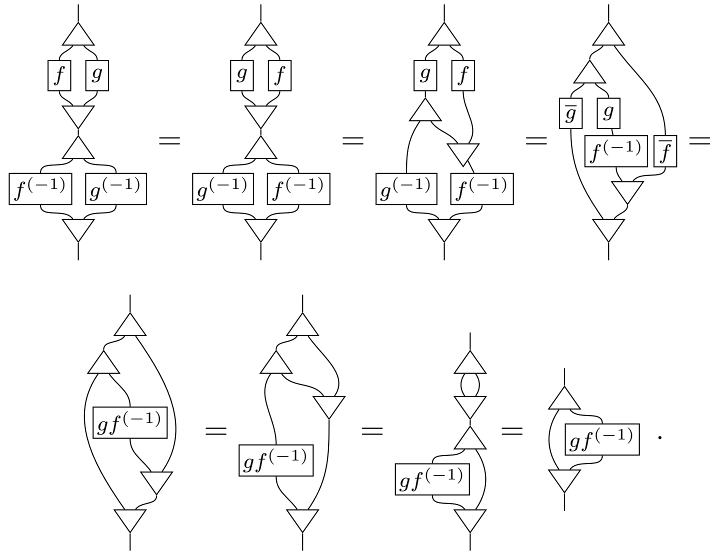


The other equalities follow for the same reasons as in (ii).

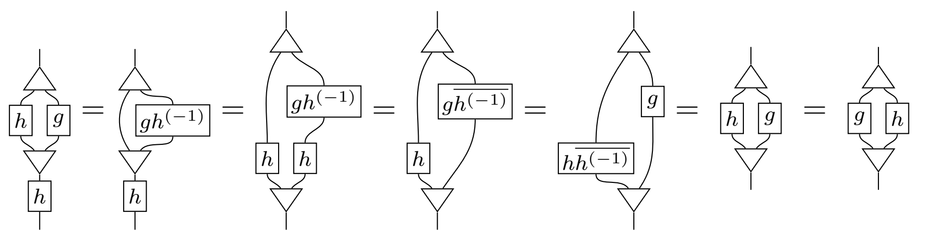
(iv) Here, we start by using the fact all maps have a partial inverse, therefore we have:

$$\overline{\Delta(f \otimes g)\Delta^{(-1)}} = \Delta(f \otimes g)\Delta^{(-1)}\Delta(f^{(-1)} \otimes g^{(-1)})\Delta^{(-1)}.$$

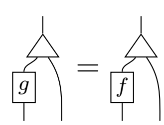
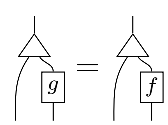
Now, we proceed with showing the rest of the equality via diagrams.



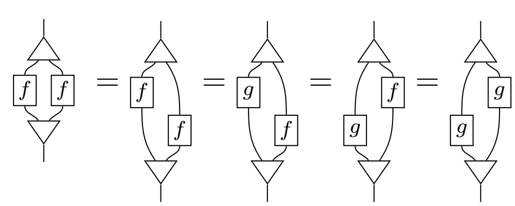
(v) Beginning with the assumption that $\Delta(h \otimes g)\Delta^{(-1)}$ equals its restriction and by item (iv), we have:



(vi) Our assumption is that:

 and by cocommutativity, .

Hence,

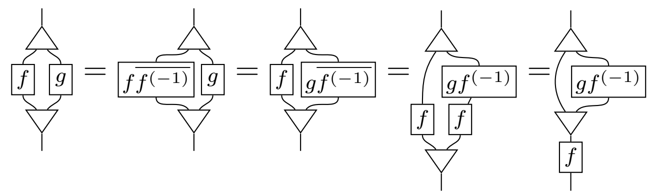
$f =$  $= g.$

(vii) Use the same diagrammatic argument as in item (vi).

$\square$

Proposition 4.3.6. *A discrete inverse category has meets, where $f \cap g = \Delta(f \otimes g)\Delta^{(-1)}$.*

Proof. $f \cap g \le f$:

$f \cap g =$  $= \overline{f \cap g} f.$

$f \cap f = f$:

$$f \cap f = \Delta(f \otimes f)\Delta^{(-1)} = f\Delta\Delta^{(-1)} = f.$$

$h(f \cap g) = hf \cap hg$:

$$\begin{aligned} h(f \cap g) &= h\Delta(f \otimes g)\Delta^{(-1)} && \text{Definition of } \cap \\ &= \Delta(h \otimes h)(f \otimes g)\Delta^{(-1)} && \Delta \text{ natural} \\ &= \Delta(hf \otimes hg)\Delta^{(-1)} && \text{compose maps} \\ &= hf \cap hg && \text{Definition of } \cap . \end{aligned}$$

□

4.3.4 The inverse subcategory of a discrete restriction category

Given a discrete restriction category, one can pick out the maps which are partial isomorphisms. Using results from Sub-Section 4.3.3 and from Section 3.10, we will show that these maps form a subcategory which is a discrete inverse category.

Proposition 4.3.7. *Given $\mathbb{X}$ is a discrete restriction category, the partial isomorphisms of $\mathbb{X}$, together with the objects of $\mathbb{X}$ form a sub-restriction category which is a discrete inverse category. For the restriction category $\mathbb{X}$, we denote this sub-category by* $\mathbf{INV}(\mathbb{X})$.

Proof. As shown in Lemma 3.5.2, partial isomorphisms are closed under composition. The identity maps are in $\mathbf{INV}(\mathbb{X})$ and restrictions of partial isomorphisms are also partial isomorphisms.

The product on the discrete restriction category $\mathbb{X}$ becomes the tensor product of the restriction category $\mathbf{INV}(\mathbb{X})$. Table 4.2 shows how each of the elements of the tensor are defined. Note that the last definition makes explicit use of the fact we are in a discrete restriction category and hence the Δ of $\mathbb{X}$ possesses a partial inverse.

$\mathbb{X}$	$\mathbf{INV}(\mathbb{X})$	Inverse map
$A\times B$	$A\otimes B$	
$\top$	1	
$\pi_1{:}\top\times A\to A$	$u^l_\otimes{:}1\otimes A\to A$	$\langle !,1\rangle$
$\pi_0{:}A\times\top\to A$	$u^r_\otimes{:}A\otimes 1\to A$	$\langle 1,!\rangle$
$a_\mathbb{X}{=}\langle\pi_0\pi_0,\langle\pi_0\pi_1,\pi_1\rangle\rangle{:}(A\times B)\times C\to A\times(B\times C)$	$a_\otimes{:}(A\otimes B)\otimes C\to A\otimes(B\otimes C)$	$\langle\langle\pi_0,\pi_1\pi_0\rangle,\pi_1\pi_1\rangle$
$c_\mathbb{X}{=}\langle\pi_1,\pi_0\rangle{:}A\times B\to B\times A$	$c_\otimes{:}A\otimes B\to B\otimes A$	$\langle\pi_1,\pi_0\rangle$
$\Delta_\mathbb{X}{:}A\to A\times A$	$\Delta{:}A\to A\otimes A$	${\Delta_\mathbb{X}}^{(-1)}$

Table 4.2: Structural maps for the tensor in $\mathbf{INV}(\mathbb{X})$

The monoid coherence diagrams follow directly from the characteristics of the product in $\mathbb{X}$. Similarly, Δ is total as it is total in $\mathbb{X}$. It remains to show cocommutativity, coassociativity and the Frobenius condition.

Cocommutativity requires $\Delta c_\otimes = c_\otimes$. We have

$$\Delta_\mathbb{X}\langle\pi_1,\pi_0\rangle = \langle\Delta_\mathbb{X}\pi_1,\Delta_\mathbb{X}\pi_0\rangle = \langle 1,1\rangle = \Delta_\mathbb{X},$$

giving us the required cocommutativity.

Coassociativity requires $\Delta(1\otimes\Delta) = \Delta(\Delta\otimes 1)a_\otimes$. Expressing this in $\mathbb{X}$, it is the requirement that

$$\Delta_\mathbb{X}(1\times\Delta_\mathbb{X}) = \Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)a_\mathbb{X}.$$

Recalling that $f\times g\pi_0 = \pi_0 f$ and $f\times g\pi_1 = \pi_1 g$, we have:

$$\begin{aligned}
\Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)a_\mathbb{X} &= \Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)\langle\pi_0\pi_0,\langle\pi_0\pi_1,\pi_1\rangle\rangle\\
&= \langle\Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)\pi_0\pi_0,\langle\Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)\pi_0\pi_1,\Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)\pi_1\rangle\rangle\\
&= \langle\Delta_\mathbb{X}\pi_0\Delta_\mathbb{X}\pi_0,\langle\Delta_\mathbb{X}\pi_0\Delta_\mathbb{X}\pi_1,\Delta_\mathbb{X}\pi_1 1\rangle\rangle\\
&= \langle 1,\langle 1,1\rangle\rangle = \Delta_\mathbb{X}(1\times\Delta_\mathbb{X})
\end{aligned}$$

and shows that we have coassociativity.

The semi-Frobenius requirement is two-fold:

$$\Delta^{(-1)}\Delta = (\Delta \otimes 1)a_{\otimes}(1 \otimes \Delta^{(-1)}), \tag{4.4}$$

$$\Delta^{(-1)}\Delta = (1 \otimes \Delta)a_{\otimes}{}^{(-1)}(\Delta^{(-1)} \otimes 1). \tag{4.5}$$

In $\mathbb{X}$, these become:

$$\Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}} = (\Delta_{\mathbb{X}} \times 1)\langle \pi_0\pi_0, \langle \pi_0\pi_1, \pi_1\rangle\rangle(1 \times \Delta_{\mathbb{X}}{}^{(-1)}) \tag{4.6}$$

$$\Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}} = (1 \times \Delta_{\mathbb{X}})\langle\langle \pi_0, \pi_1\pi_0\rangle, \pi_1\pi_1\rangle(\Delta_{\mathbb{X}}{}^{(-1)} \times 1). \tag{4.7}$$

We will give the details of the proof for Equation (4.6). Proving Equation (4.7) is similar.

Note first that $\Delta(1\times!)$ (and $\Delta(!\times 1)$) is the identity. Second, we see that maps to a product of objects may be expressed as a pairing — i.e. if $f : A \to B \times B$, then $f = \langle f(1\times!), f(!\times 1)\rangle$.

Using this we see that the left hand side of Equation (4.6) may be computed as follows:

$$\Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}} = \langle \Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}}(1\times!), \Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}}(! \times 1)\rangle = \langle \Delta_{\mathbb{X}}{}^{(-1)}, \Delta_{\mathbb{X}}{}^{(-1)}\rangle$$

Similarly, removing the associativity maps, the right hand side of the same equation becomes:

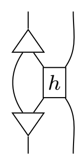
$$\begin{aligned}
(\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)}) &= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})(1\times!), (\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})(! \times 1)\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})(1\times!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})(1 \times \Delta_{\mathbb{X}})(1\times!\times!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times \overline{\Delta_{\mathbb{X}}{}^{(-1)}})(1\times!\times!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)\overline{1 \times \Delta_{\mathbb{X}}{}^{(-1)}}(1\times!\times!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle \overline{(\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})}(\Delta_{\mathbb{X}} \times 1)(1\times!\times!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle \overline{(\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})}(1\times!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle \overline{(\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})(! \times 1)}(1\times!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle \overline{\Delta_{\mathbb{X}}{}^{(-1)}}(1\times!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle \Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}}(1\times!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle = \langle \Delta_{\mathbb{X}}{}^{(-1)}, \Delta_{\mathbb{X}}{}^{(-1)}\rangle
\end{aligned}$$

and therefore we see that the first equation for the Frobenius condition is satisfied. Thus, $\mathbf{INV}(\mathbb{X})$ is a discrete inverse category. $\square$

4.4 The "slice" construction on a discrete inverse category

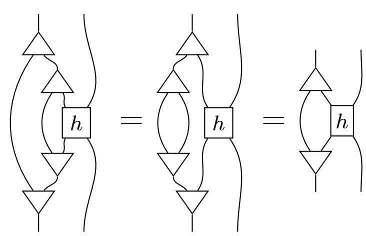
In an inverse category, we will be interested in a specific class of maps, which will be used in future chapters to connect discrete inverse categories to discrete Cartesian restriction categories. Throughout this section, we will assume $\mathbb{X}$ is a discrete inverse category.

In a discrete inverse category, suppose we are given a map $h : A \otimes B \to A \otimes C$. We define $h^{\Delta}_{\nabla} : A \otimes B \to A \otimes C$ as the composite $(\Delta \otimes 1)(1 \otimes h)(\Delta^{(-1)} \otimes 1)$. We want to consider those maps where $h = h^{\Delta}_{\nabla}$. In our graphical language, this means

$h =$  .

Lemma 4.4.1. *For any map h in a discrete inverse category, $h^{\Delta\Delta}_{\nabla\nabla} = h^{\Delta}_{\nabla}$.*

Proof.

$h^{\Delta\Delta}_{\nabla\nabla} =$  $= h^{\Delta}_{\nabla}$.

$\square$

Lemma 4.4.2. *In a discrete inverse category $\mathbb{X}$, for a fixed object A, the set of maps $h : A \otimes Y \to A \otimes Z$ where $h = h^{\Delta}_{\nabla}$ is:*

(i) Closed under partial inversion;

(ii) closed under composition;

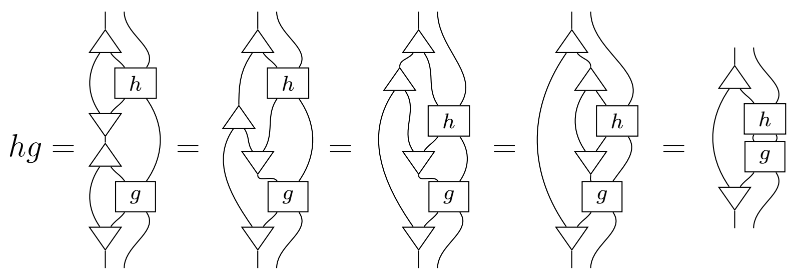
(iii) *contains all maps of the form* $1 \otimes k$ *where* $k : Y \to Z$*;*

(iv) *contains* $\Delta : A \otimes B \to A \otimes B \otimes A \otimes B$*.*

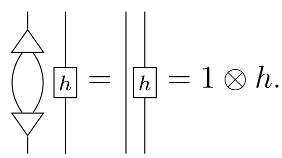
Proof. For (i), if $h = h^{\triangle}_{\triangledown}$, then

$$h^{(-1)} = ((\Delta \otimes 1)(1 \otimes h)(\Delta^{(-1)} \otimes 1))^{(-1)} =$$
$$(\Delta^{(-1)} \otimes 1)^{(-1)}(1 \otimes h)^{(-1)}(\Delta \otimes 1)^{(-1)} = (\Delta \otimes 1)(1 \otimes h^{(-1)})(\Delta^{(-1)} \otimes 1).$$

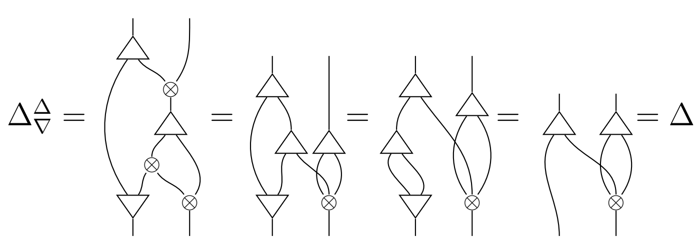
To show (ii), we compose h and g:



For (iii), this follows immediately from



Finally, for (iv), recalling the exchange rule, we have



∎

Because of these closure rules, rather than stating $h = h^{\triangle}_{\triangledown}$ we may equivalently say $h \in A^{\triangle}_{\triangledown}$ when $h = h^{\triangle}_{\triangledown} : A \otimes X \to A \otimes Y$.

From Lemma 4.4.2, we see that we will be able to form a category based on maps h such that $h = h^{\Delta}_{\nabla}$. Of course, this category is dependent upon the choice of the object A, hence we will label it $\mathbb{X}[A]$, as it is reminiscent of the simple slice category over an object A in a category. We make this precise in the following proposition:

Proposition 4.4.3. *Given a discrete inverse category $\mathbb{X}$, define $\mathbb{X}[A]$ as the restriction category:*

Objects: *The objects of $\mathbb{X}$;*

Maps: *A map $h = h^{\Delta}_{\nabla} : A \otimes X \to A \otimes Y$ in $\mathbb{X}$ is a map from X to Y in $\mathbb{X}[A]$;*

Identity: *Composition in $\mathbb{X}$;*

Composition: *$1 \otimes 1$ in $\mathbb{X}$;*

Restriction: *$\overline{h}$ in $\mathbb{X}$ is $\overline{h}$ in $\mathbb{X}[A]$.*

Then, $\mathbb{X}[A]$ is a discrete inverse category.

Proof. Given Lemma 4.4.2, we see immediately that $\mathbb{X}[A]$ is a category. We must show the restriction is in $\mathbb{X}[A]$, i.e., $h = h^{\Delta}_{\nabla}$ then $\overline{h} = \overline{h}^{\Delta}_{\nabla}$ and that $\mathbb{X}[A]$ has a tensor and a Frobenius Δ.

Recalling that $\overline{(1 \otimes h)} = 1 \otimes \overline{h}$ we have

$$(\Delta \otimes 1)\overline{(1 \otimes h)}(\Delta^{(-1)} \otimes 1) = \overline{(\Delta \otimes 1)(1 \otimes h)}(\Delta \otimes 1)(\Delta^{(-1)} \otimes 1) = \overline{(\Delta \otimes 1)(1 \otimes h)}$$
$$\leq \overline{(\Delta \otimes 1)(1 \otimes h)(\Delta^{(-1)} \otimes 1)} = \overline{h}.$$

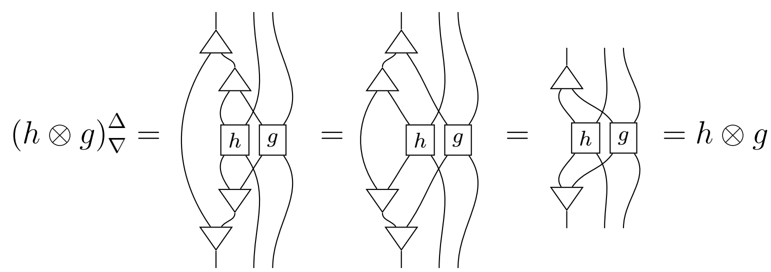
Note this means there is some k such that $\overline{k} = (\Delta \otimes 1)\overline{(1 \otimes h)}(\Delta^{(-1)} \otimes 1)$. Next, computing

$\overline{k}h$ we have:

$$\begin{aligned}
\overline{k}h &= (\Delta \otimes 1)\overline{(1 \otimes h)}(\Delta^{(-1)} \otimes 1)(\Delta \otimes 1)(1 \otimes h)(\Delta^{(-1)} \otimes 1) \\
&= (\Delta \otimes 1)\overline{(1 \otimes h)}\,\overline{(\Delta^{(-1)} \otimes 1)}(1 \otimes h)(\Delta^{(-1)} \otimes 1) \\
&= (\Delta \otimes 1)\overline{(\Delta^{(-1)} \otimes 1)}\,\overline{(1 \otimes h)}(1 \otimes h)(\Delta^{(-1)} \otimes 1) \\
&= (\Delta \otimes 1)(\widehat{\Delta \otimes 1})(1 \otimes h)(\Delta^{(-1)} \otimes 1) \\
&= (\Delta \otimes 1)(1 \otimes h)(\Delta^{(-1)} \otimes 1) \\
&= h.
\end{aligned}$$

But by Lemma 3.2.2, if $\overline{k}h = h$, we have $\overline{h} \leq \overline{k}$. As we already have $\overline{k} \leq \overline{h}$ this means they are equal and the restriction is in $\mathbb{X}[A]$, thus, $\mathbb{X}[A]$ is a restriction category.

The tensor of objects in $\mathbb{X}[A]$ is given by the tensor of the objects in $\mathbb{X}$. For two maps h, k in $\mathbb{X}[A]$, $h \otimes k$ is given by $(\Delta \otimes 1 \otimes 1)(1 \otimes c_\otimes \otimes 1)(h \otimes k)(1 \otimes c_\otimes \otimes 1)(\Delta^{(-1)} \otimes 1 \otimes 1)$ in $\mathbb{X}$. This is a map in $\mathbb{X}[A]$ as



The Δ in $\mathbb{X}[A]$ is given by the map $1 \otimes \Delta$ in $\mathbb{X}$. The various identities required of Δ hold in $\mathbb{X}[A]$ as they hold in $\mathbb{X}$, therefore $\mathbb{X}[A]$ is a discrete inverse category. $\square$

We note that there are functors between $\mathbb{X}$ and $\mathbb{X}[A]$, given by:

$$G : \mathbb{X} \to \mathbb{X}[A]; \quad G : B \mapsto B; \quad G : f \mapsto 1 \otimes f$$

and

$$F : \mathbb{X}[A] \to \mathbb{X}; \quad F : B \mapsto A \otimes B; \quad F : f \mapsto f.$$

However, these do not form an adjoint pair as the relation is

$$\frac{\mathbb{X}[A](X,Y)}{\mathbb{X}(A\otimes X, A\otimes Y)} \quad \left(\text{i.e.,} \frac{\mathbb{X}[A](X,G(Y))}{\mathbb{X}(F(X),F(Y))}\right)$$

rather than the required

$$\frac{\mathbb{X}[A](X,G(Y))}{\mathbb{X}(F(X),Y)}.$$

Chapter 5

From a discrete inverse category to a Cartesian restriction category

The purpose of this chapter is to prove that the category of discrete inverse categories is equivalent to the the category of discrete Cartesian restriction categories. We will show how to construct a discrete Cartesian restriction category, $\widetilde{\mathbb{X}}$, from a discrete inverse category, $\mathbb{X}$. Then, we will give the equivalence functors between $\mathbb{X}$ and $\widetilde{\mathbb{X}}$.

5.1 The restriction category $\widetilde{\mathbb{X}}$

We begin by giving the construction of $\widetilde{\mathbb{X}}$.

Definition 5.1.1. When $\mathbb{X}$ is an inverse category, define $\widetilde{\mathbb{X}}$ as:

Objects: objects as in $\mathbb{X}$;

Maps: A map $(f, C) : A \to B$ in $\widetilde{\mathbb{X}}$ is the equivalence class of the map $f : A \to B \otimes C$ in $\mathbb{X}$ (detailed below in Definition 5.1.2). We have the following relationship between maps in $\widetilde{\mathbb{X}}$ and $\mathbb{X}$:

$$\frac{A \xrightarrow{(f,C)} B \text{ in } \widetilde{\mathbb{X}}}{A \xrightarrow{f} B \otimes C \text{ in } \mathbb{X};}$$

Identity: by

$$\frac{A \xrightarrow{(u_\otimes^{r\,(-1)},1)} A}{A \xrightarrow{u_\otimes^{r\,(-1)}} A \otimes 1;}$$

Composition: given by

$$\begin{array}{c}
A \xrightarrow{(f,B')} B \xrightarrow{(g,C')} C \\
\hline
A \xrightarrow{f} B \otimes B',\ B \xrightarrow{g} C \otimes C' \\
\hline
A \xrightarrow{f(g\otimes 1)a_{\otimes}} C \otimes (C' \otimes B') \\
\hline
A \xrightarrow{(f(g\otimes 1)a_{\otimes}, C'\otimes B')} C.
\end{array}$$

When considering an $\widetilde{\mathbb{X}}$ map $(f, C) : A \to B$ in $\mathbb{X}$, we occasionally use the notation $f : A \to B_{|C}$ ($\equiv f : A \to B \otimes C$).

5.1.1 Equivalence classes of maps in $\mathbb{X}$

Definition 5.1.2. In a discrete inverse category $\mathbb{X}$, the map f is equivalent to f' in $\mathbb{X}$ when $\overline{f} = \overline{f'}$ in $\mathbb{X}$ and Figure 5.1 is a commutative diagram for some map $h \in B^{\triangle}_{\triangledown}$.

$$\begin{tikzcd}
 & B \otimes C \arrow[dd, dotted, "h"] \\
A \arrow[ur, "f"] \arrow[dr, "f'"'] & \\
 & B \otimes C'
\end{tikzcd}$$

Figure 5.1: Equivalence diagram for constructing maps in $\widetilde{\mathbb{X}}$.

Notation 5.1.3. When f is equivalent to g as in Definition 5.1.2 via the mediating map h, this is written as:

$$f \overset{h}{\simeq} g.$$

Lemma 5.1.4. *Definition 5.1.2 gives a symmetric, reflexive equivalence class of maps in $\mathbb{X}$.*

Proof.

Reflexivity: Choose h as the identity map.

Symmetry: Suppose $f \overset{h}{\simeq} g$. Then, $\overline{f} = \overline{g}$ and $fh = g$. Applying $h^{(-1)}$, we have

$$gh^{(-1)} = fhh^{(-1)} = f\overline{h} = \overline{fh}f = \overline{g}f = \overline{f}f = f.$$

Thus, $g \overset{h^{(-1)}}{\simeq} f$.

Transitivity: Suppose $f \overset{h}{\simeq} f'$ and $f' \overset{k}{\simeq} f''$, i.e., $fh = f'$ and $f'k = f''$. We immediately have $fhk = f''$ and by Lemma 4.4.2, we know $hk = (hk)_{\nabla}^{\Delta}$ and therefore we have an equivalence.

□

5.1.2 $\widetilde{\mathbb{X}}$ is a restriction category

Lemma 5.1.5. *$\widetilde{\mathbb{X}}$ as defined above is a category.*

Proof. The maps are well defined, as shown in Lemma 5.1.4. The existence of the identity map is due to the tensor $\otimes$ being defined on $\mathbb{X}$, an inverse category, hence ${u_\otimes^r}^{(-1)}$ is defined.

It remains to show the composition is associative and that $({u_\otimes^r}^{(-1)}, 1)$ acts as an identity in $\widetilde{\mathbb{X}}$. For all of these, we will make use of Lemma 4.4.2 (iii), which states $1 \otimes f \in A_{\nabla}^{\Delta}$ for all f.

Associativity: Consider

$$A \xrightarrow{(f,B')} B \xrightarrow{(g,C')} C \xrightarrow{(h,D')} D.$$

To show the associativity of this in $\widetilde{\mathbb{X}}$, we need to show in $\mathbb{X}$ that

$$\overline{(f(g \otimes 1)a_\otimes)(h \otimes 1)a_\otimes} = \overline{f(((g(h \otimes 1)a_\otimes) \otimes 1)a_\otimes)}$$

and that there exists a mediating map between the two of them.

To see that the restrictions are equal, first note that by the functorality of $\otimes$, for any two maps u and v, we have $uv \otimes 1 = (u \otimes 1)(v \otimes 1)$. Second, the naturality of $a_\otimes$ gives us that

$a_\otimes(h \otimes 1) = ((h \otimes 1) \otimes 1)a_\otimes$. Thus,

$$
\begin{aligned}
\overline{f(g \otimes 1)a_\otimes(h \otimes 1)a_\otimes} &= \overline{f(g \otimes 1)a_\otimes(h \otimes 1)\overline{a_\otimes}} && \text{Lemma 3.1.3} \\
&= \overline{f(g \otimes 1)a_\otimes(h \otimes 1)} && \overline{a_\otimes} = 1 \\
&= \overline{f(g \otimes 1)((h \otimes 1) \otimes 1)a_\otimes} && a_\otimes \text{ natural} \\
&= \overline{f(g \otimes 1)((h \otimes 1) \otimes 1)} && \text{Lemma 3.1.3} \\
&= \overline{f(g \otimes 1)((h \otimes 1) \otimes 1)(a_\otimes \otimes 1)} && \text{Lemma 3.1.3} \\
&= \overline{f((g(h \otimes 1)a_\otimes) \otimes 1)} && \text{see above} \\
&= \overline{f((g(h \otimes 1)a_\otimes) \otimes 1)a_\otimes} && \overline{a_\otimes} = 1.
\end{aligned}
$$

For the mediating map, see the diagram below, where the calculation is in $\mathbb{X}$. The path starting at the top left at A and going right to $D_{|D' \otimes (C' \otimes B')}$ is grouping parentheses to the left. Starting at A and then going down to $(D_{|D' \otimes C'})_{|B'}$ followed by right to $D_{|(D' \otimes C') \otimes B'}$ is grouping parentheses to the right. The commutativity of the diagram is shown by the commutativity of the internal portions, which all follow from the standard coherence diagrams for the tensor and naturality of association.

$$
\begin{tikzcd}
A \arrow[r, "f(g\otimes 1)a_\otimes"] \arrow[d, "f"'] \arrow[dr, "f(g\otimes 1)"] & C_{|C' \otimes B'} \arrow[r, "h\otimes 1"] & (D_{|D'})_{|C' \otimes B'} \arrow[r, "a_\otimes"] & D_{|D' \otimes (C' \otimes B')} \arrow[dd, dotted, "1\otimes a_\otimes{}^{(-1)}"] \\
B_{|B'} \arrow[r, "g\otimes 1"] \arrow[d, "(g(h\otimes 1)a_\otimes)\otimes 1"'] & (C_{|C'})_{|B'} \arrow[u, "a_\otimes"'] \arrow[r, "(h\otimes 1)\otimes 1"] & ((D_{|D'})_{|C'})_{|B'} \arrow[u, "a_\otimes"'] \arrow[dll, "a_\otimes\otimes 1"'] & \\
(D_{|D' \otimes C'})_{|B'} \arrow[rrr, "a_\otimes"'] & & & D_{|(D' \otimes C') \otimes B'}.
\end{tikzcd}
$$

Therefore, we can conclude

$$
(f(g \otimes 1)a_\otimes)(h \otimes 1)a_\otimes \overset{1\otimes a_\otimes{}^{(-1)}}{\simeq} f(((g(h \otimes 1)a_\otimes) \otimes 1)a_\otimes)
$$

which gives us that composition in $\widetilde{\mathbb{X}}$ is associative.

Identity: This requires:

$$(f, C)(u_\otimes^{r\,(-1)}, 1) = (f, C) = (u_\otimes^{r\,(-1)}, 1)(f, C)$$

for all maps $A \xrightarrow{(f,C)} B$ in $\widetilde{\mathbb{X}}$. By Lemma 3.1.3 we have $\overline{f(u_\otimes^{r\,(-1)} \otimes 1)a_\otimes} = \overline{f}$. Then, calculating in $\mathbb{X}$, we have a mediating map of $1 \otimes u_\otimes^l$ as shown below.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[drrr, "f"'] & B \otimes C \arrow[r, "u_\otimes^{r\,(-1)} \otimes 1"] \arrow[rr, bend right, "1 \otimes u_\otimes^{l\,(-1)}"'] \arrow[drr, equal] & (B \otimes 1) \otimes C \arrow[r, "a_\otimes"] & B \otimes (1 \otimes C) \arrow[d, dotted, "1 \otimes u_\otimes^l"] \\
 & & & B \otimes C.
\end{tikzcd}$$

$\overline{u_\otimes^{r\,(-1)}(f \otimes 1)a_\otimes} = \overline{f}$ by the naturality of $u_\otimes^{r\,(-1)}$ and Lemma 3.1.3. The diagram

$$\begin{tikzcd}
A \arrow[r, "u_\otimes^{r\,(-1)}"] \arrow[dr, "f"'] \arrow[dd, equal] & A \otimes 1 \arrow[r, "f \otimes 1"] & (B \otimes C) \otimes 1 \arrow[r, "a_\otimes"] & B \otimes (C \otimes 1) \arrow[dd, dotted, "1 \otimes u_\otimes^r"] \\
 & B \otimes C \arrow[ur, "u_\otimes^{r\,(-1)}"] \arrow[urr, "1 \otimes u_\otimes^{r\,(-1)}"'] \arrow[drr, equal] & & \\
A \arrow[rrr, "f"] & & & B \otimes C
\end{tikzcd}$$

shows our mediating map is $1 \otimes u_\otimes^r$. $\square$

Define the restriction in $\widetilde{\mathbb{X}}$ as follows:

$$\cfrac{\cfrac{A \xrightarrow{(f,C)} B}{A \xrightarrow{\overline{(f,C)}} A}}{A \xrightarrow{\overline{f} u_\otimes^{r\,(-1)}} A \otimes 1 \text{ in } \mathbb{X}.}$$

Lemma 5.1.6. *The category $\widetilde{\mathbb{X}}$ with restriction defined as above is a restriction category.*

Proof. Given the above definition the four restriction axioms must now be checked. For the remainder of this proof, all diagrams will be in $\mathbb{X}$. We make use of Lemma 4.4.2 (iii), which states $1 \otimes f \in A^{\triangle}_{\triangledown}$ for all f.

[**R.1**] ($\overline{f}f = f$). Calculating the restriction of the left hand side in $\mathbb{X}$, we have:

$$
\begin{aligned}
\overline{\overline{f}u_\otimes^{r\,(-1)}(f\otimes 1)a_\otimes} &= \overline{\overline{f}u_\otimes^{r\,(-1)}(f\otimes 1)} && \text{Lemma 3.1.3}\\
&= \overline{\overline{f}fu_\otimes^{r\,(-1)}} && u_\otimes^{r\,(-1)} \text{ natural}\\
&= \overline{fu_\otimes^{r\,(-1)}} && [\textbf{R.1}] \text{ in } \mathbb{X}\\
&= \overline{f} && \text{Lemma 3.1.3.}
\end{aligned}
$$

Then, the following diagram

$$
\begin{tikzcd}
A \arrow[r, "\overline{f}u_\otimes^{r\,(-1)}"] \arrow[drr, "\overline{f}f"'] \arrow[ddrrr, bend right, "f"'] & A\otimes 1 \arrow[r, "f\otimes 1"] & (A\otimes B)\otimes 1 \arrow[r, "a_\otimes"] \arrow[ddr, "u_\otimes^r"] & A\otimes(B\otimes 1) \arrow[dd, dotted, "1\otimes u_\otimes^r"] \\
& & A\otimes B \arrow[u, "u_\otimes^{r\,(-1)}"] \arrow[dr, equal] & \\
& & & A\otimes B
\end{tikzcd}
$$

shows $\overline{f}u_\otimes^{r\,(-1)}(f\otimes 1)a_\otimes \overset{1\otimes u_\otimes^r}{\simeq} f$ in $\mathbb{X}$ and therefore $\overline{f}f = f$ in $\widetilde{\mathbb{X}}$.

[**R.2**] ($\overline{g}\overline{f} = \overline{f}\overline{g}$). We must show

$$\overline{f}u_\otimes^{r\,(-1)}((\overline{g}u_\otimes^{r\,(-1)})\otimes 1))a_\otimes \simeq \overline{g}u_\otimes^{r\,(-1)}((\overline{f}u_\otimes^{r\,(-1)})\otimes 1)a_\otimes. \tag{5.1}$$

The restriction of the left hand side equals the restriction of the right hand side as seen below:

$$
\begin{aligned}
\overline{\overline{f}u_\otimes^{r\,(-1)}((\overline{g}u_\otimes^{r\,(-1)})\otimes 1))a_\otimes} &= \overline{\overline{f}(\overline{g}u_\otimes^{r\,(-1)})u_\otimes^{r\,(-1)}a_\otimes} && u_\otimes^{r\,(-1)} \text{ natural}\\
&= \overline{\overline{g}\overline{f}u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}a_\otimes} && [\textbf{R.2}] \text{ in } \mathbb{X}\\
&= \overline{\overline{g}u_\otimes^{r\,(-1)}((\overline{f}u_\otimes^{r\,(-1)})\otimes 1)a_\otimes} && u_\otimes^{r\,(-1)}\text{natural.}
\end{aligned}
$$

The below diagram commutes by the naturality of $u_\otimes^r$ and the tensor coherence,

$$\begin{tikzcd}
A \arrow[r, "\overline{g}u_\otimes^{r\,(-1)}"] \arrow[d, "\overline{f}u_\otimes^{r\,(-1)}"'] \arrow[dr, "\overline{g}\overline{f}", "\overline{f}\overline{g}"'] & A \otimes 1 \arrow[r, "(\overline{f}u_\otimes^{r\,(-1)})\otimes 1"] & (A \otimes 1) \otimes 1 \arrow[r, "a_\otimes"] \arrow[dl, "u_\otimes^r u_\otimes^r"'] & A \otimes (1 \otimes 1) \arrow[dd, dotted, "1\otimes id"'] \\
A \otimes 1 \arrow[d, "(\overline{g}u_\otimes^{r\,(-1)})\otimes 1"'] & A \arrow[ur, bend right, "u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}"'] & & \\
(A \otimes 1) \otimes 1 \arrow[ur, "u_\otimes^r u_\otimes^r"] \arrow[rrr, "a_\otimes"'] & & & A \otimes (1 \otimes 1)
\end{tikzcd}$$

which allows us to conclude $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ in $\widetilde{\mathbb{X}}$.

