Modeling Growth with the von Bertalanffy Model

Dr. Derek H. Ogle

Northland College

Wisconsin Age & Growth Workshop
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Objectives

1. Data Requirements

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2. Different Versions of the von Bertalanffy Model

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3. General Model Fitting

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4. Typical Model Fitting Problems

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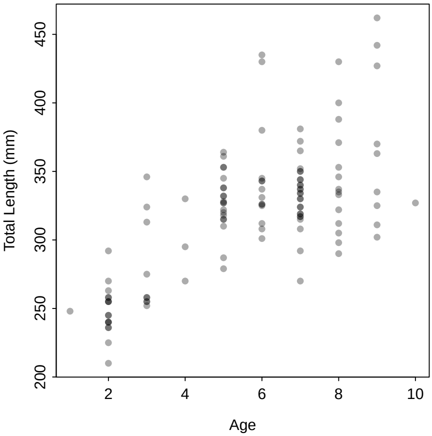
Typical Age-Length Data

- Length- and age-at-capture.

age	tl	sex
1	248	M
2	210	M
2	240	M
2	270	M
4	270	M
5	332	M
7	372	M
9	335	M
10	327	M

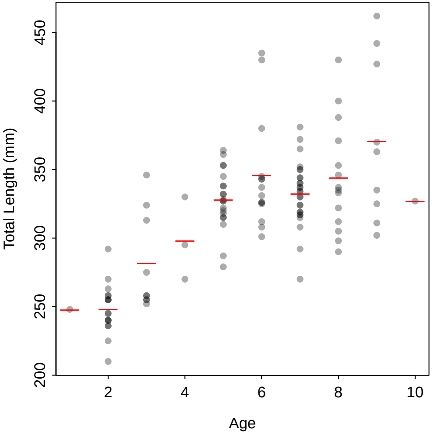
Typical Age-Length Data

- Raw observations.



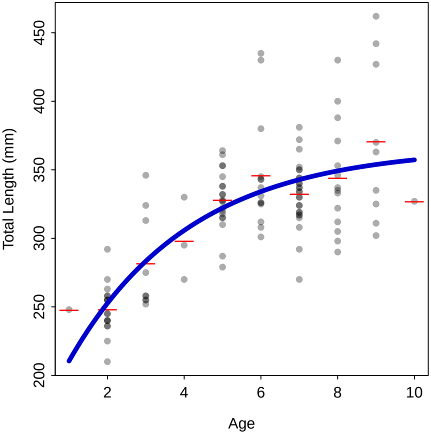
Typical Age-Length Data

- Mean lengths-at-age.



Typical Age-Length Data

- von Bertalanffy model representation of mean lengths-at-age.



Typical Age-Length Data

- Back-calculated lengths and age.

id	agecap	prvAge	bcTL
4	4	1	4.84
4	4	2	9.38
4	4	3	15.32
4	4	4	14.11
7	6	1	5.32
7	6	2	11.13
7	6	3	13.75
7	6	4	15.37
7	6	5	16.29
7	6	6	20.05

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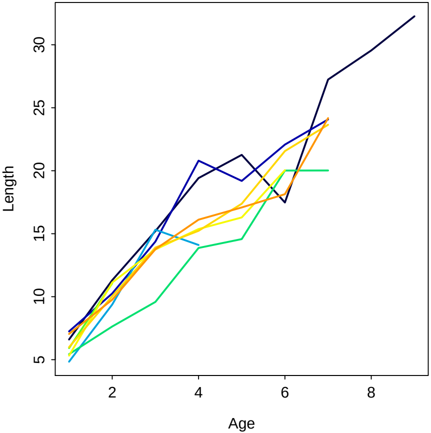

Typical Age-Length Data

- Length at time of marking and recapture.

Lrecap	Lmark	yrsAtLarge
413	336	0.91
420	353	0.32
428	427	0.16
400	397	0.21
442	430	0.14
249	239	0.17
450	258	1.29
345	269	0.82
512	485	1.09
394	394	0.10

Typical Age-Length Data

- **"At-capture" data.**
- "Back-calculated" data (i.e., longitudinal).
- "Mark-recapture" data.

