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Abstract

Acknowledgements

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List of Symbols, Abbreviations and Nomenclature

Symbol	Definition
U of C	University of Calgary
$\mathbb{N}$	The set of natural numbers, i.e., $\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	The ring of integers numbers, i.e., $\{0, \pm 1, \pm 2, \ldots\}$
$\mathbb{C}$	The field of complex numbers

Chapter 1

Introduction

1.1 Algebraic setting

We assume the basic algebraic structures of group, ring and field are known. The reader may consult [15] if further details are needed.

Chapter 2

Abstract Computability

2.1 Categories

A category as a mathematical object can be defined in a variety of equivalent ways. As much of our work will involve the exploration of partial and reversible maps, their domains and ranges, we choose a definition that highlights the algebraic nature of these. Note that ranges are normally referred to as codomains in category theory and we will use the codomain terminology in this section.

Definition 2.1.1. A *category* $\mathbb{A}$ is a collection of maps together with two functions, D and C, from $\mathbb{A}$ to $\mathbb{A}$ and a partial associative composition of maps (written by juxtaposing maps), such that:

[**C.1**] $D(f)f$ is defined and equals f,

[**C.2**] $fC(f)$ is defined and equals f,

[**C.3**] fg is defined iff $C(f) = D(g)$ and $D(fg) = D(f)$ and $C(fg) = C(g)$,

[**C.4**] $(fg)h = f(gh)$ whenever either side is defined,

[**C.5**] $D(C(x)) = C(x)$, $C(D(x)) = D(x)$ and C, D are both idempotent.

A more familiar defintion, often used in introducing categories, is given next.

Definition 2.1.2. A *category* $\mathbb{A}$ is a directed graph consisting of objects A_o and maps A_m. Each $f \in A_m$ has two associated objects in A_o, called the domain and codomain. When f has domain X and codomain Y we will write $f : X \to Y$. For $f, g \in A_m$, if $f : X \to Y$ and $g : Y \to Z$, there is a map called the *composite* of f and g, written fg such that $fg : X \to Z$.

For any $W \in A_o$ there is an *identity* map $1_W : W \to W$. Additionally, these two axioms must hold:

[**C'.1**] for $f : X \to Y$, $1_X f = f = f 1_Y$,

[**C'.2**] given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, then $f(gh) = (fg)h$.

Lemma 2.1.3. *A category as defined in Definition 2.1.1 is equivalent to a category as defined in Definition 2.1.2 and vice versa.*

Proof. Assume $\mathbb{A}$ is as in Definition 2.1.1. Then set A_o to the collection of all $D(f)$ and $C(f)$. Set A_m to all the maps in $\mathbb{A}$. The domain of any map $f \in A_m$ is $D(f)$ and the codomain is $C(f)$. By [**C.3**], for $f : X \to Y$ and $g : Y \to Z$ the composite fg is defined. The identity map of the object $D(f)$ is the map $D(f)$ and the identity map of the object $C(f)$ is $C(f)$. By [**C.5**], we see [**C'.1**] is satisfied. By [**C.4**], we see [**C'.2**] is satisfied. Therefore, $\mathbb{A}$ satisfies Definition 2.1.2.

Conversely, assume $\mathbb{Z}$ is as in Definition 2.1.2. Then, we already have the collection of maps, Z_m. For each $f : A \to B \in Z_m$, set $D(f) = 1_A$ and $C(f) = 1_B$. By the definition of the identity maps and [**C'.1**], we see [**C.1**], [**C.2**] and [**C.5**] are all satisfied. From the compostion requirements on $\mathbb{Z}$ and [**C'.2**], it follows that [**C.4**] is satisfied. For [**C.3**], assume fg is defined. Then for some $A, B, C \in Z_o$, $f : A \to B$ and $g : B \to C$. This gives us $1_B = C(f) = D(g)$, $1_A = D(fg) = D_f$ and $1_B = C(fg) = C(g)$. Next, assume we have $C(f) = D(g)$, $D(fg) = D(f)$ and $C(fg) = C(g)$. This tells us the codomain of f is some object B which is also the domain of g, hence we may form the composition fg which will have domain A, the domain of f and codomain C, the codomain of g. $\square$

As we have shown the two defintions are equivalent, it will be convenient to reference either definition and manner of referring to a category thoughout this thesis. Essentially, we will use whichever definition seems the most appropriate to use at any point.

We may also consider the notion of containment between categories.

Definition 2.1.4. Given the categories $\mathbb{C}$ and $\mathbb{D}$, we may say the following:

(i) $\mathbb{C}$ is a *sub-category* of $\mathbb{D}$ when each object of $\mathbb{C}$ is an object of $\mathbb{D}$ and when each map of $\mathbb{C}$ is a map of $\mathbb{D}$.

(ii) $\mathbb{C}$ is a *full sub-category* of $\mathbb{D}$ when it is a sub-category and given A, B objects in $\mathbb{C}$ and $f : A \to B$ in $\mathbb{D}$, then f is a map in $\mathbb{C}$.

2.1.1 Enrichement of categories

Definition 2.1.5. If $\mathbb{X}$ is a category, then $\mathbb{X}(A, B)$ is called a *hom-collection* of $\mathbb{X}$ and consists of all arrows f with $D(f) = A$ and $C(f) = B$.

In the case where the hom-objects of a category $\mathbb{X}$ are all sets, we call them hom-sets. Additionally, we say $\mathbb{X}$ is *enriched* in Sets. We may extend this to any mathematical structure, e.g., enriched in partial orders, enriched in groups, etc..

Specific types of enrichment may force a specific structure on a category. For example, if $\mathbb{X}$ is enriched in sets of cardinality of 0 or 1, then $\mathbb{X}$ must be a preorder.

2.1.2 Examples of categories

In this section, we will offer a few examples of categories. As Definition 2.1.2 tends to be a more succinct way to present the data of a category, this section will given the examples in terms of objects and maps rather than the "object-free" definition.

Categories based on Sets

There are three primary categories of interest to us where the objects are the collection of sets. The first is Sets, where the maps are given by all set functions. The second is Par, where the maps are all partial maps. In each case, the standard definition of functions suffices to ensure indenties, compositions and associativity are all satisfied. Domain and codomain are given by the domain and range respectively.

A third example, often of interest in quantum programming language semantics is Rel:

Objects: Sets

Maps: Relations: $R : X \to Y$

Identity: $1_X = \{(x, x)|x \in X\}$

Composition: $RS = \{(x, z)|\exists y, (x, y) \in R$ and $(y, z) \in S\}$

Note that Rel is enriched in posets, via set inclusion. Par can be viewed as a subcategory of Rel, with the same objects, but only allowing maps which are functions, i.e., if $(x, y), (x, y') \in R$, then $y = y'$. Par is also enriched in posets, via the same inclusion ordering as in Rel.

Matrix categories

Given a rig R (i.e., a ring minus negatives, e.g., the positive rationals), one may form the category Mat (R).

Objects: $\mathbb{N}$

Maps: $[r_{ij}] : n \to m$ where $[r_{ij}]$ is an $n \times m$ matrix over R

Identity: I_n

Composition: Matrix multiplication

Dual categories

Given a category $\mathbb{C}$, we may form the *dual* of C, written C^{op} as the following category:

Objects: The objects of $\mathbb{C}$

Maps: $f^{op} : B \to A$ in $\mathbb{C}^{op}$ when $f : A \to B$ in $\mathbb{C}$.

Identity: The identity maps of $\mathbb{C}$

Composition: If $fg = h$ in $\mathbb{C}$, $g^{op} f^{op} = h^{op}$

2.1.3 Properties of maps

Many interesting properties of maps are generalizations of notions that have been found useful in considering sets and functions. We present a few of these in a tabular format, together with their categorical definition. Throughout the table, e, f, g are maps in a category C with $e : A \to A$ and $f, g : A \to B$.

Sets	**Categorical Property**	**Definition**
Injective	Monic	f is monic whenver $hf = kf$ means that $h = k$.
Surjective	Epic	The dual notion to monic, g is epic whenever $gh = gk$ means that $h = k$. A map that is both monic and epic is called *bijic*.
Left Inverse	Section	f is a section when there is a map f^* such that $ff^* = 1_A$. f is also referred to as the *left inverse* of f^*.
Right Inverse	Retraction	f is a retraction when there is a map f_* such that $f_*f = 1_B$. f is also referred to as the *right inverse* of f_*. A map that is both a section and a retraction is called an *isomorphism*.
Idempotent	Idempotent	An endomap e is idempotent whenever $ee = e$.

We state without proof a number of properties of maps.

Lemma 2.1.6. *In a category* $\mathbb{C}$,

(i) If f, g are monic, then fg is monic.

(ii) If fg is monic, then f is monic.

(iii) f being a section means it is monic.

(iv) f, g sections implies that fg is a section.

(v) fg a section means f is a section.

Lemma 2.1.7. *If $f : A \to B$ is both a section and a retraction, then $f^* = f_*$.*

Lemma 2.1.8. *f is an isomorphisim if and only if it is an epic section.*

Note there are corresponding properties for epics and retractions, obtained by dualizing the statements of Lemma 2.1.6 and Lemma 2.1.8.

Suppose $f : A \to B$ is a retraction with left inverse $f_* : B \to A$. Note that ff_* is idempotent as $ff_*ff_* = f1_Bf_* = ff_*$. If we are given an idempotent e, we say e is *split* if there is a retraction f with $e = ff_*$.

In general, not all idempotents in a category will split. The following construction allows us to create a category based on the original one in which all idempotents do split.

Definition 2.1.9. Given a category $\mathbb{C}$ we define *Split($\mathbb{C}$)* as the following category:

Objects: (A, e), where A is an object of $\mathbb{C}$, $e : A \to A$ and $e \in E$.

Maps: $f_{d,e} : (A, d) \to (B, e)$ is given by $f : A \to B$ in $\mathbb{C}$, where $f = dfe$.

Identity: The map $e_{e,e}$ for (A, e).

Composition: Inherited from $\mathbb{C}$.

Lemma 2.1.10. *Given a category $\mathbb{C}$, then it is a full sub-category of Split($\mathbb{C}$) and all idempotents split in Split($\mathbb{C}$).*

Proof. We identify each object A in $\mathbb{C}$ with the object $(A, 1)$ in Split($\mathbb{C}$). The only maps between $(A, 1)$ and $(B, 1)$ in Split($\mathbb{C}$) are the maps between A and B in $\mathbb{C}$, hence we have a full sub-category.

Suppose we have the map $d_{e,e} : (A, e) \to (A, e)$ with $dd = d$, i.e., it is idempotent in C and Split($\mathbb{C}$). In Split($\mathbb{C}$), we have the map $d_{e,d} : (A, e) \to (A, d)$ and $d_{d,e} : (A, d) \to (A, e)$ where $d_{d,e}d_{e,d} = d_{d,d} = 1_{(A,d)}$ and $d_{e,d}d_{d,e} = d_{e,e}$, hence it is a splitting of the map $d_{e,e}$. $\square$

2.1.4 Limits and colimits in categories

We shall discuss only a few basic limits/colimits in categories. First we discuss initial and terminal objects.

Definition 2.1.11. An *initial object* in a category $\mathbb{C}$ is an object which has exactly one map to each other object in the category. The dual notion is *terminal object* which has exactly one map from each other object in the category.

Lemma 2.1.12. *Suppose I, J are initial objects in $\mathbb{C}$. Then there is an unique isomorphism $i : I \to J$.*

Proof. First, note that by definition there is only one map from I to I — which must be the identity map. As I is initial there is a map $i : I \to J$. As J is intial there is a map $j : J \to I$. But this means $ij : I \to I = 1$ and $ji : J \to J = 1$ and hence i is the unique isomorphism from I to J. $\square$

Dually, we have the corresponding result of Lemma 2.1.12 for terminal objects — they are also unique up to a unique isomorphism.

In categories, we normally designate the intial object by 0 and the terminal object by 1.

We now turn to products and co-products.

Definition 2.1.13. Let A, B be objects of the category $\mathbb{C}$. Then the object $A \times B$ is a *product* of A and B when:

- There exist maps π_0, π_1 with $\pi_0 : A \times B \to A$, $\pi_1 : A \times B \to B$;
- Given an object C with maps $f : C \to A$ and $g : C \to B$ there exists an

unique map $\langle f, g \rangle$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & A \\
C \arrow[urr, "f"] \arrow[r, dotted, "\langle f,g \rangle"] \arrow[drr, "g"'] & A \times B \arrow[ur, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
 & & B
\end{tikzcd}$$

2.1.5 Functors and natural transformations

Definition 2.1.14. A map $F : \mathbb{X} \to \mathbb{Y}$ between categories (as in Definition 2.1.1 is called a *functor*, provided it satisfies the following:

[**F.1**] $F(D(f)) = D(F(f))$ and $F(C(f)) = C(F(f))$;

[**F.2**] $F(fg) = F(f)F(g)$;

Lemma 2.1.15. *The collection of categories and functors form the category* Cat.

Proof. **Objects:** Categories.

Maps: Functors.

Identity: The identity functor which takes a map to the same map.

Composition: $FG(x) = F(G(x))$ which is clearly associative.

□

We will often restrict ourselves to specific classes of functors which either *preserve* or *reflect* certain characteristics of the domain category or codomain category. To be more precise, we provide some definitions.

Definition 2.1.16. A *diagram* in a category is a collection of objects and maps between those objects which satisfy categorical composition rules. More precisely: Given a category §, a diagram in a category $\mathbb{C}$ of *shape* § is a functor $D : \S \to \mathbb{C}$.

In practice, diagrams are pictorially represented by drawing the objects and the maps between them.

Definition 2.1.17. A *property* of a diagram D, written $P(D)$ is a logical relation expressed using the objects and maps of the diagram D.

Example 2.1.18. $P(f : A \to B) = \exists h : B \to A.hf = 1_A$ expresses that f is a retraction.

Definition 2.1.19. A functor F *preserves* the property P over maps f_i and objects A_j when $P(f_1, \ldots, f_n, A_1, \ldots, A_m)$ implies $P(F(f_1), \ldots, F(f_n), F(A_1), \ldots, F(A_m))$.

Definition 2.1.20. A functor F *reflects* the property P over maps f_i and objects A_j when $P(F(f_1), \ldots, F(f_n), F(A_1), \ldots, F(A_m))$ implies $P(f_1, \ldots, f_n, A_1, \ldots, A_m)$.

For example, all functors preserve the properties of being an idempotent or a retration or section, but in general, not the property of being monic.

A functor $F : \mathbb{C} \to \mathbb{D}$ induces a map between hom-objects in $\mathbb{C}$ and hom-objects in $\mathbb{D}$. For each object A, B in $\mathbb{C}$ we have the map:

$$F_{AB} : \mathbb{C}(A, B) \to \mathbb{D}(F(A), F(B)).$$

Definition 2.1.21. Given a functor $F : \mathbb{C} to \mathbb{D}$, we say:

- F is *faithul* when for all A, B, F_{AB} is an injective function;
- F is *full* when for all A, B, F_{AB} is an surjective function.

Definition 2.1.22. Given functors $F, G : \mathbb{X} \to \mathbb{Y}$, a *natural transformation* $\alpha : F \Rightarrow G$ is a collection of maps in $\mathbb{Y}$, $\alpha_X : F(X) \to G(X)$, indexed by the objects of $\mathbb{X}$ such that for all $f : X_1 \to X_2$ in $\mathbb{X}$ the following diagram in $\mathbb{Y}$ commutes:

$$\begin{tikzcd}
F(X_1) \arrow[r, "F(f)"] \arrow[d, "\alpha_{X_1}"'] & F(X_2) \arrow[d, "\alpha_{X_2}"] \\
G(X_1) \arrow[r, "G(f)"'] & G(X_2)
\end{tikzcd}$$

2.1.6 Categories with additional structure

Definition 2.1.23 (Symmetric Monoidal Category)**.** A *symmetric monoidal category* $\mathbb{D}$ is a category equipped with a monoid $\oplus$ (a bi-functor $\oplus : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$) together with three families of natural isomorphisms: $a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, $u_A : A \to A \oplus I$ and $c_{A,B} : A \oplus B \to B \oplus A$, which satisfy specific coherence diagrams. The isomorphisms are referred to as the *structure isomorphisms* for the symmetric monoidal category. I is the unit of the monoid.

For details on the coherence diagrams, please see e.g., [6] or [16]. The essence of the coherence diagrams is that any diagram composed solely of the structure isomorphisms will commute.

Definition 2.1.24 (Compact Closed Category)**.** A *compact closed category* $\mathbb{D}$ is a symmetric monoidal category with monoid $\otimes$ where each object A has a dual A^* and there exist families of maps $\eta_A : I \to A^* \otimes A$ (the *unit*) and $\epsilon_A : A \otimes A^* \to I$ (the *counit*) such that

$$\begin{tikzcd}
A \arrow[r, "u_A"] \arrow[d, equal] & A \otimes I \arrow[r, "1\otimes\eta_A"] & A \otimes (A^* \otimes A) \arrow[d, "a_{A,A^*,A}"] \\
A & I \otimes A \arrow[l, "u_A^{-1}"] & (A \otimes A^*) \otimes A \arrow[l, "\otimes\epsilon_B\otimes 1"]
\end{tikzcd}$$

commutes and so does the similar one based on A^*.

Given a map $f : A \to B$ in a compact closed category, define the map $f^* : B^* \to A^*$ as

$$\begin{tikzcd}
B^* \arrow[r, "u_{B^*}"] \arrow[d, "f^*"'] & I \otimes B^* \arrow[r, "\eta_A\otimes 1"] & A^* \otimes A \otimes B^* \arrow[d, "1\otimes f\otimes 1"] \\
A^* & A^* \otimes I \arrow[l, "u_{A^*}^{-1}"] & A^* \otimes B \otimes B^* \arrow[l, "1\otimes\epsilon_B"]
\end{tikzcd}$$

2.2 Restriction categories

Restriction categories were introduced in [18] as a convenient axiomatization of partial maps.

Definition 2.2.1. A *restriction category* is a category $\mathbb{X}$ together with a *restriction operator* on maps:

$$\frac{f : A \to B}{\overline{f} : A \to A}$$

where f is an map of $\mathbb{X}$ and A, B are objects of $\mathbb{X}$, such that the following four *restriction identities* hold, whenever the compositions[1] are defined.

[**R.1**] $\overline{f}f = f$ [**R.2**] $\overline{g}\,\overline{f} = \overline{f}\,\overline{g}$

[**R.3**] $\overline{\overline{f}g} = \overline{f}\,\overline{g}$ [**R.4**] $f\overline{g} = \overline{fg}f$

Definition 2.2.2. A *restriction functor* is a functor which preserves the restriction. That is, given a functor $F : \mathbb{X} \to \mathbb{Y}$ with $\mathbb{X}$ and $\mathbb{Y}$ restriction categories, F is a restriction functor if:

$$F(\overline{f}) = \overline{F(f)}.$$

Any map such that $r = \overline{r}$ is an idempotent, as $\overline{r}\,\overline{r} = \overline{\overline{r}r} = \overline{r}$, and is called a *restriction idempotent*. All maps $\overline{f}$ are restriction idempotents as $\overline{f} = \overline{\overline{f}}$. Below, we record some basic facts for restriction categories shown in [18] pp 4-5:

Lemma 2.2.3. *In a restriction category* $\mathbb{X}$*,*

(i) $\overline{f}$ *is idempotent;*

(ii) $\overline{fg} = \overline{fg}\,\overline{f}$*;*

(iii) $\overline{fg} = \overline{f\overline{g}}$ *;*

(iv) $\overline{\overline{f}} = \overline{f}$*;*

(v) $\overline{f}\,\overline{g} = \overline{\overline{f}\,\overline{g}}$*;*

(vi) f *monic implies* $\overline{f} = 1$*;*

(vii) $f = \overline{g}f \implies \overline{g}\,\overline{f} = \overline{f}$*.*

A map $f : A \to B$ in a restriction category is said to be *total* when $\overline{f} = 1_A$. The total maps in a restriction category form a subcategory $Total(\mathbb{X}) \subseteq \mathbb{X}$.

[1] Note that composition is written in diagrammatic order throughout this paper.

An example of a restriction category is PAR, the category with objects sets and arrows the partial functions between sets. In PAR, the restriction of $f : A \to B$ is:

$$\overline{f}(x) = \begin{cases} x & \text{if } f(x) \text{ is defined,} \\ \uparrow & \text{if } f(x) \text{ is } \uparrow . \end{cases}$$

(The symbol $\uparrow$ means that the function is undefined at that element). In PAR, the total maps correspond precisely to the functions that are defined on all elements of the domain.

2.2.1 Enrichment and meets

In any restriction category, there is a partial order on each hom-set, given by $f \leq g$ iff $\overline{f}g = f$, where $f, g : A \to B$.

Lemma 2.2.4. *In a restriction category $\mathbb{X}$:*

(i) $\leq$ *as defined above is a partial order on each hom-set;*

(ii) $f \leq g \implies \overline{f} \leq \overline{g}$*;*

(iii) $\overline{fg} \leq \overline{f}$*;*

(iv) $f \leq g \implies hf \leq hg$*;*

(v) $f \leq g \implies fh \leq gh$*;*

(vi) $f \leq g$ *and* $\overline{f} = \overline{g}$ *implies* $f = g$*;*

(vii) $f \leq 1 \iff f = \overline{f}$*;*

(viii) $\overline{g}f = f$ *implies* $\overline{f} \leq \overline{g}$*.*

Proof.

(i) With f, g, h parallel maps in $\mathbb{X}$, each of the requirements for a partial order is verified below:

Reflexivity: $\overline{f}f = f$ and therefore, $f \leq f$.

Anti-Symmetry: Given $\overline{f}g = f$ and $\overline{g}f = g$, it follows:

$$f = \overline{f}f = \overline{\overline{f}g}f = \overline{f}\,\overline{g}f = \overline{g}\overline{f}f = \overline{g}f = g.$$

Transitivity: Given $f \leq g$ and $g \leq h$,

$$\overline{f} h = \overline{\overline{f} g} h = \overline{f}\,\overline{g} h = \overline{f} g = f$$

showing that $f \leq h$.

(ii) The premise is that $\overline{f} g = f$. From this, $\overline{f}\,\overline{g} = \overline{\overline{f} g} = \overline{f}$, showing $\overline{f} \leq \overline{g}$.

(iii) $\overline{hf} hg = h\overline{f} g = hf$ and therefore $hf \leq hg$.

(iv) $\overline{f} g = f$, this shows $\overline{fh} gh = \overline{\overline{f} g h} gh = \overline{f}\,\overline{gh} gh = \overline{f} gh = fh$ and therefore $fh \leq gh$.

(v) $g = \overline{g} g = \overline{f} g = f$.

(vi) As $f \leq 1$ means precisely $\overline{f} 1 = f$.

(vii) Assuming $\overline{g} f = f$, we need to show $\overline{f}\,\overline{g} = \overline{f}$.

$$\begin{aligned}
\overline{f}\,\overline{g} &= \overline{g}\,\overline{f} && \textbf{[R.2]} \\
&= \overline{\overline{g} f} && \textbf{[R.3]} \\
&= \overline{f} && \text{Assumption.}
\end{aligned}$$

Hence, $\overline{f} \leq \overline{g}$.

□

Lemma 2.2.4 on the preceding page shows that restriction categories are enriched in partial orders.

In a restriction category $\mathbb{X}$, we will use the notation $\mathcal{O}(A)$ for the restriction idempotents of $A \in \operatorname{Ob}\mathbb{X}$. $\mathcal{O}(A) = \{x : A \to A | x = \overline{x}\}$. The notation $\mathcal{O}(A)$ was chosen to be suggestive of open sets.

Lemma 2.2.5. *In a restriction category* $\mathbb{X}$, $\mathcal{O}(A)$ *is a meet semi-lattice.*

Proof. The top of the meet semi-lattice is 1_A, the ordering is as above and the join is given by composition. □

Definition 2.2.6. A restriction category has *meets* if there is an operation $\cap$ on parallel maps:

$$\frac{A \overset{f}{\underset{g}{\rightrightarrows}} B}{A \xrightarrow{f \cap g} B}$$

such that $f \cap g \leq f, f \cap g \leq g, f \cap f = f, h(f \cap g) = hf \cap hg$.

Meets were introduced in [8]. The following are basic results on meets:

Lemma 2.2.7. *In a restriction category $\mathbb{X}$ with meets, where f, g, h are maps in $\mathbb{X}$, the following are true:*

(i) $f \leq g$ *and* $f \leq h \iff f \leq g \cap h$*;*

(ii) $f \cap g = g \cap f$*;*

(iii) $\overline{f \cap 1} = f \cap 1$*;*

(iv) $(f \cap g) \cap h = f \cap (g \cap h)$*;*

(v) $r(f \cap g) = rf \cap g$ *where* $r = \overline{r}$ *is a restriction idempotent;*

(vi) $(f \cap g)r = fr \cap g$ *where* $r = \overline{r}$ *is a restriction idempotent;*

(vii) $\overline{f \cap g} \leq \overline{f}$ *(and therefore* $\overline{f \cap g} \leq \overline{g}$*);*

(viii) $(f \cap 1)f = f \cap 1$*;*

(ix) $e(e \cap 1) = e$ *where* e *is idempotent.*

Proof.

(i) $f \leq g$ and $f \leq h$ means precisely $f = \overline{f}g$ and $f = \overline{f}h$. Therefore,

$$\overline{f}(g \cap h) = \overline{f}g \cap \overline{f}h = f \cap f = f$$

and so $f \leq g \cap h$. Conversely, given $f \leq g \cap h$, we have $f = \overline{f}(g \cap h) = \overline{f}g \cap \overline{f}h \leq \overline{f}g$. But $f \leq \overline{f}g$ means $f = \overline{f}\,\overline{f}g = \overline{f}g$ and therefore $f \leq g$. Similarly, $f \leq h$.

(ii) From (i) on the previous page, as by definition, $f \cap g \leq g$ and $f \cap g \leq f$.

(iii) $f \cap 1 = \overline{f \cap 1}(f \cap 1) = (\overline{f \cap 1} f) \cap (\overline{f \cap 1}) \leq \overline{f \cap 1}$ from which the result follows.

(iv) By definition and transitivity, $(f \cap g) \cap h \leq f, g, h$ therefore by (i) on the preceding page $(f \cap g) \cap h \leq f \cap (g \cap h)$. Similarly, $f \cap (g \cap h) \leq (f \cap g) \cap h$ giving the equality.

(v) Given $rf \cap g \leq rf$, calculate:

$$rf \cap g = \overline{rf \cap g} rf = \overline{r(rf \cap g)} f = \overline{rrf \cap rg} f = \overline{r(f \cap g)} f = r\overline{f \cap g} f = r(f \cap g).$$

(vi) Using the previous point with the restriction idempotent $\overline{fr}$,

$$\begin{aligned} fr \cap g = f\overline{r} \cap g = \overline{fr} f \cap g = \overline{fr}(f \cap g) = \overline{fr}\,\overline{f \cap g} f \\ = \overline{f \cap g}\,\overline{fr} f = \overline{f \cap g} f\overline{r} = (f \cap g) r. \end{aligned}$$

(vii) For the first claim,

$$\overline{f \cap g}\,\overline{f} = \overline{\overline{f}(f \cap g)} = \overline{(\overline{f} f) \cap g} = \overline{f \cap g}.$$

The second claim then follows by (ii) on the previous page.

(viii) Given $f \cap 1 \leq f$:

$$f \cap 1 \leq f \iff \overline{f \cap 1} f = f \cap 1 \iff (f \cap 1) f = f \cap 1$$

where the last step is by item (iii) on the preceding page of this lemma.

(ix) As e is idempotent, $e(e \cap 1) = (ee \cap e) = e$.

□

2.2.2 Range categories

Corresponding to Definition 2.2.1, which axiomatizes the concept of a domain of definition, we now introduce range categories which serve to axiomatize the concept of the range of a function.

Definition 2.2.8. A restriction category $\mathbb{X}$ is a *range category* when it has an operator on all maps

$$\frac{f : A \to B}{\hat{f} : B \to B}$$

where the operator satisfies the following:

$$[\mathbf{RR.1}]\ \overline{\hat{f}} = \hat{f} \qquad\qquad [\mathbf{RR.2}]\ f\hat{f} = f$$

$$[\mathbf{RR.3}]\ \widehat{f\overline{g}} = \hat{f}\overline{g} \qquad\qquad [\mathbf{RR.4}]\ \widehat{\hat{f}g} = \widehat{fg}$$

whenever the compositions are defined.

Lemma 2.2.9. *In a range category $\mathbb{X}$, the following hold:*

(i) $\hat{g}\hat{f} = \hat{f}\hat{g}$;

(ii) $\overline{f}\hat{g} = \hat{g}\overline{f}$;

(iii) $\widehat{f\hat{g}} = \hat{f}\hat{g}$;

(iv) $\hat{f} = 1$ *when f is epic, hence* $\hat{1} = 1$;

(v) $\hat{f}\hat{f} = \hat{f}$;

(vi) $\hat{\hat{f}} = \hat{f}$;

(vii) $\hat{\overline{f}} = \overline{f}$;

(viii) $\hat{g}\widehat{fg} = \widehat{fg}$;

(ix) $\widehat{\hat{f}\hat{g}} = \hat{f}\hat{g}$.

Proof. See, e.g., [11]. □

Lemma 2.2.10. *In a range category:*

(i) $\widehat{hf} \leq \hat{f}$;

(ii) $f' \leq f$ *implies* $\hat{f}' \leq \hat{f}$.

Proof.

(i) Noting that $\overline{\widehat{hf}}\hat{f} = \widehat{hf}\hat{f} = \widehat{hf\hat{f}} = \widehat{hf}$, we see $\widehat{hf} \leq \hat{f}$.

(ii) Calculating $\overline{\hat{f}'}\hat{f} = \hat{f}'\hat{f} = \widehat{\overline{f'}f}\hat{f} = \widehat{\overline{f'}f\hat{f}} = \widehat{\overline{f'}f} = \hat{f}'$, we see $\hat{f}' \leq \hat{f}$.

□

Remark 2.2.11. Note that unlike restrictions, a range is a *property* of a restriction category. To see this, assume we have two ranges $\widehat{(\_)}$ and $\widetilde{(\_)}$. Then,

$$\hat{f} = \widehat{f\tilde{f}} = \hat{f}\tilde{f} = \tilde{f}\hat{f} = \widetilde{f\hat{f}} = \tilde{f}.$$

Lemma 2.2.12. *An inverse category $\mathbb{X}$ is a range category, where $\hat{f} = f^{(-1)}f = \overline{f^{(-1)}}$.*

Proof.

[RR.1] $\overline{\hat{f}} = \overline{\overline{f^{(-1)}}} = \overline{f^{(-1)}} = \hat{f}$;

[RR.2] $f\hat{f} = f\overline{f^{(-1)}} = ff^{(-1)}f = \overline{f}f = f$;

[RR.3] $\widehat{f\overline{g}} = \overline{(f\overline{g})^{(-1)}} = \overline{\overline{g}^{(-1)}f^{(-1)}} = \overline{\overline{g}f^{(-1)}} = \overline{g}\,\overline{f^{(-1)}} = \overline{f^{(-1)}}\,\overline{g} = \hat{f}\overline{g}$;

[RR.4] $\widehat{\hat{f}g} = \overline{\left(\overline{f^{(-1)}}g\right)^{(-1)}} = \overline{g^{(-1)}\overline{f^{(-1)}}^{(-1)}} = \overline{g^{(-1)}\overline{f^{(-1)}}} = \overline{g^{(-1)}f^{(-1)}} = \overline{(fg)^{(-1)}} = \widehat{fg}$

□

2.2.3 Partial monics, sections and isomorphisms

Partial isomorphisms play a central role in this thesis and below we develop some their basic properties.

Definition 2.2.13. A map f in a restriction category $\mathbb{X}$ is said:

- To be a *partial isomorphism* when there is a *partial inverse*, written $f^{(-1)}$ with $ff^{(-1)} = \overline{f}$ and $f^{(-1)}f = \overline{f^{(-1)}}$;
- To be a *partial monic* if $hf = kf \implies h\overline{f} = k\overline{f}$;
- To be a *partial section* if there exists an h such that $fh = \overline{f}$;
- To be a *restriction monic* if it is a section s with a retraction r such that $rs = \overline{rs}$.

Note that restriction monic is a stronger notion than that of monic. Consider two objects A, B in a restriction category where we have $m : A \to B$, $r : B \to A$ with $mr = 1_A$. In this case A is called a *retract* of B, which we will write as $A \triangleleft B$. As m and r need not be unique, we will also write $(m,r)A \triangleleft B$ when the specific section and retraction are to be emphasized. Since m is a section, it is a monic and therefore total. rm is of course, an idempotent on B. A is referred to as a splitting of the idempotent rm. Note there is no requirement that $rm = \overline{rm}$ if m is simply monic.

Lemma 2.2.14. *In a restriction category:*

(i) *f, g partial monic implies fg is partial monic;*

(ii) *f a partial section implies f is partial monic;*

(iii) *f, g partial sections implies fg is a partial section;*

(iv) *The partial inverse of f, when it exists, is unique;*

(v) *If f, g have partial inverses and fg exists, then fg has a partial inverse;*

(vi) *A restriction monic s is a partial isomorphism.*

Proof.

(i) Suppose $hfg = kfg$. As g is partial monic, $hf\overline{g} = kf\overline{g}$. Therefore:

$$
\begin{array}{rcl}
h\overline{fg}f = k\overline{fg}f & & [\mathbf{R.4}] \\
h\overline{fg}\,\overline{f} = k\overline{fg}\,\overline{f} & & f\text{partial monic} \\
h\overline{fg} = k\overline{fg} & & \text{Lemma 2.2.3, (ii)}
\end{array}
$$

(ii) Suppose $gf = kf$. Then, $g\overline{f} = gfh = kfh = k\overline{f}$.

(iii) We have $fh = \overline{f}$ and $gh' = \overline{g}$. Therefore,

$$\begin{aligned}
fgh'h &= f\overline{g}h && g \text{ partial section} \\
&= \overline{fg}fh && [\mathbf{R.4}] \\
&= \overline{fg}\,\overline{f} && f \text{ partial section} \\
&= \overline{f}\,\overline{fg} && [\mathbf{R.2}] \\
&= \overline{\overline{f}fg} && [\mathbf{R.3}] \\
&= \overline{fg} && [\mathbf{R.1}]
\end{aligned}$$

(iv) Suppose both $f^{(-1)}$ and f^* are partial inverses of f. Then,

$$\begin{aligned}
f^{(-1)} &= \overline{f^{(-1)}}f^{(-1)} = f^{(-1)}ff^{(-1)} = f^{(-1)}\overline{f} = f^{(-1)}ff^* = f^{(-1)}f\overline{f^*}f^* \\
&= \overline{f^{(-1)}}\,\overline{f^*}f^* = \overline{f^*}\,\overline{f^{(-1)}}f^* = f^*f\overline{f^{(-1)}}f^* = f^*ff^{(-1)}ff^* = f^*ff^* = f^*
\end{aligned}$$

(v) For $f : A \to B,\ g : B \to C$ with partial inverses $f^{(-1)}$ and $g^{(-1)}$ respectively, the partial inverse of fg is $g^{(-1)}f^{(-1)}$. Calculating $fgg^{(-1)}f^{(-1)}$ using all the restriction identities:

$$fgg^{(-1)}f^{(-1)} = f\overline{g}f^{(-1)} = \overline{fg}ff^{(-1)} = \overline{fg}\,\overline{f} = \overline{f}\,\overline{fg} = \overline{\overline{f}fg} = \overline{fg}.$$

The calculation of $g^{(-1)}f^{(-1)}fg = \overline{g^{(-1)}f^{(-1)}}$ is similar.

(vi) The partial inverse of s is $\overline{rs}\,r$. First, note that $\overline{\overline{rs}\,r} = \overline{rs}\,\overline{r} = \overline{r}\,\overline{rs} = \overline{\overline{r}\,rs} = \overline{rs}$. Then, it follows that $(\overline{rs}\,r)s = rs = \overline{rs} = \overline{\overline{rs}r}$ and $s(\overline{rs}\,r) = sr\overline{s} = \overline{s}$.

$\square$

A restriction category in which every map is a partial isomorphism is called an *inverse category.*

An interesting property of inverse categories:

Lemma 2.2.15. *In an inverse category, all idempotents are restriction idempotents.*

Proof. Given an idempotent e,

$$\overline{e} = ee^{(-1)} = eee^{(-1)} = e\overline{e} = \overline{\overline{e}e} = \overline{e}e = e.$$

□

2.2.4 Split restriction categories

The split restriction category, $\mathrm{K}_E(\mathbb{X})$ is defined as:

Objects: (A, e), where A is an object of $\mathbb{X}$, $e : A \to A$ and $e \in E$.

Maps: $f : (A, d) \to (B, e)$ is given by $f : A \to B$ in $\mathbb{X}$, where $f = dfe$.

Identity: The map e for (A, e).

Composition: inherited from $\mathbb{X}$.

This is the standard idempotent splitting construction, also known as the Karoubi envelope.

Note that for $f : (A, d) \to (B, e)$, by definition, in $\mathbb{X}$ we have $f = dfe$, giving

$$df = d(dfe) = ddfe = dfe = f \quad \text{and} \quad fe = (dfe)e = dfee = dfe = f.$$

When $\mathbb{X}$ is a restriction category, there is an immediate candidate for a restriction in $\mathrm{K}_E(\mathbb{X})$. If $f \in \mathrm{K}_E(\mathbb{X})$ is $e_1 f e_2$ in $\mathbb{X}$, then define $\overline{f}$ as given by $e_1\overline{f}$ in $\mathbb{X}$. Note that for $f : (A, d) \to (B, e)$, in $\mathbb{X}$ we have:

$$d\overline{f} = \overline{df}d = \overline{f}d.$$

Proposition 2.2.16. *If $\mathbb{X}$ is a restriction category and E is a set of idempotents, then the restriction as defined above makes $\mathrm{K}_E(\mathbb{X})$ a restriction category.*

Proof. The restriction takes $f : (A, e_1) \to (B, e_2)$ to an endomorphism of (A, e_1). The restriction is in $\mathrm{K}_E(\mathbb{X})$ as

$$e_1(e_1\overline{f})e_1 = e_1\overline{f}e_1 = \overline{e_1 f}e_1e_1 = \overline{e_1 f}e_1 = e_1\overline{f}.$$

Checking the 4 restriction axioms:

[R.1] $[\![\overline{f}f]\!] = e_1\overline{f}f = e_1f = [\![f]\!]$

[R.2] $[\![\overline{g}\,\overline{f}]\!] = e_1\overline{g}e_1\overline{f} = e_1e_1\overline{g}\,\overline{f} = e_1e_1\overline{f}\,\overline{g} = e_1\overline{f}e_1\overline{g} = [\![\overline{f}\,\overline{g}]\!]$

[R.3] $[\![\overline{\overline{f}g} \equiv e_1\overline{e_1\overline{f}g} = \overline{e_1e_1\overline{f}ge_1} = \overline{e_1\overline{f}ge_1} = e_1\overline{\overline{f}g} = e_1\overline{f}\,\overline{g} = e_1e_1\overline{f}\,\overline{g} = e_1\overline{f}e_1\overline{g} = [\![\overline{f}\,\overline{g}]\!]$

[R.4] $[\![f\overline{g}]\!] = e_1fe_2\overline{g} = \overline{e_1fe_2g}e_1fe_2 = \overline{e_1e_1fe_2g}e_1e_1fe_2$

$$= e_1\overline{e_1fe_2g}e_1fe_2 = e_1\overline{fg}e_1fe_2 = [\![\overline{fg}f]\!]$$

□

Given this, provided all identity maps are in E, $\mathrm{K}_E(\mathbb{X})$ is a restriction category with $\mathbb{X}$ as a full sub-restriction category, via the embedding defined by taking an object A in $\mathbb{X}$ to the object $(A, 1)$ in $\mathrm{K}_E(\mathbb{X})$. Furthermore, the property of being an inverse category is preserved by splitting.

Lemma 2.2.17. *When $\mathbb{X}$ is an inverse category, $K_E(X)$ is an inverse category.*

Proof. The inverse of $f : (A, e_1) \to (B, e_2)$ in $\mathrm{K}_E(\mathbb{X})$ is $e_2f^{(-1)}e_1$ as

$$[\![ff^{(-1)}]\!] = e_1fe_2e_2f^{(-1)}e_1 = e_1e_1fe_2f^{(-1)}e_1 = e_1ff^{(-1)}e_1 = e_1e_1\overline{f}e_1 = e_1\overline{f} = [\![\overline{f}]\!]$$

and

$$[\![f^{(-1)}f]\!] = e_2f^{(-1)}e_1e_1fe_2 = e_2f^{(-1)}e_1fe_2e_2 = e_2f^{(-1)}fe_2$$

$$= e_2e_2\overline{f^{(-1)}}e_2 = e_2\overline{f^{(-1)}} = [\![\overline{f^{(-1)}}]\!]$$

□

Proposition 2.2.18. *In a restriction category $\mathbb{X}$, with meets, let R be the set of restriction idempotents. Then, $K(\mathbb{X}) \cong K_R(\mathbb{X})$(where $K(\mathbb{X})$ is the split of $\mathbb{X}$ over all idempotents). Furthermore, $K_R(\mathbb{X})$ has meets.*

Proof. The proof below first shows the equivalence of the two categories, then addresses the claim that $\mathrm{K}_R(\mathbb{X})$ has meets.

For equivalence, we require two functors,

$$U : \mathrm{K}_R(\mathbb{X}) \to \mathrm{K}(\mathbb{X}) \text{ and } V : \mathrm{K}(\mathbb{X}) \to \mathrm{K}_R(\mathbb{X}),$$

with:

$$UV \cong I_{\mathrm{K}_R(\mathbb{X})} \tag{2.1}$$

$$VU \cong I_{\mathrm{K}(\mathbb{X})}. \tag{2.2}$$

U is the standard inclusion functor. V will take the object (A, e) to $(A, e \cap 1)$ and the map $f : (A, e_1) \to (B, e_2)$ to $(e_1 \cap 1)f$.

V is a functor as:

Well Defined: If $f : (A, e_1) \to (B, e_2)$, then $(e_1 \cap 1)f$ is a map in $\mathbb{X}$ from A to B and $(e_1 \cap 1)(e_1 \cap 1)f(e_2 \cap 1) = (e_1 \cap 1)(fe_2 \cap f) = (e_1 \cap 1)(f \cap f) = (e_1 \cap 1)f$, therefore, $V(f) : V((A, e_1)) \to V((B, e_2))$.

Identities: $V(e) = (e \cap 1)e = e \cap 1$ by lemma 2.2.7 on page 15.

Composition: $V(f)V(g) = (e_1 \cap 1)f(e_2 \cap 1)g = (e_1 \cap 1)fe_2(e_2 \cap 1)g = (e_1 \cap 1)f(e_2 \cap e_2)g = (e_1 \cap 1)fe_2g = (e_1 \cap 1)fg = V(fg)$.

Recalling from Lemma 2.2.7 on page 15, $(e \cap 1)$ is a restriction idempotent. Using this fact, the commutativity of restriction idempotents and the general idempotent identities from 2.2.7 on page 15, the composite functor UV is the identity on $\mathrm{K}_r(\mathbb{X})$ as when e is a restriction idempotent, $e = e(e \cap 1) = (e \cap 1)e = (e \cap 1)$.

For the other direction, note that for a particular idempotent $e : A \to A$, this gives the maps $e : (A, e) \to (A, e \cap 1)$ and $e \cap 1 : (A, e \cap 1) \to (A, e)$, again by 2.2.7 on page 15. These maps give the natural isomorphism between I and VU as

$$\begin{tikzcd}
(A,e) \arrow[r, "e"] \arrow[dr, "e"'] & (A, e\cap 1) \arrow[d, "e\cap 1"] \\
& (A,e)
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
(A, e\cap 1) \arrow[r, "e\cap 1"] \arrow[dr, "e\cap 1"'] & (A,e) \arrow[d, "e"] \\
& (A, e\cap 1)
\end{tikzcd}$$

both commute. Therefore, $UV = I$ and $VU \cong I$, giving an equivalence of the categories.

For the rest of this proof, the bolded functions, e.g., $\mathbf{f}$ are in $\mathrm{K}_R(\mathbb{X})$. Italic functions, e.g., f are in $\mathbb{X}$.

To show that $\mathrm{K}_R(\mathbb{X})$ has meets, designate the meet in $\mathrm{K}_R(\mathbb{X})$ as $\cap_{\mathrm{K}}$ and define $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$ as the map given by the $\mathbb{X}$ map $f \cap g$, where $\mathbf{f}, \mathbf{g} : (A, d) \to (B, e)$ in $\mathrm{K}_R(\mathbb{X})$ and $f, g : A \to B$ in $\mathbb{X}$. This is a map in $\mathrm{K}_R(\mathbb{X})$ as $d(f \cap g)e = (df \cap dg)e = (f \cap g)e = (fe \cap g) = f \cap g$ where the penultimate equality is by 2.2.7 on page 15. By definition $\overline{\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}}$ is $d\overline{f \cap g}$.

It is necessary to show $\cap_{\mathrm{K}}$ satisfies the four meet properties.

- $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g} \leq \mathbf{f}$: We need to show $\overline{\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}}\mathbf{f} = \mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$. Calculating now in $\mathbb{X}$:

$$\begin{aligned}
d\overline{f \cap g}f &= \overline{d(f \cap g)}df \\
&= \overline{df \cap dg}df \\
&= \overline{f \cap g}f \\
&= f \cap g
\end{aligned}$$

  which is the definition of $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$.

- $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g} \leq \mathbf{g}$: Similarly and once again calculating in $\mathbb{X}$,

$$\begin{aligned}
d\overline{f \cap g}g &= \overline{d(f \cap g)}dg \\
&= \overline{df \cap dg}dg \\
&= \overline{f \cap g}g \\
&= f \cap g
\end{aligned}$$

  which is the definition of $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$.

- $\mathbf{f} \cap_{\mathrm{K}} \mathbf{f} = \mathbf{f}$: From the definition, this is $f \cap f = f$ which is just $\mathbf{f}$.
- $\mathbf{h}(\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}) = \mathbf{hf} \cap_{\mathrm{K}} \mathbf{hg}$: From the definition, this is given in $\mathbb{X}$ by $h(f \cap g) = hf \cap hg$ which in $\mathrm{K}_R(\mathbb{X})$ is $\mathbf{hf} \cap_{\mathrm{K}} \mathbf{hg}$.

□

2.2.5 Partial Map Categories

In [18], it is shown that split restriction categories are equivalent to *partial map categories*. The main definitions and results related to partial map categories are given below.

Definition 2.2.19. A collection $\mathcal{M}$ of monics is *a stable system of monics* when it includes all isomorphisms, is closed under composition and is pullback stable.

Stable in this definition means that if $m : A \to B$ is in $\mathcal{M}$, then for arbitrary b with codomain B, the pullback

$$\begin{tikzcd} A' \arrow[r, "a"] \arrow[d, "m'"'] & A \arrow[d, "m"] \\ B' \arrow[r, "b"'] & B \end{tikzcd}$$

exists and $m' \in \mathcal{M}$. A category that has a stable system of monics is referred to as an $\mathcal{M}$-category.

Lemma 2.2.20. *If $nm \in \mathcal{M}$, a stable system of monics, and m is monic, then $n \in \mathcal{M}$.*

Proof. The commutative square

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "n"'] & A \arrow[d, "nm"] \\ A' \arrow[r, "m"'] & B \end{tikzcd}$$

is a pullback. □

Given a category $\mathbb{C}$ and a stable system of monics, the *partial map category*, $\mathrm{Par}(\mathbb{C}, \mathcal{M})$ is:

Objects: $A \in \mathbb{C}$

Equivalence Classes of Maps: $(m, f) : A \to B$ with $m : A' \to A$ is in $\mathcal{M}$ and $f : A' \to B$ is a map in $\mathbb{C}$. i.e., $\begin{tikzcd} & A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$.

Identity: $1_A, 1_A : A \to A$

Composition: via a pullback, $(m, f)(m', g) = (m''m, f'g)$ where

$$\begin{tikzcd} & & A'' \arrow[dl, "m''"'] \arrow[dr, "f'"] & & \\ & A' \arrow[dl, "m"'] \arrow[dr, "f"'] \arrow[rr, phantom, "\text{(pb)}"] & & B' \arrow[dl, "m'"] \arrow[dr, "g"] & \\ A & & B & & C \end{tikzcd}$$

Restriction: $\overline{(m, f)} = (m, m)$

For the maps, $(m, f) \sim (m', f')$ when there is an isomorphism $\gamma : A'' \to A'$ such that $\gamma m' = m$ and $\gamma f' = f$.

In [19], it is shown that:

Theorem 2.2.21 (Cockett-Lack)**.** *Every restriction category is a full subcategory of a partial map category.*

2.2.6 Restriction products and Cartesian restriction categories

Restriction categories have analogues of products and terminal objects.

Definition 2.2.22. In a restriction category $\mathbb{X}$ a *restriction product* of two objects X, Y is an object $X \times Y$ equipped with *total* projections $\pi_0 : X \times Y \to X, \pi_1 : X \times Y \to Y$ where:

$\forall f : Z \to X, g : Z \to Y, \quad \exists$ a unique $\langle f, g \rangle : Z \to X \times Y$ such that

- $\langle f, g \rangle \pi_0 \leq f$,
- $\langle f, g \rangle \pi_1 \leq g$ and

- $\overline{\langle f, g\rangle} = \overline{f}\,\overline{g}(= \overline{g}\,\overline{f})$.

