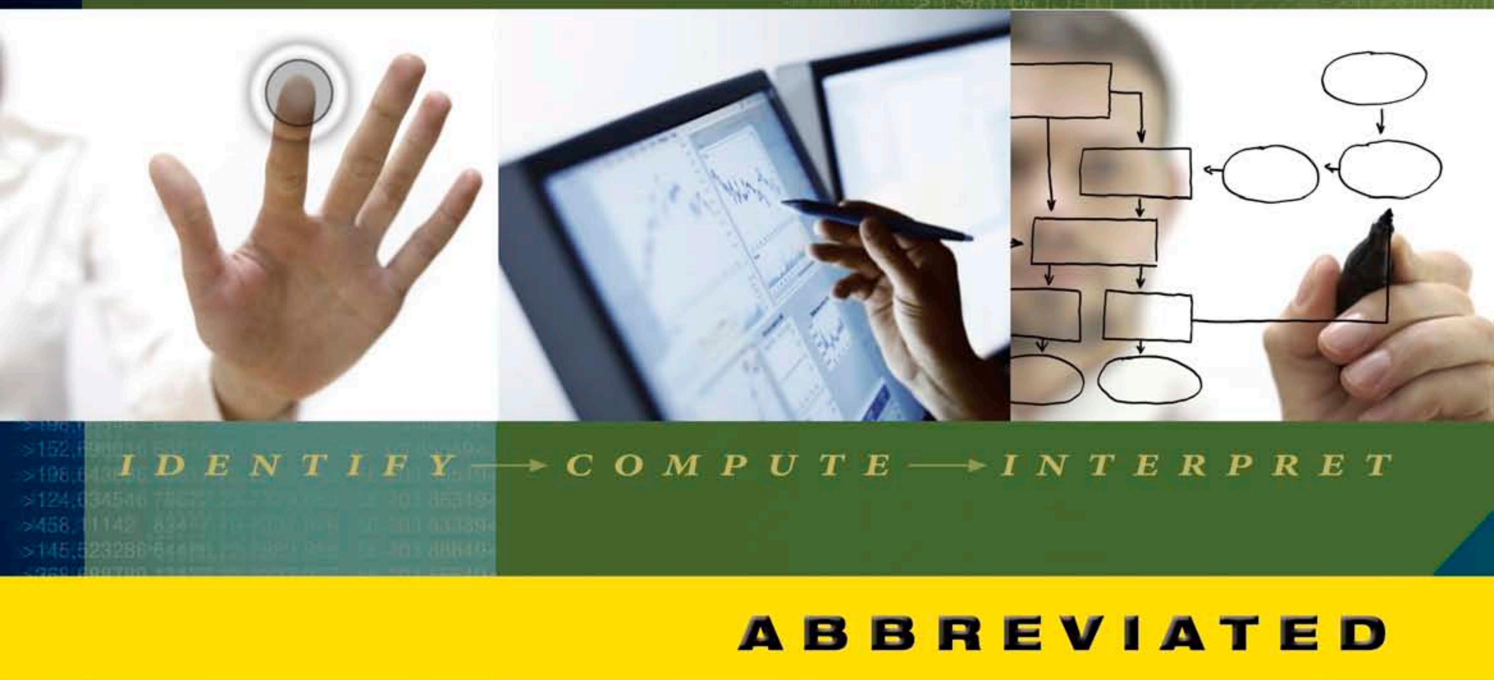
GERALD KELLER

Statistics

FOR MANAGEMENT AND ECONOMICS



IDENTIFY → COMPUTE → INTERPRET

ABBREVIATED

9e

A GUIDE TO STATISTICAL TECHNIQUES

Problem Objectives

DATA TYPES	Describe a Population	Compare Two Populations	Compare Two or More Populations	Analyze Relationship between Two Variables	Analyze Relationship among Two or More Variables
Interval	Histogram **Section 3.1** Ogive **Section 3.1** Stem-and-leaf **Section 3.1** Line chart **Section 3.2** Mean, median, and mode **Section 4.1** Range, variance, and standard deviation **Section 4.2** Percentiles and quartiles **Section 4.3** Box plot **Section 4.3** *t*-test and estimator of a mean **Section 12.1** Chi-squared test and estimator of a variance **Section 12.2**	Equal-variances *t*-test and estimator of the difference between two means: independent samples **Section 13.1** Unequal-variances *t*-test and estimator of the difference between two means: independent samples **Section 13.1** *t*-test and estimator of mean difference **Section 13.3** *F*-test and estimator of ratio of two variances **Section 13.4**	One-way analysis of variance **Section 14.1** LSD multiple comparison method **Section 14.2** Tukey's multiple comparison method **Section 14.2** Two-way analysis of variance **Section 14.4** Two-factor analysis of variance **Section 14.5**	Scatter diagram **Section 3.3** Covariance **Section 4.4** Coefficient of correlation **Section 4.4** Coefficient of determination **Section 4.4** Least squares line **Section 4.4** Simple linear regression and correlation **Chapter 16**	Multiple regression **Chapter 17**
Nominal	Frequency distribution **Section 2.2** Bar chart **Section 2.2** Pie chart **Section 2.2** *z*-test and estimator of a proportion **Section 12.3** Chi-squared goodness-of-fit test **Section 15.1**	*z*-test and estimator of the difference between two proportions **Section 13.5** Chi-squared test of a contingency table **Section 15.2**	Chi-squared test of a contingency table **Section 15.2**	Chi-squared test of a contingency table **Section 15.2**	Not covered
Ordinal	Median **Section 4.1** Percentiles and quartiles **Section 4.3** Box plot **Section 4.3**				Not covered

AMERICAN NATIONAL ELECTION SURVEY AND GENERAL SOCIAL SURVEY EXERCISES

APPLICATION SECTIONS

APPLICATION SUBSECTION

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Statistics

FOR MANAGEMENT AND ECONOMICS ABBREVIATED

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GERALD KELLER

Wilfred Laurier University

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Statistics for Management and Economics Abbreviated, Ninth Edition

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Cover Images:
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Library of Congress Control Number: 2010928228

Abbreviated Student Edition ISBN 13: 978-1-111-52708-2

Abbreviated Student Edition ISBN 10: 1-111-52708-3

Package Abbreviated Student Edition ISBN 13: 978-1-111-52732-7

Package Abbreviated Student Edition ISBN 10: 1-111-52732-6

South-Western Cengage Learning
5191 Natorp Boulevard
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Printed in the United States of America
1 2 3 4 5 6 7 14 13 12 11 10

BRIEF CONTENTS

CONTENTS

PREFACE

Businesses are increasingly using statistical techniques to convert data into information. For students preparing for the business world, it is not enough merely to focus on mastering a diverse set of statistical techniques and calculations. A course and its attendant textbook must provide a complete picture of statistical concepts and their applications to the real world. *Statistics for Management and Economics* is designed to demonstrate that statistics methods are vital tools for today's managers and economists.

Fulfilling this objective requires the several features that I have built into this book. First, I have included data-driven examples, exercises, and cases that demonstrate statistical applications that are and can be used by marketing managers, financial analysts, accountants, economists, operations managers, and others. Many are accompanied by large and either genuine or realistic data sets. Second, I reinforce the applied nature of the discipline by teaching students how to choose the correct statistical technique. Third, I teach students the concepts that are essential to interpreting the statistical results.

Why I Wrote This Book

Business is complex and requires effective management to succeed. Managing complexity requires many skills. There are more competitors, more places to sell products, and more places to locate workers. As a consequence, effective decision making is more crucial than ever before. On the other hand, managers have more access to larger and more detailed data that are potential sources of information. However, to achieve this potential requires that managers know how to convert data into information. This knowledge extends well beyond the arithmetic of calculating statistics. Unfortunately, this is what most textbooks offer—a series of unconnected techniques illustrated mostly with manual calculations. This continues a pattern that goes back many years. What is required is a complete approach to applying statistical techniques.

When I started teaching statistics in 1971, books demonstrated how to calculate statistics and, in some cases, how various formulas were derived. One reason for doing so was the belief that by doing calculations by hand, students would be able to understand the techniques and concepts. When the first edition of this book was published in 1988, an important goal was to teach students to identify the correct technique. Through the next eight editions, I refined my approach to emphasize interpretation and decision making equally. I now divide the solution of statistical problems into three stages and include them in every appropriate example: (1) *identify* the technique, (2) *compute* the statistics, and (3) *interpret* the results. The compute stage can be completed in any or all of three ways: manually (with the aid of a calculator), using Excel 2010, and using Minitab. For those courses that wish to use the computer extensively, manual calculations can be played down or omitted completely. Conversely, those that wish to emphasize manual calculations may easily do so, and the computer solutions can be selectively introduced or skipped entirely. This approach is designed to provide maximum flexibility, and it leaves to the instructor the decision of if and when to introduce the computer.

I believe that my approach offers several advantages.

- An emphasis on identification and interpretation provides students with practical skills they can apply to real problems they will face regardless of whether a course uses manual or computer calculations.
- Students learn that statistics is a method of converting data into information. With 878 data files and corresponding problems that ask students to interpret statistical results, students are given ample opportunities to practice data analysis and decision making.
- The optional use of the computer allows for larger and more realistic exercises and examples.

Placing calculations in the context of a larger problem allows instructors to focus on more important aspects of the decision problem. For example, more attention needs to be devoted to interpreting statistical results. Proper interpretation of statistical results requires an understanding of the probability and statistical concepts that underlie the techniques and an understanding of the context of the problems. An essential aspect of my approach is teaching students the concepts. I do so in two ways.

1. Nineteen Java applets allow students to see for themselves how statistical techniques are derived without going through the sometimes complicated mathematical derivations.
2. Instructions are provided about how to create Excel worksheets that allow students to perform "what-if" analyses. Students can easily see the effect of changing the components of a statistical technique, such as the effect of increasing the sample size.

Efforts to teach statistics as a valuable and necessary tool in business and economics are made more difficult by the positioning of the statistics course in most curricula. The required statistics course in most undergraduate programs appears in the first or second year. In many graduate programs, the statistics course is offered in the first semester of a three-semester program and the first year of a two-year program. Accounting, economics, finance, human resource management, marketing, and operations management are usually taught after the statistics course. Consequently, most students will not be able to understand the general context of the statistical application. This deficiency is addressed in this book by "Applications in . . ." sections, subsections, and boxes. Illustrations of statistical applications in business that students are unfamiliar with are preceded by an explanation of the background material.

- For example, to illustrate graphical techniques, we use an example that compares the histograms of the returns on two different investments. To explain what financial analysts look for in the histograms requires an understanding that risk is measured by the amount of variation in the returns. The example is preceded by an "Applications in Finance" box that discusses how return on investment is computed and used.
- Later when I present the normal distribution, I feature another "Applications in Finance" box to show why the standard deviation of the returns measures the risk of that investment.
- Thirty-six application boxes are scattered throughout the book.
- I've added Do-It-Yourself Excel exercises will teach students to compute spreadsheets on their own.

Some applications are so large that I devote an entire section or subsection to the topic. For example, in the chapter that introduces the confidence interval estimator of a proportion, I also present market segmentation. In that section, I show how the confidence interval estimate of a population proportion can yield estimates of the sizes of market segments. In other chapters, I illustrate various statistical techniques by showing how marketing managers can apply these techniques to determine the differences that exist between market segments. There are six such sections and one subsection in this book. The "Applications in . . ." segments provide great motivation to the student who asks, "How will I ever use this technique?"

New in This Edition

Six large real data sets are the sources of 150 new exercises. Students will have the opportunity to convert real data into information. Instructors can use the data sets for hundreds of additional examples and exercises.

Many of the examples, exercises, and cases using real data in the eighth edition have been updated. These include the data on wins, payrolls, and attendance in baseball, basketball, football, and hockey; returns on stocks listed on the New York Stock Exchange, NASDAQ, and Toronto Stock Exchange; and global warming.

Chapter 2 in the eighth edition, which presented graphical techniques, has been split into two chapters–2 and 3. Chapter 2 describes graphical techniques for nominal data, and Chapter 3 presents graphical techniques for interval data. Some of the material in the eighth edition Chapter 3 has been incorporated into the new Chapter 3.

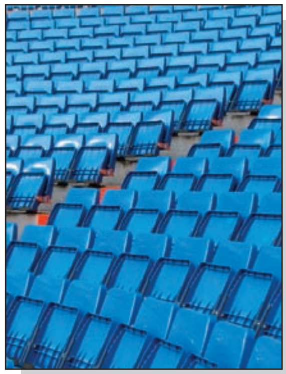
To make room for the new additional exercises we have removed Section 12.5, Applications in Accounting: Auditing.

I've created many new examples and exercises. Here are the numbers for the Abbreviated ninth edition: 116 solved examples, 1727 exercises, 26 cases, 690 data sets, 35 appendixes containing 37 solved examples, 98 exercises, and 25 data sets for a grand total of 153 solved examples, 1825 exercises, 26 cases, and 715 data sets.

GUIDED BOOK TOUR

Data Driven: The Big Picture

Solving statistical problems begins with a problem and data. The ability to select the right method by problem objective and data type is **a valuable tool for business**. Because business decisions are driven by data, students will leave this course equipped with the tools they need to make effective, informed decisions in all areas of the business world.



EXAMPLE 13.1*

DATA Xm13-01

Direct and Broker-Purchased Mutual Funds

Millions of investors buy mutual funds (see page 181 for a description of mutual funds), choosing from thousands of possibilities. Some funds can be purchased directly from banks or other financial institutions whereas others must be purchased through brokers, who charge a fee for this service. This raises the question, Can investors do better by buying mutual funds directly than by purchasing mutual funds through brokers? To help answer this question, a group of researchers randomly sampled the annual returns from mutual funds that can be acquired directly and mutual funds that are bought through brokers and recorded the net annual returns, which are the returns on investment after deducting all relevant fees. These are listed next.

Direct					Broker				
9.33	4.68	4.23	14.69	10.29	3.24	3.71	16.4	4.36	9.43
6.94	3.09	10.28	−2.97	4.39	−6.76	13.15	6.39	−11.07	8.31
16.17	7.26	7.1	10.37	−2.06	12.8	11.05	−1.9	9.24	−3.99
16.97	2.05	−3.09	−0.63	7.66	11.1	−3.12	9.49	−2.67	−4.44
5.94	13.07	5.6	−0.15	10.83	2.73	8.94	6.7	8.97	8.63
12.61	0.59	5.27	0.27	14.48	−0.13	2.74	0.19	1.87	7.06
3.33	13.57	8.09	4.59	4.8	18.22	4.07	12.39	−1.53	1.57
16.13	0.35	15.05	6.38	13.12	−0.8	5.6	6.54	5.23	−8.44
11.2	2.69	13.21	−0.24	−6.54	−5.75	−0.85	10.92	6.87	−5.72
1.14	18.45	1.72	10.32	−1.06	2.59	−0.28	−2.15	−1.69	6.95

Can we conclude at the 5% significance level that directly purchased mutual funds outperform mutual funds bought through brokers?

SOLUTION

IDENTIFY

To answer the question, we need to compare the population of returns from direct and the returns from broker-bought mutual funds. The data are obviously interval (we've recorded real numbers). This problem objective–data type combination tells us that the parameter to be tested is the difference between two means, $\mu_1 - \mu_2$. The hypothesis to be tested is that the mean net annual return from directly purchased mutual funds (μ_1) is larger than the mean of broker-purchased funds (μ_2). Hence, the alternative hypothesis is

$$H_1: \ (\mu_1 - \mu_2) > 0$$

As usual, the null hypothesis automatically follows:

$$H_0: \ (\mu_1 - \mu_2) = 0$$

To decide which of the t-tests of $\mu_1 - \mu_2$ to apply, we conduct the F-test of σ_1^2/σ_2^2.

$$H_0: \ \sigma_1^2/\sigma_2^2 = 1$$
$$H_1: \ \sigma_1^2/\sigma_2^2 \neq 1$$

COMPUTE

MANUALLY

From the data, we calculated the following statistics:

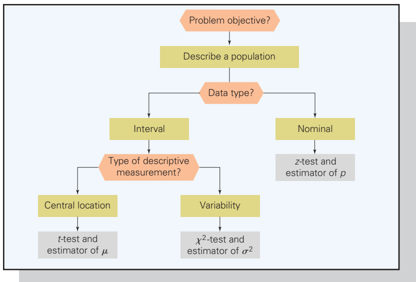
$$s_1^2 = 37.49 \quad \text{and} \quad s_2^2 = 43.34$$

$$\text{Test statistic: } F = s_1^2/s_2^2 = 37.49/43.34 = 0.86$$

$$\text{Rejection region: } F > F_{\alpha/2,\nu_1,\nu_2} = F_{.025,49,49} \approx F_{.025,50,50} = 1.75$$

Identify the Correct Technique

Examples introduce the first crucial step in this three-step (*identify–compute–interpret*) approach. Every example's solution begins by examining the data type and problem objective and then identifying the right technique to solve the problem.



