

# Modified von Bertalanffy Growth Function to Directly Estimate the Age at a Critical Length

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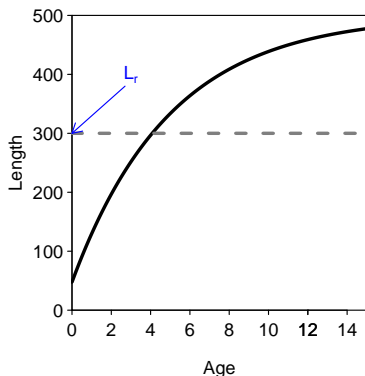
- 1 Critical Time Concept
- 2  $t_r$  Often Estimated by Inverting the VBGF
- 3 Reparameterize the VBGF to Estimate  $t_r$
- 4 An Example with Lake Michigan Lake Whitefish
- 5 Summary

# Definitions

- Critical points in a fish's life are often defined.
  - e.g., age at maturity, recruitment, a length defined by a regulation.

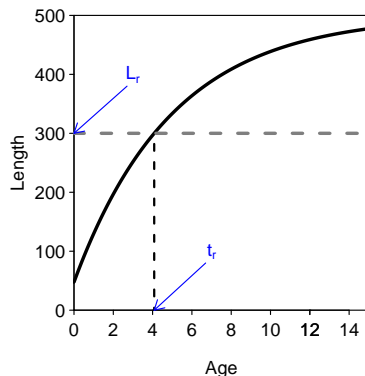
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  - e.g., age at maturity, recruitment, a length defined by a regulation.
- $L_r$ 
  - Length at critical point.
  - Defined by scientist.
- $t_r$ 
  - Age (time) at the critical point.
  - To be estimated.

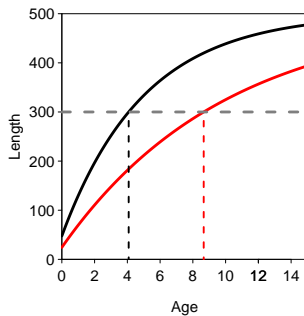


# Importance

- Understanding  $t_r$  is important.

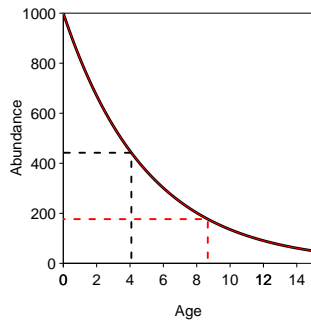
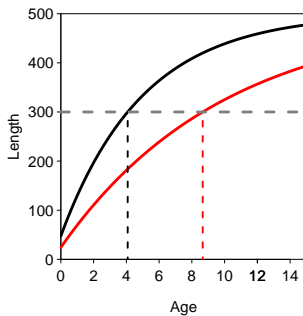
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*North American Journal of Fisheries Management* 27:918–931, 2007

[Article]

## **Yellow Perch in South Dakota: Population Variability and Predicted Effects of Creel Limit Reductions and Minimum Length Limits**

DANIEL A. ISERMANN\*<sup>1</sup> AND DAVID W. WILLIS

*Department of Wildlife and Fisheries Sciences, South Dakota State University,  
Box 2140B, Brookings, South Dakota 57007, USA*

Length limit modeling.—We conducted length limit modeling for four of the six lakes where yellow perch population characteristics were described. We excluded East 81 Slough and Enemy Swim Lake from length limit modeling because fishing mortality may not be a significant source of mortality in these populations (Blackwell 2005a; Isermann et al. 2005) and because growth of yellow perch in these systems is slow. Dynamic-pool models utilizing Jones' (1957) modification to the equilibrium yield equation of Beverton and Holt (1957), as available in Fishery Analyses and Simulation Tools (FAST version 2.0; Slipke and

intercept ( $t_0$ ) estimates due to the lack of mean length data for ages 0 and 1. Consequently,  $t_0$  was fixed at zero when deriving  $k$  and  $L_{inf}$  from the von Bertalanffy model and in all length limit simulations. Our length limit modeling assumed that growth remained constant at the designated level across year-classes and that density-dependent growth responses did not occur. To further describe growth, we used von Bertalanffy models to estimate the time (years) required to reach lengths of 178 ( $t_{178}$ ), 229 ( $t_{229}$ ), 254 ( $t_{254}$ ), and 279 mm ( $t_{279}$ ).