[**R.3**] ($\overline{\overline{f}g} = \overline{f}\,\overline{g}$). We must show

$$\overline{(\overline{f}u_\otimes^{r\,(-1)})(g \otimes 1)a_\otimes} u_\otimes^{r\,(-1)} \simeq (\overline{f}u_\otimes^{r\,(-1)})(\overline{g}u_\otimes^{r\,(-1)} \otimes 1)a_\otimes. \tag{5.2}$$

As above, the first step is to show that the restrictions of each side of Equation (5.2) are the same. Computing the restriction of the left hand side in $\mathbb{X}$:

$$\begin{aligned}
\overline{\overline{(\overline{f}u_\otimes^{r\,(-1)})(g \otimes 1)a_\otimes} u_\otimes^{r\,(-1)}} &= \overline{\overline{(\overline{f}u_\otimes^{r\,(-1)})(g \otimes 1)a_\otimes}} && \text{Lemma 3.1.3} \\
&= \overline{(\overline{f}u_\otimes^{r\,(-1)})(g \otimes 1)a_\otimes} && \text{Lemma 3.1.3} \\
&= \overline{\overline{f}gu_\otimes^{r\,(-1)}a_\otimes} && u_\otimes^{r\,(-1)} \text{ natural} \\
&= \overline{\overline{f}g} && \text{Lemma 3.1.3} \\
&= \overline{f}\,\overline{g} && [\mathbf{R.3}] \text{ in } \mathbb{X}.
\end{aligned}$$

The restriction of the right hand side computes in $\mathbb{X}$ as:

$$\begin{aligned}
&\overline{(\overline{f}u_\otimes^{r\,(-1)})(\overline{g}u_\otimes^{r\,(-1)} \otimes 1)a_\otimes} \\
&= \overline{(\overline{f}u_\otimes^{r\,(-1)})(\overline{g}u_\otimes^{r\,(-1)} \otimes 1)} && \text{Lemma 3.1.3} \\
&= \overline{\overline{f}\,\overline{g}u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}} && u_\otimes^{r\,(-1)} \text{ natural} \\
&= \overline{\overline{f}\,\overline{g}} && \text{Lemma 3.1.3} \\
&= \overline{f}\,\overline{g} && \text{Lemma 3.1.3.}
\end{aligned}$$

Additionally, we see $\overline{\overline{f}g}$ in $\widetilde{\mathbb{X}}$ is expressed in $\mathbb{X}$ as:

$$\begin{aligned}
&\overline{(\overline{f}u_\otimes^{r\,(-1)})(g\otimes 1)a_\otimes}u_\otimes^{r\,(-1)} \\
&= \overline{f}u_\otimes^{r\,(-1)}\overline{g\otimes 1} && [\mathbf{R.3}], [\mathbf{R.4}] \\
&= \overline{f}\overline{g}u_\otimes^{r\,(-1)} && \otimes\text{a restriction bi-functor, } u_\otimes^{r\,(-1)} \text{ natural.}
\end{aligned}$$

The following diagram in $\mathbb{X}$ follows the right hand side of Equation (5.2) with the top curved arrow and the left hand side of Equation (5.2) with the bottom curved arrow. Note that we are using that $\overline{(\overline{f}u_\otimes^{r\,(-1)})(g\otimes 1)a_\otimes} = \overline{f}\overline{g}$ as shown above.

$$\begin{tikzcd}
A \arrow[r, "\overline{f}"] \arrow[dr, "\overline{f}"'] \arrow[rrrrr, bend left, "\overline{f}u_\otimes^{r\,(-1)}(\overline{g}u_\otimes^{r\,(-1)}\otimes 1)a_\otimes"] \arrow[rrrrrddddd, bend right, "\overline{f}\overline{g}u_\otimes^{r\,(-1)}"'] & A \arrow[r, "u_\otimes^{r\,(-1)}"] \arrow[d, equal] & A\otimes 1 \arrow[r, "\overline{g}\otimes 1"] \arrow[dd, "u_\otimes^r"'] & A\otimes 1 \arrow[r, "u_\otimes^{r\,(-1)}\otimes 1"] \arrow[ddd, "u_\otimes^r"'] & (A\otimes 1)\otimes 1 \arrow[r, "a_\otimes"] \arrow[dddd, "u_\otimes^r"'] & A\otimes(1\otimes 1) \arrow[ddddd, dotted, "1\otimes u_\otimes^r"] \\
 & A \arrow[dr, equal] & & & & \\
 & & A \arrow[dr, "\overline{g}"'] & & & \\
 & & & A \arrow[dr, "u_\otimes^{r\,(-1)}"'] & & \\
 & & & & A\otimes 1 \arrow[dr, equal] & \\
 & & & & & A\otimes 1.
\end{tikzcd}$$

Hence, in $\mathbb{X}$, $\overline{(\overline{f}u_\otimes^{r\,(-1)})(g\otimes 1)a_\otimes}u_\otimes^{r\,(-1)} \overset{1\otimes u_\otimes^r}{\simeq} (\overline{f}u_\otimes^{r\,(-1)})(\overline{g}u_\otimes^{r\,(-1)}\otimes 1)a_\otimes$ and therefore $\overline{\overline{f}g} = \overline{f}\overline{g}$ in $\widetilde{\mathbb{X}}$.

[**R.4**] ($f\overline{g} = \overline{fg}f$). We must show

$$f(\overline{g}u_\otimes^{r\,(-1)}\otimes 1)a_\otimes \simeq \overline{f(g\otimes 1)}u_\otimes^{r\,(-1)}(f\otimes 1)a_\otimes. \tag{5.3}$$

The restriction of the left hand side of Equation (5.3) is:

$$\begin{aligned}
\overline{f(\overline{g}u_\otimes^{r\,(-1)} \otimes 1)a_\otimes} &= \overline{f(\overline{g}u_\otimes^{r\,(-1)} \otimes 1)} && \text{Lemma 3.1.3} \\
&= \overline{f\overline{g}u_\otimes^{r\,(-1)}} \otimes \overline{f} && \otimes \text{ restriction functor} \\
&= \overline{f\overline{g}} \otimes \overline{f} && \text{Lemma 3.1.3} \\
&= \overline{f(\overline{g} \otimes 1)}
\end{aligned}$$

and the restriction of the right hand side of Equation (5.3) is:

$$\begin{aligned}
\overline{\overline{f(g \otimes 1)}u_\otimes^{r\,(-1)}(f \otimes 1)a_\otimes} &= \overline{\overline{f(g \otimes 1)}u_\otimes^{r\,(-1)}(f \otimes 1)} && \text{Lemma 3.1.3} \\
&= \overline{\overline{f(g \otimes 1)}fu_\otimes^{r\,(-1)}} && u_\otimes^{r\,(-1)} \text{ natural} \\
&= \overline{f\overline{(g \otimes 1)}u_\otimes^{r\,(-1)}} && [\mathbf{R.4}] \text{ for } \mathbb{X} \\
&= \overline{f(\overline{g} \otimes 1)u_\otimes^{r\,(-1)}} && \otimes \text{ a restriction functor} \\
&= \overline{f(\overline{g} \otimes 1)} && \text{Lemma 3.1.3.}
\end{aligned}$$

Computing the right hand side of Equation (5.3) in $\mathbb{X}$,

$$\begin{aligned}
\overline{f(g \otimes 1)a_\otimes}u_\otimes^{r\,(-1)}(f \otimes 1)a_\otimes &= \overline{f(g \otimes 1)}fu_\otimes^{r\,(-1)}a_\otimes && u_\otimes^{r\,(-1)} \text{ natural,} \\
&= f(\overline{g} \otimes 1)u_\otimes^{r\,(-1)}a_\otimes && [\mathbf{R.4}].
\end{aligned}$$

Thus,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f"'] & B \otimes C \arrow[rr, "\overline{g}u_\otimes^{r\,(-1)} \otimes 1"] \arrow[d, equal] & & (B \otimes 1) \otimes C \arrow[r, "a_\otimes"] \arrow[d, "a_\otimes(1 \otimes c_\otimes)"] & B \otimes (1 \otimes C) \arrow[d, dotted, "1 \otimes c_\otimes"] \\
& B \otimes C \arrow[r, "\overline{g} \otimes 1"'] & B \otimes C \arrow[r, "u_\otimes^{r\,(-1)}"'] & (B \otimes C) \otimes 1 \arrow[r, "a_\otimes"'] & B \otimes (C \otimes 1)
\end{tikzcd}$$

and hence, $\widetilde{\mathbb{X}}$ is a restriction category. □

5.1.3 $\widetilde{\mathbb{X}}$ is a discrete Cartesian restriction category

Lemma 5.1.7. *The unit of the inverse product in $\mathbb{X}$ is the terminal object in $\widetilde{\mathbb{X}}$.*

Proof. The unique map to the terminal object for any object A in $\widetilde{\mathbb{X}}$ is the equivalence class of maps represented by $({u^l_\otimes}^{(-1)}, A)$. For this to be a terminal object, the diagram

$$\begin{tikzcd}
X \arrow[r, "\overline{(f,C)}"] \arrow[d, "(f,C)"] & X \arrow[r, "!_X"] & \top \\
Y \arrow[urr, "!_Y"'] & &
\end{tikzcd}$$

must commute for all choices of f. Translating this to $\mathbb{X}$, this is the same as requiring

$$\begin{tikzcd}
X \arrow[r, "\overline{f}"] \arrow[d, "f"] & X \arrow[r, "{u^r_\otimes}^{(-1)}"] & X \otimes 1 \arrow[r, "{u^l_\otimes}^{(-1)}"] & 1 \otimes X \otimes 1 \arrow[dl, dotted, "1\otimes(u^r_\otimes f)"'] \\
Y \otimes C \arrow[rr, "{u^l_\otimes}^{(-1)}"'] & & 1 \otimes Y \otimes C &
\end{tikzcd}$$

to commute, which is true by [**R.1**] and from the coherence diagrams for the inverse product tensor. $\square$

Next,we show that the category $\widetilde{\mathbb{X}}$ has restriction products, given by the action of $\widetilde{(\_)}$ on the $\otimes$ tensor in $\mathbb{X}$.

First, define total maps π_0, π_1 in $\widetilde{\mathbb{X}}$ by:

$$\pi_0 : \qquad A \otimes B \xrightarrow{(1,B)} A, \tag{5.4}$$

$$\pi_1 : \qquad A \otimes B \xrightarrow{(c_\otimes,A)} B. \tag{5.5}$$

Definition 5.1.8. Given a discrete inverse category $\mathbb{X}$, suppose we are given the maps $Z \xrightarrow{(f,C)} A$ and $Z \xrightarrow{(g,C')} B$ in $\widetilde{\mathbb{X}}$. Then define $\langle (f,C),(g,C')\rangle$as

$$Z \xrightarrow{(\Delta(f\otimes g)(1\otimes c_\otimes \otimes 1), C\otimes C')} A \otimes B \tag{5.6}$$

where associativity is assumed as needed. Note that with the associativity maps, this is actually:

$$Z \xrightarrow{(\Delta(f\otimes g)a_\otimes(1\otimes {a_\otimes}^{(-1)})(1\otimes(c_\otimes\otimes 1))(1\otimes a_\otimes){a_\otimes}^{(-1)}, C\otimes C')} A \otimes B. \tag{5.7}$$

Lemma 5.1.9. *On $\widetilde{\mathbb{X}}$, $\otimes$ is a restriction product with projections π_0, π_1 and the product of maps f, g being $\langle f, g\rangle$.*

Proof. From the definition above, as 1 and $c_\otimes$ are isomorphisms, the maps π_0, π_1 are total.

In order to show that $\overline{\langle f, g\rangle} = \overline{f}\,\overline{g}$, first reduce the left hand side:

$$\begin{aligned}
\overline{\langle f, g\rangle} &= \overline{\Delta(f \otimes g)(1 \otimes c_\otimes \otimes 1)u_\otimes^{r\,(-1)}} && \text{in } \mathbb{X}, \text{ definition of restriction} \\
&= \overline{\Delta(f \otimes g)u_\otimes^{r\,(-1)}} \\
&= \overline{\Delta\overline{(f \otimes g)}u_\otimes^{r\,(-1)}} && \text{from Lemma 3.1.3} \\
&= \overline{\Delta(\overline{f} \otimes \overline{g})u_\otimes^{r\,(-1)}} && \otimes \text{ is a restriction functor} \\
&= \overline{\overline{f}\,\overline{g}\,\Delta(1 \otimes 1)u_\otimes^{r\,(-1)}} && \text{Lemma 4.3.5((ii)) twice} \\
&= \overline{\overline{f}\,\overline{g}}u_\otimes^{r\,(-1)} && \text{Lemma 3.1.3} \\
&= \overline{f}\,\overline{g}u_\otimes^{r\,(-1)} && \text{Lemma 3.1.3.}
\end{aligned}$$

Then, the right hand side reduces as:

$$\begin{aligned}
\overline{f}\overline{g} &= \overline{f}u_\otimes^{r\,(-1)}(\overline{g}u_\otimes^{r\,(-1)} \otimes 1)a_\otimes && \text{in } \mathbb{X} \text{ by definitions} \\
&= \overline{f}\overline{g}u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}a_\otimes && u_\otimes^{r\,(-1)} \text{ natural.}
\end{aligned}$$

The restriction of the left hand side and the right hand side, in $\mathbb{X}$, is $\overline{\overline{f}\overline{g}}$. This is done by applying Lemma 3.1.3 once to the left hand side, $\overline{f}\,\overline{g}u_\otimes^{r\,(-1)}$ and thrice to the right hand side, $\overline{f}\overline{g}u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}a_\otimes$.

Thus, this shows $\overline{\langle f, g\rangle} = \overline{f}\overline{g}$ in $\widetilde{\mathbb{X}}$ where the mediating map in $\mathbb{X}$ is $1 \otimes u_\otimes^r$.

Next, to show $\langle f, g\rangle\pi_0 \leq f$ (and $\langle f, g\rangle\pi_1 \leq g$), it is required to show $\overline{\langle f, g\rangle\pi_0}f = \langle f, g\rangle\pi_0$. Calculating the left side, we see:

$$\begin{aligned}
\overline{\langle f, g\rangle\pi_0}f &= \overline{\langle f, g\rangle\overline{\pi_0}}f && \text{Lemma 3.1.3} \\
&= \overline{\langle f, g\rangle}f && \pi_0 \text{ is total} \\
&= \overline{f}\,\overline{g}\,f && \text{by above} \\
&= \overline{g}\overline{f}f && [\mathbf{R.2}] \\
&= \overline{g}f && [\mathbf{R.1}].
\end{aligned}$$

Now, turning to the right hand side:

$$\langle f, g\rangle\pi_0 = \Delta(f\otimes g)(1\otimes c_\otimes\otimes 1)1 \qquad\qquad \text{in } \mathbb{X}\text{, by definition.}$$

To show these are equal in $\widetilde{\mathbb{X}}$, we need to first show the restrictions are the same in $\mathbb{X}$ and then show there is a mediating map between the images in $\mathbb{X}$. The restriction of $\overline{g}f$ is $\overline{f}\overline{g}$ immediately by [**R.3**] and [**R.2**]. For the right hand side, calculate in $\mathbb{X}$:

$$\begin{aligned}
\overline{\Delta(f\otimes g)(1\otimes c_\otimes\otimes 1)} &= \overline{\Delta(f\otimes g)} && \text{Lemma 3.1.3}\\
&= \Delta(f\otimes g)(f^{(-1)}\otimes g^{(-1)})\Delta^{(-1)} && \mathbb{X}\text{ is an inverse category}\\
&= \Delta(\overline{f}\otimes\overline{g})\Delta^{(-1)} &&\\
&= \overline{f}\overline{g}\Delta\Delta^{(-1)} && \text{Lemma 4.3.5((ii)) twice}\\
&= \overline{f}\overline{g}. &&
\end{aligned}$$

The diagram below shows the required mediating map. By Lemma 4.4.2, $\Delta\in A^{\Delta}_{\nabla}$, $1\otimes k\in A^{\Delta}_{\nabla}$ and A^{Δ}_{∇} is closed under composition.

$$\begin{tikzcd}
 & & A\otimes C \arrow[d, dotted, "\Delta"] \\
 & Z \arrow[ur, "f"] & A\otimes C\otimes A\otimes C \arrow[d, dotted, "1\otimes f^{(-1)}"] \\
Z \arrow[ur, "\overline{g}"] \arrow[dr, "\Delta(f\otimes g)"'] & & A\otimes C\otimes Z \arrow[d, dotted, "1\otimes 1\otimes g"] \\
 & A\otimes C\otimes B\otimes C' \arrow[dr, "1\otimes c_\otimes\otimes 1"'] & A\otimes C\otimes B\otimes C' \arrow[d, dotted, "1\otimes c_\otimes\otimes 1"] \\
 & & A\otimes B\otimes C\otimes C'.
\end{tikzcd}$$

□

At this point, we have shown that $\widetilde{\mathbb{X}}$ is a restriction category with restriction products. This leads us to the following theorem:

Theorem 5.1.10. *For any inverse category $\mathbb{X}$, the category $\widetilde{\mathbb{X}}$ is a discrete restriction category.*

Proof. The fact that $\widetilde{\mathbb{X}}$ is a Cartesian restriction category is immediate from Lemmas 5.1.5, 5.1.6, 5.1.7 and 5.1.9.

To show that it is discrete, we need only show that the map $(\Delta {u_\otimes^r}^{(-1)}, 1)$ is in the same equivalence class as $\widetilde{\mathbb{X}}$'s $\Delta (= \langle 1, 1\rangle = \langle ({u_\otimes^r}^{(-1)}, 1), ({u_\otimes^r}^{(-1)}, 1)\rangle)$. As both Δ and ${u_\otimes^r}^{(-1)}$ are total, the restriction of each side is the same, namely 1. The diagram below uses Lemma 4.4.2 (iii) and shows that the two maps are in the same equivalence class.

$$\begin{tikzcd}
 & & A \otimes A \otimes 1 \arrow[d, dotted, "1 \otimes {u_\otimes^r}^{(-1)}"] \\
A \arrow[urr, "\Delta {u_\otimes^r}^{(-1)}"] \arrow[rr, "\Delta({u_\otimes^r}^{(-1)} \otimes {u_\otimes^r}^{(-1)})(1 \otimes c_\otimes \otimes 1)"'] & & A \otimes A \otimes 1 \otimes 1
\end{tikzcd}$$

□

5.2 Equivalence between discrete inverse and discrete Cartesian restriction categories

This section will show that the category of discrete inverse categories (with functors that preserve the inverse tensor) is equivalent to the category of discrete restriction categories (with restriction functors which preserve the product).

We will follow this path to showing the equivalence:

1. Give a functor **INV** from discrete Cartesian restriction categories to discrete inverse categories and show that it is full and faithful.

2. Draw the universal diagram:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\eta"] \arrow[dr, "F"'] & \mathbf{INV}(\widetilde{\mathbb{X}}) \arrow[d, "\mathbf{INV}(F^\#)"] \\
 & \mathbf{INV}(\mathbb{D})
\end{tikzcd} \tag{5.8}$$

and show that there exists a functor $F^{\#} : \widetilde{\mathbb{X}} \to \mathbb{D}$ which makes the diagram commute. As we have **INV** full and faithful, we may conclude it is unique.

3. Show that η in Diagram (5.8) is an isomorphism.

Once we have completed these steps, we may then conclude that there is an equivalence of discrete inverse categories to discrete Cartesian restriction categories.

In the following, $\mathbb{X}$ will always be a discrete inverse category, $\mathbb{D}$ and $\mathbb{B}$ will be discrete restriction categories.

The functor **INV** maps a discrete restriction category to its inverse subcategory and maps functors between discrete restriction categories to a functor having the same action on the partial inverses. That is, given $G : \mathbb{B} \to \mathbb{D}$, then:

$$\begin{aligned}
&\mathbf{INV}(G) : \mathbf{INV}(\mathbb{B}) \to \mathbf{INV}(\mathbb{D}) \\
&\mathbf{INV}(G)(A) = GA && \text{(all objects of } \mathbb{D} \text{ are in } Inv(\mathbb{D})\text{)} \\
&\mathbf{INV}(G)(f) = G(f) && \text{(restriction functors preserve partial inverse).}
\end{aligned}$$

Lemma 5.2.1. *The functor* **INV** *from the category of discrete restriction categories to the category of discrete inverse categories is full and faithful.*

Proof. To show fullness, we must show **INV** is surjective on hom-sets. Given a functor between two categories in the image of **INV**, i.e., $G : \mathbf{INV}(\mathbb{B}) \to \mathbf{INV}(\mathbb{D})$, construct a functor $H : \mathbb{B} \to \mathbb{D}$ as follows:

Action on objects: $H(A) = G(A)$,

Objects on maps: $H(f) = G(\langle f, 1\rangle)\pi_0$.

H is well defined as we know $\langle f, 1\rangle$ is an invertible map and therefore in the domain of G.

To see H is a functor:

$$
\begin{aligned}
H(1) &= G(\langle 1,1\rangle)\pi_0 = \Delta_{\mathbb{D}}\pi_0 = 1,\\
H(fg) &= G(\langle fg,1\rangle)\pi_0 = \langle G(fg),1\rangle\pi_0 = G(f)\langle G(g,1)\rangle\pi_0\\
&= \langle G(f),1\rangle\pi_0\langle G(g),1\rangle\pi_0 = G(\langle f,1\rangle)\pi_0 G(\langle g,1\rangle)\pi_0 = H(f)H(g).
\end{aligned}
$$

But on any invertible map, $H(f) = G(\langle f,1\rangle)\pi_0 = \langle G(f),1\rangle\pi_0 = G(f)$ and therefore $\mathbf{INV}(H) = G$, so $\mathbf{INV}$ is full.

Next, assume we have $F, G : \mathbb{B} \to \mathbb{D}$ with $\mathbf{INV}(F) = \mathbf{INV}(G)$. Considering $F(f)$ and $F(g)$, we know $F(\langle f,1\rangle) = G(\langle f,1\rangle)$ as $\langle f,1\rangle$ is invertible. Thus, as the functors preserve the product structure, we have

$$F(f) = F(\langle f,1\rangle)F(\pi_0) = G(\langle f,1\rangle)G(\pi_0) = G(f).$$

Thus, $\mathbf{INV}$ is faithful. $\square$

Next we define $\eta : \mathbb{X} \to \mathbf{INV}(\widetilde{\mathbb{X}})$ as the identity on objects functor where for a map f in $\mathbb{X}$, $\eta : f \mapsto (f{u_\otimes^r}^{(-1)}, 1)$. η is a functor as

$$
\begin{aligned}
\eta(1) &= ({u_\otimes^r}^{(-1)}, 1)\\
\eta(fg) &= (fg{u_\otimes^r}^{(-1)}, 1) \simeq (f{u_\otimes^r}^{(-1)}, 1)(g{u_\otimes^r}^{(-1)}, 1) = \eta f \eta g.
\end{aligned}
$$

Now, we may define the functor $F^\# : \widetilde{\mathbb{X}} \to \mathbb{D}$. Recall that $\mathbf{INV}(\mathbb{D})$ is a sub-category of $\mathbb{D}$ having the same objects, but only the invertible maps. Given a functor $F : \mathbb{X} \to \mathbf{INV}(\mathbb{D})$ define $F^\#$ as follows:

Objects: $F^\# : A \mapsto F(A) \in \mathbb{D}_o$

Arrows: $F^\# : (f, C) \mapsto F(f\pi_0) \in \mathbb{D}_m$

We will now show (5.8) is a universal diagram.

Lemma 5.2.2. *Diagram (5.8) above commutes and is a universal diagram. That is,*

$$\eta \mathbf{INV}(F^{\#}) = F$$

and $F^{\#}$ is unique.

Proof. Using our definitions above, given a map f in $\mathbb{X}$, then:

$$\begin{aligned}
\mathbf{INV}(F^{\#})(\eta(f) &= \mathbf{INV}(F^{\#})((f{u^r_\otimes}^{(-1)}, 1)) \\
&= F^{\#}((f{u^r_\otimes}^{(-1)}, 1)) \\
&= F(f{u^r_\otimes}^{(-1)}\pi_0) \\
&= F(f)
\end{aligned}$$

As η is identity on the objects, Diagram (5.8) commutes. The uniqueness of $F^{\#}$ follows immediately from Lemma 5.2.1, i.e., $\mathbf{INV}$ is faithful. □

Corollary 5.2.3. *The category $\widetilde{\mathbb{X}}$ and functor $\eta : \mathbb{X} \to \mathbf{INV}(\widetilde{\mathbb{X}})$ is a universal pair for the functor* $\mathbf{INV}$.

Proof. Immediate and Lemma 5.2.2. □

We may now proceed to show η is an isomorphism, but we need a lemma first showing that all invertible maps in $\widetilde{\mathbb{X}}$ are the equivalence class of the form $(f{u^r_\otimes}^{(-1)}, 1)$ for some f.

Lemma 5.2.4. *For any discrete inverse category $\mathbb{X}$, all invertible maps $(g, C) : A \to B$ in $\widetilde{\mathbb{X}}$ are in the equivalence class of $(f{u^r_\otimes}^{(-1)}, 1)$ for some $f : A \to B$.*

Proof. As (g, C) is invertible in $\widetilde{\mathbb{X}}$, the map $(g, C)^{(-1)} : B \to A$ exists. The map $(g, C)^{(-1)}$ must be in the equivalence class of some map $k : B \to A \otimes D$. By construction, the map $\overline{(k, D)}$ is in the equivalence class of the map $\bar{k}{u^r_\otimes}^{(-1)} : B \to B \otimes 1$ in $\mathbb{X}$. This means, in $\mathbb{X}$, there is an n such that

$$\begin{tikzcd} B \arrow[r, "k"] \arrow[drr, "\overline{k}u_\otimes^{r\,(-1)}"'] & A \otimes D \arrow[r, "g\otimes 1"] & B \otimes C \otimes D \arrow[d, dotted, "n"] \\ & & B \otimes 1 \end{tikzcd}$$

commutes.

Starting with $g : A \to B \otimes C$, construct the map f in $\mathbb{X}$ with the following diagram:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dddddr, dotted, "f"'] & B \otimes C \arrow[d, "\Delta\otimes 1"] \\ & B \otimes B \otimes C \arrow[d, "1\otimes k\otimes 1"] \\ & B \otimes A \otimes D \otimes C \arrow[d, "1\otimes g\otimes 1\otimes 1"] \\ & B \otimes B \otimes C \otimes D \otimes C \arrow[d, "\Delta^{(-1)}\otimes 1\otimes c_\otimes"] \\ & B \otimes C \otimes C \otimes D \arrow[d, "1\otimes\Delta^{(-1)}\otimes 1"] \\ & B \otimes C \otimes D \arrow[d, "nu_\otimes^l"] \\ & B. \end{tikzcd}$$

By its construction, $f : A \to B$ in $\mathbb{X}$ and $(fu_\otimes^{r\,(-1)}, 1)$ are in the same equivalence class as (g, C).

$\square$

Lemma 5.2.5. *The functor $\eta : \mathbb{X} \to \mathbf{INV}(\widetilde{\mathbb{X}})$ is an isomorphism.*

Proof. As η is an identity on objects functor, we need only show that it is full and faithful. Referring to Lemma 5.2.4 above, we immediately see that η is full. For faithful, if we assume $(fu_\otimes^{r\,(-1)}, 1)$ is equal in $\widetilde{\mathbb{X}}$ to $(gu_\otimes^{r\,(-1)}, 1)$. This means in $\mathbb{X}$, that $\overline{f} = \overline{g}$ and there is a $h \in B^{\Delta}_{\nabla}$ such that

$$\begin{tikzcd} & B \otimes 1 \arrow[dd, dotted, "h"] \\ A \arrow[ur, "fu_\otimes^{r\,(-1)}"] \arrow[dr, "gu_\otimes^{r\,(-1)}"'] & \\ & B \otimes 1. \end{tikzcd} \tag{5.9}$$

But as $h = (\Delta \otimes 1)(1 \otimes h)(\Delta^{(-1)} \otimes 1)$, and letting $\ell = {u^r_\otimes}^{(-1)} h u^r_\otimes$, Diagram ((5.9)) equates to $g = f\Delta(1 \otimes \ell)\Delta^{(-1)}$. But by Lemma 4.3.5(iv), $\Delta(1 \otimes \ell)\Delta^{(-1)} = \overline{\Delta(1 \otimes \ell)\Delta^{(-1)}}$. Setting $\Delta(1 \otimes \ell)\Delta^{(-1)}$ as k, we have $g = f\overline{k}$. This gives us:

$$g = f\overline{k} = \overline{fk}f = \overline{f\overline{k}}f = \overline{g}f = \overline{f}f = f.$$

This shows η is faithful and hence an isomorphism between $\mathbb{X}$ and $\mathbf{INV}(\widetilde{\mathbb{X}})$. □

Theorem 5.2.6. *The category of discrete inverse categories (objects are discrete inverse categories, maps are inverse tensor preserving functors) is equivalent to the category of discrete restriction categories (objects are discrete restriction categories, maps are the Cartesian restriction functors).*

Proof. Letting $\mathbf{T} : \mathbb{X} \to \widetilde{\mathbb{X}}$ be the functor that takes X to its $\widetilde{(\_)}$ completion, then from the above lemmas, we have shown that we have an adjoint:

$$(\eta, \varepsilon) : \mathbf{T} \vdash \mathbf{INV} : \mathrm{DInv} \to \mathrm{DCartRest}. \tag{5.10}$$

By Lemma 5.2.5 we know η is an isomorphism. But this means the functor $\mathbf{T}$ is full and faithful, as shown in, e.g., Proposition 2.2.6 of [15]. From lemma 5.2.1 we know that $\mathbf{INV}$ is full and faithful. But again by the previous reference, this means ε is an isomorphism. Thus, by Corollary 5.2.3 and Proposition 2.2.7 of [15] we have the equivalence of the two categories. □

Thus, we may now draw out the relationship between Cartesian restriction categories, discrete Cartesian restriction categories and discrete inverse categories:

$$\begin{tikzcd}
\mathrm{DCartRest} \arrow[r, hook] \arrow[d, leftrightarrow, "\cong"'] & \mathrm{CartRest} \arrow[l, bend right, "\top"'] \\
\mathrm{DInv} &
\end{tikzcd}$$

where the arrow from discrete Cartesian restriction categories to Cartesian restriction categories is the standard embedding and the reverse arrow picks out the discrete objects in the

Cartesian restriction category. Of course, the terminal object is always discrete, as noted in Example 3.10.7.

5.3 Examples of the $\widetilde{(\_)}$ construction

Example 5.3.1 (Different inverse products produce different $\widetilde{\mathbb{X}}$)**.**

Continuing from Example 4.3.4, recall the discrete category of 4 elements with two different tensors. Completing these gives two different lattices: The straight line lattice and the diamond semi-lattice. Below are the details of these constructions.

Recall $\mathbb{D}$ has four elements a, b, c and d, and there are two possible inverse product tensors, given in Table 4.1. (Repeated here for your convenience).

$\otimes$	a	b	c	d
a	a	a	a	a
b	a	b	**b**	b
c	a	**b**	c	c
d	a	b	c	d

$\odot$	a	b	c	d
a	a	a	a	a
b	a	b	**a**	b
c	a	**a**	c	c
d	a	b	c	d

Table 5.1: Two different inverse products on the same category.

Define Δ as the identity map. Then, for the first tensor, $\otimes$ of Table 4.1, $\widetilde{\mathbb{D}}$ has the following maps

$$\begin{array}{cccc}
a \xrightarrow{(id,a)\ \ (\equiv(id,b)\equiv(id,c)\equiv(id,d))} a, & a \xrightarrow{(id,a)} b, & a \xrightarrow{(id,a)} c, & a \xrightarrow{(id,a)} d \\
 & b \xrightarrow{(id,b)\ \ (\equiv(id,c)\equiv(id,d))} b, & b \xrightarrow{(id,b)} c, & b \xrightarrow{(id,b)} d \\
 & & c \xrightarrow{(id,c)\ \ (\equiv(id,d))} c, & c \xrightarrow{(id,c)} d \\
 & & & d \xrightarrow{(id,d)} d
\end{array}$$

resulting in the straight-line $(a \to b \to c \to d)$ lattice. The tensor in $\mathbb{D}$ becomes the meet and hence is a categorical product in $\widetilde{\mathbb{D}}$. Note that the only partial inverses in $\widetilde{\mathbb{D}}$ are the identity functions and that for all maps f, $\langle f, 1\rangle = id$.

With the second tensor, $\odot$ from Table 4.1, we have:

$$\begin{array}{llll}
a \xrightarrow{(id,a)\ (\equiv(id,b)\equiv(id,c)\equiv(id,d))} a, & a \xrightarrow{(id,a)} b, & a \xrightarrow{(id,a)} c, & a \xrightarrow{(id,a)} d \\
 & & b \xrightarrow{(id,b)\ (\equiv(id,d))} b, & b \xrightarrow{(id,b)} d \\
 & & c \xrightarrow{(id,c)\ (\equiv(id,d))} c, & c \xrightarrow{(id,c)} d \\
 & & & d \xrightarrow{(id,d)} d
\end{array}$$

resulting in the "diamond" lattice, $\begin{tikzcd} & b \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & d \\ & c \arrow[ur] & \end{tikzcd}$. Once again, the tensor in $\mathbb{D}$ becomes the meet.

Example 5.3.2 (Lattice completion.)**.** Suppose we have a set together with an idempotent, commutative, associative operation $\wedge$ on the set, giving us a lattice, $\mathbb{L}$. Further suppose the set is partially ordered via $\leq$ with the order being compatible with $\wedge$.

Then, we may create a pullback square for any $x' \leq x,\ y' \leq x$ with

$$\begin{tikzcd}
 & x & \\
x' \arrow[ur, "\leq"] & & y' \arrow[ul, "\leq"'] \\
 & x' \wedge y'. \arrow[ul, "\leq"] \arrow[ur, "\leq"'] &
\end{tikzcd}$$

Considering $\mathbb{L}$ as a category, we see that all maps are monic and therefore, we may create a partial map category $\mathrm{Par}(\mathbb{L}, \mathcal{M})$ where the stable system of monics are all the maps.

Then $\mathrm{Par}\widetilde{(\mathbb{L}, \mathcal{M})}$ becomes the completion of the lattice over $\wedge$.

Example 5.3.3 ($\widetilde{\textsc{Pinj}}$ is $\textsc{Par}$)**.** Noting that the objects of both $\textsc{Pinj}$ and $\textsc{Par}$ are sets, we simply need to show that any partial function is in the equivalence class of some f, a map in $\textsc{Pinj}$.

Suppose we are given $g : A \to B = \{(a, b) | a \in A, b \in B\}$, a partial function in sets. Of course, if it is a partial injective function, then g is in the equivalence class of $(g, \{*\})$ and we are done. If it is not injective, that means there are one or more elements of B which appear in the left hand element of g multiple times. Construct a new function h as follows:

$$h \doteq \{(a,(b,a))|(a,b) \in g\} \tag{5.11}$$

By its definition, $h : A \to B \otimes A$ is injective, $(h, A) : A \to B$ coincides with g and therefore we see that using the $\widetilde{(\_)}$ construction on Pinj results in Par.

Chapter 6

Disjointness in Inverse Categories

In this chapter we will explore coproduct like structure in inverse categories. We start by showing, similarly to the product, that having coproducts is too restrictive a notion for inverse categories: An inverse category with a coproduct is a preorder. Nonetheless it is possible to define coproduct like structures in an inverse category. To introduce this structure we define a "disjointness" relation between parallel maps of an inverse category and whence a "disjoint join" of disjoint maps. We will then show how a tensor product satisfying some specific conditions gives rise to both a disjointness relation and a disjoint join. Such a tensor product provides the proposed replacement for "coproducts" in an inverse category.

6.1 Coproducts in inverse categories

A restriction category can have coproducts and an initial object. For example, Par (sets and partial functions) have coproducts.

Definition 6.1.1. In a restriction category $\mathbb{R}$, a coproduct is a *restriction coproduct* when the embeddings $\amalg_1$ and $\amalg_2$ are total.

Lemma 6.1.2. *A restriction coproduct in $\mathbb{R}$ satisfies:*

(i) $\overline{f+g} = \overline{f} + \overline{g}$ *which means* $+$ *is a restriction functor.*

(ii) $\nabla : A + A \to A$ *is total.*

(iii) $? : 0 \to A$ *is total, where* 0 *is the initial object in the restriction category.*

Proof.