Appendixes 13, 14, 15, 16, and **17** reinforce this problem-solving approach and allow students to hone their skills.

Flowcharts, found within the appendixes, help students develop the logical process for choosing the correct technique, reinforce the learning process, and provide easy review material for students.

APPENDIX 14 / REVIEW OF CHAPTERS 12 TO 14

The number of techniques introduced in Chapters 12 to 14 is up to 20. As we did in Appendix 13, we provide a table of the techniques with formulas and required conditions, a flowchart to help you identify the correct technique, and 25 exercises to give you practice in how to choose the appropriate method. The table and the flowchart have been amended to include the three analysis of variance techniques introduced in this chapter and the three multiple comparison methods.

TABLE **A14.1** Summary of Statistical Techniques in Chapters 12 to 14

t-test of μ
Estimator of μ (including estimator of $N\mu$)
χ^2 test of σ^2
Estimator of σ^2
z-test of p
Estimator of p (including estimator of Np)
Equal-variances t-test of $\mu_1 - \mu_2$
Equal-variances estimator of $\mu_1 - \mu_2$
Unequal-variances t-test of $\mu_1 - \mu_2$
Unequal-variances estimator of $\mu_1 - \mu_2$
t-test of μ_D
Estimator of μ_D
F-test of σ_1^2/σ_2^2
Estimator of σ_1^2/σ_2^2
z-test of $p_1 - p_2$ (Case 1)
z-test of $p_1 - p_2$ (Case 2)
Estimator of $p_1 - p_2$
One-way analysis of variance (including multiple comparisons)
Two-way (randomized blocks) analysis of variance
Two-factor analysis of variance

Factors That Identify the t-Test and Estimator of μ_D

1. **Problem objective**: Compare two populations
2. **Data type**: Interval
3. **Descriptive measurement**: Central location
4. **Experimental design**: Matched pairs

Factors That Identify . . . boxes are found in each chapter after a technique or concept has been introduced. These boxes allow students to see a technique's essential requirements and give them a way to easily review their understanding. These essential requirements are revisited in the review chapters, where they are coupled with other concepts illustrated in flowcharts.

A GUIDE TO STATISTICAL TECHNIQUES

Problem Objectives

DATA TYPES	Describe a Population	Compare Two Populations	Compare Two or More Populations
Interval	Histogram **Section 3.1** Ogive **Section 3.1** Stem-and-leaf **Section 3.1** Line chart **Section 3.2** Mean, median, and mode **Section 4.1** Range, variance, and standard deviation **Section 4.2** Percentiles and quartiles **Section 4.3** Box plot **Section 4.3** t-test and estimator of a mean **Section 12.1** Chi-squared test and estimator of a variance **Section 12.2**	Equal-variances t-test and estimator of the difference between two means: independent samples **Section 13.1** Unequal-variances t-test and estimator of the difference between two means: independent samples **Section 13.1** t-test and estimator of mean difference **Section 13.3** F-test and estimator of ratio of two variances **Section 13.4**	One-way analysis of varianc **Section 14.1** LSD multiple comparison method **Section 14.2** Tukey's multiple compariso method **Section 14.2** Two-way analysis of varianc **Section 14.4** Two-factor analysis of varia **Section 14.5**
Nominal	Frequency distribution **Section 2.2** Bar chart **Section 2.2** Pie chart **Section 2.2** z-test and estimator of a proportion **Section 12.3** Chi-squared goodness-of-fit test **Section 15.1**	z-test and estimator of the difference between two proportions **Section 13.5** Chi-squared test of a contingency table **Section 15.2**	Chi-squared test of a contingency table **Section 15.2**
Ordinal	Median **Section 4.1** Percentiles and quartiles **Section 4.3** Box plot **Section 4.3**		

A Guide to Statistical Techniques, found on the inside front cover of the text, pulls everything together into one useful table that helps students identify which technique to perform based on the problem objective and data type.

More Data Sets

A total of 715 data sets available to be downloaded provide ample practice. These data sets often contain real data, are typically large, and are formatted for Excel, Minitab, SPSS, SAS, JMP IN, and ASCII.

DATA
Xm13-02

Prevalent use of data in examples, exercises, and cases is highlighted by the accompanying data icon, which alerts students to go to Keller's website.

that of 5 years ago, with the possible exception of the mean, can we conclude at the 5% significance level that the dean's claim is true?

11.38 Xr11-38 Past experience indicates that the monthly long-distance telephone bill is normally distributed with a mean of $17.85 and a standard deviation of $3.87. After an advertising campaign aimed at increasing long-distance telephone usage, a random sample of 25 household bills was taken.

a. Do the data allow us to infer at the 10% significance level that the campaign was successful?

b. What assumption must you make to answer part (a)?

11.39 Xr11-39 In an attempt to reduce the number of person-hours lost as a result of industrial accidents, a large production plant installed new safety equipment. In a test of the effectiveness of the equipment, a random sample of 50 departments was chosen. The number of person-hours lost in the month before and the month after the installation of the safety equipment was recorded. The percentage change was calculated and recorded. Assume that the population standard deviation is $\sigma = 6$. Can we infer at the 10% significance level that the new safety equipment is effective?

11.40 Xr11-40 A highway patrol officer believes that the average speed of cars traveling over a certain stretch of highway exceeds the posted limit of 55 mph. The speeds of a random sample of 200 cars were recorded. Do these data provide sufficient evidence at the 1% significance level to support the officer's belief? What is the *p*-value of the test? (Assume that the standard deviation is known to be 5.)

11.41 Xr11-41 An automotive expert claims that the large number of self-serve gasoline stations has resulted in poor automobile maintenance, and that the average tire pressure is more than 4 pounds per square inch (psi) below its manufacturer's specification. As a quick test, 50 tires are examined, and the number of psi each tire is below specification is recorded. If we assume that tire pressure is normally distributed with $\sigma = 1.5$ psi, can we infer at the 10% significance level that the expert is correct? What is the *p*-value?

11.42 Xr11-42 For the past few years, the number of customers of a drive-up bank in New York has averaged 20 per hour, with a standard deviation of 3 per hour.

11.43 Xr11-43 A fast-food franchiser is considering building a restaurant at a certain location. Based on financial analyses, a site is acceptable only
pedestrians passing the location av
100 per hour. The number of ped
for each of 40 hours was recorde
the population standard deviation i
can we conclude at the 1% significa
site is acceptable?

11.44 Xr11-44 Many Alpine ski centers b
tions of revenues and profits on th
the average Alpine skier skis four t
investigate the validity of this assu
sample of 63 skiers is drawn and
report the number of times he or s
vious year. If we assume that the s
is 2, can we infer at the 10% sign
the assumption is wrong?

11.45 Xr11-45 The golf professional at a private course claims that members who have taken lessons from him lowered their handicap by more than five strokes. The club manager decides to test the claim by randomly sampling 25 members who have had lessons and asking each to report the reduction in handicap, where a negative number indicates an increase in the handicap. Assuming that the reduction in handicap is approximately normally distributed with a standard deviation of two strokes, test the golf professional's claim using a 10% significance level.

11.46 Xr11-46 The current no-smoking regulations in office buildings require workers who smoke to take breaks and leave the building in order to satisfy their habits. A study indicates that such workers average 32 minutes per day taking smoking breaks.

EXAMPLE 13.9 Test Marketing of Package Designs, Part 1

DATA Xm13-09

The General Products Company produces and sells a variety of household products. Because of stiff competition, one of its products, a bath soap, is not selling well. Hoping to improve sales, General Products decided to introduce more attractive packaging. The company's advertising agency developed two new designs. The first design features several bright colors to distinguish it from other brands. The second design is light green in color with just the company's logo on it. As a test to determine which design is better, the marketing manager selected two supermarkets. In one supermarket, the soap was packaged in a box using the first design; in the second supermarket, the second design was used. The product scanner at each supermarket tracked every buyer of soap over a 1-week period. The supermarkets recorded the last four digits of the scanner code for each of the five brands of soap the supermarket sold. The code for the General Products brand of soap is 9077 (the other codes are 4255, 3745, 7118, and 8855). After the trial period, the scanner data were transferred to a computer file. Because the first

CASE 14.1 Comparing Three Methods of Treating Childhood Ear Infections*


©AP Photo/Chris Carlson

DATA C14-01

Acute otitis media, an infection of the middle ear, is a common childhood illness. There are various ways to treat the problem. To help determine the best way, researchers conducted an experiment. One hundred and eighty children between 10 months and 2 years with recurrent acute otitis media were divided into three equal groups. Group 1 was treated by surgically removing the adenoids (adenoidectomy), the second was treated with the drug Sulfafurazole, and the third with a placebo. Each child was tracked for 2 years, during which time all symptoms and episodes of acute otitis media were recorded. The data were recorded in the following way:

Column 1: ID number
Column 2: Group number
Column 3: Number of episodes of the illness
Column 4: Number of visits to a physician because of any infection
Column 5: Number of prescriptions
Column 6: Number of days with symptoms of respiratory infection

a. Are there differences between the three groups with respect to the number of episodes, number of physician visits, number of prescriptions, and number of days with symptoms of respiratory infection?

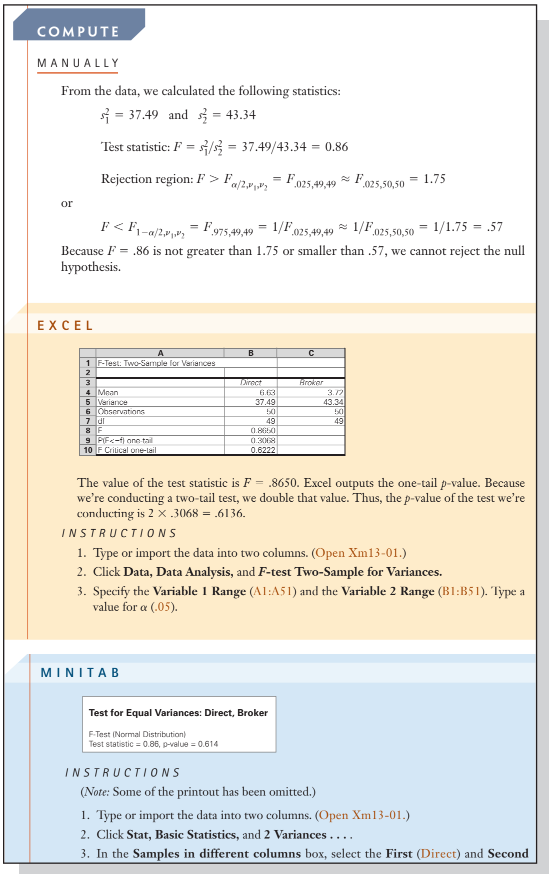
b. Assume that you are working for the company that makes the drug Sulfafurazole. Write a report to the company's executives discussing your results.

*This case is adapted from the *British Medical Journal*, February 2004.

Flexible to Use

Although many texts today incorporate the use of the computer, ***Statistics for Management and Economics*** is designed for maximum flexibility and ease of use for both instructors and students. To this end, parallel illustration of both manual and computer printouts is provided throughout the text. This approach **allows you to choose** which, if any, computer program to use. Regardless of the method or software you choose, the output and instructions that you need are provided!

Compute the Statistics



COMPUTE

MANUALLY

From the data, we calculated the following statistics:

$$s_1^2 = 37.49 \quad \text{and} \quad s_2^2 = 43.34$$

Test statistic: $F = s_1^2/s_2^2 = 37.49/43.34 = 0.86$

Rejection region: $F > F_{\alpha/2,\nu_1,\nu_2} = F_{.025,49,49} \approx F_{.025,50,50} = 1.75$

or

$$F < F_{1-\alpha/2,\nu_1,\nu_2} = F_{.975,49,49} = 1/F_{.025,49,49} \approx 1/F_{.025,50,50} = 1/1.75 = .57$$

Because $F = .86$ is not greater than 1.75 or smaller than .57, we cannot reject the null hypothesis.

EXCEL

	A	B	C
1	F-Test: Two-Sample for Variances		
2			
3		*Direct*	*Broker*
4	Mean	6.63	3.72
5	Variance	37.49	43.34
6	Observations	50	50
7	df	49	49
8	F	0.8650	
9	P(F<=f) one-tail	0.3068	
10	F Critical one-tail	0.6222	

The value of the test statistic is $F = .8650$. Excel outputs the one-tail p-value. Because we're conducting a two-tail test, we double that value. Thus, the p-value of the test we're conducting is $2 \times .3068 = .6136$.

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm13-01.)
2. Click **Data, Data Analysis,** and ***F*-test Two-Sample for Variances.**
3. Specify the **Variable 1 Range** (A1:A51) and the **Variable 2 Range** (B1:B51). Type a value for α (.05).

MINITAB

Test for Equal Variances: Direct, Broker

F-Test (Normal Distribution)
Test statistic = 0.86, p-value = 0.614

INSTRUCTIONS

(*Note:* Some of the printout has been omitted.)

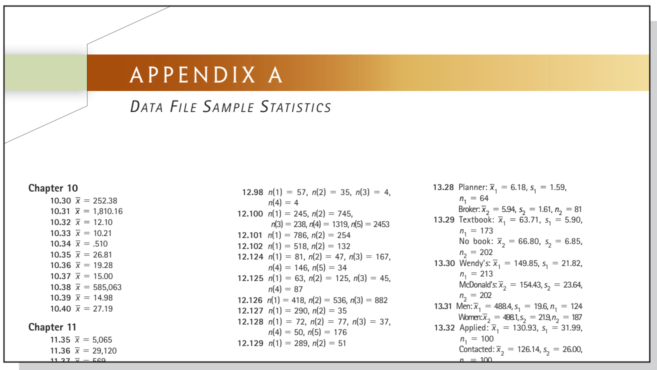
1. Type or import the data into two columns. (Open Xm13-01.)
2. Click **Stat, Basic Statistics,** and **2 Variances . . . .**
3. In the **Samples in different columns** box, select the **First** (Direct) and **Second**

Once the correct technique has been identified, examples take students to the next level within the solution by asking them to compute the statistics.

Manual calculation of the problem is presented first in each "Compute" section of the examples.

Step-by-step instructions in the use of **Excel** and **Minitab** immediately follow the manual presentation. Instruction appears in the book with the printouts—there's no need to incur the extra expense of separate software manuals. SPSS and JMP IN are also available at no cost on the Keller companion website.

Appendix A provides summary statistics that allow students to solve applied exercises with data files by hand. Offering unparalleled flexibility, this feature allows virtually *all* exercises to be solved by hand!