Definition 2.2.23. In a restriction category $\mathbb{X}$ a *restriction terminal object* is an object $\top$ such that $\forall X$, there is a unique total map $!_X : X \to \top$ and the diagram

$$\begin{tikzcd} X \arrow[r, "\overline{f}"] \arrow[d, "f"] & X \arrow[r, "!_X"] & \top \\ Y \arrow[urr, "!_Y"'] & & \end{tikzcd}$$

commutes. That is, $f\,!_Y = \overline{f}\,!_X$. Note this implies that a restriction terminal object is unique up to a unique isomorphism.

Definition 2.2.24. A restriction category $\mathbb{X}$ is *Cartesian* if it has all restriction products and a restriction terminal object.

Definition 2.2.25. An object A in a Cartesian restriction category is *discrete* when the diagonal map,

$$\Delta : A \to A \times A$$

is a partial isomorphism.

A Cartesian restriction category is *discrete* when every object is discrete.

Theorem 2.2.26. *A Cartesian restriction category $\mathbb{X}$ is discrete if and only if it has meets.*

Proof. If $\mathbb{X}$ has meets, then

$$\Delta(\pi_0 \cap \pi_1) = \Delta\pi_0 \cap \Delta\pi_1 = 1 \cap 1 = 1$$

and as $\langle \pi_0, \pi_1\rangle$ is identity,

$$\begin{aligned} \overline{\pi_0 \cap \pi_1} &= \overline{\pi_0 \cap \pi_1}\langle \pi_0, \pi_1\rangle \\ &= \langle \overline{\pi_0 \cap \pi_1}\pi_0, \overline{\pi_0 \cap \pi_1}\pi_1\rangle \\ &= \langle \pi_0 \cap \pi_1, \pi_0 \cap \pi_1\rangle \\ &= (\pi_0 \cap \pi_1)\Delta \end{aligned}$$

and therefore, $\pi_0 \cap \pi_1$ is $\Delta^{(-1)}$.

For the other direction, set $f \cap g = \langle f, g \rangle \Delta^{(-1)}$. By the definition of the restriction product:

$$f \cap g = \langle f, g \rangle \Delta^{(-1)} = \langle f, g \rangle \Delta^{(-1)} \Delta \pi_0 = \langle f, g \rangle \overline{\Delta^{(-1)}} \pi_0 \leq \langle f, g \rangle \pi_0 \leq f$$

Similarly, substituting π_1 for π_0 above, this gives $f \cap g \leq g$. For the left distributive law,

$$h(f \cap g) = h\langle f, g \rangle \Delta^{(-1)} = \langle hf, hg \rangle \Delta^{(-1)} = hf \cap hg$$

and finally an intersection of a map with itself is

$$f \cap f = \langle f, f \rangle \Delta^{(-1)} = (f\Delta)\Delta^{(-1)} = f\overline{\Delta} = f$$

as Δ is total. This shows that $\cap$ as defined above is a meet for the Cartesian restriction category $\mathbb{X}$.

□

We shall refer to a Cartesian restriction category in which every object is discrete as simply a discrete restriction category.

2.2.7 Discrete Categories

In a Cartesian restriction category, a map $A \xrightarrow{f} B$ is called *graphic* when the maps

$$A \xrightarrow{\langle f,1 \rangle} B \times A \qquad \text{and} \qquad A \xrightarrow{\langle \overline{f},1 \rangle} A \times A$$

have partial inverses. A Cartesian restriction category is *graphic* when all of its maps are graphic.

Lemma 2.2.27. *In a Cartesian restriction category:*

(i) Graphic maps are closed under composition;

(ii) Graphic maps are closed under the restriction;

(iii) An object is discrete if and only if its identity map is graphic.

Proof.

(i) To show closure, it is necessary to show that $\langle fg, 1\rangle$ has a partial inverse. By Lemma 2.2.14 on page 19, the uniqueness of the partial inverse gives

$$(\langle f, 1\rangle; \langle g, 1\rangle \times 1)^{(-1)} = \langle g, 1\rangle^{(-1)} \times 1; \langle f, 1\rangle^{(-1)}.$$

By the definition of the restriction product, $\overline{\langle fg, 1\rangle} = \overline{fg}$. Additionally, a straightforward calculation shows that $\overline{\langle f, 1\rangle; \langle g, 1\rangle \times 1} = \overline{\langle f\langle g, 1\rangle, 1\rangle} = \overline{f; \langle g, 1\rangle} = \overline{\langle f; g, f\rangle} = \overline{fg}\,\overline{f} = \overline{fg}$ where the last equality is by [**R.2**], [**R.3**] and finally [**R.1**].

Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "{\langle f,1\rangle}"] \arrow[drr, "{\langle fg,1\rangle}"'] & B \times A \arrow[r, "{\langle g,1\rangle \times 1}"] & C \times B \times A \\
& & C \times A \arrow[u, "{1 \times \langle f,1\rangle}"']
\end{tikzcd}$$

From this:

$$\begin{aligned}
\langle fg, 1\rangle(1 \times \langle f, 1\rangle)(\langle g, 1\rangle^{(-1)} \times 1)\langle f, 1\rangle^{(-1)} &= \langle f, 1\rangle(\langle g, 1\rangle \times 1)(\langle g, 1\rangle^{(-1)} \times 1)\langle f, 1\rangle^{(-1)} \\
&= \langle f, 1\rangle(\overline{g \times 1})\langle f, 1\rangle^{(-1)} \\
&= \overline{\langle f, 1\rangle(g \times 1)}\langle f, 1\rangle\langle f, 1\rangle^{(-1)} \\
&= \overline{\langle f, 1\rangle(g \times 1)\langle f, 1\rangle} \\
&= \overline{\langle f, 1\rangle\langle f, 1\rangle(g \times 1)} \\
&= \overline{\langle f, 1\rangle(g \times 1)} \\
&= \overline{\langle fg, 1\rangle}(= \overline{fg})
\end{aligned}$$

showing that $1 \times \langle f, 1\rangle(\langle g, 1\rangle^{(-1)} \times 1)\langle f, 1\rangle^{(-1)}$ is a right inverse for $\langle fg, 1\rangle$. For the other direction, note that in general $hk^{(-1)} = k^{(-1)}h^{(-1)}$ and that we have $\langle fg, 1\rangle = \langle f, 1\rangle(\langle g, 1\rangle \times 1)(1 \times \langle f, 1\rangle^{(-1)})$, thus $(1 \times \langle f, 1\rangle)(\langle g, 1\rangle^{(-1)} \times 1)\langle f, 1\rangle^{(-1)}$ will also be a left inverse and $\langle fg, 1\rangle$ is a restriction isomorphism.

(ii) This follows from the definition of graphic and that $\overline{\langle f, 1\rangle} = \overline{f} = \overline{\overline{f}}$.

(iii) Given a discrete object A, the map 1_A is graphic as $\langle 1_A, 1\rangle = \Delta$ and therefore $\langle 1, 1\rangle^{(-1)} = \Delta^{(-1)}$. Conversely, if $\langle 1_A, 1\rangle$ has an inverse, then $\Delta = \langle 1_A, 1\rangle$ has that same inverse and therefore the object is discrete.

□

Lemma 2.2.28. *A discrete restriction category is precisely a graphic Cartesian restriction category.*

Proof. The requirement is that $\langle f, 1\rangle$ (and $\langle \overline{f}, 1\rangle$) each have partial inverses. For $\langle f, 1\rangle$, the inverse is $\overline{(1 \times f)\Delta^{(-1)}}\pi_1$.

To show this, calculate the two compositions. First,

$$\langle f, 1\rangle\overline{1 \times f\Delta^{(-1)}}\pi_1 = \overline{\langle f, f\rangle\Delta^{(-1)}}\langle f, 1\rangle\pi_1 = \overline{f\Delta\Delta^{(-1)}}\langle f, 1\rangle\pi_1 = \overline{f}\langle f, 1\rangle\pi_1 = \overline{f}.$$

The other direction is:

$$\begin{aligned}
\overline{(1 \times f)\Delta^{(-1)}}\pi_1\langle f, 1\rangle &= \langle \overline{(1 \times f)\Delta^{(-1)}}\pi_1 f, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle \overline{(1 \times f)\Delta^{(-1)}}(1 \times f)\pi_1, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle (1 \times f)\overline{\Delta^{(-1)}}\pi_1, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle (1 \times f)\overline{\Delta^{(-1)}}\pi_0, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle \overline{(1 \times f)\Delta^{(-1)}}(1 \times f)\pi_0, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle \overline{(1 \times f)\Delta^{(-1)}}\pi_0, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \overline{(1 \times f)\Delta^{(-1)}}\langle \pi_0, \pi_1\rangle \\
&= \overline{(1 \times f)\Delta^{(-1)}}
\end{aligned}$$

The one tricky step is to realize

$$\begin{aligned}
\overline{\Delta^{(-1)}}\pi_1 &= \Delta^{(-1)}\Delta\pi_1 \\
&= \Delta^{(-1)} \\
&= \Delta^{(-1)}\Delta\pi_0 \\
&= \overline{\Delta^{(-1)}}\pi_0
\end{aligned}$$

For $\langle \overline{f}, 1\rangle$, the inverse is $\overline{(1 \times \overline{f})\Delta^{(-1)}}\pi_1$. Similarly to above,

$$\langle \overline{f}, 1\rangle \overline{1 \times \overline{f}\Delta^{(-1)}}\pi_1 = \overline{\langle \overline{f}, \overline{f}\rangle \Delta^{(-1)}}\langle \overline{f}, 1\rangle\pi_1 = \overline{\overline{f}\Delta\Delta^{(-1)}}\langle \overline{f}, 1\rangle\pi_1 = \overline{\overline{f}}\langle \overline{f}, 1\rangle\pi_1 = \overline{f}.$$

The other direction follows the same pattern as for $\langle f, 1\rangle$. $\square$

2.3 Turing Categories

Definition 2.3.1 (Turing category)**.** Given $\mathbb{X}$ is a cartesian restriction category:

1. For a map $\tau_{X,Y} : A \times X \to Y$, a map $f : B \times X \to Y$ *admits a $\tau_{X,Y}$-index* when there is a total $h : B \to A$ such that

$$\begin{tikzcd}
A \times X \arrow[r, "\tau_{X,Y}"] & Y \\
B \times X \arrow[u, dotted, "h \times 1_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

   commutes.

2. A map $\tau_{X,Y} : A \times X \to Y$ is called a *universal application* if all $f : B \times X \to Y$ admit a $\tau_{X,Y}$-index.

3. If A is an object in $\mathbb{X}$ such that for every pair of objects X, Y in $\mathbb{X}$ there is a universal application, then A is called a *Turing object*.

4. A cartesian restriction category that contains a Turing object is called a *Turing category*.

Note there is no requirement in the definition for the map h to be unique. When h is unique for a specific $\tau_{X,Y}$, then that $\tau_{X,Y}$ is called *extensional*. In the case where the object B is the terminal object, then the map h is a point of A (with $f = (h \times 1)\tau_{X,Y}$) and h is referred to as a *code* of f.

Definition 2.3.2. Given $\mathbb{T}$ is a Turing category and A is an object of $\mathbb{T}$,

1. If $\Upsilon = \{\tau_{X,Y} : A \times X \to Y | X, Y \in ob(\mathbb{T})\}$, then Υ is called an *applicative family* for A.
2. An applicative family Υ is called *universal for* A when each $\tau_{X,Y}$ is a universal application. This is also referred to as a *Turing structure* on A.
3. A pair (A, Υ) where Υ is universal for A is called a *Turing structure* on $\mathbb{T}$.

Lemma 2.3.3. *If $\mathbb{T}$ is a Turing category with Turing object A, then every object B in $\mathbb{T}$ is a retract of A.*

Proof. As A is a Turing object, we have a diagram for $\tau_{1,B}$ and $\pi_0 : B \times 1 \to B$:

$$\begin{tikzcd}
A \times 1 \arrow[r, "\tau_{1,B}"] & B \\
B \times 1 \arrow[u, dotted, "h \times 1_X"] \arrow[ur, "\pi_0"'] &
\end{tikzcd}$$

where h is the $\tau_{1,B}$ index for π_0. Note we also have $u_r : B \to B \times 1$ is an isomorphism and therefore we have $1_X = u_r\pi_0 = (u_r(h \times 1))\tau_{1,X}$. Hence, we have $((u_r(h \times 1)), \tau_{1,X}) : B \triangleleft A$. $\square$

This allows us to move to the primary recognition criteria for Turing categories.

Theorem 2.3.4. *A cartesian restriction category $\mathbb{C}$ is a Turing category if and only if $\mathbb{C}$ has an object T which every other object of $\mathbb{C}$ is a retract and T has a universal self-application map $\bullet$, written as $T \times T \xrightarrow{\bullet} T$.*

Proof. The "only if" portion follows immediately from setting T to be the Turing object of $\mathbb{C}$ and $\bullet = \tau_{T,T}$.

For the "if" direction, we need to construct the family of universal applications $\tau_{X,Y} : T \times X \to Y$ for each pair of objects X, Y in $\mathbb{C}$.

Let us choose pairs of maps that witness the retractions of X, Y of T, that is:

$$(m_X, r_X) : X \triangleleft T \quad \text{and} \quad (m_Y, r_Y) : Y \triangleleft T.$$

Define $\tau_{X,Y} = (1_T \times m_X) \bullet r_Y$. Suppose we are given $f : B \times X \to Y$. Consider

$$\begin{tikzcd}
T \times X \arrow[r, "1_T \times m_X"] & T \times T \arrow[r, "\bullet"] & T \arrow[r, "r_Y"] & Y \\
B \times X \arrow[u, "h \times 1_X"] \arrow[r, "1_B \times m_X"'] & B \times T \arrow[u, dotted, "h \times 1_T"] \arrow[r, "1_B \times r_X"'] & B \times X \arrow[u, "f m_Y"] \arrow[ur, "f"']
\end{tikzcd}$$

where h is the index for the composite map $(1_B \times r_X) f m_Y$. The middle square commutes as $\bullet$ is a universal application for T, T. The right triangle commutes as $m_Y r_Y = 1$. The left square commutes as each composite is $h \times m_X$. Noting that the bottom path from $B \times X$ to Y is $(1_B \times m_X)(1_B \times r_X) f = f$ and the top path from $T \times X$ to Y is our definition of $\tau_{X,Y}$, this means f admits the $\tau_{X,Y}$-index h. $\square$

Note that different splittings (choices of (m, r) pairs) would lead to different $\tau_{X,Y}$ maps. In fact there is no requirement that this is the only way to create a universal applicative family for T.

Chapter 3

Quantum computation

3.1 Linear algebra

Quantum computation requires familiarity with the basics of linear algebra. This section will give definitions of the terms used throughout this thesis.

3.1.1 Basic definitions

The first definition needed is that of a *vector space*.

Definition 3.1.1 (Vector Space)**.** Given a field F, whose elements will be referred to as scalars, a *vector space* over F is a non-empty set V with two operations, *vector addition* and *scalar multiplication*. *Vector addition* is defined as $+ : V \times V \to V$ and denoted as $\mathbf{v}+\mathbf{w}$ where $\mathbf{v}, \mathbf{w} \in V$. The set V must be an abelian group under $+$. *Scalar multiplication* is defined as $: F \times V \to V$ and denoted as $c\mathbf{v}$ where $c \in F, \mathbf{v} \in V$. Scalar multiplication distributes over both vector addition and scalar addition and is associative. F's multiplicative identity is an identity for scalar multiplication.

The specific algebraic requirements are:

1. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V,\ (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;

2. $\forall \mathbf{u}, \mathbf{v} \in V,\ \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;

3. $\exists \mathbf{0} \in V$ such that $\forall \mathbf{v} \in V, \mathbf{0} + \mathbf{v} = \mathbf{v}$;

4. $\forall \mathbf{u} \in V, \exists \mathbf{v} \in V$ such that $\mathbf{u} + \mathbf{v} = \mathbf{0}$;

5. $\forall \mathbf{u}, \mathbf{v} \in V, c \in F,\ c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$;

6. $\forall \mathbf{u} \in V, c, d \in F,\ (c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$;

7. $\forall \mathbf{u} \in V, c, d \in F,\ (cd)\mathbf{u} = c(d\mathbf{u})$;

8. $\forall \mathbf{u} \in V,\ 1\mathbf{u} = \mathbf{u}$.

Examples of vector spaces over F are: $F^{n\times m}$ – the set of $n \times m$ matrices over F; and F^n – the $n-$fold Cartesian product of F. $F^{n\times 1}$, the set of $n \times 1$ matrices over F is also called the space of column vectors, while $F^{1\times n}$, the set of row vectors. Often, F^n is identified with $F^{n\times 1}$.

This thesis shall identify F^n with the column vector space over F.

Definition 3.1.2 (Linearly independent)**.** A subset of vectors $\{\mathbf{v}_i\}$ of the vector space V is said to be *linearly independent* when no finite linear combination of them, $\sum a_j \mathbf{v}_j$ equals $\mathbf{0}$ unless all the a_j are zero.

Definition 3.1.3 (Basis)**.** A *basis* of a vector space V is a linearly independent subset of V that generates V. That is, any vector $u \in V$ is a linear combination of the basis vectors.

Definition 3.1.4. Given V, W are vector spaces over F with $v \in V$ and $s \in F$, then if $f : V \to W$ is a group homomorphism such that $f(vs) = f(v)s$, then we say f is a *linear map*. Furthermore, a map $f : V \times W \to X$ is called *bilinear* when the map $f_v : W \to T$ and $f_w : V \to T$ are linear for each $v \in V$ and $w \in W$, where f_v is the map obtained from f by fixing $v \in V$ and f_w is obtained from f by fixing $w \in W$.

Definition 3.1.5. Given a set S, the *free vector space* of S over a field F is the abelian group of formal sums $\sum a_i s_i$ where the s_i are the elements of S and $a_i \in F$. Formal sums are independent of order. Addition is defined as $(\sum a_i s_i) + (\sum b_i s_i)$ is $(\sum (a_i + b_i) s_i)$.

Definition 3.1.6. Given vector spaces V, W over the base field F, consider the free vector space of $V \times W = F(V \times W)$. Next, consider the subspace T of $F(V \times W)$ generated by the following equations:

$$(v_1, w) + (v_2, w) = (v_1 + v_2, w)$$
$$(v, w_1) + (v, w_2) = (v, w_1 + w_2)$$
$$s(v, w) = (sv, w)$$
$$s(v, w) = (v, sw),$$

where $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $s \in F$. Then the tensor product of V and W, written $V \otimes W$ is $F(V \times W)/T$.

Elements of the tensor product $V \otimes W$ are written as $v \otimes w$ and are the T-equivalence class of $(v, w) \in V \times W$. If $\{v_i\}$ is a basis for V and $\{w_j\}$ is a basis for W, then the elements $\{v_i \otimes w_j\}$ form a basis for $V \otimes W$.

3.1.2 Matrices

As mentioned above, the set of $n \times m$ matrices over a field is a vector space. Additionally, matrices compose and the tensor product of matrices is defined.

Matrix composition is defined as usual. That is, for $A = [a_{ij}] \in F^{m\times n}$, $B = [b_{jk}] \in F^{n\times p}$:

$$A\,B = \left[\left(\sum_j a_{ij} b_{jk}\right)_{ik}\right] \in F^{m\times p}.$$

Definition 3.1.7 (Diagonal matrix)**.** A *diagonal matrix* is a matrix where the only non-zero entries are those where the column index equals the row index.

The diagonal matrix $n \times n$ with only 1's on the diagonal is the identity for matrix multiplication, and is designated by I_n.

Definition 3.1.8 (Transpose)**.** The *transpose* of an $n \times m$ matrix $A = [a_{ij}]$ is an $m \times n$ matrix A^t with the i, j entry being a_{ji}.

When the base field of a matrix is $\mathbb{C}$, the complex numbers, the *conjugate transpose* (also called the *adjoint*) of an $n \times m$ matrix $A = [a_{ij}]$ is defined as the $m \times n$ matrix A^* with the i, j entry being $\overline{a}_{ji}$, where $\overline{a}$ is the complex conjugate of $a \in \mathbb{C}$.

When working with column vectors over $\mathbb{C}$, note that $\mathbf{u} \in \mathbb{C}^n \implies \mathbf{u}^* \in \mathbb{C}^{1\times n}$ and that $\mathbf{u}^* \times \mathbf{u} \in \mathbb{C}^{1\times 1}$. This thesis will use the usual identification of $\mathbb{C}$ with $\mathbb{C}^{1\times 1}$. A column vector $\mathbf{u}$ is called a *unit vector* when $\mathbf{u}^* \times \mathbf{u} = 1$.

Definition 3.1.9 (Trace)**.** The *trace*, $Tr(A)$ of a square matrix $A = [a_{ij}]$ is $\sum a_{ii}$.

Tensor Product

The tensor product of two matrices is the usual Kronecker product:

$$U \otimes V = \begin{bmatrix} u_{11}V & u_{12}V & \cdots & u_{1m}V \\ u_{21}V & u_{22}V & \cdots & u_{2m}V \\ \vdots & \vdots & \ddots & \\ u_{n1}V & u_{n2}V & \cdots & u_{nm}V \end{bmatrix} = \begin{bmatrix} u_{11}v_{11} & \cdots & u_{12}v_{11} & \cdots & u_{1m}v_{1q} \\ u_{11}v_{21} & \cdots & u_{12}v_{21} & \cdots & u_{1m}v_{2q} \\ \vdots & \vdots & \vdots & \ddots & \\ u_{n1}v_{p1} & \cdots & u_{n2}v_{p1} & \cdots & u_{nm}v_{pq} \end{bmatrix}$$

Special matrices

When working with quantum values certain types of matrices over the complex numbers are of special interest. These are:

Unitary Matrix : Any $n \times n$ matrix A with $AA^* = I$ $(= A^*A)$.

Hermitian Matrix : Any $n \times n$ matrix A with $A = A^*$.

Positive Matrix : Any Hermitian matrix A in $\mathbb{C}^{n\times n}$ where $\mathbf{u}^* A\mathbf{u} \geq 0$ for all vectors $\mathbf{u} \in \mathbb{C}^n$.

Note that for any Hermitian matrix A and vector u, $\mathbf{u}^* A\mathbf{u}$ is real.

Completely Positive Matrix : Any positive matrix A in $\mathbb{C}^{n\times n}$ where $I_m \otimes A$ is positive.

The matrix

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

is an example of a matrix that is *unitary*, *Hermitian*, *positive* and *completely positive*.

Superoperators

A *Superoperator* S is a matrix over $\mathbb{C}$ with the following restrictions:

1. S is *completely positive*. This implies that S is positive as well.

2. For all positive matrices A, $Tr(S\,A) \leq Tr(A)$.

3.2 Quantum computation overview

Quantum computation proceeds via the application of reversible transformations — Unitary transformations.

The semantics of quantum computation can be defined as a †-compact closed category as introduced in [2, 4] and completely positive maps as discussed in [23].

Definition 3.2.1 (Dagger Category)**.** A *Dagger Category*[23] is a category $\mathbb{C}$ together with an operation † that is an involutive, identity on objects, contra-variant endofunctor on $\mathbb{C}$.

Recalling first that a *symmetric monoidal category* is a category $\mathbb{B}$ with a bi-functor $\otimes$, an object I and natural isomorphisms:

$$a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$
$$c_{A,B} : A \otimes B \to B \otimes A$$
$$ul_A : A \to I \otimes A$$

with standard coherence conditions, as in [16]. Note that we also have a map $ur_A : A \to A \otimes I$ given by $ur_A = ul_A c_{I,A}$ Furthermore, a *compact closed category* $\mathbb{C}$ is a symmetric monoidal category where each object A has a dual A^* together with the maps:

$$\eta_A : I \to A^* \otimes A$$
$$\epsilon_A : A \otimes A^* \to I$$

such that

$$\begin{tikzcd}
A \arrow[r, "ur_A"] \arrow[dddrr, equal] & A \otimes I \arrow[r, "A\otimes\eta_A"] & A \otimes (A^* \otimes A) \arrow[d, "a^{-1}"] \\
& & (A \otimes A^*) \otimes A \arrow[d, "\epsilon\otimes A"] \\
& & I \otimes A \arrow[d, "ul^{-1}"] \\
& & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A^* \arrow[r, "ul_{A^*}"] \arrow[dddrr, equal] & I \otimes A^* \arrow[r, "\eta_{A^*\otimes A^*}"] & (A^* \otimes A) \otimes A^* \arrow[d, "a"] \\
& & A^* \otimes (A \otimes A^*) \arrow[d, "A^*\otimes\epsilon"] \\
& & A^* \otimes I \arrow[d, "ur^{-1}"] \\
& & A
\end{tikzcd}$$

From the above, we can define a *Dagger symmetric monoidal category* and a *Dagger compact closed category*. The latter is referred to as a *strongly compact closed category* in [2], where they were initially introduced. In each case, the † functor is added in a way that retains coherence with the bi-functor $\otimes$ and with the dualizing operator. The coherence implies that the $i^\dagger = i^{-1}$ for the SMC isomorphisms, that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all maps f, g in the symmetric monoidal category and that

$$\begin{tikzcd}
I \arrow[r, "{\epsilon_A}^\dagger"] \arrow[dr, "\eta_A"'] & A \otimes A^* \arrow[d, "c"] \\
& A^* \otimes A
\end{tikzcd}$$

commutes for all objects A in the compact closed category.

Example 3.2.2 (Rel)**.** Rel is a dagger compact closed category with the dual of an object A is A, $\otimes$ is the cartesian product and for $R : A \to B$, we have $R^* = R^\dagger = \{(y, x)|(x, y) \in R\}$.

Example 3.2.3 (FdHilb)**.** The category of finite dimensional Hilbert spaces, FdHilb is a dagger compact closed category with the dual of an object H is the normal Hilbert space dual H^*, the space of continuous linear functions from H to the base field. $\otimes$ is the normal Hilbert space tensor and and for $f : A \to B$, we have $f^\dagger$ is the unique map such that $\langle fx|y\rangle = \langle y|f^\dagger x\rangle$ for all $x \in A$, $y \in B$.

Additionally, if one has a dagger compact closed category with biproducts where the biproducts and dagger interact such that ${p_i}^\dagger = q_i$, this is called a *biproduct dagger compact closed category.*

In [23], the author continues from this point: Starting with a biproduct dagger compact closed category $\mathbb{C}$, he creates a new category, CPM($\mathbb{C}$) which has the same objects as $\mathbb{C}$, but morphisms $f : A \to B$ in CPM($\mathbb{C}$) are given by maps $f : A^* \otimes A \to B^* \otimes B$ in $\mathbb{C}$ which are *completely positive*. Note that REL and FDHILB are biproduct dagger compact closed categories.

From this, the category CPM$(\mathbb{C})^{\oplus}$, the free biproduct completion of CPM($\mathbb{C}$) is formed, which is suitable for describing quantum computation semantics. For example, given FDHILB as our starting point, the tensor unit I is the field of complex numbers. The type of **qubit** (in FDHILB and by lifting, in CPM(FDHILB)$^{\oplus}$) is given as $I \oplus I$. At this stage, the necessity of the CPM construction to model physical reality can be seen in the following as in FDHILB, the morphisms initialization of a qubit: $init : I \oplus I \to \mathbf{qubit}$ and destructive measure: $meas : \mathbf{qubit} \to I \oplus I$ are inverses. However, in CPM(FDHILB)$^{\oplus}$, these same maps are given as

$$\mathbf{qubit}^* \otimes \mathbf{qubit} \xrightarrow{meas} I \oplus I \xrightarrow{init} \mathbf{qubit}^* \otimes \mathbf{qubit}$$

by the formulae:

$$meas \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a, d), \qquad init(a, d) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

Therefore, the maps are not inverses and reflect the physical reality.

3.2.1 Density matrix representation

An alternate representation of quantum states, both pure and mixed, is via *density matrices*. If the state of a system is represented by some column vector u, then the matrix uu^* is its density matrix. Note that if $u = \nu v$ for some complex scalar ν with norm 1, then $uu^* = (\nu v)(\nu v)^* = \nu\bar{\nu} v v^* = vv^*$. For the mixed state $\sum \nu_i\{v_i\}$, the density matrix is $\sum \nu_i v_i v_i^*$. Density matrices are positive hermitian matrices with trace ≤ 1. Note that the trace of the density matrix is the probability the system has reached this particular value in the computation.

The result of applying the unitary transform U to a state u represented by the density matrix A is UAU^*. The measurement operation on a density matrix is derived from the measurement effects on the **qubit**. For example, consider the density matrix for $q = \alpha|0\rangle + \beta|1\rangle$, $\begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{pmatrix}$. Measuring this **qubit** gives either $\begin{pmatrix} \alpha\bar{\alpha} & 0 \\ 0 & 0 \end{pmatrix}$ with probability $|\alpha|^2$ or $\begin{pmatrix} 0 & 0 \\ 0 & \beta\bar{\beta} \end{pmatrix}$ with probability $|\beta|^2$. If the results of the measurment are not used, this will result in the density matrix $\begin{pmatrix} \alpha\bar{\alpha} & 0 \\ 0 & \beta\bar{\beta} \end{pmatrix}$. This extends linearly so that if a **qubit** is measured in the system whose density matrix is $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$, the result will be the mixed density matrix $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)$.

It is possible to create a complete partial order on density matrices.

Definition 3.2.4 (Löwner partial order)**.** For square complex matrices A, B of the same size, define $A \leq B$ if $B - A$ is positive.

Lemma 3.2.5. *Designate D_n to be the density matrices of size $n \times n$, then the poset $(D_n, \leq)$ is a complete partial order.*

Proof. See [21], pp 13–14. □

3.3 Dagger categories

Dagger categories generalize the concepts of Hilbert spaces that are required to model quantum computation. These were introduced in [3] as *strongly compact closed categories*, an additional structure only on compact closed categories.

3.3.1 Definitions

Although dagger categories were introduced in the context of compact closed categories, the concept of a dagger is definable independently. This was first done in [23].

Definition 3.3.1 (Dagger, dagger category)**.** A dagger operator on a category D is an involutive, identity on objects contravariant functor $\dagger : \mathbb{D} \to \mathbb{D}$. A dagger category is a category that has a dagger operator.

Typically, the dagger is written as a superscript on the morphism. So, if $f : A \to B$ is a map in $\mathbb{D}$, then $f^\dagger : B \to A$ is a map in $\mathbb{D}$ and is called the *adjoint* of f. A map where $f^{-1} = f^\dagger$ is called *unitary* and a map $f : A \to A$ with $f = f^\dagger$ is called *self-adjoint* or *hermitian*.

Definition 3.3.2 (Dagger symmetric monoidal)**.** A *dagger symmetric monoidal category* is a symmetric monoidal category $\mathbb{D}$ with a dagger operator such that the dagger interacts coherently with the monoid to preserve the symmetric monoidal structure.

The coherence requirements in definition 3.3.2 are in addition to the standard coherence diagrams for a symmetric monoidal category. The additional coherence requirements are for all maps $f : A \to B$ and $g : C \to D$, it is required that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger : B \otimes D \to A \otimes C$ and that the monoid structure isomporphisms $a_{A,B,C} : (A \otimes B) \to C$, $u_A : A \to I \otimes A$ and $c_{A,B} : A \otimes B \to B \otimes A$ are all unitary.

Definition 3.3.3 (Dagger compact closed)**.** A *dagger compact closed category* $\mathbb{D}$ is a dagger symmetric monoidal category that is compact closed where the diagram

$$\begin{tikzcd} I \arrow[r, "\epsilon_A^\dagger"] \arrow[dr, "\eta_A"'] & A \otimes A^* \arrow[d, "\sigma_{A,A^*}"] \\ & A^* \otimes A \end{tikzcd}$$

commutes for all objects A in $\mathbb{D}$.

A category $\mathbb{D}$ is said to have *finite biproducts* when it has a zero object $\mathbf{0}$ (an object that is both initial and terminal) and when each pair of objects A, B have a biproduct $A \oplus B$. In such a category the unique map $A \to \mathbf{0} \to B$ is designated as $\mathbf{0}_{A,B}$.

Note that a category with finite biproducts is enriched in commutative monoids, where if $f, g : A \to B$, define $f + g : A \to B$ as $\langle id_A, id_A \rangle (f \oplus g) [id_B, id_B]$. The unit for the addition is $\mathbf{0}_{A,B}$. In the future, $\langle id, id \rangle$ will be designated by Δ and $[id, id]$ will be designated by ∇.

Lemma 3.3.4. *If $\mathbb{D}$ is a dagger category with biproducts, with injections in_1, in_2 and projections p_1, p_2, then the following are equivalent.*

1. ${p_i}^\dagger = in_i, i = 1, 2,$
2. $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$ *and* $\Delta^\dagger = \nabla$,
3. $\langle f, g \rangle^\dagger = [f^\dagger, g^\dagger]$,
4. *the below diagram commutes.*

$$\begin{tikzcd} A^\dagger \oplus B^\dagger \arrow[d, "id"'] \arrow[dr, "{[{p_1}^\dagger, {p_2}^\dagger]}"] & \\ A \oplus B \arrow[r, "id"'] & (A \oplus B)^\dagger \end{tikzcd}$$

Proof. 1 $\Longrightarrow$ 2 To show $\Delta^\dagger = \nabla$, draw the product cone for Δ,

$$\begin{tikzcd} & A \arrow[dl, "id"'] \arrow[d, "\Delta"] \arrow[dr, "id"] & \\ A & A \oplus A \arrow[l, "p_1"] \arrow[r, "p_2"'] & A \end{tikzcd}$$

and apply the dagger functor to it. As ${p_i}^\dagger = in_i$, and $\dagger$ is identity on objects, this is now a coproduct diagram and therefore $\Delta^\dagger = \nabla$.

For $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$, start with the diagram defining $f \oplus g$ as a product of the arrows:

$$\begin{tikzcd}
A & A \oplus B \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "f \otimes g"] & A \arrow[d, "g"] \\
C & C \oplus D \arrow[l, "p_1"] \arrow[r, "p_2"'] & D
\arrow[from=1-1, to=2-1, "f"']
\end{tikzcd}$$

and once again apply the dagger functor. This is now the diagram defining the coproduct of maps and therefore $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$.

2 $\Longrightarrow$ **3** The calculation showing this is

$$\begin{aligned}
[f^\dagger, g^\dagger] &= \nabla; (f^\dagger \oplus g^\dagger) \\
&= \Delta^\dagger; (f^\dagger \oplus g^\dagger) \\
&= \Delta^\dagger; (f \oplus g)^\dagger \\
&= ((f \oplus g); \Delta)^\dagger \\
&= \langle f, g \rangle^\dagger
\end{aligned}$$

3 $\Longrightarrow$ **4** Under the assumption,

$$\begin{aligned}
[{p_1}^\dagger, {p_2}^\dagger] &= \langle p_1, p_2 \rangle^\dagger \\
&= id^\dagger \\
&= id
\end{aligned}$$

and therefore the diagram commutes.

4 $\Longrightarrow$ **1** Using the injections and under the assumption, the following diagram commutes:

$$\begin{tikzcd}
A^\dagger \oplus B^\dagger \arrow[r, "{[in_1, in_2]}"] \arrow[d, "id"'] \arrow[dr, "{[{p_1}^\dagger, {p_2}^\dagger]}"] & A^\dagger \oplus B^\dagger \arrow[d, "id"] \\
A \oplus B \arrow[r, "id"'] & (A \oplus B)^\dagger
\end{tikzcd}$$

and therefore, $p_1{}^\dagger = in_1$ and $p_2{}^\dagger = in_2$.

$\square$

Definition 3.3.5. A *biproduct dagger compact closed category* is a dagger compact closed category with biproducts where the conditions of lemma 3.3.4 hold.

3.3.2 Examples of dagger categories

FdHilb: The category of finite dimensional Hilbert spaces is the motivating example for the creation of the dagger and is, in fact, a biproduct dagger compact closed category. The biproduct is the direct sum of Hilbert spaces and the tensor for compact closure is the standard tensor of Hilbert spaces. The dual H^* of a space H is the space of all continuous linear functions from H to the base field. The dagger is defined via the adjoint as being the unique map $f^\dagger : B \to A$ such that $\langle fa|b\rangle = \langle a|f^\dagger b\rangle$ for all $a \in A, b \in B$.

Rel: The category Rel of sets and relations has the tensor $S \otimes T = S \times T$, the cartesian product and the biproduct $S \oplus T = S + T$, the disjoint union. This is compact closed under $A^* = A$ and the dagger is the relational converse, that is if the relation $R = \{(s,t)|s \in S, t \in T\} : S \to T$, then $R^\dagger = \{(t,s)|(s,t) \in R\}(= R^*)$.

Inverse categories: An inverse category $\mathbb{X}$ is also a dagger category when the dagger is defined as the partial inverse. The unitary maps are the total maps which are isomorphisms. If the inverse category $\mathbb{X}$ is also a symmetric monoidal category where the monoid $\otimes$ is actually a restriction bi-functor, then $\mathbb{X}$ is a dagger symmetric monoidal category. This follows from

$$(f \otimes g)(f \otimes g)^{(-1)} = \overline{f \otimes g} = \overline{f} \otimes \overline{g} = ff^{(-1)} \otimes gg^{(-1)} = (f \otimes g)(f^{(-1)} \otimes g^{(-1)})$$

but since the partial inverse of $f \otimes g$ is unique, $f \otimes g^{(-1)} = f^{(-1)} \otimes g^{(-1)}$. Finally, since all the structure isomorphisms are total maps, they are unitary and $\mathbb{X}$ is a dagger symmetric monoidal restriction category.

3.4 Semantics of quantum computation

3.4.1 Semantics of QPL

QPL Basics

In [21] Dr. Selinger provides a denotational semantics for a quantum programming, QPL, with the slogan of "quantum data with classical control". This slogan refers to the semantic representation described in the paper, where explicit classical branching based on a classical value is described by a **bit** value with specific probabilities of being 0 or 1.

QPL is defined via a collection of functional flowchart components, where "functional" specifically means that each flowchart is a function from its inputs to its outputs. These components describe the basic operations on **bit**s and **qubits**. Edges between the components represent the data (**bit**s and **qubits**). These edges are labelled with a typing context and annotated with a tuple of density matrices, describing the probability distribution of the classical data and the state of the quantum data. In the case of purely classical data, this annotation will be a tuple of probabilities, whereas in the case of purely quantum data, it will be a single density matrix.

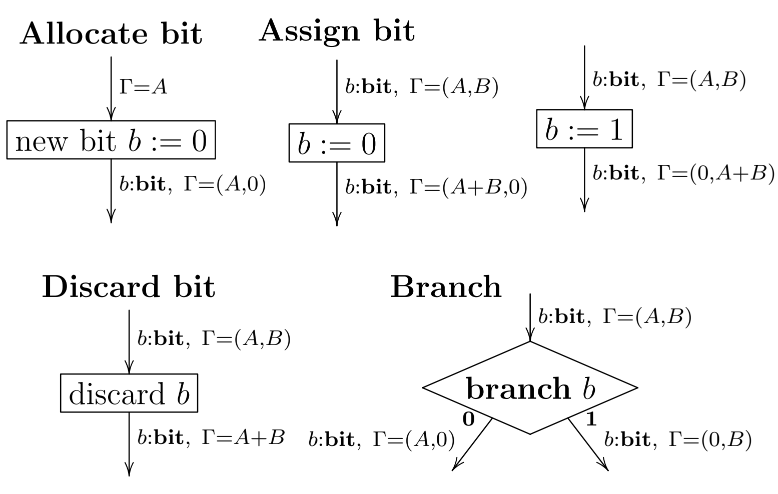


Figure 3.1: Classical flowcharts

In figure 3.1, the annotation Γ consists of a tuple of probabilities, with n **bit**s requiring

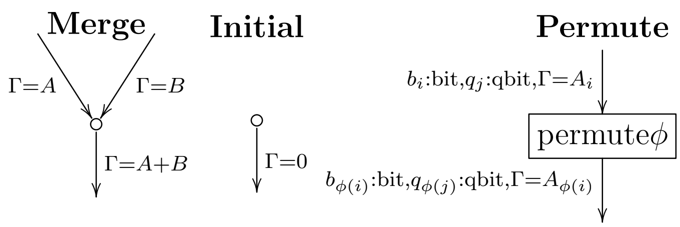


Figure 3.2: General flowcharts

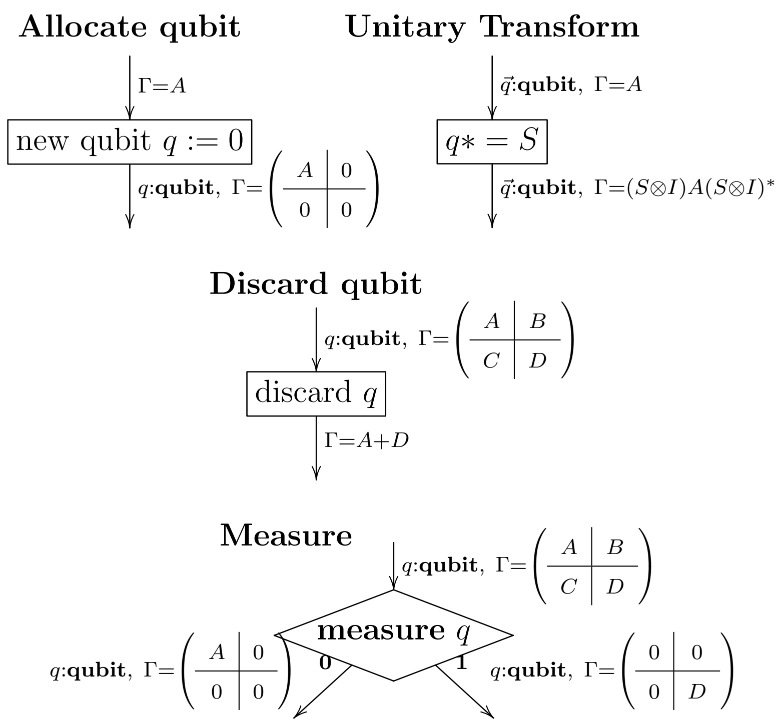


Figure 3.3: Quantum flowcharts

2^n probabilities for their description. In figure 3.3, Γ will consist of a density matrix of size $2^m \times 2^m$ for m **qubit**s. Note also that in figure 3.3, the notation $\vec{q}$ indicates an ordered set of **qubit**s.

In QPL, the classical operations consist of: *Allocate bit, Assignment, Discard bit* and *Branch*. The quantum operations are:*Allocate qubit, Unitary Transform, Discard qubit* and *Measure*. The operations applicable to both types of data are *Merge, Initial* and *Permute*. These are found in Figure 3.2.

When components are combined, the type annotation Γ consists of a tuple of density

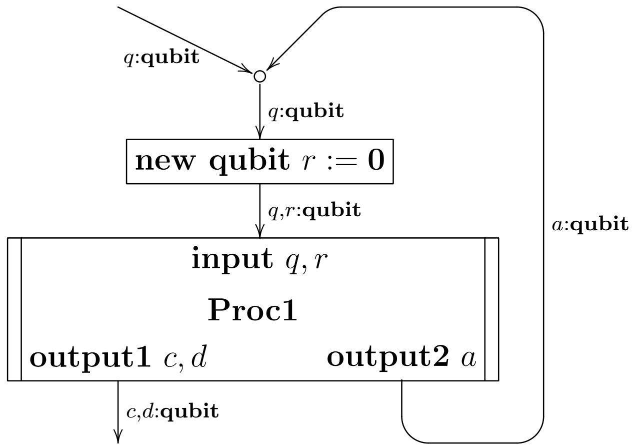


Figure 3.4: Example of a subroutine and loop

matrices. Flowchart components must be combined so that they are connected via edges with identical typing judgements. Flowcharts may have arbitrary numbers of input and output edges. By convention, component flow is from the top down and programs are read is the same manner.

The semantics of a component is the function that calculates the matrix tuple(s) of the output edges when given the matrix tuple of the input edges. Each of these functions is linear and preserves adjoints. They also preserve positivity and the sum of the traces of the output edges equals the sum of the traces of the input edges, which can be viewed as the probability of leaving a fragment is the same as the probability of entering a fragment.

Looping, subroutines and recursion

In the flow chart representation of QPL, looping occurs when one edge is connected to a component above the component originating the edge. Subroutines are represented by boxes with double left and right lines. A subroutine may have multiple input and output edges and is considered shorthand for the flowchart making up the subroutine. For example, see figure 3.4, where the subroutine $Proc1$ accepts two **qubit**s q, r as input and produces either two **qubit**s c, d or a single **qubit** q. In the case when the output is the single **qubit** q, the output is looped back to be merged with the original input.

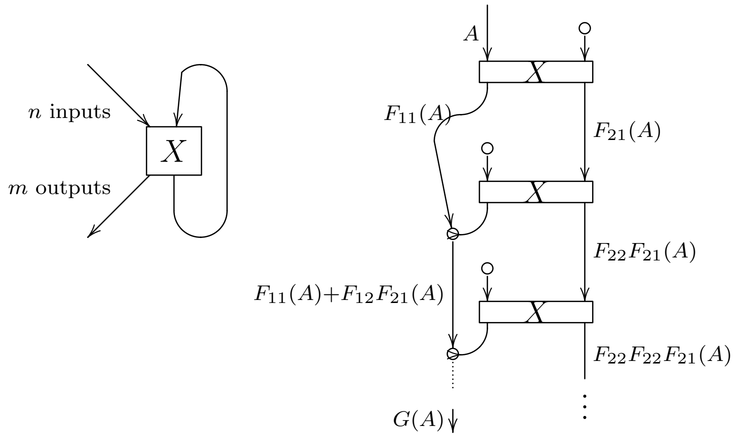


Figure 3.5: Unwinding a loop

Semantics of looping is based on "infinite unwinding". It is interesting to note this is similar to the method used in [14] where a `while` program construction is unwound to

$$\texttt{if } \neg B \texttt{ then } I \texttt{ else } S; \texttt{ while } B \texttt{ do } S \texttt{ od fi};$$

and the semantics of `while` is given as a fixpoint W of the equation $W = e_{\neg B} + (e_B; S; W)$. Referring to figure 3.5, the input to X is $n+k$ density matrices, the output is $m+k$ density matrices, where the k matrices partake in the loop. This can be written as $F(A, C) = (B, D)$ with $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_m)$, where F is the linear function giving the semantics of X. This allows creation of four component functions, where $F(A, 0) = (F_{11}(A), F_{21}(A))$ and $F(0, C) = (F_{12}(C), F_{22}(C))$.

Following the right hand side of figure 3.5, the state of the edges at the end are given by

$$G(A) = F_{11}(A) + \sum_{i=0}^{\infty} F_{12}(F_{22}^i(F_{21}(A))) \tag{3.1}$$

I will show later that this is a convergent sum.

The semantics of subroutines without recursion is the same as "in-lining" the subroutine at the place of its call. The first requirement for this is that the program handle renaming of variables as the formal parameters of the subroutine may have different names than the calling parameters. For **bit**s, renaming b to c may be accomplished by the fragment

new bit $c := 0$**;**

branch b **0¿{ } 1¿{**$c := 1$**} ;**

discard b.

For **qubits**, renaming q to r is done by the fragment

new qubit $p := 0$**;**

q,p *=CNOT;

p,q *=CNOT;

discard q.

The second requirement is that the program needs to be able to extend the semantic context. That is, suppose that a subroutine X is defined with input typing Γ, with semantic function F. This means that starting with $\Gamma = A$, applying the subroutine X gives us the typing and context $\Gamma' = F(A)$. Inlining a subroutine requires that the addition of an arbitrary number of **bit**s and **qubits** to the context and be able to derive the semantics. But, since each of the components of flow charts and looping are linear functions, this is a straightforward induction on the structure of the subroutine. The proof for one of the cases is below.

Lemma 3.4.1 (Context extension)**.** *Given a subroutine X in context $\Gamma = A$ with semantics F (i.e., applying X to $\Gamma = A$ gives $\Gamma' = F(A)$),*

- *The result of X in context b :bit, $\Gamma = (A, B)$ is b :bit, $\Gamma' = (F(A), F(B))$.*
- *The result of X in context q :qubit, $\Gamma = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is*

  q *:qubit,* $\Gamma' = \left(\begin{array}{c|c} F(A) & F(B) \\ \hline F(C) & F(D) \end{array}\right)$.

Proof. **Case** $X =$**"allocate bit"** The semantics of allocate bit is $F(A) = (A, 0)$, where the number of 0 density matrices is the same as the number of density matrices in A. When the additional context is a **bit**, the starting context is x :bit, $\Gamma = (A, B)$, where again the number and dimensions of density matrices in A and B agree. After applying X, b :bit,x :bit$\Gamma' = (A, B, 0, 0)$. Next, permute x and b, to retain x in the correct order and get x :bit,b :bit$\Gamma' = (A, 0, B, 0) = ((A, 0), (B, 0) = (F(A), F(B))$.

When the additional context is a **qubit**, the starting context is x :qubit, $\Gamma = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$. After allocation of a bit,

$$\begin{aligned} b : \text{bit}, x : \text{qubit}, \Gamma &= (\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right)) \\ &= \left(\begin{array}{c|c} (A,0) & (B,0) \\ \hline (C,0) & (D,0) \end{array}\right) \\ &= \left(\begin{array}{c|c} F(A) & F(B) \\ \hline F(C) & F(D) \end{array}\right). \end{aligned}$$

□

As the semantics of each of the components of the flowchart language are linear functions, the technique used in the example case in Lemma 3.4.1 is applicable for each of these flowchart components. Furthermore, as the semantics are compositional, this extends to looping.