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3. General Model Fitting
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Different Parameterizations of the von Bertalanffy Model

- Typical (due to Beverton (1954) and Beverton and Holt (1957)).
- Original (by von Bertalanffy).
- Gallucci and Quinn II (1979).
- Mooij *et al.* 1999.
- Francis (1988).

von Bertalanffy Model – Typical

$$E[L|t] = L_\infty \left(1 - e^{-K(t-t_0)}\right)$$

where

- $E[L|t]$ is the expected (i.e., average) length at time (or age) t,

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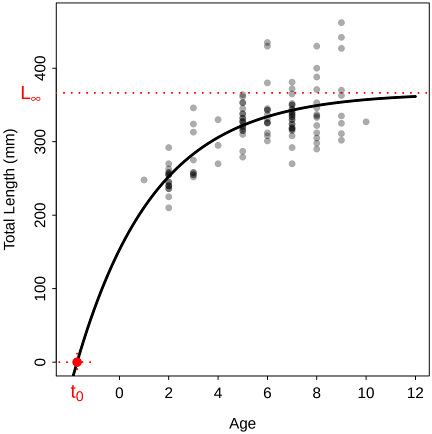
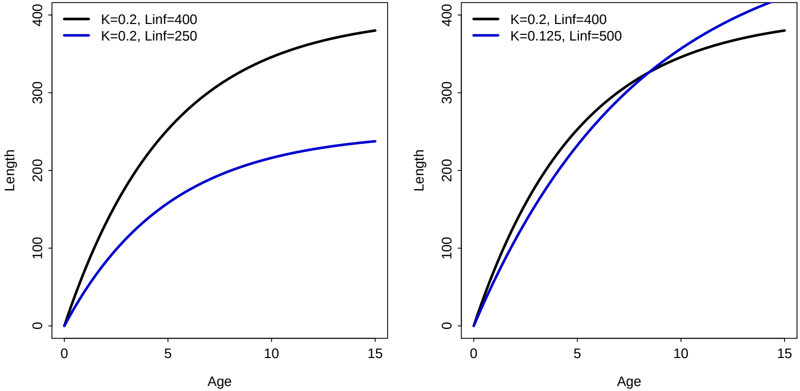


von Bertalanffy Model – Typical

Which population exhibits faster growth?

- K=0.2
- K=0.125

von Bertalanffy Model – Typical



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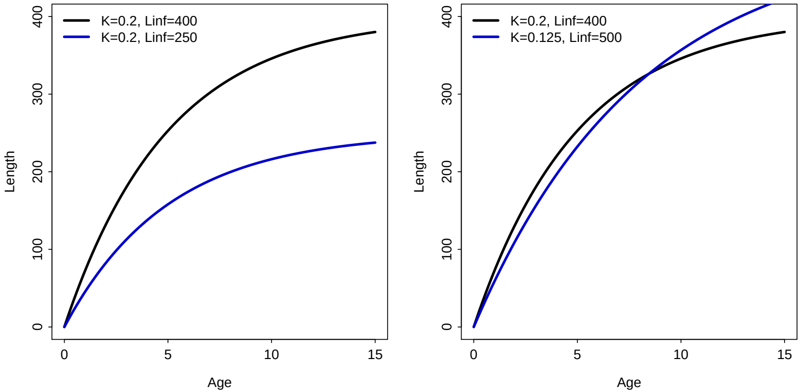

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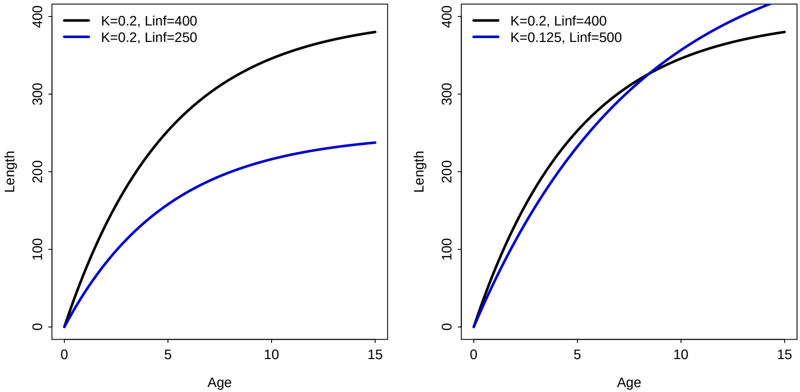

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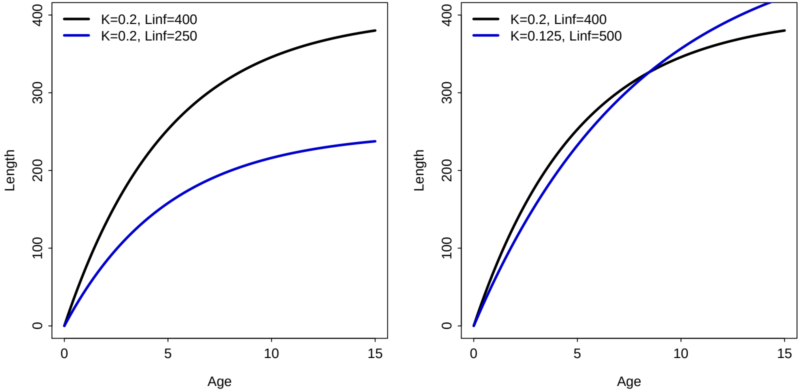