APPENDIX A

DATA FILE SAMPLE STATISTICS

Chapter 10

10.30 $\bar{x} = 252.38$
10.31 $\bar{x} = 1{,}810.16$
10.32 $\bar{x} = 12.10$
10.33 $\bar{x} = 10.21$
10.34 $\bar{x} = .510$
10.35 $\bar{x} = 26.81$
10.36 $\bar{x} = 19.28$
10.37 $\bar{x} = 15.00$
10.38 $\bar{x} = 585{,}063$
10.39 $\bar{x} = 14.98$
10.40 $\bar{x} = 27.19$

Chapter 11

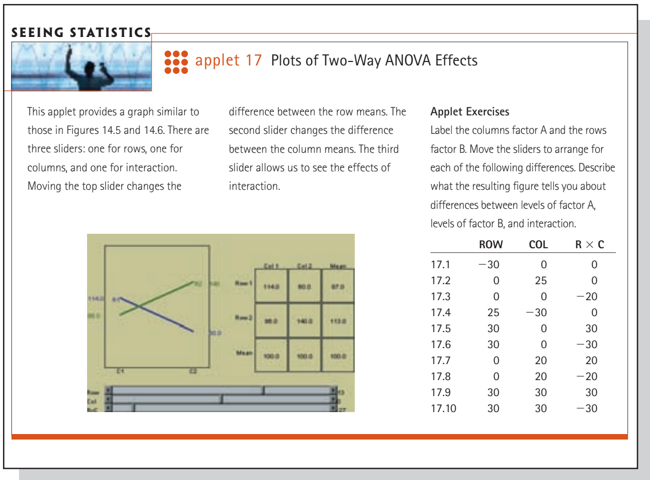
11.35 $\bar{x} = 5{,}065$
11.36 $\bar{x} = 29{,}120$
11.37 $\bar{x} = 569$

12.98 $n(1) = 57$, $n(2) = 35$, $n(3) = 4$, $n(4) = 4$
12.100 $n(1) = 245$, $n(2) = 745$, $n(3) = 238$, $n(4) = 1319$, $n(5) = 2453$
12.101 $n(1) = 786$, $n(2) = 254$
12.102 $n(1) = 518$, $n(2) = 132$
12.124 $n(1) = 81$, $n(2) = 47$, $n(3) = 167$, $n(4) = 146$, $n(5) = 34$
12.125 $n(1) = 63$, $n(2) = 125$, $n(3) = 45$, $n(4) = 87$
12.126 $n(1) = 418$, $n(2) = 536$, $n(3) = 882$
12.127 $n(1) = 290$, $n(2) = 35$
12.128 $n(1) = 72$, $n(2) = 77$, $n(3) = 37$, $n(4) = 50$, $n(5) = 176$
12.129 $n(1) = 289$, $n(2) = 51$

13.28 Planner: $\bar{x}_1 = 6.18$, $s_1 = 1.59$, $n_1 = 64$
Broker: $\bar{x}_2 = 5.94$, $s_2 = 1.61$, $n_2 = 81$
13.29 Textbook: $\bar{x}_1 = 63.71$, $s_1 = 5.90$, $n_1 = 173$
No book: $\bar{x}_2 = 66.80$, $s_2 = 6.85$, $n_2 = 202$
13.30 Wendy's: $\bar{x}_1 = 149.85$, $s_1 = 21.82$, $n_1 = 213$
McDonald's: $\bar{x}_2 = 154.43$, $s_2 = 23.64$, $n_2 = 202$
13.31 Men: $\bar{x}_1 = 488.4$, $s_1 = 19.6$, $n_1 = 124$
Women: $\bar{x}_2 = 498.1$, $s_2 = 21.9$, $n_2 = 187$
13.32 Applied: $\bar{x}_1 = 130.93$, $s_1 = 31.99$, $n_1 = 100$
Contacted: $\bar{x}_2 = 126.14$, $s_2 = 26.00$, $n_2 = 100$

Flexible Learning

For visual learners, the **Seeing Statistics** feature refers to online Java applets developed by Gary McClelland of the University of Colorado, which use the interactive nature of the web to illustrate key statistical concepts. With 19 applets and 82 follow-up exercises, students can explore and interpret statistical concepts, leading them to greater intuitive understanding. All Seeing Statistics applets can be found on CourseMate.


SEEING STATISTICS

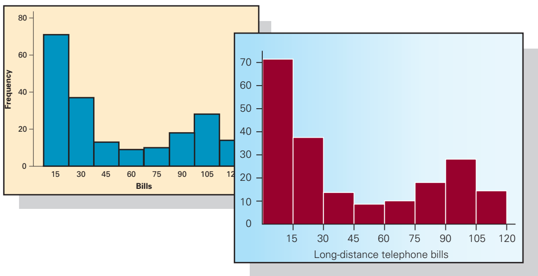
applet 17 Plots of Two-Way ANOVA Effects

This applet provides a graph similar to those in Figures 14.5 and 14.6. There are three sliders: one for rows, one for columns, and one for interaction. Moving the top slider changes the difference between the row means. The second slider changes the difference between the column means. The third slider allows us to see the effects of interaction.

Applet Exercises

Label the columns factor A and the rows factor B. Move the sliders to arrange for each of the following differences. Describe what the resulting figure tells you about differences between levels of factor A, levels of factor B, and interaction.

	ROW	COL	R × C
17.1	−30	0	0
17.2	0	25	0
17.3	0	0	−20
17.4	25	−30	0
17.5	30	0	30
17.6	30	0	−30
17.7	0	20	20
17.8	0	20	−20
17.9	30	30	30
17.10	30	30	−30



Ample use of graphics provides students many opportunities to see statistics in all its forms. In addition to manually presented figures throughout the text, Excel and Minitab graphic outputs are given for students to compare to their own results.

APPLIED: BRIDGING THE GAP

In the real world, it is not enough to know *how* to generate the statistics. To be truly effective, a business person must also know how to **interpret and articulate** the results. Furthermore, students need a framework to understand and apply statistics **within a realistic setting** by using realistic data in exercises, examples, and case studies.

Interpret the Results

INTERPRET

Examples round out the final component of the identify–compute–interpret approach by asking students to interpret the results in the context of a business-related decision. This final step motivates and shows how statistics is used in everyday business situations.

An Applied Approach

With **Applications in . . .** sections and boxes, ***Statistics for Management and Economics*** now includes 45 **applications** (in finance, marketing, operations management, human resources, economics, and accounting) highlighting how statistics is used in those professions. For example, "Applications in Finance: Portfolio Diversification and Asset Allocation" shows how probability is used to help select stocks to minimize risk. A new optional section, "Applications in Professional Sports: Baseball" contains a subsection on the success of the Oakland Athletics.

In addition to sections and boxes, **Applications in . . . exercises** can be found within the exercise sections to further reinforce the big picture.

4.5 (Optional) Applications in Professional Sports: Baseball

In the chapter-opening example, we provided the payrolls and the number of wins from the 2009 season. We discovered that there is a weak positive linear relationship between number of wins and payroll. The strength of the linear relationship tells us that some teams with large payrolls are not successful on the field, whereas some teams with small payrolls win a large number of games. It would appear that although the amount of money teams spend is a factor, another factor is *how* teams spend their money. In this section, we will analyze the eight seasons between 2002 and 2009 to see how small-payroll teams succeed.

Professional sports in North America is a multibillion-dollar business. The cost of a new franchise in baseball, football, basketball, and hockey is often in the hundreds of millions of dollars. Although some teams are financially successful during losing seasons, success on the field is often related to financial success. (Exercises 4.75 and 4.76

Applications in Medicine and Medical Insurance (Optional)

Physicians routinely perform medical tests, called *screenings*, on their patients. Screening tests are conducted for all patients in a particular age and gender group, regardless of their symptoms. For example, men in their 50s are advised to take a prostate-specific antigen (PSA) test to determine whether there is evidence of prostate cancer. Women undergo a Pap test for cervical cancer. Unfortunately, few of these tests are 100% accurate. Most can produce *false-positive* and *false-negative* results. A **false-positive** result is one in which the patient does not have the disease, but the test shows positive. A **false-negative** result is one in which the patient does have the disease, but the test produces a negative result. The consequences of each test are serious and costly. A false-negative test results in not detecting a disease in a patient, therefore postponing treatment, perhaps indefinitely. A false-positive test leads to apprehension and fear for the patient. In most cases, the patient is required to undergo further testing such as a biopsy. The unnecessary follow-up procedure can pose medical risks.

False-positive test results have financial repercussions. The cost of the follow-up procedure, for example, is usually far more expensive than the screening test. Medical insurance companies as well as government-funded plans are all adversely affected by false-positive test results. Compounding the problem is that physicians and patients are incapable of properly interpreting the results. A correct analysis can save both lives and money.

Bayes's Law is the vehicle we use to determine the true probabilities associated with screening tests. Applying the complement rule to the false-positive and false-negative rates produces the conditional probabilities that represent correct conclusions. Prior probabilities are usually derived by looking at the overall proportion of people with the diseases. In some cases, the prior probabilities may themselves have been revised

APPLICATIONS in OPERATIONS MANAGEMENT

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Quality

A critical aspect of production is quality. The quality of a final product is a function of the quality of the product's components. If the components don't fit, the product will not function as planned and likely cease functioning before its customers expect it to. For example, if a car door is not made to its specifications, it will not fit. As a result, the door will leak both water and air.

Operations managers attempt to maintain and improve the quality of products by ensuring that all components are made so that there is as little variation as possible. As you have already seen, statisticians measure variation by computing the variance.

Incidentally, an entire chapter (Chapter 21) is devoted to the topic of quality.

Education and Income: How Are They Related?

DATA GSS2008*

If you're taking this course, you're probably a student in an undergraduate or graduate business or economics program. Your plan is to graduate, get a good job, and draw a high salary. You have probably assumed that more education equals better job equals higher income. Is this true? Fortunately, the General Social Survey recorded two variables that will help determine whether education and income are related and, if so, what the value of an additional year of education might be.

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On page 663, we will provide our answer.

Chapter-opening examples and solutions present compelling discussions of how the techniques and concepts introduced in that chapter are applied to real-world problems. These examples are then revisited with a solution as each chapter unfolds, applying the methodologies introduced in the chapter.

Education and Income: How Are They Related?

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IDENTIFY

The problem objective is to analyze the relationship between two interval variables. Because we want to know how education affects income the independent variable is education (EDUC) and the dependent variable is income (INCOME).

COMPUTE

EXCEL

	A	B	C	D	E	F
1	SUMMARY OUTPUT					
2						
3	*Regression Statistics*					
4	Multiple R	0.3790				
5	R Square	0.1436				
6	Adjusted R Square	0.1429				
7	Standard Error	35,972				
8	Observations	1189				
9						
10	ANOVA					
11		*df*	*SS*	*MS*	*F*	*Significance F*
12	Regression	1	257,561,051,309	257,561,051,309	199.04	6.702E-42
13	Residual	1187	1,535,986,496,000	1,294,007,158		
14	Total	1188	1,793,547,547,309			
15						
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	
17	Intercept	–28926	5117	–5.65	1.971E-08	
18	EDUC	5111	362	14.11	6.702E-42	

MINITAB

Regression Analysis: INCOME versus EDUC

The regression equation is
Income = –28926 + 5111 EDUC
1189 cases used, 834 cases contain missing values

Predictor	Coef	SE Coef	T	P
Constant	–28926	5117	–5.65	0.000
EDUC	5110.7	362.2	14.11	0.000

S = 35972.3 R-Sq = 14.4% R-Sq(adj) = 14.3%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	2.57561E+11	2.57561E+11	199.04	0.000
Residual Error	1187	1.53599E+12	1294007158		
Total	1188	1.79355E+12			

CASE A15.1 Which Diets Work?

Every year, millions of people start new diets. There is a bewildering array of diets to choose from. The question for many people is, which ones work? Researchers at Tufts University in Boston made an attempt to point dieters in the right direction. Four diets were used:

1. Atkins low-carbohydrate diet
2. Zone high-protein, moderate-carbohydrate diet
3. Weight Watchers diet
4. Dr. Ornish's low-fat diet

The study recruited 160 overweight people and randomly assigned 40 to each diet. The average weight before dieting was 220 pounds, and all needed to lose between 30 and 80 pounds. All volunteers agreed to follow their diets for 2 months. No exercise or regular meetings were required. The following variables were recorded for each dieter using the format shown here:

Column 1: Identification number
Column 2: Diet
Column 3: Percent weight loss
Column 4: Percent low-density lipoprotein (LDL)—"bad" cholesterol—decrease
Column 5: Percent high-density lipoprotein (HDL)—"good" cholesterol—increase
Column 6: Quit after 2 months? 1 = yes, 2 = no
Column 7: Quit after 1 year? 1 = yes, 2 = no

Is there enough evidence to conclude that there are differences between the diets with respect to

a. percent weight loss?
b. percent LDL decrease?
c. percent HDL increase?
d. proportion quitting within 2 months?
e. proportion quitting after 1 year?

DATA CA15-01

Many of the **examples, exercises, and cases are based on actual studies** performed by statisticians and published in journals, newspapers, and magazines, or presented at conferences. Many data files were recreated to produce the original results.

A total of 1825 exercises, many of them new or updated, offer ample practice for students to use statistics in an applied context.

CHAPTER SUMMARY

Histograms are used to describe a single set of interval data. Statistics practitioners examine several aspects of the shapes of histograms. These are symmetry, number of modes, and its resemblance to a bell shape.

We described the difference between time-series data and cross-sectional data. Time series are graphed by line charts.

To analyze the relationship between two interval variables, we draw a scatter diagram. We look for the direction and strength of the linear relationship.

RESOURCES

Learning Resources

The Essential Textbook Resources website: At the Keller website, you'll find materials previously on the student CD, including: Interactive concept simulation exercises from ***Seeing Statistics***, the **Data Analysis Plus** add-in, 715 data sets, optional topics, and 35 appendixes (for more information, please visit www.cengage.com/bstatistics/keller).

Student Solutions Manual (ISBN: 1111531889): Students can check their understanding with this manual, which includes worked solutions of even-numbered exercises from the text.

***Seeing Statistics* by Gary McClelland**: This online product is flexible and addresses different learning styles. It presents the visual nature of statistical concepts using more than 150 Java applets to create an intuitive learning environment. Also included are relevant links to examples, exercises, definitions, and search and navigation capabilities.

Teaching Resources

To access both student and faculty resources, please visit **www.cengage.com/login.**

ACKNOWLEDGMENTS

Although there is only one name on the cover of this book, the number of people who made contributions is large. I would like to acknowledge the work of all of them, with particular emphasis on the following: Paul Baum, California State University, Northridge, and John Lawrence, California State University, Fullerton, reviewed the page proofs. Their job was to find errors in presentation, arithmetic, and composition.The following individuals played important roles in the production of this book Senior Acquisitions Editor: Charles McCormick, Jr., Developmental Editor: Elizabeth Lowry, Content Project Manager: Jacquelyn K Featherly and Project Manager Lynn Lustberg (For all remaining errors, place the blame where it belongs–on me.) Their advice and suggestions made my task considerably easier.

Fernando Rodriguez produced the test bank stored on the Instructor's Suite CD-ROM.

Trent Tucker, Wilfrid Laurier University, and Zvi Goldstein, California State University, Fullerton, each produced a set of PowerPoint slides.