Using the customized recruitment option available in

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*North American Journal of Fisheries Management* 22:1349–1357, 2002

## Predictive Evaluation of Size Restrictions as Management Strategies for Tennessee Reservoir Crappie Fisheries

DANIEL A. ISERMANN,\*<sup>1</sup> STEVE M. SAMMONS,<sup>2</sup> AND PHILLIP W. BETTOLI  
TIMOTHY N. CHURCHILL

*North American Journal of Fisheries Management* 22:1306–1313, 2002

## Effect and Acceptance of Bluegill Length Limits in Nebraska Natural Lakes

CRAIG P. PAUKERT\*<sup>1</sup> AND DAVID W. WILLIS  
DONALD W. GABELHOUSE, JR.

*North American Journal of Fisheries Management* 31:269–279, 2011

## Simulated Population Responses of Common Carp to Commercial Exploitation

Michael J. Weber,\* Matthew J. Hennen,<sup>1</sup> and Michael L. Brown

*Environ Biol Fish* (2007) 79:11–25

## The New River, Virginia, muskellunge fishery: population dynamics, harvest regulation modeling, and angler attitudes

Travis O. Brenden · Eric M. Hallerman · Brian R. Murphy · John R. Copeland ·

*North American Journal of Fisheries Management* 22:1340–1348, 2002

## Rescinding a 254-mm Minimum Length Limit on White Crappies at Ft. Supply Reservoir, Oklahoma: The Influence of Variable Recruitment, Compensatory Mortality, and Angler Dissatisfaction

JEFF BOXRUCKER\*

*North American Journal of Fisheries Management* 29:1183–1194, 2009

## Fishery and Population Characteristics of Blue Catfish and Channel Catfish and Potential Impacts of Minimum Length Limits on the Fishery in Lake Wilson, Alabama

MICHAEL P. HOLLEY,<sup>1</sup> MATTHEW D. MARSHALL, AND MICHAEL J. MACEINA\*

*Transactions of the American Fisheries Society* 134:1285–1298, 2005

## Population Characteristics and Assessment of Overfishing for an Exploited Paddlefish Population in the Lower Tennessee River

GEORGE D. SCHOLTEN\*<sup>1</sup> PHILLIP W. BETTOLI

*North American Journal of Fisheries Management* 15:766–772, 1995

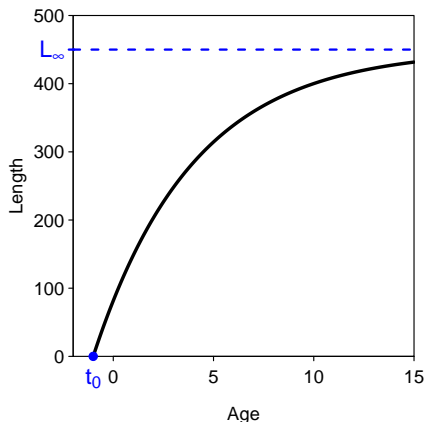
## An Evaluation of the Value of Harvest Restrictions in Managing Crappie Fisheries

M. S. ALLEN AND L. E. MIRANDA

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# von Bertalanffy Growth Function Review

$$L_t = L_{\infty} (1 - e^{-K(t-t_0)})$$



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- Then set  $L_t$  to the chosen  $L_r$  so that  $t$  will be  $t_r$ .



## Computing $t_r$ from Inverse VBGF

$$t = \frac{\log_e \left( 1 - \frac{L_t}{L_\infty} \right)}{-K} + t_0$$

- Suppose that  $L_\infty=450$ ,  $K=0.2$ , and  $t_0=-1$ .
- Suppose that the critical length of interest is  $L_r=300$ .

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- Thus, the estimated mean time to reach 300 mm is 4.5 years.

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We need a better method!



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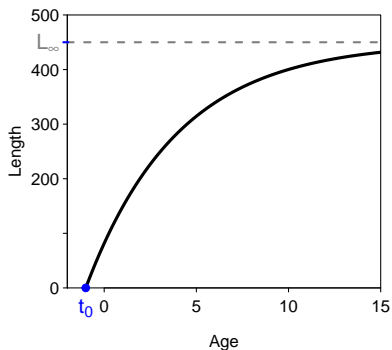
$$L_t = L_{\infty} (1 - e^{-K(t-t_0)})$$

- ... may be rewritten as ...