(i) Totality of $\amalg_1$ and $\amalg_2$.

(ii) $+$ **is a restriction functor.** Consider the diagram:

$$\begin{tikzcd}
A \arrow[rr, "f"'] \arrow[dr, "\amalg_1"'] & & A' \arrow[dr, "\amalg'_1"] & \\
& A+B \arrow[rr, dotted, "f+g"] & & A'+B' \\
B \arrow[rr, "g"'] \arrow[ur, "\amalg_2"] & & B'. \arrow[ur, "\amalg'_2"'] &
\end{tikzcd}$$

In order to show $\overline{f+g} = \overline{f}+\overline{g}$, it suffices to show that $\amalg_1\overline{f+g} = \amalg_1(\overline{f}+\overline{g}) = \overline{f}\amalg_1$.

$$\begin{aligned}
\amalg_1\overline{f+g} &= \overline{\amalg_1(f+g)}\amalg_1 && [\mathbf{R.4}] \\
&= \overline{f\amalg'_1}\amalg_1 && \text{coproduct diagram} \\
&= \overline{f\overline{\amalg'_1}}\amalg_1 && \text{Lemma 3.1.3}[(iii)] \\
&= \overline{f}\amalg_1 && \amalg'_1 \text{ total.}
\end{aligned}$$

(iii) $\nabla : A+A \to A$ **is total.** By the definition of ∇ ($= \langle 1|1\rangle$) and the co-product, the following diagram commutes,

$$\begin{tikzcd}
& A+A \arrow[d, "\nabla"'] & \\
A \arrow[ur, "\amalg_1"] \arrow[r, equal] & A \arrow[r, equal] & A \arrow[ul, "\amalg_2"']
\end{tikzcd}$$

resulting in:

$$\amalg_1\overline{\nabla} = \overline{\amalg_1\nabla}\amalg_1 = \overline{1}\amalg_1 = \amalg_1.$$

Similarly, $\amalg_2\overline{\nabla} = \amalg_2$, hence, the restriction of ∇ is 1 and therefore ∇ is total.

(iv) $? : 0 \to A$ **is total.** This follows from

$$\begin{tikzcd}
0 \arrow[r, "\amalg_2"] \arrow[dr, "?"'] & A+0 \arrow[d, equal] \\
& A
\end{tikzcd}$$

so ? can be defined as the total coproduct injection.

□

Recall that when an object is both initial and terminal, it is referred to as a zero object and denoted as $\mathbf{0}$. This gives rise to the zero map $\mathbf{0}_{A,B} : A \to \mathbf{0} \to B$ between any two objects.

Definition 6.1.3. Given a restriction category $\mathbb{X}$ with a zero object, then $\mathbf{0}$ is a *restriction zero* when for each object A in $\mathbb{X}$ we have $\overline{\mathbf{0}_{A,A}} = \mathbf{0}_{A,A}$.

Lemma 6.1.4 (Cockett-Lack [19], Lemma 2.7)**.** *For a restriction category $\mathbb{X}$, the following are equivalent:*

(i) *$\mathbb{X}$ has a restriction zero;*

(ii) *$\mathbb{X}$ has an initial object 0 and terminal object 1 and each initial map z_A is a restriction monic;*

(iii) *$\mathbb{X}$ has a terminal object 1 and each terminal map t_A is a restriction retraction.*

6.1.1 Inverse categories with restriction coproducts

Proposition 6.1.5. *An inverse category $\mathbb{X}$ with restriction coproducts is a pre-order.*

Proof. By Lemma 6.1.2, we know ∇ is total and therefore $\nabla\nabla^{(-1)} = 1$. From the coproduct diagrams, we have $\amalg_1\nabla = 1$ and $\amalg_2\nabla = 1$. But this gives us $\nabla^{(-1)}{\amalg_1}^{(-1)} = (\amalg_1\nabla)^{(-1)} = 1$ and similarly $\nabla^{(-1)}{\amalg_2}^{(-1)} = 1$. Hence, $\nabla^{(-1)} = \amalg_1$ *and* $\nabla^{(-1)} = \amalg_2$.

This means for parallel maps $f, g : A \to B$, we have

$$f = \amalg_1\langle f|g\rangle = \nabla^{(-1)}\langle f|g\rangle = \amalg_2\langle f|g\rangle = g$$

and therefore $\mathbb{X}$ is a pre-order. □

6.2 Disjointness in an inverse category

In the following, we will add a relation, *disjointness* and a construction, *disjoint join* on parallel maps in an inverse category with a restriction zero and zero maps. The disjoint join is evocative of the join as defined in Section 3.3. We will begin by defining disjointness on parallel maps and then show this is equivalent to a definition of disjointness on the restriction idempotents.

From this point forward in the thesis, we will work with a number of relations and operations on parallel pairs of maps. Suppose we have a relation $\Diamond$ between maps $f, g : B \to C$, i.e., $f \Diamond g$. We will refer to $\Diamond$ as *stable* whenever given $h : A \to B$, then $hf \Diamond hg$. We will refer to $\Diamond$ as *universal* whenever given $k : C \to D$, then $fk \Diamond gk$.

Definition 6.2.1. In an inverse category $\mathbb{X}$ with zero maps, the relation $\perp$ between two parallel maps $f, g : A \to B$ is called a *disjointness relation* when it satisfies the following properties:

- [**Dis.1**] For all $f : A \to B,\ f \perp 0$; (Zero is disjoint to all maps)
- [**Dis.2**] $f \perp g$ implies $\overline{f} g = 0$; (Disjoint maps have no intersection)
- [**Dis.3**] $f \perp g,\ f' \leq f,\ g' \leq g$ implies $f' \perp g'$; (Disjointness is down closed)
- [**Dis.4**] $f \perp g$ implies $g \perp f$; (Symmetric)
- [**Dis.5**] $f \perp g$ implies $hf \perp hg$; (Stable)
- [**Dis.6**] $f \perp g$ implies $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$; (Closed under range and restriction)
- [**Dis.7**] $\overline{f} \perp \overline{g},\ \hat{h} \perp \hat{k}$ implies $fh \perp gk$ (Determined by restriction/range).

When $f \perp g$, we will say f is *disjoint* from g.

Lemma 6.2.2. *In 6.2.1, given* [***Dis.1-5***]*, we may replace* [***Dis.6***] *and* [***Dis.7***] *by:*

$$[\textbf{\textit{Dis.6'}}] \quad f \perp g \text{ if and only if } \overline{f} \perp \overline{g} \text{ and } \hat{f} \perp \hat{g}.$$

Proof. Given [**Dis.6**] and [**Dis.7**], the *only if* direction of [**Dis.6'**] is immediate. To show the *if* direction, assume $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$. This also means that $\overline{\overline{f}} \perp \overline{\overline{g}}$. Then, by [**Dis.7**], $\overline{f}f \perp \overline{g}g$ and therefore $f \perp g$, showing [**Dis.6**] and [**Dis.7**] imply [**Dis.6'**].

Conversely, assume we are given [**Dis.6'**]. Then, [**Dis.6**] follows immediately. To show [**Dis.7**], assume we have $\overline{f} \perp \overline{g}$, $\hat{h} \perp \hat{k}$. As $\overline{fh} \leq \overline{f}$ and $\overline{gk} \leq \overline{g}$, by [**Dis.3**], we know that $\overline{fh} \perp \overline{gk}$. Similarly, $\widehat{fh} \leq \hat{h}$ and $\widehat{gk} \leq \hat{k}$, giving us $\widehat{fh} \perp \widehat{gk}$. Then, from [**Dis.6'**] we may conclude $fh \perp gk$, showing [**Dis.7**] holds. □

Lemma 6.2.3. *In an inverse category $\mathbb{X}$ with $\perp$ a disjointness relation:*

(i) $f \perp g$ if and only if $f^{(-1)} \perp g^{(-1)}$;

(ii) $f \perp g$ implies $fh \perp gh$ (Universal);

(iii) $f \perp g$ implies $f\hat{g} = 0$;

(iv) if m, n are monic, then $fm \perp gn$ implies $\overline{f} \perp \overline{g}$;

(v) if m, n are monic, then $m^{(-1)}f \perp n^{(-1)}g$ implies $\hat{f} \perp \hat{g}$.

Proof.

(i) Assume $f \perp g$. By [**Dis.6**], we have $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$. Since $\hat{f} = \overline{f^{(-1)}}$ and $\overline{f} = \widehat{f^{(-1)}}$, this means $\overline{f^{(-1)}} \perp \overline{g^{(-1)}}$ and $\widehat{f^{(-1)}} \perp \widehat{g^{(-1)}}$. By [**Dis.6'**] from Lemma 6.2.2, we have $f^{(-1)} \perp g^{(-1)}$. The converse follows with a similar argument.

(ii) Assume $f \perp g$. By (i), we have $f^{(-1)} \perp g^{(-1)}$. By [**Dis.5**], $h^{(-1)}f^{(-1)} \perp h^{(-1)}g^{(-1)}$, giving us $(fh)^{(-1)} \perp (gh)^{(-1)}$. Applying (i), we now have $fh \perp gh$.

(iii) Assume $f \perp g$. From (i) and reflexivity, we know that $g^{(-1)} \perp f^{(-1)}$ and therefore $\overline{g^{(-1)}}f^{(-1)} = \hat{g}f^{(-1)} = 0$. However, in an inverse category, $0^{(-1)} = 0$ and therefore $0 = (\hat{g}f^{(-1)})^{(-1)} = f\hat{g}^{(-1)} = f\hat{g}$.

(iv) Assume $fm \perp gn$ where m, n are monic. By [**Dis.6**], this gives us $\overline{fm} \perp \overline{gn}$. By Lemma 3.1.3, $\overline{fm} = \overline{f\overline{m}} = \overline{f1} = \overline{f}$ and therefore $\overline{f} \perp \overline{g}$.

(v) By assumption, we have $m^{(-1)}f \perp n^{(-1)}g$ and therefore $f^{(-1)}m \perp g^{(-1)}n$. By (iv), this means $\overline{f^{(-1)}} \perp \overline{g^{(-1)}}$ and hence $\hat{f} \perp \hat{g}$.

□

We may equivalently define a disjointness relation by a relation on the restriction idempotents, $\mathcal{O}(A)$.

Definition 6.2.4. Given an inverse category $\mathbb{X}$, a relation $\perp_A \subseteq \mathcal{O}(A) \times \mathcal{O}(A)$ for each $A \in \text{ob}(\mathbb{X})$, is an *open disjointness* relation when for all $e, e' \in \mathcal{O}(A)$

[$\mathcal{O}$**dis.1**] $1 \perp_A 0$; (Zero disjoint to all idempotents)

[$\mathcal{O}$**dis.2**] $e \perp_A e'$ implies $e' \perp_A e$; (Symmetric)

[$\mathcal{O}$**dis.3**] $e \perp_A e'$ implies $ee' = 0$; (Disjoint implies no intersection)

[$\mathcal{O}$**dis.4**] $e \perp_A e'$ implies $\overline{fe} \perp_B \overline{fe'}$ for all $f : B \to A$; (Stable)

[$\mathcal{O}$**dis.5**] $e \perp_A e'$ implies $\widehat{eg} \perp_C \widehat{e'g}$ for all $g : A \to C$; (Universal)

[$\mathcal{O}$**dis.6**] $e \perp_A e',\ e_1 \leq e,\ e_1' \leq e'$ implies $e_1 \perp_A e_1'$ (Down closed).

We will normally write $\perp$ rather than $\perp_A$ where the object is clear.

Proposition 6.2.5. *If $\perp$ is a disjointness relation in $\mathbb{X}$, then $\perp = \perp \cap (\cup_{A \in \mathbb{X}} \mathcal{O}(A) \times \mathcal{O}(A))$, the restriction of $\perp$ to the restriction idempotents, is an open disjointness relation.*

Proof.

[$\mathcal{O}$**dis.1**] This follows immediately from [**Dis.1**] by taking $f = 1$.

[$\mathcal{O}$**dis.2**] Reflexivity follows directly from [**Dis.4**].

[$\mathcal{O}$**dis.3**] By [**Dis.2**], $0 = \overline{e}e' = ee'$.

[$\mathcal{O}$**dis.4**] Given $e \perp e'$, we have $fe \perp fe'$ by [**Dis.5**]. Then, by [**Dis.6**] we may conclude $\overline{fe} \perp \overline{fe'}$.

[$\mathcal{O}$**dis.5**] This follows from the above item, using $g^{(-1)}$ for f. This means we have $\overline{g^{(-1)}e} \perp \overline{g^{(-1)}e'}$. But this gives us $\overline{(eg)^{(-1)}} \perp \overline{(e'g)^{(-1)}}$. Recalling from Lemma 4.1.3 that $\hat{k} = \overline{k^{(-1)}}$, we may conclude $\widehat{eg} \perp \widehat{e'g}$.

[$\mathcal{O}$**dis.6**] Assuming $e \perp e'$ and $e_1 \leq e,\ e_1' \leq e'$, by [**Dis.3**], $e_1 \perp e_1'$.

Therefore, $\underline{\perp} = \perp \cap (\cup_{A\in\mathbb{X}} \mathcal{O}(A) \times \mathcal{O}(A))$ acts as an open disjointness relation on $\mathcal{O}(A)^2$.

$\square$

For the converse, if $\underline{\perp}$ is an open disjointness relation in $\mathbb{X}$, then we may define a relation ${}_A\perp_B \subseteq \cup_{A,B\in\mathbb{X}} \mathbb{X}(A,B) \times \mathbb{X}(A,B)$ on parallel maps by

$$f\ {}_A\perp_B\ g \iff \overline{f} \,\underline{\perp}\, \overline{g} \text{ and } \hat{f} \,\underline{\perp}\, \hat{g}.$$

Note the use of the $\hat{f} \,\underline{\perp}\, \hat{g}$. By [**RR.1**], we know $\overline{\hat{f}} = \hat{f}$ and therefore it is a restriction idempotent in $\mathcal{O}(B)$.

We then have:

Proposition 6.2.6. *If $\underline{\perp}$ is an open disjointness relation in $\mathbb{X}$, then*

$$f\ {}_A\perp_B\ g \iff \overline{f} \,\underline{\perp}\, \overline{g} \text{ and } \hat{f} \,\underline{\perp}\, \hat{g}$$

is a disjointness relation in $\mathbb{X}$.

Proof.

[**Dis.1**] We need to show $f \perp 0$ for any f. We know that $1 \,\underline{\perp}\, 0$ and therefore $\overline{f} \,\underline{\perp}\, 0$ and $\hat{f} \,\underline{\perp}\, 0$, as $\overline{f} \leq 1$ and $\hat{f} \leq 1$. This gives us $f \perp 0$.

[**Dis.2**] Assume $f \perp g$, i.e., $\overline{f} \,\underline{\perp}\, \overline{g}$. Then, $\overline{f}g = \overline{f}\,\overline{g}g = 0g = 0$.

[**Dis.3**] We are given $f \perp g$, $f' \leq f$ and $g' \leq g$. By Lemma 3.2.2[(i)] $\overline{f'} \leq \overline{f}$ and $\overline{g'} \leq \overline{g}$. Then, by [$\mathcal{O}$**dis.6**], as $\overline{f} \mathrel{\underline{\perp}} \overline{g}$ we have $\overline{f'} \mathrel{\underline{\perp}} \overline{g'}$. By Lemma 3.6.3[(ii)], we have $\widehat{f'} \leq \hat{f}$ and $\widehat{g'} \leq \hat{g}$. Then, by [$\mathcal{O}$**dis.6**], as $\hat{f} \mathrel{\underline{\perp}} \hat{g}$ we have $\widehat{f'} \mathrel{\underline{\perp}} \widehat{g'}$. This means $f' \perp g'$.

[**Dis.4**] Reflexivity of $\perp$ follows immediately from the reflexivity of $\underline{\perp}$.

[**Dis.5**] Assume $f \perp g$, i.e., $\overline{f} \mathrel{\underline{\perp}} \overline{g}$ and $\hat{f} \mathrel{\underline{\perp}} \hat{g}$. Then we have $\overline{hf} \mathrel{\underline{\perp}} \overline{hg}$ by [$\mathcal{O}$**dis.4**]. By Lemma 3.6.3[(i)] we have $\widehat{hf} \leq \hat{f}$ and $\widehat{hg} \leq \hat{g}$. Therefore we have $\widehat{hf} \mathrel{\underline{\perp}} \widehat{hg}$ by [$\mathcal{O}$**dis.6**] and therefore $hf \perp hg$.

[**Dis.6**] Assume $f\,{}_{A}{\perp}_{B}\,g$. Then both $\overline{f} \mathrel{\underline{\perp}} \overline{g}$ and $\hat{f} \mathrel{\underline{\perp}} \hat{g}$. By Lemma 3.6.2 we know for any map h, $\hat{\overline{h}} = \overline{h}$ and by Lemma 3.1.3 we have $\overline{\overline{h}} = \overline{h}$. Thus, we have both $\overline{\overline{f}} \mathrel{\underline{\perp}} \overline{\overline{g}}$ and $\hat{\overline{f}} \mathrel{\underline{\perp}} \hat{\overline{g}}$ and therefore $\overline{f}\,{}_{A}{\perp}_{A}\,\overline{g}$. Similarly for any map h, [**RR.1**] gives $\overline{\hat{h}} = \hat{h}$ and Lemma 3.6.2 gives $\hat{\hat{h}} = \hat{h}$. Thus, we have $\overline{\hat{f}} \mathrel{\underline{\perp}} \overline{\hat{g}}$ and $\hat{\hat{f}} \mathrel{\underline{\perp}} \hat{\hat{g}}$ giving us $\hat{f}\,{}_{B}{\perp}_{B}\,\hat{g}$.

[**Dis.7**] We assume $\overline{f} \perp \overline{g}$ and $\hat{h} \perp \hat{k}$. Then, we have $\overline{f} \mathrel{\underline{\perp}} \overline{g}$ and $\hat{h} \mathrel{\underline{\perp}} \hat{k}$. By Lemma 3.2.2[(ii)], we have $\overline{fh} \leq \overline{f}$ and $\overline{gk} \leq \overline{g}$. Therefore, $\overline{fh} \mathrel{\underline{\perp}} \overline{gk}$ by [$\mathcal{O}$**dis.6**]. By Lemma 3.6.3[(i)], $\widehat{fh} \leq \hat{h}$ and $\widehat{gk} \leq \hat{k}$, giving us $\widehat{fh} \mathrel{\underline{\perp}} \widehat{gk}$ also by [$\mathcal{O}$**dis.6**]. This means $fh \perp gk$.

$\square$

Theorem 6.2.7. *To give a disjointness relation $\perp$ on $\mathbb{X}$ is to give an open disjointness relation $\underline{\perp}$ on $\mathbb{X}$.*

Proof. We must show there is a bijection between disjointness relations and open disjointness relations. That is, give a disjointness relation $\perp$, we know we may generate an open disjointness relation, $\underline{\perp}$, from it. We then need to show that the disjointness relation generated from $\underline{\perp}$ is in fact $\perp$.

Suppose we are given the disjointness relation $\perp$. By Proposition 6.2.5, this is an open disjointness $\underline{\perp} = \perp \cap (\cup_{A \in \mathbb{X}} \mathcal{O}(A) \times \mathcal{O}(A))$.

By Proposition 6.2.6, ${}_{A}{\perp}_{B}$ defined by

$$f\,{}_{A}{\perp}_{B}\,g \iff \overline{f} \mathrel{\underline{\perp}} \overline{g} \text{ and } \hat{f} \mathrel{\underline{\perp}} \hat{g}.$$

is a disjointness relation on $\mathbb{X}$.

Assume $f \perp g$. We have $\overline{f} \underline{\perp} \overline{g}$ and $\hat{f} \underline{\perp} \hat{g}$ by [**Dis.6**] and Proposition 6.2.5. Then by its definition, we have $f\,{}_{A}{\perp}_{B}\,g$.

Assume $f\,{}_{A}{\perp}_{B}\,g$. This means we must have had $\overline{f} \underline{\perp} \overline{g}$ and $\hat{f} \underline{\perp} \hat{g}$. Therefore $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$. By Proposition 6.2.3, we have $f \perp g$.

We have shown $f \perp g \iff f\,{}_{A}{\perp}_{B}\,g$. We may also conclude that if we started with an open disjointness relation $\underline{\perp}$ and used it to construct ${}_{A}{\perp}_{B}$, then the relation ${}_{A}{\perp}_{B} \cap (\cup_{A\in\mathbb{X}} \mathcal{O}(A) \times \mathcal{O}(A))$ would again be $\underline{\perp}$.

Hence we have shown a bijection between disjointness relations and open disjointness relations on an inverse category $\mathbb{X}$.

$\square$

Disjointness is additional structure on a restriction category, i.e., it is possible to have more than one disjointness relation on the category. To see this, consider the trivial disjointness relation, where $f \perp_0 g$ if and only if $f = 0$ or $g = 0$. As $\overline{0} = 0 = \hat{0}$, $f \leq 0 \iff f = 0$ and $h0 = 0$ for any map h, axioms [**Dis.1**] through [**Dis.7**] are immediately satisfied. $\perp_0$ is definable on any inverse category with zero maps, hence is definable on PINJ. However, PINJ does have a more useful disjointness relation:

Example 6.2.8 (PINJ has a disjointness relation)**.** Consider the inverse category PINJ, introduced in Example 3.1.8 and Example 4.3.2. We note the restriction zero is the empty set, $\emptyset$, with initial and terminal maps being $\emptyset$ and therefore $\mathbf{0}_{A,B} = \emptyset$.

We may define the disjointness relation $\perp$ by $f \perp g$ if and only if $\overline{f} \cap \overline{g} = \emptyset$ and $\hat{f} \cap \hat{g} = \emptyset$. It is then reasonably straightforward to verify [**Dis.1**] through [**Dis.7**]. For example, take [**Dis.7**]:

Proof. We are given $\overline{f} \perp \overline{g}$ and $\hat{h} \perp \hat{k}$. This means

$$\overline{f} \cap \overline{g} = \emptyset \text{ and } \hat{h} \cap \hat{k} = \emptyset.$$

As discussed in Example 3.4.4, we have $\overline{mn} \subseteq \overline{m}$ and that $\widehat{mn} \subseteq \hat{n}$. Hence we have

$$\overline{fh} \cap \overline{gk} \subseteq \overline{f} \cap \overline{g} = \emptyset$$
$$\widehat{fh} \cap \widehat{gk} \subseteq \hat{h} \cap \hat{k} = \emptyset.$$

Therefore, $fh \perp gk$. □

We see that we may define $f \perp_0 g$ on Pinj as well: $f \perp_0 g$ if and only if one of f or g is the restriction $\mathbf{0}$, $\emptyset$. As $\mathbf{0} = \overline{\mathbf{0}} = \hat{\mathbf{0}} = h\mathbf{0} = \mathbf{0}k$, all of the seven disjointness axioms are easily verifiable.

Although disjointness is additional structure on a restriction category, one can use the disjointness structure of a base category (or categories) to define a disjointness structure on the product category.

Lemma 6.2.9. *If $\mathbb{X}$ and $\mathbb{Y}$ are inverse categories with restriction zeros and respective disjointness relations $\perp$ and $\perp'$, then we may construct a disjointness relation $\perp_\times$ on $\mathbb{X} \times \mathbb{Y}$.*

Proof. The restriction, the inverse and the restriction zero on the product category are defined pointwise.

- If (f, g) is a map in $\mathbb{X} \times \mathbb{Y}$, then $(f, g)^{(-1)} = (f^{(-1)}, g^{(-1)})$;
- If (f, g) is a map in $\mathbb{X} \times \mathbb{Y}$, then $\overline{(f, g)} = (\overline{f}, \overline{g})$;
- The map $(\mathbf{0}_X, \mathbf{0}_Y)$ is the restriction zero in $\mathbb{X} \times \mathbb{Y}$.

Following this pattern, for (f, g) and (h, k) maps in $\mathbb{X} \times \mathbb{Y}$, $(f, g) \perp_\times (h, k)$ iff $f \perp h$ and $g \perp' k$.

Verifying the disjointness axioms is straightforward, we show axioms 2 and 5. Proofs of the others are similar.

[**Dis.2**] : Given $(f, g) \perp_\times (h, k)$, we have $\overline{(f, g)}(h, k) = (\overline{f}, \overline{g})(h, k) = (\overline{f}h, \overline{g}k) = (\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

[**Dis.5**] : We are given $(f,g) \perp_\times (h,k)$. Consider the map $z = (x,y)$ in $\mathbb{X} \times \mathbb{Y}$. We know that $xf \perp xh$ and $yg \perp yk$, therefore we have $z(f,g) = (xf, yg) \perp_\times (xh, yk) = z(h,k)$.

$\square$

6.3 Disjoint joins

We now consider additional structure on the inverse category, dependent upon the disjointness relation. Note that while we have worked with binary disjointness up to this point, one may extend the concept to sets of maps simply by considering disjointness pairwise: $\perp \{f_1, f_2, \ldots, f_n\} \doteq f_i \perp f_k$ whenever $i \neq j$.

Definition 6.3.1. An inverse category with $\mathbb{X}$ with a restriction 0 and a disjointness relation $\perp$ has *disjoint joins* when there is a binary operator on disjoint parallel maps:

$$\frac{f : A \to B,\ g : A \to B,\ f \perp g}{f \sqcup g : A \to B}$$

such that the following hold:

[**DJ.1**] $f \leq f \sqcup g$ and $g \leq f \sqcup g$;

[**DJ.2**] $f \leq h,\ g \leq h$ and $f \perp g$ implies $f \sqcup g \leq h$;

[**DJ.3**] $h(f \sqcup g) = hf \sqcup hg$. (Stable)

[**DJ.4**] $\perp \{f, g, h\}$ if and only if $f \perp (g \sqcup h)$.

The binary operator, $\sqcup$, is called the *disjoint join*.

Given a specific disjointness relation on a category, there is only one disjoint join:

Lemma 6.3.2. *Suppose $\mathbb{X}$ in an inverse category with a disjointness relation $\perp$, then if $\sqcup$ and $\square$ are disjoint joins for $\perp$ then $\sqcup = \square$.*

Proof. [**DJ.1**] gives us:

$$f, g \leq f \sqcup g \text{ and } f, g \leq f \,\square\, g.$$

By [**DJ.2**], we may therefore conclude $f \sqcup g \leq f \,\square\, g$ and $f \,\square\, g \leq f \sqcup g$, hence $f \sqcup g = f \,\square\, g$. $\square$

Lemma 6.3.3. *In an inverse category with disjoint joins, the disjoint join respects the restriction and is universal. Additionally, it is a partial associative and commutative operation, with identity* $\mathbf{0}$*. That is, the following hold:*

(i) $\overline{f \sqcup g} = \overline{f} \sqcup \overline{g}$;

(ii) $(f \sqcup g)k = fk \sqcup gk$ *(Universal);*

(iii) $f \perp g,\ g \perp h,\ f \perp h$ *implies that* $(f \sqcup g) \sqcup h = f \sqcup (g \sqcup h)$;

(iv) $f \perp g$ *implies* $f \sqcup g = g \sqcup f$;

(v) $f \sqcup \mathbf{0} = f$.

Proof.

(i) As $\overline{f}, \overline{g} \leq \overline{f \sqcup g}$, we immediately have $\overline{f} \sqcup \overline{g} \leq \overline{f \sqcup g}$. To show the other direction, consider

$$\begin{aligned}
\overline{f}(\overline{f} \sqcup \overline{g})(f \sqcup g) &= (\overline{f}\,\overline{f} \sqcup \overline{f}\overline{g})(f \sqcup g) && [\textbf{DJ.3}] \\
&= \overline{f}(f \sqcup g) && \text{Lemma 3.1.3, } [\textbf{Dis.2}] \\
&= f.
\end{aligned}$$

Hence, we have $f \leq (\overline{f} \sqcup \overline{g})(f \sqcup g)$ and similarly, so is g. By [**DJ.2**] and that $\overline{f} \sqcup \overline{g}$ is a restriction idempotent, we then have

$$f \sqcup g \leq (\overline{f} \sqcup \overline{g})(f \sqcup g) \leq f \sqcup g$$

and therefore $f \sqcup g = (\overline{f} \sqcup \overline{g})(f \sqcup g)$. By Lemma 3.2.2, $\overline{f \sqcup g} \leq \overline{f} \sqcup \overline{g}$ and so $\overline{f \sqcup g} = \overline{f} \sqcup \overline{g}$.

(ii) First consider when f, g and k are restriction idempotents, say e_0, e_1 and e_2. Then, we have $(e_0 \sqcup e_1)e_2 = e_2(e_0 \sqcup e_1) = e_2 e_0 \sqcup e_2 e_1 = e_0 e_2 \sqcup e_1 e_2$. Next, note that for general f, g, h, we have $fk \sqcup gk \leq (f \sqcup g)k$ as both $fk, gk \leq (f \sqcup g)k$. By Lemma 3.2.2, we need only show that their restrictions are equal:

$$
\begin{aligned}
\overline{(f \sqcup g)k} &= \overline{\overline{f \sqcup g}(f \sqcup g)k} && [\mathbf{R.1}] \\
&= \overline{f \sqcup g}\,\overline{(f \sqcup g)k} && [\mathbf{R.3}] \\
&= (\overline{f} \sqcup \overline{g})\overline{(f \sqcup g)k} && \text{previous item} \\
&= \overline{f}\,\overline{(f \sqcup g)k} \sqcup \overline{g}\,\overline{(f \sqcup g)k} && \text{idempotent universal} \\
&= \overline{\overline{f}(f \sqcup g)k} \sqcup \overline{\overline{g}(f \sqcup g)k} && [\mathbf{R.3}] \\
&= \overline{fk} \sqcup \overline{gk} \\
&= \overline{fk \sqcup gk}.
\end{aligned}
$$

Therefore, as the restrictions are equal, we have shown $(f \sqcup g)k = fk \sqcup gk$.

(iii) *Associativity*: Note that [**DJ.4**] shows that both sides of the equation exist. To show they are equal, we show that they are less than or equal to each other. From the definitions, we know that $f \sqcup g, h \leq (f \sqcup g) \sqcup h$, which also means $f, g \leq (f \sqcup g) \sqcup h$. Similarly, $g \sqcup h \leq (f \sqcup g) \sqcup h$ and then $f \sqcup (g \sqcup h) \leq (f \sqcup g) \sqcup h$. Conversely, $f, g, h \leq f \sqcup (g \sqcup h)$ and therefore $(f \sqcup g) \sqcup h \leq f \sqcup (g \sqcup h)$ and both sides are equal.

(iv) *Commutativity*: Note first that both f and g are less than or equal to both $f \sqcup g$ and $g \sqcup f$, by [**DJ.1**]. By [**DJ.2**], we have $f \sqcup g \leq g \sqcup f$ and $g \sqcup f \leq f \sqcup g$ and we may conclude $f \sqcup g = g \sqcup f$.

(v) *Identity*: By [**DJ.1**], $f \leq f \sqcup \mathbf{0}$. As $\mathbf{0} \leq f$ and $f \leq f$, by [**DJ.2**], $f \sqcup \mathbf{0} \leq f$ and we have $f = f \sqcup \mathbf{0}$.

□

Note that the previous lemma and proof of associativity allows a simple inductive argument which shows that having binary disjoint joins extends to disjoint joins of an arbitrary finite collection of disjoint maps.

Recalling our notation for disjointness of a set of maps, $\sqcup\{f_i\}$ will mean the disjoint join of all maps f_i, i.e., $f_1 \sqcup f_2 \sqcup \cdots \sqcup f_n$.

Lemma 6.3.4. *In an inverse category $\mathbb{X}$ with disjoint joins,*

(i) $\perp \{f_i\}$ *if and only if* $\sqcup\{f_i\}$ *is defined,*

(ii) if $f_i, g_j : A \to B$ *and* $\perp_{i\in I} \{f_i\}$ *and* $\perp_{j\in J} \{g_j\}$*, then* $\sqcup_{i\in I}\{f_i\} \perp \sqcup_{j\in J}\{g_j\}$ *if and only* $f_i \perp g_j$ *for all* $i \in I$ *and* $j \in J$*.*

Proof. For (i), using [**Dj.4**], proceed as in the proof of Lemma 6.3.3[(iii)], inducting on n.

To show (ii), first assume $\sqcup\{f_i\} \perp \sqcup\{g_j\}$. By [**Dj.4**] and associativity, we have $\sqcup\{f_i\} \perp g_j$ for each j. Using the reflexivity of $\perp$, [**Dj.4**] and associativity, we have $f_i \perp g_j$ for each i and j.

Next, assume $f_i \perp g_j$ for each i and j. Then by [**Dj.4**] and associativity, $f_i \perp \sqcup\{g_j\}$ for each i. Applying [**Dj.4**] again, we have $\sqcup\{f_i\} \perp \sqcup\{g_j\}$. □

Clearly the product of two inverse categories with disjoint joins has a disjoint join:

Lemma 6.3.5. *Given* $\mathbb{X}, \mathbb{Y}$ *are inverse categories with disjoint joins,* $\sqcup$ *and* $\sqcup'$ *respectively, then the category* $\mathbb{X} \times \mathbb{Y}$ *is an inverse category with disjoint joins.*

Proof. From Lemma 6.2.9, we know $\mathbb{X} \times \mathbb{Y}$ has a disjointness relation that is defined pointwise. We therefore define $\sqcup_\times$ the disjoint join on $\mathbb{X} \times \mathbb{Y}$ by

$$(f, g) \sqcup_\times (h, k) = (f \sqcup h, g \sqcup' k). \tag{6.1}$$

We now prove each of the axioms in Definition 6.3.1 hold.

[**DJ.1**] From Equation (6.1), we see that since $f, h \leq f \sqcup h$ and $g, k \leq g \sqcup' k$, we have $(f, g) \leq (f, g) \sqcup_\times (h, k)$ and $(h, k) \leq (f, g) \sqcup_\times (h, k)$.

[**DJ.2**] Suppose $(f,g) \leq (x,y)$, $(h,k) \leq (x,y)$ and $(f,g) \perp_\times (h,k)$. Then regarding it point-wise, we have $(f,g) \sqcup_\times (h,k) = (f \sqcup h, g \sqcup' k) \leq (x,y)$.

[**DJ.3**] $(x,y)\,((f,g) \sqcup_\times (h,k)) = (x(f \sqcup h), y(g \sqcup' k)) = (xf \sqcup xh, yg \sqcup' yk) = (xf, yg) \sqcup_\times (xh, yk) = ((x,y)(f,g)) \sqcup_\times ((x,y)(h,k))$.

[**DJ.4**] Given $\perp_\times [(f,g),(h,k),(x,y)]$, we know $f \perp (h \sqcup x)$ and $g \perp' (k \sqcup' y)$. Hence, $(f,g) \perp_\times ((h,k) \sqcup_\times (x,y))$. The opposite direction is similar.

$\square$

Example 6.3.6 (PINJ has a disjoint join)**.** Continuing from Example 6.2.8, we show that PINJ has disjoint joins. If $f = \{(a,b)\}$ and $g = \{(a',b'\}$ are disjoint parallel maps in PINJ from A to B, define $f \sqcup g \doteq \{(a'',b'')|(a'',b'') \in f$ or $(a'',b'') \in g\}$, i.e., the union of f and g.

This is still a partial injective map, due to the requirement of disjointness. Recall that $f \perp g$ means that $\overline{f} \cap \overline{g} = \emptyset$ and $\hat{f} \cap \hat{g} = \emptyset$ and that the respective meets will also be $\emptyset$. The empty meet of the restrictions means that $f \sqcup g$ is still a partial function, as each a'' will appear only once. The empty meet of the ranges gives us that $f \sqcup g$ is injective, because each b'' is unique.

The axioms for disjoint joins all hold:

[**DJ.1**] By construction, both f and g are less than $f \sqcup g$.

[**DJ.2**] $f \leq h,\ g \leq h$ means that h must contain all of the $(a,b) \in f$ and $(a',b') \in g$ and therefore $f \sqcup g \leq h$.

[**DJ.3**] Suppose $h : C \to A = \{(c, \dot{a})\}$. Then

$$
\begin{aligned}
h(f \sqcup g) = \{(c, \dot{b}) | (\exists a, \dot{a}. \dot{a} = a, (a, \dot{b}) \in f, (c, \dot{a}) \in h) \\
\text{or } (\exists a', \dot{a}. \dot{a} = a', (a', \dot{b}) \in g, (c, \dot{a}) \in h)\} \\
= \{(c, \dot{b}) | \exists a, \dot{a}. \dot{a} = a, (a, \dot{b}) \in f, (c, \dot{a}) \in h\} \bigcup \\
\{(c, \dot{b}) | \exists a', \dot{a}. \dot{a} = a', (a', \dot{b}) \in g, (c, \dot{a}) \in h\} \\
= hf \sqcup hg
\end{aligned}
$$

[**DJ.4**] Suppose $\perp [f, f', f'']$, $f = \{(a, b)\}$, $f' = \{(a', b')\}$, $f'' = \{(a'', b'')\}$. Then the set $\{a\}$ does not intersect either $\{a'\}$ nor $\{a''\}$ and similarly for the sets $\{b\}, \{b'\}$ and $\{b''\}'$. Thus we have $f \perp (g \sqcup h)$. The reverse direction is argued similarly.

