# MATH 160

# EXAM 4

# 20 April 2012

## Your name:

## Pledge:

There are 11 problems, and the point values of each problem are shown. A perfect score is 100 points. Notice, please, that problem 11 is worth a lot of points. Calculator use is not permitted. I'll be in my office (Arter 104B) during the test if you have questions.

**Good luck!!**

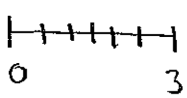
1. (4 points) Compute the sum of the first 100 positive integers. In other words, determine the integer which equals the following sum:

$$1+2+3+\cdots+100 = \frac{100(101)}{2} = 50(101) = \underline{\underline{5050}}$$

2. (12 points) Use the formal definition of the definite integral to compute the integral below. (Note: the Fundamental Theorem of Calculus is not allowed on this one. In other words, you get no credit for writing this: $\int_0^3 x^2\,dx = \frac{1}{3}x^3\Big|_0^3 = \frac{1}{3}3^3 = 9$.)

$$\int_0^3 x^2\,dx$$

$\Delta x = 3/n \,,\quad c_k = k\Delta x = 3k/n.$




$$\sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} f(3k/n)\,3/n$$

$$= \frac{3}{n}\sum_{k=1}^{n}\left(\frac{3k}{n}\right)^2$$

$$= \frac{3^3}{n^3}\sum_{k=1}^{n} k^2$$

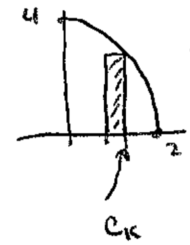
$$= \frac{3^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}.$$

$$\int_0^3 x^2\,dx = \lim_{n\to\infty}\frac{3^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{3^3}{3} = 9.$$

3. (4 points) Let $f(x) = 4 - x^2$ on the interval $0 \le x \le 2$. Suppose you used a calculator or computer to determine the value of the Riemann sum

$$\sum_{k=1}^{100} f(c_k)\,\Delta x$$

using right endpoints for the values of $c_k$. Select the correct statement from the following.




$f(x) = 4 - x^2$
(decreases)

A. $\int_0^2 f(x)\,dx = \pi$

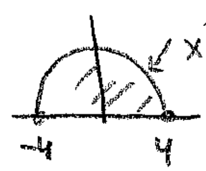
B. $\sum_{k=1}^{100} f(c_k)\,\Delta x = \int_0^2 f(x)\,dx$

C. $\sum_{k=1}^{100} f(c_k)\,\Delta x > \int_0^2 f(x)\,dx$

D. $\sum_{k=1}^{100} f(c_k)\,\Delta x < \int_0^2 f(x)\,dx$

4. (6 points) Calculate the following integral by interpreting it in terms of area:

$$\int_{-4}^{4} \sqrt{16 - x^2}\,dx \;=\; \text{area of}$$




$x^2 + y^2 = 16$

$$= \frac{1}{2} \cdot \pi (4)^2 = 8\pi$$

5. (8 points) A dam is punctured and leaks water at a rate of $r(t) = 1 + t$ gallons per minute after $t$ minutes. Determine the total amount of water leaked during the first 10 minutes after the puncture.

$$\text{Total leaked} = \int_0^{10} (1+t)\,dt$$

$$= \left(t + \tfrac{1}{2}t^2\right)\Big|_0^{10}$$

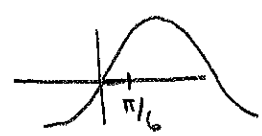
$$= 10 + \tfrac{1}{2}\cdot 100 = 60 \text{ gallons.}$$

6. (6 points) Use the comparison properties of integrals to show the following:

$$\int_0^{\pi/6} x \sin^2 x \, dx \leq \frac{\pi^2}{144}$$

You'll get an additional 4 bonus points if, instead of the above inequality, you can show this:

$$\int_0^{\pi/6} x \sin^2 x \, dx \leq \frac{\pi^2}{288}$$




$y = \sin x$

For $0 \leq x \leq \pi/6$, we have $0 \leq \sin x \leq 1/2$,

so that $\sin^2 x \leq 1/4$,

and so $x \sin^2 x \leq \frac{1}{4} x$.