The author extends thanks also to the survey participants and reviewers of the previous editions: Paul Baum, California State University, Northridge; Nagraj Balakrishnan, Clemson University; Howard Clayton, Auburn University; Philip Cross, Georgetown University; Barry Cuffe, Wingate University; Ernest Demba, Washington University–St. Louis; Neal Duffy, State University of New York, Plattsburgh; John Dutton, North Carolina State University; Erick Elder, University of Arkansas; Mohammed El-Saidi, Ferris State University; Grace Esimai, University of Texas at Arlington; Abe Feinberg, California State University, Northridge; Samuel Graves, Boston College; Robert Gould, UCLA; John Hebert, Virginia Tech; James Hightower, California State University, Fullerton; Bo Honore, Princeton University; Onisforos Iordanou, Hunter College; Gordon Johnson, California State University,

Northridge; Hilke Kayser, Hamilton College; Kenneth Klassen, California State University, Northridge; Roger Kleckner, Bowling Green State University–Firelands; Harry Kypraios, Rollins College; John Lawrence, California State University, Fullerton; Dennis Lin, Pennsylvania State University; Neal Long, Stetson University; George Marcoulides, California State University, Fullerton; Paul Mason, University of North Florida; Walter Mayer, University of Mississippi; John McDonald, Flinders University; Richard McGowan, Boston College; Richard McGrath, Bowling Green State University; Amy Miko, St. Francis College; Janis Miller, Clemson University; Glenn Milligan, Ohio State University; James Moran, Oregon State University; Patricia Mullins, University of Wisconsin; Kevin Murphy, Oakland University; Pin Ng, University of Illinois; Des Nicholls, Australian National University; Andrew Paizis, Queens College; David Pentico, Duquesne University; Ira Perelle, Mercy College; Nelson Perera, University of Wollongong; Amy Puelz, Southern Methodist University; Lawrence Ries, University of Missouri; Colleen Quinn, Seneca College; Tony Quon, University of Ottawa; Madhu Rao, Bowling Green State University; Phil Roth, Clemson University; Farhad Saboori, Albright College; Don St. Jean, George Brown College; Hedayeh Samavati, Indiana–Purdue University; Sandy Shroeder, Ohio Northern University; Jineshwar Singh, George Brown College; Natalia Smirnova, Queens College; Eric Sowey, University of New South Wales; Cyrus Stanier, Virginia Tech; Stan Stephenson, Southwest Texas State University; Arnold Stromberg, University of Kentucky; Steve Thorpe, University of Northern Iowa; Sheldon Vernon, Houston Baptist University; and W. F. Younkin, University of Miami.

Statistics

FOR MANAGEMENT AND ECONOMICS ABBREVIATED

9e

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WHAT IS STATISTICS?

INTRODUCTION

Statistics is a way to get information from data. That's it! Most of this textbook is devoted to describing how, when, and why managers and statistics practitioners* conduct statistical procedures. You may ask, "If that's all there is to statistics, why is this book (and most other statistics books) so large?" The answer is that students of applied statistics will be exposed to different kinds of information and data. We demonstrate some of these with a case and two examples that are featured later in this book.

The first may be of particular interest to you.

*The term *statistician* is used to describe so many different kinds of occupations that it has ceased to have any meaning. It is used, for example, to describe a person who calculates baseball statistics as well as an individual educated in statistical principles. We will describe the former as a *statistics practitioner* and the

(*continued*)

EXAMPLE 3.3 Business Statistics Marks (See Chapter 3)

A student enrolled in a business program is attending his first class of the required statistics course. The student is somewhat apprehensive because he believes the myth that the course is difficult. To alleviate his anxiety, the student asks the professor about last year's marks. Because this professor is friendly and helpful, like all other statistics professors, he obliges the student and provides a list of the final marks, which are composed of term work plus the final exam. What information can the student obtain from the list?

This is a typical statistics problem. The student has the data (marks) and needs to apply statistical techniques to get the information he requires. This is a function of **descriptive statistics**.

Descriptive Statistics

Descriptive statistics deals with methods of organizing, summarizing, and presenting data in a convenient and informative way. One form of descriptive statistics uses graphical techniques that allow statistics practitioners to present data in ways that make it easy for the reader to extract useful information. In Chapters 2 and 3 we will present a variety of graphical methods.

Another form of descriptive statistics uses numerical techniques to summarize data. One such method that you have already used frequently calculates the average or mean. In the same way that you calculate the average age of the employees of a company, we can compute the mean mark of last year's statistics course. Chapter 4 introduces several numerical statistical measures that describe different features of the data.

The actual technique we use depends on what specific information we would like to extract. In this example, we can see at least three important pieces of information. The first is the "typical" mark. We call this a *measure of central location.* The average is one such measure. In Chapter 4, we will introduce another useful measure of central location, the median. Suppose the student was told that the average mark last year was 67. Is this enough information to reduce his anxiety? The student would likely respond "No" because he would like to know whether most of the marks were close to 67 or were scattered far below and above the average. He needs a *measure of variability.* The simplest such measure is the *range*, which is calculated by subtracting the smallest number from the largest. Suppose the largest mark is 96 and the smallest is 24. Unfortunately, this provides little information since it is based on only two marks. We need other measures—these will be introduced in Chapter 4. Moreover, the student must determine more about the marks. In particular, he needs to know how the marks are distributed between 24 and 96. The best way to do this is to use a graphical technique, the histogram, which will be introduced in Chapter 3.

---

latter as a *statistician*. A statistics practitioner is a person who uses statistical techniques properly. Examples of statistics practitioners include the following:

1. a financial analyst who develops stock portfolios based on historical rates of return;
2. an economist who uses statistical models to help explain and predict variables such as inflation rate, unemployment rate, and changes in the gross domestic product; and
3. a market researcher who surveys consumers and converts the responses into useful information.

Our goal in this book is to convert you into one such capable individual.

The term *statistician* refers to an individual who works with the mathematics of statistics. His or her work involves research that develops techniques and concepts that in the future may help the statistics practitioner. Statisticians are also statistics practitioners, frequently conducting empirical research and consulting. If you're taking a statistics course, your instructor is probably a statistician.

Case 12.1 Pepsi's Exclusivity Agreement with a University (see Chapter 12)

In the last few years, colleges and universities have signed exclusivity agreements with a variety of private companies. These agreements bind the university to sell these companies' products exclusively on the campus. Many of the agreements involve food and beverage firms.

A large university with a total enrollment of about 50,000 students has offered Pepsi-Cola an exclusivity agreement that would give Pepsi exclusive rights to sell its products at all university facilities for the next year with an option for future years. In return, the university would receive 35% of the on-campus revenues and an additional lump sum of $200,000 per year. Pepsi has been given 2 weeks to respond.

The management at Pepsi quickly reviews what it knows. The market for soft drinks is measured in terms of 12-ounce cans. Pepsi currently sells an average of 22,000 cans per week over the 40 weeks of the year that the university operates. The cans sell for an average of one dollar each. The costs, including labor, total 30 cents per can. Pepsi is unsure of its market share but suspects it is considerably less than 50%. A quick analysis reveals that if its current market share were 25%, then, with an exclusivity agreement, Pepsi would sell 88,000 (22,000 is 25% of 88,000) cans per week or 3,520,000 cans per year. The gross revenue would be computed as follows[†]:

Gross revenue = 3,520,000 × $1.00/can = $3,520,000

This figure must be multiplied by 65% because the university would rake in 35% of the gross. Thus,

Gross revenue after deducting 35% university take
= 65% × $3,520,000 = $2,288,000

The total cost of 30 cents per can (or $1,056,000) and the annual payment to the university of $200,000 are subtracted to obtain the net profit:

Net profit = $2,288,000 − $1,056,000 − $200,000 = $1,032,000

Pepsi's current annual profit is

40 weeks × 22,000 cans/week × $.70 = $616,000

If the current market share is 25%, the potential gain from the agreement is

$1,032,000 – $616,000 = $416,000

The only problem with this analysis is that Pepsi does not know how many soft drinks are sold weekly at the university. Coke is not likely to supply Pepsi with information about its sales, which together with Pepsi's line of products constitute virtually the entire market.

Pepsi assigned a recent university graduate to survey the university's students to supply the missing information. Accordingly, she organizes a survey that asks 500 students to keep track of the number of soft drinks they purchase in the next 7 days. The responses are stored in a file C12-01 available to be downloaded. See Appendix 1 for instructions.

Inferential Statistics

The information we would like to acquire in Case 12.1 is an estimate of annual profits from the exclusivity agreement. The data are the numbers of cans of soft drinks consumed in 7 days by the 500 students in the sample. We can use descriptive techniques to

[†]We have created an Excel spreadsheet that does the calculations for this case. See Appendix 1 for instructions on how to download this spreadsheet from Keller's website plus hundreds of datasets and much more.

learn more about the data. In this case, however, we are not so much interested in what the 500 students are reporting as in knowing the mean number of soft drinks consumed by all 50,000 students on campus. To accomplish this goal we need another branch of statistics: **inferential statistics**.

Inferential statistics is a body of methods used to draw conclusions or inferences about characteristics of populations based on sample data. The population in question in this case is the university's 50,000 students. The characteristic of interest is the soft drink consumption of this population. The cost of interviewing each student in the population would be prohibitive and extremely time consuming. Statistical techniques make such endeavors unnecessary. Instead, we can sample a much smaller number of students (the sample size is 500) and infer from the data the number of soft drinks consumed by all 50,000 students. We can then estimate annual profits for Pepsi.

EXAMPLE 12.5

Exit Polls (see Chapter 12)

When an election for political office takes place, the television networks cancel regular programming to provide election coverage. After the ballots are counted, the results are reported. However, for important offices such as president or senator in large states, the networks actively compete to see which one will be the first to predict a winner. This is done through **exit polls** in which a random sample of voters who exit the polling booth are asked for whom they voted. From the data, the sample proportion of voters supporting the candidates is computed. A statistical technique is applied to determine whether there is enough evidence to infer that the leading candidate will garner enough votes to win. Suppose that the exit poll results from the state of Florida during the year 2000 elections were recorded. Although several candidates were running for president, the exit pollsters recorded only the votes of the two candidates who had any chance of winning: Republican George W. Bush and Democrat Albert Gore. The results (765 people who voted for either Bush or Gore) were stored in file Xm12-05. The network analysts would like to know whether they can conclude that George W. Bush will win the state of Florida.

Example 12.5 describes a common application of statistical inference. The population the television networks wanted to make inferences about is the approximately 5 million Floridians who voted for Bush or Gore for president. The sample consisted of the 765 people randomly selected by the polling company who voted for either of the two main candidates. The characteristic of the population that we would like to know is the proportion of the Florida total electorate that voted for Bush. Specifically, we would like to know whether more than 50% of the electorate voted for Bush (counting only those who voted for either the Republican or Democratic candidate). It must be made clear that we cannot predict the outcome with 100% certainty because we will not ask all 5 million actual voters for whom they voted. This is a fact that statistics practitioners and even students of statistics must understand. A sample that is only a small fraction of the size of the population can lead to correct inferences only a certain percentage of the time. You will find that statistics practitioners can control that fraction and usually set it between 90% and 99%.

Incidentally, on the night of the United States election in November 2000, the networks goofed badly. Using exit polls as well as the results of previous elections, all four networks concluded at about 8 P.M. that Al Gore would win Florida. Shortly after 10 P.M., with a large percentage of the actual vote having been counted, the networks reversed course and declared that George W. Bush would win the state. By 2 A.M., another verdict was declared: The result was too close to call. In the future, this experience will likely be used by statistics instructors when teaching how *not* to use statistics.

Notice that, contrary to what you probably believed, data are not necessarily numbers. The marks in Example 3.3 and the number of soft drinks consumed in a week in Case 12.1, of course, are numbers; however, the votes in Example 12.5 are not. In Chapter 2, we will discuss the different types of data you will encounter in statistical applications and how to deal with them.

1.1 Key Statistical Concepts

Statistical inference problems involve three key concepts: the population, the sample, and the statistical inference. We now discuss each of these concepts in more detail.

Population

A **population** is the group of all items of interest to a statistics practitioner. It is frequently very large and may, in fact, be infinitely large. In the language of statistics, *population* does not necessarily refer to a group of people. It may, for example, refer to the population of ball bearings produced at a large plant. In Case 12.1, the population of interest consists of the 50,000 students on campus. In Example 12.5, the population consists of the Floridians who voted for Bush or Gore.

A descriptive measure of a population is called a **parameter**. The parameter of interest in Case 12.1 is the mean number of soft drinks consumed by all the students at the university. The parameter in Example 12.5 is the proportion of the 5 million Florida voters who voted for Bush. In most applications of inferential statistics the parameter represents the information we need.

Sample

A **sample** is a set of data drawn from the studied population. A descriptive measure of a sample is called a **statistic**. We use statistics to make inferences about parameters. In Case 12.1, the statistic we would compute is the mean number of soft drinks consumed in the last week by the 500 students in the sample. We would then use the sample mean to infer the value of the population mean, which is the parameter of interest in this problem. In Example 12.5, we compute the proportion of the sample of 765 Floridians who voted for Bush. The sample statistic is then used to make inferences about the population of all 5 million votes—that is, we predict the election results even before the actual count.

Statistical Inference

Statistical inference is the process of making an estimate, prediction, or decision about a population based on sample data. Because populations are almost always very large, investigating each member of the population would be impractical and expensive. It is far easier and cheaper to take a sample from the population of interest and draw conclusions or make estimates about the population on the basis of information provided by the sample. However, such conclusions and estimates are not always going to be correct. For this reason, we build into the statistical inference a measure of reliability. There are two such measures: the **confidence level** and the **significance level**. The *confidence level* is the proportion of times that an estimating procedure will be correct. For example, in Case 12.1, we will produce an estimate of the average number of soft drinks to be consumed by all 50,000 students that has a confidence level of 95%. In other words, estimates based on this form of statistical inference will be correct 95% of the time. When the purpose of the statistical inference is to draw a conclusion about a population, the *significance level* measures how frequently the conclusion will be wrong. For example, suppose that, as a result of the analysis in Example 12.5, we conclude that more than 50% of the electorate will vote for George W. Bush, and thus he will win the state of Florida. A 5% significance level means that samples that lead us to conclude that Bush wins the election, will be wrong 5% of the time.

1.2 Statistical Applications in Business

An important function of statistics courses in business and economics programs is to demonstrate that statistical analysis plays an important role in virtually all aspects of business and economics. We intend to do so through examples, exercises, and cases. However, we assume that most students taking their first statistics course have not taken courses in most of the other subjects in management programs. To understand fully how statistics is used in these and other subjects, it is necessary to know something about them. To provide sufficient background to understand the statistical application we introduce applications in accounting, economics, finance, human resources management, marketing, and operations management. We will provide readers with some background to these applications by describing their functions in two ways.

Application Sections and Subsections

We feature five sections that describe statistical applications in the functional areas of business. For example, in Section 7.3 we show an application in finance that describes a financial analyst's use of probability and statistics to construct portfolios that decrease risk.

One section and one subsection demonstrate the uses of probability and statistics in specific industries. Section 4.5 introduces an interesting application of statistics in professional baseball. A subsection in Section 6.4 presents an application in medical testing (useful in the medical insurance industry).

Application Boxes

For other topics that require less detailed description, we provide application boxes with a relatively brief description of the background followed by examples or exercises. These boxes are scattered throughout the book. For example, in Chapter 3 we discuss a job a marketing manager may need to undertake to determine the appropriate price for a product. To understand the context, we need to provide a description of marketing management. The statistical application will follow.

1.3 Large Real Data Sets

The authors believe that you learn statistics by doing statistics. For their lives after college and university, we expect our students to have access to large amounts of real data that must be summarized to acquire the information needed to make decisions. To provide practice in this vital skill we have created six large real datasets, available to be downloaded from Keller's website. Their sources are the General Social Survey (GSS) and the American National Election Survey (ANES).

General Social Survey

Since 1972, the General Social Survey has been tracking American attitudes on a wide variety of topics. Except for the United States census, the GSS is the most frequently used sources of information about American society. The surveys now conducted every second year measure hundreds of variables and thousands of observations. We have included the results of the last four surveys (years 2002, 2004, 2006, and 2008) stored as GSS2002, GSS2004, GSS2006, and GSS2008, respectively. The survey sizes are 2,765, 2,812, 4,510, and 2,023, respectively. We have reduced the number of variables to about 60 and have deleted the responses that are known as *missing data* ("Don't know," "Refused," etc.).

We have included some demographic variables such as, age, gender, race, income, and education. Others measure political views, support for various government activities, and work. The full lists of variables for each year are stored in Appendixes GSS2002, GSS2004, GSS2006, and GSSS2008 that can be downloaded from Keller's website.

We have scattered throughout this book examples and exercises drawn from these data sets.

American National Election Survey

The goal of the American National Election Survey is to provide data about why Americans vote as they do. The surveys are conducted in presidential election years. We have included data from the 2004 and 2008 surveys. Like the General Social Survey, the ANES includes demographic variables. It also deals with interest in the presidential election as well as variables describing political beliefs and affiliations. Online Appendixes ANES2004 and ANES2008 contain the names and definitions of the variables.