Chapter 7

Disjoint sums and tensors

In this chapter, we first show how a tensor may be used to create a disjointness relation and a disjoint join in an inverse category. Conversely, if one is given an inverse category with a disjoint join and a construction we shall call a disjoint sum, this is sufficient to define a tensor in the inverse category.

Suppose $\mathbb{X}$ is an inverse category with a restriction zero, and $\oplus$ is a tensor product on $\mathbb{X}$. Given specific conditions regarding the tensor it is possible to define disjointness based upon the action of the tensor. We are assuming the following naming for the standard monoidal tensor isomorphisms:

$$u^l_\oplus : 0 \oplus A \to A \qquad\qquad u^r_\oplus : A \oplus 0 \to A$$

$$a_\oplus : (A \oplus B) \oplus C \to A \oplus (B \oplus C) \qquad\qquad c_\oplus : A \oplus B \to B \oplus A.$$

7.1 Disjointness via a tensor

Our first section will show how a disjointness tensor, defined below, may be used to create a disjointness relation in an inverse category. We will do this by noting that when it is possible for maps to work separately on the components of the tensor, this essentially will allow us to define when the restriction and range of functions are disjoint, and therefore when the maps are disjoint.

Definition 7.1.1. Given we have an inverse category $\mathbb{X}$ with restriction zero and a symmetric monoidal tensor $\oplus$, the tensor $\oplus$ is a *disjointness tensor* when:

(i) It is a restriction functor — i.e., $\_ \oplus \_ : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$.

(ii) The unit is the restriction zero. ($0 : \mathbf{1} \to \mathbb{X}$ picks out the restriction zero in $\mathbb{X}$).

(iii) Define $\amalg_1 = {u^r_\oplus}^{(-1)}(1 \oplus 0) : A \to A \oplus B$ and $\amalg_2 = {u^l_\oplus}^{(-1)}(0 \oplus 1) : A \to B \oplus A$. $\amalg_1$ and $\amalg_2$ must be jointly epic. That is, if $\amalg_1 f = \amalg_1 g$ and $\amalg_2 f = \amalg_2 g$, then $f = g$.

(iv) Define $\amalg_1^* := (1 \oplus 0)u^r_\oplus : A \oplus B \to A$ and $\amalg_2^* := (0 \oplus 1)u^l_\oplus : A \oplus B \to B$. $\amalg_1^*$ and $\amalg_2^*$ must be jointly monic. That is, whenever $f\amalg_1^* = g\amalg_1^*$ and $f\amalg_2^* = g\amalg_2^*$ then $f = g$.

Example 7.1.2 (Pinj has a disjointness tensor)**.** In Pinj, the disjoint union, $\uplus$, is a disjointness tensor. We will designate elements of the disjoint union as pairs of the elements of the original sets and the order in the disjoint join. That is, when

$$A = \{a\}, B = \{b\}, \text{ then } A \uplus B = \{(x,n) | n \in \{1,2\}, n = 1 \implies x \in A, n = 2 \implies x \in B\}.$$

Setting $\oplus$ as $\uplus$, we have the identity for the tensor is $\emptyset$. The action of the tensor on maps $f : A \to C = \{(a,c)\}$, $g : B \to D = \{(b,d)\}$ is given by:

$$f \oplus g : A \oplus B \to C \oplus D = \{((x,n),(v,m)) | (x,v) \in f \text{ or } (x,v) \in g\}.$$

From our definitions above, we may define our tensor structure maps:

$$\begin{aligned}
&u^l_\oplus : 0 \oplus A \to A, && (a, A) \mapsto a\\
&u^r_\oplus : A \oplus 0 \to A, && (a, A) \mapsto a\\
&a_\oplus : (A \oplus B) \oplus C, && ((a,1),1) \mapsto (a,1)\\
& && ((b,2),1) \mapsto ((b,1),2)\\
& && (c,2) \mapsto ((c,1),2)\\
&c_\oplus : A \oplus B \to B \oplus A && (a,1) \mapsto (a,2)\\
& && (a,2) \mapsto (a,1)
\end{aligned}$$

The map $\amalg_1 = {u^r_\oplus}^{(-1)}(1 \oplus 0)$ is given by the mapping $a \in A \mapsto (a,1) \in A \oplus B$. Similarly, $\amalg_2 = {u^l_\oplus}^{(-1)}(0 \oplus 1)$ is given by the mapping $a \in A \mapsto (a,2) \in B \oplus A$. We immediately see $\amalg_1$ and $\amalg_2$ are jointly epic. Similarly, ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ are jointly monic.

Lemma 7.1.3. *Given an inverse category $\mathbb{X}$ with restriction zero and disjointness tensor $\oplus$, then the map $\mathbf{0} \oplus \mathbf{0} : A \oplus B \to C \oplus D$ is the map $\mathbf{0} : A \oplus B \to C \oplus D$.*

Proof. Recall the zero map factors through the restriction zero, i.e. $\mathbf{0} : A \to B$ is the same as saying $A \xrightarrow{!} \mathbf{0} \xrightarrow{?} B$. Additionally, as objects, $\mathbf{0} \oplus \mathbf{0} \cong \mathbf{0}$ — the restriction zero.

Therefore the map $\mathbf{0} \oplus \mathbf{0} : A \oplus B \to C \oplus D$ is writable as

$$A \oplus B \xrightarrow{!\oplus!} \mathbf{0} \oplus \mathbf{0} \xrightarrow{?\oplus?} C \oplus D,$$

which may then be rewritten as

$$A \oplus B \xrightarrow{!\oplus!} \mathbf{0} \oplus \mathbf{0} \xrightarrow{u^l_\oplus} \mathbf{0} \xrightarrow{{u^l_\oplus}^{(-1)}} \mathbf{0} \oplus \mathbf{0} \xrightarrow{?\oplus?} C \oplus D.$$

But by the properties of the restriction zero, $(!\oplus!)u^l_\oplus =!$ and ${u^l_\oplus}^{(-1)}(? \oplus ?) = ?$ and therefore the map $\mathbf{0} \oplus \mathbf{0} : A \oplus B \to C \oplus D$ is the same as the map $\mathbf{0} : A \oplus B \to C \oplus D$. $\Box$

Lemma 7.1.4. *Given an inverse category $\mathbb{X}$ with restriction zero and disjointness tensor $\oplus$, $\amalg_1^* = {\amalg_1}^{(-1)}$ and $\amalg_2^* = {\amalg_2}^{(-1)}$ and the following identities hold:*

(i) $\amalg_i^* \amalg_i = \overline{\amalg_i^*}$ *and* $\amalg_i \amalg_i^* = \overline{\amalg_i} = 1$*;*

(ii) $\overline{\amalg_1^*}\amalg_2^* = \mathbf{0}$ *and* $\overline{\amalg_2^*}\amalg_1^* = \mathbf{0}$*;*

(iii) $\amalg_2 \amalg_1^* = \mathbf{0}$, $\amalg_2 \overline{\amalg_1^*} = \mathbf{0}$, $\amalg_1 \amalg_2^* = \mathbf{0}$ *and* $\amalg_1 \overline{\amalg_2^*} = \mathbf{0}$*;*

(iv) the maps $\amalg_1$ *and* $\amalg_2$ *are monic.*

Proof. For (i), recalling that the restriction zero is its own partial inverse, we see that

$${\amalg_1}^{(-1)} = \left({u^r_\oplus}^{(-1)}(1 \oplus 0)\right)^{(-1)} = (1 \oplus 0)^{(-1)} u^r_\oplus = (1 \oplus 0) u^r_\oplus = \amalg_1^*.$$

Similarly,

$$\amalg_2{}^{(-1)} = \left(u_\oplus^l{}^{(-1)}(0 \oplus 1)\right)^{(-1)} = (0 \oplus 1)u_\oplus^l = \amalg_2^*.$$

Hence, we may calculate the restriction of $\amalg_1$,

$$\amalg_1\amalg_1^* = u_\oplus^r{}^{(-1)}(1 \oplus 0)(1 \oplus 0)u_\oplus^r = (u_\oplus^r{}^{(-1)}(1 \oplus 0))u_\oplus^r = 1u_\oplus^r{}^{(-1)}u_\oplus^r = 1.$$

The calculation for $\amalg_2^*$ and $\amalg_2$ is analogous. For (ii), to show $\overline{\amalg_1^*}\amalg_2^* = 0$ and $\overline{\amalg_2^*}\amalg_1^* = 0$,

$$\overline{\amalg_1^*}\amalg_2^* = \overline{(1 \oplus 0)u_\oplus^r}(0 \oplus 1)u_\oplus^l = \overline{1 \oplus 0}(0 \oplus 1)u_\oplus^l = (1 \oplus 0)(0 \oplus 1)u_\oplus^l = (0 \oplus 0)u_\oplus^l = 0,$$

and

$$\overline{\amalg_2^*}\amalg_1^* = \overline{(0 \oplus 1)u_\oplus^l}(1 \oplus 0)u_\oplus^r = (0 \oplus 1)(1 \oplus 0)u_\oplus^r = (0 \oplus 0)u_\oplus^r = 0.$$

We show (iii), $\amalg_i\amalg_j^* = 0$, $\amalg_i\overline{\amalg_j^*} = 0$ when $i \neq j$,

$$\amalg_1\amalg_2^* = (u_\oplus^r{}^{(-1)}(1 \oplus 0))(0 \oplus 1)u_\oplus^l = u_\oplus^r{}^{(-1)}(0 \oplus 0)u_\oplus^l = 0$$

and

$$\amalg_2\amalg_1^* = (u_\oplus^l{}^{(-1)}(0 \oplus 1))(1 \oplus 0)u_\oplus^r = u_\oplus^l{}^{(-1)}(0 \oplus 0)u_\oplus^r = 0.$$

As $\overline{\amalg_1^*} = 1 \oplus 0$ and $\overline{\amalg_2^*} = 0 \oplus 1$, we see the other two identities hold as well.

Finally, to prove (iv), we first show $\amalg_1$ is monic. Suppose $f\amalg_1 = g\amalg_1$. Therefore we must have

$$f = f(\amalg_1\amalg_1{}^{(-1)}) = (f\amalg_1)\amalg_1{}^{(-1)} = (g\amalg_1)\amalg_1{}^{(-1)} = g(\amalg_1\amalg_1{}^{(-1)}) = g.$$

The proof that $\amalg_2$ is monic follows via a similar argument. $\square$

As we have shown that $\amalg_i^* = \amalg_i{}^{(-1)}$, we will prefer the explicit notation of $\amalg_i{}^{(-1)}$ for the remainder of this thesis.

Corollary 7.1.5. *In an inverse category $\mathbb{X}$ with a restriction zero and disjointness tensor, the following identities hold:*

(i) $\amalg_1(f \oplus g){\amalg_1}^{(-1)} = f$; *(iii)* $\amalg_2(f \oplus g){\amalg_1}^{(-1)} = 0$;

(ii) $\amalg_1(f \oplus g){\amalg_2}^{(-1)} = 0$; *(iv)* $\amalg_2(f \oplus g){\amalg_2}^{(-1)} = g$.

Additionally, if t is a map such that for $i \in \{1, 2\}$,

$$\amalg_i t {\amalg_j}^{(-1)} = \begin{cases} t_i & : \quad i \neq j \\ 0 & : \quad i = j, \end{cases}$$

then $t = t_1 \oplus t_2$.

Proof. The calculations for $f \oplus g$ follow from Lemma 7.1.4. For example, $\amalg_1(f \oplus g){\amalg_1}^{(-1)} = f\, \amalg_1\, {\amalg_1}^{(-1)} = f$.

For the second claim, note that we have $\amalg_1(t{\amalg_1}^{(-1)}) = t_1 = \amalg_1(t_1 \oplus t_2){\amalg_1}^{(-1)}$ and $\amalg_2(t{\amalg_1}^{(-1)}) = 0 = \amalg_2(t_1 \oplus t_2){\amalg_1}^{(-1)}$, hence $t{\amalg_1}^{(-1)} = (t_1 \oplus t_2){\amalg_1}^{(-1)}$. Similarly, we see $t{\amalg_2}^{(-1)} = (t_1 \oplus t_2){\amalg_2}^{(-1)}$ and therefore $t = t_1 \oplus t_2$. $\square$

Definition 7.1.6. In an inverse category $\mathbb{X}$ with a restriction zero and disjointness tensor, we define a partial pairing and a partial copairing operation on arrows in $\mathbb{X}$. First, for arrows $f : A \to B$ and $g : A \to C$, we define $f \triangledown g$ as being the map that makes Diagram (7.1) below commute, when it exists.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, dotted, "f \triangledown g"] \arrow[dr, "g"] & \\ B & B \oplus C \arrow[l, "{\amalg_1}^{(-1)}"] \arrow[r, "{\amalg_2}^{(-1)}"'] & C. \end{tikzcd} \tag{7.1}$$

Then for $h : B \to A$, $k : C \to A$, $h \vartriangle k$ is that map that makes Diagram (7.2) commute, if it exists.

$$\begin{tikzcd} B \arrow[r, "\amalg_1"] \arrow[dr, "h"'] & B \oplus C \arrow[d, dotted, "h \vartriangle k"] & C \arrow[l, "\amalg_2"'] \arrow[dl, "k"] \\ & A. & \end{tikzcd} \tag{7.2}$$

Due to $\amalg_1^{(-1)}$ and $\amalg_2^{(-1)}$ being jointly monic, $f \triangledown g$ is unique when it exists. Similarly, as $\amalg_1$ and $\amalg_2$ are jointly epic, $f \vartriangle g$ is unique when it exists.

Example 7.1.7 (PINJ)**.** Continuing from Example 6.3.6, we see that $f \triangledown g$ can only exist when $\overline{f} \cap \overline{g} = 0$, as it must be a set function, i.e., $f \triangledown g$ of some element a must be either $(b, 1)$ when $f(a) = b \in B$ or $(c, 2)$ when $g(a) = c \in C$.

Similarly, $h \vartriangle k$ can only exist when $\hat{h} \cap \hat{k} = 0$.

The partial pairing and copairing of Definition 7.1.6 will provide our mechanism for defining disjointness and eventually the disjoint join of maps. The existence of the pairing map $f \triangledown g$ allows us to ensure the restrictions of f and g are disjoint, while the copairing map $f \vartriangle d$ exists only when the ranges of f and g are disjoint.

To arrive at the disjointness we first give the following lemma detailing properties of the two operations $\triangledown$ and $\vartriangle$:

Lemma 7.1.8. *Given $\mathbb{X}$ is an inverse category with a restriction zero and a disjointness tensor $\oplus$ then the following relations hold for $\triangledown$ and $\vartriangle$:*

(i) If $f \triangledown g$ exists, then $g \triangledown f$ exists. If $f \vartriangle g$ exists, then $g \vartriangle f$ exists.

(ii) $f \triangledown 0$ and $f \vartriangle 0$ always exist.

(iii) When $f \triangledown g$ exists, $\overline{f}(f \triangledown g) = f \triangledown 0$, $\overline{f}g = 0$, $\overline{g}(f \triangledown g) = 0 \triangledown g$ and $\overline{g}f = 0$.

(iv) Dually to the previous item, when $f \vartriangle g$ exists, $(f \vartriangle g)\hat{f} = f \vartriangle 0$, $g\hat{f} = 0$, $(f \vartriangle g)\hat{g} = 0 \vartriangle g$ and $f\hat{g} = 0$.

(v) When $f \triangledown g$ exists, $f \triangledown g(h \oplus k) = fh \triangledown gk$.

(vi) Dually, when $f \vartriangle g$ exists, $(h \oplus k)f \vartriangle g = hf \vartriangle kg$.

(vii) When $f \triangledown g$ exists, then $h(f \triangledown g) = hf \triangledown hg$ and when $f \vartriangle g$ exists, $(f \vartriangle g)h = fh \vartriangle gh$.

(viii) If $\overline{f} \triangledown \overline{g}$ exists, then $\overline{f} \vartriangle \overline{g}$ exists and is the partial inverse of $\overline{f} \triangledown \overline{g}$.

(ix) If $f \triangledown g$ exists and $f' \leq f$, $g' \leq g$, then $f' \triangledown g'$ exists.

(x) When $f \vartriangle g$ exists, $(f \vartriangle g)(f \vartriangle g)^{(-1)} = \overline{f} \oplus \overline{g}$.

(xi) Given $f \triangledown g$ and $h \triangledown k$ exist, then $(f \oplus h) \triangledown (g \oplus k) = (f \triangledown g) \oplus (h \triangledown k)$. Dually, the existence of $f \vartriangle g$ and $h \vartriangle k$ implies $(f \oplus h) \vartriangle (g \oplus k) = (f \vartriangle g) \oplus (h \vartriangle k)$.

Proof.

(i) $g \triangledown f = (f \triangledown g)c_{\oplus}$ and $g \vartriangle f = c_{\oplus}(f \vartriangle g)$.

(ii) Consider $f\amalg_1$. Then $f \amalg_1 {\amalg_1}^{(-1)} = f$ and $f \amalg_1 {\amalg_2}^{(-1)} = f0 = 0$. Hence, $f\amalg_1 = f \triangledown 0$.

Consider ${\amalg_1}^{(-1)} f$. Then $\amalg_1 {\amalg_1}^{(-1)} f = f$ and $\amalg_2 {\amalg_1}^{(-1)} f = 0f = 0$ and therefore ${\amalg_1}^{(-1)} f = (f \vartriangle 0)$.

(iii) Using Lemma 7.1.4

$$\overline{f} g = \overline{(f \triangledown g){\amalg_1}^{(-1)}}(f \triangledown g){\amalg_2}^{(-1)} = (f \triangledown g)\overline{{\amalg_1}^{(-1)}}{\amalg_2}^{(-1)} = 0.$$

Similarly, $\overline{g} f = f \triangledown g\overline{{\amalg_2}^{(-1)}}{\amalg_1}^{(-1)} = 0$.

Recall that ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ are jointly monic. We have $\overline{f}(f \triangledown g){\amalg_1}^{(-1)} = \overline{f} f = f = (f \triangledown 0){\amalg_1}^{(-1)}$ and $\overline{f}(f \triangledown g){\amalg_2}^{(-1)} = \overline{f} g = 0 = (f \triangledown 0){\amalg_2}^{(-1)}$. Therefore, $\overline{f}(f \triangledown g) = f \triangledown 0$. Similarly, $\overline{g}(f \triangledown g) = 0 \triangledown g$.

(iv) Using Lemma 7.1.4

$$\begin{aligned} g\hat{f} = \amalg_2(f \vartriangle g)(\widehat{\amalg_1(f \vartriangle g)}) = \amalg_2(f \vartriangle g)\overline{(f \vartriangle g)^{(-1)}{\amalg_1}^{(-1)}} = \\ \amalg_2\,(f \vartriangle g)\overline{(f \vartriangle g)^{(-1)}\overline{{\amalg_1}^{(-1)}}} = \overline{\amalg_2\overline{(f \vartriangle g)}\overline{{\amalg_1}^{(-1)}}}\,\amalg_2\,(f \vartriangle g) = \\ \overline{\amalg_2\overline{{\amalg_1}^{(-1)}}\overline{(f \vartriangle g)}}\,\amalg_2\,(f \vartriangle g) = \overline{0}\,\amalg_2\,(f \vartriangle g) = 0 \end{aligned}$$

Similarly, $f\hat{g} = 0$.

Recall that $\amalg_1$ and $\amalg_2$ are jointly epic. We have $\amalg_1(f \vartriangle g)\hat{f} = f\hat{f} = f = \amalg_1(f \vartriangle 0)$ and $\amalg_2(f \vartriangle g)\hat{f} = g\hat{f} = 0 = \amalg_2(f \vartriangle 0)$. Therefore, $(f \vartriangle g)\hat{f} = f \vartriangle 0$. Similarly, $(f \vartriangle g)\hat{g} = 0 \vartriangle g$.

(v) Calculating, we have

$$f \triangledown g(h \oplus k){\amalg_1}^{(-1)} = f \triangledown g{\amalg_1}^{(-1)}h = fh$$

and

$$f \triangledown g(h \oplus k){\amalg_2}^{(-1)} = f \triangledown g{\amalg_2}^{(-1)}k = gk,$$

which means that $f \triangledown g(h \oplus k) = fh \triangledown gk$ by the joint monic property of ${\amalg_1}^{(-1)}$, ${\amalg_2}^{(-1)}$.

(vi) The proof for this is dual to (v), and depends on the joint epic property of $\amalg_1$ and $\amalg_2$.

(vii) We are given $f \triangledown g$ exists, therefore $f = (f \triangledown g){\amalg_1}^{(-1)}$ and $g = (f \triangledown g){\amalg_2}^{(-1)}$. But this means $hf = h(f \triangledown g){\amalg_1}^{(-1)}$ and $hg = h(f \triangledown g){\amalg_2}^{(-1)}$, from which we may conclude $hf \triangledown hg = h(f \triangledown g)$ by the fact that ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ are jointly monic. The proof of $(f \vartriangle g)h = fh \vartriangle gh$ is similar.

(viii) We are given $\overline{f} = \overline{f} \triangledown \overline{g}{\amalg_1}^{(-1)}$. Therefore,

$$\overline{f} = \overline{f}^{(-1)} = {\amalg_1}^{(-1)(-1)}(\overline{f} \triangledown \overline{g})^{(-1)} = \amalg_1(\overline{f} \triangledown \overline{g})^{(-1)}.$$

Similarly, $\overline{g} = \amalg_2(\overline{f} \triangledown \overline{g})^{(-1)}$. But this means $(\overline{f} \triangledown \overline{g})^{(-1)} = \overline{f} \vartriangle \overline{g}$.

(ix) Note that from (v), we know that $f \triangledown g = \overline{f} \triangledown \overline{g}(f \oplus g)$. We are given $f' \leq f$ and $g' \leq g$. This gives us $\overline{f'}f = f'$, $\overline{g'}g = g'$, $\overline{f'}\,\overline{f} = \overline{f'}$ and $\overline{g'}\,\overline{g} = \overline{g'}$. Consider the map $\overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(f \oplus g)$. Calculating, we see

$$\begin{aligned}
\overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(f \oplus g) &= \overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(\overline{f'} \oplus \overline{g'})(f \oplus g) \\
&= \overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(f' \oplus g') \\
&= \overline{f}\,\overline{f'} \triangledown \overline{g}\overline{g'}(f' \oplus g') \\
&= \overline{f'}\,\overline{f} \triangledown \overline{g'}\overline{g}(f' \oplus g') \\
&= \overline{f'} \triangledown \overline{g'}(f' \oplus g') \\
&= f' \triangledown g'.
\end{aligned}$$

(x) From our diagram for $\triangle$, we know:

$$f^{(-1)} = (f \triangle g)^{(-1)} \amalg_1{}^{(-1)} \text{ and}$$
$$g^{(-1)} = (f \triangle g)^{(-1)} \amalg_2{}^{(-1)}.$$

As well, we know that $\amalg_1(f \triangle g) = f$ and $\amalg_1(f \triangle g) = g$. Therefore, we have:

$$\amalg_1(f \triangle g)(f \triangle g)^{(-1)}\amalg_1{}^{(-1)} = \overline{f} \text{ and } \amalg_2\,(f \triangle g)(f \triangle g)^{(-1)}\amalg_2{}^{(-1)} = \overline{g}.$$

As $f \perp_{\oplus} g$, we know that $fg^{(-1)} = f\hat{g}g^{(-1)} = 0g^{(-1)} = 0$ and therefore,

$$\amalg_1(f \triangle g)(f \triangle g)^{(-1)}\amalg_2{}^{(-1)} = 0 \text{ and } \amalg_2\,(f \triangle g)(f \triangle g)^{(-1)}\amalg_1{}^{(-1)} = 0.$$

By Corollary 7.1.5 this means $(f \triangle g)(f \triangle g)^{(-1)} = \overline{f} \oplus \overline{g}$.

(xi) As $(f \triangledown g) \oplus (h \triangledown k)\amalg_1{}^{(-1)} = (f \triangledown g)$ and $(f \triangledown g) \oplus (h \triangledown k)\amalg_2{}^{(-1)} = (h \triangledown k)$, we see that $(f \triangledown g) \oplus (h \triangledown k)$ satisfies the diagram for $(f \oplus h) \triangledown (g \oplus k)$. Dually, as $\amalg_1(f \triangle g) \oplus (h \triangle k) = (f \triangle g)$ and $\amalg_2(f \triangle g) \oplus (h \triangle k) = (h \triangle k)$, $(f \triangle g) \oplus (h \triangle k)$ satisfies the diagram for $(f \oplus h) \triangle (g \oplus k)$.

□

We are now set up to prove that we can create a disjointness relation based on the existence of our pairing and copairing maps:

Lemma 7.1.9. *Define $f \perp_{\oplus} g$ (f is tensor disjoint to g) when $f, g : A \to B$ and both $f \triangledown g$ and $f \triangle g$ exist. If $\mathbb{X}$ is an inverse category with a restriction zero and a disjointness tensor $\oplus$ then the relation $\perp_{\oplus}$ is a disjointness relation.*

Proof. We need to show that $\perp_{\oplus}$ satisfies the disjointness axioms. We will use [**Dis.6'**] in place of [**Dis.6**] and [**Dis.7**] as discussed in Lemma 6.2.2.

[**Dis.1**] We must show $f \perp_{\oplus} 0$. This follows immediately from Lemma 7.1.8, item (ii).

[**Dis.2**] Show $f \perp_{\oplus} g$ implies $\overline{f}g = 0$. This is a direct consequence of Lemma 7.1.8, item (iii).

[**Dis.3**] We require $f \perp_{\oplus} g$, $f' \leq f$, $g' \leq g$ implies $f' \perp_{\oplus} g'$. From Lemma 7.1.8, item (ix), we immediately have $f' \triangledown g'$ exists. Using a similar argument to the proof of this item, we also have $f' \vartriangle g'$ exists and hence $f' \perp_{\oplus} g'$.

[**Dis.4**] Commutativity of $\perp_{\oplus}$ follows from the symmetry of the two required diagrams, see Lemma 7.1.8, item (i).

[**Dis.5**] Show that if $f \perp_{\oplus} g$ then $hf \perp_{\oplus} hg$ for any map h. Assuming $f \triangledown g$ exists, by Lemma 7.1.8, item (vii), we know that $hf \triangledown hg$ exists. Assuming $f \vartriangle g$, by item (vi) of the same lemma, $(hf) \vartriangle (hg)$ exists and equals $(h \oplus h)(f \vartriangle g)$ and therefore $hf \perp_{\oplus} hg$.

[**Dis.6'**] We need to show $f \perp_{\oplus} g$ if and only if $\overline{f} \perp_{\oplus} \overline{g}$ and $\hat{f} \perp_{\oplus} \hat{g}$. This follows directly from Lemma 7.1.8, items (v) and (vi), which give us $f \triangledown g = \overline{f} \triangledown \overline{g}(f \oplus g)$ and $f \vartriangle g = (f \oplus g)\hat{f} \vartriangle \hat{g}$, where the equalities hold if either side of the equation exists.

$\square$

Example 7.1.10 ($\perp_{\oplus}$ in $\textsc{Pinj}$)**.** Referring to Example 7.1.7, we noted $f \triangledown g$ exists when $\overline{f} \cap \overline{g} = 0$ and that $f \vartriangle g$ exists when $\hat{f} \cap \hat{g} = 0$. But this agrees with our initial definition of disjointness ($\perp$) in $\textsc{Pinj}$ from Example 6.2.8 and hence we have that $\perp_{\oplus}$ is the same relation as $\perp$ in $\textsc{Pinj}$.

7.2 Disjoint joins via a tensor

The operations $\triangledown$ and $\vartriangle$ are sufficient to define a disjointness relation on an inverse category. However, when we wish to extend this to a disjoint join, the two relations are insufficient to prove [**DJ.4**], That is, we need to show that $\perp_{\oplus}[f, g, h]$ implies $f \perp_{\oplus} (g \sqcup_{\oplus} h)$.

Therefore, we add one more assumption regarding our tensor in order to define disjointness.

Definition 7.2.1. Let $\mathbb{X}$ be an inverse category with a disjointness tensor $\oplus$ and a restriction zero. Consider the commutative diagrams (7.3) and (7.4).

$$\begin{tikzcd}
A \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& X \oplus Y \oplus Z \arrow[r, "{\amalg_{1,3}}^{(-1)}"'] \arrow[d, "{\amalg_{1,2}}^{(-1)}"] & X \oplus Z \arrow[d, "{\amalg_1}^{(-1)}"] \\
& X \oplus Y \arrow[r, "{\amalg_1}^{(-1)}"'] & X,
\end{tikzcd} \tag{7.3}$$

$$\begin{tikzcd}
& & A \\
X \oplus Y \arrow[r, "\amalg_{1,2}"'] \arrow[urr, bend left, "h"] & X \oplus Y \oplus Z \arrow[ur, dotted, "\beta"] & \\
X \arrow[r, "\amalg_1"'] \arrow[u, "\amalg_1"] & X \oplus Z. \arrow[u, "\amalg_{1,3}"'] \arrow[uur, bend right, "k"'] &
\end{tikzcd} \tag{7.4}$$

Then $\oplus$ is a *disjoint sum tensor* when the following two conditions hold:

- α exists if and only if $f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$ exists;
- β exists if and only if $\amalg_2 h \vartriangle \amalg_2 k$ exists.

Example 7.2.2 (In Pinj, $\oplus$ is a disjoint sum tensor)**.** In Pinj, Diagram (7.3) means that f and g must agree on those elements of A that map to $(x, 1)$ in either $X \oplus Y$ or $X \oplus Z$. The statement that $f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$ exists means that if $f(a) = (y, 2)$, then $g(a)$ must be undefined and vice versa. In such a case α exists and is defined as:

$$\alpha(a) = \begin{cases} (x,1) & f(a) = (x,1) \in X \oplus Y \text{ and } g(a) = (x,1) in X \oplus Z \\ (y,2) & f(a) = (y,2) \in X \oplus Y \text{ and } g(a)\uparrow \\ (z,3) & g(a) = (z,2) \in X \oplus Z \text{ and } f(a)\uparrow. \end{cases}$$

For the converse, assume α exists, meaning $\alpha(a)$ must be one of $(x, 1), (y, 2)$ or $(z, 3)$. As $f{\amalg_2}^{(-1)} = \alpha{\amalg_{1,2}}^{(-1)}{\amalg_2}^{(-1)}$ and $g{\amalg_2}^{(-1)} = \alpha{\amalg_{1,3}}^{(-1)}{\amalg_2}^{(-1)}$, we see this immediately requires that $\overline{f{\amalg_2}^{(-1)}} \cap \overline{g{\amalg_2}^{(-1)}} = 0$ and therefore $f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$ exists.

The reasoning for Diagram (7.4) is similar.

Lemma 7.2.3. *Let $\mathbb{X}$ be an inverse category with a disjoint sum tensor as in Definition 7.2.1 and we are given $f, g, h : A \to B$ with $\perp_{\oplus}[f, g, h]$. Then both $f \triangledown (g \triangledown h)$ and $f \vartriangle (g \vartriangle h)$ exist.*

Proof. As all the maps are disjoint, we know the maps $\triangledown$ and $\vartriangle$ exist for each pair. Consider the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[rr, bend right=15, "g \triangledown f"'] \arrow[dr, dotted, "\alpha"] \arrow[ddr, bend right, "g \triangledown h"'] & & B \\
& B \oplus B \oplus B \arrow[r, "\amalg_{1,3}{}^{(-1)}"'] \arrow[d, "\amalg_{1,2}{}^{(-1)}"] & B \oplus B \arrow[u, "\amalg_2{}^{(-1)}"'] \arrow[d, "\amalg_1{}^{(-1)}"] \\
& B \oplus B \arrow[r, "\amalg_1{}^{(-1)}"'] & B
\end{tikzcd}$$

where we claim $\alpha = (g \triangledown h) \triangledown f$.

The lower part of the diagram commutes as it fulfills the conditions of Definition 7.2.1. The upper rightmost triangle of the diagram commutes by the definition of $g \triangledown f$. Noting that $\amalg_{0,1}{}^{(-1)} : B \oplus B \oplus B \to B \oplus B$ is the same map as $\amalg_1{}^{(-1)} : (B \oplus B) \oplus B \to (B \oplus B)$ and $\amalg_{0,2}{}^{(-1)}\amalg_2{}^{(-1)} : B \oplus B \oplus B \to B \oplus B \to B$ is the same map as $\amalg_2{}^{(-1)} : (B \oplus B) \oplus B \to B$, we see α does make the $\triangledown$ diagram for $g \triangledown h$ and f commute. Therefore by Lemma 7.1.8, $f \triangledown (g \triangledown h)$ exists and is equal to $\alpha c_{\oplus\{01,2\}}$.

A dual diagram and corresponding reasoning shows $f \vartriangle (g \vartriangle h)$ exists. $\square$

Lemma 7.2.4. *In an inverse category $\mathbb{X}$ with a disjoint sum tensor, when $\perp_{\oplus}[f, g, h]$, then:*

(i) $f \triangledown (g \triangledown h) = ((f \triangledown g) \triangledown h)a_{\oplus}$ *and both exist,*

(ii) $f \vartriangle (g \vartriangle h) = ((f \vartriangle g) \vartriangle h)a_{\oplus}$ *and both exist.*

Proof. Consider the diagram

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[drr, bend left=10, "f \triangledown h"] \arrow[dr, dotted, "\alpha"] \arrow[ddr, bend right, "f \triangledown g"'] & & B \\
& B \oplus B \oplus B \arrow[r, "{\amalg_{1,3}}^{(-1)}"'] \arrow[d, "{\amalg_{1,2}}^{(-1)}"] & B \oplus B \arrow[u, "{\amalg_2}^{(-1)}"'] \arrow[d, "{\amalg_1}^{(-1)}"] \\
& B \oplus B \arrow[r, "{\amalg_1}^{(-1)}"'] & B
\end{tikzcd} \tag{7.5}$$

which gives us $\alpha = (f \triangledown g) \triangledown h : A \to (B \oplus B) \oplus B$ and $\alpha a_\oplus : A \to B \oplus (B \oplus B)$. Next consider the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[drr, bend left=10, "g \triangledown f"] \arrow[dr, dotted, "\gamma"] \arrow[ddr, bend right, "g \triangledown h"'] & & B \\
& B \oplus B \oplus B \arrow[r, "{\amalg_{1,3}}^{(-1)}"'] \arrow[d, "{\amalg_{1,2}}^{(-1)}"] & B \oplus B \arrow[u, "{\amalg_2}^{(-1)}"'] \arrow[d, "{\amalg_1}^{(-1)}"] \\
& B \oplus B \arrow[r, "{\amalg_1}^{(-1)}"'] & B
\end{tikzcd} \tag{7.6}$$

which gives us $\gamma c_\oplus = f \triangledown (g \triangledown h) : A \to B \oplus (B \oplus B)$.

Note from Diagrams (7.5) and (7.6) we have

$$\gamma c_\oplus {\amalg_0}^{(-1)} = f = \alpha a_\oplus {\amalg_1}^{(-1)}$$

$$\gamma c_\oplus {\amalg_1}^{(-1)} {\amalg_0}^{(-1)} = g = \alpha a_\oplus {\amalg_2}^{(-1)} {\amalg_1}^{(-1)}$$

$$\gamma c_\oplus {\amalg_1}^{(-1)} {\amalg_2}^{(-1)} = h = \alpha a_\oplus {\amalg_2}^{(-1)} {\amalg_2}^{(-1)}.$$

By the assumption that ${\amalg_1}^{(-1)}, {\amalg_2}^{(-1)}$ are jointly monic, we have $\alpha = \gamma c_\oplus a_\oplus$. Therefore $f \triangledown (g \triangledown h) = (f \triangledown g) \triangledown h$, up to the associativity isomorphism. $\square$

We may now state the main result of this chapter:

Proposition 7.2.5. *An inverse category with a restriction zero and a disjoint sum tensor has disjoint joins.*

Given two maps f, g with $f \perp_{\oplus} g$, we claim the map $f \sqcup_{\oplus} g = \overline{f} \triangledown \overline{g}(f \oplus g)\hat{f} \vartriangle \hat{g}$ is a disjoint join. We will prove this after giving an example and a lemma detailing the properties of $\sqcup_{\oplus}$.