Thus, $\int_0^{\pi/6} x \sin^2 x\,dx \leq \int_0^{\pi/6} \frac{1}{4} x\,dx = \frac{1}{4}\cdot\frac{1}{2}x^2\Big|_0^{\pi/6} = \frac{1}{8}\frac{\pi^2}{36} = \frac{\pi^2}{288}$.

7. (8 points) Given that the graph of $f$ passes through the point $(1,5)$ and that the slope of its tangent line at $(x, f(x))$ is $2x+1$, find $f(2)$.

$f(1) = 5$

$f'(x) = 2x+1$

$f(x) = x^2+x+C$

$5 = f(1) = 1+1+C$.

$C = 3$

$f(x) = x^2+x+3$

$f(2) = 4+2+3 = \underline{\underline{9}}$

8. (6 points) Find the derivative of the function

$$F(x) = \int_{x^3}^{1} \sqrt{1+\cos t}\, dt$$

$F'(x) = -3x^2\sqrt{1+\cos x^3}$

9. (6 points) Determine a function $f$ and a number $a$ for which

$$4+\int_a^x \frac{f(t)}{t}\,dt = \sqrt{x} \qquad \text{for } x > 0.$$

$$\frac{d}{dx}\left[4+\int_a^x \frac{f(t)}{t}\,dt\right] = \frac{d}{dx}\sqrt{x}$$

$$\frac{f(x)}{x} = \frac{1}{2\sqrt{x}} \,. \quad \boxed{f(x) = \frac{\sqrt{x}}{2}}$$

$$\sqrt{x} = 4+\int_a^x \frac{\sqrt{t}}{2t}\,dt = 4+\int_a^x \tfrac{1}{2}t^{-1/2}\,dt = 4+\left(t^{1/2}\Big|_a^x\right) = 4+\sqrt{x}-\sqrt{a}\,.$$

Then $4=\sqrt{a}$, $\boxed{a = 16}$

10. (4 points) Give an example of any nonconstant continuous function $f$ for which

$$\int_{-20}^{20} f(x)\,dx = 0\,. \qquad \boxed{f(x) = x}$$

Any odd continuous non constant function will do.

11. (6+6+6+6+6+6=36 points) Compute the following 6 integrals:

$$\int_0^{\pi} \sec^2(x/4)\, dx = 4\int_0^{\pi/4} \sec^2 u\, du$$

$u = x/4$

$4du = dx$

$u(0) = 0$

$u(\pi) = \pi/4$

$$= 4 \tan u \Big|_0^{\pi/4}$$

$$= 4(\tan \pi/4 - \tan 0)$$

$$= 4$$

$$\int \frac{1+\sin^2\theta}{\sin^2\theta}\, d\theta = \int (\csc^2\theta + 1)\, d\theta$$

$$= -\cot\theta + \theta + C$$

$$\int \sec^3 x \tan x\, dx = \int \sec^2 x \sec x \tan x\, dx = \int u^2\, du$$

$u = \sec x$

$du = \sec x \tan x\, dx$

$$= \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}\sec^3 x + C$$

$$\int_{1/3}^{1} \frac{\cos(x^{-2})}{x^3}\,dx = -\frac{1}{2}\int_{9}^{1} \cos u\,du$$

$$= \frac{1}{2}\int_{1}^{9} \cos u\,du$$

$$= \frac{1}{2}\sin u\Big|_{1}^{9}$$

$$= \frac{1}{2}(\sin 9 - \sin 1)$$

$u = x^{-2} = \frac{1}{x^2}$

$du = -2x^{-3}dx$

$-\frac{1}{2}du = \frac{1}{x^3}dx$

$u(1/3) = 9$

$u(1) = 1$

$$\int x^2(4+x^3)^6\,dx = \frac{1}{3}\int u^6\,du$$

$$= \frac{1}{3}\cdot\frac{1}{7}u^7 + C$$

$$= \frac{1}{21}(4+x^3)^7 + C$$

$u = 4+x^3$

$du = 3x^2dx$

$\frac{1}{3}du = x^2dx$

$$\int x\sqrt{1+x}\,dx = \int (u-1)u^{1/2}du$$

$$= \int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + C$$

$u = 1+x$

$x = u-1$

$du = dx$