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- K does represent how fast L approaches L_∞.
  - $\frac{log(2)}{K}$ is "half-life" (time to reach $\frac{L_\infty}{2}$).

von Bertalanffy Model – Original

$$E[L|t] = L_\infty - (L_\infty - L_0)\, e^{-Kt}$$

where

- L_∞ is the asymptotic average length,
- K is the Brody growth rate coefficient (units are yr^{-1}), and
- L_0 is the mean length at time zero (i.e., birth).

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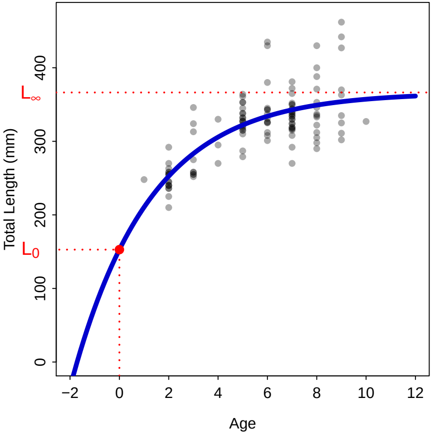

von Bertalanffy Model – Gallucci & Quinn (1979)

$$E[L|t] = \frac{\omega}{K}\left(1 - e^{-K(t-t_0)}\right)$$

where

- ω is a contrived parameter (i.e., $= KL_\infty$) that is representative of the instantaneous growth rate near t_0,
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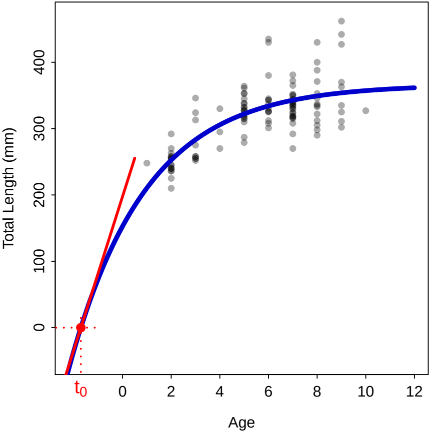

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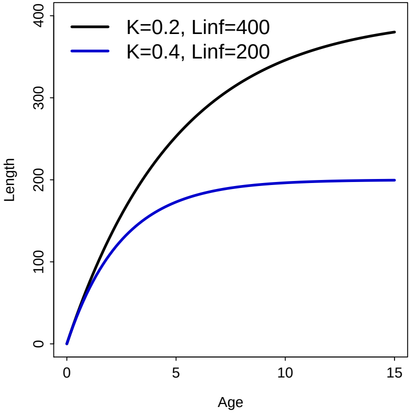

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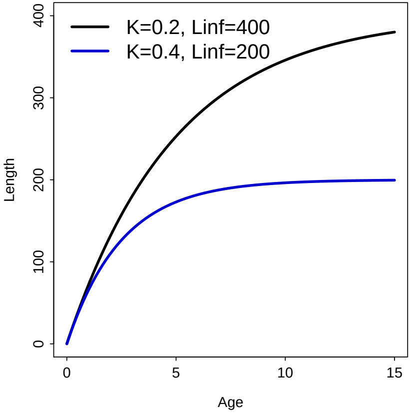

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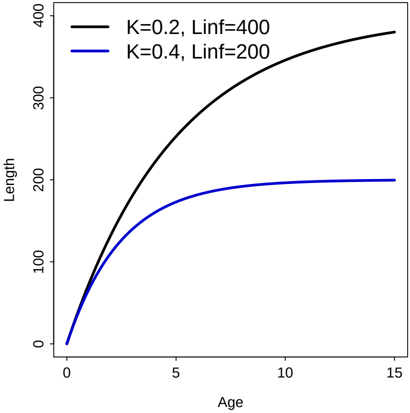

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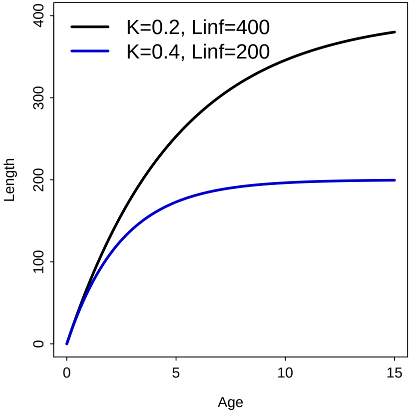