For reference, the map $f \sqcup_{\oplus} g$ may be visualized as follows:

$$\begin{tikzcd}
 & A & B \arrow[d, "\amalg_1"'] \arrow[dr, "\hat{f}"] & \\
A \arrow[ur, "\overline{f}"] \arrow[r, dotted, "\overline{f} \triangledown \overline{g}"] \arrow[dr, "\overline{g}"'] & A \oplus A \arrow[u, "\amalg_1{}^{(-1)}"'] \arrow[d, "\amalg_2{}^{(-1)}"] \arrow[r, "f \oplus g"] & B \oplus B \arrow[r, dotted, "\hat{f} \vartriangle \hat{g}"] & B. \\
 & A & B \arrow[u, "\amalg_2"] \arrow[ur, "\hat{g}"'] &
\end{tikzcd}$$

Using Lemma 7.1.8, we may rewrite this in a variety of equivalent ways:

$$\begin{aligned}
f \sqcup_{\oplus} g &= \overline{f} \triangledown \overline{g}(f \oplus g)\hat{f} \vartriangle \hat{g} \\
&= f \triangledown g\, \hat{f} \vartriangle \hat{g} \\
&= \overline{f} \triangledown \overline{g}\, f \vartriangle g \\
&= f \triangledown g(f^{(-1)} \oplus g^{(-1)}) f \vartriangle g.
\end{aligned}$$

In particular, note that $\overline{f} \sqcup_{\oplus} \overline{g} = (\overline{f} \triangledown \overline{g})(\overline{f} \vartriangle \overline{g})$ as $\hat{\overline{g}} = \overline{g}$.

Example 7.2.6 (Tensor join in $\textsc{Pinj}$)**.** We will use the third equality of those above for $f \sqcup_{\oplus} g$, i.e., $\overline{f} \triangledown \overline{g}\, f \vartriangle g$.

We have:

$$\overline{f} \triangledown \overline{g}(a) = \begin{cases} (a, 1) & \overline{f}(a) = a, \overline{g}\uparrow \\ (a, 2) & \overline{g}(a) = a, \overline{f}\uparrow \end{cases} \tag{7.7}$$

and

$$f \vartriangle g((a, n)) = \begin{cases} f(a) & n = 1 \\ g(a) & n = 2. \end{cases} \tag{7.8}$$

Combining Equation (7.7) with Equation (7.8) then gives us the same definition as that of $\sqcup$ as given in Example 6.3.6.

Lemma 7.2.7. *Let $\mathbb{X}$ be an inverse category with a disjointness tensor and restriction zero. Let $\mathbb{X}$ have the maps $f, g : A \to B$ with $f \perp_{\oplus} g$. Then $\sqcup_{\oplus}$ has the following properties.*

(i) For all maps $h : A \to B$, $\overline{f}h \sqcup_{\oplus} \overline{g}h = (\overline{f} \sqcup_{\oplus} \overline{g})h$.

(ii) $\overline{f} \sqcup_{\oplus} \overline{g} = \overline{f \sqcup_{\oplus} g}$.

Proof.

(i) By Lemma 6.2.3, item (ii), we know that $\overline{f}h \perp_{\oplus} \overline{g}h$, hence we can form $\overline{f}h \sqcup_{\oplus} \overline{g}h$. Also, noting that

$$h\widehat{\overline{f}h} = h\overline{h^{(-1)}\overline{f}} = \overline{hh^{(-1)}\overline{f}}h = \overline{\overline{h}\overline{f}}h = \overline{\overline{f}h}h = \overline{f}h,$$

we may then calculate from the left hand side as follows:

$$\begin{aligned}
\overline{f}h \sqcup_{\oplus} \overline{g}h &= (\overline{f}h \triangledown \overline{g}h)(\widehat{\overline{f}h} \vartriangle \widehat{\overline{g}h}) \\
&= (\overline{f} \triangledown \overline{g})(h\widehat{\overline{f}h} \vartriangle h\widehat{\overline{g}h}) \\
&= (\overline{f} \triangledown \overline{g})(\overline{f}h \vartriangle \overline{g}h) \\
&= (\overline{f} \triangledown \overline{g})(\overline{f} \vartriangle \overline{g})h \\
&= (\overline{f} \sqcup_{\oplus} \overline{g})h.
\end{aligned}$$

(ii) Using Lemma 7.1.8, item (x), we can compute:

$$\begin{aligned}
\overline{f \sqcup_{\oplus} g} &= f \sqcup_{\oplus} g(f \sqcup_{\oplus} g)^{(-1)} \\
&= \left((\overline{f} \triangledown \overline{g})(f \vartriangle g)\right) \left((f \triangledown g)^{(-1)}(\overline{f} \triangledown \overline{g})^{(-1)}\right) \\
&= \overline{f} \triangledown \overline{g}(f \triangledown g)(f \triangledown g)^{(-1)}\overline{f} \vartriangle \overline{g} \\
&= \overline{f} \triangledown \overline{g}(\overline{f} \oplus \overline{g})\overline{f} \vartriangle \overline{g} \\
&= \overline{f} \triangledown \overline{g}\,\overline{f} \vartriangle \overline{g} \\
&= \overline{f} \sqcup_{\oplus} \overline{g}.
\end{aligned}$$

$\square$

We may now complete the proof of Proposition 7.2.5:

Proof. [**DJ.1**] We must show $f, g \leq f \sqcup_{\oplus} g$.

$$
\begin{aligned}
\overline{f}\,(\overline{f} \triangledown \overline{g}) f \vartriangle g &= (\overline{f} \triangledown \overline{g}) \amalg_1{}^{(-1)} (\overline{f} \triangledown \overline{g}) f \vartriangle g \\
&= \overline{(\overline{f} \triangledown \overline{g}) \amalg_1{}^{(-1)}} (\overline{f} \triangledown \overline{g}) f \vartriangle g \\
&= (\overline{f} \triangledown \overline{g}) \overline{\amalg_1{}^{(-1)}} f \vartriangle g \\
&= (\overline{f} \triangledown \overline{g}) \amalg_1{}^{(-1)} \amalg_1 f \vartriangle g \\
&= ((\overline{f} \triangledown \overline{g}) \amalg_1{}^{(-1)})(\amalg_1 (f \vartriangle g)) \\
&= \overline{f} f \\
&= f.
\end{aligned}
$$

Thus, we see $f \leq f \sqcup_{\oplus} g$. Showing $g \leq f \sqcup_{\oplus} g$ proceeds in the same manner.

[**DJ.2**] We must show that $f \leq h,\ g \leq h$ and $f \perp_{\oplus} g$ implies $f \sqcup_{\oplus} g \leq h$.

$$
\begin{aligned}
\overline{f \sqcup_{\oplus} g}\, h &= \overline{\overline{f} h \sqcup_{\oplus} \overline{g} h}\, h \\
&= \overline{(\overline{f} \sqcup_{\oplus} \overline{g}) h}\, h \\
&= \overline{\overline{(\overline{f} \sqcup_{\oplus} \overline{g})} h}\, h \\
&= \overline{\overline{(\overline{f} \sqcup_{\oplus} \overline{g})} h}\, (\overline{f} \sqcup_{\oplus} \overline{g}) h \\
&= \overline{(\overline{f} \sqcup_{\oplus} \overline{g}) h}\, (\overline{f} \sqcup_{\oplus} \overline{g}) h \\
&= (\overline{f} \sqcup_{\oplus} \overline{g}) h \\
&= (\overline{f} h \sqcup_{\oplus} \overline{g} h) \\
&= (f \sqcup_{\oplus} g).
\end{aligned}
$$

[**DJ.3**] We must show stability of $\sqcup_{\oplus}$, i.e., that $h(f \sqcup_{\oplus} g) = hf \sqcup_{\oplus} hg$.

$$
\begin{aligned}
h(f \sqcup_{\oplus} g) &= h((\overline{f} \triangledown \overline{g})(f \vartriangle g)) \\
&= (h\overline{f} \triangledown h\overline{g})(f \vartriangle g) \\
&= (\overline{hf}h \triangledown \overline{hg}h)(f \vartriangle g) \\
&= (\overline{hf} \triangledown \overline{hg})(h \oplus h)(f \vartriangle g) \\
&= (\overline{hf} \triangledown \overline{hg})(hf \vartriangle hg) \\
&= hf \sqcup_{\oplus} hg.
\end{aligned}
$$

[**DJ.4**] We need to show $\perp_{\oplus}[f, g, h]$ if and only if $f \perp_{\oplus} (g \sqcup_{\oplus} h)$. For the right to left implication, note that the existence of $g \sqcup_{\oplus} h$ implies $g \perp_{\oplus} h$. We also know $g, h \leq g \sqcup_{\oplus} h$ by [**DJ.4**], as shown in this lemma. This gives us that $f \perp_{\oplus} g$ and $f \perp_{\oplus} h$, hence $\perp_{\oplus}[f, g, h]$.

For the left to right implication, we use Lemma 7.2.3. As we have $\perp_{\oplus}[f, g, h]$, we know $f \triangledown (g \triangledown h)$ and $f \vartriangle (g \vartriangle h)$ exist.

Recall that $g \sqcup_{\oplus} h = (g \triangledown h)(\hat{g} \vartriangle \hat{h})$. Then the map

$$A \xrightarrow{f \triangledown (g \triangledown h)} B \oplus B \oplus B \xrightarrow{1 \oplus (\hat{g} \vartriangle \hat{h})} B \oplus B$$

makes the diagram for $f \triangledown (g \sqcup_{\oplus} h)$ commute.

Recalling that $g \sqcup_{\oplus} h = (\overline{g} \triangledown \overline{h})(g \vartriangle h)$, we also see that

$$A \oplus A \xrightarrow{1 \oplus (\overline{g} \triangledown \overline{h})} A \oplus A \oplus A \xrightarrow{f \vartriangle (g \vartriangle h)} B$$

provides the witness map for $f \vartriangle (g \sqcup_{\oplus} h)$ and hence $f \perp_{\oplus} (g \sqcup_{\oplus} h)$.

$\square$

7.3 Disjoint sums

As noted at the beginning of this chapter, we now take the reverse direction and determine what conditions will allow us to define a disjoint sum tensor in an inverse category which already has a disjoint join. As we shall see, it is sufficient to have "enough" *disjoint sum* objects, as defined below.

First, we show that the disjoint sum tensor is universal with respect to the disjoint join.

Lemma 7.3.1. *Given an inverse category $\mathbb{X}$ with a disjoint sum tensor $\oplus$, then $\oplus$ preserves the disjoint join. That is,*

$$f \perp g,\ h \perp k \text{ implies } f \oplus h \perp g \oplus k \tag{7.9}$$

$$f \perp g,\ h \perp k \text{ implies } (f \sqcup g) \oplus (h \sqcup k) = (f \oplus h) \sqcup (g \oplus k). \tag{7.10}$$

Proof. For Condition (7.9), suppose we have $f \perp_{\oplus} g$ and $h \perp_{\oplus} k$. From Lemma 7.1.8(xi), we know both $(f \oplus h) \triangledown (g \oplus k)$ and $(f \oplus h) \vartriangle (g \oplus k)$ exist, hence $(f \oplus h) \perp_{\oplus} (g \oplus k)$.

For Condition (7.10), we compute from the right hand side:

$$\begin{aligned}
(f \oplus h) \sqcup_{\oplus} (g \oplus k) &= (f \oplus h) \triangledown (g \oplus k) (\widehat{f \oplus h}) \vartriangle (\widehat{g \oplus k}) \\
&= ((f \triangledown g) \oplus (h \triangledown k)) \Big((\hat{f} \oplus \hat{h}) \vartriangle (\hat{g} \oplus \hat{k}) \Big) \\
&= ((f \triangledown g) \oplus (h \triangledown k)) \Big((\hat{f} \vartriangle \hat{g}) \oplus (\hat{h} \vartriangle \hat{k}) \Big) \\
&= \Big((f \triangledown g)(\hat{f} \vartriangle \hat{g}) \Big) \oplus \Big((h \triangledown k)(\hat{h} \vartriangle \hat{k}) \Big) \\
&= (f \sqcup_{\oplus} g) \oplus (h \sqcup_{\oplus} k).
\end{aligned}$$

The second and third lines above again use Lemma 7.1.8(xi).

□

Now, we define an abstract "disjoint sum", which will be our candidate for creating a disjoint sum tensor.

Definition 7.3.2. In an inverse category with disjoint joins, an object X is the *disjoint sum* of A and B when there exist maps $i_1,\ i_2,\ x_1,\ x_2$ such that:

(i) i_1 and i_2 are monic;

(ii) $i_1 : A \to X$, $i_2 : B \to X$, $x_1 : X \to A$ and $x_2 : X \to B$.

(iii) ${i_1}^{(-1)} = x_1$ and ${i_2}^{(-1)} = x_2$.

(iv) ${i_1}^{(-1)} i_1 \perp {i_2}^{(-1)} i_2$ and ${i_1}^{(-1)} i_1 \sqcup {i_2}^{(-1)} i_2 = 1_X$.

i_1 and i_2 will be referred to as the *injection* maps of the disjoint sum.

Lemma 7.3.3. *The disjoint sum X of A and B is unique up to isomorphism.*

Proof. Assume we have two disjoint sums over A and B:

$$\begin{tikzcd} A \arrow[r, "i_1"] & X \arrow[l, bend left, "x_1"] \arrow[r, bend right, "x_2"'] & B \arrow[l, "i_2"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "j_1"] & Y \arrow[l, bend left, "y_1"] \arrow[r, bend right, "y_2"'] & B \arrow[l, "j_2"'] \end{tikzcd}.$$

We will show that the map $x_1 j_1 \sqcup x_2 j_2 : X \to Y$ is an isomorphism.

Note by the fact that i_2 is monic, we may conclude from the definition that $0 = \overline{x_1 i_1 x_2}$ and therefore $0 = x_1 i_1 x_2$. Then, given that x_1 is the inverse of the monic i_1, we may calculate $0 = \hat{0} = \widehat{x_2 i_1 x_2} = \overline{{x_2}^{(-1)}\, {i_1}^{(-1)}\, i_1} = \overline{{x_2}^{(-1)}\, {i_1}^{(-1)}} = \widehat{i_1 x_2}$. From this we see $i_1 x_2 = 0$. Similarly, we have $i_2 x_2 = 0$, $j_1 y_2 = 0$ and $j_2 y_1 = 0$.

Next, by Lemma 6.2.3, we know that $\overline{x_1} \perp \overline{x_2}$ as both i_1 and i_2 are monic. By the same lemma, $\hat{j_1} \perp \hat{j_2}$ as y_1, y_2 are the inverses of monic maps. Then, from [**Dis.7**], we have $x_2 j_1 \perp x_2 j_2$, hence we may form $x_2 j_1 \sqcup x_2 j_2 : X \to Y$.

Similarly, we may form the map $y_1 i_1 \sqcup y_2 i_2 : Y \to X$. Computing their composition:

$$\begin{aligned}
(x_2 j_1 \sqcup x_2 j_2)(y_1 i_1 \sqcup y_2 i_2) &= (x_2 j_1 (y_1 i_1 \sqcup y_2 i_2)) \sqcup (x_2 j_2 (y_1 i_1 \sqcup y_2 i_2)) \\
&= x_2 j_1 y_1 i_1 \sqcup x_2 j_1 y_2 i_2 \sqcup x_2 j_2 y_1 i_1 \sqcup x_2 j_2 y_2 i_2 \\
&= x_2\, 1\, i_1 \sqcup x_2\, 0\, i_2 \sqcup x_2\, 0\, i_1 \sqcup x_2\, 1\, i_2 \\
&= x_2 i_1 \sqcup x_2 i_2 = 1.
\end{aligned}$$

Computing the other direction,

$$\begin{aligned}(y_1 i_1 \sqcup y_2 i_2)(x_2 j_1 \sqcup x_2 j_2) &= (y_1 i_1(x_2 j_1 \sqcup x_2 j_2)) \sqcup (y_2 i_2(x_2 j_1 \sqcup x_2 j_2)) \\ &= y_1 i_1 x_2 j_1 \sqcup y_1 i_1 x_2 j_2 \sqcup y_2 i_2 x_2 j_1 \sqcup y_2 i_2 x_2 j_2 \\ &= y_1\, 1\, j_1 \sqcup y_1\, 0\, j_2 \sqcup y_2\, 0\, j_1 \sqcup y_2\, 1\, j_2 \\ &= y_1 j_1 \sqcup y_2 j_2 = 1.\end{aligned}$$

This shows that the map between any two disjoint sums over the same two objects is an isomorphism. $\square$

Proposition 7.3.4. *A disjoint sum tensor in an inverse category* $\mathbb{X}$ *gives disjoint sums, i.e., for each pair of objects* A, B*,* $A \oplus B$ *is a disjoint sum.*

Proof. We claim that setting $i_i = \amalg_i$ and $x_i = {\amalg_i}^{(-1)}$ and setting $X = A \oplus B$ produces a disjoint sum in $\mathbb{X}$. We show this satisfies the four conditions of Definition 7.3.2.

(i) From Lemma 7.1.4, we know that $\amalg_1$ and $\amalg_2$ are monic maps.

(ii) $\amalg_1 : A \to A \oplus B$, $\amalg_2 : B \to A \oplus B$, ${\amalg_1}^{(-1)} : A \oplus B \to A$ and ${\amalg_2}^{(-1)} : A \oplus B \to B$.

(iii) ${\amalg_1}^{(-1)} = {\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)} = {\amalg_2}^{(-1)}$.

(iv) ${i_1}^{(-1)} i_1 = 1 \oplus 0 \perp_{\oplus} 0 \oplus 1 = {i_2}^{(-1)} i_2$ as $1 \oplus 0 \mathbin{\triangledown} 0 \oplus 1 = ({u^r_{\oplus}}^{(-1)} \oplus {u^l_{\oplus}}^{(-1)})$ and $1 \oplus 0 \mathbin{\vartriangle} 0 \oplus 1 = ({\amalg_1}^{(-1)} \oplus {\amalg_2}^{(-1)})$. For their join, $(1 \oplus 0) \sqcup_{\oplus} (0 \oplus 1) = ({u^r_{\oplus}}^{(-1)} \oplus {u^l_{\oplus}}^{(-1)})({\amalg_1}^{(-1)} \oplus {\amalg_2}^{(-1)}) = {u^r_{\oplus}}^{(-1)} {\amalg_1}^{(-1)} \oplus {u^l_{\oplus}}^{(-1)} {\amalg_2}^{(-1)} = 1 \oplus 1 = 1$.

$\square$

Now, we show the other direction, that is, when there are "enough" disjoint sums in a category, we may construct a disjoint sum tensor. In anticipation of the result, henceforth we will write $A \oplus B$ for the disjoint sum of A and B.

We begin by showing that we may construct maps between the disjoint sums:

Lemma 7.3.5. *Given $\mathbb{X}$ an inverse category with all disjoint sums and suppose $f : A \to C$ and $g : B \to D$ in $\mathbb{X}$. Then $i_1{}^{(-1)} f i_1 \perp i_2{}^{(-1)} g i_2 : A \oplus B \to C \oplus D$ and therefore $i_1{}^{(-1)} f i_1 \sqcup i_2{}^{(-1)} g i_2 : A \oplus B \to C \oplus D$.*

Proof. Note that $\overline{i_1{}^{(-1)} f i_1} = \overline{i_1{}^{(-1)} f} \leq \overline{i_1{}^{(-1)}}$ and similarly $\overline{i_2{}^{(-1)} g i_2} \leq \overline{i_2{}^{(-1)}}$. Then, by [**Dis.3**], we have $\overline{i_1{}^{(-1)} f i_1} \perp \overline{i_2{}^{(-1)} g i_2}$. As $\widehat{i_1{}^{(-1)} f i_1} \leq \widehat{i_1}$ and $\widehat{i_2{}^{(-1)} g i_2} \leq \widehat{i_2}$, we have $\widehat{i_1{}^{(-1)} f i_1} \perp \widehat{i_2{}^{(-1)} g i_2}$ and by Lemma 6.2.3, this means $i_1{}^{(-1)} f i_1 \perp i_2{}^{(-1)} g i_2$. □

Note that the argument in the above lemma is one we will make frequent use of in the following. Specifically, that $\overline{i_1{}^{(-1)} g} \perp \overline{i_2{}^{(-1)} h}$ for arbitrary g, h and that $\widehat{h i_1} \perp \widehat{k i_2}$.

Proposition 7.3.6. *Given $\mathbb{X}$ is an inverse category where every pair of objects has a disjoint sum, then the tensor $\oplus$ where $A \oplus B$ is a disjoint sum of A, B is a disjoint sum tensor.*

Proof. We must first show, of course, that $\oplus$ is a symmetric monoidal tensor. We first note that $f \oplus g$ is given by $i_1{}^{(-1)} f i_1 \sqcup i_2{}^{(-1)} g i_2$ as shown in Lemma 7.3.5. By the definition of the disjoint sum, identity maps are taken to identity maps and by the stability and universality of the disjoint join, composition is preserved and therefore $\oplus$ is a bi-functor as required.

We must now show the restriction zero is the unit of $\oplus$ and give the structural maps.

If we consider the disjoint sum $A \oplus 0$, we see that $i_2 = \mathbf{0}$, the zero map. This gives us $1_{A \oplus \mathbf{0}} = i_1{}^{(-1)} i_1 \sqcup i_2{}^{(-1)} i_2 = i_1{}^{(-1)} i_1 \sqcup 0 = i_1{}^{(-1)} i_1 = \overline{i_1{}^{(-1)}}$, meaning $i_1{}^{(-1)}$ is total. Thus, $i_1{}^{(-1)}$ is an isomorphism and is $u_\oplus^r$. Similarly, $i_2{}^{(-1)} = u_\oplus^l : \mathbf{0} \oplus A \to A$ is an isomorphism.

For the symmetry map, we see that $c_\oplus = i_1{}^{(-1)} i_2 \sqcup i_2{}^{(-1)} i_1 : A \oplus B \to B \oplus A$ is a symmetry map and that

$$
\begin{aligned}
&(i_1{}^{(-1)} i_2 \sqcup i_2{}^{(-1)} i_1)(i_1{}^{(-1)} i_2 \sqcup i_2{}^{(-1)} i_1) \\
&\qquad = i_1{}^{(-1)} i_2 (i_1{}^{(-1)} i_2 \sqcup i_2{}^{(-1)} i_1) \sqcup i_2{}^{(-1)} i_1 (i_1{}^{(-1)} i_2 \sqcup i_2{}^{(-1)} i_1) \\
&\qquad = i_1{}^{(-1)} i_2 i_1{}^{(-1)} i_2 \sqcup i_1{}^{(-1)} i_2 i_2{}^{(-1)} i_1 \sqcup i_2{}^{(-1)} i_1 i_1{}^{(-1)} i_2 \sqcup i_2{}^{(-1)} i_1 i_2{}^{(-1)} i_1 \\
&\qquad = 0 \sqcup 1 \sqcup 1 \sqcup 0 \\
&\qquad = 1.
\end{aligned}
$$

Thus, we have $c_{\oplus}c_{\oplus} = 1$.

For associativity, we set $a_{\oplus} = {i_1}^{(-1)}i_1i_1 \sqcup {i_2}^{(-1)}{i_1}^{(-1)}i_2i_1 \sqcup {i_2}^{(-1)}{i_2}^{(-1)}i_2 : A \oplus (B \oplus C) \to (A \oplus B) \oplus C$. To visualize this, we have

$$\begin{tikzcd}
B \oplus C \arrow[r, "{i_1}^{(-1)}"] \arrow[dr, "{i_2}^{(-1)}"'] & B \arrow[r, "i_2"] & A \oplus B \arrow[dd, "i_1"] \\
 & C \arrow[dr, "i_2"] & \\
A \oplus (B \oplus C) \arrow[uu, "{i_2}^{(-1)}"] \arrow[rr, dotted, "a_{\oplus}"] \arrow[d, "{i_1}^{(-1)}"'] & & (A \oplus B) \oplus C \\
A \arrow[rr, "i_1"'] & & A \oplus B \arrow[u, "i_1"']
\end{tikzcd}$$

The inverse of the $a_{\oplus}$ is obtained by taking the inverses of the arrows in the above diagram, yielding ${i_1}^{(-1)}{i_1}^{(-1)}i_1 \sqcup {i_1}^{(-1)}{i_2}^{(-1)}i_1i_1 \sqcup {i_2}^{(-1)}i_2i_2$.

Thus, $\oplus$ is a symmetric monoidal tensor on $\mathbb{X}$.

To show it is a disjoint sum tensor, we first note that we immediately have that it is a disjointness tensor as:

(i) We have shown above it is a restriction functor.

(ii) We have shown above that the unit is the restriction zero.

(iii) From the above, we see $\amalg_1 = i_1$ and $\amalg_2 = i_2$. Assume we have $\amalg_1 f = \amalg_1 g$ and $\amalg_2 f = \amalg_2 g$. As ${\amalg_1}^{(-1)} \amalg_1 \sqcup {\amalg_2}^{(-1)}\amalg_2 = 1$, we have

$$f = ({\amalg_1}^{(-1)} \amalg_1 \sqcup {\amalg_2}^{(-1)}\amalg_2)f = {\amalg_1}^{(-1)} \amalg_1 f \sqcup {\amalg_2}^{(-1)} \amalg_2 f =$$
$${\amalg_1}^{(-1)} \amalg_1 g \sqcup {\amalg_2}^{(-1)} \amalg_2 g = ({\amalg_1}^{(-1)} \amalg_1 \sqcup {\amalg_2}^{(-1)}\amalg_2)g = g$$

and therefore $\amalg_1$ and $\amalg_2$ are jointly monic.

(iv) ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ are jointly epic by a similar argument as (iii).

Note that we may also immediately conclude that the original disjointness relation on $\mathbb{X}$ is the same as the one generated by $\oplus$. This is because $f \triangledown g$ will be equal to $f \amalg_1 \sqcup g\amalg_2$ and

$f \vartriangle g$ will be equal to ${\amalg_1}^{(-1)} f \sqcup {\amalg_2}^{(-1)} g$, therefore, $f \triangledown g$ and $f \vartriangle g$ will exist if and only if $f \perp g$.

To show it is a disjoint sum tensor, we must show the existence of the maps α and β from Definition 7.2.1.

For Diagram (7.3), assuming we have $f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$ exists, then $\alpha = (f{\amalg_1}^{(-1)})\, \amalg_1 \sqcup ((f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)})\amalg_2)$ satisfies the diagram. If α does exist, then $\alpha{\amalg_2}^{(-1)}$ satisfies the diagram for $f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$. The argument for satisfying Diagram (7.4) is analogous.

Therefore, $\oplus$ is a disjoint sum tensor. $\square$

Chapter 8

Commutative Frobenius Algebras

8.1 The category of Commutative Frobenius Algebras

8.1.1 Dagger categories

Dagger categories generalize the concepts of Hilbert spaces that are required to model quantum computation. These were introduced in [3] as *strongly compact closed categories*, an additional structure only on compact closed categories.

Before introducing dagger categories, we define compact closed categories.

Definition 8.1.1. A *compact closed category* $\mathbb{D}$ is a symmetric monoidal category with tensor $\otimes$ where each object A has a dual A^*. Additionally, there must exist families of maps $\eta_A : I \to A^* \otimes A$ (the *unit*) and $\epsilon_A : A \otimes A^* \to I$ (the *counit*) such that

$$\begin{tikzcd}
A \arrow[r, "u_A"] \arrow[d, equal] & A \otimes I \arrow[r, "1\otimes\eta_A"] & A \otimes (A^* \otimes A) \arrow[d, "a_{A,A^*,A}"] \\
A & I \otimes A \arrow[l, "u_A^{-1}"] & (A \otimes A^*) \otimes A \arrow[l, "\otimes\epsilon_B\otimes 1"]
\end{tikzcd}$$

commutes and so does the similar one based on A^*.

Given a map $f : A \to B$ in a compact closed category, define the map $f^* : B^* \to A^*$ as

$$\begin{tikzcd}
B^* \arrow[r, "u_{B^*}"] \arrow[d, "f^*"'] & I \otimes B^* \arrow[r, "\eta_A\otimes 1"] & A^* \otimes A \otimes B^* \arrow[d, "1\otimes f\otimes 1"] \\
A^* & A^* \otimes I \arrow[l, "u_{A^*}^{-1}"] & A^* \otimes B \otimes B^*. \arrow[l, "1\otimes\epsilon_B"]
\end{tikzcd}$$

Although dagger categories were introduced in the context of compact closed categories, the concept of a dagger is definable independently. This was first done in [50].

Definition 8.1.2. A *dagger operator* on a category D is a functor $\dagger : \mathbb{D}^{op} \to \mathbb{D}$, which is involutive in the sense that it is the identity on objects. A *dagger category* is a category that has a dagger operator.

Typically, the dagger is written as a superscript on the morphism. So, if $f : A \to B$ is a map in $\mathbb{D}$, then $f^\dagger : B \to A$ is a map in $\mathbb{D}$ and is called the *adjoint* of f. A map where $f^{-1} = f^\dagger$ is called *unitary.* A map $f : A \to A$ with $f = f^\dagger$ is called *self-adjoint* or *Hermitian.*

Definition 8.1.3. A *dagger symmetric monoidal category* is a symmetric monoidal category $\mathbb{D}$ with a dagger operator such that:

(i) For all maps $f : A \to B$ and $g : C \to D$, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger : B \otimes D \to A \otimes C$;

(ii) The monoid structure isomorphisms $a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $u^l_A : I \otimes A \to A$, $u^r_A : A \otimes I \to A$ and $c_{A,B} : A \otimes B \to B \otimes A$ are unitary.

Definition 8.1.4. A *dagger compact closed category* $\mathbb{D}$ is a dagger symmetric monoidal category that is compact closed where the diagram

$$\begin{tikzcd} I \arrow[r, "\epsilon_A^\dagger"] \arrow[dr, "\eta_A"'] & A \otimes A^* \arrow[d, "c_{A,A^*}"] \\ & A^* \otimes A \end{tikzcd}$$

commutes for all objects A in $\mathbb{D}$.

Lemma 8.1.5. *If $\mathbb{D}$ is a dagger category with biproducts, with injections in_1, in_2 and projections p_1, p_2, then the following are equivalent:*

(i) ${p_i}^\dagger = in_i, i = 1, 2,$

(ii) $(f \boxplus g)^\dagger = f^\dagger \boxplus g^\dagger$ *and* $\Delta^\dagger = \nabla$,

(iii) $\langle f, g\rangle^\dagger = [f^\dagger, g^\dagger]$,

(iv) The map $[{p_1}^\dagger, {p_2}^\dagger] : A^\dagger \boxplus B^\dagger \to (A \boxplus B)^\dagger$ *is the identity map.*

Proof. **(i)** $\implies$ **(ii)** To show $\Delta^\dagger = \nabla$, draw the product cone for Δ,

$$\begin{tikzcd} & A \arrow[dl, "id"'] \arrow[d, "\Delta"] \arrow[dr, "id"] & \\ A & A \boxplus A \arrow[l, "p_1"] \arrow[r, "p_2"'] & A \end{tikzcd}$$

and apply the dagger functor to it. As ${p_i}^\dagger = in_i$, and $\dagger$ is identity on objects, this is now a coproduct diagram and therefore $\Delta^\dagger = \nabla$.

For $(f \boxplus g)^\dagger = f^\dagger \boxplus g^\dagger$, start with the diagram defining $f \boxplus g$ as a product of the arrows:

$$\begin{tikzcd} A \arrow[d, "f"'] & A \boxplus B \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "f \boxplus g"] & A \arrow[d, "g"] \\ C & C \boxplus D \arrow[l, "p_1"] \arrow[r, "p_2"'] & D. \end{tikzcd}$$

Then, apply the dagger functor to this diagram. This is now the diagram defining the co-product of maps and therefore $(f \boxplus g)^\dagger = f^\dagger \boxplus g^\dagger$.

(ii) $\implies$ **(iii)** The calculation showing this is

$$\begin{aligned} [f^\dagger, g^\dagger] &= \nabla; (f^\dagger \boxplus g^\dagger) \\ &= \Delta^\dagger; (f^\dagger \boxplus g^\dagger) \\ &= \Delta^\dagger; (f \boxplus g)^\dagger \\ &= ((f \boxplus g); \Delta)^\dagger \\ &= \langle f, g\rangle^\dagger \end{aligned}$$

(iii) $\implies$ **(iv)** Under the assumption,

$$[{p_1}^\dagger, {p_2}^\dagger] = \langle p_1, p_2\rangle^\dagger = id^\dagger = id.$$

(iv) $\implies$ **(i)** As $[in_1, in_2] : A^\dagger \boxplus B^\dagger \to A^\dagger \boxplus B^\dagger = id = [{p_1}^\dagger, {p_2}^\dagger]$, we immediately have ${p_1}^\dagger = in_1$ and ${p_2}^\dagger = in_2$.

$\square$

Definition 8.1.6. A *biproduct dagger compact closed category* is a dagger compact closed category with biproducts where the conditions of lemma 8.1.5 hold.

8.1.2 Examples of dagger categories

Example 8.1.7 (FdHilb)**.** The category of finite dimensional Hilbert spaces is the motivating example for the creation of the dagger and is, in fact, a biproduct dagger compact closed category. The biproduct is the direct sum of Hilbert spaces and the tensor for compact closure is the standard tensor of Hilbert spaces. The dual H^* of a space H is the space of all continuous linear functions from H to the base field. The dagger is defined via the adjoint as being the unique map $f^\dagger : B \to A$ such that $\langle fa|b\rangle = \langle a|f^\dagger b\rangle$ for all $a \in A, b \in B$.

Example 8.1.8 (Rel)**.** The category Rel of sets and relations has the tensor $S \otimes T = S \times T$, the Cartesian product and the biproduct $S \boxplus T = S + T$, the disjoint union. This is compact closed under $A^* = A$ and the dagger is the * operation, the relational converse. That is, if the relation $R = \{(s,t)|s \in S, t \in T\} : S \to T$, then $R^\dagger = R^* = \{(t,s)|(s,t) \in R\}$.

Example 8.1.9 (Inverse categories)**.** An inverse category $\mathbb{X}$ is also a dagger category when the dagger is defined as the partial inverse. The unitary maps are the total maps. When the inverse category $\mathbb{X}$ is also a symmetric monoidal category where the monoid $\otimes$ is actually a restriction bi-functor, then $\mathbb{X}$ is a dagger symmetric monoidal category.

Requirement (i) of Definition 8.1.3 is fulfilled, as

$$(f \otimes g)(f \otimes g)^{(-1)} = \overline{f \otimes g} = \overline{f} \otimes \overline{g} = ff^{(-1)} \otimes gg^{(-1)} = (f \otimes g)(f^{(-1)} \otimes g^{(-1)})$$

and since the partial inverse of $f \otimes g$ is unique, $(f \otimes g)^{(-1)} = f^{(-1)} \otimes g^{(-1)}$. Requirement (ii) is that the structure isomorphisms are unitary. This is, of course, true as each of them are isomorphisms, hence total and therefore unitary.

8.1.3 Frobenius Algebras

Frobenius algebras are defined as a finite dimensional algebra over a field together with a non-degenerate pairing operation. We will continue with the definitions that make this precise.

Definition 8.1.10. Given a symmetric monoidal category $\mathbb{D}$, a *Frobenius algebra* is an object X of $\mathbb{D}$ and four maps, $\nabla : X \otimes X \to X$, $\eta : I \to X$, $\Delta : X \to X \otimes X$ and $\epsilon : X \to I$, with the conditions that (X, ∇, η) forms a commutative monoid, (X, Δ, ϵ) forms a commutative comonoid and the diagrams

$$\begin{tikzcd}
X \otimes X \arrow[rr, "1\otimes\Delta"] \arrow[dr, "\nabla"] \arrow[dd, "\Delta\otimes 1"'] & & X \otimes X \otimes X \arrow[dd, "\nabla\otimes 1"] \\
& X \arrow[dr, "\Delta"] & \\
X \otimes X \otimes X \arrow[rr, "1\otimes\nabla"'] & & X \otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[ddr, equal] & A \otimes A \arrow[d, "\epsilon\otimes 1"] \\
& I \otimes A \arrow[d, "u^l_\otimes"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u^{r\,(-1)}_\otimes"] \arrow[ddr, equal] & A \otimes I \arrow[d, "1\otimes\eta"] \\
& A \otimes A \arrow[d, "\nabla"] \\
& A
\end{tikzcd}$$

all commute. The Frobenius algebra is *separable* when $\Delta\nabla = 1_X$ and *commutative* when $\Delta c_{X,X} = \Delta$. Note that separable is sometimes referred to as *special*.