For recursive subroutines, a variant of the infinite unwinding is used. A recursive subroutine is one that calls itself in some way. If we have a subroutine X, let $X(Y)$ be the flowchart defined as X with the recursive call to itself replaced with a call to Y. As the semantics are compositional, there is some function Θ such that give the semantics of $X(Y)$ from the semantics of Y. Let Y_0 be a non-terminating program and define Y_i by the equation $Y_{i+1} = X(Y_i)$. Denote the semantics of Y_i by F_i. Then $F_0 = 0$ and $F_{i+1} = \Theta(F_i)$. From this, define

$$X = \lim_{i\to\infty} F_i. \qquad (3.2)$$

The existence of this limit will be discussed below in section 3.4.1.

An important point to note here is that the semantics of an arbitrary G may actually reduce the trace, that is, there may be a non-zero chance the program will not terminate.

Categorical semantics of QPL

While the exposition above referred to the functions described in the flowchart components as the semantics of the program, this section will give a formal definition of the categorical semantics.

Definition 3.4.2 (Signature)**.** A *signature* is a list of positive non-zero integers, $\sigma = n_1, \ldots, n_s$ which is associated with the complex vector space $V_\sigma = \mathbb{C}^{n_1 \times n_1} \times \cdots \times \mathbb{C}^{n_s \times n_s}$

Designate the elements of V_σ by tuples of matrices, $A = (A_1, \ldots, A_s)$. The trace of A will be the sum of the traces of the tuple matrices. A will be said to have a specific property when all matrices in the tuple have that property, e.g., positive, hermitian.

From the above, now define the category V with objects being signatures σ and maps from σ to τ being any complex linear function from V_σ to V_τ. Define $\oplus$ by concatenation of signatures. Then, $\oplus$ is both a product and coproduct in V. The co-pair map $[F, G] : \sigma \oplus \sigma' \to \tau$ is defined as $[F, G](A, B) = F(A) + F(B)$, while the pairing map $\langle F, G\rangle : \sigma \to \tau \oplus \tau'$ is defined as $\langle F, G\rangle(A) = (FA, GA)$. Additionally, define the tensor $\otimes$ on $\sigma = n_1, \ldots, n_s$ and $\tau = m_1, \ldots, m_t$ as

$$\sigma \otimes \tau = n_1 m_1, n_1 m_2, \ldots, n_s, m_t.$$

This tensor, together with the unit $I = 1$ makes V a symmetric monoidal category. Note that it is also distributive with $\tau \otimes (\sigma \oplus \sigma') = (\tau \otimes \sigma) \oplus (\tau \otimes \sigma')$.

As V is equivalent to the category of finite dimensional vector spaces, we will need to restrict the morphisms to those that can occur as programs in QPL. V has too many morphisms, for instance the signature $1, 1$ (which will be designated as **bit**) is isomorphic to the signature 2 (which will be designated as **qubit**).

Definition 3.4.3 (Superoperator)**.** Given $F : V_\sigma \to V_\tau$, define:

- F as *positive* if $F(A)$ is positive for all positive A;
- F as *completely positive* if $id_\rho \otimes F : V_{\rho\otimes\sigma} \to V_{\rho\otimes\tau}$ is positive for all ρ;
- F as a *superoperator* if it is completely positive and tr $F(A) \leq$ tr A for all positive A.

The definition of a superoperator is trace *non-increasing* rather than trace *preserving* due to the possibility of non-termination in programs.

Considering superoperators in the category V, there a number of properties that hold. It is immediate to see that an identity map is a superoperator and that compositions of superoperators are again superoperators. The canonical injections $i_1 : \sigma \to \sigma \oplus \tau$ and $i_2 : \tau \to \sigma \oplus \tau$ are superoperators. The remain properties of interest are detailed in the following lemma.

Lemma 3.4.4. *In the category V, the following hold:*

1. *If $F : \sigma \to \tau$ and $G : \sigma' \to \tau$ are superoperators, so is $[F, G] : \sigma \oplus \sigma' \to \tau$.*
2. *If $F : \sigma \to \sigma'$ and $G : \tau \to \tau'$ are superoperators, then so are $F \oplus G$ and $F \otimes G$.*
3. *if $id_\nu \otimes F$ is positive for all one element signatures ν, then F is completely positive.*
4. *Given U, a unitary $n \times n$ matrix, then $F : n \to n$ defined as $F(A) = UAU^*$ is a superoperator.*
5. *If T_1, T_2 are $n \times n$ matrices such that $T_1^* T_1 + T_2^* T_2 = I$, then $F : n \to n, n$ defined as $F(A) = (T_1 A T_1^*, T_2 A T_2^*)$ is a superoperator.*

Proof. For statement 1, as $tr\ (F(A) + F(B)) = (tr\ F(A)) + (tr\ F(B)) \leq tr\ A + tr\ B = tr\ (A, B)$, note that $[F, G]$ satisfies the trace condition. Secondly, because of distributivity $id_\rho \otimes [F, G] = [id_\rho \otimes F, id_\rho \otimes G]$ and the complete positivity follows. The first assertion of statement 2 follows in a similar manner. As $F \otimes G = (F \otimes id_\tau)(id_{\sigma'} \otimes G)$, note that each element of the composition is a superoperator, hence so is $F \otimes G$. In statement 3, note that any signature ν of length n is equal to a coproduct of n single element signatures, $\nu_1 \oplus \cdots \oplus \nu_n$. Then using distributivity, $id_\nu \otimes F = (id_{\nu_1} \otimes F) \oplus \cdots \oplus (id_{\nu_1} \otimes F)$ which by assumption and statement 2 is positive. Hence, F is completely positive. For statement 4, as U is unitary, it is immediate that F is positive and preserves the trace. Note also that $(id_n \otimes F)(A) = (I \otimes U)A(I \otimes U)^*$ where I is the $n \times n$ identity matrix. However, $I \otimes U$ is also a unitary matrix, therefore $(id_n \otimes F)$ is positive for all n and by the previous point, F is completely positive and therefore a superoperator. For the last statement, by construction F preserves both positivity and trace and by a similar argument to the previous point, it is a superoperator. □

At this point there is now sufficient machinery to define the category Q which will be used for the categorical semantics for QPL. Define Q as the subcategory of V having the same objects, but only superoperators as morphisms. By lemma 3.4.4, this is a valid subcategory, which inherits $\oplus$ as a coproduct. It is not a product as the diagonal morphism increases the trace and is therefore not a superoperator.

Q is also a CPO enriched category. First, note that superoperators send density matrices to density matrices (positive hermitian matrices with trace ≤ 1). Designating D_σ to be the subset of density matrix tuples contained in V_σ, then for any superoperator F, it can be restricted to the density matrices. This restricted function preserves the Löwner order from definition 3.2.4 and it preserves the least upper bounds of sequences. From this, given signatures σ and τ, define a partial order on $Q(\sigma, \tau)$ by $F \leq G$ when $\forall \nu, A \in D_{\nu\otimes\sigma}$: $(id_\nu \otimes F)(A) \leq (id_\nu \otimes G)(A)$.

Lemma 3.4.5. *The poset $Q(\sigma, \tau)$ is a complete partial order. Composition, co-pairing and tensor are Scott-continuous and therefore Q is CPO-enriched.*

As Q is a CPO enriched category, it is possible to define a monoidal trace over the coproduct monoid. Recall that if $F : \sigma \oplus \tau \to \sigma' \oplus \tau$, then $tr\ F$ is a map, $\sigma \to \sigma'$. Given such an F, construct the trace as follows:

- Define T_0 as the constant zero function.
- Define $T_{i+1} = F; [id_{\sigma'}, i_2 H_i] : \sigma \oplus \tau \to \sigma'$.

Then, $T_0 \leq T_1$ as 0 is the least element in the partial order. $T_i \leq T_{i+1}$ for all i as all the categorical operations are monotonic due to the CPO enrichment. Therefore, now define $T = \vee_i T_i : \sigma \oplus \tau \to \sigma'$. Finally, define $tr\ F = i_1; T : \sigma \to \sigma'$.

This trace construction may be compared to the loop semantics construction in section 3.4.1. For F as above, we may decompose it into components $F_{11} : \sigma \to \sigma'$, $F_{21} : \sigma \to \tau$, $F_{12} : \tau \to \sigma'$ and $F_{22} : \tau \to \tau$. This gives us

$$\begin{aligned} T_0(A, 0) &= 0, \\ T_1(A, 0) &= F_{11}(A), \\ T_2(A, 0) &= F_{11}(A) + F_{12}F_{21}(A), \\ &\vdots \end{aligned}$$

which brings us to

$$(Tr\ F)(A) = T(A, 0) = F_{11}(A) + \sum_{i=0}^{\infty} F_{12}(F_{22}^{i}(F_{21}(A))).$$

This is the same construction as equation ((3.1)) and will be used for the interpretation of loops. In particular, this justifies the convergence of the infinite sum in that equation.

At this point, we now have the information required to give an interpretation of the quantum flow charts of QPL in the category Q. There are two types, $[\![\mathbf{bit}]\!] = 1, 1$ and $[\![\mathbf{qubit}]\!] = 2$. The interpretation of basic operations is given in Table 3.1.

Additionally, if a type context Γ is $x_i : T_i$, then $[\![\Gamma]\!] = \otimes_i [\![A_i]\!]$. If $\bar{\Theta}$ is a list of typing context, Θ_i, then $[\![\bar{\Theta}]\!] = \oplus_i [\![\Theta_i]\!]$. For the various types of composite flowcharts, the interpretation is as follows:

Table 3.1: Interpretation of QPL operations

$$
\begin{array}{rcl}
[\![\text{new bit } b := 0]\!] & = newbit : I \to \mathbf{bit} : & a \mapsto (a, 0) \\
[\![\text{discard } b]\!] & = discardbit : \mathbf{bit} \to I : & (a, b) \mapsto a + b \\
[\![b := 0]\!] & = set_0 : \mathbf{bit} \to \mathbf{bit} : & (a, b) \mapsto (a + b, 0) \\
[\![b := 1]\!] & = set_1 : \mathbf{bit} \to \mathbf{bit} : & (a, b) \mapsto (0, a + b) \\
[\![\text{branch } b]\!] & = branch : \mathbf{bit} \to \mathbf{bit} \oplus \mathbf{bit} : & (a, b) \mapsto (a, 0, 0, b) \\
[\![\text{new qbit } q := 0]\!] & = newqbit : I \to \mathbf{qubit} : & a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \\
[\![\text{discard } q]\!] & = discardqbit : \mathbf{qubit} \to I : & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d \\
[\![\vec{q} * = U]\!] & = unitary_U : \mathbf{qubit}^n \to \mathbf{qubit}^n : & A \mapsto UAU^* \\
[\![\text{measure } q]\!] & = measure : \mathbf{qubit} \to \mathbf{qubit} \oplus \mathbf{qubit} : & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}) \\
[\![\text{merge}]\!] & = merge : I \oplus I \to I : & (a, b) \mapsto (a + b) \\
[\![\text{initial}]\!] & = initial : 0 \to I : & () \mapsto 0 \\
[\![\text{permute } \Phi]\!] & = permute_\Phi : \oplus_i A_i \to \oplus_i A_{\Phi(i)} &
\end{array}
$$

- If the context Γ is added to flowchart A, producing B, then $[\![B]\!] = [\![A]\!] \otimes [\![\Gamma]\!]$.
- If the outputs of flowchart A are connected to the inputs of B, giving flowchart C, then $[\![C]\!] = [\![A]\!]; [\![B]\!]$.
- If flowchart C is made up of parallel flowcharts A and B, then $[\![C]\!] = [\![A]\!] \oplus [\![B]\!]$.
- if flowchart C is a loop on flowchart A, then $[\![C]\!] = tr\,([\![X]\!])$.

For procedures, it is necessary to consider abstract variable flowcharts with specified types, $R_i : \overline{\Theta_i} \to \overline{\Theta_i'}$. With these variable flowcharts allowed, these may be interpreted in a specified environment κ which maps the R_i to specific morphisms of Q with the appropriate type. Then, $[\![R_i]\!]_\kappa = \kappa R_i$ and if A is a flow chart using R_i, its interpretation relative to κ may be built up inductively via the operations above, giving a function Ω_A which will map the environments to a specific map in Q.

For recursion, consider the recursive subroutine defined as $P = T(P)$ for a flowchart T. Then $\Omega_T : Q(\sigma, \tau) \to Q(\sigma, \tau)$ will be a Scott-continuous function. In this case, $[\![Y]\!]$ will be the least fixed point of Ω_T. There is an increasing sequence for all $i \geq 0$, $S_i \leq S_{i+1}$ given by $S_0 = 0$ and $S_{i+1} = \Omega_T(S_i)$. This gives the interpretation of P as

$$[\![P]\!] = \vee_i S_i = \lim_i S_i.$$

This corresponds to equation ((3.2)) above, which shows this is the correct interpretation for recursive procedures and since Q is a CPO enriched category with a least point, therefore this limit exists and therefore the the limit in equation ((3.2)) will exist.

Conclusions for QPL

Data types As can be seen by the proceeding pages, creating the categorical machinery for a semantic interpretation of QPL is quite detailed. The paper [21] goes on to prove soundness, completeness and provides some alternative syntaxes for QPL. The subject of structured types is discussed briefly. Tuple types (Γ, Λ) may be constructed as $[\![\Gamma]\!] \otimes [\![\Lambda]\!]$. This immediately provides types such as fixed length classical or quantum integers, characters, and so forth. Sum types can similarly be added as $\Gamma \oplus \Lambda$, noting that the "choice" between the two types remains classical. The one primary weakness in the type system is not allowing structured recursive types such as **List**. This weakness is addressed in a follow on paper, [12]. In this paper, the major change is that rather than restricting to a tuple of integers for the objects of V and Q, they consider arbitrary families of integers as the objects, defining a new category Q^∞. Definitions such as positive, Hermitian, the Löwner order and trace follow in a straightforward manner, as does the definition $V_\sigma = \prod_{i \in |\sigma|} \mathbb{C}^{\sigma_i \times \sigma_i}$, noting this is now an infinite product. Note that in the infinite dimensional case there is no canonical basis for V_σ and therefore no canonical isomorphism between $V_{\sigma\otimes\tau}$ and $V_\sigma \otimes V_\tau$. To rectify this, the authors refine the allowed morphisms in the category Q^∞. First, the define the category $\overline{Q^\infty}$ as having infinite signatures as objects, but maps $f : \sigma \to \tau$ are maps $f : D_\sigma \to D_\tau$ (D_σ are the density matrix tuples of V_σ). These f are called *positive operators*. They are required to extend to linear maps $\overline{f} : V_\sigma \to V_\tau$ and be continuous for the Löwner order. These maps are definable as a matrix of maps over finite dimensional spaces $f_{ij} : \mathbb{C}^{\sigma_i \times \sigma_i} \to \mathbb{C}^{\tau_i \times \tau_i}$, calling this the *operator matrix*.

One can now define the tensor of two positive operators f, g by tensoring their respective operator matrices. Then, following the finite case, the positive operator $f : \sigma \to \tau$ with operator matrix F is a superoperator, if when ID_γ is the operator matrix for the identity on the signature γ, $ID_\gamma \otimes F$ is a positive operator. The infinite case superoperators follow the desired properties as in the finite case and the category Q^∞ is defined as the category with objects being infinite signatures and morphisms these superoperators.

From this, the authors show that any endofunctor definable via an "arithmetic" equation involving the coproduct and tensor will give rise to a data type in the category. In particular, one can define **qubit** lists as

$$QList = 1 \oplus (\mathbf{qubit} \otimes QList)$$

and trees of **qubits** as

$$QTree = \mathbf{qubit} \oplus (QTree \otimes QTree).$$

Quantum communication   QPL makes no attempts to handle communication or transmission of quantum data. This will not be addressed in this thesis.

Higher order functions   QPL is defined as a functional language. One of the expectations of modern functional languages is that programs themselves are first class objects, that is, they may be operated on by the program. Typical uses are partial evaluation and passing a subroutine of a specified type for use by another subroutine. In quantum computation, the primary issue with this in how does one guarantee the no-cloning, no-erasing rules with respect to quantum data. Work on a quantum lambda calculus, [25, 24], has attempted to address this, albeit primarily with operational rather than denotational semantics. In, [22], the author explores the use of cones rather than vector spaces to create a denotational semantics, but finds that the candidates fail to provide the correct answer over the base types. We will not be considering the higher-order issues further in this paper.

3.4.2 Semantics of pure quantum computations

In [3], the authors approach the creation of a categorical semantics for quantum computation independently of a specific language. Rather, they use finitary quantum mechanics as their reference point.

Finitary quantum mechanics consists of the following:

1. The system's state space is represented by a finite dimensional Hilbert space H.

2. The basic type of the system is that of **qubit**— 2-dimensional Hilbert space — with the computational basis $\{|0\rangle, |1\rangle\}$.

3. Compound systems are tensor products of the components. This is what enables *entanglement* as the general form of the system $H \otimes J$ where H and J are Hilbert spaces is
$$\sum_{i=1}^{n} \alpha_i (u_i \otimes v_i)$$
where u_i is a basis element of H and v_i is a basis element of J.

4. The basic transforms are *unitary transformations*.

5. The measurements performable are *self-adjoint* (hermitian) operators - with two sub-steps:

    (a) The actual act of measurement. (Preparation).

    (b) The communication of the results of the measurement. (Observation).

The above definition does allow for the possibility of mixed states, as described in section 3.2.1, but for the remainder of this section, it is assumed both steps of the measurement are carried out, resulting in pure states only.

[3] gives the interpretation of finitary quantum mechanics in the context of a biproduct dagger compact closed category, $\mathbb{D}$.

1. An $n-$dimensional state space S is an object of $\mathbb{D}$, together with a unitary isomorphism

$$base_A : \oplus^n I \to A.$$

2. A **qubit** is a 2 dimensional state space Q with the computational basis $base_Q : I \oplus I \to Q$.

3. Compound systems A, B are described by $A \otimes B$ and $base_{A\otimes B} = \phi(base_A \otimes base_B)$ where $\phi : \oplus^{nm} I \cong (\oplus^n I) \otimes (\oplus^m I)$ is the isomorphism obtained by repeated application of distributivity isomporphisms.

4. The basic transformations are unitary transformations, i.e., f, where $f^\dagger = f^{-1}$.

5a. A preparation is a morphism $P : I \to A$ which has a corresponding unitary morphism $f_P : \oplus^n I \to \oplus^n I$ and

$$\begin{tikzcd} I \arrow[r, "P"] \arrow[d, "i_1"'] & A \\ \oplus^n I \arrow[r, "f_P"'] & \oplus^n I \arrow[u, "base_A"'] \end{tikzcd}$$

commutes.

5b. An observation is an isomorphism $O = \oplus^n O_i$ with components $O_i : A \to I$ which has an unitary automorphism $f_O : \oplus^n I \to \oplus^n I$ such that

$$\begin{tikzcd} A \arrow[r, "O_i"] & I \\ \oplus^n I \arrow[u, "base_A"] \arrow[r, "f_O"'] & \oplus^n I \arrow[u, "p_i"'] \end{tikzcd}$$

commutes for all $i = 1, \ldots, n$. The observational branches are the individual $O_i : A \to I$.

Additionally, the biproduct $\oplus$ represents distinct branches resulting from measurement. Accordingly, any operation on a biproduct must be an explicit biproduct, that is $f : A \oplus B \to C \oplus D$ will be $f_1 \oplus f_2$ with $f_1 : A \to C$ and $f_2 : B \to D$.

The authors go on to show how this interpretation is sufficient to model quantum teleportation, logic gate teleportation and entanglement swapping.

3.4.3 Complete positivity

Given a †-compact closed category, it is possible to construct its category of completely positive maps.

Definition 3.4.6 (Positive map)**.** A map $f : A \to A$ in a dagger category is called *positive* if there is an object B and a map $g : A \to B$ with $f = gg^\dagger$

Definition 3.4.7 (Trace)**.** For $f : A \to A$ in a compact closed category, its *trace* is defined as $tr\, f : I \to I = \eta_A; c_{A^*,A}; (f \otimes A^*); \epsilon$.

The following lemma gives some properties of positive maps:

Lemma 3.4.8. *In any biproduct dagger compact closed category, the following hold:*

1. f *positive* $\implies$ $hfh^\dagger$ *is positive for all maps* h.
2. id_A *is positive.*
3. *If* $f : A \to A$ *and* $g : B \to B$ *are positive, so are* $f \otimes g$ *and* $f \oplus g$.
4. $0_{A,A}$ *is positive. If* $f, g : A \to A$ *is positive, so is* $f + g$.
5. f *positive* $\implies$ $f^\dagger = f$.
6. f *positive* $\implies$ f^* *and* $tr\ f$ *are positive.*
7. $f, g : A \to A$ *positive* $\implies$ $tr(g\, f)$ *is positive.*

Proof. The first six items follow immediately from the definitions and how structure is preserved for $(\_)^\dagger$. For item 6, note that $g = h\,h^\dagger$ and $tr(g\,f) = tr(h^\dagger\,f\,h)$ which is positive by points 1 and 5. $\square$

Definition 3.4.9. In a compact closed category, the *name* of a map $f : A \to B$ is the map $\ulcorner f \urcorner : I \to A^* \otimes B$ defined as $\eta_A; (1 \otimes f)$. This is also called the *matrix* of f.

In the case of a positive map f, $\ulcorner f \urcorner$ is referred to as a *positive matrix.*

Definition 3.4.10. In a dagger compact closed category, a map $f : A^* \otimes A \to B^* \otimes B$ is *completely positive* if for all objects C and all positive matrices $f : I \to C^* \otimes A^* \otimes A \otimes C$ the morphism $g; (1 \otimes f \otimes 1) : I \to C^* \otimes B^* \otimes B \otimes C$ is a positive matrix.

This now allows us to define the CPM construction.

Definition 3.4.11. Given a dagger compact closed category $\mathbb{D}$, define $\mathrm{CPM}(\textsc{d})$ as the category with the same objects as $\mathbb{D}$, and a map $f : A \to B$ in $\mathrm{CPM}(\textsc{d})$ is a completely positive map $f : A^* \otimes A \to B^* \otimes B$ in $\mathbb{D}$.

$\mathrm{CPM}(\textsc{d})$ is also a dagger compact closed structure, inheriting its tensor from $\mathbb{D}$. There is a functor $F : \mathbb{D} \to \mathrm{CPM}(\textsc{d})$ defined as $F(A) = A$ on objects and $F(f) = f_* \otimes f$ on maps. The image of the structure maps under F are structure maps for $\mathrm{CPM}(\textsc{d})$. The dagger of a map f is the same as its dagger in $\mathbb{D}$.

Biproduct completion

When the CPM construction is applied to a biproduct dagger compact closed category, it will not in general retain biproducts. However, it will be monoid enriched by lemma 3.4.8. This allows us to create the biproduct completion.

The biproduct completion of a category $\mathbb{D}$, which is enriched in commutative monoids is the category $\mathbb{D}^\oplus$ which has as objects finite sequences $\langle A_1, \ldots, A_n\rangle$ where $n \geq 0$. The morphisms of $\mathbb{D}^\oplus$ are matrices of the morphisms of $\mathbb{D}$. Application and composition of morphisms is via matrix multiplication. The functor $F(A) = \langle A\rangle$, $F(f) = [f]$ is an embedding of

$\mathbb{D}$ in $\mathbb{D}^{\oplus}$. If $\mathbb{D}$ is compact closed and the tensor is linear (i.e., interacts with the enrichment in a linear fashion), then $\mathbb{D}^{\oplus}$ is also compact closed.

Furthermore, if $\mathbb{D}$ is a dagger category and the dagger is linear, then $\mathbb{D}^{\oplus}$ will be a dagger category. The dagger of a map $(f_{i,j})$ in $\mathbb{D}^{\oplus}$ is $((f_{j,i})^{\dagger})$.

This gives us the following theorem:

Theorem 3.4.12. *Given $\mathbb{D}$, a biproduct dagger compact closed category,* $\mathrm{CPM}(\textsc{d})$ *is enriched in commutative monoids as a dagger compact closed category. Therefore, it is possible to construct its biproduct completion,* $\mathrm{CPM}(\textsc{d})^{\oplus}$.

Note that the canonical embedding from above, F, while it preserves the dagger compact closed structure, it does *not* preserve biproducts.

Chapter 4

Inverse categories

4.1 Inverse products

Our goal is now to add "products", to an inverse category. Because an inverse category that has a restriction product is a restriction preorder, what is meant by "product" must be specialized for the inverse setting. These we call *inverse products*, which are defined in sub-section 4.1.2 below.

Inverse products are given by a tensor product which supports a diagonal, but lack projections. The diagonal map is required to give a natural Frobenius structure to each object.

4.1.1 Inverse categories with restriction products

We start by showing than an inverse category with restriction products is a restriction preorder. Thus simply using restriction products provides a notion which is too narrow.

Definition 4.1.1. Two parallel maps $f, g : A \to B$ in a restriction category are *compatible*, written as $f \smile g$, when $\overline{f}g = \overline{g}f$.

Definition 4.1.2. A restriction category $\mathbb{X}$ is a *restriction preorder* when all parallel pairs of maps are compatible.

Lemma 4.1.3. *Given an inverse category $\mathbb{X}$, if it has restriction products, it is a restriction preorder. That is,*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \implies f \smile g.$$

Proof. Notice,

$$\begin{aligned}{\pi_1}^{(-1)} &= \Delta\pi_1{\pi_1}^{(-1)} \\ &= \Delta\overline{\pi_1} \\ &= \Delta.\end{aligned}$$

This gives $\overline{{\pi_1}^{(-1)}} = 1$ and therefore π_1 (and similarly, π_0) is an isomorphism.

Starting with the product map $\langle f, g\rangle$,

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle f, g\rangle = \langle f, g\rangle}{\langle f, g\rangle\pi_1{\pi_1}^{(-1)} = \langle f, g\rangle\pi_0{\pi_0}^{(-1)}}}{\overline{f}g{\pi_1}^{(-1)} = \overline{g}f{\pi_0}^{(-1)}}}{\overline{f}g\Delta = \overline{g}f\Delta}}{\overline{f}g = \overline{g}f}$$

which shows that f and g are compatible. □

Corollary 4.1.4. *$\mathbb{X}$ is an Cartesian inverse category if and only if Total($K_r(\mathbb{X})$) is a meet preorder.*

Proof. Total($\mathbb{X}$), the subcategory of total maps on $\mathbb{X}$, has products and therefore every pair of parallel maps is compatible. However, total compatible maps are simply equal, therefore there is at most one map between any two objects. Hence, it is a preorder with the meet being the product.

Similarly, from [18] and [20], Total($\mathrm{K}_r(\mathbb{X})$) is an inverse category and has products and is therefore also a meet preorder.

When Total($\mathrm{K}_r(\mathbb{X})$) is a meet preorder, define the product as the meet of the maps and the terminal object as the supremum of all maps. □

Corollary 4.1.5. *Every Cartesian inverse category is a full subcategory of a partial map category of a meet semi-lattice.*

4.1.2 Inverse products

An *inverse product* on a restriction category $\mathbb{X}$ is given by a tensor $\otimes$ together with a natural "Frobenius" diagonal map, Δ. The data for the tensor is:

$$\begin{aligned} \_ \otimes \_ &: \mathbb{X} \times \mathbb{X} \to \mathbb{X} \ \text{ (a restriction functor)} \\ 1 &: \mathbf{1} \to \mathbb{X} \\ u^l_\otimes &: 1 \otimes A \to A \\ u^r_\otimes &: A \otimes 1 \to A \\ a_\otimes &: (A \otimes B) \otimes C \to A \otimes (B \otimes C) \\ c_\otimes &: A \otimes B \to B \otimes A \end{aligned}$$

where $u^l_\otimes, u^r_\otimes, a_\otimes, c_\otimes$ are all natural isomorphisms and the standard symmetric monoidal equations and coherence diagrams hold (see, e.g., [16]). Note that as all the coherence maps are isomorphisms, they are total. Additionally, we define the map $ex_\otimes : (A \otimes B) \otimes (C \otimes D) \to (A \otimes C) \otimes (B \otimes D)$

$$ex_\otimes = a_\otimes (1 \otimes {a_\otimes}^{(-1)})(1 \otimes (c_\otimes \otimes 1))(1 \otimes a_\otimes){a_\otimes}^{(-1)}).$$

The diagonal map $\Delta_A : A \to A \otimes A$ must be total and must satisfy the following:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & A \otimes A \arrow[d, "c_\otimes"] \\ & A \otimes A \end{tikzcd}$$

Co-commutative

$$\begin{tikzcd}
A \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] & & A \otimes A \arrow[d, "1\otimes\Delta"] \\
A \otimes A \arrow[dr, "\Delta\otimes 1"'] & & A \otimes (A \otimes A) \\
& (A \otimes A) \otimes A \arrow[ur, "a_\otimes"'] &
\end{tikzcd}$$

Co-associative

$$\begin{tikzcd}
A \otimes B \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "\Delta"'] & (A \otimes A) \otimes (B \otimes B) \arrow[d, "ex_\otimes"] \\
(A \otimes B) \otimes (A \otimes B) \arrow[r, equal] & (A \otimes B) \otimes (A \otimes B)
\end{tikzcd}$$

Exchange

$$\begin{tikzcd}
A \otimes A \arrow[rr, "(\Delta\otimes 1)a_\otimes"] \arrow[dr, "\Delta^{(-1)}"] \arrow[dd, "(1\otimes\Delta)a_\otimes{}^{(-1)}"'] & & A \otimes (A \otimes A) \arrow[dd, "1\otimes\Delta^{(-1)}"] \\
& A \arrow[dr, "\Delta"] & \\
(A \otimes A) \otimes A \arrow[rr, "\Delta^{(-1)}\otimes 1"'] & & A \otimes A
\end{tikzcd}$$

Frobenius

Thus, Δ is a co-commutative, coassociative map which together with $\Delta^{(-1)}$ forms a Frobenius algebra.

Remark 4.1.6. Note also, co-commutativity implies that $c_\otimes \Delta^{(-1)} = \Delta^{(-1)}$. One can see this as:

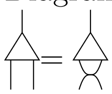
$$\Delta(c_\otimes \Delta^{(-1)}) = (\Delta c_\otimes)\Delta^{(-1)} = \Delta\Delta^{(-1)} = \overline{\Delta} \text{ and}$$

$$(c_\otimes \Delta^{(-1)})\Delta = (c_\otimes \Delta^{(-1)})(\Delta c_\otimes) = \overline{c_\otimes \Delta^{(-1)}}.$$

But this means that both $\Delta^{(-1)}$ and $c_\otimes\Delta^{(-1)}$ are partial inverses for Δ and are therefore equal.

Similarly, one can show that $(\Delta^{(-1)} \otimes 1)\Delta^{(-1)} = a_\otimes(\Delta^{(-1)} \otimes 1)\Delta^{(-1)}$.

Diagrammatic language



Inverse products are extra structure on an inverse category, rather than a property. A concrete category showing this is given in the following example.

Example 4.1.7 (Showing that inverse product is additional structure.)**.**

Any discrete category (i.e., a category with only the identity arrows) is a trivial inverse category. To create an inverse product on the category, add a commutative, associative, idempotent multiplication, with a unit, on the objects.

Label the four objects of $\mathbb{D}$ as a, b, c and d. Then, define two different inverse product tensors by:

$\otimes$	a	b	c	d
a	a	a	a	a
b	a	b	**b**	b
c	a	**b**	c	c
d	a	b	c	d

$\otimes$	a	b	c	d
a	a	a	a	a
b	a	b	**a**	b
c	a	**a**	c	c
d	a	b	c	d

The fact that these operations are idempotent(commutative and associative) implies there is a trivial Frobenius structure.

4.1.3 Discrete inverse categories

An inverse category with inverse products is a *discrete inverse category.* This paper will now present some properties of discrete inverse categories. These properties will be used later when describing a functor that lifts the inverse category to a Cartesian restriction category.

Lemma 4.1.8. *In a discrete inverse category $\mathbb{X}$ with the tensor $\otimes$ and Δ defined as above, where $e = \bar{e}$ is a restriction idempotent and f, g, h are arrows in $\mathbb{X}$, the following are true:*

(i) $e = \Delta(e \otimes 1)\Delta^{(-1)}$.

(ii) $e\Delta(f \otimes g) = \Delta(ef \otimes g)$ *(and* $= \Delta(f \otimes eg)$ *and* $= \Delta(ef \otimes eg)$*.)*

(iii) $(f \otimes ge)\Delta^{(-1)} = (f \otimes g)\Delta^{(-1)}e$ *(and* $= (fe \otimes g)\Delta^{(-1)}$ *and* $= (fe \otimes ge)\Delta^{(-1)}$*.)*

(iv) $\overline{\Delta(f \otimes g)\Delta^{(-1)}} = \Delta(1 \otimes gf^{(-1)})\Delta^{(-1)}$.

(v) If $\Delta(h \otimes g)\Delta^{(-1)} = \overline{\Delta(h \otimes g)\Delta^{(-1)}}$ *then* $(\Delta(h \otimes g)\Delta^{(-1)})h = \Delta(h \otimes g)\Delta^{(-1)}$.

(vi) $\Delta(f \otimes 1) = \Delta(g \otimes 1) \implies f = g$.

(vii) $(f \otimes 1) = (g \otimes 1) \implies f = g$.

Proof.

(i) This is shown by proving both sides equal $\Delta(e \otimes 1)\Delta^{(-1)}\Delta(e \otimes 1)\Delta^{(-1)}$.

$$
\begin{aligned}
&\Delta(e \otimes 1)\Delta^{(-1)}\Delta(e \otimes 1)\Delta^{(-1)} \\
&\quad = \Delta(e \otimes 1)\Delta^{(-1)}\Delta(1 \otimes e)\Delta^{(-1)} && \text{cocommutativity} \\
&\quad = \Delta(e\Delta \otimes 1)(1 \otimes \Delta^{(-1)}e)\Delta^{(-1)} && \text{Frobenius} \\
&\quad = \Delta(\Delta \otimes 1)(e \otimes e \otimes 1)(1 \otimes \Delta^{(-1)}e)\Delta^{(-1)} && \Delta\text{natural} \\
&\quad = \Delta(\Delta \otimes 1)(e \otimes e \otimes 1) \\
&\qquad\quad (1 \otimes e \otimes e)(1 \otimes \Delta^{(-1)})\Delta^{(-1)} && \Delta^{(-1)} \text{ natural} \\
&\quad = \Delta(\Delta \otimes 1)(e \otimes e \otimes e)(1 \otimes \Delta^{(-1)})\Delta^{(-1)} && e \text{ idempotent} \\
&\quad = \Delta(\Delta \otimes 1)(e \otimes \Delta^{(-1)}e)\Delta^{(-1)} && \Delta^{(-1)} \text{ natural} \\
&\quad = \Delta(\Delta \otimes 1)(1 \otimes \Delta^{(-1)})\Delta^{(-1)}e && \Delta^{(-1)} \text{ natural} \\
&\quad = \Delta\Delta^{(-1)}\Delta\Delta^{(-1)}e && \text{Frobenius} \\
&\quad = e && \Delta \text{ total.}
\end{aligned}
$$

At the same time,

$$\begin{aligned}
\Delta(e\otimes 1)\Delta^{(-1)}\Delta(e\otimes 1)\Delta^{(-1)} &= \Delta(e\Delta\otimes 1)(e\otimes\Delta^{(-1)}1)\Delta^{(-1)} && \text{Frobenius}\\
&= \Delta(\Delta\otimes 1)(e\otimes e\otimes 1)(e\otimes\Delta^{(-1)})\Delta^{(-1)} && \Delta\text{natural}\\
&= \Delta(\Delta\otimes 1)(e\otimes e\otimes 1)(e\otimes 1\otimes 1)(1\otimes\Delta^{(-1)})\Delta^{(-1)} && \Delta^{(-1)}\text{ natural}\\
&= \Delta(\Delta\otimes 1)(e\otimes e\otimes 1)(1\otimes\Delta^{(-1)})\Delta^{(-1)} && e\text{ idempotent}\\
&= \Delta(e\Delta\otimes 1)(1\otimes\Delta^{(-1)})\Delta^{(-1)} && \Delta\text{natural}\\
&= \Delta(e\otimes 1)\Delta^{(-1)}\Delta\Delta^{(-1)} && \text{Frobenius}\\
&= \Delta(e\otimes 1)\Delta^{(-1)} && \Delta\text{ total}
\end{aligned}$$

which gives $e = \Delta(e\otimes 1)\Delta^{(-1)}$.

(ii) This equality starts by using the previous equality:

$$\begin{aligned}
e\Delta(f\otimes g) &= \Delta(e\otimes 1)\Delta^{(-1)}\Delta(f\otimes g) && \text{by part (i)}\\
&= \Delta(e\otimes 1)\overline{\Delta^{(-1)}}(f\otimes g)\\
&= \Delta\overline{\Delta^{(-1)}}(e\otimes 1)(f\otimes g) && [\mathbf{R.2}]\\
&= \Delta(ef\otimes g) && (ff^{(-1)} = f).
\end{aligned}$$

The second and third equalities follow by cocommutativity, naturality of Δ and e being a restriction idempotent.

(iii) As in ((ii)), details are only given for the first equality.

$$\begin{aligned}
&(f\otimes g)\Delta^{(-1)}e\\
&\quad= (f\otimes g)\Delta^{(-1)}\Delta(1\otimes e)\Delta^{(-1)} && \text{part (i)}\\
&\quad= (f\otimes g)\overline{\Delta^{(-1)}}(1\otimes e)\Delta^{(-1)}\\
&\quad= (f\otimes g)(1\otimes e)\overline{\Delta^{(-1)}}\Delta^{(-1)} && [\mathbf{R.2}]\\
&\quad= (f\otimes ge)\Delta^{(-1)} && [\mathbf{R.1}]
\end{aligned}$$

The other equalities follow from co-commutativity, naturality of Δ and e being a restriction idempotent.

(iv) Here, we start by using the fact all maps have a partial inverse:

$$
\begin{aligned}
&\overline{\Delta(f\otimes g)\Delta^{(-1)}} \\
&= \Delta(f\otimes g)\Delta^{(-1)}\Delta(f^{(-1)}\otimes g^{(-1)})\Delta^{(-1)} \\
&= \Delta(g\otimes f)\Delta^{(-1)}\Delta(g^{(-1)}\otimes f^{(-1)})\Delta^{(-1)} && \text{co-commutative} \\
&= \Delta(g\Delta\otimes f)(g^{(-1)}\otimes \Delta^{(-1)}f^{(-1)})\Delta^{(-1)} && \text{Frobenius} \\
&= \Delta(\Delta\otimes 1)(g\otimes g\otimes f)(g^{(-1)}\otimes \Delta^{(-1)}f^{(-1)})\Delta^{(-1)} && \Delta \text{ natural} \\
&= \Delta(\Delta\otimes 1)(g\otimes g\otimes f) \\
&\qquad (g^{(-1)}\otimes f^{(-1)}\otimes f^{(-1)})(1\otimes \Delta^{(-1)})\Delta^{(-1)} && \Delta^{(-1)} \text{ natural} \\
&= \Delta(\Delta\otimes 1)(\overline{g}\otimes gf^{(-1)}\otimes \overline{f})(1\otimes \Delta^{(-1)})\Delta^{(-1)} && \text{combine maps} \\
&= \Delta(\Delta\otimes 1)(\overline{g}\otimes \overline{g}\,gf^{(-1)}\overline{f}\otimes \overline{f})(1\otimes \Delta^{(-1)})\Delta^{(-1)} && \text{restriction axioms} \\
&= \Delta(\overline{g}\Delta\otimes 1)(1\otimes gf^{(-1)}\overline{f}\otimes \overline{f})(1\otimes \Delta^{(-1)})\Delta^{(-1)} && \Delta \text{ natural} \\
&= \Delta(\overline{g}\Delta\otimes 1)(1\otimes gf^{(-1)}\otimes 1)(1\otimes \Delta^{(-1)}\overline{f})\Delta^{(-1)} && \Delta^{(-1)} \text{ natural} \\
&= \Delta(\Delta\otimes 1)(1\otimes \overline{g}gf^{(-1)}\otimes 1)(1\otimes \Delta^{(-1)}\overline{f})\Delta^{(-1)} && \text{This lemma((ii))} \\
&= \Delta(\Delta\otimes 1)(1\otimes \overline{g}gf^{(-1)}\overline{f}\otimes 1)(1\otimes \Delta^{(-1)})\Delta^{(-1)} && \text{This lemma((iii))} \\
&= \Delta(\Delta\otimes 1)(1\otimes gf^{(-1)}\otimes 1)(1\otimes \Delta^{(-1)})\Delta^{(-1)} && \text{restriction axioms} \\
&= \Delta c_{A,A}(\Delta\otimes 1)(1\otimes gf^{(-1)}\otimes 1)(1\otimes \Delta^{(-1)})\Delta^{(-1)} && \text{co-commutative} \\
&= \Delta(1\otimes \Delta)c_{A,A\otimes A}(1\otimes gf^{(-1)}\otimes 1)(1\otimes \Delta^{(-1)})\Delta^{(-1)} && c_\otimes \text{natural} \\
&= \Delta(1\otimes \Delta)(1\otimes 1\otimes gf^{(-1)})c_{A,A\otimes A}(1\otimes \Delta^{(-1)})\Delta^{(-1)} && c_\otimes \text{natural} \\
&= \Delta(1\otimes \Delta)(1\otimes 1\otimes gf^{(-1)})(\Delta^{(-1)}\otimes 1)c_{A,A}\Delta^{(-1)} && c_\otimes \text{natural} \\
&= \Delta(1\otimes \Delta)(1\otimes 1\otimes gf^{(-1)})(\Delta^{(-1)}\otimes 1)\Delta^{(-1)} && c_\otimes \text{co-commutative} \\
&= \Delta\Delta^{(-1)}\Delta(1\otimes gf^{(-1)})\Delta^{(-1)} && \text{Frobenius} \\
&= \Delta(1\otimes gf^{(-1)})\Delta^{(-1)} && \Delta \text{ total}
\end{aligned}
$$

Note the pattern in the last few lines of using the co-commutativity of Δ, the naturality of the commutativity isomorphism and finishing with the co-commutativity of $\Delta^{(-1)}$. In future proofs, these steps will be combined to a single line and referred to as commutativity.

(v) Beginning with the assumption,

$$
\begin{aligned}
(\Delta(h \otimes g)\Delta^{(-1)})h &= \overline{\Delta(h \otimes g)\Delta^{(-1)}}h \\
&= \Delta(1 \otimes gh^{(-1)})\Delta^{(-1)}h && \text{This lemma((iv))} \\
&= \Delta(1 \otimes gh^{(-1)})\Delta^{(-1)}\Delta(h \otimes h)\Delta^{(-1)} && \Delta \text{ total and natural} \\
&= \Delta(1 \otimes gh^{(-1)})(\Delta \otimes 1)(1 \otimes \Delta^{(-1)})(h \otimes h)\Delta^{(-1)} && \text{Frobenius} \\
&= \Delta(\Delta \otimes 1)(1 \otimes 1 \otimes gh^{(-1)}) \\
&\qquad (1 \otimes \Delta^{(-1)})(h \otimes h)\Delta^{(-1)} && \Delta \text{ natural} \\
&= \Delta(\Delta \otimes 1)(1 \otimes 1 \otimes gh^{(-1)}) \\
&\qquad (h \otimes h \otimes h)(1 \otimes \Delta^{(-1)})\Delta^{(-1)} && \Delta^{(-1)} \text{ natural} \\
&= \Delta(\Delta \otimes 1)(h \otimes h \otimes gh^{(-1)}h) \\
&\qquad (1 \otimes \Delta^{(-1)})\Delta^{(-1)} && \text{combine terms} \\
&= \Delta(h \otimes g\overline{h^{(-1)}})(\Delta \otimes 1)(1 \otimes \Delta^{(-1)})\Delta^{(-1)} && \Delta \text{ natural} \\
&= \Delta(h \otimes g\overline{h^{(-1)}})\Delta^{(-1)}\Delta\Delta^{(-1)} && \text{Frobenius} \\
&= \Delta(h \otimes g\overline{h^{(-1)}})\Delta^{(-1)} && \Delta \text{ total} \\
&= \Delta(h \otimes g)\Delta^{(-1)}\overline{h^{(-1)}} && \text{part ((ii))} \\
&= \Delta(h\overline{h^{(-1)}} \otimes g)\Delta^{(-1)} && \text{part ((ii))} \\
&= \Delta(h \otimes g)\Delta^{(-1)} && \text{property of inverse.}
\end{aligned}
$$

(vi) As Δ is total and natural, we start with:

$$\begin{aligned}
f &= \Delta(f \otimes f)\Delta^{(-1)} & \\
&= \Delta(f \otimes 1)(1 \otimes f)\Delta^{(-1)} & \\
&= \Delta(g \otimes 1)(1 \otimes f)\Delta^{(-1)} & \text{assumption} \\
&= \Delta(1 \otimes f)(g \otimes 1)\Delta^{(-1)} & \text{Identities commute} \\
&= \Delta(1 \otimes g)(g \otimes 1)\Delta^{(-1)} & \text{assumption, co-commutative} \\
&= \Delta(g \otimes g)\Delta^{(-1)} & \\
&= g\Delta\Delta^{(-1)} & \Delta \text{ natural} \\
&= g & \Delta \text{ total.}
\end{aligned}$$

(vii) Immediate from part (vi).

$\square$

Proposition 4.1.9. *A discrete inverse category has meets, where* $f \cap g = \Delta(f \otimes g)\Delta^{(-1)}$.

Proof. $f \cap g \leq f$:

$$\begin{aligned}
f \cap g &= \Delta(f \otimes g)\Delta^{(-1)} & \text{Definition of } \cap \\
&= \Delta(f\overline{f^{(-1)}} \otimes g)\Delta^{(-1)} & \text{property of inverse} \\
&= \Delta(f \otimes g\overline{f^{(-1)}})\Delta^{(-1)} & \text{by lemma 4.1.8((iii))} \\
&= \Delta(f \otimes gf^{(-1)}f)\Delta^{(-1)} & \text{definition of inverse} \\
&= \Delta(1 \otimes gf^{(-1)})\Delta^{(-1)}f & \Delta^{(-1)} \text{ natural} \\
&= \overline{f \cap g}f & \text{by lemma 4.1.8((iv))}
\end{aligned}$$

$f \cap f = f$:

$$\begin{aligned} f \cap f &= \Delta(f \otimes f)\Delta^{(-1)} && \\ &= f\Delta\Delta^{(-1)} && \Delta \text{ natural} \\ &= f && \Delta \text{ total.} \end{aligned}$$

$h(f \cap g) = hf \cap hg$:

$$\begin{aligned} h(f \cap g) &= h\Delta(f \otimes g)\Delta^{(-1)} && \text{Definition of } \cap \\ &= \Delta(h \otimes h)(f \otimes g)\Delta^{(-1)} && \Delta \text{ natural} \\ &= \Delta(hf \otimes hg)\Delta^{(-1)} && \text{compose maps} \\ &= hf \cap hg && \text{Definition of } \cap. \end{aligned}$$

□

4.1.4 The inverse subcategory of a discrete restriction category

Given a discrete restriction category, one can pick out the maps which are partial isomorphisms. Using results from the previous sub-section and from sub-section 2.2.7, this section will show that these maps form a restriction subcategory and in fact, form a discrete inverse category.

Lemma 4.1.10. *Given $\mathbb{X}$ is a discrete restriction category, the invertible maps of $\mathbb{X}$, together with the objects of $\mathbb{X}$ form a sub restriction category which is a discrete inverse category, denoted by $Inv(\mathbb{X})$.*

Proof. As shown in Lemma 2.2.14, partial isomorphisms are closed under composition. The identity maps are in $Inv(\mathbb{X})$. Trivially, restrictions of partial isomorphisms are also partial isomorphisms.

The product on the discrete restriction category $\mathbb{X}$ becomes the tensor product of the restriction category $Inv(\mathbb{X})$. Table 4.1 shows how each of the elements of the tensor are defined. Note that the last definition makes explicit use of the fact we are in a discrete restriction category and hence the Δ of $\mathbb{X}$ possesses a partial inverse.

$\mathbb{X}$	$Inv(\mathbb{X})$	Inverse map
$A \times B$	$A \otimes B$	
$\top$	1	
$\pi_1 : \top \times A \to A$	$u_\otimes^l : 1 \otimes A \to A$	$\langle !, 1 \rangle$
$\pi_0 : A \times \top \to A$	$u_\otimes^r : A \otimes 1 \to A$	$\langle 1, ! \rangle$
$\langle \pi_0\pi_0, \langle \pi_0\pi_1, \pi_1 \rangle \rangle : (A \times B) \times C \to A \times (B \times C)$	$a_\otimes : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$	$\langle \langle \pi_0, \pi_1\pi_0 \rangle, \pi_1\pi_1 \rangle$
$\langle \pi_1, \pi_0 \rangle : A \times B \to B \times A$	$c_\otimes : A \otimes B \to B \otimes A$	$\langle \pi_1, \pi_0 \rangle$
$\Delta_{\mathbb{X}} : A \to A \times A$	$\Delta : A \to A \otimes A$	${\Delta_{\mathbb{X}}}^{(-1)}$

Table 4.1: Structural maps for the tensor in $Inv(\mathbb{X})$

The monoid coherence diagrams and Δ being total follow directly from the characteristics of the product in $\mathbb{X}$. It remains to show co-commutativity, co-associativity and the Frobenius condition.