- ω represents "instantaneous growth" near t_0 – i.e., growth very early in life.
- It is NOT a useful metric of overall "growth."
- It is NOT useful, by itself, for comparing among groups.

von Bertalanffy Model – Mooij *et al.* (1999)

$$E[L|t] = L_\infty - (L_\infty - L_0)\, e^{-\frac{\omega}{L_\infty}t}$$

where

- ω is a contrived parameter (i.e., $= KL_\infty$) that is representative of the instantaneous growth rate near t_0,
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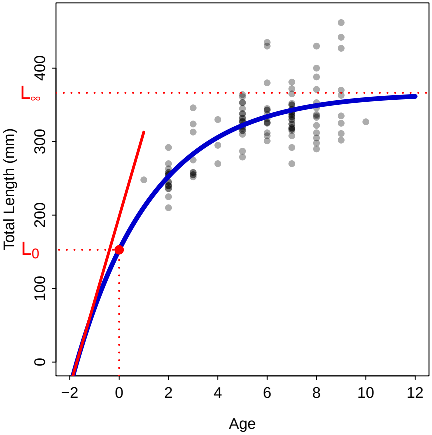

$$E[L|t] = L_1 + (L_3 - L_1)\,\frac{1-r^{2\frac{t-t_1}{t_3-t_1}}}{1-r^2}$$

where

$$r = \frac{L_3-L_2}{L_2-L_1}$$

and

- t_1 is a user chosen “young” age,
- t_3 is a user chosen “old” age,
- L_1 is the modeled mean length at t_1,
- L_2 is the modeled mean length at $\frac{t_1+t_3}{2}$, and
- L_3 is the modeled mean length at t_3.

von Bertalanffy Model – Francis (1988)

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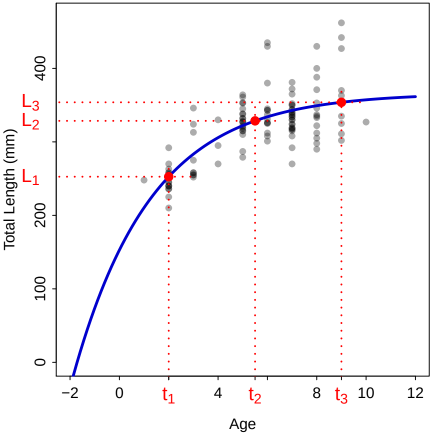

Different Parameterizations, Same Fit

Model	RSS	SE	Iterations	Linf	K	omega
Typical	123974	33.42	4	366.4	0.31	-
Original	123974	33.42	4	366.4	0.31	-
Gallucci Quinn	123974	33.42	4	-	0.31	115.3
Mooij	123974	33.42	4	366.4	-	115.3
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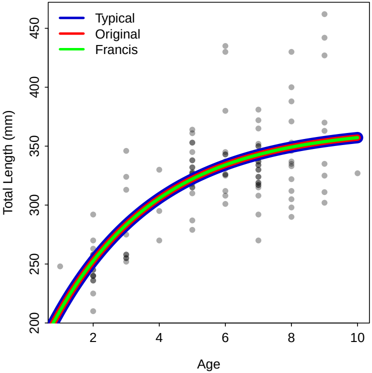

Different Parameterizations, Different Correlations

Typical Parameterization

	K	t0
Linf	-0.951	-0.871
K		0.972

Francis Parameterization

L2	L3
-0.026	0.033
	-0.080

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-0.026	0.033
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- Correlation (absolute value) summaries from each model

Model	Mean	Max
Typical	0.931	0.972
Original	0.889	0.951
Gallucci Quinn	0.983	0.999
Mooij	0.888	0.935
Francis	0.046	0.080

Model Parameterizations – Recommendations for Use

- **"Typical" Parameterization**
  - ✓ Need parameters for further modeling (e.g., yield models).
  - ✓ Following historical precedence.

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- **Francis Parameterization**
  - ✓ Interest is in mean length at specific ages.
  - ✓ Comparing "growth" among groups (using mean length-at-age).
  - ✓ Trouble fitting other models because of correlated parameters (*later*).

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3 General Model Fitting

4 Typical Model Fitting Problems

5 Example Output

Non-Linear Least-Squares

- von Bertalanffy growth model is non-linear.