The 2008 surveys overly sampled African American and Hispanic voters. We have "adjusted" these data by randomly deleting responses from these two racial groups.

As is the case with the General Social Surveys, we have removed missing data.

1.4 Statistics and the Computer

In virtually all applications of statistics, the statistics practitioner must deal with large amounts of data. For example, Case 12.1 (Pepsi-Cola) involves 500 observations. To estimate annual profits, the statistics practitioner would have to perform computations on the data. Although the calculations do not require any great mathematical skill, the sheer amount of arithmetic makes this aspect of the statistical method time-consuming and tedious.

Fortunately, numerous commercially prepared computer programs are available to perform the arithmetic. We have chosen to use Microsoft Excel, which is a spreadsheet program, and Minitab, which is a statistical software package. (We use the latest versions of both software: Office 2010 and Minitab 16.) We chose Excel because we believe that it is and will continue to be the most popular spreadsheet package. One of its drawbacks is that it does not offer a complete set of the statistical techniques we introduce in this book. Consequently, we created add-ins that can be loaded onto your computer to enable you to use Excel for all statistical procedures introduced in this book. The add-ins can be downloaded and, when installed, will appear as *Data Analysis Plus*© on Excel's Add-Ins menu. Also available are introductions to Excel and Minitab, and detailed instructions for both software packages.

Appendix 1 describes the material that can be downloaded and provides instructions on how to acquire the various components.

A large proportion of the examples, exercises, and cases feature large data sets. These are denoted with the file name on an orange background. We demonstrate the solution to the statistical examples in three ways: manually, by employing Excel, and by using Minitab. Moreover, we will provide detailed instructions for all techniques.

The files contain the data needed to produce the solution. However, in many real applications of statistics, additional data are collected. For instance, in Example 12.5, the pollster often records the voter's gender and asks for other information including race, religion, education, and income. Many other data sets are similarly constructed. In later chapters, we will return to these files and require other statistical techniques to extract the needed information. (Files that contain additional data are denoted by an asterisk on the file name.)

The approach we prefer to take is to minimize the time spent on manual computations and to focus instead on selecting the appropriate method for dealing with a problem and on interpreting the output after the computer has performed the necessary computations. In this way, we hope to demonstrate that statistics can be as interesting and as practical as any other subject in your curriculum.

Applets and Spreadsheets

Books written for statistics courses taken by mathematics or statistics majors are considerably different from this one. It is not surprising that such courses feature mathematical proofs of theorems and derivations of most procedures. When the material is covered in this way, the underlying concepts that support statistical inference are exposed and relatively easy to see. However, this book was created for an applied course in business and economics statistics. Consequently, we do not address directly the mathematical principles of statistics. However, as we pointed out previously, one of the most important functions of statistics practitioners is to properly interpret statistical results, whether produced manually or by computer. And, to correctly interpret statistics, students require an understanding of the principles of statistics.

To help students understand the basic foundation, we offer two approaches. First, we will teach readers how to create Excel spreadsheets that allow for *what-if* analyses. By changing some of the input value, students can see for themselves how statistics works. (The term is derived from *what* happens to the statistics *if* I change this value?) These spreadsheets can also be used to calculate many of the same statistics that we introduce later in this book. Second, we offer *applets*, which are computer programs that perform similar what-if analyses or simulations. The applets and the spreadsheet applications appear in several chapters and explained in greater detail.

Chapter Summary

IMPORTANT TERMS

Descriptive statistics 2
Inferential statistics 4
Exit polls 4
Population 5
Parameter 5
Sample 5
Statistic 5
Statistical inference 5
Confidence level 5
Significance level 5

CHAPTER EXERCISES

1.1 In your own words, define and give an example of each of the following statistical terms.

a. population
b. sample
c. parameter
d. statistic
e. statistical inference

1.2 Briefly describe the difference between descriptive statistics and inferential statistics.

1.3 A politician who is running for the office of mayor of a city with 25,000 registered voters commissions a survey. In the survey, 48% of the 200 registered voters interviewed say they plan to vote for her.

a. What is the population of interest?
b. What is the sample?
c. Is the value 48% a parameter or a statistic? Explain.

1.4 A manufacturer of computer chips claims that less than 10% of its products are defective. When 1,000 chips were drawn from a large production, 7.5% were found to be defective.

a. What is the population of interest?
b. What is the sample?
c. What is the parameter?
d. What is the statistic?
e. Does the value 10% refer to the parameter or to the statistic?
f. Is the value 7.5% a parameter or a statistic?
g. Explain briefly how the statistic can be used to make inferences about the parameter to test the claim.

1.5 Suppose you believe that, in general, graduates who have majored in *your* subject are offered higher salaries upon graduating than are graduates of other programs. Describe a statistical experiment that could help test your belief.

1.6 You are shown a coin that its owner says is fair in the sense that it will produce the same number of heads and tails when flipped a very large number of times.

a. Describe an experiment to test this claim.
b. What is the population in your experiment?
c. What is the sample?
d. What is the parameter?
e. What is the statistic?
f. Describe briefly how statistical inference can be used to test the claim.

1.7 Suppose that in Exercise 1.6 you decide to flip the coin 100 times.

a. What conclusion would you be likely to draw if you observed 95 heads?
b. What conclusion would you be likely to draw if you observed 55 heads?
c. Do you believe that, if you flip a perfectly fair coin 100 times, you will always observe exactly 50 heads? If you answered "no," then what numbers do you think are possible? If you answered "yes," how many heads would you observe if you flipped the coin twice? Try flipping a coin twice and repeating this experiment 10 times and report the results.

1.8 Xm01-08 The owner of a large fleet of taxis is trying to estimate his costs for next year's operations. One major cost is fuel purchases. To estimate fuel purchases, the owner needs to know the total distance his taxis will travel next year, the cost of a gallon of fuel, and the fuel mileage of his taxis. The owner has been provided with the first two figures (distance estimate and cost of a gallon of fuel). However, because of the high cost of gasoline, the owner has recently converted his taxis to operate on propane. He has measured and recorded the propane mileage (in miles per gallon) for 50 taxis.

a. What is the population of interest?
b. What is the parameter the owner needs?
c. What is the sample?
d. What is the statistic?
e. Describe briefly how the statistic will produce the kind of information the owner wants.

APPENDIX 1 / Instructions for Keller's website

The Keller website that accompanies this book contains the following features:

Data Analysis Plus 9.0 in VBA, which works with new and earlier versions of Excel (Office 1997, 2000, XP, 2003, 2007, and 2010 Office for Mac 2004)

A help file for Data Analysis Plus 9.0 in VBA

Data files in the following formats: ASCII, Excel, JMP, Minitab, SAS, and SPSS

Excel workbooks

Seeing Statistics (Java applets that teach a number of important statistical concepts)

Appendices (40 additional topics that are not covered in the book)

Formula card listing every formula in the book

Keller website Instructions

"Data Analysis Plus 9.0 in VBA" can be found on the Keller website. It will be installed into the XLSTART folder of the most recent version of Excel on your computer. If properly installed Data Analysis Plus will be a menu item in Excel. The help file for Data Analysis Plus will be stored directly in your computer's My Documents folder. It will appear when you click the Help button or when you make a mistake when using Data Analysis Plus.

The Data Sets will also be installed from a link within the Keller website.

The Excel workbooks, Seeing Statistics Applets, and Appendixes will be accessed from the Keller website. Alternatively, you can store the Excel workbooks and Appendixes to your hard drive.

The Keller website is available using the student access code accompanying all new books. For more information on how to access the Keller website, please visit www.cengage.com/bstatistics/keller.

For technical support, please visit www.cengage.com/support for contact options. Refer to Statistics for Management and Economics, Ninth edition, by Gerald Keller (ISBN 0-538-47749-0).

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2

GRAPHICAL DESCRIPTIVE TECHNIQUES I

Do Male and Female American Voters Differ in Their Party Affiliation?

DATA
ANES2008*

In Chapter 1, we introduced the American National Election Survey (ANES), which is conducted every 4 years with the objective of developing information about how Americans vote. One question in the 2008 survey was "Do you think of yourself as Democrat, Republican, Independent, or what?"

Responses were

1. Democrat
2. Republican
3. Independent

© AP Photo/David Smith

On page 37 we will provide our answer.

4. Other party
5. No preference

Respondents were also identified by gender: 1 = male, and 2 = female. The responses are stored in file ANES2008* on our Keller's website. The asterisk indicates that there are variables that are not needed for this example but which will be used later in this book. For Excel users, GENDER AND PARTY are in columns B and BD, respectively. For Minitab users, GENDER AND PARTY are in columns 2 and 56, respectively. Some of the data are listed here.

ID	GENDER	PARTY
1	1	3
2	2	1
3	2	2
.	.	.
.	.	.
1795	1	1
1796	1	2
1797	1	1

Determine whether American female and male voters differ in their political affiliations.

Introduction

In Chapter 1, we pointed out that statistics is divided into two basic areas: descriptive statistics and inferential statistics. The purpose of this chapter, together with the next, is to present the principal methods that fall under the heading of descriptive statistics. In this chapter, we introduce graphical and tabular statistical methods that allow managers to summarize data visually to produce useful information that is often used in decision making. Another class of descriptive techniques, numerical methods, is introduced in Chapter 4.

Managers frequently have access to large masses of potentially useful data. But before the data can be used to support a decision, they must be organized and summarized. Consider, for example, the problems faced by managers who have access to the databases created by the use of debit cards. The database consists of the personal information supplied by the customer when he or she applied for the debit card. This information includes age, gender, residence, and the cardholder's income. In addition, each time the card is used the database grows to include a history of the timing, price, and brand of each product purchased. Using the appropriate statistical technique, managers can determine which segments of the market are buying their company's brands. Specialized marketing campaigns, including telemarketing, can be developed. Both descriptive and inferential statistics would likely be employed in the analysis.

Descriptive statistics involves arranging, summarizing, and presenting a set of data in such a way that useful information is produced. Its methods make use of graphical techniques and numerical descriptive measures (such as averages) to summarize and present the data, allowing managers to make decisions based on the information generated. Although descriptive statistical methods are quite straightforward, their importance should not be underestimated. Most management, business, and economics students will encounter numerous opportunities to make valuable use of graphical and numerical descriptive techniques when preparing reports and presentations in the workplace. According to a Wharton Business School study, top managers reach a consensus 25% more quickly when responding to a presentation in which graphics are used.

In Chapter 1, we introduced the distinction between a population and a sample. Recall that a **population** is the entire set of observations under study, whereas a **sample** is a subset of a population. The descriptive methods presented in this chapter and in Chapters 3 and 4 apply to both a set of data constituting a population and a set of data constituting a sample.

In both the preface and Chapter 1, we pointed out that a critical part of your education as statistics practitioners includes an understanding of not only *how* to draw graphs and calculate statistics (manually or by computer) but also *when* to use each technique that we cover. The two most important factors that determine the appropriate method to use are (1) the type of data and (2) the information that is needed. Both are discussed next.

2.1 Types of Data and Information

The objective of statistics is to extract information from data. There are different types of data and information. To help explain this important principle, we need to define some terms.

A **variable** is some characteristic of a population or sample. For example, the mark on a statistics exam is a characteristic of statistics exams that is certainly of interest to readers of this book. Not all students achieve the same mark. The marks will vary from student to student, thus the name *variable*. The price of a stock is another variable. The prices of most stocks vary daily. We usually represent the name of the variable using uppercase letters such as X, Y, and Z.

The **values** of the variable are the possible observations of the variable. The values of statistics exam marks are the integers between 0 and 100 (assuming the exam is marked out of 100). The values of a stock price are real numbers that are usually measured in dollars and cents (sometimes in fractions of a cent). The values range from 0 to hundreds of dollars.

Data* are the observed values of a variable. For example, suppose that we observe the following midterm test marks of 10 students:

67 74 71 83 93 55 48 82 68 62

These are the data from which we will extract the information we seek. Incidentally, *data* is plural for **datum**. The mark of one student is a datum.

When most people think of data, they think of sets of numbers. However, there are three types of data: interval, nominal, and ordinal.†

*Unfortunately, the term *data*, like the term *statistician*, has taken on several different meanings. For example, dictionaries define data as facts, information, or statistics. In the language of computers, data may refer to any piece of information such as this textbook or an essay you have written. Such definitions make it difficult for us to present *statistics* as a method of converting *data* into *information*. In this book, we carefully distinguish among the three terms.

†There are actually four types of data, the fourth being *ratio* data. However, for statistical purposes there is no difference between ratio and interval data. Consequently, we combine the two types.

Interval data are real numbers, such as heights, weights, incomes, and distances. We also refer to this type of data as **quantitative** or **numerical**.

The values of **nominal** data are categories. For example, responses to questions about marital status produce nominal data. The values of this variable are single, married, divorced, and widowed. Notice that the values are not numbers but instead are words that describe the categories. We often record nominal data by arbitrarily assigning a number to each category. For example, we could record marital status using the following codes:

single = 1, married = 2, divorced = 3, widowed = 4

However, any other numbering system is valid provided that each category has a different number assigned to it. Here is another coding system that is just as valid as the previous one.

Single = 7, married = 4, divorced = 13, widowed = 1

Nominal data are also called **qualitative** or **categorical**.

The third type of data is ordinal. **Ordinal** data appear to be nominal, but the difference is that the order of their values has meaning. For example, at the completion of most college and university courses, students are asked to evaluate the course. The variables are the ratings of various aspects of the course, including the professor. Suppose that in a particular college the values are

poor, fair, good, very good, and excellent

The difference between nominal and ordinal types of data is that the order of the values of the latter indicate a higher rating. Consequently, when assigning codes to the values, we should maintain the order of the values. For example, we can record the students' evaluations as

Poor = 1, Fair = 2, Good = 3, Very good = 4, Excellent = 5

Because the only constraint that we impose on our choice of codes is that the order must be maintained, we can use any set of codes that are in order. For example, we can also assign the following codes:

Poor = 6, Fair = 18, Good = 23, Very good = 45, Excellent = 88

As we discuss in Chapter 19, which introduces statistical inference techniques for ordinal data, the use of any code that preserves the order of the data will produce exactly the same result. Thus, it's not the magnitude of the values that is important, it's their order.

Students often have difficulty distinguishing between ordinal and interval data. The critical difference between them is that the intervals or differences between values of interval data are consistent and meaningful (which is why this type of data is called *interval*). For example, the difference between marks of 85 and 80 is the same five-mark difference that exists between 75 and 70—that is, we can calculate the difference and interpret the results.

Because the codes representing ordinal data are arbitrarily assigned except for the order, we cannot calculate and interpret differences. For example, using a 1-2-3-4-5 coding system to represent poor, fair, good, very good, and excellent, we note that the difference between excellent and very good is identical to the difference between good and fair. With a 6-18-23-45-88 coding, the difference between excellent and very good is 43, and the difference between good and fair is 5. Because both coding systems are valid, we cannot use either system to compute and interpret differences.

Here is another example. Suppose that you are given the following list of the most active stocks traded on the NASDAQ in descending order of magnitude:

Order	Most Active Stocks
1	Microsoft
2	Cisco Systems
3	Dell Computer
4	Sun Microsystems
5	JDS Uniphase

Does this information allow you to conclude that the difference between the number of stocks traded in Microsoft and Cisco Systems is the same as the difference in the number of stocks traded between Dell Computer and Sun Microsystems? The answer is "no" because we have information only about the order of the numbers of trades, which are ordinal, and not the numbers of trades themselves, which are interval. In other words, the difference between 1 and 2 is not necessarily the same as the difference between 3 and 4.

Calculations for Types of Data

Interval Data

All calculations are permitted on interval data. We often describe a set of interval data by calculating the average. For example, the average of the 10 marks listed on page 13 is 70.3. As you will discover, there are several other important statistics that we will introduce.