Co-commutativity requires $\Delta c_\otimes = c_\otimes$. From the definitions, this means we need

$$\Delta_{\mathbb{X}} \langle \pi_1, \pi_0 \rangle = \Delta_{\mathbb{X}}.$$

Once again, this follows immediately from the definition of restriction product.

Co-associativity requires $\Delta(1 \otimes \Delta) = \Delta(\Delta \otimes 1)a_\otimes$. Expressing this in $\mathbb{X}$, we require

$$\Delta_{\mathbb{X}}(1 \times \Delta_{\mathbb{X}}) = \Delta_{\mathbb{X}}(\Delta_{\mathbb{X}} \times 1)(\langle \pi_0\pi_0, \langle \pi_0\pi_1, \pi_1 \rangle \rangle).$$

Again each is equal based on the properties of the restriction product.

The Frobenius requirement is two-fold:

$$\Delta^{(-1)}\Delta = (\Delta \otimes 1)a_\otimes(1 \otimes \Delta^{(-1)}) \tag{4.1}$$

$$\Delta^{(-1)}\Delta = (1 \otimes \Delta){a_\otimes}^{(-1)}(\Delta^{(-1)} \otimes 1), \tag{4.2}$$

but in $\mathbb{X}$, this becomes:

$${\Delta_{\mathbb{X}}}^{(-1)}\Delta_{\mathbb{X}} = (\Delta_{\mathbb{X}} \times 1)\langle \pi_0\pi_0, \langle \pi_0\pi_1, \pi_1 \rangle \rangle (1 \times {\Delta_{\mathbb{X}}}^{(-1)}) \tag{4.3}$$

$${\Delta_{\mathbb{X}}}^{(-1)}\Delta_{\mathbb{X}} = (1 \times \Delta_{\mathbb{X}})\langle \langle \pi_0, \pi_1\pi_0 \rangle, \pi_1\pi_1 \rangle ({\Delta_{\mathbb{X}}}^{(-1)} \times 1). \tag{4.4}$$

We will detail the proof of Equation ((4.3)). Equation ((4.4)) is proved similarly.

To show the equation, note first that $\Delta(1\times !)$ (and $\Delta(! \times 1)$) is the identity and secondly that maps to a product of objects may be split into a product map — e.g. if $f : A \to B \times B$, then $f = \langle f(1\times !), f(! \times 1)\rangle$.

Using this we see that the left hand side of Equation ((4.3)) computes as follows:

$$\begin{aligned}
{\Delta_{\mathbb{X}}}^{(-1)}\Delta_{\mathbb{X}} &= \langle {\Delta_{\mathbb{X}}}^{(-1)}\Delta_{\mathbb{X}}(1\times !), {\Delta_{\mathbb{X}}}^{(-1)}\Delta_{\mathbb{X}}(! \times 1)\rangle \\
&= \langle {\Delta_{\mathbb{X}}}^{(-1)}, {\Delta_{\mathbb{X}}}^{(-1)}\rangle
\end{aligned}$$

Similarly, removing the associativity maps, the right hand side of the same equation becomes:

$$\begin{aligned}
(\Delta_{\mathbb{X}} \times 1)(1 \times {\Delta_{\mathbb{X}}}^{(-1)}) &= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times {\Delta_{\mathbb{X}}}^{(-1)})(1\times !), (\Delta_{\mathbb{X}} \times 1)(1 \times {\Delta_{\mathbb{X}}}^{(-1)})(! \times 1)\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times {\Delta_{\mathbb{X}}}^{(-1)})(1\times !), {\Delta_{\mathbb{X}}}^{(-1)}\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times {\Delta_{\mathbb{X}}}^{(-1)})(1 \times \Delta_{\mathbb{X}})(1\times !\times !), {\Delta_{\mathbb{X}}}^{(-1)}\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times \overline{{\Delta_{\mathbb{X}}}^{(-1)}})(1\times !\times !), {\Delta_{\mathbb{X}}}^{(-1)}\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)\overline{1 \times {\Delta_{\mathbb{X}}}^{(-1)}}(1\times !\times !), {\Delta_{\mathbb{X}}}^{(-1)}\rangle \\
&= \langle \overline{(\Delta_{\mathbb{X}} \times 1)(1 \times {\Delta_{\mathbb{X}}}^{(-1)})}(\Delta_{\mathbb{X}} \times 1)(1\times !\times !), {\Delta_{\mathbb{X}}}^{(-1)}\rangle \\
&= \langle \overline{(\Delta_{\mathbb{X}} \times 1)(1 \times {\Delta_{\mathbb{X}}}^{(-1)})}(1\times !), {\Delta_{\mathbb{X}}}^{(-1)}\rangle \\
&= \langle \overline{(\Delta_{\mathbb{X}} \times 1)(1 \times {\Delta_{\mathbb{X}}}^{(-1)})(! \times 1)}(1\times !), {\Delta_{\mathbb{X}}}^{(-1)}\rangle \\
&= \langle \overline{{\Delta_{\mathbb{X}}}^{(-1)}}(1\times !), {\Delta_{\mathbb{X}}}^{(-1)}\rangle \\
&= \langle {\Delta_{\mathbb{X}}}^{(-1)}\Delta_{\mathbb{X}}(1\times !), {\Delta_{\mathbb{X}}}^{(-1)}\rangle \\
&= \langle {\Delta_{\mathbb{X}}}^{(-1)}, {\Delta_{\mathbb{X}}}^{(-1)}\rangle
\end{aligned}$$

and therefore we see that the first equation for the Frobenius condition is satisfied. Thus, $Inv(\mathbb{X})$ is a discrete inverse category. $\square$

4.2 Completing a discrete inverse category

The purpose of this section is to prove that the category of discrete inverse categories is equivalent to the the category of discrete restriction categories. In order to prove this, we show how to construct a discrete restriction category, $\widetilde{\mathbb{X}}$, from a discrete inverse category, $\mathbb{X}$.

4.2.1 The restriction category $\widetilde{\mathbb{X}}$

Definition 4.2.1. When $\mathbb{X}$ is an inverse category, define $\widetilde{\mathbb{X}}$ as:

Objects: objects as in $\mathbb{X}$

Maps: equivalence classes of maps (the equivalence class is defined below in Definition 4.2.2) with the following structure in $\mathbb{X}$:

$$\frac{A \xrightarrow{(f,C)} B \text{ in } \widetilde{\mathbb{X}}}{A \xrightarrow{f} B \otimes C \text{ in } \mathbb{X}}$$

Identity: by

$$\frac{A \xrightarrow{(u_\otimes^{r\,(-1)},1)} A}{A \xrightarrow{u_\otimes^{r\,(-1)}} A \otimes 1}$$

Composition: given by

$$\cfrac{\cfrac{A \xrightarrow{(f,B')} B \xrightarrow{(g,C')} C}{A \xrightarrow{f(g\otimes 1)a_\otimes} C \otimes (C' \otimes B')}}{A \xrightarrow{(f(g\otimes 1)a_\otimes, C'\otimes B')} C}$$

When considering an $\widetilde{\mathbb{X}}$ map $(f,C) : A \to B$ in $\mathbb{X}$, we occasionally use the notation $f : A \to B_{|C}$ ($\equiv f : A \to B \otimes C$).

Equivalence classes of maps in $\mathbb{X}$

Definition 4.2.2. In a discrete inverse category $\mathbb{X}$ as defined above, the map f is equivalent to f' in $\mathbb{X}$ when $\overline{f} = \overline{f'}$ in $\mathbb{X}$ and the below diagram commutes for some map h:

$$\begin{tikzcd}
& B \otimes C \arrow[r, dotted, "(\Delta\otimes 1)\, a_\otimes"] & B \otimes (B \otimes C) \arrow[dd, dotted, "1\otimes h"] \\
A \arrow[ur, "f"] \arrow[dr, "f'"'] & & \\
& B \otimes C' & B \otimes (B \otimes C') \arrow[l, dotted, "{a_\otimes}^{(-1)}\, (\Delta^{(-1)}\otimes 1)"]
\end{tikzcd}$$

Notation 4.2.3. When f is equivalent to g via the mediating map h, this is written as

$$f \overset{h}{\simeq} g.$$

Lemma 4.2.4. *Definition 4.2.2 gives a symmetric, reflexive equivalence class of maps in* $\mathbb{X}$*.*

Proof.

Reflexivity: Choose h as the identity map.

Symmetry: Suppose $f \overset{h}{\simeq} g$. Then, $\overline{f} = \overline{g}$ and $fk = g$ where

$$k = (\Delta \otimes 1)\, a_\otimes\, (1 \otimes h)\, {a_\otimes}^{(-1)}\, (\Delta^{(-1)} \otimes 1).$$

Applying $k^{(-1)}$, which is

$$(\Delta \otimes 1)\, a_\otimes\, (1 \otimes h^{(-1)})\, {a_\otimes}^{(-1)}\, (\Delta^{(-1)} \otimes 1),$$

we have

$$gk^{(-1)} = fkk^{(-1)} = f\overline{k} = \overline{fk}f = \overline{g}f = \overline{f}f = f.$$

Thus, $g \overset{h^{(-1)}}{\simeq} f$.

Transitivity: Suppose $f \overset{h}{\simeq} f'$ and $f' \overset{k}{\simeq} f''$. Then, consider the compositions of the mediating portions of the equivalences:

$$\ell = ((\Delta \otimes 1)a_\otimes(1 \otimes h){a_\otimes}^{(-1)}(\Delta^{(-1)} \otimes 1))((\Delta \otimes 1)a_\otimes(1 \otimes k){a_\otimes}^{(-1)}(\Delta^{(-1)} \otimes 1)).$$

By pasting the diagrams which give the above equivalences, we see that $f\ell = f''$. However, it is not in the form of a mediating map as presented.

The claim is that ℓ is the actual mediating map for f and f''. That is, that we have $f(\Delta \otimes 1)a_\otimes(1 \otimes \ell){a_\otimes}^{(-1)}(\Delta^{(-1)} \otimes 1) = f''$. In the interest of some brevity, this is shown below with the associativity maps elided from the equations.

We need to show that $(\Delta \otimes 1)(1 \otimes \ell)(\Delta^{(-1)} \otimes 1) = \ell$.

$$\begin{aligned}
&(\Delta \otimes 1)(1 \otimes \ell)(\Delta^{(-1)} \otimes 1) \\
&\quad = (\Delta \otimes 1)(1 \otimes \Delta \otimes 1)(1 \otimes 1 \otimes h)(1 \otimes \Delta^{(-1)} \otimes 1)) \\
&\qquad (1 \otimes \Delta \otimes 1)(1 \otimes 1 \otimes k)(1 \otimes \Delta^{(-1)} \otimes 1)(\Delta^{(-1)} \otimes 1) \\
&\quad = (\Delta \otimes 1)(\Delta \otimes 1 \otimes 1)(1 \otimes 1 \otimes h)(1 \otimes \Delta^{(-1)} \otimes 1)) \\
&\qquad (1 \otimes \Delta \otimes 1)(1 \otimes 1 \otimes k)(\Delta^{(-1)} \otimes 1 \otimes 1)(\Delta^{(-1)} \otimes 1) && \text{co-associativity} \\
&\quad = (\Delta \otimes 1)(1 \otimes h)(\Delta \otimes 1 \otimes 1)(1 \otimes \Delta^{(-1)} \otimes 1)) \\
&\qquad (1 \otimes \Delta \otimes 1)(\Delta^{(-1)} \otimes 1 \otimes 1)(1 \otimes k)(\Delta^{(-1)} \otimes 1) && \text{Naturality} \\
&\quad = (\Delta \otimes 1)(1 \otimes h)(\Delta^{(-1)} \otimes 1)(\Delta \otimes 1)) \\
&\qquad (\Delta^{(-1)} \otimes 1)(\Delta \otimes 1)(1 \otimes k)(\Delta^{(-1)} \otimes 1) && \text{Frobenius} \\
&\quad = (\Delta \otimes 1)(1 \otimes h)(\Delta^{(-1)} \otimes 1)(\Delta \otimes 1)(1 \otimes k)(\Delta^{(-1)} \otimes 1) && \Delta\ \text{Total} \\
&\quad = \ell
\end{aligned}$$

and therefore $f \overset{\ell}{\simeq} f''$. $\square$

Corollary 4.2.5. *If $\overline{f} = \overline{g}$ in $\mathbb{X}$, a discrete inverse category, and the diagram*

$$\begin{tikzcd} & B \otimes C \arrow[d, dotted, "1\otimes h"] \\ A \arrow[ur, "f"] \arrow[r, "g"'] & B \otimes C' \end{tikzcd}$$

commutes for some h, then there is a h' such that $f \overset{h'}{\simeq} g$.

Proof. Consider

$$\begin{aligned}
&(\Delta \otimes 1)\, a_\otimes\, (1 \otimes (1 \otimes h))\, {a_\otimes}^{(-1)}(\Delta^{(-1)} \otimes 1) \\
&\qquad = (\Delta \otimes 1)\, ((1 \otimes 1) \otimes h)\, a_\otimes {a_\otimes}^{(-1)}(\Delta^{(-1)} \otimes 1) && \text{Naturality} \\
&\qquad = (\Delta \otimes 1)\, ((1 \otimes 1) \otimes h)\, (\Delta^{(-1)} \otimes 1) && \text{Isomorphism Inverse} \\
&\qquad = (\Delta(1 \otimes 1)\Delta^{(-1)}) \otimes h && \text{Naturality of } \otimes \\
&\qquad = (1 \otimes h) && \Delta\Delta^{(-1)} = 1
\end{aligned}$$

and therefore we can set $h' = 1 \otimes h$. □

Lemma 4.2.6. *$\widetilde{\mathbb{X}}$ as defined above is a category.*

Proof. The maps are well defined, as shown in Lemma 4.2.4. The existence of the identity map is due to the tensor $\otimes$ being defined on $\mathbb{X}$, an inverse category, hence ${u_\otimes^r}^{(-1)}$ is defined.

It remains to show the composition is associative and that $({u_\otimes^r}^{(-1)}, 1)$ acts as an identity in $\widetilde{\mathbb{X}}$.

Associativity: Consider

$$A \xrightarrow{(f,B')} B \xrightarrow{(g,C')} C \xrightarrow{(h,D')} D.$$

To show the associativity of this in $\widetilde{\mathbb{X}}$, we need to show in $\mathbb{X}$ that

$$\overline{(f(g \otimes 1)a_\otimes)(h \otimes 1)a_\otimes} = \overline{f(((g(h \otimes 1)a_\otimes) \otimes 1)a_\otimes)}$$

and that there exists a mediating map between the two of them.

To see that the restrictions are equal, first note that by the functorality of $\otimes$, for any two maps u and v, we have $uv \otimes 1 = (u \otimes 1)(v \otimes 1)$. Second, the naturality of $a_\otimes$ gives us that $a_\otimes(h \otimes 1) = ((h \otimes 1) \otimes 1)a_\otimes$. Thus,

$$
\begin{aligned}
\overline{f(g \otimes 1)a_\otimes(h \otimes 1)a_\otimes} &= \overline{f(g \otimes 1)a_\otimes(h \otimes 1)\overline{a_\otimes}} && \text{Lemma 2.2.3} \\
&= \overline{f(g \otimes 1)a_\otimes(h \otimes 1)} && \overline{a_\otimes} = 1 \\
&= \overline{f(g \otimes 1)((h \otimes 1) \otimes 1)a_\otimes} && a_\otimes \text{ natural} \\
&= \overline{f(g \otimes 1)((h \otimes 1) \otimes 1)} && a_\otimes \text{ iso, Lemma 2.2.3} \\
&= \overline{f(g \otimes 1)((h \otimes 1) \otimes 1)(a_\otimes \otimes 1)} && a_\otimes \otimes 1 \text{ iso, Lemma 2.2.3} \\
&= \overline{f((g(h \otimes 1)a) \otimes 1)} && \text{see above} \\
&= \overline{f((g(h \otimes 1)a) \otimes 1)a_\otimes} && a_\otimes \text{ iso}
\end{aligned}
$$

For the mediating map, see the diagram below, where calculation is in $\mathbb{X}$. The path starting at the top left at A and going right to $D_{|D' \otimes (C' \otimes B')}$ is grouping parentheses to the left, while starting in the same place but going down to $(D_{|D' \otimes C'})_{|B'}$ and then right to $D_{|(D' \otimes C') \otimes B'}$ is grouping parentheses to the right. The commutativity of the diagram is shown by the commutativity of the internal portions, which all follow from the standard coherence diagrams for the tensor and naturality of association.

$$
\begin{tikzcd}
A \arrow[r, "f(g\otimes 1)a_\otimes"] \arrow[d, "f"'] \arrow[dr, "f(g\otimes 1)"] & C_{|C' \otimes B'} \arrow[r, "h\otimes 1"] & (D_{|D'})_{|C' \otimes B'} \arrow[r, "a_\otimes"] & D_{|D' \otimes (C' \otimes B')} \arrow[dd, dotted, "1\otimes a_\otimes{}^{(-1)}"] \\
B_{|B'} \arrow[r, "g\otimes 1"'] \arrow[d, "(g(h\otimes 1)a_\otimes)\otimes 1"'] & (C_{|C'})_{|B'} \arrow[u, "a_\otimes"'] \arrow[r, "(h\otimes 1)\otimes 1"'] & ((D_{|D'})_{|C'})_{|B'} \arrow[u, "a_\otimes"'] \arrow[dll, "a_\otimes \otimes 1"'] & \\
(D_{|D' \otimes C'})_{|B'} \arrow[rrr, "a_\otimes"'] & & & D_{|(D' \otimes C') \otimes B'}
\end{tikzcd}
$$

From this, we can conclude

$$(f(g \otimes 1)a_\otimes)(h \otimes 1)a_\otimes \overset{1 \otimes a_\otimes{}^{(-1)}}{\simeq} f(((g(h \otimes 1)a_\otimes) \otimes 1)a_\otimes)$$

which gives us that composition in $\widetilde{\mathbb{X}}$ is associative.

Identity: This requires:

$$(f, C)(u_\otimes^r{}^{(-1)}, 1) = (f, C) = (u_\otimes^r{}^{(-1)}, 1)(f, C)$$

for all maps $A \xrightarrow{(f,C)} B$ in $\widetilde{\mathbb{X}}$.

First, we see $\overline{f(u_\otimes^r{}^{(-1)} \otimes 1)a_\otimes} = \overline{f}$ by Lemma 2.2.3. Then, calculating in $\mathbb{X}$, we have a mediating map of $1 \otimes u_\otimes^l$ as shown below.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[drrr, "f"'] & B \otimes C \arrow[r, "u_\otimes^r{}^{(-1)} \otimes 1"] \arrow[rr, bend right, "1 \otimes u_\otimes^l{}^{(-1)}"'] \arrow[drr, equal] & (B \otimes 1) \otimes C \arrow[r, "a_\otimes"] & B \otimes (1 \otimes C) \arrow[d, dotted, "1 \otimes u_\otimes^l"] \\
 & & & B \otimes C
\end{tikzcd}$$

Next, $\overline{u_\otimes^r{}^{(-1)}(f \otimes 1)a_\otimes} = \overline{f}$ by the naturality of $u_\otimes^r{}^{(-1)}$ and Lemma 2.2.3. The diagram below

$$\begin{tikzcd}
A \arrow[r, "u_\otimes^r{}^{(-1)}"] \arrow[dr, "f"'] \arrow[dd, equal] & A \otimes 1 \arrow[r, "f \otimes 1"] & (B \otimes C) \otimes 1 \arrow[r, "a_\otimes"] & B \otimes (C \otimes 1) \arrow[dd, dotted, "1 \otimes u_\otimes^r"] \\
 & B \otimes C \arrow[ur, "u_\otimes^r{}^{(-1)}"] \arrow[urr, "1 \otimes u_\otimes^r{}^{(-1)}"'] \arrow[drr, equal] & & \\
A \arrow[rrr, "f"] & & & B \otimes C
\end{tikzcd}$$

shows our mediating map is $1 \otimes u_\otimes^r$. $\square$

Defining the restriction on $\widetilde{\mathbb{X}}$

Define the restriction in $\widetilde{\mathbb{X}}$ as follows:

$$\cfrac{\cfrac{A \xrightarrow{(f,C)} B}{A \xrightarrow{\overline{(f,C)}} A}}{A \xrightarrow{\overline{f}u_\otimes^{r\,(-1)}} A \otimes 1 \text{ in } \mathbb{X}}$$

Lemma 4.2.7. *The category $\widetilde{\mathbb{X}}$ with restriction defined as above is a restriction category.*

Proof. Given the above definition, the four restriction axioms must now be checked. (Diagrams are in $\mathbb{X}$).

[**R.1**] $(\overline{f}f = f)$ Calculating the restriction of the left hand side in $\mathbb{X}$, we have:

$$\begin{aligned}
\overline{\overline{f}u_\otimes^{r\,(-1)}(f \otimes 1)a_\otimes} &= \overline{\overline{f}u_\otimes^{r\,(-1)}(f \otimes 1)} && a_\otimes \text{ iso, Lemma 2.2.3} \\
&= \overline{\overline{f}fu_\otimes^{r\,(-1)}} && u_\otimes^{r\,(-1)} \text{ natural} \\
&= \overline{fu_\otimes^{r\,(-1)}} && [\mathbf{R.1}] \text{ in } \mathbb{X} \\
&= \overline{f} && u_\otimes^{r\,(-1)} \text{ iso, Lemma 2.2.3.}
\end{aligned}$$

Then, the following diagram

$$\begin{tikzcd}
A \arrow[r, "\overline{f}u_\otimes^{r\,(-1)}"] \arrow[rrd, "\overline{f}f"] \arrow[rrrdd, bend right, "f"'] & A \otimes 1 \arrow[r, "f\otimes 1"] & (A \otimes B) \otimes 1 \arrow[r, "a_\otimes"] \arrow[rdd, "u_\otimes^r"] & A \otimes (B \otimes 1) \arrow[dd, dotted, "1\otimes u_\otimes^r"] \\
& & A \otimes B \arrow[u, "u_\otimes^{r\,(-1)}"] \arrow[rd, equal] & \\
& & & A \otimes B
\end{tikzcd}$$

shows $\overline{f}u_\otimes^{r\,(-1)}(f \otimes 1)a_\otimes \overset{1\otimes u_\otimes^r}{\simeq} f$ in $\mathbb{X}$ and therefore $\overline{f}f = f$ in $\widetilde{\mathbb{X}}$.

[**R.2**] $(\overline{g}\,\overline{f} = \overline{f}\,\overline{g})$ The restriction of the left hand side equals the restriction of the right hand side as seen below:

$$\begin{aligned}
\overline{\overline{f}u_\otimes^{r\,(-1)}((\overline{g}u_\otimes^{r\,(-1)}) \otimes 1))a_\otimes} &= \overline{\overline{f}(\overline{g}u_\otimes^{r\,(-1)})u_\otimes^{r\,(-1)}a_\otimes} && u_\otimes^{r\,(-1)} \text{ natural} \\
&= \overline{\overline{g}\,\overline{f}u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}a_\otimes} && [\mathbf{R.2}] \text{ in } \mathbb{X} \\
&= \overline{\overline{g}u_\otimes^{r\,(-1)}((\overline{f}u_\otimes^{r\,(-1)}) \otimes 1)a_\otimes} && u_\otimes^{r\,(-1)}\text{natural.}
\end{aligned}$$

The below diagram commutes by the naturality of $u^r_\otimes$ and the tensor coherence,

$$\begin{tikzcd}
A \arrow[r, "\overline{g}u_\otimes^{r\,(-1)}"] \arrow[d, "\overline{f}u_\otimes^{r\,(-1)}"'] \arrow[dr, "\overline{g}\overline{f}", "\overline{f}\overline{g}"'] & A \otimes 1 \arrow[r, "(\overline{f}u_\otimes^{r\,(-1)})\otimes 1"] & (A \otimes 1) \otimes 1 \arrow[r, "a_\otimes"] \arrow[dl, "u^r_\otimes u^r_\otimes"'] & A \otimes (1 \otimes 1) \arrow[dd, dotted, "1\otimes id"'] \\
A \otimes 1 \arrow[d, "(\overline{g}u_\otimes^{r\,(-1)})\otimes 1"'] & A \arrow[ur, bend right, "u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}"'] & & \\
(A \otimes 1) \otimes 1 \arrow[ur, "u^r_\otimes u^r_\otimes"] \arrow[rrr, "a_\otimes"'] & & & A \otimes (1 \otimes 1)
\end{tikzcd}$$

which allows us to conclude $\overline{f}\overline{g} = \overline{g}\overline{f}$ in $\widetilde{\mathbb{X}}$.

R.3 ($\overline{\overline{f}g} = \overline{f}\overline{g}$). As above, the first step is to show that the restrictions of each side are the same. Computing the restriction of the left hand side in $\mathbb{X}$:

$$\begin{aligned}
\overline{\overline{\overline{(\overline{f}u_\otimes^{r\,(-1)})(g\otimes 1)a_\otimes}u_\otimes^{r\,(-1)}}} &= \overline{\overline{(\overline{f}u_\otimes^{r\,(-1)})(g\otimes 1)a_\otimes}} && u_\otimes^{r\,(-1)} \text{ iso, Lemma 2.2.3} \\
&= \overline{(\overline{f}u_\otimes^{r\,(-1)})(g\otimes 1)a_\otimes} && \text{Lemma 2.2.3} \\
&= \overline{\overline{f}gu_\otimes^{r\,(-1)}a_\otimes} && u_\otimes^{r\,(-1)} \text{ natural} \\
&= \overline{\overline{f}g} && u_\otimes^{r\,(-1)}, a_\otimes \text{ iso, Lemma 2.2.3} \\
&= \overline{f}\overline{g} && [\textbf{R.3}] \text{ in } \mathbb{X}.
\end{aligned}$$

The restriction of the right hand side computes in $\mathbb{X}$ as:

$$\begin{aligned}
&\overline{(\overline{f}u_\otimes^{r\,(-1)})}(\overline{g}u_\otimes^{r\,(-1)}\otimes 1)a_\otimes \\
&\qquad = \overline{(\overline{f}u_\otimes^{r\,(-1)})(\overline{g}u_\otimes^{r\,(-1)}\otimes 1)} && a_\otimes \text{ iso, Lemma 2.2.3} \\
&\qquad = \overline{\overline{f}\overline{g}u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}} && u_\otimes^{r\,(-1)} \text{ natural} \\
&\qquad = \overline{\overline{f}\overline{g}} && u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)} \text{ iso, Lemma 2.2.3} \\
&\qquad = \overline{f}\overline{g} && \text{Lemma 2.2.3.}
\end{aligned}$$

Additionally, we see $\overline{\overline{f}g}$ in $\widetilde{\mathbb{X}}$ is expressed in $\mathbb{X}$ as:

$$
\begin{aligned}
&\overline{(\overline{f}u_{\otimes}^{r\,(-1)})}(g\otimes 1)a_{\otimes}u_{\otimes}^{r\,(-1)} \\
&\qquad = \overline{f}u_{\otimes}^{r\,(-1)}\overline{g\otimes 1} && [\mathbf{R.3}],\ [\mathbf{R.4}],\ a_{\otimes} \text{ iso} \\
&\qquad = \overline{f}\overline{g}u_{\otimes}^{r\,(-1)} && \otimes \text{a restriction bi-functor, } u_{\otimes}^{r\,(-1)} \text{ natural.}
\end{aligned}
$$

The following diagram in $\mathbb{X}$ follows the above right hand side with the top curved arrow and the left hand side with the bottom curved arrow. Note that we are using that $\overline{(\overline{f}u_{\otimes}^{r\,(-1)})(g\otimes 1)a_{\otimes}} = \overline{f}\overline{g}$ as shown above.

$$
\begin{tikzcd}
A \arrow[r, "\overline{f}"] \arrow[dr, "\overline{f}"'] \arrow[rrrrr, bend left, "\overline{f}u_{\otimes}^{r\,(-1)}(\overline{g}u_{\otimes}^{r\,(-1)}\otimes 1)a_{\otimes}"] \arrow[rrrrrddddd, bend right, "\overline{f}\,\overline{g}u_{\otimes}^{r\,(-1)}"'] & A \arrow[r, "u_{\otimes}^{r\,(-1)}"] \arrow[d, equal] & A\otimes 1 \arrow[r, "\overline{g}\otimes 1"] \arrow[dd, "u_{\otimes}^{r}"] & A\otimes 1 \arrow[r, "u_{\otimes}^{r\,(-1)}\otimes 1"] \arrow[ddd, "u_{\otimes}^{r}"] & (A\otimes 1)\otimes 1 \arrow[r, "a_{\otimes}"] \arrow[dddd, "u_{\otimes}^{r}"] & A\otimes(1\otimes 1) \arrow[ddddd, dotted, "1\otimes u_{\otimes}^{r}"] \\
& A \arrow[dr, equal] & & & & \\
& & A \arrow[dr, "\overline{g}"] & & & \\
& & & A \arrow[dr, "u_{\otimes}^{r\,(-1)}"] & & \\
& & & & A\otimes 1 \arrow[dr, equal] & \\
& & & & & A\otimes 1
\end{tikzcd}
$$

Hence, in $\mathbb{X}$, $\overline{(\overline{f}u_{\otimes}^{r\,(-1)})(g\otimes 1)a_{\otimes}}u_{\otimes}^{r\,(-1)} \overset{1\otimes u_{\otimes}^{r}}{\simeq} (\overline{f}u_{\otimes}^{r\,(-1)})(\overline{g}u_{\otimes}^{r\,(-1)}\otimes 1)a_{\otimes}$ and therefore $\overline{\overline{f}g} = \overline{f}\overline{g}$ in $\widetilde{\mathbb{X}}$.

R.4 $f\overline{g} = \overline{fg}f$ The restriction of the left hand side is:

$$
\begin{aligned}
\overline{f(\overline{g}u_{\otimes}^{r\,(-1)}\otimes 1)a_{\otimes}} &= \overline{f(\overline{g}u_{\otimes}^{r\,(-1)}\otimes 1)} && a_{\otimes} \text{ iso, Lemma 2.2.3} \\
&= \overline{f\overline{g}u_{\otimes}^{r\,(-1)}}\otimes\overline{f} && \otimes \text{ restriction functor} \\
&= \overline{f\overline{g}}\otimes\overline{f} && u_{\otimes}^{r\,(-1)} \text{ iso, Lemma 2.2.3} \\
&= \overline{f(\overline{g}\otimes 1)}
\end{aligned}
$$

and the restriction of the right hand side is:

$$
\begin{aligned}
\overline{\overline{f(g\otimes 1)}u_{\otimes}^{r\,(-1)}(f\otimes 1)a_{\otimes}} &= \overline{\overline{f(g\otimes 1)}u_{\otimes}^{r\,(-1)}(f\otimes 1)} && a_{\otimes} \text{ iso, Lemma 2.2.3}\\
&= \overline{\overline{f(g\otimes 1)}fu_{\otimes}^{r\,(-1)}} && u_{\otimes}^{r\,(-1)} \text{ natural}\\
&= \overline{f\overline{(g\otimes 1)}u_{\otimes}^{r\,(-1)}} && [\mathbf{R.4}] \text{ for } \mathbb{X}\\
&= \overline{f(\overline{g}\otimes 1)u_{\otimes}^{r\,(-1)}} && \otimes \text{ is a restriction functor}\\
&= \overline{f(\overline{g}\otimes 1)} && u_{\otimes}^{r\,(-1)} \text{ iso, Lemma 2.2.3}
\end{aligned}
$$

Computing the right hand side in $\mathbb{X}$,

$$
\begin{aligned}
\overline{f(g\otimes 1)a_{\otimes}}u_{\otimes}^{r\,(-1)}(f\otimes 1)a_{\otimes} &= \overline{f(g\otimes 1)}fu_{\otimes}^{r\,(-1)}a_{\otimes} && a_{\otimes} \text{ iso, } u_{\otimes}^{r\,(-1)} \text{ natural.}\\
&= f(\overline{g}\otimes 1)u_{\otimes}^{r\,(-1)}a_{\otimes} && [\mathbf{R.3}], \otimes \text{ a restriction functor.}
\end{aligned}
$$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f"'] & B\otimes C \arrow[rr, "\overline{g}u_{\otimes}^{r\,(-1)}\otimes 1"] & & (B\otimes 1)\otimes C \arrow[r, "a_{\otimes}"] & B\otimes(1\otimes C) \arrow[d, dotted, "1\otimes c_{\otimes}"] \\
& B\otimes C \arrow[r, "\overline{g}\otimes 1"'] & B\otimes C \arrow[r, "u_{\otimes}^{r\,(-1)}"'] & (B\otimes C)\otimes 1 \arrow[r, "a_{\otimes}"'] & B\otimes(C\otimes 1)
\end{tikzcd}
$$

and hence, $\widetilde{\mathbb{X}}$ is a restriction category. □

4.2.2 The category $\widetilde{\mathbb{X}}$ is a discrete restriction category

Lemma 4.2.8. *The unit of the inverse product in $\mathbb{X}$ is the terminal object in $\widetilde{\mathbb{X}}$.*

Proof. The unique map to the terminal object for any object A in $\widetilde{\mathbb{X}}$ is the equivalence class of maps represented by $(u_{\otimes}^{l\,(-1)}, A)$. For this to be a terminal object, the diagram

$$
\begin{tikzcd}
X \arrow[r, "\overline{(f,C)}"] \arrow[d, "(f{,}C)"] & X \arrow[r, "!_X"] & \top \\
Y \arrow[urr, "!_Y"'] & &
\end{tikzcd}
$$

must commute for all choices of f. Translating this to $\mathbb{X}$, this is the same as requiring

$$
\begin{tikzcd}
X \arrow[r, "\overline{f}"] \arrow[d, "f"] & X \arrow[r, "u_{\otimes}^{r\,(-1)}"] & X\otimes 1 \arrow[r, "u_{\otimes}^{l\,(-1)}"] & 1\otimes X\otimes 1 \arrow[dl, dotted, "1\otimes(u_{\otimes}^{r}f)"'] \\
Y\otimes C \arrow[rr, "u_{\otimes}^{l\,(-1)}"'] & & 1\otimes Y\otimes C &
\end{tikzcd}
$$

commute, which is true by [**R.1**] and from the coherence diagrams for the inverse product tensor. ☐

Next,we show that the category $\widetilde{\mathbb{X}}$ has restriction products, given by the the action of $\widetilde{(\_)}$ on the $\otimes$ tensor in $\mathbb{X}$.

First, define total maps π_0, π_1 in $\widetilde{\mathbb{X}}$ by:

$$\pi_0 : \qquad A \otimes B \xrightarrow{(1,B)} A \tag{4.5}$$

$$\pi_1 : \qquad A \otimes B \xrightarrow{(c_\otimes,A)} B \tag{4.6}$$

Given the maps $Z \xrightarrow{(f,C)} A$ and $Z \xrightarrow{(g,C')} B$, define $\langle (f,C),(g,C')\rangle$ as

$$Z \xrightarrow{(\Delta(f\otimes g)(1\otimes c_\otimes \otimes 1),C\otimes C')} A \otimes B \tag{4.7}$$

where associativity is assumed as needed. Note that with the associativity maps, this is actually:

$$Z \xrightarrow{(\Delta(f\otimes g)a_\otimes(1\otimes a_\otimes{}^{(-1)})(1\otimes(c_\otimes\otimes 1))(1\otimes a_\otimes)a_\otimes{}^{(-1)},C\otimes C')} A \otimes B \tag{4.8}$$

Lemma 4.2.9. *On $\widetilde{\mathbb{X}}$, $\otimes$ is a restriction product with projections π_0, π_1 with the product of maps f, g being $\langle f, g\rangle$.*

Proof. From the definition above, as 1 and $c_\otimes$ are isomorphisms, the maps π_0, π_1 are total.

In order to show that $\overline{\langle f, g\rangle} = \overline{f}\,\overline{g}$, first reduce the left hand side:

$$\begin{aligned}
\overline{\langle f,g\rangle} &= \overline{\Delta(f\otimes g)(1\otimes c_\otimes \otimes 1)u_\otimes^{r}{}^{(-1)}} && \text{in } \mathbb{X}, \text{ definition of restriction}\\
&= \overline{\Delta(f\otimes g)u_\otimes^{r}{}^{(-1)}} && c_\otimes \text{ is iso}\\
&= \overline{\Delta\overline{(f\otimes g)}u_\otimes^{r}{}^{(-1)}} && \text{from Lemma 2.2.3}\\
&= \overline{\Delta(\overline{f}\otimes\overline{g})u_\otimes^{r}{}^{(-1)}} && \otimes \text{ is a restriction functor}\\
&= \overline{\overline{f}\,\overline{g}\,\Delta(1\otimes 1)u_\otimes^{r}{}^{(-1)}} && \text{Lemma 4.1.8((ii)) twice}\\
&= \overline{\overline{f}\,\overline{g}}u_\otimes^{r}{}^{(-1)} && \text{Lemma 2.2.3}\\
&= \overline{f}\,\overline{g}u_\otimes^{r}{}^{(-1)} && \text{Lemma 2.2.3.}
\end{aligned}$$

Then, the right hand side reduces as:

$$\overline{f}\overline{g} = \overline{f} u_\otimes^{r\,(-1)}(\overline{g} u_\otimes^{r\,(-1)} \otimes 1)a_\otimes \qquad \text{in } \mathbb{X} \text{ by definitions}$$

$$= \overline{f}\overline{g} u_\otimes^{r\,(-1)} u_\otimes^{r\,(-1)} a_\otimes \qquad u_\otimes^{r\,(-1)} \text{ natural.}$$

The restriction of the left hand side and the right hand side, in $\mathbb{X}$, is $\overline{\overline{f}\overline{g}}$. This is done by applying Lemma 2.2.3 once on the left and thrice on the right.

Thus, this shows $\overline{\langle f, g\rangle} = \overline{f}\overline{g}$ in $\widetilde{\mathbb{X}}$ where the mediating map in $\mathbb{X}$ is $1 \otimes u_\otimes^r$.

Next, to show $\langle f, g\rangle\pi_0 \leq f$ (and $\langle f, g\rangle\pi_1 \leq g$), it is required to show $\overline{\langle f, g\rangle\pi_0} f = \langle f, g\rangle\pi_0$. Calculating the left side, we see:

$$\begin{aligned}
\overline{\langle f, g\rangle\pi_0} f &= \overline{\langle f, g\rangle\overline{\pi_0}} f && \text{Lemma 2.2.3}\\
&= \overline{\langle f, g\rangle} f && \pi_0 \text{ is total}\\
&= \overline{f}\,\overline{g}\,f && \text{by above}\\
&= \overline{g}\overline{f} f && [\mathbf{R.2}]\\
&= \overline{g} f && [\mathbf{R.1}].
\end{aligned}$$

Now, turning to the right hand side:

$$\langle f, g\rangle\pi_0 = \Delta(f \otimes g)(1 \otimes c_\otimes \otimes 1)1 \qquad \text{in } \mathbb{X}\text{, by definition.}$$

To show these are equal in $\widetilde{\mathbb{X}}$, we need to first show the restrictions are the same in $\mathbb{X}$ and then show there is a mediating map between the images in $\mathbb{X}$. The restriction of $\overline{g}f$ is $\overline{f}\overline{g}$ immediately by [**R.3**] and [**R.2**]. For the right hand side, calculate in $\mathbb{X}$:

$$\begin{aligned}
\overline{\Delta(f \otimes g)(1 \otimes c_\otimes \otimes 1)} &= \overline{\Delta(f \otimes g)} && \text{Lemma 2.2.3}\\
&= \Delta(f \otimes g)(f^{(-1)} \otimes g^{(-1)})\Delta^{(-1)} && \mathbb{X} \text{ is an inverse category}\\
&= \Delta(\overline{f} \otimes \overline{g})\Delta^{(-1)} &&\\
&= \overline{f}\overline{g}\Delta\Delta^{(-1)} && \text{Lemma 4.1.8((ii)) twice}\\
&= \overline{f}\overline{g}. &&
\end{aligned}$$

The diagram below, shows the required mediating map.

$$
\begin{tikzcd}
 & & & A \otimes C \arrow[dr, "\Delta \otimes 1"] & \\
 & & & & A \otimes A \otimes C \arrow[d, dotted, "1 \otimes \Delta"] \\
 & Z \arrow[uurr, "f"] & & & A \otimes A \otimes C \otimes A \otimes C \arrow[d, dotted, "1 \otimes 1 \otimes f^{(-1)}"] \\
Z \arrow[ur, "\overline{g}"] \arrow[dr, "\Delta"'] & & & & A \otimes A \otimes C \otimes Z \arrow[d, dotted, "1 \otimes 1 \otimes 1 \otimes g"] \\
 & Z \otimes Z \arrow[dr, "f \otimes g"'] & & & A \otimes A \otimes C \otimes B \otimes C' \arrow[d, dotted, "1 \otimes 1 \otimes c_{\otimes} \otimes 1"] \\
 & & A \otimes C \otimes B \otimes C' \arrow[dr, "1 \otimes c_{\otimes} \otimes 1"'] & & A \otimes A \otimes B \otimes C \otimes C' \arrow[dl, "\Delta^{(-1)} \otimes 1 \otimes 1 \otimes 1"] \\
 & & & A \otimes B \otimes C \otimes C' &
\end{tikzcd}
$$

□

At this point, we have shown that $\widetilde{\mathbb{X}}$ is a restriction category with restriction products. This leads us to the following theorem:

Theorem 4.2.10. *For any inverse category $\mathbb{X}$, the category $\widetilde{\mathbb{X}}$ is a discrete restriction category.*

Proof. The fact that $\widetilde{\mathbb{X}}$ is a Cartesian restriction category is immediate from Lemmas 4.2.6, 4.2.7, 4.2.8 and 4.2.9.

To show that it is discrete, we need only show that the map $(\Delta {u^r_\otimes}^{(-1)}, 1)$ is in the same equivalence class as $\widetilde{\mathbb{X}}$'s $\Delta (= \langle 1, 1 \rangle = \langle ({u^r_\otimes}^{(-1)}, 1), ({u^r_\otimes}^{(-1)}, 1) \rangle)$. As both Δ and ${u^r_\otimes}^{(-1)}$ are total, the restriction of each side is the same, namely 1. The diagram below uses Corollary

4.2.5 and shows that the two maps are in the same equivalence class.

$$\begin{tikzcd}
 & & A \otimes A \otimes 1 \arrow[d, dotted, "{u_\otimes^{r\,(-1)}}"] \\
A \arrow[urr, "{\Delta u_\otimes^{r\,(-1)}}"] \arrow[rr, "{\Delta(u_\otimes^{r\,(-1)} \otimes u_\otimes^{r\,(-1)})(1 \otimes c_\otimes \otimes 1)}"'] & & A \otimes A \otimes 1 \otimes 1
\end{tikzcd}$$

□

4.2.3 Equivalence of categories

This section will show that the category of discrete inverse categories (maps being restriction functors that preserve the inverse tensor) is equivalent to the category of discrete restriction categories (maps being the restriction functors which preserve the product). In the following, $\mathbb{X}$ will always be a discrete inverse category, $\mathbb{D}$ and $\mathbb{C}$ will be discrete restriction categories.

We approach the equivalence proof by exhibiting the universal property for discrete inverse categories for the functor **INV** from discrete restriction categories to discrete inverse categories. The functor **INV** maps a discrete restriction category to its inverse subcategory and maps functors between discrete restriction categories to a functor having the same action on the partial inverses. That is, given $G : \mathbb{C} \to \mathbb{D}$, then:

$$\mathbf{INV}(G) : \mathbf{INV}(\mathbb{C}) \to \mathbf{INV}(\mathbb{D})$$

$$\mathbf{INV}(G)(A) = GA \qquad \text{(all objects of } \mathbb{D} \text{ are in } Inv(\mathbb{D}))$$

$$\mathbf{INV}(G)(f) = G(f) \qquad \text{(restriction functors preserve partial inverse)}$$

We continue by showing the η and ε of the universal property are isomorphisms. First, let $\eta : \mathbb{X} \to \mathbf{INV}(\widetilde{\mathbb{X}})$ be an identity on objects functor. For maps f in $\mathbb{X}$, $\eta(f) = (f u_\otimes^{r\,(-1)}, 1)$.

Next, consider a functor $F : \mathbb{X} \to \mathbf{INV}(\mathbb{D})$ defined as follows:

Objects: $F^{\#} : A \mapsto F(A)$

Arrows: $F^{\#} : (f, C) \mapsto F(f)\pi_0$

This allows us to write the diagram:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\eta"] \arrow[dr, "F"'] & \mathbf{INV}(\widetilde{\mathbb{X}}) \arrow[d, "\mathbf{INV}(F^{\#})"] \\
& \mathbf{INV}(\mathbb{D})
\end{tikzcd} \tag{4.9}$$

In order to show this is a universal diagram, we proceed with a series of lemmas building to the result.

Lemma 4.2.11. *For any discrete inverse category $\mathbb{X}$, all invertible maps $(g, C) : A \to B$ in $\widetilde{\mathbb{X}}$ are in the equivalence class of $(f u_\otimes^{r\,(-1)}, 1)$ for some $f : A \to B$.*

Proof. As (g, C) is invertible in $\widetilde{\mathbb{X}}$, the map $(g, C)^{(-1)} : B \to A$ exists. $(g, C)^{(-1)}$ must be in the equivalence class of some map $k : B \to A \otimes D$, and also note that $\overline{(g, C)}$ is by construction the equivalence class of the map $\overline{g} u_\otimes^{r\,(-1)} : A \to A \otimes 1$ in $\mathbb{X}$. This means, diagramming in $\mathbb{X}$, there is an n such that

$$\begin{tikzcd}
B \arrow[r, "k"] \arrow[ddddrr, "\overline{k} u_\otimes^{r\,(-1)}"'] & A \otimes D \arrow[r, "f \otimes 1"] & B \otimes C \otimes D \arrow[d, "\Delta \otimes 1"] \\
& & B \otimes B \otimes C \otimes D \arrow[d, dotted, "1 \otimes n"] \\
& & B \otimes B \otimes 1 \arrow[d, "\Delta^{(-1)} \otimes 1"] \\
& & B \otimes 1
\end{tikzcd}$$

commutes.

Starting with $g : A \to B \otimes C$, construct the map f in $\mathbb{X}$ with the following diagram:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[ddddddddr, dotted, "f"'] & B \otimes C \arrow[d, "\Delta \otimes 1"] \\
 & B \otimes B \otimes C \arrow[d, "1 \otimes \Delta \otimes 1"] \\
 & B \otimes B \otimes B \otimes C \arrow[d, "1 \otimes 1 \otimes k \otimes 1"] \\
 & B \otimes B \otimes A \otimes D \otimes C \arrow[d, "1 \otimes 1 \otimes g \otimes 1 \otimes 1"] \\
 & B \otimes B \otimes B \otimes C \otimes D \otimes C \arrow[d, "1 \otimes \Delta^{(-1)} \otimes 1 \otimes c_{\otimes}"] \\
 & B \otimes B \otimes C \otimes C \otimes D \arrow[d, "1 \otimes 1 \otimes \Delta^{(-1)} \otimes 1"] \\
 & B \otimes B \otimes C \otimes D \arrow[d, "1 \otimes n"] \\
 & B \otimes B \otimes 1 \arrow[d, "(\Delta^{(-1)} \otimes 1) u^l_{\otimes}"] \\
 & B
\end{tikzcd}$$

By its construction, $f : A \to B$ in $\mathbb{X}$ and $(f {u^r_{\otimes}}^{(-1)}, 1)$ is in the same equivalence class as (g, C).

$\square$

Lemma 4.2.12. *Diagram ((4.9)) above is a commutative diagram.*

Proof. Chasing maps around the diagram, we have:

$$\begin{tikzcd}
f \arrow[rr, mapsto, "\eta"] \arrow[dr, mapsto, "F"'] & & (f {u^r_{\otimes}}^{(-1)}, 1) \arrow[d, mapsto, "\mathbf{INV}(F^{\#})"] \\
 & F(f) \arrow[r, equal] & F(f {u^r_{\otimes}}^{(-1)}) \pi_0
\end{tikzcd}$$

As η is identity on the objects, Diagram ((4.9)) commutes. $\square$

Lemma 4.2.13. *The functor* **INV** *from the category of discrete restriction categories to the category of discrete inverse categories is full and faithful.*

Proof. To show fullness, we must show **INV** is surjective on hom-sets. Given a functor between two categories in the image of **INV**, i.e., $G : \mathbf{INV}(\mathbb{C}) \to \mathbf{INV}(\mathbb{D})$, construct a functor $H : \mathbb{C} \to \mathbb{D}$ as follows:

Action on objects: $H(A) = G(A)$,

Objects on maps: $H(f) = G(\langle f, 1\rangle)\pi_0$.

H is well defined as we know $\langle f, 1\rangle$ is an invertible map and therefore in the domain of G. To see H is a functor:

$$H(1) = G(\langle 1, 1\rangle)\pi_0 = \Delta_{\mathbb{D}}\pi_0 = 1$$

$$H(fg) = G(\langle fg, 1\rangle)\pi_0 = G(\langle f, 1\rangle)\pi_0 G(\langle g, 1\rangle)\pi_0 = H(f)H(g)$$

But on any invertible map, $H(f) = G(\langle f, 1\rangle)\pi_0 = \langle G(f), 1\rangle\pi_0 = G(f)$ and therefore $\mathbf{INV}(()H) = G$, so **INV** is full.