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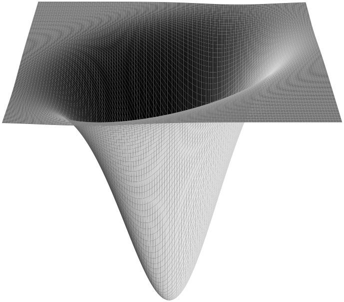
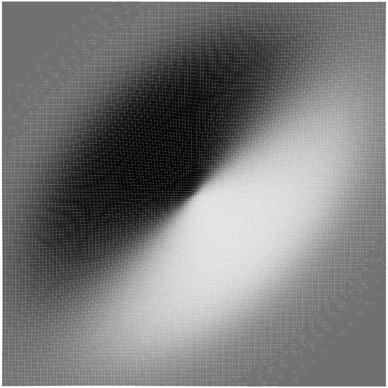
- Non-linear least-squares methods minimize RSS.

RSS Surface (side view)



Non-Linear Least-Squares

RSS Surface (top view)



- No closed-form solution as in linear least-squares.

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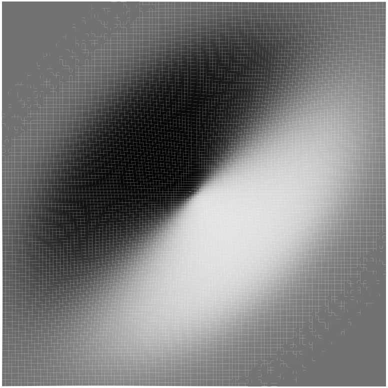

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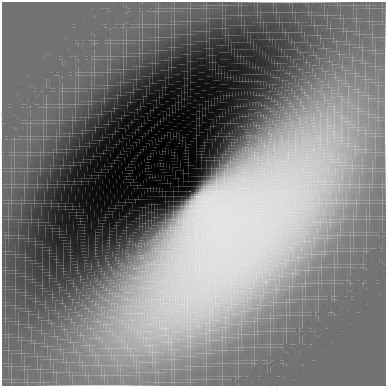

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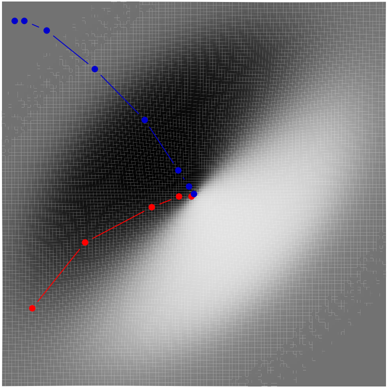

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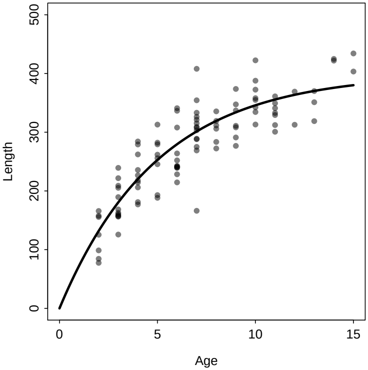
Additive Errors

$$L_i = L_\infty \left(1 - e^{-K(t_i - t_0)}\right) + \epsilon_i$$

where $\epsilon \sim N(0, \sigma)$



Error Structures

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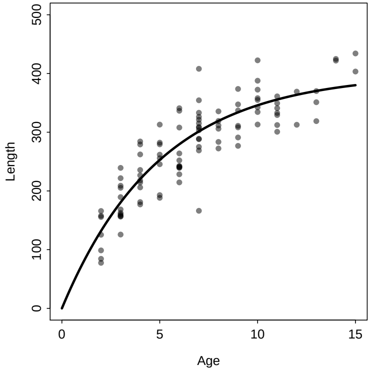

Multiplicative Errors

$$L_i = L_\infty \left(1 - e^{-K(t_i - t_0)}\right) e^{\epsilon_i}$$

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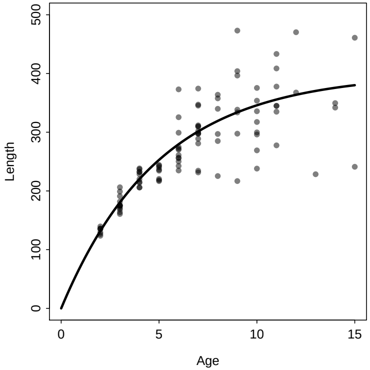