Nominal Data

Because the codes of nominal data are completely arbitrary, we cannot perform any calculations on these codes. To understand why, consider a survey that asks people to report their marital status. Suppose that the first 10 people surveyed gave the following responses:

single, married, married, married, widowed, single, married, married, single, divorced

Using the codes

Single = 1, married = 2, divorced = 3, widowed = 4

we would record these responses as

1 2 2 2 4 1 2 2 1 3

The average of these numerical codes is 2.0. Does this mean that the average person is married? Now suppose four more persons were interviewed, of whom three are widowed and one is divorced. The data are given here:

1 2 2 2 4 1 2 2 1 3 4 4 4 3

The average of these 14 codes is 2.5. Does this mean that the average person is married—but halfway to getting divorced? The answer to both questions is an emphatic "no." This example illustrates a fundamental truth about nominal data: Calculations based on the codes used to store this type of data are meaningless. All that we are permitted to do with nominal data is count or compute the percentages of the occurrences of each category. Thus, we would describe the 14 observations by counting the number of each marital status category and reporting the frequency as shown in the following table.

Category	Code	Frequency
Single	1	3
Married	2	5
Divorced	3	2
Widowed	4	4

The remainder of this chapter deals with nominal data only. In Chapter 3, we introduce graphical techniques that are used to describe interval data.

Ordinal Data

The most important aspect of ordinal data is the order of the values. As a result, the only permissible calculations are those involving a ranking process. For example, we can place all the data in order and select the code that lies in the middle. As we discuss in Chapter 4, this descriptive measurement is called the *median*.

Hierarchy of Data

The data types can be placed in order of the permissible calculations. At the top of the list, we place the interval data type because virtually *all* computations are allowed. The nominal data type is at the bottom because *no* calculations other than determining frequencies are permitted. (We are permitted to perform calculations using the frequencies of codes, but this differs from performing calculations on the codes themselves.) In between interval and nominal data lies the ordinal data type. Permissible calculations are ones that rank the data.

Higher-level data types may be treated as lower-level ones. For example, in universities and colleges, we convert the marks in a course, which are interval, to letter grades, which are ordinal. Some graduate courses feature only a pass or fail designation. In this case, the interval data are converted to nominal. It is important to point out that when we convert higher-level data as lower-level we lose information. For example, a mark of 83 on an accounting course exam gives far more information about the performance of that student than does a letter grade of A, which might be the letter grade for marks between 80 and 90. As a result, we do not convert data unless it is necessary to do so. We will discuss this later.

It is also important to note that we cannot treat lower-level data types as higher-level types.

The definitions and hierarchy are summarized in the following box.

Types of Data

Interval

Values are real numbers.

All calculations are valid.

Data may be treated as ordinal or nominal.

Ordinal

Values must represent the ranked order of the data.

Calculations based on an ordering process are valid.

Data may be treated as nominal but not as interval.

Nominal

Values are the arbitrary numbers that represent categories.

Only calculations based on the frequencies or percentages of occurrence are valid.

Data may not be treated as ordinal or interval.

Interval, Ordinal, and Nominal Variables

The variables whose observations constitute our data will be given the same name as the type of data. Thus, for example, interval data are the observations of an interval variable.

Problem Objectives and Information

In presenting the different types of data, we introduced a critical factor in deciding which statistical procedure to use. A second factor is the type of information we need to produce from our data. We discuss the different types of information in greater detail in Section 11.4 when we introduce *problem objectives*. However, in this part of the book (Chapters 2–5), we will use statistical techniques to describe a set of data, compare two or more sets of data, and describe the relationship between two variables. In Section 2.2, we introduce graphical and tabular techniques employed to describe a set of nominal data. Section 2.3 shows how to describe the relationship between two nominal variables and compare two or more sets of nominal data.

Exercises

2.1 Provide two examples each of nominal, ordinal, and interval data.

2.2 For each of the following examples of data, determine the type.
a. The number of miles joggers run per week
b. The starting salaries of graduates of MBA programs
c. The months in which a firm's employees choose to take their vacations
d. The final letter grades received by students in a statistics course

2.3 For each of the following examples of data, determine the type.
a. The weekly closing price of the stock of Amazon.com
b. The month of highest vacancy rate at a La Quinta motel
c. The size of soft drink (small, medium, or large) ordered by a sample of McDonald's customers
d. The number of Toyotas imported monthly by the United States over the last 5 years
e. The marks achieved by the students in a statistics course final exam marked out of 100

2.4 The placement office at a university regularly surveys the graduates 1 year after graduation and asks for the following information. For each, determine the type of data.
a. What is your occupation?
b. What is your income?
c. What degree did you obtain?
d. What is the amount of your student loan?
e. How would you rate the quality of instruction? (excellent, very good, good, fair, poor)

2.5 Residents of condominiums were recently surveyed and asked a series of questions. Identify the type of data for each question.
a. What is your age?
b. On what floor is your condominium?

c. Do you own or rent?
d. How large is your condominium (in square feet)?
e. Does your condominium have a pool?

2.6 A sample of shoppers at a mall was asked the following questions. Identify the type of data each question would produce.
a. What is your age?
b. How much did you spend?
c. What is your marital status?
d. Rate the availability of parking: excellent, good, fair, or poor
e. How many stores did you enter?

2.7 Information about a magazine's readers is of interest to both the publisher and the magazine's advertisers. A survey of readers asked respondents to complete the following:
a. Age
b. Gender
c. Marital status
d. Number of magazine subscriptions
e. Annual income
f. Rate the quality of our magazine: excellent, good, fair, or poor

For each item identify the resulting data type.

2.8 Baseball fans are regularly asked to offer their opinions about various aspects of the sport. A survey asked the following questions. Identify the type of data.
a. How many games do you attend annually?
b. How would you rate the quality of entertainment? (excellent, very good, good, fair, poor)
c. Do you have season tickets?
d. How would you rate the quality of the food? (edible, barely edible, horrible)

2.9 A survey of golfers asked the following questions. Identify the type of data each question produces.
a. How many rounds of golf do you play annually?
b. Are you a member of a private club?
c. What brand of clubs do you own?

2.10 At the end of the term, university and college students often complete questionnaires about their courses. Suppose that in one university, students were asked the following.
a. Rate the course (highly relevant, relevant, irrelevant)
b. Rate the professor (very effective, effective, not too effective, not at all effective)
c. What was your midterm grade (A, B, C, D, F)?

Determine the type of data each question produces.

2.2 Describing a Set of Nominal Data

As we discussed in Section 2.1, the only allowable calculation on nominal data is to count the frequency or compute the percentage that each value of the variable represents. We can summarize the data in a table, which presents the categories and their counts, called a **frequency distribution** A **relative frequency distribution** lists the categories and the proportion with which each occurs. We can use graphical techniques to present a picture of the data. There are two graphical methods we can use: the **bar chart** and the **pie chart**.

Example 2.1 Work Status in the GSS 2008 Survey

DATA
GSS2008*

In Chapter 1, we briefly introduced the General Social Survey. In the 2008 survey respondents were asked the following.

"Last week were you working full time, part time, going to school, keeping house, or what"? The responses were

1. Working full-time
2. Working part-time
3. Temporarily not working
4. Unemployed, laid off
5. Retired

6. School
7. Keeping house
8. Other

The responses were recorded using the codes 1, 2, 3, 4, 5, 6, 7, and 8, respectively. The first 150 observations are listed here. The name of the variable is WRKSTAT, and the data are stored in the 16th column (column P in Excel, column 16 in Minitab).

Construct a frequency and relative frequency distribution for these data and graphically summarize the data by producing a bar chart and a pie chart.

1	1	1	1	1	7	7	1	1	5	1	5	7	1	1
5	7	1	5	2	5	1	5	8	1	5	7	1	4	2
7	1	2	1	1	2	1	7	1	7	1	2	1	1	1
1	1	6	5	1	1	1	1	1	2	5	2	7	2	7
8	1	8	1	7	1	6	7	6	1	5	1	2	2	4
1	1	1	1	1	6	5	5	3	2	1	1	8	1	5
1	1	1	1	5	5	1	5	4	7	1	1	1	4	5
2	5	6	7	7	1	4	2	1	2	6	1	1	1	1
1	1	7	4	1	1	1	7	8	1	3	1	1	3	1
1	1	1	1	1	2	1	5	1	1	1	1	1	2	1

SOLUTION

Scan the data. Have you learned anything about the responses of these 150 Americans? Unless you have special skills you have probably learned little about the numbers. If we had listed all 2,023 observations you would be even less likely to discover anything useful about the data. To extract useful information requires the application of a statistical or graphical technique. To choose the appropriate technique we must first identify the type of data. In this example the data are nominal because the numbers represent categories. The only calculation permitted on nominal data is to count the number of occurrences of each category. Hence, we count the number of 1s, 2s, 3s, 4s, 5s, 6s, 7s, and 8s. The list of the categories and their counts constitute the frequency distribution. The relative frequency distribution is produced by converting the frequencies into proportions. The frequency and relative frequency distributions are combined in Table 2.1.

TABLE 2.1 Frequency and Relative Frequency Distributions for Example 2.1

WORK STATUS	CODE	FREQUENCY	RELATIVE FREQUENCY (%)
Working full-time	1	1003	49.6
Working part-time	2	211	10.4
Temporarily not working	3	53	2.6
Unemployed, laid off	4	74	3.7
Retired	5	336	16.6
School	6	57	2.8
Keeping house	7	227	11.2
Other	8	60	3.0
Total		2021	100

There were two individuals who refused to answer hence the number of observations is the sample size 2,023 minus 2, which equals 2,021.

As we promised in Chapter 1 (and the preface), we demonstrate the solution of all examples in this book using three approaches (where feasible): manually, using Excel, and using Minitab. For Excel and Minitab, we provide not only the printout but also instructions to produce them.

EXCEL

INSTRUCTIONS

(Specific commands for this example are highlighted.)

1. Type or import the data into one or more columns. (Open GSS2008.)
2. Activate any empty cell and type

=COUNTIF ([Input range], [Criteria])

Input range are the cells containing the data. In this example, the range is P1:P2024. The criteria are the codes you want to count: (1) (2) (3) (4) (5) (6) (7) (8). To count the number of 1s (“Working full-time”), type

=COUNTIF (P1:P2024, 1)

and the frequency will appear in the dialog box. Change the criteria to produce the frequency of the other categories.

MINITAB

WRKSTAT	Count	Percent
1	1003	49.63
2	211	10.44
3	53	2.62
4	74	3.66
5	336	16.63
6	57	2.82
7	227	11.23
8	60	2.97

N= 2021
*= 2

INSTRUCTIONS

(Specific commands for this example are highlighted.)

1. Type or import the data into one column. (Open GSS2008.)
2. Click **Stat**, **Tables**, and **Tally Individual Variables**.
3. Type or use the **Select** button to specify the name of the variable or the column where the data are stored in the **Variables** box (WRKSTAT). Under **Display**, click **Counts** and **Percents.**

INTERPRET

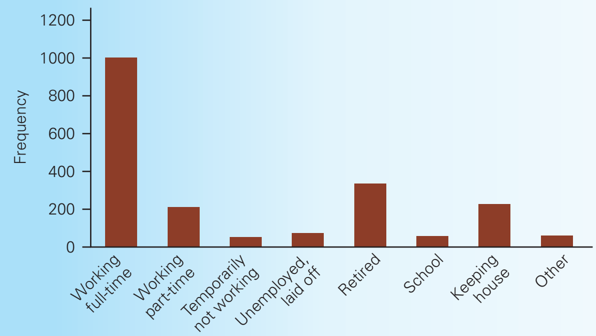
Almost 50% of respondents are working full-time, 16.6% are retired, 11.2% are keeping house, 10.4% are working part-time, and the remaining 12.1% are divided almost equally among the other four categories.

Bar and Pie Charts

The information contained in the data is summarized well in the table. However, graphical techniques generally catch a reader's eye more quickly than does a table of numbers. Two graphical techniques can be used to display the results shown in the table. A **bar chart** is often used to display frequencies; a **pie chart** graphically shows relative frequencies.

The bar chart is created by drawing a rectangle representing each category. The height of the rectangle represents the frequency. The base is arbitrary. Figure 2.1 depicts the manually drawn bar chart for Example 2.1.

FIGURE **2.1** Bar Chart for Example 2.1



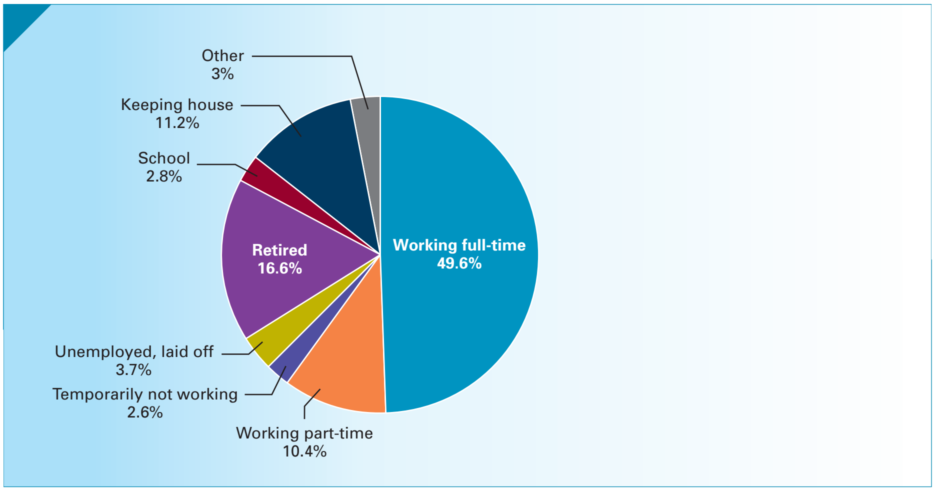
If we wish to emphasize the relative frequencies instead of drawing the bar chart, we draw a pie chart. A pie chart is simply a circle subdivided into slices that represent the categories. It is drawn so that the size of each slice is proportional to the percentage corresponding to that category. For example, because the entire circle is composed of 360 degrees, a category that contains 25% of the observations is represented by a slice of the pie that contains 25% of 360 degrees, which is equal to 90 degrees. The number of degrees for each category in Example 2.1 is shown in Table 2.2.

TABLE **2.2** Proportion in Each Category in Example 2.1

WORK STATUS	RELATIVE FREQUENCY (%)	SLICE OF THE PIE (°)
Working full-time	49.6	178.7
Working part-time	10.4	37.6
Temporarily not working	2.6	9.4
Unemployed, laid off	3.7	13.2
Retired	16.6	59.9
School	2.8	10.2
Keeping house	11.2	40.4
Other	3.0	10.7
Total	100.0	360

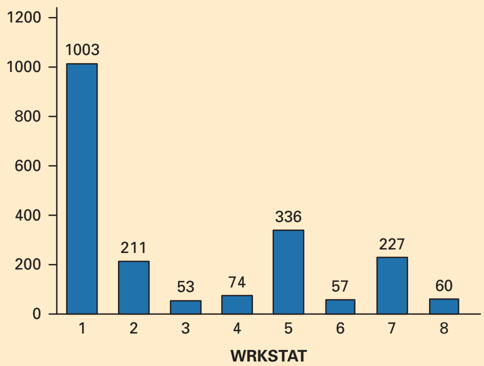
Figure 2.2 was drawn from these results.

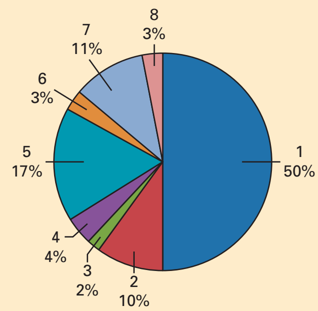
FIGURE 2.2 Pie Chart for Example 2.1



EXCEL

Here are Excel's bar and pie charts.