Next, assume we have $F, G : \mathbb{C} \to \mathbb{D}$ with $\mathbf{INV}(F) = \mathbf{INV}(G)$. Considering $F(f)$ and $F(g)$, we know $F(\langle f, 1\rangle) = G(\langle f, 1\rangle)$ as $\langle f, 1\rangle$ is invertible. Thus, as the functors preserve the product structure, we have

$$F(f) = F(\langle f, 1\rangle)F(\pi_0) = G(\langle f, 1\rangle)G(\pi_0) = G(f).$$

Thus, **INV** is faithful. $\square$

Corollary 4.2.14. *The functor $F^{\#}$ in Diagram (4.9) is unique.*

Proof. This follows immediately from Lemma 4.2.13, **INV** is faithful. $\square$

Corollary 4.2.15. *The category $\widetilde{\mathbb{X}}$ and functor $\eta : \mathbb{X} \to \mathbf{INV}(\widetilde{\mathbb{X}})$ is a universal pair for the functor* **INV**.

Proof. Immediate from Corollary 4.2.14 and Lemma 4.2.12. $\square$

Lemma 4.2.16. *The functor $\eta : \mathbb{X} \to \mathbf{INV}(\widetilde{\mathbb{X}})$ is an isomorphism.*

Proof. As η is an identity on objects functor, we need only show that it is full and faithful. Referring to Lemma 4.2.11 above, we immediately see that η is full. For faithful, if we assume $(fu_{\otimes}^{r\,(-1)}, 1)$ is equal in $\widetilde{\mathbb{X}}$ to $(gu_{\otimes}^{r\,(-1)}, 1)$. This means in $\mathbb{X}$, that $\overline{f} = \overline{g}$ and there is a h such that

$$\begin{tikzcd}
 & B \otimes 1 \arrow[dr, dotted, "(\Delta\otimes 1)\,a_{\otimes}"] & \\
 & & B \otimes (B \otimes 1) \arrow[dd, dotted, "1\otimes h"] \\
A \arrow[uur, "fu_{\otimes}^{r\,(-1)}"] \arrow[ddr, "gu_{\otimes}^{r\,(-1)}"'] & & \\
 & & B \otimes (B \otimes 1) \arrow[dl, dotted, "a_{\otimes}^{(-1)}\,(\Delta^{(-1)}\otimes 1)"] \\
 & B \otimes 1 &
\end{tikzcd}$$

This simplifies out to $g = f\Delta(1 \otimes h)\Delta^{(-1)}$. But by Lemma 4.1.8, part (iv), $\Delta(1 \otimes h)\Delta^{(-1)} = \overline{\Delta(1 \otimes h)\Delta^{(-1)}}$. Setting $\Delta(1 \otimes h)\Delta^{(-1)}$ as k, we have $g = f\overline{k}$. But this gives us:

$$g = f\overline{k} = \overline{fk}f = \overline{f\overline{k}}f = \overline{g}f = \overline{f}f = f.$$

This shows η is faithful and hence an isomorphism between $\mathbb{X}$ and $\mathbf{INV}(\widetilde{\mathbb{X}})$. □

Theorem 4.2.17. *The category of discrete inverse categories (objects are discrete inverse categories, maps are inverse tensor preserving functors) is equivalent to the category of discrete restriction categories (objects are discrete restriction categories, maps are the Cartesian restriction functors).*

Proof. From the above lemmas, we have shown that we have an adjoint:

$$(\eta, \varepsilon) : \mathbf{T} \vdash \mathbf{INV} : D_{ic} \to D_{rc} \tag{4.10}$$

By Lemma 4.2.16 we know η is an isomorphism. But this means the functor $\mathbf{T}$ is full and faithful, as shown in, e.g., Proposition 2.2.6 of [17]. From lemma 4.2.13 we know that $\mathbf{INV}$ is full and faithful. But again by the previous reference, this means ε is an isomorphism. Thus, by Corollary 4.2.15 and Proposition 2.2.7 of [17] we have the equivalence of the two categories. □

4.2.4 Examples of the $\widetilde{(\_)}$ construction

Example 4.2.18 (Completing a finite discrete inverse category)**.**

Continuing from Example 4.1.7, recall the discrete category of 4 elements with two different tensors. Completing these gives two different lattices. They are either the straight line lattice, or the diamond semilattice. Below are the details of these constructions.

Recall $\mathbb{D}$ has four elements a, b, c and d, and there are two possible inverse product tensors:

⊗	a	b	c	d
a	a	a	a	a
b	a	b	**b**	b
c	a	**b**	c	c
d	a	b	c	d

⊗	a	b	c	d
a	a	a	a	a
b	a	b	**a**	b
c	a	**a**	c	c
d	a	b	c	d

Define Δ as the identity map. Then, for the first tensor, $\widetilde{\mathbb{D}}$ has the following maps

$$
\begin{array}{llll}
a \xrightarrow{(id,a)\ \ (\equiv(id,b)\equiv(id,c)\equiv(id,d))} a, & a \xrightarrow{(id,a)} b, & a \xrightarrow{(id,a)} c, & a \xrightarrow{(id,a)} d \\
 & b \xrightarrow{(id,b)\ \ (\equiv(id,c)\equiv(id,d))} b, & b \xrightarrow{(id,b)} c, & b \xrightarrow{(id,b)} d \\
 & & c \xrightarrow{(id,c)\ \ (\equiv(id,d))} c, & c \xrightarrow{(id,c)} d \\
 & & & d \xrightarrow{(id,d)} d
\end{array}
$$

resulting in the straight-line ($a \to b \to c \to d$) lattice. The tensor in $\mathbb{D}$ becomes the meet and hence is a categorical product in $\widetilde{\mathbb{D}}$. Note that the only partial inverses in $\widetilde{\mathbb{D}}$ are the identity functions and that for all maps f, $\langle f, 1\rangle = id$.

With the second tensor table, we have:

$$
\begin{array}{llll}
a \xrightarrow{(id,a)\ \ (\equiv(id,b)\equiv(id,c)\equiv(id,d))} a, & a \xrightarrow{(id,a)} b, & a \xrightarrow{(id,a)} c, & a \xrightarrow{(id,a)} d \\
 & & b \xrightarrow{(id,b)\ \ (\equiv(id,d))} b, & b \xrightarrow{(id,b)} d \\
 & & c \xrightarrow{(id,c)\ \ (\equiv(id,d))} c, & c \xrightarrow{(id,c)} d \\
 & & & d \xrightarrow{(id,d)} d
\end{array}
$$

resulting in the "diamond" lattice, $\begin{tikzcd}[sep=small] & b \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & d \\ & c \arrow[ur] & \end{tikzcd}$. Once again, the tensor in $\mathbb{D}$ is the meet.

Example 4.2.19. Lattice completion. Suppose we have a set together with an idempotent, commutative, associative operation $\wedge$ on the set, giving us a lattice, $\mathbb{L}$. Further suppose the set is partially ordered via $\leq$ with the order being compatible with $\wedge$.

Then, we may create a pullback square for any $x' \leq x,\ y' \leq x$ with

$$\begin{tikzcd} & x & \\ x' \arrow[ur, "\leq"'] & & y' \arrow[ul, "\leq"] \\ & x' \wedge y' \arrow[ul, "\leq"] \arrow[ur, "\leq"'] & \end{tikzcd}$$

Considering $\mathbb{L}$ as a category, we see that all maps are monic and therefore, we may create a partial map category $\mathrm{Par}(\mathbb{L}, \mathcal{M})$ where the stable system of monics are all the maps.

Then $\mathrm{Par}\widetilde{(\mathbb{L}, \mathcal{M})}$ becomes the completion of the lattice over $\wedge$.

4.3 Inverse Turing Categories

4.4 Reversible computation

Bennet, in [7], showed that it was possible to emulate a standard Turing machine via a reversible Turing machine and vice-versa. This showed the equivalence of standard and reversible Turing machines. We reproduce the essence of this proof below.

4.4.1 Reversible Turing machines

Turing machines consist of a tape, a read-write head positioned over the tape, a machine state and a set of instructions. The set of instructions may be given as a set of transitions determining the movement of the read-write head, what it writes and the resulting state of the machine.

Definition 4.4.1. Given an alphabet A which does not contain a space, a tape is in *standard format* when:

[**T.1**] The tape head is positioned directly over a blank space;

[**T.2**] The spaces to the left (the $+1$ direction) contain only elements of A.

[**T.3**] All other spaces of the tape are blank.

Definition 4.4.2. A *turing quintuple* is a quintuple $(s, \alpha, \alpha', \delta, s')$ where:

[**Q.1**] $s, s' \in S$, where S is a predefined set of states;

[**Q.2**] $\alpha, \alpha' \in A$ is predefined set of glyphs;

[**Q.3**] $\delta \in \{-1, 0, 1\}$.

Definition 4.4.3. A *standard turing quintuple set* Q consists of a set of turing quintuples such that:

(i) If $q_1 = (s_1, \alpha_1, \alpha'_1, \delta_1, s'_1)$ and $q_2 = (s_2, \alpha_2, \alpha'_2, \delta_2, s'_2)$ are in Q, then either $s_1 \neq s_2$ or $\alpha_1 \neq \alpha_2$ or both are not equal.

(ii) There are two special quintuples contained in Q:

(a) $(s_1, \text{␣}, \text{␣}, +1, s_2)$[1], the *start quintuple*;

(b) $(s_{t-1}, \text{␣}, \text{␣}, 0, s_t)$, the *end quintuple* where t is the number of states and is the final state of the machine.

Definition 4.4.4. A *standard Turing machine* is given by

[**TM.1**] a standard turing quintuple set;

[**TM.2**] a tape that starts in standard format;

[**TM.3**] and the condition that and if the machine halts, it will halt in state s_t, the final state of the end quintuple and the output will be in standard format.

[1]Here, ␣ is used to signify a blank.

The turing quintuples may also be regarded as giving the data for a partial function in SETS: $\tau : S \times A \to A \times \{-1, 0, 1\} \times S$.

Remark 4.4.5. A multi-tape Turing machine with n tapes and read-write heads can be described by modifying definition 4.4.4 on the previous page such that α is an n-tuple of the set of glyphs for the Turing machine and δ is an n-tuple of movement directions.

Example 4.4.6. Suppose $S = \{start, run, reset, done\}$, $A = \{0, 1, \llcorner\!\lrcorner\}$ and the Turing machine program is given by the quintuples

$$(start, \llcorner\!\lrcorner, \llcorner\!\lrcorner, +1, run),$$
$$(run, 0, 1, +1, run), (run, 1, 0, +1, run),$$
$$(run, \llcorner\!\lrcorner, \llcorner\!\lrcorner, -1, reset),$$
$$(reset, 0, 0, -1, reset), (reset, 1, 1, -1, reset),$$
$$(reset, \llcorner\!\lrcorner, \llcorner\!\lrcorner, 0, done).$$

This program will perform a "bit-flip" of all 0s and 1s on the tape until it reads a space, reposition the read head to the standard format and then it will halt.

As we see in example 4.4.6, it is *possible* that a Turing machine program is reversible. If we had chosen the second quintuple to be $(run, 0, 0, +1, run)$ instead, the program would not have been reversible.

The essential property that a Turing machine program needs to be reversible is that the function τ defined from the quintuples is injective. In order to simplify the discovery the function being injective, we reformulate the turing quintuples as quadruples.

Definition 4.4.7. A *turing quadruple* is given by a quadruple

$$(s, [b_1, b_2, \dots, b_n], [b'_1, b'_2, \dots, b'_n], s')$$

such that:

(i) $s, s' \in S$, some set of states;

(i) $b_j \in A \cup \{\phi\}$ where A is some alphabet;

(i) $b'_j \in A + \{-1, 0, 1\}$;

(i) $b'_j \in \{-1, 0, 1\}$ if and only if $b_j = \phi$.

In this definition, $b_j = \phi$ means that the value of tape j is ignored.

A turing quadruple explicitly splits the read/write action of the Turing machine away from the movement. In a particular step for tape k, the turing machine will either read and write an item or it will move.

Remark 4.4.8. Any turing quintuple q of n tapes may be split into two turing quadruples, q_r and q_m by the addition of a new state a'' in A. The quadruple q_r will consist of all the read-write operations and leave the Turing machine in state a''. The quadruple q_m will start in state a'' with all the b_j set to ϕ and b'_j being movement on each of the n tapes.

Definition 4.4.9. A set of turing quadruples Q is called *reversible set of turing quadruples* when given $q_1, q_2 \in Q$, with $q_1 = (a, [b_j], [b'_j], a')$ and $q_2 = (c, [d_j], [d'_j], c')$:

[**RTM.1**] if $a = c$, then there is a k where $b_k, d_k \in A$ and $b_k \neq d_k$;

[**RTM.2**] if $a' = c'$, then there is a j with $b'_j, d'_j \in A$ and $b'_j \neq d'_j$.

Similarly to turing quintuples, turing quadruples may be taken as the data for a function in Sets:

$$\rho : S \times (A \cup \{\phi\}) \to (A + \{-1, 0, 1\}) \times S.$$

We can see by inspection that ρ is a reversible partial function when the set of turing quadruples that give ρ is a reversible set of turing quadruples.

Definition 4.4.10. A *reversible Turing machine* is one that is described by a set of reversible turing quadruples.

We will show that a reversible Turing machine with three tapes can emulate a Turing machine.

Theorem 4.4.11 (Bennet[7])**.** *Given a standard Turing Machine M, it may be emulated by a three tape reversible Turing machine R. In this case, emulated means:*

(i) M halts on standard input I if and only if R halts on standard input $(I, \llcorner\lrcorner, \llcorner\lrcorner)$.

(i) M halts on standard input I producint standard output O, if and only if R halts on input $(I, \llcorner\lrcorner, \llcorner\lrcorner)$ producing standard output $(I, \llcorner\lrcorner, O)$.

Proof. (Sketch only).

The crux of the proof is to convert the quintuples of M to the quadruples of R as noted in remark 4.4.8 on the previous page. Explicitly for a single tape machine, we have

$$(s, a, a, \delta, s') \mapsto ((s, a, a', s''), (s'', \phi, \delta, s')) . \tag{4.11}$$

In (4.11), s'' is a new state for the machine M, not in the current set of states.

Assign an order to the n quintuples of M, where the start quintuple is the first in the order and the end quintuple comes last. Convert these to quadruples as in (4.11).

We then proceed to create three groups of quadruples for R. We call these *emulation, copy,* and *restore*.

To create the emulation phase quadruples, we examine the pairs of quadruples of M in the sorted order and produce a pair of quadruples for R.

$$
\begin{aligned}
\text{Pair } 1 \quad (s_1, \sqcup, \sqcup, s_1'') &\mapsto (s_1, [\sqcup, \phi, \sqcup], [\sqcup, +1, \sqcup], e_1) \\
(s_1'', \phi, \delta, s_2) &\mapsto (e_1, [\phi, \sqcup, \phi], [\delta, 1, 0], s_2) \\
&\vdots \\
\text{Pair } j \quad (s_k, a_j, a_j', s_k'') &\mapsto (s_k, [a_j, \phi, \sqcup], [a_j', +1, \sqcup], e_j) \\
(s_k'', \phi, \delta, s_i) &\mapsto (e_j, [\phi, \sqcup, \phi], [\delta_j, j, 0], s_i) \\
&\vdots \\
\text{Pair } n \quad (s_\ell, \sqcup, \sqcup, s_\ell'') &\mapsto (s_\ell, [\sqcup, \phi, \sqcup], [\sqcup, +1, \sqcup], e_n) \\
(s_\ell'', \phi, 0, s_f) &\mapsto (e_n, [\phi, \sqcup, \phi], [0, n, 0], s_f).
\end{aligned}
$$

By inspection, one can see that even if the quadruples of M were not a reversible set, the set created for R is a reversible set, due to the writing of the quadruple index on tape 2. Upon completion of the emulation phase, tape 1 will be the same as M would have produced on its single tape, tape 2 will be $[1, 2, \ldots, n]$ and tape 3 will be blanks.

For the copy phase, we create the following quadruples:

$$
\begin{aligned}
&(s_f, [\sqcup, n, \sqcup], [\sqcup, n, \sqcup], c_1) \\
&(c_1, [\phi, \phi, \phi], [+1, 0, +1], c_1') \\
&(c_1', [x, n, \sqcup], [x, n, x], c_1) \quad \text{when } x \neq \sqcup \\
&(c_1', [\sqcup, n, \sqcup], [\sqcup, n, x], c_2) \\
&(c_2, [\phi, \phi, \phi], [-1, 0, -1], c_2') \\
&(c_2', [x, n, x], [x, n, x], c_2) \quad \text{when } x \neq \sqcup \\
&(c_2', [\sqcup, n,] , [\sqcup, n, \sqcup], r_\ell).
\end{aligned}
$$

In these quadruples, the states $\{c_1, c_1', c_2, c_2'\}$ should be chosen to be distinct from the states in the emulation phase. As an example, set them as follows:

$$
c_1 = (\{c\}, s_1) \quad c_1' = (\{c'\}, s_1) \quad c_2 = (\{c\}, s_f) \quad c_1' = (\{c'\}, s_f).
$$

At the completion of this phase, tapes 1 and 2 will be unchanged and tape 3 will be a copy of tape 1.

Finally we perform the restore phase where the history will be erased and tape 1 reset to the input. The quadruples that will accomplish this are:

$$\begin{aligned}
\text{Pair } n \quad & (r_n, [\phi, n, \phi], [0, \llcorner\!\lrcorner, 0], r'_n) \\
& (r'_n, [\llcorner\!\lrcorner, \phi, \lrcorner], [\llcorner\!\lrcorner, -1, \lrcorner], r_{n-1}) \\
& \vdots \\
\text{Pair } j \quad & (r_k, [\phi, j, \phi], [-\delta_j, \llcorner\!\lrcorner, 0], r'_j) \\
& (r'_j, [a'_j, \phi, \lrcorner], [a_j, -1, \lrcorner], r_i) \\
& \vdots \\
\text{Pair } 1 \quad & (r_2, [\phi, 1, \phi], [-1, \llcorner\!\lrcorner, 0], r'_1) \\
& (r'_1, [\llcorner\!\lrcorner, \phi, \lrcorner], [\llcorner\!\lrcorner, -1, \lrcorner], r_1).
\end{aligned}$$

The r states are derived from the s states of the emulation phase.

$$r_j = (\{r\}, s_j) \qquad r'_j = (\{r'\}, s_j).$$

In this restore phase, the indexes of the states r match up to the indexes of states s. The quadruples reverse the actions of the emulate phase on tape 1, erase the history on tape 2 and make no change to tape 3.

$\square$

4.4.2 Reversible automata and linear combinatory algebras

While reversible Turing machines, as described in 4.4.1 on page 97, show that reversible computing is as powerful as standard computing, they do not give us a sense of what may be considered to be happening at a higher level.

To accomplish that task we examine the results of the paper "A Structural Approach to Reversible Computation"[1]. In this paper, Abramsky gives a description of a reversible automaton together with a linear combinatory algebra. We will begin by revisiting some definitions and constructions necessary for discussing automata. The next subsection will introduce combinatory algebras, after which we will describe the reversible automata of [1] and add a short proof that it can emulate a reversible turing machine.

Automata

We will describe the automata as a term-rewriting system. This requires, of course, giving a few basic definitions. See, e.g., [5].

Definition 4.4.12. An *arity* is a function from a function to the natural numbers. The arity of F is the number of inputs (arguments) required by F.

Definition 4.4.13. A *signature* Σ is a set of *function symbols* $F, G, \ldots$, each of which has an arity.

Remark 4.4.14. We refer to functions with low arity in the following ways:

- $Arity = 0$. These are known as *nullary* functions or constants.
- $Arity = 1$. These are known as *unary* functions.
- $Arity = 2$. These are known as *binary* functions.

Definition 4.4.15. A *term alphabet* is a set A containing a signature Σ and a countably infinite set X, the variables. Furthermore, $\Sigma \cap X = \phi$.

Definition 4.4.16. A *term algebra* of the term alphabet $\Sigma \cup X$ is denoted by $T_\Sigma(X)$ and defined as follows:

- $x \in V \implies x \in T_\Sigma$ and
- For any $F \in \Sigma$, with $arity(F) = n$, and $\{t_1, \ldots, t_n\} \subseteq T_\Sigma$, then $F(t_1, \ldots, t_n) \in T_\Sigma$. In the case where $arity(F) = 0$, we write $F \in T_\Sigma$.

Definition 4.4.17. The *ground terms* of a term algebra are those terms that do not contain any variable. The set of these terms is designated as T_Σ.

Remark 4.4.18. Note the ground terms consist of the constants and recursively applying the function symbols of Σ to them.

As we are considering rewrite systems, we will need to consider aspects of substitution and unification.

Definition 4.4.19. A *substitution* is a map $\sigma : T_\Sigma(X) \to T_\Sigma(X)$ which is natural for all function symbols in Σ. In particular if $arity(c) = 0$ then $\sigma(c) = c$.

Note that given the above definition a substitution σ is completely determined by its action on variables. If $\sigma : X \to X$ and is injective, we call σ a renaming. Moreover, if σ restricted to the variables in a term t is an injective map of X on those variables, we call *sigma* a renaming of t.

Substitution allows us to define a partial order on $T_\Sigma(X)$, as follows:

Definition 4.4.20. In $T_\Sigma(X)$, let $\sigma(t) = s$. Then we say s is an *instance* of t, written $s \preceq t$. Moreover, if σ is not just a renaming for t, then we write $s \prec t$. If σ is a renaming of t, we write $s \simeq t$.

Lemma 4.4.21. *Subsumption, as defined in 4.4.20 is a partial order, i.e., it is transitive and reflexive.*

Proof. ◻

Lemma 4.4.22. *Given terms r, t such that there is at least one s with $s \preceq r$ and $s \preceq t$, then there exists a g such that $g \preceq r$ and $g \preceq t$ and for any s' with $s' \preceq r$ and $s' \preceq t$ we will have $s' \preceq g$.*

Proof.

1. Algorithm to compute supremum of p, q terms.

2. Strict $\prec$ has no infinite ascending chains.

3. Shows main part - there exists.

4. Can now show it is unique up to renaming.

$\square$

The subsumption ordering can be used to derive a similar ordering on substitutions:

Definition 4.4.23. $\sigma \preceq \tau$ if and only if there is a ρ with $\sigma = \tau\rho$, where $\tau\rho$ is the diagrammatic order composition of the two substitutions.

Definition 4.4.24. For terms s, t, if $\sigma(t) = \sigma(s)$, then the substitution σ is called a *unifier* for the terms s, t.

Lemma 4.4.25. *If s, t are terms with a unifier σ, there exists a substitution τ that unifies s, t such that $\tau \preceq \rho$ whenever ρ unifies s, t. ρ is called the* most general unifier *of s and t.*

Proof. Follows from 4.4.22 on the preceding page. $\square$

Notation 4.4.26. Following [1], we write $\mathcal{U}(t, u){\downarrow}\sigma$ if σ is the most general unifier of terms t, u.

Combinatory Algebra

Definition 4.4.27. A *combinatory algebra* is an algebra with one binary operation, $\cdot$ written in infix notation. The operation is not assumed to be associative. Multi-element expressions such as $a \cdot b \cdot c$ are to be taken as associating to the left, that is,

$$a \cdot b \cdot c = (a \cdot b) \cdot c.$$

The combinatory algebra may possess distinguished elements that are subject to specific rewrite rules.

Definition 4.4.28. *Combinatory logic* is the combinatory algebra with two distinguished elements, K and S, such that the following hold:

$$\begin{aligned} \mathrm{K} \cdot x \cdot y &= x \\ \mathrm{S} \cdot x \cdot y \cdot z &= x \cdot z \cdot (y \cdot z). \end{aligned}$$

Note that combinatory logic does not require a specific set that must be used for the algebra, simply that it has the two distinguished elements.

Combinatory logic was shown to be equivalent to the λ calculus by

For example, we may define the identity combinator I as $\mathrm{I} = \mathrm{S} \cdot \mathrm{K} \cdot \mathrm{K}$. Further combinators may be defined, such as the B combinator, defined by $\mathrm{B} \cdot a \cdot b \cdot c = a \cdot (b \cdot c)$. The S and K combinators are complete, in that other combinators such as B may be defined from them. E.g., $\mathrm{B} = \mathrm{S} \cdot (\mathrm{K} \cdot \mathrm{S}) \cdot \mathrm{K}$. In fact, we may define an alternate combinatory algebra that is equivalent to Combinatory Logic.

Definition 4.4.29. A *BCKW-Combinatory algebra* is a Combinatory Algebra with four distinguish elements, B, C, K, and W subject to the following equations:

$$\begin{aligned} \mathrm{B} \cdot a \cdot b \cdot c &= a \cdot (b \cdot c) \\ \mathrm{C} \cdot a \cdot b \cdot c &= a \cdot c \cdot b \\ \mathrm{K} \cdot a \cdot b &= a \\ \mathrm{W} \cdot a \cdot b &= a \cdot b \cdot b \end{aligned}$$

In fact, a BCKW-Combinatory algebra is equivalent to a Combinatory logic.

Lemma 4.4.30. *The distinguished elements of a BCKW-Combinatory algebra may be represented by S and K. Conversly, the S and K of a Combinatory logic may be created from B,C,K and W.*

Proof. For the first statement, we have:

$$\begin{aligned}
\mathrm{B} &= \mathrm{S}\cdot(\mathrm{K}\cdot\mathrm{S})\cdot\mathrm{K}\\
\mathrm{C} &= \mathrm{S}\cdot(\mathrm{S}\cdot(\mathrm{K}\cdot(\mathrm{S}\cdot(\mathrm{K}\cdot\mathrm{S})\cdot\mathrm{K}))\cdot\mathrm{S})\cdot(\mathrm{K}\cdot\mathrm{K})\\
\mathrm{K} &= \mathrm{K}\\
\mathrm{W} &= \mathrm{S}\cdot\mathrm{S}\cdot(\mathrm{S}\cdot\mathrm{K}).
\end{aligned}$$

Going the other direction, we have:

$$\begin{aligned}
\mathrm{I} &= \mathrm{W}\cdot\mathrm{K}\\
\mathrm{K} &= \mathrm{K}\\
\mathrm{S} &= \mathrm{B}\cdot(\mathrm{B}\cdot(\mathrm{B}\cdot\mathrm{W})\cdot\mathrm{C})\cdot(\mathrm{B}\cdot\mathrm{B})\text{ and}\\
&= \mathrm{B}\cdot(\mathrm{B}\cdot\mathrm{W})\cdot(\mathrm{B}\cdot\mathrm{B}\cdot\mathrm{C}).
\end{aligned}$$

We show the computations of B and S in detail.

$$\begin{aligned}
\mathrm{B}\cdot a\cdot b\cdot c &= \mathrm{S}\cdot(\mathrm{K}\cdot\mathrm{S})\cdot\mathrm{K}\cdot a\cdot b\cdot c\\
&= (\mathrm{K}\cdot\mathrm{S})\cdot a\cdot(\mathrm{K}\cdot a)\cdot b\cdot c\\
&= \mathrm{S}\cdot(\mathrm{K}\cdot a)\cdot b\cdot c\\
&= \mathrm{K}\cdot a\cdot c\cdot(b\cdot c)\\
&= a\cdot(b\cdot c)
\end{aligned}$$

$$\begin{aligned}
\mathrm{S}\cdot a\cdot b\cdot c &= \mathrm{B}\cdot(\mathrm{B}\cdot\mathrm{W})\cdot(\mathrm{B}\cdot\mathrm{B}\cdot\mathrm{C})\cdot a\cdot b\cdot c\\
&= (\mathrm{B}\cdot\mathrm{W})\cdot((\mathrm{B}\cdot\mathrm{B}\cdot\mathrm{C})\cdot a)\cdot b\cdot c\\
&= \mathrm{B}\cdot\mathrm{W}\cdot((\mathrm{B}\cdot\mathrm{B}\cdot\mathrm{C})\cdot a)\cdot b\cdot c\\
&= \mathrm{W}\cdot(((\mathrm{B}\cdot\mathrm{B}\cdot\mathrm{C})\cdot a)\cdot b)\cdot c\\
&= (((\mathrm{B}\cdot\mathrm{B}\cdot\mathrm{C})\cdot a)\cdot b)\cdot c\cdot c\\
&= \mathrm{B}\cdot\mathrm{B}\cdot\mathrm{C}\cdot a\cdot b\cdot c\cdot c\\
&= \mathrm{B}\cdot(\mathrm{C}\cdot a)\cdot b\cdot c\cdot c\\
&= (\mathrm{C}\cdot a)\cdot(b\cdot c)\cdot c\\
&= \mathrm{C}\cdot a\cdot(b\cdot c)\cdot c\\
&= a\cdot c\cdot(b\cdot c)
\end{aligned}$$

□

If we use the notation $a^n\cdot b$ to mean $a\cdot a\cdot\dots\cdot a\cdot b$ where a is repeated n times, then we can terms which correspond to the Church numbers of lambda calculus:

$$\bar{n}\equiv(\mathrm{S}\cdot\mathrm{B})^n\cdot(\mathrm{K}\cdot\mathrm{I})$$

Definition 4.4.31. A partial function $f : \mathbb{N} \to \mathbb{N}$ is *representable* in combinatory logic if there is a term M_f such that $M_f \cdot \bar{n} = \bar{m}$ whenever $f(n) = m$ and $M_f \cdot \bar{n}$ does not have a normal form if $f(n) \uparrow$.

When we say that combinatory logic with S and K is complete, we mean the following theorem:

Theorem 4.4.32. *The partial functions that are representable in combinatory logic are exactly the partial recursive functions.*

Linear Combinatory Algebra

Definition 4.4.33. A *Linear Combinatory Algebra* $(A, \cdot, !)$ is an algebra A with an applicative binary operation $\cdot$, an unary operator $! : A \to A$ and eight distinquished elements: B, C, I, K, D, δ, F and W in A which satisfy the following rules:

1. $\mathrm{B} \cdot a \cdot b \cdot c \quad = a \cdot (b \cdot c)$
2. $\mathrm{C} \cdot a \cdot b \cdot c \quad = a \cdot c \cdot b$
3. $\mathrm{I} \cdot a \quad = a$
4. $\mathrm{K} \cdot a \cdot !b \quad = a$
5. $\mathrm{D} \cdot !a \quad = a$
6. $\delta \cdot !a \quad = !!a$
7. $\mathrm{F} \cdot !a \cdot !b \quad = !(a \cdot b)$
8. $\mathrm{W} \cdot a \cdot !b \quad = a \cdot !b \cdot !b$

Note that a Linear Combinatory Algebra always contains a BCKW-Combinatory algebra.

Define $\mathrm{D}' = \mathrm{C} \cdot (\mathrm{B} \cdot \mathrm{B} \cdot \mathrm{I}) \cdot (\mathrm{B} \cdot \mathrm{D} \cdot \mathrm{I})$ and the binary operator $\bullet$ on A such that $a \bullet b \equiv a \cdot !b$. Then, define the following:

$$\begin{aligned}
\mathrm{B}_s &= \mathrm{C} \cdot (\mathrm{B} \cdot (\mathrm{B} \cdot \mathrm{B} \cdot \mathrm{B}) \cdot (\mathrm{D}' \cdot \mathrm{I})) \cdot (\mathrm{C} \cdot ((\mathrm{B} \cdot \mathrm{B}) \cdot \mathrm{F}) \cdot \delta) \\
\mathrm{C}_s &= \mathrm{D}' \cdot \mathrm{C} \\
\mathrm{K}_s &= \mathrm{D}' \cdot \mathrm{K} \\
\mathrm{W}_s &= \mathrm{D}' \cdot \mathrm{W}.
\end{aligned}$$

Lemma 4.4.34. *Given and Linear Combinatory Algebra $(A, \cdot, !)$, then $(A, \bullet)$ is a BCKW-Combinatory algebra with B, C, K, W set to B_s, C_s, K_s, W_s from above.*

Proof. We show the calculation for K_s, the others are similar.

$$\begin{aligned}
\mathrm{K}_s \bullet a \bullet b &\equiv \mathrm{D}' \cdot \mathrm{K} \cdot !a \cdot !b \\
&= \mathrm{C} \cdot (\mathrm{B} \cdot \mathrm{B} \cdot \mathrm{I}) \cdot (\mathrm{B} \cdot \mathrm{D} \cdot \mathrm{I}) \cdot \mathrm{K} \cdot !a \cdot !b \\
&= (\mathrm{B} \cdot \mathrm{B} \cdot \mathrm{I} \cdot \mathrm{K}) \cdot (\mathrm{B} \cdot \mathrm{D} \cdot \mathrm{I}) \cdot !a \cdot !b \\
&= \mathrm{B} \cdot (\mathrm{I} \cdot \mathrm{K}) \cdot (\mathrm{B} \cdot \mathrm{D} \cdot \mathrm{I}) \cdot !a \cdot !b \\
&= (\mathrm{I} \cdot \mathrm{K}) \cdot ((\mathrm{B} \cdot \mathrm{D} \cdot \mathrm{I}) \cdot !a) \cdot !b \\
&= \mathrm{K} \cdot ((\mathrm{B} \cdot \mathrm{D} \cdot \mathrm{I}) \cdot !a) \cdot !b \\
&= (\mathrm{B} \cdot \mathrm{D} \cdot \mathrm{I}) \cdot !a \\
&= \mathrm{D} \cdot (\mathrm{I} \cdot !a) \\
&= \mathrm{D} \cdot !a \\
&= a
\end{aligned}$$

$\square$

Chapter 5

Frobenius Algebras and Quantum Computation

5.1 Definition of a Frobenius Algebra

In their most general setting, Frobenius algebras are defined as a finite dimensional algebra over a field together with a non-degenerate pairing operation. We will continue with the definitions that make this precise.

5.1.1 Frobenius algebra definitions

Definition 5.1.1 (Frobenius algebra)**.** Given a symmetric monoidal category $\mathbb{D}$, a *Frobenius algebra* is an object X of $\mathbb{D}$ and four maps, $\nabla : X \otimes X \to X$, $e : I \to X$, $\Delta : X \to X \otimes X$ and $\epsilon : X \to I$, with the conditions that (X, ∇, e) forms a commutative monoid, (X, Δ, ϵ) forms a commutative comonoid and the diagram

$$\begin{tikzcd}
X \otimes X \arrow[rr, "X\otimes\Delta"] \arrow[dd, "\Delta\otimes X"'] \arrow[dr, "\nabla"] & & X \otimes X \otimes X \arrow[dd, "\nabla\otimes X"] \\
& X \arrow[dr, "\Delta"] & \\
X \otimes X \otimes X \arrow[rr, "X\otimes\nabla"'] & & X \otimes X
\end{tikzcd}$$

commutes. The Frobenius algebra is *special* when $\Delta;\nabla = 1_X$ and *commutative* when $\Delta; c_{X,X} = \Delta$.

Definition 5.1.2 (†-Frobenius algebra)**.** A Frobenius algebra in a dagger symmetric monoidal category where $\Delta = \nabla^\dagger$ and $\epsilon = u^\dagger$ is a †-*Frobenius algebra.*

For an example of a †-Frobenius algebra, consider a finite dimensional Hilbert space H with an orthonormal basis $\{|\phi_i\rangle\}$ and define $\Delta : H \to H \otimes H : |\phi_i\rangle \mapsto |\phi_i\rangle \otimes |\phi_i\rangle$ and

$\epsilon : H \to \mathbb{C} : |\phi_i\rangle \mapsto 1$. Then $(H, \nabla = \Delta^\dagger, u = \epsilon^\dagger, \Delta, \epsilon)$ forms a commutative special †-Frobenius algebra.

5.2 The category of Commutative Frobenius Algebras

Example 5.2.1 (Commutative Frobenius algebras)**.** Let $\mathbb{X}$ be a symmetric monoidal category and form CFrob($\mathbb{X}$) as follows:

Objects: Commutative Frobenius algebras[13]: A quintuple $(X, \nabla, \eta, \Delta, \epsilon)$ where X is a k-algebra for some field k, and $\nabla : A \otimes A \to A$, $\eta : k \to A$, $\Delta : A \to A \otimes A$, $\epsilon : A \to k$ are natural maps in the algebra. Additionally, these satisfy

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\Delta\otimes 1"] \arrow[dr, "\nabla"] \arrow[dd, "1\otimes\Delta"'] & & A \otimes (A \otimes A) \arrow[dd, "1\otimes\nabla"] \\
& A \arrow[dr, "\Delta"] & \\
(A \otimes A) \otimes A \arrow[rr, "\nabla\otimes 1"'] & & A \otimes A
\end{tikzcd}$$

together with the additional property that $\Delta\nabla = 1$.

Maps: Multiplication (∇) and co-multiplication (Δ) preserving homomorphisms which do not necessarily preserve the unit.

Theorem 5.2.2. *When $\mathbb{X}$ is a symmetric monoidal category, CFrob($\mathbb{X}$) is a discrete inverse category.*

Proof. For $f : X \to Y$, define $f^{(-1)}$ as

$$Y \xrightarrow{1\otimes\eta} Y \otimes X \xrightarrow{1\otimes\Delta} Y \otimes X \otimes X \xrightarrow{1\otimes f\otimes 1} Y \otimes Y \otimes X \xrightarrow{\nabla\otimes 1} Y \otimes X \xrightarrow{\epsilon\otimes 1} X$$

Using a result from [18], we need only show:

$$\left(f^{(-1)}\right)^{(-1)} = f$$

$$ff^{(-1)}f = f$$

$$ff^{(-1)}gg^{(-1)} = gg^{(-1)}ff^{(-1)}$$

We also use the following two identities from [13]:

$$(1 \otimes \eta)\nabla = id \tag{5.1}$$

$$\Delta(1 \otimes \epsilon) = id. \tag{5.2}$$

$$\begin{aligned}
f^{(-1)^{(-1)}} &= (1 \otimes \eta)(1 \otimes \Delta)(1 \otimes (f^{(-1)}) \otimes 1)(\nabla \otimes 1)(\epsilon \otimes 1) \\
&= (1 \otimes \eta)(1 \otimes \Delta)(1 \otimes ((1 \otimes \eta)(1 \otimes \Delta)(1 \otimes f \otimes 1)(\nabla \otimes 1)(\epsilon \otimes 1)) \otimes 1) \\
&\qquad\quad (\nabla \otimes 1)(\epsilon \otimes 1) \\
&= (1 \otimes \eta)(1 \otimes \Delta)(1 \otimes 1 \otimes \eta)(1 \otimes 1 \otimes f \otimes 1 \otimes 1)(1 \otimes \nabla \otimes 1 \otimes 1) \\
&\qquad\quad (1 \otimes \epsilon \otimes 1 \otimes 1)(\nabla \otimes 1)(\epsilon \otimes 1) \\
&= (\eta \otimes 1)(\Delta \otimes 1)(1 \otimes \nabla)(f \otimes 1)(((\eta)(\Delta \otimes 1)(1 \otimes \nabla)(1 \otimes \epsilon)) \otimes 1)((1 \otimes \epsilon) \\
&= (1 \otimes \eta)\nabla\Delta(1 \otimes \epsilon)f(\eta \otimes 1)\nabla\Delta(1 \otimes \epsilon) \\
&= id_x id_x \ f \ id_y id_y \\
&= f
\end{aligned}$$

$$
\begin{aligned}
ff^{(-1)}f &= f(1\otimes\eta)(1\otimes\Delta)(1\otimes f\otimes 1)(\nabla\otimes 1)(\epsilon\otimes 1)f \\
&= (1\otimes\eta)(1\otimes\Delta)(f\otimes f\otimes 1)(\nabla\otimes 1)(1\otimes f)(\epsilon\otimes 1) \\
&= (1\otimes\eta)(1\otimes\Delta)(\nabla\otimes 1)(f\otimes f)(\epsilon\otimes 1) \\
&= (1\otimes\eta)\nabla\Delta(f\otimes f)(\epsilon\otimes 1) \\
&= \Delta(f\otimes f)(\epsilon\otimes 1) \\
&= f\Delta(\epsilon\otimes 1) \\
&= f
\end{aligned}
$$

Finally, to show $ff^{(-1)}$ and $gg^{(-1)}$ commute:

$$
\begin{aligned}
&f(1\otimes\eta)(1\otimes\Delta)(1\otimes f\otimes 1)(\nabla\otimes 1)(\epsilon\otimes 1)g(1\otimes\eta)(1\otimes\Delta)(1\otimes g\otimes 1)(\nabla\otimes 1)(\epsilon\otimes 1) \\
&\quad = (1\otimes\eta)(1\otimes\Delta)(\nabla\otimes 1)(f\otimes 1)(\epsilon\otimes 1)(1\otimes\eta)(1\otimes\Delta)(\nabla\otimes 1)(g\otimes 1)(\epsilon\otimes 1) \\
&\quad = (1\otimes\eta)\nabla\Delta(f\otimes 1)(\epsilon\otimes 1)(1\otimes\eta)\nabla\Delta(g\otimes 1)(\epsilon\otimes 1) \\
&\quad = \Delta(f\otimes 1)(\epsilon\otimes 1)\Delta(g\otimes 1)(\epsilon\otimes 1) \\
&\quad = \Delta(1\otimes\Delta)(f\otimes g\otimes 1)(\epsilon\otimes\epsilon\otimes 1) \\
&\quad = \Delta(1\otimes\Delta)(g\otimes f\otimes 1)(\epsilon\otimes\epsilon\otimes 1) \qquad \text{co-commutativity} \\
&\quad = gg^{(-1)}ff^{(-1)}
\end{aligned}
$$

□

5.3 Bases and Frobenius Algebras

In [10], the authors provide an algebraic description of orthogonal bases in finite dimensional Hilbert spaces. As noted in section 5.2, an orthonormal basis for such a space is a special commutative †-Frobenius algebra. To show the other direction, given a commutative †-Frobenius algebra, (H, ∇, u) and for each element $\alpha \in H$, define the right action of α as $R_\alpha := (id \otimes \alpha)\,\nabla : H \to H$. Note the use of the fact that elements $\alpha \in H$ can be considered as linear maps $\alpha : \mathbb{C} \to H : 1 \mapsto |\alpha\rangle$. The dagger of a right action is also a right action, ${R_\alpha}^\dagger = R_{\alpha'}$ where $\alpha' = u\,\nabla\,(id \otimes \alpha^\dagger)$, which is a consequence of the Frobenius identities.

The $(\_)'$ construction is actually an involution:

$$\begin{aligned}
(\alpha')' &= u\nabla(id \otimes {\alpha'}^\dagger) \\
&= u\nabla(id \otimes (u\nabla(id \otimes \alpha^\dagger))^\dagger \\
&= u\nabla(id \otimes ((id \otimes \alpha)\Delta\epsilon)) \\
&= (u \otimes \alpha)(\nabla \otimes id)(id \otimes \Delta)(id \otimes \epsilon) \\
&= (u \otimes \alpha)(id \otimes \Delta)(\nabla \otimes id)(id \otimes \epsilon) \\
&= (u \otimes \alpha)(id \otimes \epsilon) \\
&= \alpha
\end{aligned}$$

Lemma 5.3.1. *Any †-Frobenius algebra in* FdHilb *is a C^*-algebra.*

Proof. The endomorphism monoid of FdHilb (H,H) is a C^*-algebra. From the proceeding,

$$H \cong \text{FdHilb}(\mathbb{C}, H) \cong R_{[\text{FdHilb}(\mathbb{C},H)]} \subseteq \text{FdHilb}(H, H).$$

This inherits the algebra structure from FdHilb (H,H). Furthermore, since any finite dimensional involution-closed subalgebra of a C^*-algebra is also a C^*-algebra, this shows the †-Frobenius algebra is a C^*-algebra. □

Using the fact that the involution preserving homomorphisms from a finite dimensional commutative C^*-algebra to $\mathbb{C}$ form a basis for the dual of the underlying vector space, write these homomorphisms as ${\phi_i}^\dagger : H \to \mathbb{C}$. Then their adjoints, $\phi_i : \mathbb{C} \to H$ will form a basis for the space H. These are the copyable elements in H.

This, together with continued applications of the Frobenius rules and linear algebra allow the authors to prove:

Theorem 5.3.2. *Every commutative †-Frobenius algebra in* FdHilb *determines an orthogonal basis consisting of its copyable elements. Conversely, every orthogonal basis* $\{|\phi_i\rangle\}_i$

determines a commutative †-Frobenius algebra via

$$\Delta : H \to H \otimes H : |\phi_i\rangle \mapsto |\phi_i\rangle \otimes |\phi_i\rangle \qquad \epsilon : H \to \mathbb{C} : |\phi_i\rangle \mapsto 1$$

and these constructions are inverse to each other.

5.3.1 Quantum and classical data

In [9], the authors build on the results above, to start from a †-symmetric monoidal category and construct the minimal machinery needed to model quantum and classical computations. For the rest of this section, $\mathbb{D}$ will be assumed to be such a category, with $\otimes$ the monoid tensor and I the unit of the monoid.

Definition 5.3.3. A compact structure on an object A in the category $\mathbb{D}$ is given by the object A, an object A^* called its *dual* and the maps $\eta : I \to A^* \otimes A$, $\epsilon : A \otimes A^* \to I$ such that the diagrams

$$\begin{tikzcd} A^* \arrow[d, "\eta\otimes A^*"'] \arrow[dr, "id"] & \\ A^* \otimes A \otimes A^* \arrow[r, "A^*\otimes\epsilon"'] & A^* \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "A\otimes\eta"] \arrow[dr, "id"'] & A \otimes A^* \otimes A \arrow[d, "\epsilon\otimes A"] \\ & A \end{tikzcd}$$

commute.

Definition 5.3.4 (Quantum Structure)**.** A *quantum structure* is an object A and map $\eta : I \to A \otimes A$ such that $(A, A, \eta, \eta^\dagger)$ form a compact structure.

Note that A is self-dual in definition 5.3.4.

This allows the creation of the category $\mathbb{D}_q$ which has as objects quantum structures and maps are the maps in $\mathbb{D}$ between the objects in the quantum structures.

In the category $\mathbb{D}_q$, it is now possible to define the upper and lower $*$ operations on maps, such that $(f_*)^* = (f^*)_* = f^\dagger$.

$$\begin{aligned} f^* &:= (\eta_A \otimes 1)(1 \otimes f \otimes 1)(1 \otimes \eta_B^\dagger) \\ f_* &:= (\eta_B \otimes 1)(1 \otimes f^\dagger \otimes 1)(1 \otimes \eta_A^\dagger) \end{aligned}$$

Interestingly, $\mathbb{D}_q$ possesses enough structure to be axiomatized in the same manner as above in section 3.4.2, excepting the portions dependent upon biproducts.

Next, define a classical structure on $\mathbb{D}$.

Definition 5.3.5 (Classical structure)**.** A *classical structure* in $\mathbb{D}$ is an objects X and two maps, $\Delta : X \to X \otimes X$, $\epsilon : X \to I$ such that $X, \Delta^\dagger, \epsilon^\dagger, \Delta, \epsilon$ forms a special Frobenius algebra.

As above, this allows us to define $\mathbb{D}_c$, the category whose objects are the classical structures of $\mathbb{D}$ with maps between classical structures being the maps in $\mathbb{D}$ between the objects of the classical structure.

Note that a classical structure will induce a quantum structure, setting η_X to be ${\epsilon_X}^\dagger \Delta_X$.

Chapter 6

Inverse sum categories

Remark 6.0.6. Throughout this chapter, we will work with a number of relations between maps and operations on pairs of maps. Suppose we have a relation $\Diamond$ between maps $f, g : B \to C$, i.e., $f \Diamond g$. We will refer to $\Diamond$ as *stable* whenever given a $h : A \to B$, then $hf \Diamond hg$. We will refer to $\Diamond$ as *universal* whenever given a $k : C \to D$, then $fk \Diamond gk$.

6.1 Coproducts in restriction categories

6.1.1 Coproducts

Restriction categories may also have coproducts and initial objects.

Definition 6.1.1. In a restriction category $\mathbb{X}$, a coproduct is a *restriction coproduct* when the embeddings $\amalg_1$ and $\amalg_2$ are total.

Lemma 6.1.2. *The definition of restriction coproduct implies the following:*

(i) $\overline{f+g} = \overline{f} + \overline{g}$ *which means* $+$ *is a restriction functor.*

(ii) $\nabla : A + A \to A$ *is total.*

(iii) $? : 0 \to A$ *is total, where* 0 *is the initial object in the restriction category.*

Proof.

(i) $+$ **is a restriction functor.** Consider the diagram:

$$\begin{tikzcd}
A \arrow[rr, "f"'] \arrow[dr, "\amalg_1"'] & & A' \arrow[dr, "\amalg'_1"] & \\
& A+B \arrow[rr, dotted, "f+g"] & & A'+B' \\
B \arrow[rr, "g"'] \arrow[ur, "\amalg_2"] & & B' \arrow[ur, "\amalg'_2"'] &
\end{tikzcd}$$

In order to show $\overline{f+g} = \overline{f} + \overline{g}$, it suffices to show that $\amalg_1 \overline{f+g} = \amalg_1(\overline{f} + \overline{g}) = \overline{f}\amalg_1$.

$$\begin{aligned}
\amalg_1 \overline{f+g} &= \overline{\amalg_1(f+g)}\amalg_1 && [\textbf{R.4}] \\
&= \overline{f\amalg'_1}\amalg_1 && \text{coproduct diagram} \\
&= \overline{f\overline{\amalg'_1}}\amalg_1 && \text{Lemma 2.2.3}[(iii)] \\
&= \overline{f}\amalg_1 && \amalg'_1 \text{ total}
\end{aligned}$$

(ii) $\nabla : A + A \to A$ **is total.** By the definition of ∇ $(= \langle 1|1 \rangle)$ and the co-product, the following diagram commutes,

$$\begin{tikzcd}
& A+A \arrow[d, "\nabla"'] & \\
A \arrow[ur, "\amalg_1"] \arrow[r, equal] & A \arrow[r, equal] & A \arrow[ul, "\amalg_2"']
\end{tikzcd}$$

resulting in:

$$\begin{aligned}
\amalg_1 \overline{\nabla} &= \overline{\amalg_1 \nabla}\amalg_1 \\
&= \overline{1}\amalg_1 \\
&= \amalg_1
\end{aligned}$$

Similarly, $\amalg_2 \overline{\nabla} = \amalg_2$, hence, the restriction of ∇ is 1 and therefore ∇ is total.