Definition 8.1.11. A Frobenius algebra in a dagger symmetric monoidal category where $\Delta = \nabla^\dagger$ and $\epsilon = \eta^\dagger$ is a $\dagger$-*Frobenius algebra*.

For an example of a †-Frobenius algebra, consider a finite dimensional Hilbert space H with an orthonormal basis $\{|\phi_i\rangle\}$ and define $\Delta : H \to H \otimes H : |\phi_i\rangle \mapsto |\phi_i\rangle \otimes |\phi_i\rangle$ and $\epsilon : H \to \mathbb{C} : |\phi_i\rangle \mapsto 1$. Then $(H, \nabla = \Delta^\dagger, \eta = \epsilon^\dagger, \Delta, \epsilon)$ forms a commutative special †-Frobenius algebra.

In [22], Coecke et. al. provide an algebraic description of orthogonal bases in finite dimensional Hilbert spaces. Additionally, an orthonormal basis for such a space is a separable commutative †-Frobenius algebra. To show the other direction, given a commutative †-Frobenius algebra, (H, ∇, η) and for each element $\alpha \in H$, define the right action of α as $R_\alpha := (id \otimes \alpha)\,\nabla : H \to H$. Note the use of the fact that elements $\alpha \in H$ can be considered as linear maps $\alpha : \mathbb{C} \to H : 1 \mapsto |\alpha\rangle$. The dagger of a right action is also a right action, ${R_\alpha}^\dagger = R_{\alpha'}$ where $\alpha' = \eta\,\nabla\,(id \otimes \alpha^\dagger)$, which is a consequence of the Frobenius identities.

The $(\_)'$ construction is actually an involution:

$$\begin{aligned}
(\alpha')' &= \eta\nabla(id \otimes \alpha'^{\dagger}) \\
&= u\nabla(id \otimes (\eta\nabla(id \otimes \alpha^{\dagger}))^{\dagger} \\
&= u\nabla(id \otimes ((id \otimes \alpha)\Delta\epsilon)) \\
&= (u \otimes \alpha)(\nabla \otimes id)(id \otimes \Delta)(id \otimes \epsilon) \\
&= (u \otimes \alpha)(id \otimes \Delta)(\nabla \otimes id)(id \otimes \epsilon) \\
&= (u \otimes \alpha)(id \otimes \epsilon) \\
&= \alpha
\end{aligned}$$

Lemma 8.1.12. *Any †-Frobenius algebra in* FdHilb *is a C^*-algebra.*

Proof. The endomorphism monoid of FdHilb (H,H) is a C^*-algebra. From the proceeding, we have

$$H \cong \text{FdHilb}(\mathbb{C}, H) \cong R_{[\text{FdHilb}(\mathbb{C},H)]} \subseteq \text{FdHilb}(H, H).$$

This inherits the algebra structure from FdHilb (H,H). Furthermore, since any finite dimensional involution-closed sub-algebra of a C^*-algebra is also a C^*-algebra, this shows the †-Frobenius algebra is a C^*-algebra. □

Using the fact that the involution preserving homomorphisms from a finite dimensional commutative C^*-algebra to $\mathbb{C}$ form a basis for the dual of the underlying vector space, write these homomorphisms as ${\phi_i}^{\dagger} : H \to \mathbb{C}$. Then their adjoints, $\phi_i : \mathbb{C} \to H$ will form a basis for the space H. These are the copyable elements in H.

This, together with continued applications of the Frobenius rules and linear algebra allow the authors to prove the following Theorem.

Theorem 8.1.13. *Every commutative †-Frobenius algebra in* FdHilb *determines an orthogonal basis consisting of its copyable elements. Conversely, every orthogonal basis* $\{|\phi_i\rangle\}_i$

determines a commutative †-Frobenius algebra via

$$\Delta : H \to H \otimes H : |\phi_i\rangle \mapsto |\phi_i\rangle \otimes |\phi_i\rangle \qquad \epsilon : H \to \mathbb{C} : |\phi_i\rangle \mapsto 1$$

and these constructions are inverse to each other.

In [21], Coecke et.al. build on the results of [22] to start from a †-symmetric monoidal category and construct the minimal machinery needed to model quantum and classical computations. For the rest of this section, $\mathbb{D}$ will be assumed to be such a category, with $\otimes$ the monoid tensor and I the unit of the monoid.

Definition 8.1.14. A compact structure on an object A in the category $\mathbb{D}$ is given by the object A, an object A^* called its *dual* and the maps $\eta : I \to A^* \otimes A$, $\epsilon : A \otimes A^* \to I$ such that the diagrams

$$\begin{tikzcd} A^* \arrow[d, "\eta\otimes A^*"'] \arrow[dr, "id"] & \\ A^* \otimes A \otimes A^* \arrow[r, "A^*\otimes\epsilon"'] & A^* \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "A\otimes\eta"] \arrow[dr, "id"'] & A \otimes A^* \otimes A \arrow[d, "\epsilon\otimes A"] \\ & A \end{tikzcd}$$

commute.

Definition 8.1.15. A *quantum structure* is an object A and map $\eta : I \to A \otimes A$ such that $(A, A, \eta, \eta^\dagger)$ form a compact structure.

Note that A is self-dual in definition 8.1.15.

This allows the creation of the category $\mathbb{D}_q$ which has as objects quantum structures and maps are the maps in $\mathbb{D}$ between the objects in the quantum structures.

In the category $\mathbb{D}_q$, it is now possible to define the upper and lower $*$ operations on maps, such that $(f_*)^* = (f^*)_* = f^\dagger$:

$$\begin{aligned} f^* &:= (\eta_A \otimes 1)(1 \otimes f \otimes 1)(1 \otimes \eta_B^\dagger), \\ f_* &:= (\eta_B \otimes 1)(1 \otimes f^\dagger \otimes 1)(1 \otimes \eta_A^\dagger). \end{aligned}$$

8.1.4 CFrob($\mathbb{X}$) is an inverse category

Example 8.1.16 (Commutative separable Frobenius algebras)**.** Let $\mathbb{X}$ be a symmetric monoidal category and form CFrob($\mathbb{X}$) as follows:

Objects: Commutative separable Frobenius algebras [34]: A quintuple $(A, \nabla, \eta, \Delta, \epsilon)$ where A is an object of $\mathbb{X}$ with the following maps: $\nabla : A \otimes A \to A$, $\eta : I \to A$, $\Delta : A \to A \otimes A$, $\epsilon : A \to I$ which are natural maps in $\mathbb{X}$, with (A, ∇, η) a monoid and (A, Δ, ϵ) a co-monoid. Additionally, these satisfy

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\Delta\otimes 1"] \arrow[d, "1\otimes\Delta"'] \arrow[dr, "\nabla"] & & A \otimes (A \otimes A) \arrow[d, "1\otimes\nabla"] \\
 & A \arrow[dr, "\Delta"] & \\
(A \otimes A) \otimes A \arrow[rr, "\nabla\otimes 1"'] & & A \otimes A
\end{tikzcd}$$

Frobenius

together with the additional property that $\Delta\nabla = 1$ (separable).

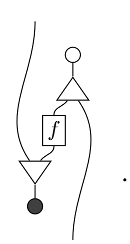
Maps: The maps of $\mathbb{X}$ between the objects of $\mathbb{X}$ which preserve multiplication (∇) and co-multiplication (Δ) but do not necessarily preserve the units.

Lemma 8.1.17. *When $\mathbb{X}$ is a symmetric monoidal category, CFrob($\mathbb{X}$) is a inverse category.*

Proof. We need to show that CFrob($\mathbb{X}$) has restrictions and that each map has a partial inverse. We do this by exhibiting the partial inverse of a map. For $f : X \to Y$, define $f^{(-1)}$ as

$$Y \xrightarrow{1\otimes\eta} Y \otimes X \xrightarrow{1\otimes\Delta} Y \otimes X \otimes X \xrightarrow{1\otimes f\otimes 1} Y \otimes Y \otimes X \xrightarrow{\nabla\otimes 1} Y \otimes X \xrightarrow{\epsilon\otimes 1} X.$$

As a string diagram, this looks like:



If $f^{(-1)}$ is truly a partial inverse, we may then define $\overline{f} = ff^{(-1)}$. Using Theorem 2.20 from [17], we need only show:

$$(f^{(-1)})^{(-1)} = f \tag{8.1}$$

$$ff^{(-1)}f = f \tag{8.2}$$

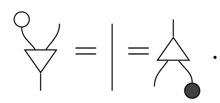
$$ff^{(-1)}gg^{(-1)} = gg^{(-1)}ff^{(-1)}. \tag{8.3}$$

We also use the following two identities from [34]:

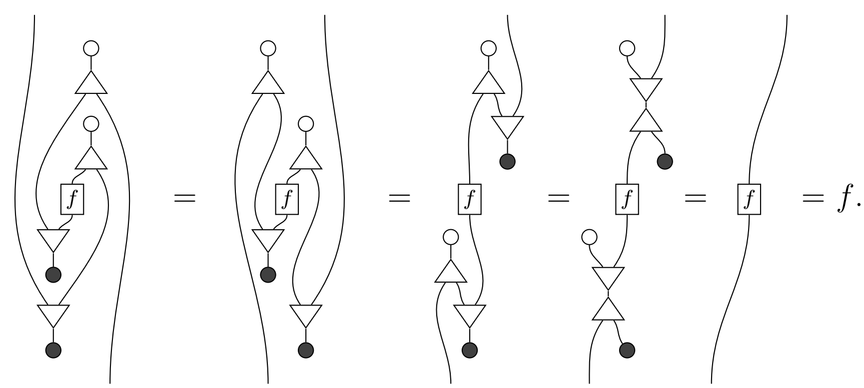
$$(1 \otimes \eta)\nabla = id \tag{8.4}$$

$$\Delta(1 \otimes \epsilon) = id. \tag{8.5}$$

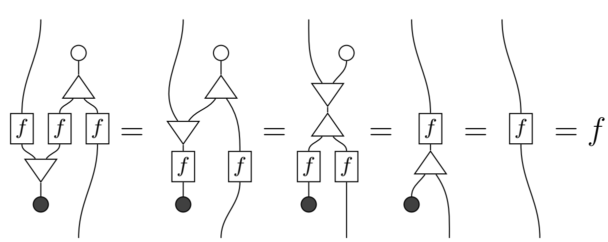
Diagrammatically, this is:



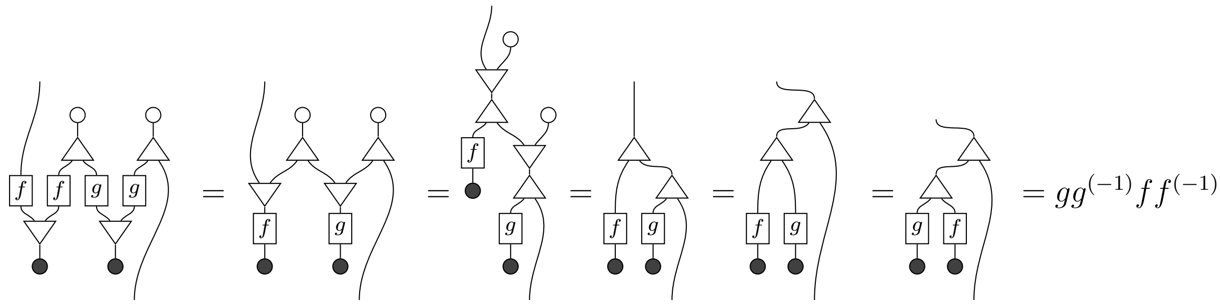
Proof of Equation (8.1): $(f^{(-1)})^{(-1)} =$



Proof of Equation (8.2): $ff^{(-1)}f =$



Proof of Equation (8.3): $ff^{(-1)}gg^{(-1)} =$



where the last step is accomplished by reversing all the previous diagrammatic steps. Hence, CFrob($\mathbb{X}$) is an inverse category. □

Theorem 8.1.18. *When $\mathbb{X}$ is a symmetric monoidal category, CFrob($\mathbb{X}$) is a discrete inverse category.*

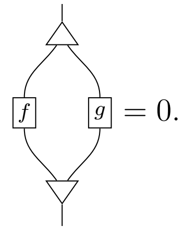
Proof. Lemma 8.1.17 shows CFrob($\mathbb{X}$) is an inverse category. We need only show the conditions of Definition 4.3.1 are met.

By assumption, as we are working with Frobenius algebras over a symmetric tensor $\otimes$, we immediately have the Frobenius rule, co-commutativity, co-associativity and that Δ is canonical with respect to the tensor. Hence the tensor fulfills the conditions of being an inverse product and therefore CFrob($\mathbb{X}$) is a discrete inverse category. □

8.2 Disjointness in Frobenius Algebras

An important example of a disjointness system arises naturally from Frobenius Algebras.

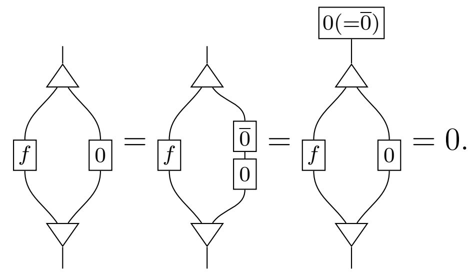
Lemma 8.2.1. *As shown in Lemma 8.1.17 $CFrob(\mathbb{X})$ is a discrete inverse category. For $f, g : A \to B$, define $f \perp g$ when*



The relation $\perp$ is a disjointness relation.

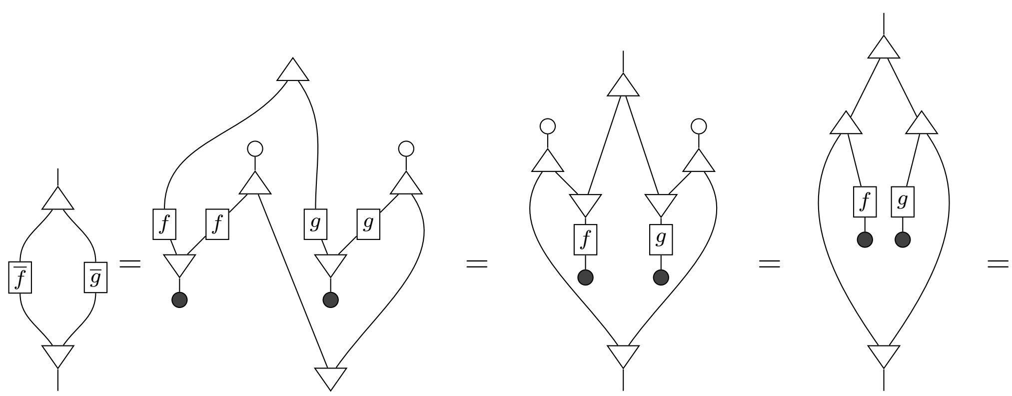
Proof. We need to show the seven axioms of the disjointness relation hold. Note that we will show [**Dis.6**] early on as its result will be used to establish some of the other axioms.

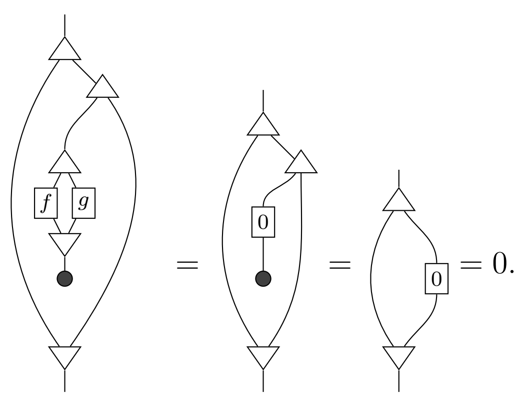
[**Dis.1**]: For all $f : A \to B$, $f \perp 0$.



[**Dis.6**]: $f \perp g$ implies $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$.

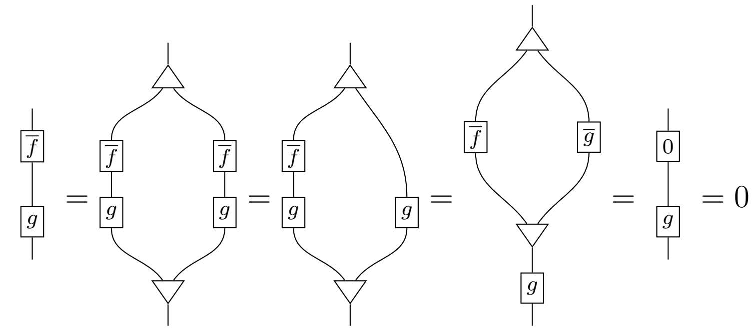
We will show the details of $\overline{f} \perp \overline{g}$, using $\overline{f} = ff^{(-1)}$ and the definition of $f^{(-1)}$ as given in Theorem 8.1.18. The proof of $(f^{(-1)}f =)\hat{f} \perp \hat{g}(= g^{(-1)}g)$ is similar.



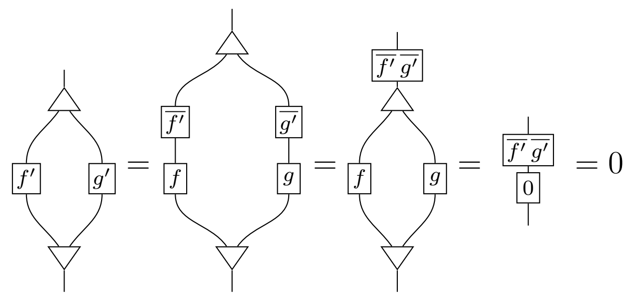


[**Dis.2**]: $f \perp g$ implies $\overline{f}g = 0$.

In this proof, we use the result of [**Dis.6**], i.e., that $\overline{f} \perp \overline{g}$.



[**Dis.3**]: $f \perp g,\ f' \leq f,\ g' \leq g$ implies $f' \perp g'$.



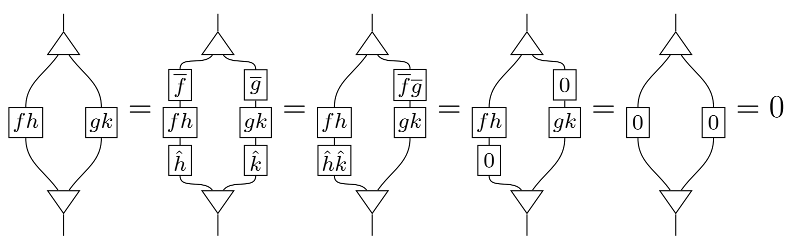
[**Dis.4**]: $f \perp g$ implies $g \perp f$.

This follows directly from the co-commutativity of Δ.

[**Dis.5**]: $f \perp g$ implies $hf \perp hg$.

This follows directly from the naturality of Δ.

[**Dis.7**]: $\overline{f} \perp \overline{g},\ \hat{h} \perp \hat{k}$ implies $fh \perp gk$.



$\square$

8.3 Disjoint sums in Frobenius algebras

In the previous sections, we have shown that CFrob($\mathbb{X}$) is a discrete inverse category, with a disjointness relation whenever $\mathbb{X}$ is a symmetric monoidal category.

We now show that if $\mathbb{X}$ is an additive category, i.e., a symmetric monoidal category where the hom-sets are enriched in additive monoids, then CFrob($\mathbb{X}$) will have a disjoint join and moreover, a disjoint sum.

We begin by showing that given an additive category, we may form an equivalent category which has biproducts.

Lemma 8.3.1. *Suppose $\mathbb{X}$ is an additive symmetric monoidal category. If we have the diagram*

$$\begin{tikzcd} A \arrow[r, "\sigma_1"] & X \arrow[l, bend left, "\pi_1"] \arrow[r, bend right, "\pi_2"'] & B \arrow[l, "\sigma_2"'] \end{tikzcd}$$

with

$$\sigma_1\pi_1 = 1_A, \quad \sigma_2\pi_2 = 1_B, \qquad \sigma_1\pi_2 = 0 = \sigma_2\pi_2$$

and $\pi_1\sigma_1 + \pi_2\sigma_2 = 1$, then X is a biproduct of A and B.

Corollary 8.3.2. *If $F : \mathbb{X} \to \mathbb{Y}$ is an additively enriched functor, then F preserves all biproducts.*

Corollary 8.3.3. *The functor $A \otimes \_ : \mathbb{X} \to \mathbb{X}$ preserves biproducts.*

Thus, if $\mathbb{X}$ is additively enriched, we may add biproducts by moving to the matrix category of $\mathbb{X}$, defined as:

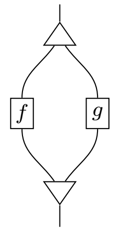
Objects: Lists of the objects $[A_i]$ of $\mathbb{X}$;

Maps: Matrices of maps in $\mathbb{X}$, $[f_{i,j}] : [A_i] \to [B_j]$;

Identity: The diagonal matrix I ($f_{i,i} = 1_{A_i}$ and $f_{i,j} = 0, i \neq j$);

Composition: Matrix multiplication.

Now, let us consider CFrob($\mathbb{X}$) where $\mathbb{X}$ is an additive symmetric monoidal category with biproducts. We know from Lemma 8.2.1 that

$f \perp g \iff$  $= 0.$

We will now show that the biproduct is the disjoint join of any two disjoint maps.

Lemma 8.3.4. *Given $\mathbb{X}$ is an additive symmetric monoidal category with biproduct $+$, and $f \perp g$, then $f \sqcup g \doteq f + g$ is a disjoint join.*

Proof. We need to show the four axioms of disjoint join from Definition 6.3.1.

[**DJ.1**] : $f \leq f \sqcup g$ and $g \leq f \sqcup g$. As $f + g = g + f$, we need only show the first part of the axiom. As $f \perp g$, we know $\overline{f}g = 0$ by the definition of disjointness and therefore we have:

$$\overline{f}(f + g) = \overline{f}f + \overline{f}g = f + 0 = f$$

and thus $f \leq f \sqcup g$ and [**DJ.1**] is true.

[**DJ.2**]: $f \leq h,\ g \leq h$ and $f \perp g$ implies $f \sqcup g \leq h$. We calculate

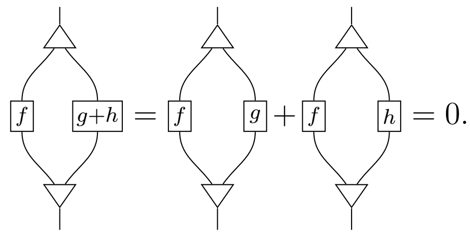
$$\overline{f + g}h = (\overline{f} + \overline{g})h = \overline{f}h + \overline{g}h = f + g$$

giving us the required inequality and [**DJ.1**] is true.

[**DJ.3**]: Disjoint join is stable, i.e., $h(f \sqcup g) = hf \sqcup hg$. This is immediate as $h(f + g) = hf + hg$.

[**DJ.4**]: $\perp \{f, g, h\} \iff f \perp (g \sqcup h)$. Consider the $\Longleftarrow$ direction first. We immediately have $g \perp h$ as we are able to form the disjoint join. By [**DJ.1**], we have both $g \leq g \sqcup h$ and $h \leq g \sqcup h$ and therefore by [**Dis.3**], $f \perp g$ and $f \perp h$. Thus the $\Longleftarrow$ direction is true.

For the $\implies$ direction, we compute



This gives us both directions and all of the axioms have been shown to be true, hence, the biproduct is a disjoint join.

□

Now that we have shown we have a disjoint join, our last remaining task is to add a disjoint sum.

Proposition 8.3.5. *Suppose A and B are Frobenius algebras in CFrob($\mathbb{X}$) where $\mathbb{X}$ has the biproduct $+$. Then we define $A \oplus B$ to be the Frobenius algebra on $A + B$ with the following maps:*

$$\epsilon_{A+B} \doteq [\epsilon_A, \epsilon_B] : A + B \to I,$$

$$\eta_{A+B} \doteq \langle \eta_A, \eta_B \rangle : I \to A + B,$$

$$\Delta_{A+B} \doteq (\Delta_A i_1 + \Delta_B i_2) : A + B \to A \otimes A + A \otimes B + B \otimes A + B \otimes B,$$

$$\nabla_{A+B} \doteq (\pi_1 \nabla_A + \pi_2 \nabla_B) : (A \otimes A + A \otimes B) + (B \otimes A + B \otimes B) \to A + B,$$

noting that $A \otimes A + A \otimes B + B \otimes A + B \otimes B \cong (A + B) \otimes (A + B)$. Then $A \oplus B$ is a disjoint sum in CFrob($\mathbb{X}$).

Proof. First, we must prove that $A \oplus B$ is actually a Frobenius algebra and therefore in CFrob($\mathbb{X}$). To do this, we must show it is separable, the unit laws hold and the Frobenius condition hold for $A \oplus B$.

For the requirement that it is separable, we have

$$\begin{aligned}
\Delta_{A+B}\nabla_{A+B} &= (\Delta_A i_1 + \Delta_B i_2)(\pi_1\nabla_A + \pi_2\nabla_B) \\
&= (\Delta_A i_1\pi_1\nabla_A + \Delta_B i_2\pi_2\nabla_B) \\
&= (\Delta_A\nabla_A + \Delta_B\nabla_B) \\
&= (1_A + 1_B) = 1_{A+B}
\end{aligned}$$

To show the comultiplication unit law, consider

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "i_1"'] & A\otimes A \arrow[d, "i_1\otimes i_1"] \arrow[dr, equal] & \\
A+B \arrow[r, "\Delta"] \arrow[ddr, equal] & (A+B)\otimes(A+B) \arrow[d, "1\otimes\epsilon"] & A\otimes A \arrow[l, "i_1\otimes i_1"] \arrow[d, "1\otimes\epsilon"] \\
& (A+B)\otimes I \arrow[d, "u^l_\otimes"] & A\otimes I \arrow[l, "i_1\otimes 1"] \arrow[d, "u^l_\otimes"] \\
& A+B & A. \arrow[l, "i_1"]
\end{tikzcd}$$

The outer path shows the unit law for the Frobenius algebra A and is what happens to the A component in the inner path. There is a similar diagram where the outer path As are replaced with a B and the i_1 with i_2. As these outer paths commute, they show that the inner path commutes as each component of it commutes.

For the Frobenius law, as we are in a commutative world, we need only show $\nabla\Delta = (1\otimes\Delta)(\nabla\otimes 1)$.

$$\begin{tikzcd}
& A\otimes A \arrow[rr, "1\otimes\Delta"] \arrow[d, "i_1\otimes i_1"] \arrow[dl, equal] & & A\otimes A\otimes A \arrow[d, "i_1\otimes i_1\otimes i_1"] \arrow[dr, equal] & \\
A\otimes A \arrow[r, "i_1\otimes i_1"] \arrow[dr, "\nabla"'] & (A+B)\otimes(A+B) \arrow[rr, "1\otimes\Delta"] \arrow[dr, "\nabla"] & & (A+B)\otimes(A+B)\otimes(A+B) \arrow[dd, "\nabla\otimes 1"] & A\otimes A\otimes A \arrow[l, "i_1\otimes i_1\otimes i_1"] \arrow[dd, "\nabla\otimes 1"] \\
& A \arrow[r, "i_1"] \arrow[ddrr, "\Delta"'] & A+B \arrow[dr, "\Delta"] & & \\
& & & (A+B)\otimes(A+B) & A\otimes A \arrow[l, "i_1\otimes i_1"'] \arrow[dl, equal] \\
& & & A\otimes A \arrow[u, "i_1\otimes i_1"'] &
\end{tikzcd}$$

By the same reasoning as the previous two arguments, the diagram commutes and $A+B$ is a Frobenius algebra.

Next, we must give maps in CFrob($\mathbb{X}$) i_1, i_2, x_1, x_2 that satisfy the disjoint sum diagram,

$$\begin{tikzcd} A \arrow[r, "i_1"] & A+B \arrow[l, bend left, "x_1"] \arrow[r, bend right, "x_2"'] & B \arrow[l, "i_2"'] \end{tikzcd}$$

where

(i) i_1 and i_2 are monic,

(ii) ${i_1}^{(-1)} = x_1$ and ${i_2}^{(-1)} = x_2$, and

(iii) ${i_1}^{(-1)} i_1 \perp {i_2}^{(-1)} i_2$ and ${i_1}^{(-1)} i_1 \sqcup {i_2}^{(-1)} i_2 = 1_X$.

Since $A+B$ is a biproduct and we are in an inverse category, setting i_1, i_2 to be the injections of the biproduct and x_1, x_2 to be the projections will immediately give us this diagram, provided all those maps are in CFrob($\mathbb{X}$). Therefore we must show each of the injections and projections are in CFrob($\mathbb{X}$), that is, they must preserve Δ and ∇.

Consider

$$\begin{tikzcd} A \arrow[rr, "i_1"] \arrow[d, "\Delta"'] & & A+B \arrow[d, "\Delta_{A+B}(=\Delta_A i_1 + \Delta_B i_1)"] \\ A \otimes A \arrow[r, "i_1"'] & A \otimes A + A \otimes B \arrow[r, "i_1"'] & A \otimes A + A \otimes B + B \otimes A + B \otimes B. \end{tikzcd}$$

This commutes as $i_1(f+g) = f i_1$. Similarly, i_2 preserves Δ. Conversely,

$$\begin{tikzcd} A \otimes A + A \otimes B + B \otimes A + B \otimes B \arrow[r, "\pi_1"] \arrow[d, "(\pi_1 \nabla_A + \pi_2 \nabla_B =)\nabla_{A+B}"'] & A \otimes A + A \otimes B \arrow[r, "\pi_1"] & A \otimes A \arrow[d, "\nabla_A"] \\ A+B \arrow[rr, "\pi_1"'] & & A \end{tikzcd}$$

which shows that ∇ is preserved by π_1 and π_2.

For the three conditions, as i_1 and i_2 are total maps, they are monic in the inverse category. We know from above that $i_j \pi_j = 1$ and therefore ${i_j}^{(-1)} = \pi_j$. Additionally we know that $\pi_1 i_1 + \pi_2 i_2 = 1$, but as $+$ is the disjoint join, this shows the third condition is true and $A \oplus B$ is a disjoint sum. $\square$

Chapter 9

Matrix categories

Inverse categories with enough disjoint sums are equivalent to their matrix categories and are, in fact, Unique Decomposition Categories [28].

9.1 Matrices

In this section, we will show that when given an inverse category $\mathbb{X}$ with a disjoint sums, one can define a matrix category based on $\mathbb{X}$. We will call this category $iMat(\mathbb{X})$. Furthermore, we will show that $iMat(\mathbb{X})$ is an inverse category and that $\mathbb{X}$ embeds within this category.

The types of matrices allowed in the matrix category are subject to certain constraints:

Definition 9.1.1. Suppose we have $\mathbb{X}$ is an inverse category with disjoint sums. Then an *disjoint sum matrix* in $\mathbb{X}$ is a matrix of maps $[f_{ij}]$ where $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ with $f_{ij} : A_i \to B_j$ which satisfy the two conditions:

$$\text{For each } i,\ \perp \{f_{ij} \amalg_j\}_{j=1,\ldots,m} \text{ where } \amalg_j : B_j \to B_1 \oplus B_2 \oplus \cdots \oplus B_m. \tag{9.1}$$

$$\text{For each } j,\ \perp \{{\amalg_i}^{(-1)} f_{ij}\}_{i=1,\ldots,n} \text{ where } {\amalg_i}^{(-1)} : A_1 \oplus A_2 \oplus \cdots \oplus A_n \to A_i. \tag{9.2}$$

In the above and following we will use the notation $\amalg_i$ for the i^{th} injection map of the disjoint sum, with i starting at 1. This simply extends the notation introduced in Definition 7.1.1.

We will show that this type of matrix corresponds to maps in the category $iMat(\mathbb{X})$, where composition is given by "matrix multiplication", where the operations of multiplication and addition are replaced with composition in $\mathbb{X}$ and the disjoint join respectively.

Definition 9.1.2. Given an inverse category $\mathbb{X}$ with disjoint sums, we define the *inverse matrix category* of $\mathbb{X}$, $iMat(\mathbb{X})$, as follows:

Objects: Non-empty lists of the objects of $\mathbb{X}$;

Maps: disjoint sum matrices $[f_{ij}] : [A_i] \to [B_j]$. In such a matrix each individual map $f_{ij} : A_i \to B_j$ is a map in $\mathbb{X}$. For each j, B_j is given by applying the map $\sqcup_i \amalg_i^{(-1)} f_{ij}$ to the object $\oplus_i A_i$;

Identity: The disjoint sum matrix I;

Composition: Given $[f_{ij}] : [A_i] \to [B_j]$ and $[g_{jk}] : [B_j] \to [C_k]$, then $[h_{ik}] = [f_{ij}][g_{jk}] : [A_i] \to [C_k]$ is defined as $h_{ik} = \bigsqcup_j f_{ij} g_{jk}$;

Restriction: We set $\overline{[f_{ij}]}$ to be $[f'_{ij}]$ where $f'_{ij} = 0$ when $i \neq j$ and $f'_{ii} = \sqcup_j \overline{f_{ij}}$.

In the following, we will use the notation $\text{diag}[d_j]$ for diagonal matrices where the j, j entry is d_j.

Lemma 9.1.3. *When $\mathbb{X}$ is an inverse category with disjoint sums, $iMat(\mathbb{X})$ is an inverse category.*

Proof. We need to show the following:

- Composition is well defined and associative.
- The restriction is well defined.

Composition is well defined: Consider $[h_{ik}] = [f_{ij}][g_{jk}]$ where $[f_{ij}] : [A_1, \ldots, A_n] \to [B_1, \ldots, B_m]$ and $[g_{jk}] : [B_1, \ldots, B_m] \to [C_1, \ldots, C_\ell]$. By supposition, we know $h_{ik} = \bigsqcup_j f_{ij} g_{jk}$. As each of the maps are disjoint sum matrices, we know that $\perp \{f_{ij} \amalg_j\}$ and $\perp \{\amalg_j^{(-1)} g_{jk}\}$. Hence, for each j we know the composition $f_{ij} \amalg_j \amalg_j^{(-1)} g_{jk} = f_{ij} g_{jk}$ is defined and from A_i to C_k. By the stability and universality of $\sqcup$, we know h_{ik} exists and by the definition of $\sqcup$, we have each $h_{ik} : A_i \to C_k$ and hence composition is well-defined.

Associativity of composition. We have

$$
\begin{aligned}
([f_{ij}][g_{jk}])[h_{k\ell}] &= \left[(\bigsqcup_j f_{ij}g_{jk})\right][h_{k\ell}] \\
&= \left[\bigsqcup_k(\bigsqcup_j f_{ij}g_{jk})h_{k\ell}\right] \\
&= \left[\bigsqcup_j f_{ij}(\bigsqcup_k g_{jk}h_{k\ell})\right] \\
&= [f_{ij}]([g_{jk}][h_{k\ell}]).
\end{aligned}
$$

The restriction axioms.

$$
[\mathbf{R.1}] \quad \overline{[f_{ij}]}[f_{ij}] = \begin{bmatrix} (\sqcup_j\overline{f_{1j}})f_{11} & \cdots & (\sqcup_j\overline{f_{1j}})f_{1n} \\ & \vdots & \\ (\sqcup_j\overline{f_{mj}})f_{m1} & \cdots & (\sqcup_j\overline{f_{mj}})f_{mn} \end{bmatrix} = [f_{ij}].
$$

$[\mathbf{R.2}]$ $\overline{[f_{ij}]}\overline{g_{ij}} = \overline{g_{ij}}\overline{[f_{ij}]}$ as diagonal matrices commute and $\sqcup$ is commutative.

$$
\begin{aligned}
[\mathbf{R.3}] \quad \overline{\overline{[f_ij]}[g_{jk}]} &= \overline{\text{diag}[\sqcup_j\overline{f_{1j}},\ldots,\sqcup_j\overline{f_{nj}}][g_{jk}]} \\
&= \overline{\begin{bmatrix} \sqcup_j\overline{f_{1j}}g_{11} & \ldots & \sqcup_j\overline{f_{1j}}g_{1k} \\ & \vdots & \\ \sqcup_j\overline{f_{nj}}g_{n1} & \ldots & \sqcup_j\overline{f_{nj}}g_{nk} \end{bmatrix}} \\
&= \text{diag}[\sqcup_k(\overline{\sqcup_j(\overline{f_{1j}}g_{1k})}),\ldots,\sqcup_k(\overline{\sqcup_j(\overline{f_{nj}})g_{nk}})] \\
&= \text{diag}[\sqcup_k(\sqcup_j(\overline{f_{1j}})\overline{g_{1k}}),\ldots,\sqcup_k(\sqcup_j(\overline{f_{nj}})\overline{g_{nk}})] \\
&= \text{diag}[(\sqcup_j(\overline{f_{1j}})\sqcup_k\overline{g_{1k}}),\ldots,(\sqcup_j(\overline{f_{nj}})\sqcup_k\overline{g_{nk}})] = \overline{[f_ij]}\,\overline{[g_{jk}]}.
\end{aligned}
$$

[R.4] $[f_{ij}]\overline{[g_{jk}]} = [f_{ij}]\ \mathrm{diag}_j[\sqcup_k \overline{g_{jk}}]$

$$= \begin{bmatrix} f_{11} \sqcup_k \overline{g_{1k}} & \dots & f_{1n} \sqcup_k \overline{g_{nk}} \\ & \vdots & \\ f_{m1} \sqcup_k \overline{g_{1k}} & \dots & f_{mn} \sqcup_k \overline{g_{nk}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqcup_k f_{11}\overline{g_{1k}} & \dots & \sqcup_k f_{1n}\overline{g_{nk}} \\ & \vdots & \\ \sqcup_k f_{m1}\overline{g_{1k}} & \dots & \sqcup_k f_{mn}\overline{g_{nk}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqcup_k \overline{f_{11}g_{1k}} f_{11} & \dots & \sqcup_k \overline{f_{1n}g_{nk}} f_{1n} \\ & \vdots & \\ \sqcup_k \overline{f_{m1}g_{1k}} f_{m1} & \dots & \sqcup_k \overline{f_{mn}g_{nk}} f_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \sqcup_j \sqcup_k \overline{f_{1j}g_{jk}} f_{11} & \dots & \sqcup_j \sqcup_k \overline{f_{1j}g_{jk}} f_{1n} \\ & \vdots & \\ \sqcup_j \sqcup_k \overline{f_{mj}g_{jk}} f_{m1} & \dots & \sqcup_j \sqcup_k \overline{f_{mj}g_{jk}} f_{mn} \end{bmatrix}$$

$$= \overline{[f_{ij}][g_{jk}]}[f_{ij}].$$

Thus, $iMat(\mathbb{X})$ is a restriction category.