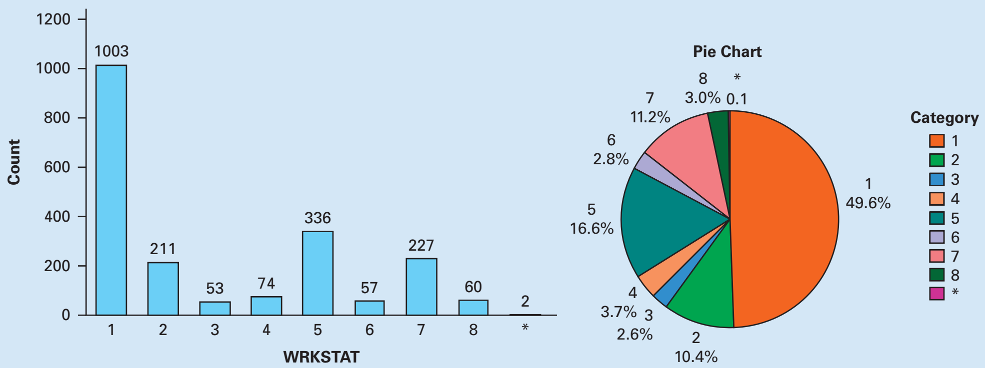




INSTRUCTIONS

1. After creating the frequency distribution, highlight the column of frequencies.
2. For a bar chart, click **Insert, Column,** and the first **2-D Column.**
3. Click **Chart Tools** (if it does not appear, click inside the box containing the bar chart) and **Layout.** This will allow you to make changes to the chart. We removed the **Gridlines**, the **Legend**, and clicked the **Data Labels** to create the titles.
4. For a pie chart, click **Pie** and **Chart Tools** to edit the graph.

MINITAB



INSTRUCTIONS

1. Type or import the data into one column. (Open GSS2008.)

For a bar chart:

2. Click **Graph** and **Bar Chart**.
3. In the **Bars represent** box, click **Counts of unique values** and select **Simple**.
4. Type or use the **Select** button to specify the variable in the **Variables** box (WRKSTAT).

We clicked **Labels** and added the title and clicked **Data Labels** and **Use y-value labels** to display the frequencies at the top of the columns.

For a pie chart:

2. Click **Graph** and **Pie Chart**.
3. Click **Chart, Counts of unique values,** and in the **Categorical variables** box type or use the **Select** button to specify the variable (WRKSTAT).

We clicked **Labels** and added the title. We clicked **Slice Labels** and clicked **Category name** and **Percent.**

INTERPRET

The bar chart focuses on the frequencies and the pie chart focuses on the proportions.

Other Applications of Pie Charts and Bar Charts

Pie and bar charts are used widely in newspapers, magazines, and business and government reports. One reason for this appeal is that they are eye-catching and can attract the reader's interest whereas a table of numbers might not. Perhaps no one understands this better than the newspaper *USA Today*, which typically has a colored graph on the front page and others inside. Pie and bar charts are frequently used to simply present numbers associated with categories. The only reason to use a bar or pie chart in such a situation would be to enhance the reader's ability to grasp the substance of the data. It might, for example, allow the reader to more quickly recognize the relative sizes of the categories, as in the breakdown of a budget. Similarly, treasurers might use pie charts to show the breakdown of a firm's revenues by department, or university students might use pie charts to show the amount of time devoted to daily activities (e.g., eat 10%, sleep 30%, and study statistics 60%).

APPLICATIONS in **ECONOMICS**

Macroeconomics

Macroeconomics is a major branch of economics that deals with the behavior of the economy as a whole. Macroeconomists develop mathematical models that predict variables such as gross domestic product, unemployment rates, and inflation. These are used by governments and corporations to help develop strategies. For example, central banks attempt to control inflation by lowering or raising interest rates. To do this requires that economists determine the effect of a variety of variables, including the supply and demand for energy.

APPLICATIONS in **ECONOMICS**

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Energy Economics

One variable that has had a large influence on the economies of virtually every country is energy. The 1973 oil crisis in which the price of oil quadrupled over a short period of time is generally considered to be one of the largest financial shocks to our economy. In fact, economists often refer to two different economies: before the 1973 oil crisis and after.

Unfortunately, the world will be facing more shocks to our economy because of energy for two primary reasons. The first is the depletion of nonrenewable sources of energy and the resulting price increases. The second is the possibility that burning fossil fuels and the creation of carbon dioxide may be the cause of global warming. One economist predicted that the cost of global warming will be calculated in the trillions of dollars. Statistics can play an important role by determining whether Earth's temperature has been increasing and, if so, whether carbon dioxide is the cause. (See Case 3.1.)

In this chapter, you will encounter other examples and exercises that involve the issue of energy.

EXAMPLE 2.2

DATA
Xm02-02

Energy Consumption in the United States in 2007

Table 2.3 lists the total energy consumption of the United States from all sources in 2007 (latest data available at publication). To make it easier to see the details, the table measures the energy in quadrillions of British thermal units (BTUs). Use an appropriate graphical technique to depict these figures.

TABLE **2.3** Energy Consumption in the United States by Source, 2007

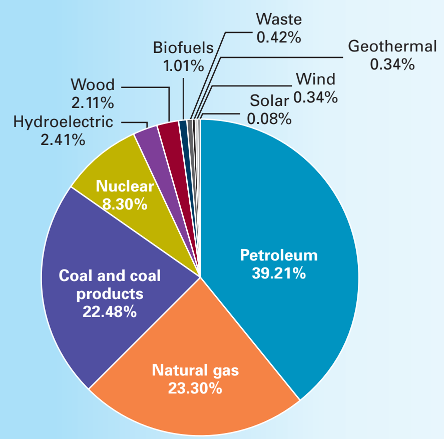
ENERGY SOURCES	QUADRILLIONS OF BTUS
Nonrenewable	
Petroleum	39.773
Natural Gas	23.637
Coal and coal products	22.801
Nuclear	8.415
Renewable Energy Sources	
Hydroelectric	2.446
Wood derived fuels	2.142
Biofuels	1.024
Waste	0.430
Geothermal	0.349
Wind	0.341
Solar/photovoltaic	0.081
Total	101.439

Sources: Non-renewable energy: Energy Information Administration (EIA), Monthly Energy Review (MER) December 2008, DOE/EIA-0035 (2008/12) (Washington, DC: December 2008) Tables 1.3, 1.4a, and 1.4b; Renewable Energy: Table 1.2 of this report.

SOLUTION

We're interested in describing the proportion of total energy consumption for each source. Thus, the appropriate technique is the pie chart. The next step is to determine the proportions and sizes of the pie slices from which the pie chart is drawn. The following pie chart was created by Excel. Minitab's would be similar.

FIGURE **2.3** Pie Chart for Example 2.2



INTERPRET

The United States depends heavily on petroleum, coal, and, natural gas. About 85% of national energy use is based on these sources. The renewable energy sources amount to less than 7%, of which about a third is hydroelectric and probably cannot be expanded much further. Wind and solar barely appear in the chart.

See Exercises 2.11 to 2.15 for more information on the subject.

EXAMPLE 2.3 Per Capita Beer Consumption (10 Selected Countries)

DATA
Xm02-03

Table 2.4 lists the per capita beer consumption for each of 20 countries around the world. Graphically present these numbers.

TABLE **2.4** Per Capita Beer Consumption 2008

COUNTRY	BEER CONSUMPTION(L/YR)
Australia	119.2
Austria	106.3
Belgium	93.0
Canada	68.3
Croatia	81.2
Czech Republic	138.1
Denmark	89.9
Finland	85.0
Germany	147.8
Hungary	75.3
Ireland	138.3
Luxembourg	84.4
Netherlands	79.0
New Zealand	77.0
Poland	69.1
Portugal	59.6
Slovakia	84.1
Spain	83.8
United Kingdom	96.8
United States	81.6

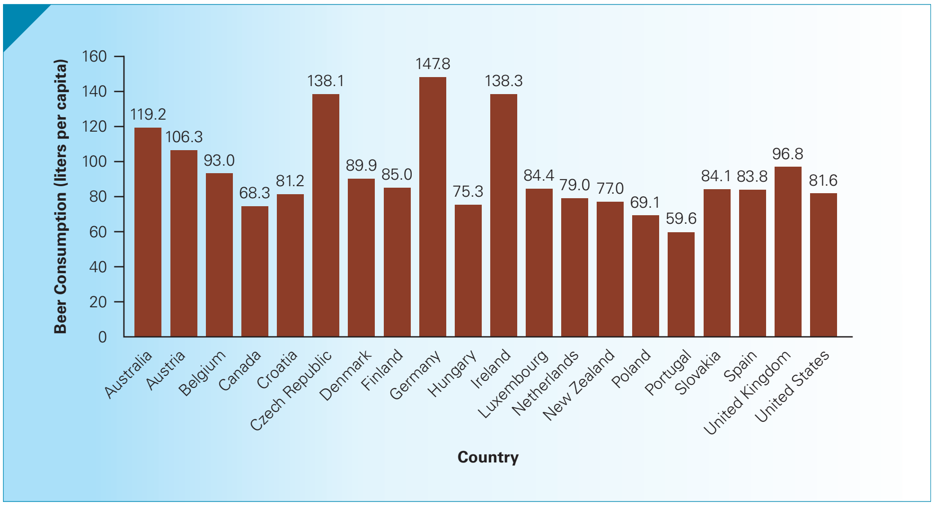
Source: www.beerinfo.com

SOLUTION

In this example, we're primarily interested in the numbers. There is no use in presenting proportions here.

The following is Excel's bar chart.

FIGURE **2.4** EXCEL Bar Chart for Example 2.3



INTERPRET

Germany, the Czech Republic, Ireland, Australia, and Austria head the list. Both the United States and Canada rank far lower. Surprised?

Describing Ordinal Data

There are no specific graphical techniques for ordinal data. Consequently, when we wish to describe a set of ordinal data, we will treat the data as if they were nominal and use the techniques described in this section. The only criterion is that the bars in bar charts should be arranged in ascending (or descending) ordinal values; in pie charts, the wedges are typically arranged clockwise in ascending or descending order.

We complete this section by describing when bar and pie charts are used to summarize and present data.

Factors That Identify When to Use Frequency and Relative Frequency Tables, Bar and Pie Charts

1. **Objective**: Describe a single set of data.
2. **Data type**: Nominal or ordinal

EXERCISES

2.11 Xr02-11 When will the world run out of oil? One way to judge is to determine the oil reserves of the countries around the world. The next table displays the known oil reserves of the top 15 countries. Graphically describe the figures.

Country	Reserves
Brazil	12,620,000,000
Canada	178,100,000,000
China	16,000,000,000

(Continued)

Country	Reserves
Iran	136,200,000,000
Iraq	115,000,000,000
Kazakhstan	30,000,000,000
Kuwait	104,000,000,000
Libya	43,660,000,000
Nigeria	36,220,000,000
Qatar	15,210,000,000
Russia	60,000,000,000
Saudi Arabia	266,700,000,000
United Arab Emirates	97,800,000,000
United States	21,320,000,000
Venezuela	99,380,000,000

Source: CIA World Factbook.

2.12 Refer to Exercise 2.11. The total reserves in the world are 1,348,528,420,000 barrels. The total reserves of the top 15 countries are 1,232,210,000,000 barrels. Use a graphical technique that emphasizes the percentage breakdown of the top 15 countries plus others. Briefly describe your findings.

2.13 Xr02-13 The following table lists the average oil consumption per day for the top 15 oil-consuming countries. Use a graphical technique to present these figures.

Country	Consumption (barrels per day)
Brazil	2,520,000
Canada	2,260,000
China	7,850,000
France	1,986,000
Germany	2,569,000
India	2,940,000
Iran	1,755,000
Italy	1,639,000
Japan	4,785,000
Mexico	2,128,000
Russia	2,900,000
Saudi Arabia	2,380,000
South Korea	2,175,000
United Kingdom	1,710,000
United States	19,500,000

Source: CIA World Factbook.

2.14 Xr02-14 There are 42 gallons in a barrel of oil. The number of products produced and the proportion of the total are listed in the following table. Draw a graph to depict these numbers. What can you conclude from your graph?

Product	Percent of Total (%)
Gasoline	51.4
Distillate fuel oil	15.3
Jet fuel	12.6
Still gas	5.4
Marketable coke	5.0
Residual fuel oil	3.3
Liquefied refinery gas	2.8
Asphalt and road oil	1.9
Lubricants	.9
Other	1.5

Source: California Energy Commission based on 2004 data.

2.15 Xr02-15* The following table displays the energy consumption pattern of Australia. The figures measure the heat content in metric tons (1,000 kilograms) of oil equivalent. Draw a graph that depicts these numbers and explain what you have learned.

Energy Sources	Heat Content
Nonrenewable	
Coal and coal products	55,385
Oil	33,185
Natural Gas	20,350
Nuclear	0
Renewable Energy Sources	
Hydroelectric	1,388
Solid Biomass	4,741
Other (Liquid biomass, geothermal, solar, wind, and tide, wave, and ocean)	347
Total	115,396

Source: International Energy Association.

2.16 Xr02-16 The planet may be threatened by global warming, which may be caused by the burning of fossil fuels (petroleum, natural gas, and coal) that produces carbon dioxide (CO_2). The following table lists the top 15 producers of CO_2 and the annual amounts (million of metric tons) from fossil fuels. Graphically depict these figures. Explain what you have learned.

Country	CO_2	Country	CO_2
Australia	406.6	Japan	1230.4
Canada	631.3	Korea, South	499.6
China	5322.7	Russia	1696.0
France	415.3	Saudi Arabia	412.4
Germany	844.2	South Africa	423.8
India	1165.7	United Kingdom	577.2
Iran	450.7	United States	5957.0
Italy	466.6		

Source: Statistical Abstract of the United States, 2009, Table 1304.

2.17 Xr02-17 The production of steel has often been used as a measure of the economic strength of a country. The following table lists the steel production in the 20 largest steel-producing nations in 2008. The units are millions of metric tons. Use a graphical technique to display these figures.

Country	Steel production	Country	Steel production
Belgium	10.7	Mexico	17.2
Brazil	33.7	Poland	9.7
Canada	14.8	Russia	68.5
China	500.5	South Korea	53.6
France	17.9	Spain	18.6
Germany	45.8	Taiwan	19.9
India	55.2	Turkey	26.8
Iran	10	Ukraine	37.1
Italy	30.6	United Kingdom	13.5
Japan	118.7	United States	91.4

Source: World Steel Association.

2.18 Xr02-18 In 2003 (latest figures available) the United States generated 251.3 million tons of garbage. The following table lists the amounts by source. Use one or more graphical techniques to present these figures.

Source	Amount (millions of tons)
Paper and paperboard	85.2
Glass	13.3
Metals	19.1
Plastics	29.4
Rubber and leather	6.5
Textiles	11.8
Wood	13.8
Food scraps	31.2
Yard trimmings	32.4
Other	8.6

Source: Statistical Abstract of the United States, 2009, Table 361.

2.19 Xr02-19 In the last five years, the city of Toronto has intensified its efforts to reduce the amount of garbage that is taken to landfill sites. [Currently, the Greater Toronto Area (GTA) disposes of its garbage in a dump site in Michigan.] A current analysis of GTA reveals that 36% of waste collected is taken from residences and 64% from businesses and public institutions (hospitals, schools, universities, etc.). A further breakdown is listed below. (*Source:* Toronto City Summit Alliance.)

a. Draw a pie chart for residential waste including both recycled and disposed waste.
b. Repeat part (a) for nonresidential waste.

Residential

Recycled	Pct	Disposed	Pct
Recycled Plastic	1%	Plastic	7%
Recycled Glass	3%	Paper	12%
Recycled Paper	14%	Metal	2%
Recycled Metal	1%	Organic	23%
Recycled Organic/Food	7%	Other	17%
Recycled Organic/Yard	10%		
Recycled Other	4%		

Non-Residential

Recycled	Pct	Disposed	Pct
Recycled Glass	1%	Plastic	10%
Recycled Paper	11%	Glass	3%
Recycled Metal	3%	Paper	31%
Recycled Organic	1%	Metal	8%
Recycled Construction/Demolition	1%	Organic	18%
Recycled Other	1%	Construction/Demolition	7%
		Other	6%

2.20 Xr02-20 The following table lists the top 10 countries and amounts of oil (millions of barrels annually) they exported to the United States in 2007.