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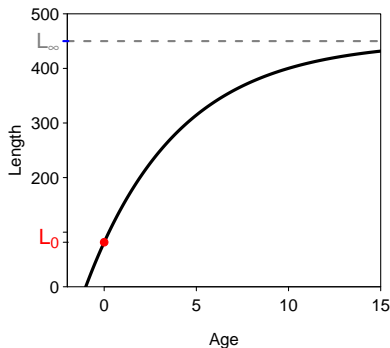
$$L_t = L_0 + (L_\infty - L_0) (1 - e^{-Kt})$$

- ... can be rewritten ...

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# Example with Lake Michigan Lake Whitefish

- Data from Belnap (2014).<sup>1</sup>
  - Fish from commercial trapnets in six management zones.
  - Measured total length (TL; mm).
  - Estimated age (yrs) from otolith thin sections.
  - Interested in  $t_{480}$  (480 mm is length at full vulnerability to harvest).

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- Fit traditional and modified VBGF to fish from Big Bay de Noc.
  - Compared parameter estimates and predicted mean lengths-at-age.

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- Fit traditional and modified VBGF to fish from Big Bay de Noc.
  - Compared parameter estimates and predicted mean lengths-at-age.
- Fit modified VBGF to fish from Big Bay de Noc and Green Bay to illustrate  $t_r$  comparison (following methods in Ogle (2016)<sup>2</sup>).

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# Comparison of VBGF Results

## Traditional VBGF

	Estimate	LCI	UCI
$L_{\infty}$	549.51	541.56	560.99
$K$	0.29	0.19	0.39
$t_0$	2.70	-0.32	4.32
$t_{480}$	9.88	-	-

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Age	Pred Len
8.00	429.97
15.00	533.56
25.00	548.61
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Model	Params	AICc	$\Delta$ AICc	Weight
$L_{\infty}, t_{480}$	6	3203.3	0.00	0.38
$K, t_{480}$	6	3203.5	0.19	0.34
$L_{\infty}, K, t_{480}$	7	3204.7	1.40	0.19
$t_{480}$	5	3206.2	2.90	0.09
$L_{\infty}$	5	3221.2	17.91	0.00
$K$	5	3222.5	19.14	0.00
$L_{\infty}, K$	6	3223.3	19.97	0.00
None	4	3230.0	26.68	0.00

# Comparing $t_r$ Between Groups

## Big Bay de Noc

	Estimate	LCI	UCI
$L_\infty$	549.51	541.79	560.50
$K$	0.29	0.20	0.38
$t_{480}$	9.88	9.44	10.27

## Green Bay

	Estimate	LCI	UCI
$L_\infty$	555.91	546.84	571.16
$K$	0.39	0.18	1.23
$t_{480}$	7.56	6.20	8.48

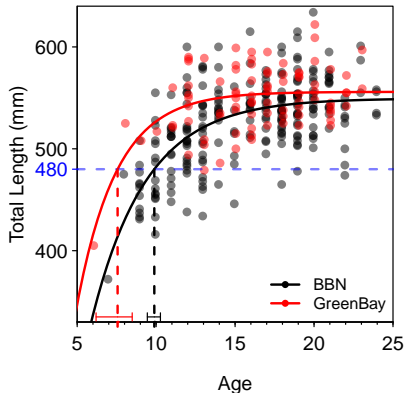
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	Estimate	LCI	UCI
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$$L_t = L_r + (L_\infty - L_r) (1 - e^{-K(t-t_r)})$$

- A simple modification of the VBGF allows direct estimation of a parameter of interest,  $t_r$ .
  - Same estimates of  $L_\infty$  and  $K$  as with the traditional VBGF.
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  - Easy interval estimates of  $t_r$ .
  - Compare  $t_r$  between groups with standard (ANCOVA-like) methods.



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- Benefits
  - Easy interval estimates of  $t_r$ .
  - Compare  $t_r$  between groups with standard (ANCOVA-like) methods.
- Costs
  - No direct estimate of  $t_0$  (in the traditional VBGF).

## Recommendation

$$L_t = L_r + (L_\infty - L_r) (1 - e^{-K(t-t_r)})$$

Use this modified VBGF in place of the traditional VBGF.

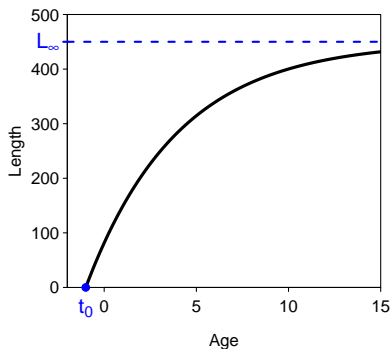
# Acknowledgments

- Matthew Belnap for collection and initial processing of Whitefish data.
- Ben Wegleitner, Zach Kleemann, Andrew Gullickson, and Connie Isermann for help processing Whitefish otoliths.
- David Staples (Minnesota DNR) and Joshua McCormick (Oregon Department of Fisheries & Wildlife) for comments on modified VBGF.