The inverse of the map $f = [f_{ij}]$ is the map $f^{(-1)} \doteq [{f_{ji}}^{(-1)}]$. Recalling that the rows and columns of f are each disjoint, we see that the composition $ff^{(-1)} = \mathrm{diag}_i[\sqcup_j \overline{f_{ij}}] = \overline{f}$. □

Furthermore, $iMat(\mathbb{X})$ is actually a disjoint sum category:

Theorem 9.1.4. *Given $\mathbb{X}$ an inverse category with disjoint sums and restriction zero, $iMat(\mathbb{X})$ has disjoint sums.*

We will prove this in a series of lemmas.

Lemma 9.1.5. *Given $\mathbb{X}$ is an inverse restriction category with a restriction zero and a disjoint join, then $iMat(\mathbb{X})$ has a restriction zero.*

Proof. The restriction zero in $iMat(\mathbb{X})$ is the list $[0]$ where 0 is the restriction zero in $\mathbb{X}$.

For the object $A = [A_1, \ldots, A_n]$, the 0 map is given by the $n \times 1$ matrix $[0, \ldots, 0]$. The map from 0 is given by the $1 \times n$ matrix $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

□

Lemma 9.1.6. *Given $\mathbb{X}$ is an inverse restriction category with a restriction zero, 0, and a disjoint join, then the monoid $\oplus$ defined by list catenation of objects is a disjointness tensor.*

Proof. We first note the monoidal isomorphisms:

$$u^l_\oplus : [0, A_1, A_2, \ldots, A_n] \to [A_1, A_2, \ldots, A_n] \qquad u^l_\oplus := \begin{bmatrix} 0 & \cdots & 0 \\ & I_{n\times n} & \end{bmatrix}$$

$$u^r_\oplus : [A_1, A_2, \ldots, A_n, 0] \to [A_1, A_2, \ldots, A_n] \qquad u^r_\oplus := \begin{bmatrix} & I_{n\times n} & \\ 0 & \cdots & 0 \end{bmatrix}$$

$$a_\oplus : (A \oplus B) \oplus C \to A \oplus (B \oplus C) \qquad a_\oplus := id$$

$$c_\oplus : [A_1, \ldots, A_n, B_1, \ldots, B_m] \to [B_1, \ldots, B_m, A_1, \ldots, A_n] \qquad c_\oplus := \begin{bmatrix} 0_{m\times n} & I_{n\times n} \\ I_{m\times m} & 0_{n\times m} \end{bmatrix}.$$

The action of $\oplus$ on maps is given by:

$$[f_{ij}] \oplus [g_{\ell k}] = \begin{bmatrix} [f_{ij}] & 0 \\ 0 & [g_{\ell k}] \end{bmatrix}.$$

With this definition, we see that $\oplus$ is a restriction functor:

$$id_X \oplus id_Y = id_{X\oplus Y},$$

$$f_1g_1 \oplus f_2g_2 = h_1 \oplus h_2 = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ 0 & g_1 \end{bmatrix} \begin{bmatrix} f_2 & 0 \\ 0 & g_2 \end{bmatrix} = (f_1 \oplus g_1)(f_2 \oplus g_2).$$

Following Definition 7.1.1, we note ${\amalg_1}^{(-1)} = (1 \oplus 0)u_\oplus^r = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and similarly ${\amalg_2}^{(-1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose we have $f = [f_{ij}]$ and $g = [g_{ij}]$ where $i \in \{1, \dots, n\}$ and $j \in \{1, 2\}$. Further suppose $f{\amalg_1}^{(-1)} = g{\amalg_1}^{(-1)}$ and $f{\amalg_1}^{(-1)} = g{\amalg_1}^{(-1)}$. Therefore, $f{\amalg_1}^{(-1)} = [f_{i1}] = [g_{i1}] = g{\amalg_1}^{(-1)}$ and $f{\amalg_2}^{(-1)} = [f_{i2}] = [g_{i2}] = g{\amalg_2}^{(-1)}$, but this means that $f = g$ and we may conclude ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ are jointly monic. Similarly, $\amalg_1 = [1\ 0]$ and $\amalg_2 = [0\ 1]$ are jointly epic. $\square$

Lemma 9.1.7. *Given $\mathbb{X}$ is an inverse category with a disjoint join and restriction zero, then $iMat(\mathbb{X})$ has a disjoint sum tensor.*

Proof. By Lemma 9.1.6, we know that the tensor defined by list catenation is a disjointness tensor. To show that it is a disjoint sum tensor, we must show the diagrams and conditions of Definition 7.2.1 hold.

First, for the diagram below, we show that α exists if and only if $f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$.

$$\begin{tikzcd}
{[A]} \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& {[X,Y,Z]} \arrow[r, "{\amalg_{1,3}}^{(-1)}"'] \arrow[d, "{\amalg_{1,2}}^{(-1)}"] & {[X,Z]} \arrow[d, "{\amalg_1}^{(-1)}"] \\
& {[X,Y]} \arrow[r, "{\amalg_1}^{(-1)}"'] & {[X]}.
\end{tikzcd}$$

The existence of $f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$ means there is an $h = [h_1, h_2] : [A] \to [Y, Z]$ such that $h{\amalg_1}^{(-1)} = f{\amalg_2}^{(-1)}$ and $h{\amalg_2}^{(-1)} = g{\amalg_2}^{(-1)}$. From the diagram, given that $f = [f_1, f_2]$ and $g = [g_1, g_2]$, we know that $f_1 = f{\amalg_1}^{(-1)} = g{\amalg_1}^{(-1)} = g_1$. We also have $h_1 = f_2$ and $h_2 = g_2$. If we set α to the matrix $[f_1, f_2, g_2]$, the diagram above commutes. We need only show that

α is a map in $iMat(\mathbb{X})$. As f, g and h are maps in $iMat(\mathbb{X})$, we know that:

$$
\begin{aligned}
f_1\amalg_1 &\perp f_2\amalg_2 \\
(f_1\amalg_1 =)g_1\amalg_1 &\perp g_2\amalg_2 \\
(f_2\amalg_2 =)h_1\amalg_1 &\perp h_2\,\amalg_2\,(= g_2\amalg_2).
\end{aligned}
$$

From this, we can conclude $\perp [f_1\amalg_1, f_2\amalg_2, g_2\amalg_3]$.

Conversely, suppose we have an $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ that makes the above diagram commute. Then $h := [\alpha_2, \alpha_3]$ is a map in $\mathbb{X}$. Since $[\alpha_1, \alpha_3] = g$ and $[\alpha_1, \alpha_2] = f$, we have $h{\amalg_1}^{(-1)} = f{\amalg_2}^{(-1)}$ and $h{\amalg_2}^{(-1)} = g{\amalg_2}^{(-1)}$, hence $h = f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$.

The proof that β in the diagram below exists if and only if $\amalg_2 h \vartriangle \amalg_2 k$ follows similar reasoning.

$$
\begin{tikzcd}
 & & {[A]} \\
{[X,Y]} \arrow[r, "\amalg_{1,2}"'] \arrow[urr, bend left, "h"] & {[X,Y,Z]} \arrow[ur, dotted, "\beta"] & \\
{[X]} \arrow[u, "\amalg_1"] \arrow[r, "\amalg_1"'] & {[X,Z].} \arrow[u, "\amalg_{1,3}"'] \arrow[uur, bend right, "k"'] &
\end{tikzcd}
$$

□

We are now ready to prove Theorem 9.1.4, that $iMat(\mathbb{X})$ has disjoint sums.

Proof. By Lemma 9.1.7, we know $iMat(\mathbb{X})$ has a disjoint sum tensor and therefore by Proposition 7.2.5, it has a disjoint join. By Proposition 7.3.4 we know that $[A, B] = A \oplus B$ is a disjoint sum of A and B for any two objects in $iMat(\mathbb{X})$, and hence, $iMat(\mathbb{X})$ has disjoint sums. □

9.2 Equivalence between a disjoint sum category and its matrix category

In this section we will provide restriction functors between a disjoint sum category $\mathbb{X}$ and its matrix category $iMat(\mathbb{X})$. Furthermore, we will show these functors form an equivalence between these two categories. That is,

$$\begin{tikzcd} \text{Disjoint Sum Cats} \arrow[r, hook] \arrow[r, phantom, shift right=3, "\top"] & \text{Disjoint Join Cats.} \arrow[l, bend left, "\text{Matrix}"] \end{tikzcd}$$

Definition 9.2.1. Given $\mathbb{X}$ has disjoint sums with disjoint join $\perp$ and restriction zero, define $M : \mathbb{X} \to iMat(\mathbb{X})$ by:

$$\text{Objects: } M(A) := [A]$$

$$\text{Maps: } M(f) := [f] - \text{The } 1 \times 1 \text{ matrix with entry } f.$$

Lemma 9.2.2. *The map M from Definition 9.2.1 is a restriction functor.*

Proof. From the definition of $iMat(\mathbb{X})$, we have

$$f : A \to B \text{ if and only if } M(f) : M(A) \to M(B) \quad ([f] : [A] \to [B])$$

$$M(id_A) = [id_A] = id_M(A)$$

$$M(fg) = [fg] = [f][g] = M(f)M(g)$$

$$M(\overline{f}) = [\overline{f}] = \overline{[f]} = \overline{M(f)}.$$

□

Definition 9.2.3. Given $\mathbb{X}$ has disjoint sums with disjoint join $\perp$ and restriction zero 0, and disjoint sum tensor $\oplus$ define $S : iMat(\mathbb{X}) \to \mathbb{X}$ by:

$$\text{Objects: } S([A_1, A_2, \ldots, A_n]) := A_1 \oplus A_2 \oplus \cdots \oplus A_n$$

$$\text{Maps: } S([f_{ij}]) := \bigsqcup_i \amalg_i^{(-1)}(\sqcup_j f_{ij} \amalg_j).$$

Lemma 9.2.4. *The map S from Definition 9.2.3 is a restriction functor.*

Proof. From the definition of $iMat(\mathbb{X})$, where $A = [A_1, A_2, \ldots, A_n]$, $B = [B_1, B_2, \ldots, B_M]$, and $f = [f_{ij}]$ we have

$$S(id_A) = S([id_{A_i}]) = \bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j \amalg_j) = id_{S(A)}$$

$$f : A \to B \iff S(f) : S(A) \to S(B) \iff$$

$$\bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j) : A_1 \oplus \cdots \oplus A_n \to B_1 \oplus \cdots \oplus B_m$$

$$M(\overline{f}) = [\overline{f}] = \overline{[f]} = \overline{M(f)}.$$

For composition, we have

$$\begin{aligned}
S(f)S(g) &= (\bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j))(\bigsqcup_{j'} \amalg_{j'}{}^{(-1)}(\sqcup_k g_{jk} \amalg_k)) \\
&= \bigsqcup_i \amalg_i{}^{(-1)} \bigsqcup_j \bigsqcup_{j'} f_{ij}\ \amalg_j\ \amalg_{j'}{}^{(-1)}(\sqcup_k g_{j'k} \amalg_k) \\
&= \bigsqcup_i \amalg_i{}^{(-1)} \bigsqcup_j f_{ij}(\sqcup_k g_{jk} \amalg_k) \\
&= \bigsqcup_i \amalg_i{}^{(-1)} \bigsqcup_k (\sqcup_j f_{ij} g_{jk} \amalg_k) \\
&= S([\sqcup_j f_{ij} g_{jk}]) \\
&= S(fg).
\end{aligned}$$

□

Proposition 9.2.5. *Given an inverse category $\mathbb{X}$ with a disjoint sum tensor $\oplus$ and restriction zero, then the categories $\mathbb{X}$ and $iMat(\mathbb{X})$ are equivalent.*

Proof. The functors of the equivalence are S from Definition 9.2.3 and M from Definition 9.2.1.

First, we see that $MS : \mathbb{X} \to \mathbb{X}$ is the identity functor as

$$\text{Objects: } S(M(A)) = S([A]) = A,$$

$$\text{Maps: } S(M(f)) = S([f]) = f.$$

Next, we need to show that there is a natural transformation and isomorphism ρ such that $\rho(SM) = I_{iMat(\mathbb{X})}$. For each object $[A_1, A_2, \ldots, A_n]$, set $\rho A = \begin{bmatrix} \amalg_1{}^{(-1)} & \cdots & \amalg_n{}^{(-1)} \end{bmatrix}$.

Note that the functor SM has the following effect:

$$\begin{aligned} \text{Objects:} \quad & M(S([A_1, \ldots, A_n])) = M(A_1 \oplus \cdots \oplus A_n) = [A_1 \oplus \cdots \oplus A_n] \\ \text{Maps:} \quad & M(S([f_{ij}]) = M(\bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j)) = [\bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j)]. \end{aligned}$$

We can now draw the commuting naturality square for $f = [f_{ij}] : [A_i] \to [B_j]$:

$$\begin{tikzcd} SM([A_i]) \arrow[r, equal] & {[\oplus_i A_i]} \arrow[rr, "{\begin{bmatrix} \amalg_1{}^{(-1)} & \cdots & \amalg_n{}^{(-1)} \end{bmatrix}}"] \arrow[d, "SM(f)"'] & & {[A_i]} \arrow[d, "f"] \\ SM([B_j]) \arrow[r, equal] & {[\oplus_j B_j]} \arrow[rr, "{\begin{bmatrix} \amalg_1{}^{(-1)} & \cdots & \amalg_m{}^{(-1)} \end{bmatrix}}"'] & & {[B_j]}. \end{tikzcd}$$

Following the square by the top–right path from $[\oplus_i A_i]$ to $[B_j]$, by the definition of the maps in the category $iMat(\mathbb{X})$, we see each $B_j = \sqcup_i \amalg_i{}^{(-1)} f_{ij}(\oplus_i A_i)$. Following the left–bottom path, composing $SM(f)$ with $\begin{bmatrix} \amalg_1{}^{(-1)} & \cdots & \amalg_m{}^{(-1)} \end{bmatrix}$ gives us the map

$$\begin{bmatrix} \sqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j)\amalg_1{}^{(-1)} & \cdots & \sqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j)\amalg_m{}^{(-1)} \end{bmatrix} = \begin{bmatrix} \sqcup_i \amalg_i{}^{(-1)} f_{i1} & \cdots & \sqcup_i \amalg_i{}^{(-1)} f_{im} \end{bmatrix}.$$

Applying this to $[\oplus_i A_i]$, we see each $B_j = \sqcup_i \amalg_i{}^{(-1)} f_{ij}(\oplus_i A_i)$ and the two directions are equal.

Finally, we know that $\rho_{A_i}{}^{(-1)} = \begin{bmatrix} \amalg_1 \\ \vdots \\ \amalg_n \end{bmatrix}$ and defines an isomorphism between any object of the form $[\oplus_i A_i]$ and the object $[A_1, \ldots, A_n]$. □

Example 9.2.6. We may obtain a matrix representative of any map $f : A \oplus B \to C \oplus D$ by applying the construction of Definition 9.2.3 in reverse.

Then given a function $f : A \oplus B \to C \oplus D$ define

$$f_M = \begin{bmatrix} \amalg_1 f {\amalg_1}^{(-1)} & \amalg_1 f {\amalg_2}^{(-1)} \\ \amalg_2 f {\amalg_1}^{(-1)} & \amalg_2 f {\amalg_2}^{(-1)} \end{bmatrix}.$$

Thus, applying the functor S from Definition 9.2.4, we have

$$\begin{aligned} S(f_M) &= {\amalg_1}^{(-1)}(\amalg_1 f {\amalg_1}^{(-1)} \amalg_1 \sqcup \amalg_1 f {\amalg_2}^{(-1)} \amalg_2) \sqcup {\amalg_2}^{(-1)}(\amalg_2 f {\amalg_1}^{(-1)} \amalg_1 \sqcup \amalg_2 f {\amalg_2}^{(-1)} \amalg_2) \\ &= {\amalg_1}^{(-1)} \amalg_1 f({\amalg_1}^{(-1)} \amalg_1 \sqcup {\amalg_2}^{(-1)} \amalg_2) \sqcup {\amalg_2}^{(-1)} \amalg_2 f({\amalg_1}^{(-1)} \amalg_1 \sqcup {\amalg_2}^{(-1)} \amalg_2) \\ &= {\amalg_1}^{(-1)} \amalg_1 f \sqcup {\amalg_2}^{(-1)} \amalg_2 f \\ &= ({\amalg_1}^{(-1)} \amalg_1 \sqcup {\amalg_2}^{(-1)} \amalg_2) f \\ &= f. \end{aligned}$$

In particular, we note that we may represent $f : A \to B$ by the matrix

$$\begin{bmatrix} 1f1 & 1f0 \\ 0f1 & 0f0 \end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$$

as $A \cong A \oplus 0$ and $B \cong B \oplus 0$.

Definition 9.2.7. A unique decomposition category [28] is a category where any

$$h : A \oplus B \to C \oplus D$$

is uniquely determined by:

$$\begin{matrix} \amalg_1 h {\amalg_1}^{(-1)} : A \to C & \amalg_1 h {\amalg_2}^{(-1)} : A \to D \\ \amalg_2 h {\amalg_1}^{(-1)} : B \to C & \amalg_2 h {\amalg_2}^{(-1)} : B \to D \end{matrix}$$

i.e., is writable as the matrix:

$$\begin{bmatrix} \amalg_1 h \amalg_1^{(-1)} & \amalg_1 h \amalg_2^{(-1)} \\ \amalg_2 h \amalg_1^{(-1)} & \amalg_2 h \amalg_2^{(-1)} \end{bmatrix} : A \oplus B \to C \oplus D.$$

The 0 map in the category corresponds to the 0-dimensional matrix,

$$0 \xrightarrow{[]} B.$$

Corollary 9.2.8. *If $\mathbb{X}$ is an inverse category with disjoint sums, then it is a unique decomposition category.*

Chapter 10

Distributive inverse categories

We now consider inverse categories with both an disjoint sum and inverse product, where the inverse product distributes overs the disjoint sum in a specific way. We show that completing such a category, in the sense of Chapter 5, produces a distributive restriction category where the product distributes over the coproduct.

10.1 Distributive restriction categories

Definition 10.1.1. A Cartesian category $\mathbb{B}$ with coproducts is called a *distributive* [10] category when

$$(A \times B) + (A \times C) \cong A \times (B + C).$$

Definition 10.1.2. A Cartesian restriction category $\mathbb{D}$ with a restriction zero and coproducts is called a *distributive restriction category* [19] when there is an isomorphism ρ such that

$$A \times (B + C) \xrightarrow{\rho} (A \times B) + (A \times C).$$

If $\mathbb{D}$ is a distributive restriction category, then $\mathrm{Total}(\mathbb{D})$ is a distributive category as in Definition 10.1.1.

In the following, we will present distributivity in inverse categories.

10.2 Distributive inverse categories

Definition 10.2.1. A *distributive inverse category* $\mathbb{D}$ consists of the following:

- $\mathbb{D}$ is an inverse category;
- $\mathbb{D}$ has an inverse product with tensor $\otimes$, per Definition 4.3.1;

- $\mathbb{D}$ has an disjoint sum tensor, $\oplus$, per Definition 7.2.1 and
- There is a family of isomorphisms, d, such that

$$\begin{tikzcd}
 & & A \otimes B \\
A \otimes (B \oplus C) \arrow[urr, "1\otimes\amalg_1{}^{(-1)}"] \arrow[rr, "d"] \arrow[drr, "1\otimes\amalg_2{}^{(-1)}"'] & & (A \otimes B) \oplus (A \otimes C) \arrow[u, "\amalg_1{}^{(-1)}"'] \arrow[d, "\amalg_2{}^{(-1)}"] \\
 & & A \otimes C,
\end{tikzcd} \tag{10.1}$$

  commutes in $\mathbb{D}$ for any choices of objects A, B, C.

Example 10.2.2 (PINJ is a distributive inverse category)**.** The isomorphism d of Diagram (10.1) is given by the following:

$$d((a,(x,n)) = \begin{cases} ((a,x),1) & n = 1 \\ ((a,x),2) & n = 2. \end{cases} \tag{10.2}$$

Note that as we are operating in an inverse category, we also have the inverse of diagram (10.1) available to us. That is,

$$\begin{tikzcd}
A \otimes B \arrow[d, "\amalg_1"'] \arrow[drr, "1\otimes\amalg_1"] & & \\
(A \otimes B) \oplus (A \otimes C) \arrow[rr, "d^{(-1)}"] & & A \otimes (B \oplus C) \\
A \otimes C \arrow[u, "\amalg_2"] \arrow[urr, "1\otimes\amalg_2"'] & &
\end{tikzcd} \tag{10.3}$$

is also a commuting diagram in $\mathbb{D}$.

Definition 10.2.3. Suppose $\mathbb{X}$ is an inverse category with a disjoint join tensor $\oplus$ and a restriction zero. Then for maps $f : A \to B$ and $g : A \to C$ with $\overline{f} \perp \overline{g}$, define the map $[f, g] : A \to B \oplus C$ as $(f\amalg_1) \sqcup (g\amalg_2)$. This is well defined as $\widehat{\amalg_1} \perp \widehat{\amalg_2}$ and therefore by [**Dis.7**], $f\amalg_1 \perp g\amalg_2$.

Lemma 10.2.4. *Suppose $\mathbb{X}$ is an inverse category $\mathbb{X}$ with:*

- *a disjoint join tensor $\oplus$,*
- *a restriction zero, and*
- *an inverse product tensor $\otimes$ which distributes over disjoint joins, (that is, $f \otimes (g \sqcup h) = (f \otimes g) \sqcup (f \otimes h)$).*

Then, $\mathbb{X}$ is an inverse distributive category.

Proof. By assumption, we have the first three items of Definition 10.2.1. Therefore, we need to construct an isomorphism d such that diagram (10.1) commutes. We claim that the map $d = [1 \otimes {\amalg_1}^{(-1)}, 1 \otimes {\amalg_2}^{(-1)}]$ does this.

First, note that the typing of d is correct. By Definition 10.2.3,

$$d = ((1 \otimes {\amalg_1}^{(-1)})\amalg_1) \sqcup ((1 \otimes {\amalg_2}^{(-1)})\amalg_2) : A \otimes (B \oplus C) \to (A \otimes B) \oplus (A \otimes C)$$

as

$$A \otimes (B \oplus C) \xrightarrow{(1 \otimes {\amalg_1}^{(-1)})} A \otimes B \xrightarrow{{\amalg_1}^{(-1)}} (A \otimes B) \oplus (A \otimes C),$$
$$A \otimes (B \oplus C) \xrightarrow{(1 \otimes {\amalg_2}^{(-1)})} A \otimes C \xrightarrow{{\amalg_2}^{(-1)}} (A \otimes B) \oplus (A \otimes C).$$

Next, we need to show d is an isomorphism. We will do this by showing both $\overline{d} = 1$ and $\overline{d^{(-1)}} = 1$. As a consequence of Lemma 6.3.3, we know the inverse of d is

$$((1 \otimes {\amalg_1}^{(-1)})\amalg_1)^{(-1)} \sqcup ((1 \otimes {\amalg_2}^{(-1)})\amalg_2)^{(-1)} = ({\amalg_1}^{(-1)}(1 \otimes \amalg_1)) \sqcup ({\amalg_2}^{(-1)}(1 \otimes \amalg_2)).$$

Having $\otimes$ distribute over the disjoint sum means that for any maps f, h, k with $h \perp k$, we have $f \otimes (h \sqcup k) = (f \otimes h) \sqcup (f \otimes k)$. We use this in the calculation of the restriction of f:

$$\begin{aligned}
\overline{((1 \otimes {\amalg_1}^{(-1)})\amalg_1) \sqcup ((1 \otimes {\amalg_2}^{(-1)})\amalg_2)} &= \overline{((1 \otimes {\amalg_1}^{(-1)})\amalg_1)} \sqcup \overline{((1 \otimes {\amalg_2}^{(-1)})\amalg_2)} \\
&= (1 \otimes \overline{{\amalg_1}^{(-1)}}) \sqcup (1 \otimes \overline{{\amalg_2}^{(-1)}}) \\
&= (1 \otimes (\overline{{\amalg_1}^{(-1)}} \sqcup \overline{{\amalg_2}^{(-1)}}) \\
&= 1 \otimes ((1 \oplus 0) \sqcup (0 \oplus 1)) \\
&= 1 \otimes 1 = 1
\end{aligned}$$

and for the inverse,

$$\begin{aligned}
\overline{({\amalg_1}^{(-1)}(1\otimes\amalg_1))\sqcup({\amalg_2}^{(-1)}(1\otimes\amalg_2))} &= \overline{({\amalg_1}^{(-1)}(1\otimes\amalg_1))}\sqcup\overline{({\amalg_2}^{(-1)}(1\otimes\amalg_2))}\\
&= \overline{({\amalg_1}^{(-1)}\overline{(1\otimes\amalg_1)})}\sqcup\overline{({\amalg_2}^{(-1)}\overline{(1\otimes\amalg_2)})}\\
&= \overline{({\amalg_1}^{(-1)})}\sqcup\overline{({\amalg_2}^{(-1)})}\\
&= (1\oplus 0)\sqcup(0\oplus 1)\\
&= 1.
\end{aligned}$$

Hence, $[1\otimes{\amalg_1}^{(-1)}, 1\otimes{\amalg_2}^{(-1)}]$ is an isomorphism. Finally, we must show that diagram (10.1) commutes:

$$\begin{aligned}
d{\amalg_1}^{(-1)} &= \Big(((1\otimes{\amalg_1}^{(-1)})\amalg_1)\sqcup((1\otimes{\amalg_2}^{(-1)})\amalg_2)\Big)\,{\amalg_1}^{(-1)}\\
&= \Big(((1\otimes{\amalg_1}^{(-1)})\amalg_1){\amalg_1}^{(-1)}\Big)\sqcup\Big(((1\otimes{\amalg_2}^{(-1)})\amalg_2){\amalg_1}^{(-1)}\Big)\\
&= \Big((1\otimes{\amalg_1}^{(-1)})1\Big)\sqcup\Big((1\otimes{\amalg_2}^{(-1)})0\Big)\\
&= (1\otimes{\amalg_1}^{(-1)})\sqcup 0\\
&= 1\otimes{\amalg_1}^{(-1)}
\end{aligned}$$

and

$$\begin{aligned}
d{\amalg_2}^{(-1)} &= \Big(((1\otimes{\amalg_1}^{(-1)})\amalg_1)\sqcup((1\otimes{\amalg_2}^{(-1)})\amalg_2)\Big)\,{\amalg_2}^{(-1)}\\
&= \Big(((1\otimes{\amalg_1}^{(-1)})\amalg_1){\amalg_2}^{(-1)}\Big)\sqcup\Big(((1\otimes{\amalg_2}^{(-1)})\amalg_2){\amalg_2}^{(-1)}\Big)\\
&= 0\sqcup(1\otimes{\amalg_2}^{(-1)})\\
&= 1\otimes{\amalg_2}^{(-1)}.
\end{aligned}$$

This shows the fourth condition is satisfied and $\mathbb{X}$ is a distributive inverse category. □

We have seen that a second tensor distributing over the disjoint joins implies that we have an inverse distributive category. We now show the converse is true.

Lemma 10.2.5. *Given an inverse distributive category $\mathbb{X}$, then $h\otimes(f\triangledown g) = (h\otimes f)\triangledown(h\otimes g)$ whenever $f\triangledown g$ exists and $h\otimes(f\vartriangle g) = (h\otimes f)\vartriangle(h\otimes g)$ whenever $f\vartriangle g$ exists.*

Proof. Let $h : A \to C$, $f : B \to D$ and $g : B \to E$. Consider the following diagram:

$$\begin{tikzcd}
 & & & B \otimes E \\
A \otimes C \arrow[rrru, "h \otimes f"] \arrow[r, "h\otimes(f \triangledown g)"] \arrow[rrrd, "h\otimes g"'] & B \otimes (D \oplus E) \arrow[rru, "1\otimes\amalg_1{}^{(-1)}"] \arrow[rr, "\cong"] \arrow[rrd, "1\otimes\amalg_2{}^{(-1)}"'] & & (B \otimes D) \oplus (B \otimes E) \arrow[u, "\amalg_1{}^{(-1)}"'] \arrow[d, "\amalg_2{}^{(-1)}"] \\
 & & & B \otimes D.
\end{tikzcd}$$

The two leftmost triangles commute by the diagram for $f \triangledown g$. The right hand triangles commute as per Definition 10.2.1. By the uniqueness of the $\triangledown$ operation we see $h \otimes (f \triangledown g) = (h \otimes f) \triangledown (h \otimes g)$,

The argument for showing $h \otimes (f \vartriangle g) = (h \otimes f) \vartriangle (h \otimes g)$ follows the same pattern. □

Lemma 10.2.6. *Given an inverse distributive category $\mathbb{X}$, then $\otimes$ distributes over the disjoint join.*

Proof. First recall the definition of $f \sqcup g = (\overline{f} \triangledown \overline{g})(f \vartriangle g)$. In order to show $h \otimes (f \sqcup g) = (h \otimes f) \sqcup (h \otimes g)$, we need to show that

$$h \otimes (\overline{f} \triangledown \overline{g})(f \vartriangle g) = (\overline{h \otimes f} \triangledown \overline{h \otimes g})(h \otimes f \vartriangle h \otimes g). \tag{10.4}$$

Since $h \otimes (\overline{f} \triangledown \overline{g})(f \vartriangle g) = (\overline{h} \otimes (\overline{f} \triangledown \overline{g}))(h \otimes (f \vartriangle g))$, Equation (10.4) follows directly from Lemma 10.2.5 and the fact that $\otimes$ is a restriction functor. □

Corollary 10.2.7. *Suppose we have an inverse distributive category $\mathbb{X}$. Then,*

(i) if $f \perp g$, then $h \otimes f \perp h \otimes g$ for any h,

(ii) if $f \perp g : A \to B$ and $h \perp k : C \to D$, then $(f \otimes h) \perp (g \otimes k)$.

Proof.

(i) As $f \perp g$, we have $f \bigtriangleup g$ and $f \bigtriangledown g$. By Lemma 10.2.5, both $h \otimes f \bigtriangleup h \otimes g$ and $h \otimes f \bigtriangledown h \otimes g$ exist and therefore $h \otimes f \perp h \otimes g$.

(ii) By the previous item, we have that $((f \sqcup g) \otimes h) \perp ((f \sqcup g) \otimes k)$. Then, by [**DJ.1**] and [**Dis.3**] we have $(f \otimes h) \perp (g \otimes k)$.

$\square$

10.3 Discrete inverse categories with disjoint sums

We now consider the case where we have a discrete inverse category with inverse product tensor $\otimes$ and a disjoint join tensor $\oplus$, with the $\otimes$ tensor preserves the disjoint join.

A map in $\widetilde{\mathbb{X}}$ is related to a map in $\mathbb{X}$ in the following way:

$$\frac{A \xrightarrow{(f,C)} B \text{ in } \widetilde{\mathbb{X}}}{A \xrightarrow{f} B \otimes C \text{ in } \mathbb{X}.}$$

Our goal is to show that an disjoint sum in a distributive inverse category becomes a co-product in $\widetilde{\mathbb{X}}$.

Lemma 10.3.1. *Given $\mathbb{X}$ is a distributive inverse category, then $\widetilde{\mathbb{X}}$, the discrete inverse category created from $\mathbb{X}$, has a restriction zero.*

Proof. Recall from Theorem 5.2.6 that $\mathbb{X}$ is equivalent as a category to $\widetilde{\mathbb{X}}$ under the identity on objects functor

$$\mathbf{T} : \mathbb{X} \to \widetilde{\mathbb{X}}; \quad \begin{array}{c} A \\ \big\downarrow f \\ B \end{array} \mapsto \begin{array}{c} A \\ \big\downarrow (f u_{\otimes}^{r\,(-1)}, 1) \\ B \end{array}.$$

In $\mathbb{X}$, we know 0 is a terminal and initial object, with maps $A \xrightarrow{t_A} 0$ and $0 \xrightarrow{z_A} A$, where $\overline{0_{A,A}} = 0_{A,A} = t_A z_A$.

First we note that 0 is both initial and terminal in $\widetilde{\mathbb{X}}$, with the terminal maps being $\mathbf{T}(t_A)$ and initial maps being $\mathbf{T}(z_A)$.

As was also shown in Theorem 5.2.6, $\mathbf{T}$ is a restriction functor, so in $\widetilde{\mathbb{X}}$ we have

$$0_{A,A} = \mathbf{T}(t_A)\mathbf{T}(z_A) = \mathbf{T}(t_A z_A) = \mathbf{T}(0_{A,A}) = \mathbf{T}(\overline{0_{A,A}}) = \overline{\mathbf{T}(0_{A,A})} = \overline{0_{A,A}}.$$

Hence, $0_{A,A}$ is a restriction zero in $\widetilde{\mathbb{X}}$. ☐

Lemma 10.3.2. *In a distributive inverse category $\mathbb{X}$, the following hold:*

(i) Given $f : A \to Y \otimes C$, we can construct $f' : A \to Y \otimes (C \oplus D)$ for some object D such that $f \simeq f'$.

(ii) Given $f : A \to Y \otimes C$, $g : A \to Y \otimes D$, then the $f' : A \to Y \otimes (C \oplus D)$, $g' : B \to Y \otimes (C \oplus D)$ as constructed in (i) satisfies ${\amalg_1}^{(-1)} f' \perp {\amalg_2}^{(-1)} g'$.

Proof.

(i) Set $f' = f(1 \otimes \amalg_1)$. To show $f \simeq f'$, we must first show their restriction is the same.

$$\overline{f(1 \otimes \amalg_1)} = \overline{f\overline{(1 \otimes \amalg_1)}} = \overline{f1} = \overline{f}$$

The mediating map between f and f' is, of course, $1 \otimes \amalg_1$:

$$\begin{tikzcd} & Y \otimes C \arrow[d, dotted, "1\otimes\amalg_1"] \\ A \arrow[ur, "f"] \arrow[r, "f'"'] & Y \otimes (C \oplus D). \end{tikzcd}$$

By the same reasoning we may also create $f' : A \to Y \otimes (D \oplus C)$ by setting $f' = f(1 \otimes \amalg_2)$.

(ii) First note we have ${\amalg_1}^{(-1)} f'$, ${\amalg_2}^{(-1)} g' : A \oplus B \to Y \otimes (C \oplus D)$. In order to show ${\amalg_1}^{(-1)} f' \perp {\amalg_2}^{(-1)} g'$, we will proceed by showing their restrictions and ranges are disjoint. As $\overline{{\amalg_1}^{(-1)}} \perp \overline{{\amalg_2}^{(-1)}}$ and $\overline{{\amalg_1}^{(-1)} f'} \leq \overline{{\amalg_1}^{(-1)}}$ and $\overline{{\amalg_2}^{(-1)} g'} \leq \overline{{\amalg_2}^{(-1)}}$, we immediately have $\overline{{\amalg_1}^{(-1)} f'} \perp \overline{{\amalg_2}^{(-1)} g'}$.