(iii) $? : 0 \to A$ **is total.** This follows from

$$\begin{tikzcd} 0 \arrow[r, "\amalg_2"] \arrow[dr, "?"'] & A+0 \arrow[d, equal] \\ & A \end{tikzcd}$$

so ? can be defined as the total coproduct injection.

$\square$

Recall that when an object is both initial and terminal, it is referred to as a zero object and denoted as 0. This gives rise to the zero map $0_{A,B} : A \to 0 \to B$ between any two objects.

Definition 6.1.3. Given a restriction category $\mathbb{X}$ with a zero object, then 0 is a *restriction zero* when for each object A in $\mathbb{X}$ we have $\overline{0_{A,A}} = 0_{A,A}$.

Lemma 6.1.4 (Cockett-Lack)**.** *For a restriction category $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ has a restriction zero;

(ii) $\mathbb{X}$ has an initial object 0 and terminal object 1 and each initial map z_A is a restriction monic;

(iii) $\mathbb{X}$ has a terminal object 1 and each terminal map t_A is a restriction retraction.

6.1.2 Inverse categories with restriction coproducts

Proposition 6.1.5. *An inverse category $\mathbb{X}$ with restriction coproducts is a preorder.*

Proof. By Lemma 6.1.2, we know ∇ is total and therefore $\nabla\nabla^{(-1)} = 1$. From the coproduct diagrams, we have $\amalg_1\nabla = 1$ and $\amalg_2\nabla = 1$. But this gives us $\nabla^{(-1)}{\amalg_1}^{(-1)} = (\amalg_1\nabla)^{(-1)} = 1$ and similarly $\nabla^{(-1)}{\amalg_2}^{(-1)} = 1$. Hence, $\nabla^{(-1)} = \amalg_1$ *and* $\nabla^{(-1)} = \amalg_2$.

This means for parallel maps $f, g : A \to B$, we have

$$f = \amalg_1\langle f|g\rangle = \nabla^{(-1)}\langle f|g\rangle = \amalg_2\langle f|g\rangle = g$$

and therefore $\mathbb{X}$ is a preorder. $\square$

6.2 Disjointness in an inverse category

In the following, we will add two related structures to an inverse category with a restriction zero. This structure is meant to be evocative of the concept of *join* in a restriction category. We will first re-iterate some of the basic definitions and lemmas of join.

6.2.1 Joins in restriction categories

Definition 6.2.1. Given $\mathbb{R}$ is a restriction category with a restriction zero, then $\mathbb{R}$ is said to have *joins* whenever there is an operator $\vee$ defined between compatible maps (from Definition 4.1.1 on page 64) such that:

- $f \leq f \vee g$ and $g \leq f \vee g$,
- $\overline{f \vee g} = \overline{f} \vee \overline{g}$,
- $f, g \leq h$ implies that $f \vee g \leq h$ and
- $h(f \vee g) = hf \vee hg$.

For example, in the restriction category $\textsc{Par}$, the join is given by:

$$(f \vee g)(x) = \begin{cases} f(x)(= g(x)) & \text{when both } f \text{ and } g \text{ are defined;} \\ f(x) & \text{when only } f \text{ is defined;} \\ g(x) & \text{when only } g \text{ is defined;} \\ \uparrow & \text{when both } f \text{ and } g \text{ are undefined.} \end{cases}$$

Lemma 6.2.2. *If $\mathbb{R}$ is a meet restriction category with joins, then the meet distributes over the join, i.e.,*

$$h \cap (f \vee g) = (h \cap f) \vee (h \cap g).$$

Proof.

$$\begin{aligned} h \cap (f \vee g) &= \overline{(f \vee g)} h \cap (f \vee g) \\ &= (\overline{f} \vee \overline{g}) h \cap (f \vee g) \\ &= (\overline{f}(h \cap (f \vee g))) \vee (\overline{g}(h \cap (f \vee g))) \\ &= (h \cap \overline{f}(f \vee g)) \vee (h \cap \overline{g}(f \vee g))) \\ &= (h \cap (f \vee g)) \vee (h \cap (f \vee g))). \end{aligned}$$

□

6.2.2 Disjointness relations

In this subsection, we will define a disjointness relationship between maps and explore alternate characterizations of this relation on the restriction idempotents of objects.

Definition 6.2.3. In an inverse category $\mathbb{X}$ with a restriction zero, the relation $\perp$ between two parallel maps $f, g : A \to B$ is called a *disjointess relation* when it satisfies the following properties:

- [**Dis.1**] For all $f : A \to B,\ f \perp 0$;
- [**Dis.2**] $f \perp g$ implies $\overline{f} g = 0$;
- [**Dis.3**] $f \perp g,\ f' \leq f,\ g' \leq g$ implies $f' \perp g'$;
- [**Dis.4**] $f \perp g$ implies $g \perp f$;
- [**Dis.5**] $f \perp g$ implies $hf \perp hg$; (Stable)
- [**Dis.6**] $f \perp g$ implies $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$.
- [**Dis.7**] $\overline{f} \perp \overline{g},\ \hat{h} \perp \hat{k}$ implies $fh \perp gk$;

Lemma 6.2.4. *In Definition 6.2.3, provided we retain [**Dis.1-5**], we may replace [**Dis.6**] and [**Dis.7**] by:*

[***Dis.6'***] *$f \perp g$ if and only if $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$.*

Proof. Given [**Dis.6**] and [**Dis.7**], the *only if* direction of [**Dis.6'**] is immediate. To show the *if* direction, assume $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$. This also means that $\overline{\overline{f}} \perp \overline{\overline{g}}$. Then, by [**Dis.7**], $\overline{f}f \perp \overline{g}g$ and therefore $f \perp g$.

Conversely, assume we are given [**Dis.6'**]. Then, [**Dis.6**] follows immediately. To show [**Dis.7**], assume we have $\overline{f} \perp \overline{g}$, $\hat{h} \perp \hat{k}$. As $\overline{fh} \leq \overline{f}$ and $\overline{gk} \leq \overline{g}$, by [**Dis.3**], we know that $\overline{fh} \perp \overline{gk}$. Similarly, $\widehat{fh} \leq \hat{h}$ and $\widehat{gk} \leq \hat{k}$, giving us $\widehat{fh} \perp \widehat{gk}$. Then, from [**Dis.6'**] we may conclude $fh \perp gk$, showing [**Dis.7**] holds. $\square$

Lemma 6.2.5. *In an inverse category* $\mathbb{X}$ *with* $\perp$ *a disjointness relation:*

(i) $f \perp g$ *if and only if* $f^{(-1)} \perp g^{(-1)}$*;*

(ii) $f \perp g$ *implies* $fh \perp gh$ *(Universal);*

(iii) $f \perp g$ *implies* $f\hat{g} = 0$*;*

(iv) *if* m, n *are monic, then* $fm \perp gn$ *implies* $\overline{f} \perp \overline{g}$*;*

(v) *if* m, n *are monic, then* $m^{(-1)}f \perp n^{(-1)}g$ *implies* $\hat{f} \perp \hat{g}$*;*

Proof.

(i) Assume $f \perp g$. Then we know that $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$. But since $\hat{f} = \overline{f^{(-1)}}$ and $\overline{f} = \widehat{f^{(-1)}}$, this means $\overline{f^{(-1)}} \perp \overline{g^{(-1)}}$ and $\widehat{f^{(-1)}} \perp \widehat{g^{(-1)}}$ and again by the first item of this lemma, we have $f^{(-1)} \perp g^{(-1)}$. The converse follows with a similar argument.

(ii) Assume $f \perp g$. By the previous item, we have $f^{(-1)} \perp g^{(-1)}$. By [**Dis.5**], $h^{(-1)}f^{(-1)} \perp h^{(-1)}g^{(-1)}$, giving us $(fh)^{(-1)} \perp (gh)^{(-1)}$. Again by the previous item, we now have $fh \perp gh$.

(iii) Assume $f \perp g$. From item (i) and reflexivity, we know that $g^{(-1)} \perp f^{(-1)}$ and therefore $\overline{g^{(-1)}}f^{(-1)} = \hat{g}f^{(-1)} = 0$. However, in an inverse category, $0^{(-1)} = 0$ and therefore $0 = (\hat{g}f^{(-1)})^{(-1)} = f\hat{g}^{(-1)} = f\hat{g}$.

(iv) Assume $fm \perp gn$ where m, n are monic. By [**Dis.6**], this gives us $\overline{fm} \perp \overline{gn}$. By Lemma 2.2.3, $\overline{fm} = \overline{f\overline{m}} = \overline{f1} = \overline{f}$ and therefore $\overline{f} \perp \overline{g}$.

(v) This is a corollary to the previous item. By assumption, we have $m^{(-1)}f \perp n^{(-1)}g$ and therefore $f^{(-1)}m \perp g^{(-1)}n$. By the previous item, this means $\overline{f^{(-1)}} \perp \overline{g^{(-1)}}$ and hence $\hat{f} \perp \hat{g}$.

$\square$

We may define the disjointness relation via its action in $\mathcal{O}(a)$.

Definition 6.2.6. Given an inverse category $\mathbb{X}$, a relation $\perp_A \subseteq \mathcal{O}(A)^2$ for each $A \in \text{ob}(\mathbb{X})$, is an *open disjointness* relation when for all $e, e' \in \mathcal{O}(A)$

[$\mathcal{O}$**dis.1**] $1 \perp_A 0$;

[$\mathcal{O}$**dis.2**] $e \perp_A e'$ implies $e' \perp_A e$;

[$\mathcal{O}$**dis.3**] $e \perp_A e'$ implies $ee' = 0$;

[$\mathcal{O}$**dis.4**] $e \perp_A e'$ implies $\overline{fe} \perp_B \overline{fe'}$ for all $f : B \to A$;

[$\mathcal{O}$**dis.5**] $e \perp_A e'$ implies $\widehat{eg} \perp_C \widehat{e'g}$ for all $g : A \to C$;

[$\mathcal{O}$**dis.6**] $e \perp_A e',\ e_1 \leq e,\ e_1' \leq e'$ implies $e_1 \perp_A e_1'$.

We will normally write $\perp$ rather than $\perp_A$ where the object is either clear or not germane to the point under discussion.

Proposition 6.2.7. *If $\perp$ is a disjointness relation in $\mathbb{X}$, it is an open disjointness relation on the restriction idempotents.*

Proof.

[$\mathcal{O}$**dis.1**] This follows immediately from [**Dis.1**] by taking $f = 1$.

[$\mathcal{O}$**dis.2**] Reflexivity follows directly from [**Dis.4**].

[$\mathcal{O}$**dis.3**] By [**Dis.2**], $0 = \overline{e}e' = ee'$.

[$\mathcal{O}$**dis.4**] Given $e \perp e'$, we have $fe \perp fe'$ by [**Dis.5**]. Then, by [**Dis.6**] we may conclude $\overline{fe} \perp \overline{fe'}$.

[$\mathcal{O}$**dis.5**] This follows from the above item, using $g^{(-1)}$ for f. This means we have $\overline{g^{(-1)}e} \perp \overline{g^{(-1)}e'}$. But this gives us $\overline{(eg)^{(-1)}} \perp \overline{(e'g)^{(-1)}}$. Recalling from Lemma 2.2.12 that $\hat{k} = \overline{k^{(-1)}}$, we may conclude $\widehat{eg} \perp \widehat{e'g}$.

[$\mathcal{O}$**dis.6**] Assuming $e \perp e'$ and $e_1 \leq e,\ e'_1 \leq e'$, by [**Dis.3**], $e_1 \perp e'_1$.

Therefore, $\perp$ acts as an open disjointness relation on $\mathcal{O}(A)^2$.

$\square$

Definition 6.2.8. If $\underline{\perp}$ is an open disjointness relation in $\mathbb{X}$, then we may define a relation ${}_{A}\!\perp_{B} \in \mathbb{X}(A,B)^2$ by

$$\frac{f,g : A \to B,\ \overline{f} \,\underline{\perp}\, \overline{g},\ \hat{f} \,\underline{\perp}\, \hat{g}}{f\ {}_{A}\!\perp_{B}\, g}.$$

We call ${}_{A}\!\perp_{B}$ an *extended disjointness relation.*

Proposition 6.2.9. *If $\perp$ is an extended disjointess relation based on $\underline{\perp}$ in $\mathbb{X}$, then $\perp$ is a disjointness relation in $\mathbb{X}$.*

Proof.

[**Dis.1**] We need to show $f \perp 0$ for any f. We know that $1 \,\underline{\perp}\, 0$ and therefore $\overline{f} \,\underline{\perp}\, 0$ and $\hat{f} \,\underline{\perp}\, 0$, as $\overline{f} \leq 1$ and $\hat{f} \leq 1$. This gives us $f \perp 0$.

[**Dis.2**] Assume $f \perp g$, i.e., $\overline{f} \,\underline{\perp}\, \overline{g}$. Then, $\overline{f}g = \overline{f}\,\overline{g}g = 0g = 0$.

[**Dis.3**] We are given $f \perp g$, $f' \leq f$ and $g' \leq g$. By Lemma 2.2.4[(ii)] $\overline{f'} \leq \overline{f}$ and $\overline{g'} \leq \overline{g}$. Then, by [$\mathcal{O}$**dis.6**], as $\overline{f} \,\underline{\perp}\, \overline{g}$ we have $\overline{f'} \,\underline{\perp}\, \overline{g'}$. By Lemma 2.2.10[(ii)], we have $\widehat{f'} \leq \hat{f}$ and $\widehat{g'} \leq \hat{g}$. Then, by [$\mathcal{O}$**dis.6**], as $\hat{f} \,\underline{\perp}\, \hat{g}$ we have $\widehat{f'} \,\underline{\perp}\, \widehat{g'}$. This means $f' \perp g'$.

[**Dis.4**] Reflexivity of $\perp$ follows immediately from the reflexivity of $\underline{\perp}$.

[**Dis.5**] Assume $f \perp g$, i.e., $\overline{f} \mathrel{\underline{\perp}} \overline{g}$ and $\hat{f} \mathrel{\underline{\perp}} \hat{g}$. Then we have $\overline{hf} \mathrel{\underline{\perp}} \overline{hg}$ by [$\mathcal{O}$**dis.4**]. By Lemma 2.2.10[(i)] we have $\widehat{hf} \leq \hat{f}$ and $\widehat{hg} \leq \hat{g}$. Therefore we have $\widehat{hf} \mathrel{\underline{\perp}} \widehat{hg}$ by [$\mathcal{O}$**dis.6**] and therefore $hf \perp hg$.

[**Dis.6**] This follows directly from definition 6.2.8.

[**Dis.7**] We assume $\overline{f} \perp \overline{g}$ and $\hat{h} \perp \hat{k}$. By definition 6.2.8 we have $\overline{f} \mathrel{\underline{\perp}} \overline{g}$ and $\hat{h} \mathrel{\underline{\perp}} \hat{k}$. By Lemma 2.2.4[(iii)], we have $\overline{fh} \leq \overline{f}$ and $\overline{gk} \leq \overline{g}$. Therefore, $\overline{fh} \mathrel{\underline{\perp}} \overline{gk}$ by [$\mathcal{O}$**dis.6**]. By Lemma 2.2.10[(i)], $\widehat{fh} \leq \hat{h}$ and $\widehat{gk} \leq \hat{k}$, giving us $\widehat{fh} \mathrel{\underline{\perp}} \widehat{gk}$ also by [$\mathcal{O}$**dis.6**]. This means $fh \perp gk$.

□

Theorem 6.2.10. *To give a disjointness relation $\perp$ on $\mathbb{X}$ is to give an open disjointess relation $\underline{\perp}$ on $\mathbb{X}$.*

Proof. Suppose we are given the disjointess relation $\perp$. By Proposition 6.2.7, this is an open disjointness relation on each of the sets of idempotents, $\mathcal{O}(A)$. We will label that relation $\underline{\perp}$.

Use Definition 6.2.8 to create an extended disjointness relation based on $\underline{\perp}$, signify it by $\underline{\underline{\perp}}$. By Proposition 6.2.9, $\underline{\underline{\perp}}$ is a disjointness relation on $\mathbb{X}$.

Assume $f \perp g$. Then we have $\overline{f} \mathrel{\underline{\perp}} \overline{g}$ and $\hat{f} \mathrel{\underline{\perp}} \hat{g}$ by [**Dis.6**] and Proposition 6.2.7. Then, from Definition 6.2.8, we have $f \mathrel{\underline{\underline{\perp}}} g$.

Assume $f \mathrel{\underline{\underline{\perp}}} g$. Then we must have had $\overline{f} \mathrel{\underline{\perp}} \overline{g}$ and $\hat{f} \mathrel{\underline{\perp}} \hat{g}$ by Definition 6.2.8 and therefore $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$. Then, by Proposition 6.2.5, we have $f \perp g$.

Now, suppose we are given the open disjointness relation $\underline{\perp}$. Similar to above, we can construct the extended disjointess relation $\perp$ by Definition 6.2.8. From the disjointness relation $\perp$, we have the open disjointess relation $\overline{\perp}$ by Lemma 6.2.7.

Assume $e \mathrel{\underline{\perp}} e'$. As this means both $\overline{e} \mathrel{\underline{\perp}} \overline{e'}$ and $\hat{e} \mathrel{\underline{\perp}} \widehat{e'}$, we have $e \perp e'$. By Proposition 6.2.7 this means $e \mathrel{\overline{\perp}} e'$.

If we are given that $e \overline{\perp} e'$, then we know that $e \perp e'$ by Proposition 6.2.7. From Definition 6.2.8, this requires that $\overline{e} \perp \overline{e'}$ and $\hat{e} \perp \widehat{e'}$, but that just means $e \underline{\perp} e'$. ☐

Note that while we have worked with binary disjointness throughout this section, one may extend the concept to lists of maps simply by considering disjointness pairwise. I.e., we have $\perp [f_1, f_2, \ldots, f_n]$ if and only if $f_i \perp f_k$ whenever $i \neq j$.

Disjointness is additional structure on a restriction category, i.e., it is possible to have more than one disjointness relation on the category.

Example 6.2.11. Consider the restriction category $\textsc{Inj}$. Here, the objects are sets and maps are the partial injective set functions, where $\overline{f} = id_{|\operatorname{dom}(f)}$. The restriction zero is the empty map (i.e., $\operatorname{dom}(0) = \operatorname{range}(0) = \emptyset$).

We may define the disjointness relation $\perp$ by $f \perp g$ if and only if $\operatorname{dom}(f) \bigcap \operatorname{dom}(g) = \emptyset$ and $\operatorname{range}(f) \bigcap \operatorname{range}(g) = \emptyset$. It is reasonably straightforward to verify [**Dis.1**] through [**Dis.7**]. For example, take [**Dis.7**]:

Proof. We are given $\overline{f} \perp \overline{g}$ and $\hat{h} \perp \hat{k}$. This means

$$\operatorname{dom} f \bigcap \operatorname{dom} g = \emptyset \text{ and } \operatorname{range} h \bigcap \operatorname{range} k = \emptyset.$$

Note that in general for partial injective functions m and n we have $\operatorname{dom} mn \subseteq \operatorname{dom} m$ and that $\operatorname{range} mn \subseteq \operatorname{range} n$. Hence we have

$$\operatorname{dom} fh \bigcap \operatorname{dom} gk \subseteq \operatorname{dom} f \bigcap \operatorname{dom} g = \emptyset$$
$$\operatorname{range} fh \bigcap \operatorname{range} gk \subseteq \operatorname{range} h \bigcap \operatorname{range} k = \emptyset.$$

Therefore, $fh \perp gk$. ☐

We may define a different disjointness relation, $\perp'$, on the same restriction category. Define $f \perp' g$ if and only if one of f or g is the restriction 0, $\emptyset$. As $0 = \overline{0} = \hat{0} = h0 = 0k$, all of the seven disjointness axioms are easily verifiable.

Although disjointness is additional structure on a restriction category, one can use the disjointness structure of a base category (or categories) to define a disjointness structure on derived categories, such as the product category.

Lemma 6.2.12. *If $\mathbb{X}$ and $\mathbb{Y}$ are inverse categories with restriction zeros and respective disjointness relations $\perp$ and $\perp'$, then we may construct a disjointness relation $\perp_\times$ on $\mathbb{X}\times\mathbb{Y}$.*

Proof. Recall that product categories are defined component-wise. These definitions extend to the restriction, the inverse and the restriction zero. That is:

- If (f,g) is a map in $\mathbb{X}\times\mathbb{Y}$, then $(f,g)^{(-1)} = (f^{(-1)}, g^{(-1)})$;
- If (f,g) is a map in $\mathbb{X}\times\mathbb{Y}$, then $\overline{(f,g)} = (\overline{f},\overline{g})$;
- The map $(0_X, 0_Y)$ is the restriction zero in $\mathbb{X}\times\mathbb{Y}$.

Following this pattern, for (f,g) and (h,k) maps in $\mathbb{X}\times\mathbb{Y}$, $(f,g) \perp_\times (h,k)$ iff $f \perp h$ and $g \perp' k$.

Verifying the disjointness axioms is straightforward, we show axioms 2 and 5. Proofs of the others are similar.

[**Dis.2**] : Given $(f,g)\perp_\times (h,k)$, we have $\overline{(f,g)}(h,k) = (\overline{f},\overline{g})(h,k) = (\overline{f}h, \overline{g}k) = (0,0) = 0$.

[**Dis.5**] : We are given $(f,g)\perp_\times (h,k)$. Consider the map $z = (x,y)$ in $\mathbb{X}\times\mathbb{Y}$. We know that $xf \perp xh$ and $yg \perp yk$, therefore we have $z(f,g) = (xf, yg) \perp_\times (xh, yk) = z(h,k)$.

$\square$

6.2.3 Disjoint joins

We now consider additional structure on the inverse category, dependant upon the disjointness relation.

Definition 6.2.13. An *inverse category with disjoint joins* is an inverse category $\mathbb{X}$, with a restriction 0, a disjointness relation $\perp$ and a binary operator on disjointness parallel maps:

$$\frac{f : A \to B,\ g : A \to B,\ f \perp g}{f \sqcup g : A \to B}$$

where the following hold:

[**DJ.1**] $f \le f \sqcup g$ and $g \le f \sqcup g$;

[**DJ.2**] $f \le h,\ g \le h$ and $f \perp g$ implies $f \sqcup g \le h$;

[**DJ.3**] $h(f \sqcup g) = hf \sqcup hg$. (Stable)

[**DJ.4**] $\perp [f, g, h]$ if and only if $f \perp (g \sqcup h)$.

The binary operator, $\sqcup$, is referred to as the *disjoint join.*

Note that [**DJ.1**] with [**DJ.2**] immediately gives us that there is only one disjoint join given a specific disjointness relation.

Lemma 6.2.14. *Suppose $\mathbb{X}$ in an inverse category with disjoint joins, with the join $\sqcup$ and that it has a second disjoint join, $\square$. Then $f \sqcup g = f \square g$ for all maps f, g in $\mathbb{X}$.*

Proof. The first axiom tells us:

$$f, g \le f \sqcup g \text{ and } f, g \le f \square g.$$

Using the second axiom, we may therefore conclude $f \sqcup g \le f \square g$ and $f \square g \le f \sqcup g$, hence $f \sqcup g = f \square g$. $\square$

Lemma 6.2.15. *In an inverse category with disjoint joins, the disjoint join respects the restriction and is universal. Additionally, it is a partial associative and commutative operation, with identity* 0*. That is, the following hold:*

(i) $\overline{f \sqcup g} = \overline{f} \sqcup \overline{g}$*;*

(ii) $(f \sqcup g)k = fk \sqcup gk$ *(Universal);*

(iii) $f \perp g,\ g \perp h,\ f \perp h$ *implies that* $(f \sqcup g) \sqcup h = f \sqcup (g \sqcup h)$*;*

(iv) $f \perp g$ *implies* $f \sqcup g = g \sqcup f$*;*

(v) $f \sqcup 0 = f$.

Proof.

(i) As $\overline{f}, \overline{g} \leq \overline{f \sqcup g}$, we immediately have $\overline{f} \sqcup \overline{g} \leq \overline{f \sqcup g}$. To show the other direction, consider

$$\begin{aligned} \overline{f}(\overline{f} \sqcup \overline{g})(f \sqcup g) &= (\overline{f}\,\overline{f} \sqcup \overline{f}\overline{g})(f \sqcup g) && [\textbf{DJ.3}] \\ &= \overline{f}(f \sqcup g) && \text{Lemma 2.2.3, } [\textbf{Dis.2}] \\ &= f. \end{aligned}$$

Hence, we have $f \leq (\overline{f} \sqcup \overline{g})(f \sqcup g)$ and similarly, so is g. By [**DJ.2**] and that $\overline{f} \sqcup \overline{g}$ is a restriction idempotent, we then have

$$f \sqcup g \leq (\overline{f} \sqcup \overline{g})(f \sqcup g) \leq f \sqcup g$$

and therefore $f \sqcup g = (\overline{f} \sqcup \overline{g})(f \sqcup g)$. By Lemma 2.2.4, $\overline{f \sqcup g} \leq \overline{f} \sqcup \overline{g}$ and so $\overline{f \sqcup g} = \overline{f} \sqcup \overline{g}$.

(ii) First consider when f, g and k are restriction idempotents, say e_0, e_1 and e_2. Then, we have $(e_0 \sqcup e_1)e_2 = e_2(e_0 \sqcup e_1) = e_2e_0 \sqcup e_2e_1 = e_0e_2 \sqcup e_1e_2$. Next, note that for general f, g, h, we have $fk \sqcup gk \leq (f \sqcup g)k$ as both $fk, gk \leq (f \sqcup g)k$.

By Lemma 2.2.4, we need only show that their restrictions are equal:

$$
\begin{aligned}
\overline{(f \sqcup g)k} &= \overline{\overline{f \sqcup g}(f \sqcup g)k} && [\mathbf{R.1}] \\
&= \overline{f \sqcup g}\,\overline{(f \sqcup g)k} && [\mathbf{R.3}] \\
&= (\overline{f} \sqcup \overline{g})\overline{(f \sqcup g)k} && \text{previous item} \\
&= \overline{f}\,\overline{(f \sqcup g)k} \sqcup \overline{g}\,\overline{(f \sqcup g)k} && \text{idempotent universal} \\
&= \overline{\overline{f}(f \sqcup g)k} \sqcup \overline{\overline{g}(f \sqcup g)k} && [\mathbf{R.3}] \\
&= \overline{fk} \sqcup \overline{gk} \\
&= \overline{fk \sqcup gk}.
\end{aligned}
$$

Therefore, as the restrictions are equal, we have shown $(f \sqcup g)k = fk \sqcup gk$.

(iii) *Associativity*: Note that [**DJ.4**] shows that both sides of the equation exist. To show they are equal, we show that they are less than or equal to each other. From the definitions, we know that $f \sqcup g, h \leq (f \sqcup g) \sqcup h$, which also means $f, g \leq (f \sqcup g) \sqcup h$. Similarly, $g \sqcup h \leq (f \sqcup g) \sqcup h$ and then $f \sqcup (g \sqcup h) \leq (f \sqcup g) \sqcup h$. Conversely, $f, g, h \leq f \sqcup (g \sqcup h)$ and therefore $(f \sqcup g) \sqcup h \leq f \sqcup (g \sqcup h)$ and both sides are equal.

(iv) *Commutativity*: Note first that both f and g are less than or equal to both $f \sqcup g$ and $g \sqcup f$, by [**DJ.1**]. By [**DJ.2**], we have $f \sqcup g \leq g \sqcup f$ and $g \sqcup f \leq f \sqcup g$ and we may conclude $f \sqcup g = g \sqcup f$.

(v) *Identity*: By [**DJ.1**], $f \leq f \sqcup 0$. As $0 \leq f$ and $f \leq f$, by [**DJ.2**], $f \sqcup 0 \leq f$ and we have $f = f \sqcup 0$.

□

Note that the previous lemma and proof of associativity allows a simple inductive argument which shows that having binary disjoint joins extends unambiguously to disjoint joins of an arbitrary finite collection of disjoint maps.

We will write $[f_i]$ to signify a list of maps, where each $f_i : A \to B$. For disjointness, $\perp [f_i]$ will mean that $f_j \perp f_k$ where $j \neq k$ and $f_j, f_k \in [f_i]$. Finally, $\sqcup[f_i]$ will mean the disjoint join of all maps f_i, i.e., $f_1 \sqcup f_2 \sqcup \cdots \sqcup f_n$.

Lemma 6.2.16. *In an inverse category with disjoint joins, $\perp [f_i]$ if and only if $\sqcup[f_i]$ is defined unambiguously.*

Proof. Using [**Dj.4**], proceed as in the proof of Lemma 6.2.15[(iii)], inducting on n. $\square$

Lemma 6.2.17. *Given $\mathbb{X}$ is an inverse category with a disjoint join, then if $f_i, g_j : A \to B$ and $\perp [f_i]$ and $\perp [g_j]$, then $\sqcup[f_i] \perp \sqcup[g_j]$ if and only $f_i \perp g_j$ for all i, j;*

Proof. Assume $\sqcup[f_i] \perp \sqcup[g_j]$. Then by [**Dj.4**] and associativity, we have $\sqcup[f_i] \perp g_j$ for each j. Then, applying the reflexivity of $\perp$, [**Dj.4**] and associativity, we have $f_i \perp g_j$ for each i and j.

Assume $f_i \perp g_j$ for each i and j. Then by [**Dj.4**] and associativity, $f_i \perp \sqcup[g_j]$ for each i. Applying [**Dj.4**] again, we have $\sqcup[f_i] \perp \sqcup[g_j]$. $\square$

Following the same method as in the previous section, we show that the product of two inverse categories with disjoint joins has a disjoint join.

Lemma 6.2.18. *Given $\mathbb{X}, \mathbb{Y}$ are inverse categories with disjoint joins, $\sqcup$ and $\sqcup'$ respectively, then the category $\mathbb{X} \times \mathbb{Y}$ is an inverse category with disjoint joins.*

Proof. From Lemma 6.2.12, we know $\mathbb{X} \times \mathbb{Y}$ has a disjointness relation that is defined pointwise. We therefore define $\sqcup_\times$ the disjoint join on $\mathbb{X} \times \mathbb{Y}$ by

$$(f, g) \sqcup_\times (h, k) = (f \sqcup h, g \sqcup' k) \tag{6.1}$$

We now prove each of the axioms in Definition 6.2.13 hold.

[**DJ.1**] From Equation ((6.1)), we see that since $f, h \leq f \sqcup h$ and $g, k \leq g \sqcup' k$, we have $(f, g) \leq (f, g) \sqcup_\times (h, k)$ and $(h, k) \leq (f, g) \sqcup_\times (h, k)$.

[DJ.2] Suppose $(f,g) \leq (x,y)$, $(h,k) \leq (x,y)$ and $(f,g) \perp_\times (h,k)$. Then regarding it point-wise, we have $(f,g) \sqcup_\times (h,k) = (f \sqcup h, g \sqcup' k) \leq (x,y)$.

[DJ.3] $(x,y)\,((f,g) \sqcup_\times (h,k)) = (x(f \sqcup h), y(g \sqcup' k)) = (xf \sqcup xh, yg \sqcup' yk) = (xf, yg) \sqcup_\times (xh, yk) = ((x,y)(f,g)) \sqcup_\times ((x,y)(h,k))$.

[DJ.4] Given $\perp_\times [(f,g),(h,k),(x,y)]$, we know $f \perp (h \sqcup x)$ and $g \perp' (k \sqcup' y)$. Hence, $(f,g) \perp_\times ((h,k) \sqcup_\times (x,y))$. The opposite direction is similar.

$\square$

6.2.4 Monoidal Tensors for disjointness

Suppose we are given a monoidal tensor $\oplus$ on $\mathbb{X}$, an inverse category with a restriction zero. Under certain conditions, it is possible to define disjointness based upon the action of the tensor. Note that throughout, we are assuming the following naming for the standard monoidal tensor isomorphisms.

$$u_\oplus^l : 0 \oplus A \to A$$

$$u_\oplus^r : A \oplus 0 \to A$$

$$a_\oplus : (A \oplus B) \oplus C \to A \oplus (B \oplus C)$$

$$c_\oplus : A \oplus B \to B \oplus A.$$

Note we also require the tensor isomorphisms above be be natural.

Definition 6.2.19. Suppose we are given an inverse category $\mathbb{X}$ with restriction zero and a symmetric monoidal tensor $\oplus$. Then $\oplus$ is a *disjointness tensor* when:

- It is a restriction functor — i.e., $\_ \oplus \_ : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$.
- The unit is the restriction zero. ($0 : \mathbf{1} \to \mathbb{X}$ picks out the restriction zero in $\mathbb{X}$).

- Define $\amalg_1 = u_\oplus^{r\,(-1)}(1 \oplus 0) : A \to A \oplus B$ and $\amalg_2 = u_\oplus^{l\,(-1)}(0 \oplus 1) : A \to B \oplus A$. Then $\amalg_1$ and $\amalg_2$ are jointly epic. That is, if $\amalg_1 f = \amalg_1 g$ and $\amalg_2 f = \amalg_2 g$, then $f = g$.
- Define $\amalg_1^* := (1 \oplus 0)u_\oplus^r : A \oplus B \to A$ and $\amalg_2^* := (0 \oplus 1)u_\oplus^l : A \oplus B \to B$. Then $\amalg_1^*$ and $\amalg_2^*$ are jointly monic. That is, whenever $f\amalg_1^* = g\amalg_1^*$ and $f\amalg_2^* = g\amalg_2^*$ then $f = g$.

Lemma 6.2.20. *Given an inverse category $\mathbb{X}$ with restriction zero and disjointness tensor $\oplus$, then the map $0 \oplus 0 : A \oplus B \to C \oplus D$ is the map $0 : A \oplus B \to C \oplus D$.*

Proof. Recall the zero map factors through the restriction zero, i.e. $0 : A \to B$ is the same as saying $A \xrightarrow{!} 0 \xrightarrow{?} B$. Additionally, as objects, $0 \oplus 0 \cong 0$ — the restriction zero.

Therefore the map $0 \oplus 0 : A \oplus B \to C \oplus D$ is writable as

$$A \oplus B \xrightarrow{!\oplus!} 0 \oplus 0 \xrightarrow{?\oplus?} C \oplus D,$$

which may then be rewritten as

$$A \oplus B \xrightarrow{!\oplus!} 0 \oplus 0 \xrightarrow{u_\oplus^l} 0 \xrightarrow{u_\oplus^{l\,(-1)}} 0 \oplus 0 \xrightarrow{?\oplus?} C \oplus D.$$

But by the properties of the restriction zero, $(!\oplus!)u_\oplus^l =!$ and $u_\oplus^{l\,(-1)}(?\oplus?) =?$ and therefore the map $0 \oplus 0 : A \oplus B \to C \oplus D$ is the same as the map $0 : A \oplus B \to C \oplus D$. □

Lemma 6.2.21. *Given an inverse category $\mathbb{X}$ with a restriction zero and a disjointness tensor, the map $\amalg_1$ is natural in the left component and $\amalg_2$ is natural in the right, up to isomorphism. This means:*

$$\amalg_1(f \oplus g) = f\,\amalg_1 \quad \textit{and} \quad \amalg_2\,(f \oplus g) = g\,\amalg_2\,.$$

Proof. For the left and right naturality, we see:

$$\amalg_1(f \oplus g) = u_\oplus^{r\,(-1)}(1 \oplus 0)(f \oplus g) = u_\oplus^{r\,(-1)}(f \oplus 0) = f u_\oplus^{r\,(-1)}(1 \oplus 0) = f\amalg_1,$$

and

$$\amalg_2(f \oplus g) = {u^l_\oplus}^{(-1)}(0 \oplus 1)(f \oplus g) = {u^l_\oplus}^{(-1)}(0 \oplus g) = g{u^l_\oplus}^{(-1)}(0 \oplus 1) = g\,\amalg_2\,.$$

$\square$

Lemma 6.2.22. *Given an inverse category $\mathbb{X}$ with restriction zero and disjointness tensor $\oplus$, $\amalg_1^* = {\amalg_1}^{(-1)}$ and $\amalg_2^* = {\amalg_2}^{(-1)}$ and the following hold:*

1. $\amalg_i^*\amalg_i = \overline{\amalg_i^*}$ *and* $\amalg_i\amalg_i^* = \overline{\amalg_i} = 1$;
2. $\overline{\amalg_1^*}\amalg_2^* = 0$ *and* $\overline{\amalg_2^*}\amalg_1^* = 0$;
3. $\amalg_2\amalg_1^* = 0$, $\amalg_2\overline{\amalg_1^*} = 0$, $\amalg_1\amalg_2^* = 0$ *and* $\amalg_1\overline{\amalg_2^*} = 0$;
4. *the maps* $\amalg_1$ *and* $\amalg_2$ *are monic.*

Accordingly,

Proof. For item 1, recalling that the restriction zero is its own partial inverse, we see that

$${\amalg_1}^{(-1)} = \left({u^r_\oplus}^{(-1)}(1 \oplus 0)\right)^{(-1)} = (1 \oplus 0)^{(-1)}u^r_\oplus = (1 \oplus 0)u^r_\oplus = \amalg_1^*.$$

Similarly,

$${\amalg_2}^{(-1)} = \left({u^l_\oplus}^{(-1)}(0 \oplus 1)\right)^{(-1)} = (0 \oplus 1)u^l_\oplus = \amalg_2^*.$$

Hence, we may calculate the restriction of $\amalg_1$,

$$\amalg_1\amalg_1^* = {u^r_\oplus}^{(-1)}(1 \oplus 0)(1 \oplus 0)u^r_\oplus = ({u^r_\oplus}^{(-1)}(1 \oplus 0))u^r_\oplus = 1{u^r_\oplus}^{(-1)}u^r_\oplus = 1.$$

The calculation for $\amalg_2^*$ and $\amalg_2$ is analogous.

To show $\overline{\amalg_1^*}\amalg_2^* = 0$ and $\overline{\amalg_2^*}\amalg_1^* = 0$,

$$\begin{aligned}\overline{\amalg_1^*}\amalg_2^* &= \overline{(1 \oplus 0)u^r_\oplus}(0 \oplus 1)u^l_\oplus \\ &= \overline{1 \oplus 0}(0 \oplus 1)u^l_\oplus \\ &= (1 \oplus 0)(0 \oplus 1)u^l_\oplus \\ &= (0 \oplus 0)u^l_\oplus = 0,\end{aligned}$$

and

$$\begin{aligned}\overline{\amalg_2^*}\amalg_1^* &= \overline{(0 \oplus 1)u_\oplus^l}(1 \oplus 0)u_\oplus^r \\ &= (0 \oplus 1)(1 \oplus 0)u_\oplus^r \\ &= (0 \oplus 0)u_\oplus^r \\ &= 0.\end{aligned}$$

To show $\amalg_i \amalg_j^* = 0$, $\amalg_i \overline{\amalg_j^*} = 0$ when $i \neq j$,

$$\begin{aligned}\amalg_1\amalg_2^* &= ({u_\oplus^r}^{(-1)}(1 \oplus 0))(0 \oplus 1)u_\oplus^l \\ &= {u_\oplus^r}^{(-1)}(0 \oplus 0)u_\oplus^l \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\amalg_2\amalg_1^* &= ({u_\oplus^l}^{(-1)}(0 \oplus 1))(1 \oplus 0)u_\oplus^r \\ &= {u_\oplus^l}^{(-1)}(0 \oplus 0)u_\oplus^r \\ &= 0.\end{aligned}$$

As $\overline{\amalg_1^*} = 1 \oplus 0$ and $\overline{\amalg_2^*} = 0 \oplus 1$, we see the other two identities hold as well.

To prove $\amalg_1$ is monic, suppose $f\amalg_1 = g\amalg_1$. Therefore we must have

$$f = f(\amalg_1{\amalg_1}^{(-1)}) = (f\amalg_1){\amalg_1}^{(-1)} = (g\amalg_1){\amalg_1}^{(-1)} = g(\amalg_1{\amalg_1}^{(-1)}) = g.$$

The proof that $\amalg_2$ is monic is similar. □

As we have shown that $\amalg_i^* = {\amalg_i}^{(-1)}$, we will prefer the explicit notation of ${\amalg_i}^{(-1)}$ for rest of this paper.

Corollary 6.2.23. *In an inverse category $\mathbb{X}$ with a restriction zero and disjointness tensor, the following hold:*

(i) $\amalg_1(f \oplus g){\amalg_1}^{(-1)} = f$; *(iii)* $\amalg_2(f \oplus g){\amalg_1}^{(-1)} = 0$;

(ii) $\amalg_1(f \oplus g){\amalg_2}^{(-1)} = 0$; *(iv)* $\amalg_2(f \oplus g){\amalg_2}^{(-1)} = g$.

Additionally, if t is a map such that for $i \in \{1, 2\}$,

$$\amalg_i t {\amalg_j}^{(-1)} = \begin{cases} t_i & : \quad i \neq j \\ 0 & : \quad i = j, \end{cases}$$

then $t = t_1 \oplus t_2$

Proof. The calculations for $f \oplus g$ follow from Lemma 6.2.21 and Lemma 6.2.22. For example, $\amalg_1(f \oplus g){\amalg_1}^{(-1)} = f \amalg_1 {\amalg_1}^{(-1)} = f$.

For the second claim, note that we have $\amalg_1(t{\amalg_1}^{(-1)}) = t_1 = \amalg_1(t_1 \oplus t_2){\amalg_1}^{(-1)}$ and $\amalg_2(t{\amalg_1}^{(-1)}) = 0 = \amalg_2(t_1 \oplus t_2){\amalg_1}^{(-1)}$, hence $t{\amalg_1}^{(-1)} = (t_1 \oplus t_2){\amalg_1}^{(-1)}$. Similarly, we see $t{\amalg_2}^{(-1)} = (t_1 \oplus t_2){\amalg_2}^{(-1)}$ and therefore $t = t_1 \oplus t_2$. $\square$

Disjointess via a monoidal tensor

Definition 6.2.24. In an inverse category $\mathbb{X}$ with a restriction zero and disjointness tensor, we define two partial operations on pairs of arrows in $\mathbb{X}$ to another arrow in $\mathbb{X}$. First, for arrows $f : A \to B$ and $g : A \to C$, we define $f \triangledown g$ as being the map that makes diagram ((6.2)) below commute, when it exists.

$$\begin{tikzcd}
& A \arrow[dl, "g"'] \arrow[d, dotted, "f \triangledown g"] \arrow[dr, "f"] & \\
B & B \oplus C \arrow[l, "{\amalg_1}^{(-1)}"] \arrow[r, "{\amalg_2}^{(-1)}"'] & C
\end{tikzcd} \tag{6.2}$$

Then for $h : B \to A$, $k : C \to A$, $h \vartriangle k$ is that map that makes diagram ((6.3)) commute, if it exists.

$$\begin{tikzcd}
B \arrow[r, "\amalg_1"'] \arrow[dr, "h"] & B \oplus C \arrow[d, dotted, "h \vartriangle k"] & C \arrow[l, "\amalg_2"] \arrow[dl, "k"'] \\
& A &
\end{tikzcd} \tag{6.3}$$

Due to $\amalg_1{}^{(-1)}$ and $\amalg_2{}^{(-1)}$ being jointly monic, $f \triangledown g$ is unique when it exists. Similarly, as $\amalg_1$ and $\amalg_2$ are jointly epic, $f \vartriangle g$ is unique when it exists.

We give a lemma exploring the behaviour of the two operations: $\triangledown$ and $\vartriangle$.

Lemma 6.2.25. *Given $\mathbb{X}$ is an inverse category with a restriction zero and a disjointness tensor $\oplus$ then the following relations hold for $\triangledown$ and $\vartriangle$:*

(i) *If $f \triangledown g$ exists, then $g \triangledown f$ exists. If $f \vartriangle g$ exists, then $g \vartriangle f$ exists.*

(ii) *$f \triangledown 0$ and $f \vartriangle 0$ always exist.*

(iii) *When $f \triangledown g$ exists, $\overline{f}(f \triangledown g) = f \triangledown 0$, $\overline{f}g = 0$, $\overline{g}(f \triangledown g) = 0 \triangledown g$ and $\overline{g}f = 0$.*

(iv) *Dually to the previous item, when $f \vartriangle g$ exists, $(f \vartriangle g)\hat{f} = f \vartriangle 0$, $g\hat{f} = 0$, $(f \vartriangle g)\hat{g} = 0 \vartriangle g$ and $f\hat{g} = 0$.*

(v) *When $f \triangledown g$ exists, $f \triangledown g(h \oplus k) = fh \triangledown gk$.*

(vi) *Dually, when $f \vartriangle g$ exists, $(h \oplus k)f \vartriangle g = hf \vartriangle kg$.*

(vii) *When $f \triangledown g$ exists, then $h(f \triangledown g) = hf \triangledown hg$ and when $f \vartriangle g$ exists, $(f \vartriangle g)h = fh \vartriangle gh$.*

(viii) *If $\overline{f} \triangledown \overline{g}$ exists, then $\overline{f} \vartriangle \overline{g}$ exists and is the partial inverse of $\overline{f} \triangledown \overline{g}$.*

(ix) *If $f \triangledown g$ exists and $f' \leq f$, $g' \leq g$, then $f' \triangledown g'$ exists.*

(x) *When $f \vartriangle g$ exists, $(f \vartriangle g)(f \vartriangle g)^{(-1)} = \overline{f} \oplus \overline{g}$.*

(xi) *Given $f \triangledown g$ and $h \triangledown k$ exist, then $(f \oplus h) \triangledown (g \oplus k) = (f \triangledown g) \oplus (h \triangledown k)$. Dually, the existence of $f \vartriangle g$ and $h \vartriangle k$ implies $(f \oplus h) \vartriangle (g \oplus k) = (f \vartriangle g) \oplus (h \vartriangle k)$.*

Proof.

(i) $g \triangledown f = (f \triangledown g)c_\oplus$ and $g \vartriangle f = c_\oplus(f \vartriangle g)$.

(ii) Consider $f\amalg_1$. Then $f\ \amalg_1\ {\amalg_1}^{(-1)} = f$ and $f\ \amalg_1\ {\amalg_2}^{(-1)} = f0 = 0$. Hence, $f\amalg_1 = f \triangledown 0$.

Consider ${\amalg_1}^{(-1)}f$. Then $\amalg_1{\amalg_1}^{(-1)}f = f$ and $\amalg_2{\amalg_1}^{(-1)}f = 0f = 0$ and therefore ${\amalg_1}^{(-1)}f = (f \vartriangle 0)$.

(iii) Using Lemma 6.2.22

$$\overline{f}g = \overline{(f \triangledown g){\amalg_1}^{(-1)}}(f \triangledown g){\amalg_2}^{(-1)} = (f \triangledown g)\overline{{\amalg_1}^{(-1)}}{\amalg_2}^{(-1)} = 0.$$

Similarly, $\overline{g}f = f \triangledown g\overline{{\amalg_2}^{(-1)}}{\amalg_1}^{(-1)} = 0$.

Recall that ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ are jointly monic. We have $\overline{f}(f \triangledown g){\amalg_1}^{(-1)} = \overline{f}f = f = (f \triangledown 0){\amalg_1}^{(-1)}$ and $\overline{f}(f \triangledown g){\amalg_2}^{(-1)} = \overline{f}g = 0 = (f \triangledown 0){\amalg_2}^{(-1)}$. Therefore, $\overline{f}(f \triangledown g) = f \triangledown 0$. Similarly, $\overline{g}(f \triangledown g) = 0 \triangledown g$.

(iv) Using Lemma 6.2.22

$$\begin{aligned} g\hat{f} = \amalg_2(f \vartriangle g)(\widehat{\amalg_1(f \vartriangle g)}) = \amalg_2(f \vartriangle g)\overline{(f \vartriangle g)^{(-1)}{\amalg_1}^{(-1)}} = \\ \amalg_2\ (f \vartriangle g)\overline{(f \vartriangle g)^{(-1)}\overline{{\amalg_1}^{(-1)}}} = \overline{\amalg_2\overline{(f \vartriangle g)}\,\overline{{\amalg_1}^{(-1)}}}\ \amalg_2\ (f \vartriangle g) = \\ \overline{\amalg_2\overline{{\amalg_1}^{(-1)}}\,\overline{(f \vartriangle g)}}\ \amalg_2\ (f \vartriangle g) = \overline{0}\ \amalg_2\ (f \vartriangle g) = 0 \end{aligned}$$

Similarly, $f\hat{g} = 0$.

Recall that $\amalg_1$ and $\amalg_2$ are jointly epic. We have $\amalg_1(f \vartriangle g)\hat{f} = f\hat{f} = f = \amalg_1(f \vartriangle 0)$ and $\amalg_2(f \vartriangle g)\hat{f} = g\hat{f} = 0 = \amalg_2(f \vartriangle 0)$. Therefore, $(f \vartriangle g)\hat{f} = f \vartriangle 0$. Similarly, $(f \vartriangle g)\hat{g} = 0 \vartriangle g$.