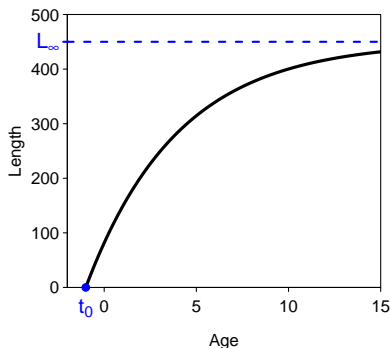
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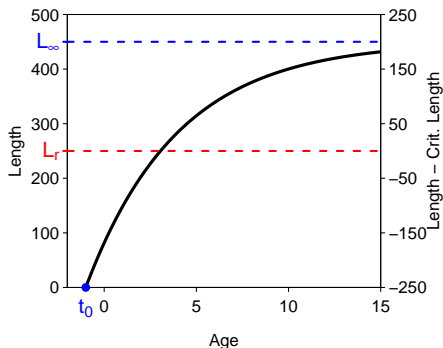
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- Let's define  $L_t^* = L_t - L_r$  (difference in length from critical length).

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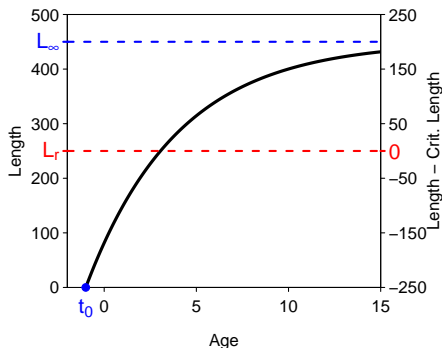
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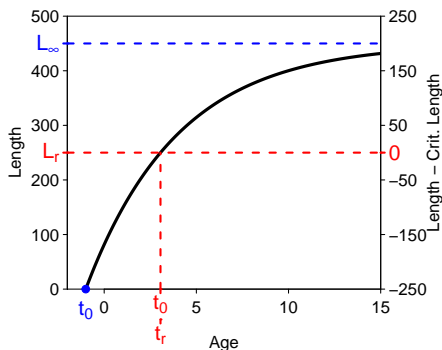


- Let's define  $L_t^* = L_t - L_r$  (difference in length from critical length).
- Thus,  $Y = 0$  means that  $L_t^* = 0$ ,  $L_t - L_r = 0$ , and  $L_t = L_r$ .



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- Thus,  $Y = 0$  means that  $L_t^* = 0$ ,  $L_t - L_r = 0$ , and  $L_t = L_r$ .
- Thus, when using  $L_t^*$ ,  $t_0 = t_r$ .

# Modified VBGF

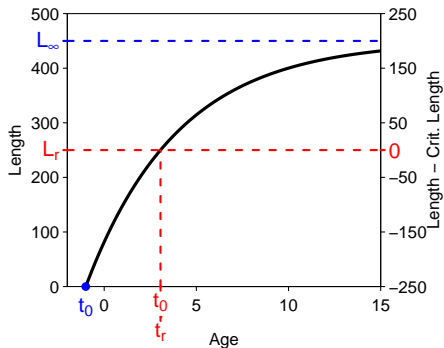
- Therefore, this simple adjustment allows direction estimation of  $t_r$ .

$$L_t - L_r = L_\infty (1 - e^{-K(t-t_r)})$$

$$L_t = L_r + L_\infty (1 - e^{-K(t-t_r)})$$

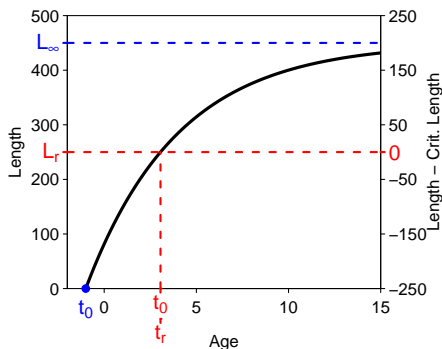
# Modified VBGF

- However,  $L_{\infty}$  is now incorrect.



# Modified VBGF

- However,  $L_{\infty}$  is now incorrect.



- But this is easily corrected.

$$L_t = L_r + (L_{\infty} - L_r) (1 - e^{-K(t-t_r)})$$