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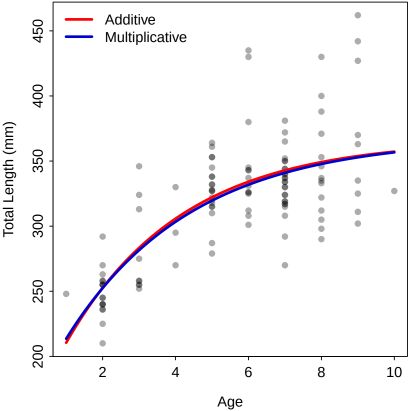
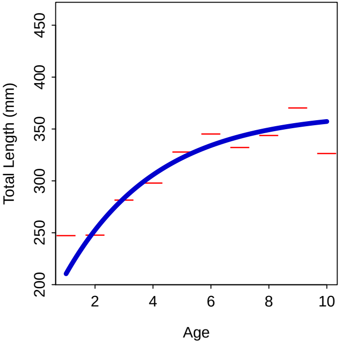


Fit to Mean Lengths-at-Age?



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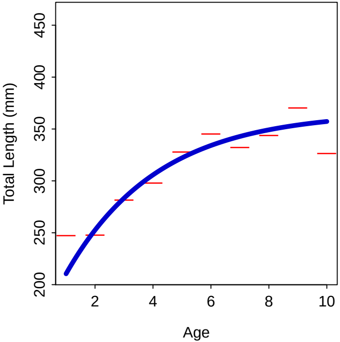

- Point estimates can be made with weights proportional to n_i.
- Variability and interval estimates can NOT be made.

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  4. 95% CI is values of ordered parameter estimates with 2.5% of values lesser and 2.5% of values greater.

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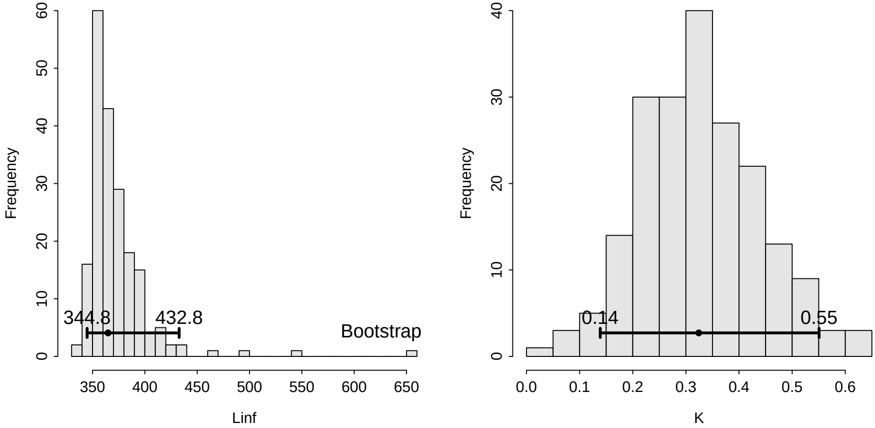

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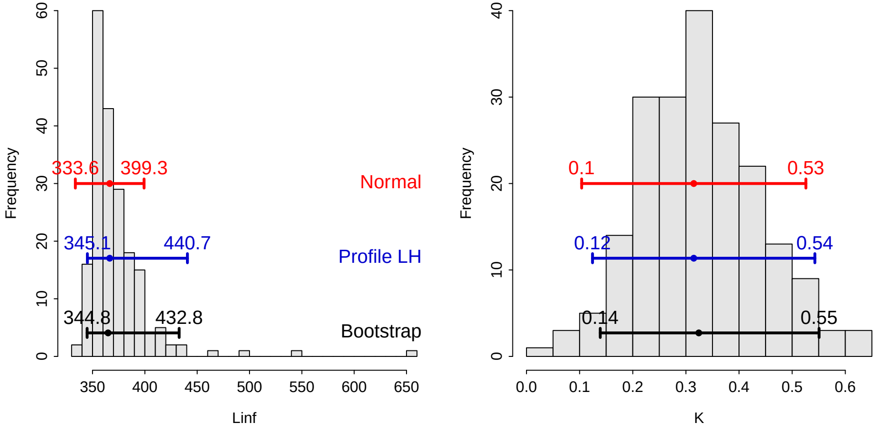

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Typical Model Fitting Problems

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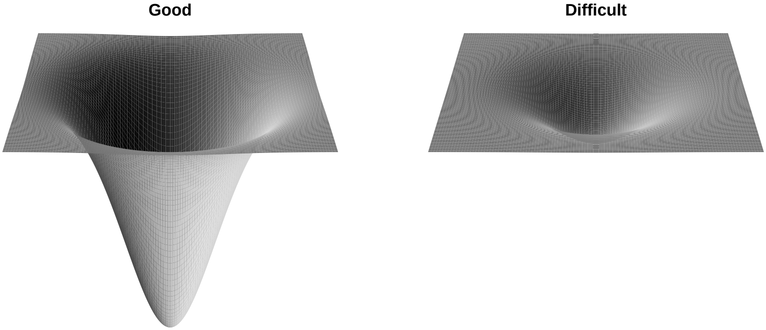
- Model values fail at an iteration.
- Best-fit values are unrealistic.