For the ranges, we have

$$
\begin{aligned}
\amalg_1{}^{(-1)}\widehat{(f(1 \otimes \amalg_1))} &= \overline{((1 \otimes \amalg_1{}^{(-1)})f^{(-1)})\amalg_1} \\
&= \overline{((1 \otimes \amalg_1{}^{(-1)})f^{(-1)}} \\
&\le \overline{(1 \otimes \amalg_1{}^{(-1)})}
\end{aligned}
$$

and similarly

$$\widehat{\amalg_2{}^{(-1)}}g' \le \overline{(1 \otimes \amalg_2{}^{(-1)})}.$$

Using Lemma 6.2.3 we know that $\overline{(\amalg_1{}^{(-1)})} \perp \overline{(\amalg_2{}^{(-1)})}$. From Corollary 10.2.7 we conclude that $\overline{(1 \otimes \amalg_1{}^{(-1)})} \perp \overline{(1 \otimes \amalg_2{}^{(-1)})}$ and giving us $\widehat{\amalg_1{}^{(-1)}}f' \perp \widehat{\amalg_2{}^{(-1)}}g'$ and therefore $\amalg_1{}^{(-1)}f' \perp \amalg_2{}^{(-1)}g'$.

□

Proposition 10.3.3. *Given $\mathbb{X}$ is an distributive inverse category, then the category $\widetilde{\widetilde{\mathbb{X}}}$ has coproducts.*

Proof. The tensor object $A \oplus B$ in $\mathbb{X}$ will become the co-product of A, B in $\widetilde{\widetilde{\mathbb{X}}}$.

The injection maps of the co-product are $i_1 = (\amalg_1 u_\otimes^r{}^{(-1)}, 1)$ and $i_2 = (\amalg_2 u_\otimes^r{}^{(-1)}, 1)$.

Consider the following diagram in $\widetilde{\widetilde{\mathbb{X}}}$:

$$
\begin{tikzcd}
A \arrow[dr, "i_1"'] \arrow[drr, "{(f,C)}"] & & \\
& A \oplus B \arrow[r, dotted, "h"] & Y. \\
B \arrow[ur, "i_2"] \arrow[urr, "{(g,D)}"'] & &
\end{tikzcd}
$$

In $\mathbb{X}$, this comes from the diagram:

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\amalg_1"'] & & Y \otimes C \\
& A \oplus B & \\
B \arrow[rr, "g"] \arrow[ur, "\amalg_2"] & & Y \otimes D
\end{tikzcd}
\tag{10.5}
$$

where the extraneous unit isomorphisms are removed.

This corresponds to the conditions of Lemma 10.3.2. Hence by that lemma we may revise Diagram (10.5) as

$$
\begin{tikzcd}
A \arrow[dr, "\amalg_1"'] \arrow[drr, "f'"] & & \\
& A \oplus B \arrow[r, "{\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g'"] & Y \otimes (C \oplus D) \\
B \arrow[ur, "\amalg_2"] \arrow[urr, "g'"'] & &
\end{tikzcd}
\tag{10.6}
$$

where f' and g' are respectively equivalent to f, g.

Lifting Diagram (10.6) to $\mathbb{X}$, we see this corresponds to the desired co-product diagram, where h in $\widetilde{\mathbb{X}}$ is the map $({\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g', (C \oplus D))$.

By construction, in $\mathbb{X}$, we have

$$\amalg_1({\amalg_1}^{(-1)} f' \sqcup \amalg_1 {\amalg_2}^{(-1)} g') = (\amalg_1 {\amalg_1}^{(-1)} f') \sqcup (\amalg_1 {\amalg_2}^{(-1)} g') = f' \sqcup 0 = f'$$

and

$$\amalg_2({\amalg_1}^{(-1)} f' \sqcup \amalg_1 {\amalg_2}^{(-1)} g') = g'.$$

Hence, in $\widetilde{\mathbb{X}}$, we have $(i_1 u_\otimes^r, 1)h = f$ and $(i_2 u_\otimes^r, 1)h = g$.

All that remains to be shown is that h is unique.

Suppose there is another (k, E) in $\widetilde{\mathbb{X}}$ such that it satisfies the coproduct properties, i.e., that $i_1(k, E) = (f', C \oplus D)$ and $i_2(k, E) = (g', C \oplus D)$. In $\mathbb{X}$, $k : A \oplus B \to Y \otimes E$ and we have

$$\amalg_1 k \simeq f' \quad \text{and}$$

$$\amalg_2 k \simeq g'.$$

Since equivalence is a transitive relation, this means we have

$$f \overset{q_1}{\simeq} \amalg_1 k \quad \text{and}$$
$$g \overset{q_2}{\simeq} \amalg_2 k.$$

where the maps $q_1 : Y \otimes C \to Y \otimes E$ and $q_2 : Y \otimes D \to Y \otimes E$ fulfill the respective equivalence diagrams.

Drawing this explicitly for k, f and q_1,

$$\begin{tikzcd} & Y \otimes C \arrow[dd, dotted, "1\otimes q_1"] \\ A \arrow[ur, "f"] \arrow[dr, "\amalg_1 k"'] & \\ & Y \otimes E. \end{tikzcd}$$

Now, we turn out attention to showing that $k \simeq h = {\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g'$. Consider the diagram

$$\begin{tikzcd} & Y \otimes (C \oplus D) \arrow[dd, dotted, "t"] \\ A \arrow[ur, "{\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g'"] \arrow[dr, "k"'] & \\ & Y \otimes E. \end{tikzcd} \tag{10.7}$$

As we are in an inverse category, we know there is a map t that makes this diagram commute, namely $t = ({\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g')^{(-1)} k$. However, we must show this is in Y_{∇}^{Δ}.

Next, recalling Definition 7.1.6 consider

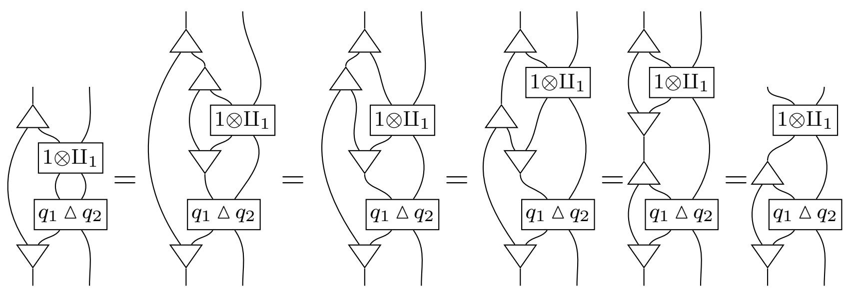
$$\begin{tikzcd} Y \otimes C \arrow[r, "1\otimes\amalg_1"] \arrow[dr, "q_1"'] & Y \otimes (C \oplus D) \arrow[d, dotted, "q_1 \,\Delta\, q_2"] & Y \otimes D \arrow[l, "1\otimes\amalg_2"'] \arrow[dl, "q_2"] \\ & Y \otimes E & \end{tikzcd}$$

We know the map $q_1 \,\Delta\, q_2$ exists iff $\widehat{q_1} \perp \widehat{q_2}$. But, the map t from above does make the diagram commute. We can see this as

$$
\begin{aligned}
&(1 \otimes \amalg_1)({\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g')^{(-1)} k \\
&\qquad = (1 \otimes \amalg_1)({\amalg_1}^{(-1)} f(1 \otimes \amalg_1) \sqcup {\amalg_2}^{(-1)} g(1 \otimes \amalg_2))^{(-1)} k \\
&\qquad = (1 \otimes \amalg_1)((1 \otimes {\amalg_1}^{(-1)}) f^{(-1)} \amalg_1 \sqcup (1 \otimes {\amalg_2}^{(-1)}) g^{(-1)} \amalg_2) k \\
&\qquad = ((1 \otimes 1) f^{(-1)} \amalg_1 \sqcup (1 \otimes 0) g^{(-1)} \amalg_2) k \\
&\qquad = f^{(-1)} \amalg_1 k \\
&\qquad = q_1
\end{aligned}
$$

and similarly for q_2. Therefore, we have $\widehat{q_1} \perp \widehat{q_2}$.

This means we may form the map $q_1 \vartriangle q_2 : Y \otimes (C \oplus D) \to Y \otimes E$. But then the map $(q_1 \vartriangle q_2)$ makes Diagram (10.7) commute. We also have $q_1, q_2 \in Y^{\Delta}_{\nabla}$, but this means $q_1 \vartriangle q_2 \in Y^{\Delta}_{\nabla}$ as

$q_1 =$ 

$= (1 \otimes \amalg_1)(q_1 \vartriangle q_2)^{\Delta}_{\nabla}.$

and similarly $q_2 = (1 \otimes \amalg_2)(q_1 \vartriangle q_2)^{\Delta}_{\nabla}$ giving us $q_1 \vartriangle q_2 \in Y^{\Delta}_{\nabla}$ by the uniqueness of the $\vartriangle$ map. Therefore $q_1 \vartriangle q_2$ provides an equivalence between k and ${\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g'$, meaning the coproduct is unique. □

Corollary 10.3.4. *When $\mathbb{X}$ is a distributive inverse category, $\widetilde{\widetilde{\mathbb{X}}}$ is a distributive restriction category.*

Proof. As $\widetilde{\widetilde{\mathbb{X}}}$ has restriction products by Lemma 5.1.9, restriction co-products by Theorem 10.3.3 and the equations for distributivity follow directly from the distributivity of the base tensors, we see $\widetilde{\widetilde{\mathbb{X}}}$ is a distributive restriction category. □

Chapter 11

Turing Categories and PCAs

In this chapter, we review the definition and properties of a Turing category and Partial Combinatory Algebras [14, 16]. Because of the theorems of the earlier chapters, we will be able to transfer these ideas in a straightforward way from Cartesian restriction categories over to discrete inverse categories. We will show which structures are needed in a discrete inverse category $\mathbb{X}$ so that the constructed category, $\widetilde{\mathbb{X}}$, is a Turing category or has a PCA.

11.1 Turing Categories

Definition 11.1.1 (Turing category)**.** Given $\mathbb{X}$ is a Cartesian restriction category:

1. For a map $\tau_{X,Y} : A \times X \to Y$, a map $f : B \times X \to Y$ *admits a* $\tau_{X,Y}$*-index* when there is a total $\ulcorner f \urcorner : B \to A$ such that

$$\begin{tikzcd} A \times X \arrow[r, "\tau_{X,Y}"] & Y \\ B \times X \arrow[u, dotted, "\ulcorner f \urcorner \times 1_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

2. A map $\tau_{X,Y} : A \times X \to Y$ is called a *universal application* if all $f : B \times X \to Y$ admit a $\tau_{X,Y}$-index.

3. If A is an object in $\mathbb{X}$ such that for every pair of objects X, Y in $\mathbb{X}$ there is a universal application, then A is called a *Turing object.*

4. A Cartesian restriction category that contains a Turing object is called a *Turing category.*

Note there is no requirement in the definition for the map $\ulcorner f \urcorner$ to be unique. When $\ulcorner f \urcorner$ is unique for a specific $\tau_{X,Y}$, then that $\tau_{X,Y}$ is called *extensional*. In the case where the object B is the terminal object, then the map $\ulcorner f \urcorner$ is a point of A (with $f = (\ulcorner f \urcorner \times 1)\tau_{X,Y}$) and $\ulcorner f \urcorner$ is referred to as a *code* of f.

Definition 11.1.2. Given $\mathbb{T}$ is a Turing category and A is an object of $\mathbb{T}$,

1. If $\Upsilon = \{\tau_{X,Y} : A \times X \to Y | X, Y \in ob(\mathbb{T})\}$, then Υ is called an *applicative family* for A.
2. An applicative family Υ is called *universal for A* when each $\tau_{X,Y}$ is a universal application. This is also referred to as a *Turing structure* on A.
3. A pair (A, Υ) where Υ is universal for A is called a *Turing structure* on $\mathbb{T}$.

Lemma 11.1.3. *If $\mathbb{T}$ is a Turing category with Turing object T, then every object B in $\mathbb{T}$ is a retract of T.*

Proof. As T is a Turing object, we have a diagram for $\tau_{1,B}$ and $\pi_0 : B \times 1 \to B$:

$$\begin{tikzcd} T \times 1 \arrow[r, "\tau_{1,B}"] & B \\ B \times 1. \arrow[u, dotted, "\ulcorner \pi_0 \urcorner \times 1_1"] \arrow[ur, "\pi_0"'] & \end{tikzcd}$$

Note we also have $u_r : B \to B \times 1$ is an isomorphism and therefore we have $1_X = u_r\pi_0 = (u_r(\ulcorner \pi_0 \urcorner \times 1))\tau_{1,X}$. Hence, we have $((u_r(\ulcorner \pi_0 \urcorner \times 1)), \tau_{1,X}) : B \triangleleft T$. $\square$

This allows for various recognition criteria for Turing categories.

Theorem 11.1.4. *A Cartesian restriction category $\mathbb{D}$ is a Turing category if and only if $\mathbb{T}$ has an object T for which every other object of $\mathbb{D}$ is a retract and T has a universal self-application map $\bullet$, written as $T \times T \xrightarrow{\bullet} T$.*

Proof. The "only if" portion follows immediately from setting T to be the Turing object of $\mathbb{D}$ and $\bullet = \tau_{T,T}$.

For the "if" direction, we need to construct the family of universal applications $\tau_{X,Y} : T \times X \to Y$ for each pair of objects X, Y in $\mathbb{D}$.

Let us choose pairs of maps that witness the retractions of X, Y of T, that is:

$$(m_X, r_X) : X \triangleleft T \quad \text{and} \quad (m_Y, r_Y) : Y \triangleleft T.$$

Define $\tau_{X,Y} = (1_T \times m_X) \bullet r_Y$. Suppose we are given $f : B \times X \to Y$. Consider

$$\begin{tikzcd}
T \times X \arrow[r, "1_T \times m_X"] & T \times T \arrow[r, "\bullet"] & T \arrow[r, "r_Y"] & Y \\
B \times X \arrow[u, "h \times 1_X"] \arrow[r, "1_B \times m_X"'] & B \times T \arrow[u, dotted, "h \times 1_T"] \arrow[r, "1_B \times r_X"'] & B \times X \arrow[u, "f m_Y"] \arrow[ur, "f"'] &
\end{tikzcd}$$

where h is the index for the composite map $(1_B \times r_X) f m_Y$. The middle square commutes as $\bullet$ is a universal application for T, T. The right triangle commutes as $m_Y r_Y = 1$. The left square commutes as each composite is $h \times m_X$. Noting that the bottom path from $B \times X$ to Y is $(1_B \times m_X)(1_B \times r_X) f = f$ and the top path from $T \times X$ to Y is our definition of $\tau_{X,Y}$, this means f admits the $\tau_{X,Y}$-index h. $\square$

Note that different splittings (choices of (m, r) pairs) would lead to different $\tau_{X,Y}$ maps. In fact there is no requirement that this is the only way to create a universal applicative family for T.

There is another criteria that also gives a Turing category:

Lemma 11.1.5. *A Cartesian restriction category $\mathbb{T}$ is a Turing category if:*

(i) $\mathbb{T}$ has an object T for which every other object of $\mathbb{D}$ is a retract;

(ii) $T \times T$ has a map $T \times T \xrightarrow{\circ} T$ and for all $f : T \to T$ there exists an element, $f_\bullet : 1 \to T$ (which is total) such that

$$\begin{tikzcd}
T \times T \arrow[r, "\circ"] & T \\
T \arrow[u, "\langle ! f_\bullet, 1 \rangle"] \arrow[ur, "f"'] &
\end{tikzcd}$$

is a commutative diagram.

Proof. We need only show that T has a universal self-application map and then use Theorem 11.1.4.

T having a universal self-application map, $\bullet$, means for every map $f : B \times T \to T$ there is a map, $\ulcorner f \urcorner : B \to T$ such that

$$\begin{tikzcd}
T \times T \arrow[r, "\bullet"] & T \\
B \times T \arrow[u, "\ulcorner f \urcorner \times 1"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes.

Let $(m, r) : T \times T \triangleleft T$. Then, consider

$$\begin{tikzcd}
T \times T \arrow[r, "r \times 1"] & T \times T \times T \arrow[r, "1 \times m"] & T \times T \arrow[rr, "\circ"] & & T \\
T \times T \times T \arrow[u, "m \times 1"] \arrow[ur, equal] & & & & \\
T \times T \arrow[u, dotted, "\langle (rf)_\bullet, \pi_0, \pi_1 \rangle"] \arrow[uur, dotted, "\langle (rf)_\bullet, \pi_0, \pi_1 \rangle"'] \arrow[rr, "m"'] \arrow[rrr, equal, bend right] & & T \arrow[uu, dotted, "\langle (rf)_\bullet, 1 \rangle"'] \arrow[r, "r"'] & T \times T. \arrow[uur, "f"'] &
\end{tikzcd}$$

The rightmost quadrilateral commutes by assumption of this lemma. The middle quadrilateral commutes due to the properties of the product map and π_0 and π_1. The top left triangle commutes as $mr = 1$ and the remaining triangle has the same map on both dotted lines.

Thus, we may conclude that $\bullet \doteq (r \times 1)(1 \times m)\circ$ and $\ulcorner f \urcorner \doteq \langle !f_\bullet, 1 \rangle m$ satisfy the requirements of Theorem 11.1.4 and therefore T is a Turing object in a Turing category. $\square$

11.2 Inverse Turing categories

Now, we define inverse Turing categories. Essentially, an inverse Turing category is a discrete inverse category $\mathbb{X}$ where $\widetilde{\mathbb{X}}$ is a Turing category. Let us proceed with a more concrete definition of this.

Definition 11.2.1. A discrete inverse category $\mathbb{X}$ is an *inverse Turing category* when there is a universal object T (i.e., every $B \in \mathbb{X}$ is a retract of T) in $\mathbb{X}$ with a map $\diamond : T \otimes T \to T \otimes T$ such that for every map $f : T \to T \otimes T$ there is a total map $\ulcorner f \urcorner : I \to T$ and a map $h_f : T \otimes T \to T \otimes T$ with $h_f \in T^{\triangle}_{\triangledown}$ such that $f \overset{h_f}{\simeq} {u^l_\otimes}^{(-1)}(\ulcorner f \urcorner \otimes 1)\diamond$, i.e., the diagram

$$\begin{tikzcd}
 & & T \otimes T \arrow[dd, dotted, "h_f"] \\
 & T \otimes T \arrow[ur, "\diamond"] & \\
T \arrow[ur, "{u^l_\otimes}^{(-1)}(\ulcorner f \urcorner \otimes 1)"] \arrow[rr, "f"'] & & T \otimes T
\end{tikzcd}$$

commutes.

We know from Chapter 5 that when $\mathbb{X}$ is an inverse Turing category, we know $\widetilde{\mathbb{X}}$ is a discrete Cartesian restriction category. Moreover, if $(m_A, r_A) : A \triangleleft T$ in $\mathbb{X}$, then $(m_A {u^r_\otimes}^{(-1)}, r_A {u^r_\otimes}^{(-1)}) A \triangleleft T$ in $\widetilde{\mathbb{X}}$ and hence T will remain universal in $\widetilde{\mathbb{X}}$. Hence, we have the basic requirements for a Turing category as specified in Theorem 11.1.4 and Lemma 11.1.5. All that remains to be shown is that we have a self-application map and a code for each map $f : 1 \to T$ as in Lemma 11.1.5.

Theorem 11.2.2. *When $\mathbb{X}$ is an inverse Turing category, $\widetilde{\mathbb{X}}$ is a Turing category.*

Proof. From the discussion, we need to specify the self-application map $\circ : T \times T \to T$ and $f_\bullet : 1 \to T$ in $\widetilde{\mathbb{X}}$.

The diagram of Definition 11.2.1, when raised to $\widetilde{\mathbb{X}}$ translates to:

$$\begin{tikzcd}
T \times T \arrow[r, "{(\diamond, T)}"] & T \\
T. \arrow[u, "{\langle \ulcorner f \urcorner, 1 \rangle}"] \arrow[ur, "{(f, T)}"'] &
\end{tikzcd}$$

But this corresponds exactly to the requirement of Lemma 11.1.5 with $\circ = (\diamond, T)$ and $(f, T)_\bullet = \langle \ulcorner f \urcorner, 1 \rangle$. Finally, noting that T is universal in $\mathbb{X}$, if we have $(f, B) : T \to T$ in $\widetilde{\mathbb{X}}$, where $(m_B, r_B) : B \triangleleft T$ in $\mathbb{X}$, we see $(f, B) \simeq (fr_B, T)$ in $\mathbb{X}$ and therefore may be written as above and we therefore have shown that $\widetilde{\mathbb{X}}$ is a Turing category. $\square$

11.3 Partial combinatory algebras

In a Cartesian restriction category, for any operation $f : A \times A \to A$ define $f^{(n)}$ for $n \geq 1$ recursively by:

(i) $f^{(1)} = f$

(ii) $f^{(n+1)} = (f \times 1)f^{(n)}$

Definition 11.3.1. A Cartesian restriction category has a *partial combinatory algebra* when it has an object A together with:

(i) A partial map $\bullet : A \times A \to A$,

(ii) two total elements $1 \xrightarrow{k} A$ and $1 \xrightarrow{s} A$ which satisfy

$$\begin{tikzcd}
A \times A \times A \arrow[r, "(\bullet\times 1)\bullet"] & A \\
A \times A \arrow[u, "k\times 1\times 1"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times A \times A \times A \arrow[r, "\bullet^{(3)}"] & A \\
 & A \times A \arrow[u, "\bullet"'] \\
A \times A \times A \arrow[uu, "s\times 1\times 1\times 1"] \arrow[r, "\theta'_A"'] & (A \times A) \times (A \times A). \arrow[u, "\bullet\times\bullet"']
\end{tikzcd}$$

(iii) $A \times A \xrightarrow{s\times 1\times 1} A \times A \times A \xrightarrow{\bullet^2} A$ is total.

In the above $\theta' = (1 \times 1 \times \Delta)(1 \times c \times 1)a$ where a sets the parenthesis as in the diagram.

Of course, this is more familiarly given equationally by:

$$(k \bullet x) \bullet y = x \qquad ((s \bullet x) \bullet y) \bullet z = (x \bullet z) \bullet (y \bullet z)$$

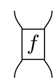
These are the equations of a combinatory algebra where partiality is not considered. As we have partiality, we also add the requirement that $s \bullet x \bullet y$ is a total map for any x, y.

Note that if we have a Turing object T in a Cartesian restriction category, it is a partial combinatory algebra. All we need to do is to actually define the element k and s by using the commuting diagrams of Definition 11.3.1.

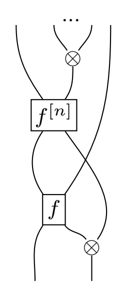
Now, we want to consider what are the conditions required for an inverse category $\mathbb{X}$ such that $\widetilde{\mathbb{X}}$ has a partial combinatory algebra.

In a discrete inverse category, we define the notation $f^{[n]}$. For any operation $f : A \otimes A \to A \otimes A$ define $f^{[n]}$ recursively by:

(i)

$$f^{[1]} : A \otimes A \to A \otimes A = f =$$  .

(ii)

$$f^{[n+1]} : A \otimes (\otimes_n A) \otimes A \to A \otimes (\otimes_{n+1} A) =$$  .

Definition 11.3.2. A discrete inverse category $\mathbb{X}$ has a *inverse partial combinatory algebra* when there is an object A in $\mathbb{X}$ with a map $A \otimes A \xrightarrow{\bullet} A \otimes A$ and two total elements:

$$1 \xrightarrow{k} A \qquad 1 \xrightarrow{s} A$$

and maps $h_k : A \otimes A \otimes A \to A \otimes A, h_s : A \otimes A \otimes A \otimes A \to A \otimes A \otimes A \otimes A$ in $A^{\triangle}_{\nabla}$ which satisfy the following three axioms:

[**iCPA.1**]

$$\begin{tikzcd}
 & & A \otimes A \arrow[dd, dotted, "h_k"] \\
 & A \otimes A \otimes A \arrow[ur, "\bullet^{[2]}"] & \\
A \otimes A \otimes A \arrow[ur, "{u^l_\otimes}^{(-1)}(k\otimes 1\otimes 1)"] \arrow[rr, "1"'] & & A \otimes A.
\end{tikzcd}$$

[**iCPA.2**]

$$\begin{tikzcd}
 & & & A \otimes A \otimes A \otimes A \arrow[ddddd, dotted, "h_s"] \\
 & & A \otimes A \otimes A \otimes A \arrow[ur, "\bullet^{[3]}"] & \\
A \otimes A \otimes A \arrow[urr, "{u^l_\otimes}^{(-1)}(s\otimes 1)"] \arrow[dr, "\theta'_A"'] & & & \\
 & (A \otimes A) \otimes (A \otimes A) \arrow[dr, "(\bullet\otimes\bullet)(1\otimes c\otimes 1)"'] & & \\
 & & A \otimes A \otimes A \otimes A \arrow[dr, "(\bullet\otimes 1)"'] & \\
 & & & A \otimes A \otimes A \otimes A.
\end{tikzcd}$$

[**iCPA.3**] $I \otimes A \otimes A \xrightarrow{s\otimes 1\otimes 1} A \otimes A \otimes A \xrightarrow{\bullet^{[2]}} A \otimes A \otimes A$ is total.

Proposition 11.3.3. *A discrete inverse category $\mathbb{X}$ has an inverse partial combinatory algebra if and only if $\widetilde{\mathbb{X}}$ has a partial combinatory algebra.*

Proof. When we have a discrete inverse category $\mathbb{X}$ with an inverse partial combinatory algebra, we see immediately the map $\bullet : A \otimes A \to A \otimes A$ in $\mathbb{X}$ becomes the map $(\bullet, A) : A \times A \to A$, satisfying (i) of Definition 11.3.1. The commutative diagrams [**iCPA.1**] and [**iCPA.2**], when lifted to $\widetilde{\mathbb{X}}$, become the diagrams for a partial combinatory algebra as given in (ii), where $(k, {u^l_\otimes}^{(-1)})$ and $(s, {u^l_\otimes}^{(-1)})$ are the k, s of the partial combinatory algebra. Finally, the totality requirement, [**iCPA.3**], gives (iii) of the partial combinatory algebra definition.

Hence, we have shown that an inverse partial combinatory algebra in $\mathbb{X}$ gives a partial combinatory algebra in $\widetilde{\mathbb{X}}$.

For the reverse, when we have a partial combinatory algebra over A in $\widetilde{\mathbb{X}}$, a discrete Cartesian restriction category, we know that the map $\langle \bullet, 1 \rangle$ is invertible and hence is in $\mathbb{X}$. The two maps from the terminal object k, s are also invertible (as they are total) and are therefore also in $\mathbb{X}$.

Given this, the diagrams of the partial combinatory algebra in $\widetilde{\mathbb{X}}$ translate directly to the [**iCPA.1**] and [**iCPA.2**] where $\bullet$ in $\mathbb{X}$ is the invertible map $\langle \bullet, 1 \rangle$.

The totality of $s\bullet^{(2)}$ in $\widetilde{\mathbb{X}}$ then immediately gives us [**iCPA.3**], the totality of $s\bullet^{[2]}$ in $\mathbb{X}$.

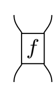
$\square$

However, we can simplify the definition of an inverse partial combinatory algebra when A is powerful in $\mathbb{X}$. (Here, powerful means that $1 \triangleleft A$, $A \otimes A \triangleleft A$, $A \otimes A \otimes A \triangleleft A$, $\ldots$). Note that if A is a partial combinatory algebra in $\widetilde{\mathbb{X}}$, that guarantees it is powerful in $\widetilde{\mathbb{X}}$. Assuming the retractions are $(m_n, r_n) : A^n \triangleleft A$, we have $m_j r_j = 1$ and $r_j m_j = \overline{r_j m_j}$ for each j. Thus, each of the maps are partial inverses in $\widetilde{\mathbb{X}}$ and therefore in $\mathbb{X}$. Hence we have A is a powerful object in $\mathbb{X}$.

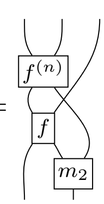
We redefine the meaning of the notation $f^{(n)}$ when in a discrete inverse category.

In a discrete inverse category, we redefine the notation $f^{(n)}$. For any map $f : A \otimes A \to A \otimes A$ where A is a powerful object define $f^{(n)}$ recursively by:

(i)

$f^{(1)} : A \otimes A \to A \otimes A = f =$ 

(ii)

$f^{(n+1)} : \otimes_{n+2} A \to A \otimes A =$ 

Lemma 11.3.4. *Suppose a discrete inverse category $\mathbb{X}$ has a inverse partial combinatory algebra over A and A is a powerful object in $\mathbb{X}$, with $(m_n, r_n) \otimes^n A \triangleleft A$. Then [**iCPA.1**], [**iCPA.2**] and [**iCPA.3**] may be simplified to:*

[***iCPA′.1***]

$$\begin{tikzcd}
& & A \otimes A \arrow[ddd, dotted, "h'_k"] \\
& A \otimes A \otimes A \arrow[ur, "\bullet^{(2)}"] & \\
A \otimes A \arrow[ur, "{u_\otimes^l}^{(-1)}(k \otimes 1 \otimes 1)"] \arrow[drr, "1"'] & & \\
& & A \otimes A
\end{tikzcd}$$

[***iCPA′.2***]

$$\begin{tikzcd}
& & & A \otimes A \arrow[ddddd, dotted, "h'_s"] \\
& A \otimes A \otimes A \otimes A \arrow[urr, "\bullet^{(3)}"] & & \\
A \otimes A \otimes A \arrow[ur, "{u_\otimes^l}^{(-1)}(s \otimes 1)"] \arrow[dr, "\theta'_A"'] & & & \\
& (A \otimes A) \otimes (A \otimes A) \arrow[dr, "(\bullet \otimes \bullet)(1 \otimes c \otimes 1)(1 \otimes 1 \otimes m_2)"'] & & \\
& & A \otimes A \otimes A \arrow[dr, "(\bullet \otimes 1)(1 \otimes m_2)"'] & \\
& & & A \otimes A
\end{tikzcd}$$

[***iCPA′.3***] *$I \otimes A \otimes A \xrightarrow{s \otimes 1 \otimes 1} A \otimes A \otimes A \xrightarrow{\bullet^{(2)}} A \otimes A$ is total.*

11.4 Computable functions

Given a partial combinatory algebra, A, in a Cartesian restriction category, one can form Comp(A), the category of computable partial functions generated by A. These are the maps with an index:

$$\begin{tikzcd} A \times (\times_n A) \arrow[r, "\bullet^{(n)}"] & A \\ (\times_n A). \arrow[u, "\ulcorner f \urcorner \times 1"] \arrow[ur, "f"'] & \end{tikzcd}$$

We would like Comp(A) to be a discrete Turing category so that **INV**(Comp(A)) is an inverse Turing category. However, this is not guaranteed. However, we can define conditions so that it is true:

Definition 11.4.1. Given a discrete object A in a Cartesian restriction category, A has a *discrete partial combinatory algebra* when:

(i) A has a partial combinatory algebra;

(ii) there exists $e : 1 \to A$, a total element, such that

$$\begin{tikzcd} A \times A \times A \arrow[r, "\bullet^{(2)}"] & A \\ A \times A \arrow[u, "e \times 1"] \arrow[ur, "\Delta^{(-1)}"'] & \end{tikzcd}$$

meaning there is a code for $\Delta^{(-1)}$.

We immediately have:

Lemma 11.4.2. *When A has a discrete partial combinatory algebra, then Comp(A) is a discrete Cartesian restriction category.*

This, of course, means that **INV**(Comp(A)) is an inverse Turing category. Note it is still the case that there can be a map of Comp(A) which is invertible in $\widetilde{\mathbb{X}}$ (i.e., is in $\mathbb{X}$), but is *not* invertible in Comp(A).

Chapter 12

Conclusions and future work

This thesis has studied inverse categories [17], providing conditions for a tensor to act like a product — the inverse product — and similarly for a second tensor to behave like a sum — the disjoint sum.

The following are the main results of this thesis.

1. **Inverse Categories**

   We showed in Proposition 4.2.1 that an inverse category with restriction products is a restriction pre-order. Similarly, in Proposition 6.1.5, an inverse category with coproducts is a pre-order.

2. **Discrete Inverse Categories**

   We introduced the inverse product, Definition 4.3.1 and showed that this provided meets for the inverse category in Proposition 4.3.6. We showed that the inverse sub-category of a discrete Cartesian restriction category is a Discrete Inverse Category in Lemma 4.3.7. Commutative Frobenius Algebras are given as an example of a discrete inverse category.

   We then provided an equivalence relation on the maps of a discrete inverse category $\mathbb{X}$ in Definition 5.1.2. This allowed us to introduce the category $\widetilde{\mathbb{X}}$ constructed from $\mathbb{X}$ by using the equivalence relation to generate more maps between each object in Definition 5.1.1. In Lemmas 5.1.5 and 5.1.6 we showed $\widetilde{\mathbb{X}}$ is a restriction category and in Theorem 5.1.10 we showed it is in fact a discrete Cartesian restriction category.

   Finally, in Theorem 5.2.6 we provided an equivalence functor between the categories of discrete inverse categories and discrete Cartesian restriction categories.

3. **Disjointness Relations and Disjoint Joins**

   In Definition 6.2.1 we defined what a disjointness relation is in an inverse category, followed by Definition 6.2.4 which defined disjointness on the restriction idempotents of the inverse category. Theorem 6.2.7 shows that these two definitions are equivalent, allowing us to define disjointness in whichever way is most convenient.

   Disjoint joins are introduced with Definition 6.3.1, providing a correspondence to joins in a restriction category.

   In Chapter 7 we explore the properties a symmetric monoidal tensor on an inverse category requires in order to generate a disjointness relation and a disjoint join. These are referred to as disjoint join tensors.

   We show that in the category of commutative Frobenius algebras that $\Delta(f \otimes g)\Delta^{(-1)} = 0$ gives a disjointness relation.

4. **Disjoint Sum Categories**

   disjoint sum categories are given by inverse categories that have a disjoint join tensor. We define $iMat(\mathbb{X})$, a matrix category over the disjoint sum category $\mathbb{X}$ in Definition 9.1.2 and show that it is an disjoint sum category in Theorem 9.1.4. Furthermore, we show that an disjoint sum category $\mathbb{X}$ is equivalent to a $iMat(\mathbb{X})$ in Proposition 9.2.5.

5. **Distributive Inverse Categories**

   We define distributive inverse categories in Definition 10.2.1 and show that distributivity of the inverse product over the disjoint join is equivalent to distributing over an disjoint sum tensor in Lemma 10.2.6.

Then, in Theorem 10.3.3 we show that the **T** construction of Chapter 5 lifts an disjoint sum tensor into a coproduct. From this we conclude that $\widetilde{\mathbb{X}}$ is a distributive restriction category in Corollary 10.3.4.

6. **Inverse Turing Categories and Inverse PCAs**

   Inverse Turing categories and inverse partial combinatory algebras are defined in Definition 11.2.1 and Definition 11.3.2 respectively. We show that an inverse Turing category gives a Turing category under the **T** construction in Theorem 11.2.2. In Proposition 11.3.3 we show that a discrete inverse category $\mathbb{X}$ has an inverse PCA if and only if $\widetilde{\mathbb{X}}$ has a PCA. Furthermore, we discuss the category of computable function based on a PCA in a Cartesian restriction category and show the conditions required for that to be a discrete Cartesian restriction category in Lemma 11.4.2.

12.1 Whither next?

The obvious next step is to apply these results to existing reversible languages such as **Inv** [42], and Theseus [29,30]. In each case, there do not appear to be any issues in doing this work, it simply requires careful attention to the language definitions. Note that our examples about the category of partial injective functions done throughout this thesis provide the basis for the semantics of **Inv**.

A more interesting direction, in my opinion, would be to further investigate the work on reversible CCS as in [23] and [45]. This could be combined with the work of Chakraborty on MPL [9] to describe the communication of multiple processes and generalize that to a series of reversible processes.

A third direction would be to formalize types in reversible languages using the results of this thesis as a basis. The obvious correspondences are from product types to the inverse product and between sum types and the disjoint sum. From there, one would study the trace and explore infinite and recursive types. Note that some of this work would follow naturally from exploring the semantics of Π^0 and Π from [29, 30].

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