(v) Calculating, we have

$$f \triangledown g(h \oplus k){\amalg_1}^{(-1)} = f \triangledown g{\amalg_1}^{(-1)}h = fh$$

and

$$f \triangledown g(h \oplus k){\amalg_2}^{(-1)} = f \triangledown g{\amalg_2}^{(-1)}k = gk,$$

which means that $f \triangledown g(h \oplus k) = fh \triangledown gk$ by the joint monic property of ${\amalg_1}^{(-1)}$, ${\amalg_2}^{(-1)}$.

(vi) The proof for this is dual to the previous item, and depends on the joint epic property of $\amalg_1$ and $\amalg_2$.

(vii) We are given $f \triangledown g$ exists, therefore $f = (f \triangledown g){\amalg_1}^{(-1)}$ and $g = (f \triangledown g){\amalg_2}^{(-1)}$. But this means $hf = h(f \triangledown g){\amalg_1}^{(-1)}$ and $hg = h(f \triangledown g){\amalg_2}^{(-1)}$, from which we may conclude $hf \triangledown hg = h(f \triangledown g)$ by the fact that ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ are jointly monic. The proof of $(f \vartriangle g)h = fh \vartriangle gh$ is similar.

(viii) We are given $\overline{f} = \overline{f} \triangledown \overline{g}{\amalg_1}^{(-1)}$. Therefore,

$$\overline{f} = \overline{f}^{(-1)} = {\amalg_1}^{(-1)(-1)}(\overline{f} \triangledown \overline{g})^{(-1)} = \amalg_1(\overline{f} \triangledown \overline{g})^{(-1)}.$$

Similarly, $\overline{g} = \amalg_2(\overline{f} \triangledown \overline{g})^{(-1)}$. But this means $(\overline{f} \triangledown \overline{g})^{(-1)} = \overline{f} \vartriangle \overline{g}$.

(ix) Note that from item (v), we know that $f \triangledown g = \overline{f} \triangledown \overline{g}(f \oplus g)$. We are given $f' \leq f$ and $g' \leq g$. This gives us $\overline{f'}f = f'$, $\overline{g'}g = g'$, $\overline{f'}\,\overline{f} = \overline{f'}$ and $\overline{g'}\,\overline{g} = \overline{g'}$. Consider the map $\overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(f \oplus g)$. Calculating, we see

$$\begin{aligned}
\overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(f \oplus g) &= \overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(\overline{f'} \oplus \overline{g'})(f \oplus g)\\
&= \overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(f' \oplus g')\\
&= \overline{f}\,\overline{f'} \triangledown \overline{g}\overline{g'}(f' \oplus g')\\
&= \overline{f'}\,\overline{f} \triangledown \overline{g'}\overline{g}(f' \oplus g')\\
&= \overline{f'} \triangledown \overline{g'}(f' \oplus g')\\
&= f' \triangledown g'.
\end{aligned}$$

(x) From our diagram for $\vartriangle$, we know:

$$\begin{aligned}
f^{(-1)} &= (f \vartriangle g)^{(-1)}{\amalg_1}^{(-1)} \text{ and}\\
g^{(-1)} &= (f \vartriangle g)^{(-1)}{\amalg_2}^{(-1)}.
\end{aligned}$$

As well, we know that $\amalg_1(f \vartriangle g) = f$ and $\amalg_1(f \vartriangle g) = g$. Therefore, we have:

$$\amalg_1(f \vartriangle g)(f \vartriangle g)^{(-1)}{\amalg_1}^{(-1)} = \overline{f} \text{ and } \amalg_2\,(f \vartriangle g)(f \vartriangle g)^{(-1)}{\amalg_2}^{(-1)} = \overline{g}.$$

As $f \perp_{\oplus} g$, we know that $fg^{(-1)} = f\hat{g}g^{(-1)} = 0g^{(-1)} = 0$ and therefore,

$$\amalg_1(f \vartriangle g)(f \vartriangle g)^{(-1)}{\amalg_2}^{(-1)} = 0 \text{ and } \amalg_2\,(f \vartriangle g)(f \vartriangle g)^{(-1)}{\amalg_1}^{(-1)} = 0.$$

By Corollary 6.2.23 this means $(f \vartriangle g)(f \vartriangle g)^{(-1)} = \overline{f} \oplus \overline{g}$.

(xi) As $(f \triangledown g) \oplus (h \triangledown k){\amalg_1}^{(-1)} = (f \triangledown g)$ and $(f \triangledown g) \oplus (h \triangledown k){\amalg_2}^{(-1)} = (h \triangledown k)$, we see that $(f \triangledown g) \oplus (h \triangledown k)$ satisfies the diagram for $(f \oplus h) \triangledown (g \oplus k)$. Dually, as $\amalg_1(f \vartriangle g) \oplus (h \vartriangle k) = (f \vartriangle g)$ and $\amalg_2(f \vartriangle g) \oplus (h \vartriangle k) = (h \vartriangle k)$, $(f \vartriangle g) \oplus (h \vartriangle k)$ satisfies the diagram for $(f \oplus h) \vartriangle (g \oplus k)$.

□

Definition 6.2.26. Define $f \perp_{\oplus} g$ when $f, g : A \to B$ and both $f \triangledown g$ and $f \vartriangle g$.

Lemma 6.2.27. *If $\mathbb{X}$ is an inverse category with a restriction zero and a disjointness tensor $\oplus$ then the relation $\perp_{\oplus}$ is a disjointness relation.*

Proof. We need to show that $\perp_{\oplus}$ satisfies the disjointness axioms. We will use [**Dis.6'**] in place of [**Dis.6**] and [**Dis.7**] as discussed in Lemma 6.2.4.

[**Dis.1**] We must show $f \perp_{\oplus} 0$. This follows immediately from Lemma 6.2.25, item (ii).

[**Dis.2**] Show $f \perp_{\oplus} g$ implies $\overline{f}g = 0$. This is a direct consequence of Lemma 6.2.25, item (iii).

[**Dis.3**] We require $f \perp_{\oplus} g$, $f' \le f$, $g' \le g$ implies $f' \perp_{\oplus} g'$. From Lemma 6.2.25, item (ix), we immediately have $f' \triangledown g'$ exists. Using a similar argument to the proof of this item, we also have $f' \vartriangle g'$ exists and hence $f' \perp_{\oplus} g'$.

[**Dis.4**] Commutativity of $\perp_{\oplus}$ follows from the symmetry of the two required diagrams, see Lemma 6.2.25, item (i).

[**Dis.5**] Show that if $f \perp_{\oplus} g$ then $hf \perp_{\oplus} hg$ for any map h. By Lemma 6.2.25, item (vii), we know that $hf \triangledown hg$ exists. By item (vi), $(hf) \vartriangle (hg) = (h \oplus h)(f \vartriangle g)$ and therefore $hf \perp_{\oplus} hg$.

[**Dis.6'**] We need to show $f \perp_{\oplus} g$ if and only if $\overline{f} \perp_{\oplus} \overline{g}$ and $\hat{f} \perp_{\oplus} \hat{g}$. This follows directly from Lemma 6.2.25, items (v) and (vi), which give us $f \triangledown g = \overline{f} \triangledown \overline{g}(f \oplus g)$ and $f \vartriangle g = (f \oplus g)\hat{f} \vartriangle \hat{g}$, where the equalities hold if either side of the equation exists.

□

Disjoint join via a monoidal tensor

The operations $\triangledown$ and $\vartriangle$ are sufficient to define a disjointness relation on an inverse category. However, when we wish to extend this to a disjoint join, we run into problems when trying to prove [**DJ.4**]. Specifically, there is not enough information to show that $\perp_{\oplus}[f, g, h]$ implies $f \perp_{\oplus} (g \sqcup_{\oplus} h)$.

Therefore, we add one more assumption regarding our tensor in order to define disjointness.

Definition 6.2.28. Let $\mathbb{X}$ be an inverse category with a disjointness tensor $\oplus$ and a restriction zero. Consider diagrams (6.4) and (6.5).

$$\begin{tikzcd}
A \arrow[dr, dotted, "\alpha"] \arrow[rr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& X \oplus Y \oplus Z \arrow[r, "\amalg_{1,3}{}^{(-1)}"'] \arrow[d, "\amalg_{1,2}{}^{(-1)}"] & X \oplus Z \arrow[d, "\amalg_{1}{}^{(-1)}"] \\
& X \oplus Y \arrow[r, "\amalg_{1}{}^{(-1)}"'] & X
\end{tikzcd} \tag{6.4}$$

$$\begin{tikzcd}
& & A \\
X \oplus Y \arrow[r, "\amalg_{1,2}"'] \arrow[urr, bend left, "h"] & X \oplus Y \oplus Z \arrow[ur, dotted, "\beta"] & \\
X \arrow[u, "\amalg_{1}"] \arrow[r, "\amalg_{1}"'] & X \oplus Z \arrow[u, "\amalg_{1,3}"] \arrow[uur, bend right, "k"'] &
\end{tikzcd} \tag{6.5}$$

Then $\oplus$ is a *disjoint sum tensor* when the following two conditions hold:

- α exists if and only if $f\amalg_2{}^{(-1)} \triangledown g\amalg_2{}^{(-1)}$ exists;

- β exists if and only if $\amalg_2 h \vartriangle \amalg_2 k$ exists.

Lemma 6.2.29. *Let $\mathbb{X}$ be an inverse category with a disjoint sum tensor as in Definition 6.2.28 and we are given $f, g, h : A \to B$ with $\perp_\oplus[f, g, h]$. Then both $f \triangledown (g \triangledown h)$ and $f \vartriangle (g \vartriangle h)$ exist.*

Proof. As all the maps are disjoint, we know that each pair's $\triangledown$ map and $\vartriangle$ maps exist. Consider the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[drr, bend left=10, "g \triangledown f"] \arrow[dr, dotted, "\alpha"] \arrow[ddr, bend right, "g \triangledown h"'] & & B \\
& B \oplus B \oplus B \arrow[r, "{\amalg_{0,2}}^{(-1)}"'] \arrow[d, "{\amalg_{0,1}}^{(-1)}"] & B \oplus B \arrow[u, "{\amalg_2}^{(-1)}"'] \arrow[d, "{\amalg_1}^{(-1)}"] \\
& B \oplus B \arrow[r, "{\amalg_1}^{(-1)}"'] & B
\end{tikzcd}$$

where we claim $\alpha = (g \triangledown h) \triangledown f$.

The lower part of the diagram commutes as it fulfills the conditions of Definition 6.2.28. The upper rightmost triangle of the diagram commutes by the definition of $g \triangledown f$. Noting that ${\amalg_{0,1}}^{(-1)} : B \oplus B \oplus B \to B \oplus B$ is the same map as ${\amalg_1}^{(-1)} : (B \oplus B) \oplus B \to (B \oplus B)$ and ${\amalg_{0,2}}^{(-1)}{\amalg_2}^{(-1)} : B \oplus B \oplus B \to B \oplus B \to B$ is the same map as ${\amalg_2}^{(-1)} : (B \oplus B) \oplus B \to B$, we see α does make the $\triangledown$ diagram for $g \triangledown h$ and f commute. Therefore by Lemma 6.2.25, $f \triangledown (g \triangledown h)$ exists and is equal to $\alpha c_{\oplus\{01,2\}}$.

A dual diagram and reasoning shows $f \vartriangle (g \vartriangle h)$ exists. $\square$

Lemma 6.2.30. *In an inverse category $\mathbb{X}$ with a disjoint sum tensor, when $\perp_\oplus[f, g, h]$, then:*

1. $f \triangledown (g \triangledown h) = ((f \triangledown g) \triangledown h)a_\oplus$ *and both exist;*
2. $f \vartriangle (g \vartriangle h) = ((f \vartriangle g) \vartriangle h)a_\oplus$ *and both exist;*

Proof. Consider the diagram

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[drr, bend left=10, "f \triangledown h"] \arrow[dr, dotted, "\alpha"] \arrow[ddr, bend right, "f \triangledown g"'] & & B \\
& B \oplus B \oplus B \arrow[r, "\amalg_{0,2}{}^{(-1)}"'] \arrow[d, "\amalg_{0,1}{}^{(-1)}"] & B \oplus B \arrow[u, "\amalg_2{}^{(-1)}"'] \arrow[d, "\amalg_1{}^{(-1)}"] \\
& B \oplus B \arrow[r, "\amalg_1{}^{(-1)}"'] & B
\end{tikzcd}$$

which gives us $\alpha = (f \triangledown g) \triangledown h : A \to (B \oplus B) \oplus B$ and $\alpha a_{\oplus} : A \to B \oplus (B \oplus B)$. Next consider the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[drr, bend left=10, "g \triangledown f"] \arrow[dr, dotted, "\gamma"] \arrow[ddr, bend right, "g \triangledown h"'] & & B \\
& B \oplus B \oplus B \arrow[r, "\amalg_{0,2}{}^{(-1)}"'] \arrow[d, "\amalg_{0,1}{}^{(-1)}"] & B \oplus B \arrow[u, "\amalg_2{}^{(-1)}"'] \arrow[d, "\amalg_1{}^{(-1)}"] \\
& B \oplus B \arrow[r, "\amalg_1{}^{(-1)}"'] & B
\end{tikzcd}$$

which gives us $\gamma c_{\oplus} = f \triangledown (g \triangledown h) : A \to B \oplus (B \oplus B)$.

Note from the diagrams we have:

$$\gamma c_{\oplus} \amalg_0{}^{(-1)} = f = \alpha a_{\oplus} \amalg_1{}^{(-1)}$$

$$\gamma c_{\oplus} \amalg_1{}^{(-1)} \amalg_0{}^{(-1)} = g = \alpha a_{\oplus} \amalg_2{}^{(-1)} \amalg_1{}^{(-1)}$$

$$\gamma c_{\oplus} \amalg_1{}^{(-1)} \amalg_2{}^{(-1)} = h = \alpha a_{\oplus} \amalg_2{}^{(-1)} \amalg_2{}^{(-1)}.$$

Hence, by the assumption that $\amalg_1{}^{(-1)}, \amalg_2{}^{(-1)}$ are jointly monic, we have $\alpha = \gamma c_{\oplus} a_{\oplus}$ and hence $f \triangledown (g \triangledown h) = (f \triangledown g) \triangledown h$, up to the associativity isomorphism. $\square$

Definition 6.2.31. Let $\mathbb{X}$ be an inverse category with a disjointness tensor and restriction zero. Assume we have two maps $f, g : A \to B$ with $f \perp_{\oplus} g$. Then define the map $f \sqcup_{\oplus} g = \overline{f} \triangledown \overline{g} (f \oplus g) \hat{f} \vartriangle \hat{g}$.

For reference, the map $f \sqcup_\oplus g$ may be visualized as follows:

$$
\begin{tikzcd}
 & A & & B \arrow[d, "\amalg_1"'] \arrow[dr, "\hat{f}"] & \\
A \arrow[ur, "\overline{f}"] \arrow[r, dotted, "\overline{f} \triangledown \overline{g}"] \arrow[dr, "\overline{g}"'] & A \oplus A \arrow[u, "{\amalg_1}^{(-1)}"'] \arrow[d, "{\amalg_2}^{(-1)}"] \arrow[rr, "f \oplus g"] & & B \oplus B \arrow[r, dotted, "\hat{f} \vartriangle \hat{g}"] & B \\
 & A & & B \arrow[u, "\amalg_2"] \arrow[ur, "\hat{g}"'] &
\end{tikzcd}
$$

Using Lemma 6.2.25, we may rewrite this in a variety of equivalent ways:

$$
\begin{aligned}
f \sqcup_\oplus g &= \overline{f} \triangledown \overline{g}(f \oplus g)\hat{f} \vartriangle \hat{g} \\
&= f \triangledown g \hat{f} \vartriangle \hat{g} \\
&= \overline{f} \triangledown \overline{g} f \vartriangle g \\
&= f \triangledown g (f^{(-1)} \oplus g^{(-1)}) f \vartriangle g
\end{aligned}
$$

In particular, note that $\overline{f} \sqcup_\oplus \overline{g} = (\overline{f} \triangledown \overline{g})(\overline{f} \vartriangle \overline{g})$ as $\hat{\overline{g}} = \overline{g}$.

Lemma 6.2.32. *Let $\mathbb{X}$ be an inverse category with a disjointness tensor and restriction zero. Let $\mathbb{X}$ have the maps $f, g : A \to B$ with $f \perp_\oplus g$. Then $\sqcup_\oplus$ has the following properties.*

(i) *For all maps $h : A \to B$, $\overline{f}h \sqcup_\oplus \overline{g}h = (\overline{f} \sqcup_\oplus \overline{g})h$.*

(ii) *$\overline{f} \sqcup_\oplus \overline{g} = \overline{f \sqcup_\oplus g}$.*

Proof. (i) By Lemma 6.2.5, item (ii), we know that $\overline{f}h \perp_\oplus \overline{g}h$, hence we can form $\overline{f}h \sqcup_\oplus \overline{g}h$. Also, noting that

$$h\widehat{\overline{f}h} = h\overline{h^{(-1)}\overline{f}} = \overline{hh^{(-1)}\overline{f}}h = \overline{\overline{hf}}h = \overline{fh}h = \overline{f}h,$$

we may then calculate from the left hand side as follows:

$$
\begin{aligned}
\overline{f}h \sqcup_{\oplus} \overline{g}h &= (\overline{f}h \triangledown \overline{g}h)(\widehat{\overline{f}h} \vartriangle \widehat{\overline{g}h}) \\
&= (\overline{f} \triangledown \overline{g})(h\widehat{\overline{f}h} \vartriangle h\widehat{\overline{g}h}) \\
&= (\overline{f} \triangledown \overline{g})(\overline{f}h \vartriangle \overline{g}h) \\
&= (\overline{f} \triangledown \overline{g})(\overline{f} \vartriangle \overline{g})h \\
&= (\overline{f} \sqcup_{\oplus} \overline{g})h.
\end{aligned}
$$

(ii) Using Lemma 6.2.25, item (x), we can compute:

$$
\begin{aligned}
\overline{f \sqcup_{\oplus} g} &= f \sqcup_{\oplus} g(f \sqcup_{\oplus} g)^{(-1)} \\
&= \left((\overline{f} \triangledown \overline{g})(f \vartriangle g)\right)\left((f \triangledown g)^{(-1)}(\overline{f} \triangledown \overline{g})^{(-1)}\right) \\
&= \overline{f} \triangledown \overline{g}(f \triangledown g)(f \triangledown g)^{(-1)}\overline{f} \vartriangle \overline{g} \\
&= \overline{f} \triangledown \overline{g}(\overline{f} \oplus \overline{g})\overline{f} \vartriangle \overline{g} \\
&= \overline{f} \triangledown \overline{g}\overline{f} \vartriangle \overline{g} \\
&= \overline{f} \sqcup_{\oplus} \overline{g}
\end{aligned}
$$

□

Proposition 6.2.33. *Let $\mathbb{X}$ be an inverse category with a disjoint sum tensor and restriction zero. Assume we have two maps f, g with $f \perp_{\oplus} g$. Then the map $f \sqcup_{\oplus} g$ from Definition 6.2.31 is a disjoint join.*

Proof. [**DJ.1**] We must show $f, g \leq f \sqcup_{\oplus} g$. Computing,

$$
\begin{aligned}
\overline{f}\,(\overline{f} \triangledown \overline{g}) f \vartriangle g &= (\overline{f} \triangledown \overline{g})\amalg_1{}^{(-1)}(\overline{f} \triangledown \overline{g}) f \vartriangle g \\
&= \overline{(\overline{f} \triangledown \overline{g})\amalg_1{}^{(-1)}}(\overline{f} \triangledown \overline{g}) f \vartriangle g \\
&= (\overline{f} \triangledown \overline{g})\overline{\amalg_1{}^{(-1)}} f \vartriangle g \\
&= (\overline{f} \triangledown \overline{g})\amalg_1{}^{(-1)}\ \amalg_1\, f \vartriangle g \\
&= ((\overline{f} \triangledown \overline{g})\amalg_1{}^{(-1)})(\amalg_1(f \vartriangle g)) \\
&= \overline{f} f \\
&= f
\end{aligned}
$$

we see $f \leq f \sqcup_{\oplus} g$. Showing $g \leq f \sqcup_{\oplus} g$ proceeds in the same manner.

[**DJ.2**] We must show that $f \leq h,\ g \leq h$ and $f \perp_{\oplus} g$ implies $f \sqcup_{\oplus} g \leq h$. First, note that

$$
\begin{aligned}
\overline{f \sqcup_{\oplus} g}\, h &= \overline{\overline{f} h \sqcup_{\oplus} \overline{g} h}\, h \\
&= \overline{(\overline{f} \sqcup_{\oplus} \overline{g})h}\, h \\
&= \overline{\overline{(\overline{f} \sqcup_{\oplus} \overline{g})}h}\, h \\
&= \overline{\overline{(\overline{f} \sqcup_{\oplus} \overline{g})}h}\, (\overline{f} \sqcup_{\oplus} \overline{g})h \\
&= \overline{(\overline{f} \sqcup_{\oplus} \overline{g})h}\, (\overline{f} \sqcup_{\oplus} \overline{g})h \\
&= (\overline{f} \sqcup_{\oplus} \overline{g})h \\
&= (\overline{f} h \sqcup_{\oplus} \overline{g} h) \\
&= (f \sqcup_{\oplus} g)
\end{aligned}
$$

[**DJ.3**] We must show stability of $\sqcup_{\oplus}$, i.e., that $h(f \sqcup_{\oplus} g) = hf \sqcup_{\oplus} hg$.

$$
\begin{aligned}
h(f \sqcup_{\oplus} g) &= h((\overline{f} \triangledown \overline{g})(f \vartriangle g)) \\
&= (h\overline{f} \triangledown h\overline{g})(f \vartriangle g) \\
&= (\overline{hf}h \triangledown \overline{hg}h)(f \vartriangle g) \\
&= (\overline{hf} \triangledown \overline{hg})(h \oplus h)(f \vartriangle g) \\
&= (\overline{hf} \triangledown \overline{hg})(hf \vartriangle hg) \\
&= hf \sqcup_{\oplus} hg
\end{aligned}
$$

[**DJ.4**] We need to show $\perp_{\oplus}[f, g, h]$ if and only if $f \perp_{\oplus} (g \sqcup_{\oplus} h)$. For the right to left implication, note that the existence of $g \sqcup_{\oplus} h$ implies $g \perp_{\oplus} h$. We also know $g, h \leq g \sqcup_{\oplus} h$ by item 1 of this lemma. This gives us that $f \perp_{\oplus} g$ and $f \perp_{\oplus} h$, hence $\perp_{\oplus}[f, g, h]$. For the left to right implication, we use Lemma 6.2.29. As we have $\perp_{\oplus}[f, g, h]$, we know $f \triangledown (g \triangledown h)$ and $f \vartriangle (g \vartriangle h)$.

Recall that $g \sqcup_{\oplus} h = (g \triangledown h)(\hat{g} \vartriangle \hat{h})$. Then the map

$$A \xrightarrow{f \triangledown (g \triangledown h)} B \oplus B \oplus B \xrightarrow{1 \oplus (\hat{g} \vartriangle \hat{h})} B \oplus B$$

makes the diagram for $f \triangledown (g \sqcup_{\oplus} h)$ commute.

Recalling that $g \sqcup_{\oplus} h = (\overline{g} \triangledown \overline{h})(g \vartriangle h)$, we also see that

$$A \oplus A \xrightarrow{1 \oplus (\overline{g} \triangledown \overline{h})} A \oplus A \oplus A \xrightarrow{f \vartriangle (g \vartriangle h)} B$$

provides the witness map for $f \vartriangle (g \sqcup_{\oplus} h)$ and hence $f \perp_{\oplus} (g \sqcup_{\oplus} h)$.

□

6.3 Inverse sum categories

6.3.1 Inverse sums

Definition 6.3.1. In an inverse category with disjoint joins, an object X is the *inverse sum* of A and B when there exist maps $i_1,\ i_2,\ x_1,\ x_2$ such that:

(i) i_1 and i_2 are monic;

(ii) $i_1 : A \to X$, $i_2 : B \to X$, $x_1 : X \to A$ and $x_2 : X \to B$.

(iii) ${i_1}^{(-1)} = x_1$ and ${i_2}^{(-1)} = x_2$.

(iv) ${i_1}^{(-1)}i_1 \perp {i_2}^{(-1)}i_2$ and ${i_1}^{(-1)}i_1 \sqcup {i_2}^{(-1)}i_2 = 1_X$.

i_1 and i_2 will be referred to as the *injection* maps of the inverse sum.

Lemma 6.3.2. *The inverse sum X of A and B is unique up to isomorphism.*

Proof. Assume we have two inverse sums over A and B:

$$\begin{tikzcd} A \arrow[r, shift left, "i_1"] & X \arrow[l, shift left, bend left, "x_0"] \arrow[r, shift right, bend right, "x_1"'] & B \arrow[l, "i_2"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, shift left, "j_1"] & Y \arrow[l, shift left, bend left, "y_1"] \arrow[r, shift right, bend right, "y_2"'] & B \arrow[l, "j_2"'] \end{tikzcd}.$$

We will show that the map $x_1 j_1 \sqcup x_2 j_2 : X \to Y$ is an isomorphism.

Note by the fact that i_2 is monic, we may conclude from the definition that $0 = \overline{x_1 i_1 x_2}$ and therefore $0 = x_1 i_1 x_2$. Then, given that x_1 is the inverse of the monic i_1, we may calculate $0 = \hat{0} = \widehat{x_2 i_1 x_2} = \overline{{x_2}^{(-1)}\, {i_1}^{(-1)}\, i_1} = \overline{{x_2}^{(-1)}\, {i_1}^{(-1)}} = \widehat{i_1 x_2}$. From this we see $i_1 x_2 = 0$. Similarly, we have $i_2 x_2 = 0$, $j_1 y_2 = 0$ and $j_2 y_1 = 0$.

Next, by Lemma 6.2.5, we know that $\overline{x_1} \perp \overline{x_2}$ as both i_1 and i_2 are monic. By the same lemma, $\hat{j_1} \perp \hat{j_2}$ as y_1, y_2 are the inverses of monic maps. Then, from [**Dis.7**], we have $x_2 j_1 \perp x_2 j_2$, hence we may form $x_2 j_1 \sqcup x_2 j_2 : X \to Y$.

Similarly, we may form the map $y_1 i_1 \sqcup y_2 i_2 : Y \to X$. Computing their composition:

$$\begin{aligned} (x_2 j_1 \sqcup x_2 j_2)(y_1 i_1 \sqcup y_2 i_2) &= (x_2 j_1 (y_1 i_1 \sqcup y_2 i_2)) \sqcup (x_2 j_2 (y_1 i_1 \sqcup y_2 i_2)) \\ &= x_2 j_1 y_1 i_1 \sqcup x_2 j_1 y_2 i_2 \sqcup x_2 j_2 y_1 i_1 \sqcup x_2 j_2 y_2 i_2 \\ &= x_2\, 1\, i_1 \sqcup x_2\, 0\, i_2 \sqcup x_2\, 0\, i_1 \sqcup x_2\, 1\, i_2 \\ &= x_2 i_1 \sqcup x_2 i_2 = 1. \end{aligned}$$

Computing the other direction,

$$
\begin{aligned}
(y_1 i_1 \sqcup y_2 i_2)(x_2 j_1 \sqcup x_2 j_2) &= (y_1 i_1 (x_2 j_1 \sqcup x_2 j_2)) \sqcup (y_2 i_2 (x_2 j_1 \sqcup x_2 j_2)) \\
&= y_1 i_1 x_2 j_1 \sqcup y_1 i_1 x_2 j_2 \sqcup y_2 i_2 x_2 j_1 \sqcup y_2 i_2 x_2 j_2 \\
&= y_1\, 1\, j_1 \sqcup y_1\, 0\, j_2 \sqcup y_2\, 0\, j_1 \sqcup y_2\, 1\, j_2 \\
&= y_1 j_1 \sqcup y_2 j_2 = 1.
\end{aligned}
$$

This shows that the map between any two inverse sums over the same two objects is an isomorphism. □

Lemma 6.3.3. *Suppose X is the inverse sum of A and B in the inverse category $\mathbb{X}$. Then for all maps $f : C \to A$ and $g : C \to B$, the composition with the injections is disjoint, that is, $fi_1 \perp gi_2$. (This is not right - need further thought...)*

Proof. First note $fi_1 = fi_1\widehat{i_1} = fi_1 {i_1}^{(-1)} i_1$ and similarly, $gi_2 = gi_2 {i_2}^{(-1)} i_2$. □

Definition 6.3.4. Suppose $\mathbb{X}$ is an inverse category with with disjoint joins $\sqcup$ based on a disjointness relation $\perp$ and a restriction zero. If every pair of objects has an inverse sum as in Definition 6.3.1, we call the category an *inverse sum* category. For any two objects A, B in $\mathbb{X}$, we write their inverse sum as $A + B$.

Lemma 6.3.5. *Let $\mathbb{X}$ be an inverse category with a restriction 0 and a disjoint sum tensor $\oplus$. Then $\mathbb{X}$ is an inverse sum category.*

Proof. We claim that setting $i_i = \amalg_i$ and $x_i = {\amalg_i}^{(-1)}$ and setting $X = A \oplus B$ produces inverse sums in $\mathbb{X}$ and show this satisfies the four conditions of Definition 6.3.1.

(i) From Lemma 6.2.22, we know that $\amalg_1$ and $\amalg_2$ are monic maps.

(ii) $\amalg_1 : A \to A \oplus B$, $\amalg_2 : B \to A \oplus B$, ${\amalg_1}^{(-1)} : A \oplus B \to A$ and ${\amalg_2}^{(-1)} : A \oplus B \to B$.

(iii) ${\amalg_1}^{(-1)} = {\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)} = {\amalg_2}^{(-1)}$.

(iv) ${i_1}^{(-1)} i_1 = 1 \oplus 0 \perp_{\oplus} 0 \oplus 1 = {i_2}^{(-1)} i_2$ as $1 \oplus 0 \triangledown 0 \oplus 1 = ({u^r_{\oplus}}^{(-1)} \oplus {u^l_{\oplus}}^{(-1)})$ and $1 \oplus 0 \vartriangle 0 \oplus 1 = ({\amalg_1}^{(-1)} \oplus {\amalg_2}^{(-1)})$. For their join, $(1 \oplus 0) \sqcup_{\oplus} (0 \oplus 1) = ({u^r_{\oplus}}^{(-1)} \oplus {u^l_{\oplus}}^{(-1)})({\amalg_1}^{(-1)} \oplus {\amalg_2}^{(-1)}) = {u^r_{\oplus}}^{(-1)} {\amalg_1}^{(-1)} \oplus {u^l_{\oplus}}^{(-1)} {\amalg_2}^{(-1)} = 1 \oplus 1 = 1$.

□

Lemma 6.3.6. *If A is an object in $\mathbb{X}$, an inverse sum category, then $A + 0$ is isomorphic to A.*

Proof. We write the inverse sum diagram:

$$\begin{tikzcd} A \arrow[r, "1"] & A \arrow[l, bend left, "1"] \arrow[r, bend right, "0"'] & 0. \arrow[l, "0"'] \end{tikzcd}$$

□

Lemma 6.3.7. *Suppose $\mathbb{X}$ is an inverse sum category and $\mathbb{Y}$ is an inverse category with a restriction zero. Further, suppose $F : \mathbb{X} \to \mathbb{Y}$ is a restriction functor which preserves disjoint joins. Then, F preserves inverse sums.*

Proof. In $\mathbb{X}$, consider the inverse sum over A and B,

$$\begin{tikzcd} A \arrow[r, "i_1"] & X \arrow[l, bend left, "x_0"] \arrow[r, bend right, "x_1"'] & B \arrow[l, "i_2"'] \end{tikzcd}.$$

The functor F maps this as follows:

$$\begin{tikzcd} F(A) \arrow[r, "F(i_1)"] & F(X) \arrow[l, bend left, "F(x_0)"] \arrow[r, bend right, "F(x_1)"'] & F(B) \arrow[l, "F(i_2)"'] \end{tikzcd}.$$

As F is a restriction functor, we immediately have $F(x_0) = F({i_1}^{(-1)}) = F(i_1)^{(-1)}$ and $F(x_1) = F(i_2)^{(-1)}$. Since F preserves the disjoint join, we also have $F(i_1)^{(-1)} F(i_1) \perp F(i_2)^{(-1)} F(i_2)$ and $F(i_1)^{(-1)} F(i_1) \sqcup F(i_2)^{(-1)} F(i_2) = 1$.

Finally, as F is a restriction functor, it preserves monics, hence $F(i_1)$ and $F(i_2)$ are both monic and therefore $F(X)$ is the inverse sum of $F(A)$ and $F(B)$.

□

Lemma 6.3.8. *Given $\mathbb{X}$ an inverse sum category and maps $f : A \to C$ and $g : B \to D$ in $\mathbb{X}$. Then $i_1{}^{(-1)} f i_1 \perp i_2{}^{(-1)} g i_2 : A + B \to A + B$.*

Proof. Note that $\overline{i_1{}^{(-1)} f i_1} = \overline{i_1{}^{(-1)} f} \leq \overline{i_1{}^{(-1)}}$ and similarly $\overline{i_2{}^{(-1)} g i_2} \leq \overline{i_2{}^{(-1)}}$. Then, by [**Dis.3**], we have $\overline{i_1{}^{(-1)} f i_1} \perp \overline{i_2{}^{(-1)} g i_2}$. As $\widehat{i_1{}^{(-1)} f i_1} \leq \widehat{i_1}$ and $\widehat{i_2{}^{(-1)} g i_2} \leq \widehat{i_2}$, we have $\widehat{i_1{}^{(-1)} f i_1} \perp \widehat{i_2{}^{(-1)} g i_2}$ and by Lemma 6.2.5, this means $i_1{}^{(-1)} f i_1 \perp i_2{}^{(-1)} g i_2$. □

Lemma 6.3.9. *Given $\mathbb{X}$ is an inverse sum category. Denote the inverse sum of objects A, B of $\mathbb{X}$ by $A + B$. Then for objects A, B and X with maps $f : A \to X$ and $g : B \to X$ such that $\hat{f} \perp \hat{g}$, there exists a unique map h making the following diagram commute.*

$$\begin{tikzcd}
A \arrow[d, "i_1"'] \arrow[dr, "f"] & \\
A + B \arrow[r, dotted, "h"] & X. \\
B \arrow[u, "i_2"] \arrow[ur, "g"'] &
\end{tikzcd}$$

We use the notation $f \hat{+} g$ for the unique map h.

Proof. As $\hat{f} \perp \hat{g}$ and $\overline{i_1{}^{(-1)}} \perp \overline{i_2{}^{(-1)}}$ we may form the map $h' = i_1{}^{(-1)} f \sqcup i_2{}^{(-1)} g$. By its construction, h' is a map from $A + B$ to X which makes the diagram commute. Suppose now that both maps v and w are such maps. Then we have

$$(i_1{}^{(-1)} i_1) v = (i_1{}^{(-1)} i_1) w \quad \text{and} \quad (i_2{}^{(-1)} i_2) v = (i_2{}^{(-1)} i_2) w.$$

As $i_1{}^{(-1)} i_1 \perp i_2{}^{(-1)} i_2$, by Lemmas 6.2.5 and 6.2.15, we know that $(i_1{}^{(-1)} i_1) v \perp (i_2{}^{(-1)} i_2) v$ and $(i_1{}^{(-1)} i_1) w \perp (i_2{}^{(-1)} i_2) w$ allowing us to form their respective disjoint joins. As the disjoint joins of equal maps remains equal, we have

$$(i_1{}^{(-1)} i_1) v \sqcup (i_2{}^{(-1)} i_2) v = (i_1{}^{(-1)} i_1) w \sqcup (i_2{}^{(-1)} i_2) w$$

$$(i_1{}^{(-1)} i_1 \sqcup i_2{}^{(-1)} i_2) v = (i_1{}^{(-1)} i_1 \sqcup i_2{}^{(-1)} i_2) w$$

$$(1) v = (1) w$$

$$v = w.$$

$\square$

Corollary 6.3.10. *Given $\mathbb{X}$ is an inverse sum category. Then for objects A, B and X with maps $f : X \to A$ and $g : X \to B$ such that $\overline{f} \perp \overline{g}$, there exists a unique map h making the following diagram commute.*

$$\begin{tikzcd}
 & A \arrow[d, "i_1"] \\
X \arrow[ur, "f"] \arrow[r, dotted, "h"] \arrow[dr, "g"'] & A + B \\
 & B. \arrow[u, "i_2"']
\end{tikzcd}$$

We use the notation $f \mathbin{\overline{+}} g$ for the unique map h.

Proof. This is simply the dual of Lemma 6.3.9. The unique map h in this case is $fi_1 \sqcup gi_2$. $\square$

Corollary 6.3.11. *Suppose $\mathbb{X}$ is an inverse sum category. Then for objects A, B, C and D with maps $f : A \to C$ and $g : B \to D$, there exists a unique map h making the following diagram commute.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_1"'] & C \arrow[d, "i_1"'] \\
A + B \arrow[r, dotted, "h"] & C + D \\
B \arrow[u, "i_2"'] \arrow[r, "g"'] & D. \arrow[u, "i_2"']
\end{tikzcd}$$

We use the notation $f + g$ for the map h.

Proof. This follows directly from Lemma 6.3.9 by setting $X = C + D$. The unique map in this case is ${i_1}^{(-1)} f i_1 \sqcup {i_2}^{(-1)} g i_2$. $\square$

Lemma 6.3.12. *Suppose $\mathbb{X}$ and $\mathbb{Y}$ are inverse sum categories and $F : \mathbb{X} \to \mathbb{Y}$ is a restriction functor which preserves inverse sums. Then, F preserves disjoint joins.*

Proof. By stating that F preserves the inverse sum, we mean it preserves diagrams derived via the properties of the inverse sum, and specifically, it will preserve the diagrams of Lemma 6.3.9 and Corollaries 6.3.10 and 6.3.11.

Suppose we are given $f, g : A \to B$ with $f \perp g$. In the inverse sum category, we know that $f \sqcup g = (\overline{f} i_1 \sqcup \overline{g} i_2)(i_1{}^{(-1)} f i_1 \sqcup i_2{}^{(-1)} g i_2)(i_1{}^{(-1)} \hat{f} \sqcup i_2{}^{(-1)} \hat{g})$, as this follows by:

1. Apply Corollary 6.3.10 to $\overline{f}$ and $\overline{g}$;
2. then apply Corollary 6.3.11 to f, g;
3. finally apply Lemma 6.3.9 to $\hat{f}, \hat{g}$.

In the notation above, we have that $f \sqcup g = (\overline{f} \mathbin{\overline{+}} \overline{g})(f + g)(\hat{f} \mathbin{\hat{+}} \hat{g})$. This gives us, as F preserves the inverse sum:

$$\begin{aligned}
F(f \sqcup g) &= F(\overline{f} \mathbin{\overline{+}} \overline{g}) F(f + g) F(\hat{f} \mathbin{\hat{+}} \hat{g}) \\
&= (F(\overline{f}) \mathbin{\overline{+}} F(\overline{g}))(F(f) + F(g))(F(\hat{f}) \mathbin{\hat{+}} F(\hat{g})) \\
&= (\overline{F(f)} \mathbin{\overline{+}} \overline{F(g)})(F(f) + F(g))(\widehat{F(f)} \mathbin{\hat{+}} \widehat{F(g)}) \\
&= F(f) \sqcup F(g).
\end{aligned}$$

The last line in the above is due to $\mathbb{Y}$ being an inverse sum category as well.

□

6.3.2 Inverse sum tensor

Inverse sum tensor definitions

Definition 6.3.13. An *inverse sum tensor* in an inverse category $\mathbb{X}$ with disjoint joins $\sqcup$ based on a disjointness relation $\perp$ and a restriction zero is given by a tensor combined with two restriction monics, $\amalg_1$ and $\amalg_2$. The data for the tensor is:

$$\begin{aligned}
_ \oplus _ &: \mathbb{X} \times \mathbb{X} \to \mathbb{X} \quad \text{(a restriction functor preserving disjoint joins)}\\
0 &: \mathbf{1} \to \mathbb{X}\\
u_\oplus^l &: 0 \oplus A \to A\\
u_\oplus^r &: A \oplus 0 \to A\\
a_\oplus &: (A \oplus B) \oplus C \to A \oplus (B \oplus C)\\
c_\oplus &: A \oplus B \to B \oplus A\\
\amalg_1 &: A \to A \oplus B\\
\amalg_2 &: B \to A \oplus B
\end{aligned}$$

where $u_\oplus^l, u_\oplus^r, a_\oplus, c_\oplus$ are all isomorphisms and the standard symmetric monoidal equations and coherence diagrams hold. The unit of the tensor, $0 : \mathbf{1} \to \mathbb{X}$, is the restriction zero of the category. We specifically note that preserving disjoint joins means the tensor obeys the following two equations:

$$f \perp g,\ h \perp k \text{ implies } f \oplus h \perp g \oplus k \tag{6.6}$$

$$f \perp g,\ h \perp k \text{ implies } (f \sqcup g) \oplus (h \sqcup k) = (f \oplus h) \sqcup (g \oplus k). \tag{6.7}$$

Disjointness and the inverse sum tensor

Lemma 6.3.14. *Given an inverse category $\mathbb{X}$ with a disjoint sum tensor $\oplus$ as in Definition 6.2.28, then $\oplus$ is an inverse sum tensor.*

Proof. From the data of the disjoint sum tensor, the only thing remaining to show is that the tensor preserves the disjoint join.

Suppose we have $f \perp_\oplus g$ and $h \perp_\oplus k$. From Lemma 6.2.25, item (xi), we know both $(f \oplus h) \triangledown (g \oplus k)$ and $(f \oplus h) \vartriangle (g \oplus k)$ exist, hence $(f \oplus h) \perp_\oplus (g \oplus k)$. This shows condition ((6.6)).

For condition ((6.7)), we compute from the right hand side:

$$
\begin{aligned}
(f \oplus h) \sqcup_{\oplus} (g \oplus k) &= (f \oplus h) \triangledown (g \oplus k) (\widehat{f \oplus h}) \vartriangle (\widehat{g \oplus k}) \\
&= ((f \triangledown g) \oplus (h \triangledown k)) \left((\hat{f} \oplus \hat{h}) \vartriangle (\hat{g} \oplus \hat{k})\right) \\
&= ((f \triangledown g) \oplus (h \triangledown k)) \left((\hat{f} \vartriangle \hat{g}) \oplus (\hat{h} \vartriangle \hat{k})\right) \\
&= \left((f \triangledown g)(\hat{f} \vartriangle \hat{g})\right) \oplus \left((h \triangledown k)(\hat{h} \vartriangle \hat{k})\right) \\
&= (f \sqcup_{\oplus} g) \oplus (h \sqcup_{\oplus} k).
\end{aligned}
$$

The second and third lines above again use Lemma 6.2.25, item (xi).

$\square$

Inverse sums and the inverse sum tensor

Lemma 6.3.15. *If $\oplus$ is an inverse sum tensor in the inverse category $\mathbb{X}$, then $A \oplus B \cong A + B$, an inverse sum of A and B.*

Proof. As $\oplus$ is a restriction functor from $\mathbb{X} \times \mathbb{X}$ to $\mathbb{X}$, this actually follows immediately from Lemma 6.3.7. It may also be proven directly:

Draw the inverse sum diagram:

$$
\begin{tikzcd}
A \arrow[r, "i_1 = {u^r_{\oplus}}^{(-1)}(1 \oplus 0)"] & A \oplus B \arrow[l, bend left, "x_0 = (1 \oplus 0) u^r_{\oplus}"] \arrow[r, bend right, "x_1 = (0 \oplus 1) u^l_{\oplus}"'] & B \arrow[l, "i_2 = {u^l_{\oplus}}^{(-1)}(0 \oplus 1)"']
\end{tikzcd} .
$$

Therefore, we have ${i_1}^{(-1)} i_1 = (1 \oplus 0) u^r_{\oplus} {u^r_{\oplus}}^{(-1)} (1 \oplus 0) = (1 \oplus 0)(1 \oplus 0) = (1 \oplus 0)$. Similarly, ${i_2}^{(-1)} i_2 = (0 \oplus 1)$. Since $0 \perp 1$, we have ${i_1}^{(-1)} i_1 \perp {i_2}^{(-1)} i_2$.

By the functorality of $\oplus$ and that it preserves disjoint joins, we have $(1 \oplus 0) \sqcup (0 \oplus 1) = (1 \sqcup 0) \oplus (0 \sqcup 1) = 1 \oplus 1 = 1_{A \oplus B}$. Hence $A \oplus B$ is an inverse sum of A and B and by Lemma 6.3.2 it is isomorphic to $A + B$. $\square$

Conversely, we can show that given a tensor which produces inverse sums, that tensor will be an inverse sum tensor.

Lemma 6.3.16. *Given an inverse category $\mathbb{X}$ with restriction zero, a disjointness relation $\perp$, a disjoint join $\sqcup$ and a symmetric monoidal tensor $\oplus$, with natural restriction monics $\amalg_1 : A \to A \oplus B$ and $\amalg_2 : B \to A \oplus B$ such that $A \oplus B$ is an inverse sum under $\amalg_1$ and $\amalg_2$, then when $f, g : A \to B$ and $h, k : C \to D$ with $f \perp g$ and $h \perp k$, then $f \oplus h \perp g \oplus k$ and $(f \oplus h) \sqcup (g \oplus k) = (f \sqcup g) \oplus (h \sqcup k)$.*

Proof. Similarly, this follows immediately from Lemma 6.3.12. We show it directly below:

$$\begin{tikzcd}
A \arrow[r, "\overline{f}"] \arrow[d, "\amalg_1"'] \arrow[rr, bend left, "f"] & A \arrow[r, "f \sqcup g"] \arrow[d, "\amalg_1"'] & B \arrow[d, "\amalg_1"'] \\
A \oplus C \arrow[r, "\overline{f} \oplus \overline{h}", "(1)"'] & A \oplus C \arrow[r, "f \sqcup g \oplus h \sqcup k", "(2)"'] & B \oplus D \\
C \arrow[r, "\overline{h}"'] \arrow[u, "\amalg_2"] \arrow[rr, bend right, "h"'] & C \arrow[r, "h \sqcup k"'] \arrow[u, "\amalg_2"] & D \arrow[u, "\amalg_2"]
\end{tikzcd}$$

Consider $\amalg_1{}^{(-1)}\overline{f}\amalg_1$. As this is idempotent and we are in an inverse category, we know that $\amalg_1{}^{(-1)}\overline{f}\amalg_1 = \overline{\amalg_1{}^{(-1)}\overline{f}\amalg_1} = \overline{\amalg_1{}^{(-1)}\overline{f}} = \widehat{\overline{f}\amalg_1}$. Similarly, $\amalg_2{}^{(-1)}\overline{h}\amalg_2 = \widehat{\overline{h}\amalg_2}$. By [**Dis.5**] and [**Dis.6**], we know that $\widehat{\overline{f}\amalg_1} \perp \widehat{\overline{g}\amalg_1}$ and $\widehat{\overline{h}\amalg_2} \perp \widehat{\overline{k}\amalg_2}$. Additionally, as shown in the proof of Lemma 6.3.2, we know $\widehat{\amalg_1} \perp \widehat{\amalg_2}$. Hence, by [**Dis.3**], we have $\widehat{\overline{x}\amalg_1} \perp \widehat{\overline{y}\amalg_2}$ for any maps x, y.

Hence, we can form the map $\widehat{\overline{f}\amalg_1} \sqcup \widehat{\overline{h}\amalg_2}$. Referring to the commutative diagram above, by Corollary 6.3.11 there is an unique map at location (1) which makes the diagram commute — currently given as $\overline{f} \oplus \overline{h}$. But, the map $\widehat{\overline{f}\amalg_1} \sqcup \widehat{\overline{h}\amalg_2}$ also satisfies this, hence we have $\widehat{\overline{f}\amalg_1} \sqcup \widehat{\overline{h}\amalg_2} = \overline{f} \oplus \overline{h}$. Similarly, $\widehat{\overline{g}\amalg_1} \sqcup \widehat{\overline{k}\amalg_2} = \overline{g} \oplus \overline{k}$. But, by Lemma 6.2.17, this means $\overline{f \oplus h} \perp \overline{g \oplus k}$.

Using a similar argument based on the diagram

$$\begin{tikzcd}
A \arrow[r, "f\sqcup g"] \arrow[d, "\amalg_1"'] \arrow[rr, bend left, "f"] & B \arrow[r, "f"] \arrow[d, "\amalg_1"'] & B \arrow[d, "\amalg_1"'] \\
A \oplus C \arrow[r, "f\sqcup g \oplus h\sqcup k", "(2)"'] & B \oplus D \arrow[r, "\hat{f}\oplus\hat{h}", "(3)"'] & B \oplus D \\
C \arrow[u, "\amalg_2"] \arrow[r, "h\sqcup k"'] \arrow[rr, bend right, "h"'] & D \arrow[u, "\amalg_2"] \arrow[r, "\hat{h}"'] & D \arrow[u, "\amalg_2"]
\end{tikzcd}$$

we can show $\widehat{f \oplus h} \perp \widehat{g \oplus k}$ and therefore $f \oplus h \perp g \oplus k$.