Algorithm Failure – Possible Reasons

- Failure to converge – RSS surface is flat



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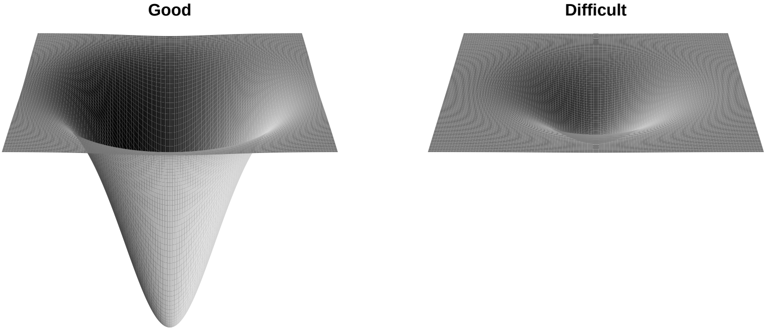

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- Often occurs with highly correlated parameters.

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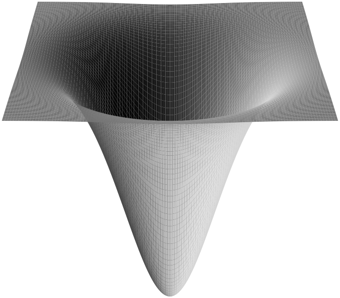

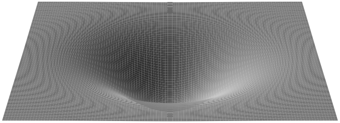

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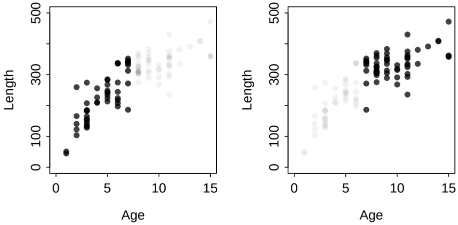

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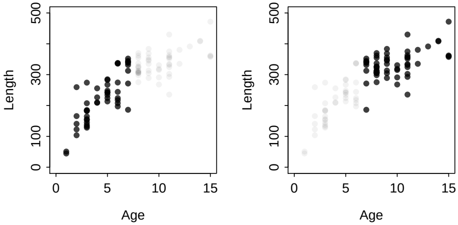

✓ *Fitting multiplicative errors (if appropariate) may help.*

✓ *Try different starting values.*

✓ *Use an algorithm that allows parameter constraints.*

✓ *Work hard to sample all ages.*

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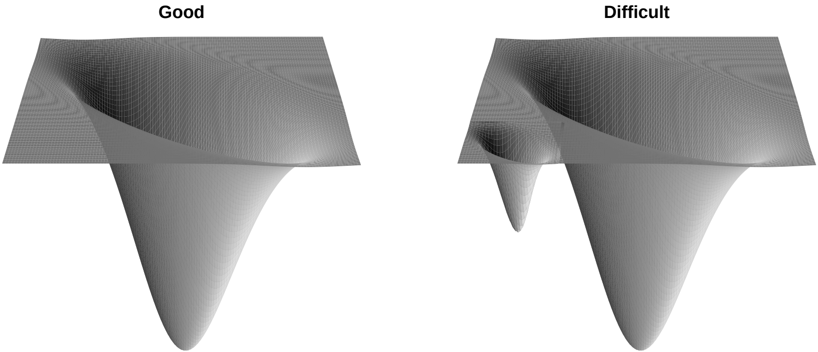

Objectives

1. Data Requirements
2. Different Versions of the von Bertalanffy Model
3. General Model Fitting
4. Typical Model Fitting Problems
5. Example Output

Example Output – Mooij Parameterization

1. Declare the model parameterization (e.g., Mooij)

```
> ## returns predicted length given age and values for parameters
> vb <- function(age,Linf,omega,L0)
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3. Run default non-linear algorithm

```
> ## variables in crm data frame are tl and age
> fit <- nls(tl~vb(age,Linf,omega,L0),data=crm,
             start=stvals,trace=TRUE)