This allows us to form the map $(f \oplus h) \sqcup (g \oplus k)$. Once again, as the objects are inverse sums, the map at (2) is unique. However, we see that both $f \sqcup g \oplus h \sqcup k$ and $(f \oplus h) \sqcup (g \oplus k)$ fulfill this requirement and hence they are equal. $\square$

Definition 6.3.17. An inverse category $\mathbb{X}$ with restriction zero, a disjointness relation $\perp$, a disjoint join $\sqcup$ and an inverse sum tensor $\oplus$ is called an *inverse sum tensor category.*

Corollary 6.3.18. *In an inverse sum tensor category, $f \oplus g$ is given by ${i_1}^{(-1)} f i_1 \sqcup {i_2}^{(-1)} g i_2$.*

Proof. Recall that in the proof of Lemma 6.3.2 that we showed $\overline{{i_1}^{(-1)}} \perp \overline{{i_2}^{(-1)}}$ and $\widehat{i_1} \perp \widehat{i_2}$. Since $\overline{xf} \leq \overline{x}$, by [**Dis.3**] and [**Dis.7**], we know that ${i_1}^{(-1)} f i_1 \perp {i_2}^{(-1)} g i_2$ and we can therefore form the disjoint join. $\square$

Matrices

In this section, we will show that when given an inverse category $\mathbb{X}$ with a disjoint sum tensor, one can define a matrix category based on $\mathbb{X}$. We will call this category $iMat(\mathbb{X})$. Furthermore, we will show that $iMat(\mathbb{X})$ has is an inverse category and that $\mathbb{X}$ embeds within this category.

Definition 6.3.19. Any matrix of maps $[f_{ij}]$ where $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ with $f_{ij} : A_i \to B_j$ in an inverse category $\mathbb{X}$ with a disjoint sum tensor which satisfy the two conditions:

$$\text{For each } i,\ \perp [f_{ij}\amalg_j]_{j=1,\ldots,m} \text{ where } \amalg_j : B_j \to B_1 \oplus B_2 \oplus \cdots \oplus B_m \tag{6.8}$$

$$\text{For each } j,\ \perp [\amalg_i^{(-1)} f_{ij}]_{i=1,\ldots,n} \text{ where } \amalg_i^{(-1)} = \amalg_i^{(-1)} : A_1 \oplus A_2 \oplus \cdots \oplus A_n \to A_i \tag{6.9}$$

is called an *inverse sum matrix*.

In the above and following we will use the notation $\amalg_i$ for the i^{th} injection map of the disjoint sum tensor, with i starting at 1. This simply extends the notation introduced in Definition 6.2.19.

We will show that this type of matrix corresponds to maps in a the category $iMat(\mathbb{X})$, where composition is given by "matrix multiplication", where the operations of multiplication and addition are replaced with composition in $\mathbb{X}$ and the disjoint join respectively.

Definition 6.3.20. Given an inverse category $\mathbb{X}$ with a disjoint sum tensor, we define the *inverse matrix category* of $\mathbb{X}$, $iMat(\mathbb{X})$, as follows:

Objects: Non-empty lists of the objects of $\mathbb{X}$.

Maps: Inverse sum matrices $[f_{ij}] : [A_i] \to [B_j]$. In such a matrix each individual map $f_{ij} : A_i \to B_j$ is a map in $\mathbb{X}$. For each j, B_j is given by applying the map $\sqcup_i \amalg_i^{(-1)} f_{ij}$ to the object $\oplus_i A_i$

Identity: The inverse sum matrix I.

Composition: Given $[f_{ij}] : [A_i] \to [B_j]$ and $[g_{jk}] : [B_j] \to [C_k]$, then $[h_{ik}] = [f_{ij}][g_{jk}] : [A_i] \to [C_k]$ is defined as $h_{ik} = \bigsqcup_j f_{ij} g_{jk}$.

Restriction: We set $\overline{[f_{ij}]}$ to be $[f'_{ij}]$ where $f'_{ij} = 0$ when $i \neq j$ and $f'_{ii} = \sqcup_j \overline{f_{ij}}$.

In the following, we will use the notation $\text{diag}[d_1, d_2, \ldots, d_n]$ for a diagonal $n \times n$ matrix with entries $[d_1, d_2, \ldots, d_n]$ and $\text{diag}_j[d_j]$ for diagonal matrices where the j, j entry is d_j.

Lemma 6.3.21. *When $\mathbb{X}$ is an inverse sum category, $iMat(\mathbb{X})$ is a restriction category.*

Proof. We need to show the following:

- Composition is well defined and associative;
- The restriction is well defined.

Composition is well defined: Consider $[h_{ik}] = [f_{ij}][g_{jk}]$ where $[f_{ij}] : [A_1, \ldots, A_n] \to [B_1, \ldots, B_m]$ and $[g_{jk}] : [B_1, \ldots, B_m] \to [C_1, \ldots, C_\ell]$. By supposition, we know $h_{ik} = \bigsqcup_j f_{ij} g_{jk}$. As each of the maps are inverse sum matrices, we know that $\perp [f_{ij} \amalg_j]$ and $\perp [{\amalg_j}^{(-1)} g_{jk}]$. Hence, For each j we know the composition $f_{ij} \amalg_j {\amalg_j}^{(-1)} g_{jk} = f_{ij} g_{jk}$ is defined and from A_i to C_k. By the the stability and universality of $\sqcup$, we know h_{ik} exists and by the definition of $\sqcup$, we have each $h_{ik} : A_i \to C_k$ and hence composition is well-defined.

Associativity of composition. We have

$$
\begin{aligned}
([f_{ij}][g_{jk}])[h_{k\ell}] &= \left[(\bigsqcup_j f_{ij} g_{jk}) \right] [h_{k\ell}] \\
&= \left[\bigsqcup_k (\bigsqcup_j f_{ij} g_{jk}) h_{k\ell} \right] \\
&= \left[\bigsqcup_j f_{ij} (\bigsqcup_k g_{jk} h_{k\ell}) \right] \\
&= [f_{ij}]([g_{jk}][h_{k\ell}])
\end{aligned}
$$

Proof the restriction axioms are held:

$$
[\mathbf{R.1}] \quad \overline{[f_{ij}]}[f_{ij}] = \begin{bmatrix} (\sqcup_j \overline{f_{1j}}) f_{11} & \cdots & (\sqcup_j \overline{f_{1j}}) f_{1n} \\ & \vdots & \\ (\sqcup_j \overline{f_{mj}}) f_{m1} & \cdots & (\sqcup_j \overline{f_{mj}}) f_{mn} \end{bmatrix} = [f_{ij}].
$$

$[\mathbf{R.2}]$ $\overline{[f_{ij}]}\overline{g_{ij}} = \overline{g_{ij}}\overline{[f_{ij}]}$ as diagonal matrices commute and $\sqcup$ is also commutative.

[R.3] $\overline{\overline{[f_i j]}[g_{jk}]} = \overline{\mathrm{diag}[\sqcup_j \overline{f_{1j}}, \ldots, \sqcup_j \overline{f_{nj}}][g_{jk}]}$

$$= \overline{\begin{bmatrix} \sqcup_j \overline{f_{1j}} g_{11} & \ldots & \sqcup_j \overline{f_{1j}} g_{1k} \\ & \vdots & \\ \sqcup_j \overline{f_{nj}} g_{n1} & \ldots & \sqcup_j \overline{f_{nj}} g_{nk} \end{bmatrix}}$$

$$= \mathrm{diag}[\sqcup_k(\overline{\sqcup_j(\overline{f_{1j}} g_{1k})}), \ldots, \sqcup_k(\overline{\sqcup_j(\overline{f_{nj}}) g_{nk}})]$$

$$= \mathrm{diag}[\sqcup_k(\sqcup_j(\overline{f_{1j}})\overline{g_{1k}}), \ldots, \sqcup_k(\sqcup_j(\overline{f_{nj}})\overline{g_{nk}})]$$

$$= \mathrm{diag}[(\sqcup_j(\overline{f_{1j}}) \sqcup_k \overline{g_{1k}}), \ldots, (\sqcup_j(\overline{f_{nj}}) \sqcup_k \overline{g_{nk}})]$$

$$= \overline{[f_i j]}\,\overline{[g_{jk}]}$$

[**R.4**] $\quad [f_{ij}]\overline{[g_{jk}]} = [f_{ij}]\,\mathrm{diag}_j[\sqcup_k \overline{g_{jk}}]$

$$= \begin{bmatrix} f_{11} \sqcup_k \overline{g_{1k}} & \dots & f_{1n} \sqcup_k \overline{g_{nk}} \\ & \vdots & \\ f_{m1} \sqcup_k \overline{g_{1k}} & \dots & f_{mn} \sqcup_k \overline{g_{nk}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqcup_k f_{11}\overline{g_{1k}} & \dots & \sqcup_k f_{1n}\overline{g_{nk}} \\ & \vdots & \\ \sqcup_k f_{m1}\overline{g_{1k}} & \dots & \sqcup_k f_{mn}\overline{g_{nk}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqcup_k \overline{f_{11}g_{1k}} f_{11} & \dots & \sqcup_k \overline{f_{1n}g_{nk}} f_{1n} \\ & \vdots & \\ \sqcup_k \overline{f_{m1}g_{1k}} f_{m1} & \dots & \sqcup_k \overline{f_{mn}g_{nk}} f_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \sqcup_j \sqcup_k \overline{f_{1j}g_{jk}} f_{11} & \dots & \sqcup_j \sqcup_k \overline{f_{1j}g_{jk}} f_{1n} \\ & \vdots & \\ \sqcup_j \sqcup_k \overline{f_{mj}g_{jk}} f_{m1} & \dots & \sqcup_j \sqcup_k \overline{f_{mj}g_{jk}} f_{mn} \end{bmatrix}$$

$$= \overline{[f_{ij}][g_{jk}]}[f_{ij}].$$

$\square$

Note that when $\mathbb{X}$ is an inverse category with a disjoint join, $iMat(\mathbb{X})$ is also an inverse category. The inverse of the map $f = [f_{ij}]$ is the map $f^{(-1)} := [f_j i^{(-1)}]$. Recalling that the rows and columns of f are each disjoint, we see that the composition $ff^{(-1)} = \mathrm{diag}_i[\sqcup_j \overline{f_{ij}}] = \overline{f}$.

Lemma 6.3.22. *Given $\mathbb{X}$ is an inverse restriction category with a restriction zero, 0, and a disjoint join, then $iMat(\mathbb{X})$ has a restriction zero.*

Proof. The restriction zero in $iMat(\mathbb{X})$ is the list $[0]$.

For the ojbect $A = [A_1, \ldots, A_n]$, the 0 map to it is given by the $n \times 1$ matrix $[0, \ldots, 0]$.

The map from 0 is given by the $1 \times n$ matrix $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

$\square$

Lemma 6.3.23. *Given $\mathbb{X}$ is an inverse restriction category with a restriction zero, 0, and a disjoint join, then the monoid $\oplus$ defined by list concatenation of objects is a disjointness tensor.*

Proof. We first note the monoidal isomorphisms:

$$u^l_\oplus : [0, A_1, A_2, \ldots, A_n] \to [A_1, A_2, \ldots, A_n] \qquad u^l_\oplus := \begin{bmatrix} 0 & \cdots & 0 \\ & I_{n\times n} & \end{bmatrix}$$

$$u^r_\oplus : [A_1, A_2, \ldots, A_n, 0] \to [A_1, A_2, \ldots, A_n] \qquad u^r_\oplus := \begin{bmatrix} & I_{n\times n} & \\ 0 & \cdots & 0 \end{bmatrix}$$

$$a_\oplus : (A \oplus B) \oplus C \to A \oplus (B \oplus C) \qquad a_\oplus := id$$

$$c_\oplus : [A_1, \ldots, A_n, B_1, \ldots, B_m] \to [B_1, \ldots, B_m, A_1, \ldots, A_n] \qquad c_\oplus := \begin{bmatrix} 0_{m\times n} & I_{n\times n} \\ I_{m\times m} & 0_{n\times m} \end{bmatrix}$$

The action of $\oplus$ on maps is given by:

$$[f_{ij}] \oplus [g_{\ell k}] = \begin{bmatrix} [f_{ij}] & 0 \\ 0 & [g_{\ell k}] \end{bmatrix}.$$

With this definition, we see that $\oplus$ is a restriction functor:

$$id_X \oplus id_Y = id_{X \oplus Y}$$

$$f_1 g_1 \oplus f_2 g_2 = h_1 \oplus h_2 = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ 0 & g_1 \end{bmatrix} \begin{bmatrix} f_2 & 0 \\ 0 & g_2 \end{bmatrix} = (f_1 \oplus g_1)(f_2 \oplus g_2)$$

Following Definition 6.2.19, we note ${\amalg_1}^{(-1)} = (1 \oplus 0)u^r_\oplus = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and similarly ${\amalg_2}^{(-1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose we have $f = [f_{ij}]$ and $g = [g_{ij}]$ where $i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$. Further suppose $f{\amalg_1}^{(-1)} = g{\amalg_1}^{(-1)}$ and $f{\amalg_1}^{(-1)} = g{\amalg_1}^{(-1)}$. Therefore, $f{\amalg_1}^{(-1)} = [f_{i1}] = [g_{i1}] = g{\amalg_1}^{(-1)}$ and $f{\amalg_2}^{(-1)} = [f_{i2}] = [g_{i2}] = g{\amalg_2}^{(-1)}$, but this means that $f = g$ and we may conclude ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ are jointly monic. Similarly, $\amalg_1 = [1\ 0]$ and $\amalg_2 = [0\ 1]$ are jointly epic. $\square$

Lemma 6.3.24. *Given $\mathbb{X}$ is an inverse category with a disjoint join and restriction zero, then $iMat(\mathbb{X})$ has a disjoint sum tensor.*

Proof. By Lemma 6.3.23, we know that the tensor defined by list catenation is a disjoint tensor. To show that it is disjoint sum tensor, we must fulfill Definition 6.2.28.

First, for the diagram below, we show that α exists if and only if $f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$. Note that diagram assumes all the solid arrows exist and make the diagram a commutative diagram.

$$\begin{tikzcd}
{[A]} \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& {[X,Y,Z]} \arrow[r, "{\amalg_{1,3}}^{(-1)}"'] \arrow[d, "{\amalg_{1,2}}^{(-1)}"] & {[X,Z]} \arrow[d, "{\amalg_1}^{(-1)}"] \\
& {[X,Y]} \arrow[r, "{\amalg_1}^{(-1)}"'] & {[X]}
\end{tikzcd}$$

$f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$ means there is an $h = [h_1, h_2] : [A] \to [Y, Z]$ such that $h{\amalg_1}^{(-1)} = f{\amalg_2}^{(-1)}$ and $h{\amalg_2}^{(-1)} = g{\amalg_2}^{(-1)}$. From the diagram, given that $f = [f_1, f_2]$ and $g = [g_1, g_2]$, we know that $f_1 = f{\amalg_1}^{(-1)} = g{\amalg_1}^{(-1)} = g_1$. We also have $h_1 = f_2$ and $h_2 = g_2$. If we set α to the matrix $[f_1, f_2, g_2]$, the diagram above commutes. We need only show that α is a map in

$iMat(\mathbb{X})$. As f, g and h are maps in $iMat(\mathbb{X})$, we know that:

$$f_1\amalg_1 \perp f_2\amalg_2$$
$$f_1\amalg_1 = g_1\amalg_1 \perp g_2\amalg_2$$
$$f_2\amalg_2 = h_1\amalg_1 \perp h_2\amalg_2 = g_2\amalg_2$$

From this, we can conclude $\perp [f_1\amalg_1, f_2\amalg_2, g_2\amalg_3]$.

Conversely, suppose we have an $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ that makes the above diagram commute. Then $h := [\alpha_2, \alpha_3]$ is a map in $\mathbb{X}$. Since $[\alpha_1, \alpha_3] = g$ and $[\alpha_1, \alpha_2] = f$, we have $h\amalg_1{}^{(-1)} = f\amalg_2{}^{(-1)}$ and $h\amalg_2{}^{(-1)} = g\amalg_2{}^{(-1)}$, hence $h = f\amalg_2{}^{(-1)} \triangledown g\amalg_2{}^{(-1)}$.

The proof that β in the diagram below exists if and only if $\amalg_2 h \vartriangle \amalg_2 k$ is similar.

$$\begin{tikzcd}
 & & {[A]} \\
{[X,Y]} \arrow[urr, bend left, "h"] \arrow[r, "\amalg_{1,2}"'] & {[X,Y,Z]} \arrow[ur, dotted, "\beta"] & \\
{[X]} \arrow[u, "\amalg_1"] \arrow[r, "\amalg_1"'] & {[X,Z].} \arrow[u, "\amalg_{1,3}"] \arrow[uur, bend right, "k"'] &
\end{tikzcd}$$

□

Theorem 6.3.25. *Given $\mathbb{X}$ an inverse category with a disjoint sum tensor and restriction zero, $iMat(\mathbb{X})$ is an inverse sum category.*

Proof. By Lemma 6.3.24, we know $iMat(\mathbb{X})$ has a disjoint sum tensor and therefore by Proposition 6.2.33, it has a disjoint join. By Lemmas 6.3.15 and 6.3.14, we know that $[A, B] = A \oplus B$ is an inverse sum of A and B for any two objects in $iMat(\mathbb{X})$, and hence, $iMat(\mathbb{X})$ is an inverse sum category. □

6.3.3 Equivalence between an inverse sum category and its matrix category

In this sub-section we will provide restriction functors between an inverse sum category $\mathbb{X}$ and its matrix category $iMat(\mathbb{X})$. Furthermore, we will show these functors form an equivalence between these two categories.

Definition 6.3.26. Given $\mathbb{X}$ is an inverse sum category with disjoint join $\perp$ and restriction zero 0, define $M : \mathbb{X} \to iMat(\mathbb{X})$ by:

$$\text{Objects: } M(A) := [A]$$
$$\text{Maps: } M(f) := [f] - \text{The } 1 \times 1 \text{ matrix with entry } f.$$

Lemma 6.3.27. *The map M from Definition 6.3.26 is a restriction functor.*

Proof. From the definition of $iMat(\mathbb{X})$, we have

$$f : A \to B \iff M(f) : M(A) \to M(B) \quad ([f] : [A] \to [B])$$
$$M(id_A) = [id_A] = id_M(A)$$
$$M(fg) = [fg] = [f][g] = M(f)M(g)$$
$$M(\overline{f}) = [\overline{f}] = \overline{[f]} = \overline{M(f)}$$

□

Definition 6.3.28. Given $\mathbb{X}$ is an inverse sum category with disjoint join $\perp$ and restriction zero 0, and inverse sum tensor $\oplus$ define $S : iMat(\mathbb{X}) \to \mathbb{X}$ by:

$$\text{Objects: } S([A_1, A_2, \ldots, A_n]) := A_1 \oplus A_2 \oplus \cdots \oplus A_n$$
$$\text{Maps: } S([f_{ij}]) := \bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j)$$

Lemma 6.3.29. *The map S from Definition 6.3.28 is a restriction functor.*

Proof. From the definition of $iMat(\mathbb{X})$, where $A = [A_1, A_2, \ldots, A_n]$, $B = [B_1, B_2, \ldots, B_M]$, and $f = [f_{ij}]$ we have

$$S(id_A) = S([id_{A_i}]) = \bigsqcup_i \amalg_i^{(-1)}(\sqcup_j \amalg_j) = id_{S(A)}$$

$$f : A \to B \iff S(f) : S(A) \to S(B) \iff$$

$$\bigsqcup_i \amalg_i^{(-1)}(\sqcup_j f_{ij} \amalg_j) : A_1 \oplus \cdots \oplus A_n \to B_1 \oplus \cdots \oplus B_m$$

$$M(\overline{f}) = [\overline{f}] = \overline{[f]} = \overline{M(f)}$$

and for composition, we have

$$\begin{aligned}
S(f)S(g) &= (\bigsqcup_i \amalg_i^{(-1)}(\sqcup_j f_{ij} \amalg_j))(\bigsqcup_{j'} \amalg_{j'}^{(-1)}(\sqcup_k g_{jk} \amalg_k)) \\
&= \bigsqcup_i \amalg_i^{(-1)} \bigsqcup_j \bigsqcup_{j'} f_{ij}\, \amalg_j\, \amalg_{j'}^{(-1)}(\sqcup_k g_{j'k} \amalg_k) \\
&= \bigsqcup_i \amalg_i^{(-1)} \bigsqcup_j f_{ij}(\sqcup_k g_{jk} \amalg_k) \\
&= \bigsqcup_i \amalg_i^{(-1)} \bigsqcup_k (\sqcup_j f_{ij} g_{jk} \amalg_k) \\
&= S([\sqcup_j f_{ij} g_{jk}]) \\
&= S(fg)
\end{aligned}$$

$\square$

Proposition 6.3.30. *Given an inverse category $\mathbb{X}$ with a disjoint sum tensor $\oplus$ and restriction zero, then the categories $\mathbb{X}$ and $iMat(\mathbb{X})$ are equivalent.*

Proof. The functors of the equivalence are S from Definition 6.3.28 and M from Definition 6.3.26.

First, we see that $MS : \mathbb{X} \to \mathbb{X}$ is the identity functor as

Objects: $S(M(A)) = S([A]) = A$

Maps: $S(M(f)) = S([f]) = f$

Next, we need to show that there is a natural transformation and isomorphism ρ such that $\rho(SM) = I_{iMat(\mathbb{X})}$. For each object $[A_1, A_2, \dots, A_n]$, $\rho A = \begin{bmatrix} \amalg_1{}^{(-1)} & \dots & \amalg_n{}^{(-1)} \end{bmatrix}$.

Note that the functor SM has the following effect:

$$\begin{aligned} \text{Objects:} \quad & M(S([A_1, \dots, A_n])) = M(A_1 \oplus \dots \oplus A_n) = [A_1 \oplus \dots \oplus A_n] \\ \text{Maps:} \quad & M(S([f_{ij}]) = M(\bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j)) = [\bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j)] \end{aligned}$$

We can now draw the commuting naturality square for $f = [f_{ij}] : [A_i] \to [B_j]$:

$$\begin{tikzcd} SM([A_i]) \arrow[r, equal] & {[\oplus_i A_i]} \arrow[rr, "{\begin{bmatrix} \amalg_1{}^{(-1)} & \dots & \amalg_n{}^{(-1)} \end{bmatrix}}"] \arrow[d, "SM(f)"'] & & {[A_i]} \arrow[d, "f"] \\ SM([B_j]) \arrow[r, equal] & {[\oplus_j B_j]} \arrow[rr, "{\begin{bmatrix} \amalg_1{}^{(-1)} & \dots & \amalg_m{}^{(-1)} \end{bmatrix}}"'] & & {[B_j]} \end{tikzcd}$$

Following the square by the top–right path from $[\oplus_i A_i]$ to $[B_j]$, by the definition of the maps in the category $iMat(\mathbb{X})$, we see each $B_j = \sqcup_i \amalg_i{}^{(-1)} f_{ij}(\oplus_i A_i)$. Following the left–bottom path, composing $SM(f)$ with $\begin{bmatrix} \amalg_1{}^{(-1)} & \dots & \amalg_m{}^{(-1)} \end{bmatrix}$ gives us the map

$$\begin{bmatrix} \sqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j) \amalg_1{}^{(-1)} & \dots & \sqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j) \amalg_m{}^{(-1)} \end{bmatrix} = \begin{bmatrix} \sqcup_i \amalg_i{}^{(-1)} f_{i1} & \dots & \sqcup_i \amalg_i{}^{(-1)} f_{im} \end{bmatrix}.$$

Applying this to $[\oplus_i A_i]$, we see each $B_j = \sqcup_i \amalg_i{}^{(-1)} f_{ij}(\oplus_i A_i)$ and the two directions are equal.

Finally, we know that $\rho_{A_i}{}^{(-1)} = \begin{bmatrix} \amalg_1 \\ \vdots \\ \amalg_n \end{bmatrix}$ and defines an isomorphism between any object of the form $[\oplus_i A_i]$ and the object $[A_1, \dots, A_n]$ □

Lemma 6.3.31. *In an inverse sum tensor category, any map $f : A \oplus B \to C \oplus D$ may be represented in a matrix form. Composition of maps may be computed by multiplication of* *the matrices, with composition taking the place of base level multiplication and $\sqcup$ in the place of addition.*

Proof. Recall from Lemma 6.3.15 that $A \oplus B$ and $C \oplus D$ are inverse sums. Referencing Definition 6.3.1, define $e_0 = {i_1}^{(-1)} i_1$ and $e_1 = {i_2}^{(-1)} i_2$ and recall that $e_0 \perp e_1$, $e_0 \sqcup e_1 = 1$. Then given a function $f : A \oplus B \to C \oplus D$ define

$$f_M = \begin{bmatrix} e_0 f e_0 & e_0 f e_1 \\ e_1 f e_0 & e_1 f e_1 \end{bmatrix}.$$

Note first that since $e_0 \perp e_1$, the maps in the rows of f_M are disjoint by the stability of the disjointness relation. Similarly, the maps in the columns are disjoint by universality. We have $e_0 f e_0 \sqcup e_0 f e_1 = e_0 f$ and $e_1 f e_0 \sqcup e_1 f e_1 = e_1 f$. Each of these maps are disjoint by universality. Finally, $e_0 f \sqcup e_1 f = (e_0 \sqcup e_1) f = f$ and hence we may recover the initial map whenever we have a matrix of this from. We will call this computation the distinct join of f_M.

Next, consider $f_M \times g_M$. As each e_i is idempotent, this is

$$f_M \times g_M = \begin{bmatrix} e_0 f e_0 g e_0 \sqcup e_0 f e_1 g e_0 & e_0 f e_0 g e_1 \sqcup e_0 f e_1 g e_1 \\ e_1 f e_0 g e_0 \sqcup e_1 f e_1 g e_0 & e_1 f e_0 g e_1 \sqcup e_1 f e_1 g e_1 \end{bmatrix} = \begin{bmatrix} e_0 f g e_0 & e_0 f g e_1 \\ e_1 f g e_0 & e_1 g f e_1 \end{bmatrix}$$

where the distinct joins are well defined due to the stability and universality of the join. We can see that the distinct join of $f_M \times g_M = fg$ and as such we have composition.

$\square$

In particular, we note that we may represent $f : A \to B$ by the matrix

$$\begin{bmatrix} 1f1 & 1f0 \\ 0f1 & 0f0 \end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$$

as $A \cong A \oplus 0$ and $B \cong B \oplus 0$.

We now turn to examining a category of specialized matrices over an inverse sum category. In general, the matrix category $iMat((\mathbb{X}))$ will have objects that are lists of objects in $\mathbb{X}$,

$X = (X_1, \ldots, X_m)$. Maps between lists will be matrices $[f_{ij}] : (X_1, \ldots, X_m) \to (Y_1, \ldots, Y_n)$. We will only consider maps whose matrices have disjoint rows, i.e., if $[f_{ij}]$ is a matrix, it must have $f_{ij} \perp f_{ik}$ for all i whenever $j \neq k$.

6.4 Completing a distributive inverse category

6.4.1 Distributive restriction categories

Definition 6.4.1. A Cartesian restriction category with a restriction zero and coproducts is called *distributive* when there is an isomorphism ρ such that

$$A \times (B + C) \xrightarrow{\rho} (A \times B) + (A \times C).$$

In a distributive inverse category, we lack:

$$A \xrightarrow{!} 1$$

and

$$A + A \xrightarrow{\nabla} A.$$

6.4.2 Distributive inverse categories

Definition 6.4.2. A *distributive inverse category* $\mathbb{D}$ consists of the following:

- $\mathbb{D}$ is an inverse category;
- $\mathbb{D}$ has an inverse product tensor, $\otimes$, per SubSection 4.1.2;
- $\mathbb{D}$ has an inverse sum tensor, $\oplus$, per Definition 6.3.13 and
- The below diagram,

$$\begin{tikzcd}
& A \otimes B \\
A \otimes (B \oplus C) \arrow[ur, "1\otimes\amalg_1^{(-1)}"] \arrow[r, "d"] \arrow[dr, "1\otimes\amalg_2^{(-1)}"'] & (A \otimes B) \oplus (A \otimes C) \arrow[u, "\amalg_1^{(-1)}"'] \arrow[d, "\amalg_2^{(-1)}"] \\
& A \otimes C,
\end{tikzcd} \tag{6.10}$$

commutes in $\mathbb{D}$ for any choices of objects A, B, C, where the map d is an isomorphism.

Note that as we are operating in an inverse category, we also have the inverse of diagram (6.10) available to us. That is,

$$\begin{tikzcd} A \otimes B \arrow[d, "\amalg_1"'] \arrow[drr, "1\otimes\amalg_1"] & & \\ (A \otimes B) \oplus (A \otimes C) \arrow[rr, "d^{(-1)}"] & & A \otimes (B \oplus C) \\ A \otimes C \arrow[u, "\amalg_2"] \arrow[urr, "1\otimes\amalg_2"'] & & \end{tikzcd} \tag{6.11}$$

is also a commuting diagram in $\mathbb{D}$.

Definition 6.4.3. Suppose $\mathbb{X}$ is an inverse category with a disjoint join tensor $\oplus$ and a restriction zero. Then for maps $f : A \to B$ and $g : A \to C$ with $\overline{f} \perp \overline{g}$, define the map $[f, g] : A \to B \oplus C$ as $(f\amalg_1) \sqcup (g\amalg_2)$. This is well defined as $\widehat{\amalg_1} \perp \widehat{\amalg_2}$ and therefore by [**Dis.7**], $f\amalg_1 \perp g\amalg_2$.

Lemma 6.4.4. *Given an inverse category $\mathbb{X}$ with a disjoint join tensor $\oplus$, a restriction zero, and an inverse product tensor $\otimes$ which distributes over disjoint joins, (i.e., $f \otimes (g \sqcup h) = (f \otimes g) \sqcup (f \otimes h)$ it is an inverse distributive category.*

Proof. By assumption, we have the first three items of Definition 6.4.2. Therefore, we need to construct an isomorphism d such that diagram (6.10) commutes. We claim that the map $d = [1 \otimes {\amalg_1}^{(-1)}, 1 \otimes {\amalg_2}^{(-1)}]$ does this.

First, note that the typing of d is correct. By Definition 6.4.3,

$$d = ((1 \otimes {\amalg_1}^{(-1)})\amalg_1) \sqcup ((1 \otimes {\amalg_2}^{(-1)})\amalg_2) : A \otimes (B \oplus C) \to (A \otimes B) \oplus (A \otimes C)$$

as

$$A \otimes (B \oplus C) \xrightarrow{(1\otimes{\amalg_1}^{(-1)})} A \otimes B \xrightarrow{{\amalg_1}^{(-1)}} (A \otimes B) \oplus (A \otimes C),$$
$$A \otimes (B \oplus C) \xrightarrow{(1\otimes{\amalg_2}^{(-1)})} A \otimes C \xrightarrow{{\amalg_2}^{(-1)}} (A \otimes B) \oplus (A \otimes C).$$

Next, we need to show d is an isomorphism. We will do this by showing both $\overline{d} = 1$ and $\overline{d^{(-1)}} = 1$. As a consequence of Lemma 6.2.15, we know the inverse of d is

$$((1 \otimes \amalg_1{}^{(-1)})\amalg_1)^{(-1)} \sqcup ((1 \otimes \amalg_2{}^{(-1)})\amalg_2)^{(-1)} = (\amalg_1{}^{(-1)}(1 \otimes \amalg_1)) \sqcup (\amalg_2{}^{(-1)}(1 \otimes \amalg_2)).$$

Having $\otimes$ distribute over the disjoint sum means that for any maps f, h, k with $h \perp k$, we have $f \otimes (h \sqcup k) = (f \otimes h) \sqcup (f \otimes k)$. We use this in the calculation of the restriction of f:

$$\begin{aligned}
\overline{((1 \otimes \amalg_1{}^{(-1)})\amalg_1) \sqcup ((1 \otimes \amalg_2{}^{(-1)})\amalg_2)} &= \overline{((1 \otimes \amalg_1{}^{(-1)})\amalg_1)} \sqcup \overline{((1 \otimes \amalg_2{}^{(-1)})\amalg_2)} \\
&= (1 \otimes \overline{\amalg_1{}^{(-1)}}) \sqcup (1 \otimes \overline{\amalg_2{}^{(-1)}}) \\
&= (1 \otimes (\overline{\amalg_1{}^{(-1)}} \sqcup \overline{\amalg_2{}^{(-1)}}) \\
&= 1 \otimes ((1 \oplus 0) \sqcup (0 \oplus 1)) \\
&= 1 \otimes 1 = 1
\end{aligned}$$

and for the inverse,

$$\begin{aligned}
\overline{(\amalg_1{}^{(-1)}(1 \otimes \amalg_1)) \sqcup (\amalg_2{}^{(-1)}(1 \otimes \amalg_2))} &= \overline{(\amalg_1{}^{(-1)}(1 \otimes \amalg_1))} \sqcup \overline{(\amalg_2{}^{(-1)}(1 \otimes \amalg_2))} \\
&= \overline{(\amalg_1{}^{(-1)}\overline{(1 \otimes \amalg_1)})} \sqcup \overline{(\amalg_2{}^{(-1)}\overline{(1 \otimes \amalg_2)})} \\
&= \overline{(\amalg_1{}^{(-1)})} \sqcup \overline{(\amalg_2{}^{(-1)})} \\
&= (1 \oplus 0) \sqcup (0 \oplus 1) \\
&= 1
\end{aligned}$$

Hence, $[1 \otimes \amalg_1{}^{(-1)}, 1 \otimes \amalg_2{}^{(-1)}]$ is an isomorphism. Finally, we must show that diagram (6.10) commutes.

$$\begin{aligned}
d\amalg_1{}^{(-1)} &= \Big(((1 \otimes \amalg_1{}^{(-1)})\amalg_1) \sqcup ((1 \otimes \amalg_2{}^{(-1)})\amalg_2)\Big)\, \amalg_1{}^{(-1)} \\
&= \Big(((1 \otimes \amalg_1{}^{(-1)})\amalg_1)\amalg_1{}^{(-1)}\Big) \sqcup \Big(((1 \otimes \amalg_2{}^{(-1)})\amalg_2)\amalg_1{}^{(-1)}\Big) \\
&= \Big((1 \otimes \amalg_1{}^{(-1)})1\Big) \sqcup \Big((1 \otimes \amalg_2{}^{(-1)})0\Big) \\
&= (1 \otimes \amalg_1{}^{(-1)}) \sqcup 0 \\
&= 1 \otimes \amalg_1{}^{(-1)}
\end{aligned}$$

and

$$
\begin{aligned}
d\amalg_2{}^{(-1)} &= \Big(((1 \otimes \amalg_1{}^{(-1)})\amalg_1) \sqcup ((1 \otimes \amalg_2{}^{(-1)})\amalg_2) \Big) \amalg_2{}^{(-1)} \\
&= \Big(((1 \otimes \amalg_1{}^{(-1)})\amalg_1)\amalg_2{}^{(-1)} \Big) \sqcup \Big(((1 \otimes \amalg_2{}^{(-1)})\amalg_2)\amalg_2{}^{(-1)} \Big) \\
&= 0 \sqcup (1 \otimes \amalg_2{}^{(-1)}) \\
&= 1 \otimes \amalg_2{}^{(-1)}.
\end{aligned}
$$

This shows the fourth condition is satisfied and $\mathbb{X}$ is a distributive inverse category. ☐

We have seen that a second tensor distributing over the disjoint joins implies that we have an inverse distributive category. We now show the converse is true.

Lemma 6.4.5. *Given an inverse distributive category $\mathbb{X}$, then $h \otimes (f \triangledown g) = (h \otimes f) \triangledown (h \otimes g)$ whenever $f \triangledown g$ exists and $h \otimes (f \vartriangle g) = (h \otimes f) \vartriangle (h \otimes g)$ whenever $f \vartriangle g$ exists.*

Proof. Let $h : A \to C$, $f : B \to D$ and $g : B \to E$. Consider the following diagram:

$$
\begin{tikzcd}
 & & B \otimes E \\
A \otimes C \arrow[urr, "h\otimes f"] \arrow[r, "h\otimes(f \triangledown g)"] \arrow[drr, "h\otimes g"'] & B \otimes (D \oplus E) \arrow[ur, "1\otimes\amalg_1{}^{(-1)}"'] \arrow[r, "\cong"] \arrow[dr, "1\otimes\amalg_2{}^{(-1)}"] & (B \otimes D) \oplus (B \otimes E) \arrow[u, "\amalg_1{}^{(-1)}"'] \arrow[d, "\amalg_2{}^{(-1)}"] \\
 & & B \otimes D.
\end{tikzcd}
$$

The two leftmost triangles commute by the diagram for $f \triangledown g$. The right hand triangles commute as per Definition 6.4.2. From this, we see $h \otimes (f \triangledown g) = (h \otimes f) \triangledown (h \otimes g)$, by the uniqueness of the $\triangledown$ operation.

The argument for showing $h \otimes (f \vartriangle g) = (h \otimes f) \vartriangle (h \otimes g)$ follows the same methodology. ☐

Lemma 6.4.6. *Given an inverse distributive category $\mathbb{X}$, then $\otimes$ distributes over the disjoint join.*

Proof. First recall the definition of $f \sqcup g = (\overline{f} \triangledown \overline{g})(f \vartriangle g)$. So, in order to show $h \otimes (f \sqcup g) = (h \otimes f) \sqcup (h \otimes g)$, we need to show that

$$h \otimes (\overline{f} \triangledown \overline{g})(f \vartriangle g) = (\overline{h \otimes f} \triangledown \overline{h \otimes g})(h \otimes f \vartriangle h \otimes g). \tag{6.12}$$

Since $h \otimes (\overline{f} \triangledown \overline{g})(f \vartriangle g) = (\overline{h} \otimes (\overline{f} \triangledown \overline{g}))(h \otimes (f \vartriangle g))$, this follows directly from Lemma 6.4.5 and the fact that $\otimes$ is a restriction functor. □

Corollary 6.4.7. *Suppose we have an inverse distributive category $\mathbb{X}$. Then,*

(i) if $f \perp g$, then $h \otimes f \perp h \otimes g$ for any h,

(ii) if $f \perp g : A \to B$ and $h \perp k : C \to D$, then $(f \otimes h) \perp (g \otimes k)$.

Proof.

(i) As $f \perp g$, we have $f \vartriangle g$ and $f \triangledown g$. By Lemma 6.4.5, both $h \otimes f \vartriangle h \otimes g$ and $h \otimes f \triangledown h \otimes g$ exist and therefore $h \otimes f \perp h \otimes g$.

(ii) By the previous item, we have that $((f \sqcup g) \otimes h) \perp ((f \sqcup g) \otimes k)$. Then, by [**DJ.1**] and [**Dis.3**] we have $(f \otimes h) \perp (g \otimes k)$.

□

6.4.3 Discrete inverse categories with inverse sums

We now consider the case where we have a discrete inverse category with inverse product tensor $\otimes$ and a disjoint join tensor $\oplus$, with the $\otimes$ tensor preserves the disjoint join.

A map in $\widetilde{\mathbb{X}}$ is related to map in $\mathbb{X}$ in the following way:

$$\frac{A \xrightarrow{(f,C)} B \text{ in } \widetilde{\mathbb{X}}}{A \xrightarrow{f} B \otimes C \text{ in } \mathbb{X}}$$

Our goal is to show that an inverse sum in a distributive inverse category becomes a co-product in $\widetilde{\mathbb{X}}$.

Lemma 6.4.8. *Given $\mathbb{X}$ is a distributive inverse category, then $\widetilde{\mathbb{X}}$, the discrete inverse category created from $\mathbb{X}$, has a restriction zero.*

Proof. Recall from Theorem 4.2.17 that $\mathbb{X}$ is equivalent as a category to $\widetilde{\mathbb{X}}$ under the identity on objects functor

$$\mathbf{T} : \mathbb{X} \to \widetilde{\mathbb{X}}; \qquad \begin{array}{c} A \\ \big\downarrow f \\ B \end{array} \mapsto \begin{array}{c} A \\ \big\downarrow (f u_\otimes^{r\,(-1)}, 1) \\ B \end{array} \text{ given by } \begin{array}{c} A \\ \big\downarrow f u_\otimes^r \\ B \otimes 1 \end{array}).$$

In $\mathbb{X}$, we know 0 is a terminal and initial object, with maps $A \xrightarrow{t_A} 0$ and $0 \xrightarrow{z_A} A$, with $\overline{0_{A,A}} = 0_{A,A} = t_A z_A$.

First we note that 0 is both initial and terminal in $\widetilde{\mathbb{X}}$, with the terminal maps being $\mathbf{T}(t_A)$ and initial maps being $\mathbf{T}(z_A)$.

As was also shown in ..., $\mathbf{T}$ is a restriction functor, so in $\widetilde{\mathbb{X}}$ we have

$$0_{A,A} = \mathbf{T}(t_A)\mathbf{T}(z_A) = \mathbf{T}(t_A z_A) = \mathbf{T}(0_{A,A}) = \mathbf{T}(\overline{0_{A,A}}) = \overline{\mathbf{T}(0_{A,A})} = \overline{0_{A,A}}.$$

Hence, $0_{A,A}$ is a restriction zero in $\widetilde{\mathbb{X}}$. $\square$

Lemma 6.4.9. *In a distributive inverse category $\mathbb{X}$:*

(i) Given $f : A \to Y \otimes C$, we can construct $f' : A \to Y \otimes (C \oplus D)$ for some object D such that $f \simeq f'$.

(ii) Given $g : B \to Y \otimes D$, we can construct $g'' : B \to Y \otimes (C \oplus D)$ for some object C such that $g \simeq g''$.

(iii) Given $f : A \to Y \otimes C$, $g : A \to Y \otimes D$, then the f', g'' as constructed by the previous points satisfy ${\amalg_1}^{(-1)} f' \perp {\amalg_2}^{(-1)} g''$, where icpa$f'$, ${\amalg_2}^{(-1)} g'' : A \oplus B \to Y \otimes (C \oplus D)$.

Proof.

(i) Set $f' = f(1 \otimes \amalg_1)$. To show $f \equiv f'$, we must first show their restriction is the same.

$$\begin{aligned}\overline{f(1 \otimes \amalg_1)} &= \overline{f\overline{(1 \otimes \amalg_1)}}\\ &= \overline{f1}\\ &= \overline{f}\end{aligned}$$

Now we detail the mediating map between f and f':

$$\begin{tikzcd}
 & Y \otimes C \arrow[dr, dotted, "\Delta\otimes 1"] & \\
 & & Y \otimes (Y \otimes C) \arrow[dd, dotted, "1\otimes 1\otimes \amalg_1"] \\
A \arrow[uur, "f"] \arrow[ddr, "f'"'] & & \\
 & & Y \otimes Y \otimes (C \oplus D) \arrow[dl, dotted, "\Delta^{(-1)}\otimes 1"] \\
 & Y \otimes (C \oplus D) \otimes X &
\end{tikzcd}$$

To show this commutes, we primarily use that $\Delta Delta^{(-1)} = 1$

$$\begin{aligned}&f(\Delta \otimes 1)(1 \otimes 1 \otimes \amalg_1(\Delta^{(-1)}\otimes 1)\\ &\qquad = f(\Delta \otimes 1)(\Delta^{(-1)} \otimes \amalg_1)\\ &\qquad = f(1 \otimes \amalg_1)\\ &\qquad = f'\end{aligned}$$

and hence $f \overset{h}{\simeq} f'$ where $h = (1 \otimes \amalg_1)$.

(ii) For this item, we set $g'' = g(1\otimes\amalg_2)$. The proof that $g \overset{k}{\simeq} g''$, where $k = (1\otimes\amalg_2)$ is done in the same way as the previous point.

(iii) In order to show ${\amalg_1}^{(-1)}f' \perp {\amalg_2}^{(-1)}g''$, we will proceed by showing their restrictions and ranges are disjoint. As $\overline{{\amalg_1}^{(-1)}} \perp \overline{{\amalg_2}^{(-1)}}$ and $\overline{{\amalg_1}^{(-1)}f'} \leq \overline{{\amalg_1}^{(-1)}}$ and $\overline{{\amalg_2}^{(-1)}g''} \leq \overline{{\amalg_2}^{(-1)}}$, we immediately have $\overline{{\amalg_1}^{(-1)}f'} \perp \overline{{\amalg_2}^{(-1)}g'}$.

For the ranges, we have

$$\begin{aligned}
\amalg_1{}^{(-1)}\widehat{(f(1\otimes\amalg_1))} &= \overline{((1\otimes\amalg_1{}^{(-1)})f^{(-1)})\amalg_1} \\
&= \overline{((1\otimes\amalg_1{}^{(-1)})f^{(-1)}} \\
&\leq \overline{(1\otimes\amalg_1{}^{(-1)})}
\end{aligned}$$

and similarly

$$\widehat{\amalg_2{}^{(-1)}g''} \leq \overline{(1\otimes\amalg_2{}^{(-1)})}.$$

Using Lemma 6.2.5 we know that $\overline{(\amalg_1{}^{(-1)})} \perp \overline{(\amalg_2{}^{(-1)})}$. From Corollary 6.4.7 we conclude that $\overline{(1\otimes\amalg_1{}^{(-1)})} \perp \overline{(1\otimes\amalg_2{}^{(-1)})}$ and giving us $\widehat{\amalg_1{}^{(-1)}f'} \perp \widehat{\amalg_2{}^{(-1)}g''}$ and therefore $\amalg_1{}^{(-1)}f' \perp \amalg_2{}^{(-1)}g''$.

□

Proposition 6.4.10. *Given $\mathbb{X}$ is an distributive inverse category, then the category $\widetilde{\mathbb{X}}$ has co-products.*

Proof. The tensor object, $A\oplus B$ in $\mathbb{X}$ will become the co-product of A, B in $\widetilde{\mathbb{X}}$.

The injection maps of the co-product are $i_1 = (\amalg_1 u_\otimes^r, 1)$ and $i_2 = (\amalg_2 u_\otimes^r, 1)$.

Consider the following diagram in $\widetilde{\mathbb{X}}$:

$$\begin{tikzcd}
A \arrow[dr, "i_1"'] \arrow[drr, "{(f,C)}"] & & \\
& A\oplus B \arrow[r, dotted, "h"] & Y \\
B \arrow[ur, "i_2"] \arrow[urr, "{(g,D)}"'] & &
\end{tikzcd}$$

In $\mathbb{X}$, this comes from the diagram:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\amalg_1"'] & & Y\otimes C \\
& A\oplus B & \\
B \arrow[ur, "\amalg_2"] \arrow[rr, "g"] & & Y\otimes D
\end{tikzcd} \tag{6.13}$$

where the extraneous unit isomorphisms are removed.

This corresponds to the conditions of Lemma 6.4.9. Hence by that lemma we may revise Diagram ((6.13)) as

$$\begin{tikzcd}
A \arrow[d, "\amalg_1"] \arrow[drr, "f'"] & & \\
A \oplus B \arrow[rr, "{\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g''"] & & Y \otimes (C \oplus D) \\
B \arrow[u, "\amalg_2"'] \arrow[urr, "g''"'] & &
\end{tikzcd} \tag{6.14}$$

where f' and g'' are respectively equivalent to f, g.

Lifting Diagram (6.14) to $\mathbb{X}$, we see this corresponds to the desired co-product diagram, where h in $\widetilde{\mathbb{X}}$ is the map $({\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g'', (C \oplus D))$.

By construction, in $\mathbb{X}$,

$$\amalg_1({\amalg_1}^{(-1)} f' \sqcup \amalg_1{\amalg_2}^{(-1)} g'') = (\amalg_1{\amalg_1}^{(-1)} f') \sqcup (\amalg_1{\amalg_2}^{(-1)} g'') = f' \sqcup 0 = f'$$

and

$$\amalg_2({\amalg_1}^{(-1)} f' \sqcup \amalg_1{\amalg_2}^{(-1)} g'') = g''.$$

Hence, in $\widetilde{\mathbb{X}}$, we have $(i_1 u_{\otimes}^r, 1)h = f$ and $(i_2 u_{\otimes}^r, 1)h = g$.

All that remains to be shown is that h is unique.

Suppose there is another (k, E) in $\widetilde{\mathbb{X}}$ such that it satisfies the coproduct properties, i.e., that $i_1(k, E) = (f', C \oplus D)$ and $i_2(k, E) = (g'', C \oplus D)$. In $\mathbb{X}$, $k : A \oplus B \to Y \oplus E$ and we have

$$\amalg_1 k \overset{x}{\simeq} f' \quad \text{and}$$
$$\amalg_2 k \overset{y}{\simeq} g''$$

where the maps $x, y : Y \otimes E \to Y \otimes (C \oplus D)$ fulfill the respective equivalence diagrams.

The above gives us ${\amalg_1}^{(-1)} \amalg_1 k \simeq {\amalg_1}^{(-1)} f'$ and ${\amalg_2}^{(-1)} \amalg_2 k \simeq {\amalg_2}^{(-1)} g''$. As we know that $\otimes$ preserves the disjoint join, and the equivalence diagrams consist of maps under $\otimes$, we now have:

$$
\begin{aligned}
h &= \amalg_1{}^{(-1)} f' \sqcup \amalg_2{}^{(-1)} g'' \\
&\simeq \amalg_1{}^{(-1)} \amalg_1 k \sqcup \amalg_2{}^{(-1)} \amalg_2 k \\
&= (\amalg_1{}^{(-1)} \amalg_1 \sqcup \amalg_2{}^{(-1)} \amalg_2) k \\
&= 1k = k.
\end{aligned}
$$

Hence, $h \simeq k$ in $\mathbb{X}$ and $(h, C \oplus D) = (k, E)$ in $\widetilde{\mathbb{X}}$, meaning $A \oplus B$ in $\mathbb{X}$ is the co-product of A and B in $\widetilde{\mathbb{X}}$. $\square$

Chapter 7

Sums in Frobenius Algebras

Chapter 8

Conclusions and future work

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