130529 :  380 114 150
123980 :  366.1 115.3 152.6
123974 :  366.4 115.3 152.8
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```

Example Output – Mooij Parameterization

4. Examine coefficients

```
> ## Best-fit parameter estimates
> (cf <- coef(fit) )

 Linf omega    L0
366.4 115.3 152.8

> ## profile LH CIs for parameters
> confint(fit)

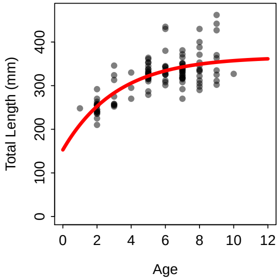
        2.5% 97.5%
Linf  345.09 440.7
omega  54.51 188.7
L0     54.51 212.5

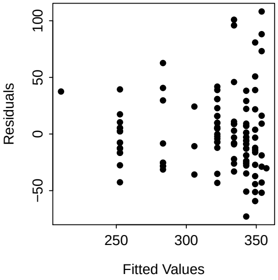
> ## Correlations among parameter estimates
> summary(fit,correlation=TRUE)$correlation

         Linf   omega      L0
Linf   1.0000 -0.9348  0.7962
omega -0.9348  1.0000 -0.9321
L0     0.7962 -0.9321  1.0000
```

5. Examine fit

```
> ## Best-fit line -- LEFT
> plot(tl~age,data=crm,xlab="Age",ylab="Total Length (mm)",
      ylim=c(0,470),xlim=c(0,12),pch=16,col=rgb(0,0,0,0.5))
> curve(vb(x,Linf=cf["Linf"],omega=cf["omega"],L0=cf["L0"]),
        from=0,to=12,lwd=4,add=TRUE,col="red")
> ## Residual plot -- RIGHT
> plot(residuals(fit)~fitted(fit),pch=16,
      xlab="Fitted Values",ylab="Residuals")
```





Example Output – Mooij Parameterization

6. Construct bootstrap samples

```
> bres <- nlsBoot(fit,niter=200)
> head(bres$coefboot,n=10)

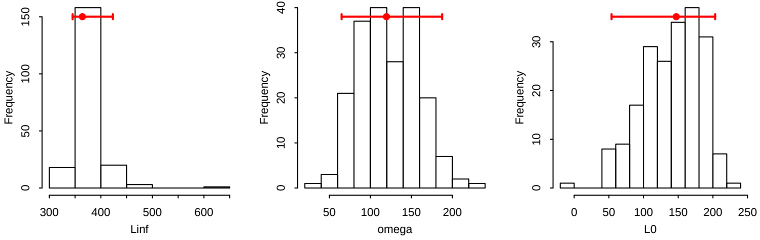
       Linf  omega    L0
 [1,] 371.3 107.81 149.4
 [2,] 378.6  85.41 198.4
 [3,] 357.8 109.19 190.0
 [4,] 370.4 139.35 113.4
 [5,] 413.1  66.39 202.4
 [6,] 362.0 104.62 177.9
 [7,] 363.7 128.56 159.5
 [8,] 379.5  87.99 177.7
 [9,] 354.3 142.30 121.7
[10,] 372.2 103.39 164.1
```

Example Output – Mooij Parameterization

7. Examine bootstrap distributions and confidence intervals

```
> confint(bres,plot=TRUE)
```

```
      95% LCI 95% UCI
Linf   345.17   423.2
omega   64.91   187.6
L0      53.90   203.3
```



References

Beverton, R. J. H. 1954. Notes on the use of theoretical models in the study of the dynamics of exploited fish populations. Miscellaneous Contribution 2, United States Fishery Laboratory, Beaufort, North Carolina.

Beverton, R. J. H. and S. J. Holt. 1957. On the dynamics of exploited fish populations, *Fisheries Investigations (Series 2)*, volume 19. United Kingdom Ministry of Agriculture and Fisheries, 533 pp.

Francis, R. 1988. Are growth parameters estimated from tagging and age-length data comparable? Canadian Journal of Fisheries and Aquatic Sciences 45:936–942.

Gallucci, V. F. and T. Quinn II. 1979. Reparameterizing, fitting, and testing a simple growth model. Transactions of the American Fisheries Society 108:14–25.

von Bertalanffy, L. 1934. Untersuchungen ueber die gesetzlichkeity des wachstums. Wilhelm Roux' Arch Entwick Organ 131:613–652.

von Bertalanffy, L. 1938. A quantitative theory of organic growth (inquiries on growth laws ii). Human Biology 10:181–213.