Country	Oil Imports
Algeria	162
Angola	181
Canada	681
Iraq	177
Kuwait	64
Mexico	514
Nigeria	395
Saudi Arabia	530
United Kingdom	37
Venezuela	420

Source: Statistical Abstract of the United States, 2009, Table 895.

a. Draw a bar chart.
b. Draw a pie chart.
c. What information is conveyed by each chart?

2.21 Xr02-21 The following table lists the percentage of males and females in five age groups that did not have health insurance in the United States in September 2008. Use a graphical technique to present these figures.

Age Group	Male	Female
Under 18	8.5	8.5
18–24	32.3	24.9
25–34	30.4	21.4
35–44	21.3	17.1
45–64	13.5	13.0

Source: National Health Interview Survey.

2.22 Xr02-22 The following table lists the average costs for a family of four to attend a game at a National Football League (NFL) stadium compared to a Canadian Football League (CFL) stadium. Use a graphical technique that allows the reader to compare each component of the total cost.

	NFL	CFL
Four tickets	274.12	171.16
Parking	19.75	10.85
Two ball caps	31.12	44.26
Two beers	11.90	11.24
Two drinks	7.04	7.28
Four hot dogs	15.00	16.12

Source: Team Market Report, Matthew Coutts.

2.23 Xr02-23 Productivity growth is critical to the economic well-being of companies and countries. In the table below we list the average annual growth rate (in percent) in productivity for the Organization for Economic Co-Operation and Development (OECD) countries. Use graphical technique to present these figures.

Country	Productivity Growth	Country	Productivity Growth
Australia	1.6	Japan	2.775
Austria	1.5	Korea	5.55
Belgium	1.975	Luxembourg	2.6
Canada	1.25	Mexico	1.2
Czech Republic	3.3	Netherlands	2
		New Zealand	1.5
Denmark	2.175	Norway	2.575
Finland	2.775	Portugal	2.8
France	2.35	Slovak Republic	4.8
Germany	2.3	Spain	2
Greece	1.6	Sweden	1.775
Hungary	3.6	Switzerland	1.033
Iceland	0.775	United Kingdom	2.2
Ireland	3.775	United States	1.525
Italy	1.525		

Source: OECD Labor Productivity Database July 2007.

The following exercises require a computer and software.

2.24 Xr02-24 In an attempt to stimulate the economy in 2008, the U.S. government issued rebate checks totaling $107 billion. A survey conducted by the National Retail Federation (NRF) asked recipients what they intended to do with their rebates. The choices are:

1. Buy something
2. Pay down debt
3. Invest
4. Pay medical bills
5. Save
6. Other

Use a graphical technique to summarize and present these data. Briefly describe your findings.

2.25 Xr02-25 Refer to Exercise 2.24. Those who responded that they planned to buy something were asked what they intended to buy. Here is a list of their responses.

1. Home improvement project
2. Purchase appliances
3. Purchase automobles
4. Purchase clothing
5. Purchase electronics
6. Purchase furniture
7. Purchase gas
8. Spa or salon time
9. Purchase vacation
10. Purchase groceries
11. Impulse purchase
12. Down payment on house

Graphically summarize these data. What can you conclude from the chart?

2.26 Xr02-26 What are the most important characteristics of colleges and universities? This question was asked of a sample of college-bound high school seniors. The responses are:

1. Location
2. Majors
3. Academic reputation
4. Career focus
5. Community
6. Number of students

The results are stored using the codes. Use a graphical technique to summarize and present the data.

2.27 Xr02-27 Where do consumers get information about cars? A sample of recent car buyers was asked to identify the most useful source of information about the cars they purchased. The responses are:

1. Consumer guide
2. Dealership
3. Word of mouth
4. Internet

The responses were stored using the codes. Graphically depict the responses. *Source: Automotive Retailing Today*, The Gallup Organization.

2.28 Xr02-28 A survey asked 392 homeowners which area of their homes they would most like to renovate. The responses and frequencies are shown next. Use a graphical technique to present these results. Briefly summarize your findings.

Area	Code
Basement	1
Bathroom	2
Bedroom	3
Kitchen	4
Living/dining room	5

Source: Toronto Star, November 23, 2004.

2.29 Xr02-29 Subway train riders frequently pass the time by reading a newspaper. New York City has a subway and four newspapers. A sample of 360 subway riders who regularly read a newspaper was asked to identify that newspaper. The responses are:

1. *New York Daily News*
2. *New York Post*
3. *New York Times*
4. *Wall Street Journal*

The responses were recorded using the numerical codes shown.

a. Produce a frequency distribution and a relative frequency distribution.
b. Draw an appropriate graph to summarize the data. What does the graph tell you?

2.30 Xr02-30 Who applies to MBA programs? To help determine the background of the applicants, a sample of 230 applicants to a university's business school was asked to report their undergraduate degrees. The degrees were recorded using these codes.

1. BA
2. BBA
3 BEng
4. BSc
5. Other

a. Determine the frequency distribution.
b. Draw a bar chart.
c. Draw a pie chart.
d. What do the charts tell you about the sample of MBA applicants?

2.31 Xr02-31 Many business and economics courses require the use of computer, so students often must buy their own computers. A survey asks students to identify which computer brands they have purchased. The responses are:

1. IBM
2. Compaq
3. Dell
4. Other

a. Use a graphical technique that depicts the frequencies.
b. Graphically depict the proportions.
c. What do the charts tell you about the brands of computers used by the students?

2.32 Xr02-32 An increasing number of statistics courses use a computer and software rather than manual calculations. A survey of statistics instructors asked them to report the software their courses use. The responses are:

1. Excel
2. Minitab
3. SAS
4. SPSS
5. Other

a. Produce a frequency distribution.
b. Graphically summarize the data so that the proportions are depicted.
c. What do the charts tell you about the software choices?

2.33 Xr02-33* The total light beer sales in the United States are approximately 3 million gallons annually. With this large of a market, breweries often need to know more about who is buying their product. The marketing manager of a major brewery wanted to analyze the light beer sales among college and university students who drink light beer. A random sample of 285 graduating students was asked to report which of the following is their favorite light beer:

1. Bud Light
2. Busch Light
3. Coors Light
4. Michelob Light
5. Miller Lite
6. Natural Light
7. Other brands

The responses were recorded using the codes 1, 2, 3, 4, 5, 6, and 7, respectively. Use a graphical to summarize these data. What can you conclude from the chart?

General Social Survey Exercises

The following exercises are based on the GSS described above.

2.34 GSS2008* In the 2008 General Social Survey, respondents were asked to identify their race (RACE) using the following categories:

1. White
2. Black
3. Other

Summarize the results using an appropriate graphical technique and interpret your findings.

2.35 GSS2008* Several questions deal with education. One question in the 2008 survey asked respondents to indicate their highest degree (DEGREE). The responses are:

0. Left high school
1. Completed high school
2. Completed junior college
3. Completed bachelor's degree
4. Completed graduate degree

Use a graphical technique to summarize the data. Describe what the graph tells you.

2.36 GSS2006* Refer to the GSS in 2006. Responses to the question about marital status (MARITAL) were:

1. Married
2. Widowed
3. Divorced
4. Separated
5. Never Married

a. Create a frequency distribution
b. Use a graphical method to present these data and briefly explain what the graph reveals.

2.37 GSS2004* Refer to the 2004 GSS, which asked about individual's class (CLASS). The responses were:

1. Lower class
2. Working class
3. Middle class
4. Upper class

Summarize the data using a graphical method and describe your findings.

2.3 Describing the Relationship between Two Nominal Variables and Comparing Two or More Nominal Data Sets

In Section 2.2, we presented graphical and tabular techniques used to summarize a set of nominal data. Techniques applied to single sets of data are called **univariate**. There are many situations where we wish to depict the relationship between variables; in such cases, **bivariate** methods are required. A **cross-classification table** (also called a **cross-tabulation table**) is used to describe the relationship between two nominal variables. A variation of the bar chart introduced in Section 2.2 is employed to graphically describe the relationship. The same technique is used to compare two or more sets of nominal data.

Tabular Method of Describing the Relationship between Two Nominal Variables

To describe the relationship between two nominal variables, we must remember that we are permitted only to determine the frequency of the values. As a first step, we need to produce a cross-classification table that lists the frequency of each combination of the values of the two variables.

EXAMPLE 2.4

DATA
Xm02-04

Newspaper Readership Survey

A major North American city has four competing newspapers: the *Globe and Mail (G&M)*, *Post*, *Star and Sun*. To help design advertising campaigns, the advertising managers of the newspapers need to know which segments of the newspaper market are reading their papers. A survey was conducted to analyze the relationship between newspapers read and occupation. A sample of newspaper readers was asked to report which newspaper they read—*Globe and Mail* (1), *Post* (2), *Star* (3), *Sun* (4)—and indicate whether they were blue-collar workers (1), white-collar workers (2), or professionals (3). Some of the data are listed here.

Reader	Occupation	Newspaper
1	2	2
2	1	4
3	2	1
.	.	.
.	.	.
352	3	2
353	1	3
354	2	3

Determine whether the two nominal variables are related.

SOLUTION

By counting the number of times each of the 12 combinations occurs, we produced the Table 2.5.

TABLE **2.5** Cross-Classification Table of Frequencies for Example 2.4

	NEWSPAPER				
OCCUPATION	*G&M*	*POST*	*STAR*	*SUN*	TOTAL
Blue collar	27	18	38	37	120
White collar	29	43	21	15	108
Professional	33	51	22	20	126
Total	89	112	81	72	354

If occupation and newspaper are related, there will be differences in the newspapers read among the occupations. An easy way to see this is to convert the frequencies in each row (or column) to relative frequencies in each row (or column). That is, compute the row (or column) totals and divide each frequency by its row (or column) total, as shown in Table 2.6. Totals may not equal 1 because of rounding.

TABLE **2.6** Row Relative Frequencies for Example 2.4

	NEWSPAPER				
OCCUPATION	*G&M*	*POST*	*STAR*	*SUN*	TOTAL
Blue collar	.23	.15	.32	.31	1.00
White collar	.27	.40	.19	.14	1.00
Professional	.26	.40	.17	.16	1.00
Total	.25	.32	.23	.20	1.00

EXCEL

Excel can produce the cross-classification table using several methods. We will use and describe the PivotTable in two ways: (1) to create the cross-classification table featuring the counts and (2) to produce a table showing the row relative frequencies.

Count of Reader	Newspaper				
Occupation	G&M	Post	Star	Sun	Grand Total
Blue collar	27	18	38	37	120
White collar	29	43	21	15	108
Professional	33	51	22	20	126
Grand Total	89	112	81	72	354

Count of Reader	Newspaper				
Occupation	G&M	Post	Star	Sun	Grand Total
Blue collar	0.23	0.15	0.32	0.31	1.00
White collar	0.27	0.40	0.19	0.14	1.00
Professional	0.26	0.40	0.17	0.16	1.00
Grand Total	0.25	0.32	0.23	0.20	1.00

INSTRUCTIONS

The data must be stored in (at least) three columns as we have done in Xm02-04. Put the cursor somewhere in the data range.

1. Click **Insert** and **PivotTable**.
2. Make sure that the Table/Range is correct.
3. Drag the Occupation button to the **ROW** section of the box. Drag the Newspaper button to the **COLUMN** section. Drag the Reader button to the **DATA** field. Right-click any number in the table, click **Summarize Data By**, and check **Count.** To convert to row percentages, right-click any number, click **Summarize Data By, More options . . .** , and **Show values as**. Scroll down and click **% of rows**. (We then formatted the data into decimals.) To improve both tables, we substituted the names of the occupations and newspapers.

MINITAB

Tabulated statistics: Occupation, Newspaper

Rows: Occupation			Columns: Newspaper		
	1	2	3	4	All
1	27	18	38	37	120
	22.50	15.00	31.67	30.83	100.00
2	29	43	21	15	108
	26.85	39.81	19.44	13.89	100.00
3	33	51	22	20	126
	26.19	40.48	17.46	15.87	100.00
All	89	112	81	72	354
	25.14	31.64	22.88	20.34	100.00

Cell Contents: Count
% of Row

INSTRUCTIONS

1. Type or import the data into two columns. (Open xM02-04)
2. Click **Stat**, **Tables**, and **Cross Tabulation and Chi-square**.
3. Type or use the **Select** button to specify the **Categorical variables**: **For rows** (Occupation) and **For columns** (Newspaper)
4. Under **Display**, click **Counts** and **Row percents** (or any you wish)

INTERPRET

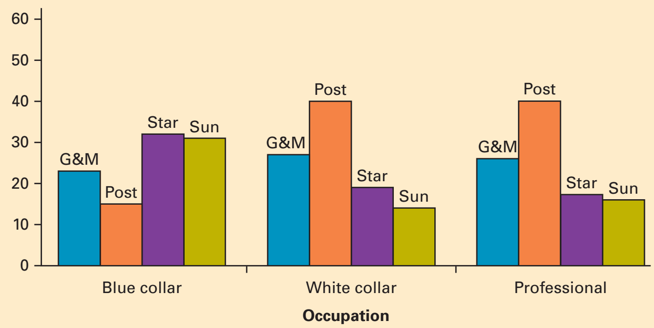
Notice that the relative frequencies in the second and third rows are similar and that there are large differences between row 1 and rows 2 and 3. This tells us that blue-collar workers tend to read different newspapers from both white-collar workers and professionals and that white-collar workers and professionals are quite similar in their newspaper choices.

Graphing The Relationship between Two Nominal Variables

We have chosen to draw three bar charts, one for each occupation depicting the four newspapers. We'll use Excel and Minitab for this purpose. The manually drawn charts are identical.

EXCEL

There are several ways to graphically display the relationship between two nominal variables. We have chosen two dimensional bar charts for each of the three occupations. The charts can be created from the output of the PivotTable (either counts as we have done) or row proportions.

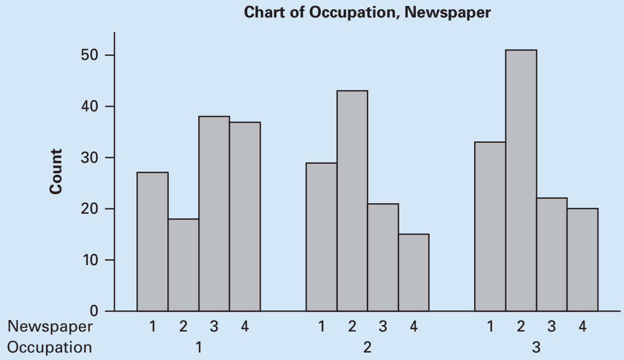


INSTRUCTIONS

From the cross-classification table, click **Insert** and **Column.** You can do the same from any completed cross-classification table.

MINITAB

Minitab can draw bar charts from the raw data.



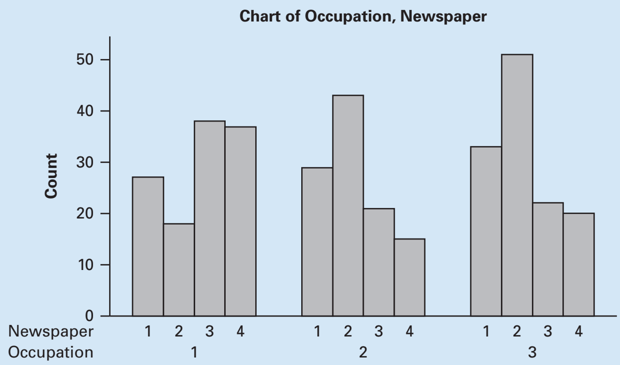
INSTRUCTIONS

1. Click **Graph** and **Bar Chart**.
2. In the **Bars represent** box, specify **Counts of unique values**. Select **Cluster**.
3. In the **Categorical variables** box, type or select the two variables (Newspaper Occupation).

If you or someone else has created the cross-classification table, Minitab can draw bar charts directly from the table.

INSTRUCTIONS

1. Start with a completed cross-classification table such as Table 2.9.
2. Click **Graph** and Bar **Chart**
3. In the **Bars represent** box click **Values from a table.** Choose **Two-way table Cluster**.
4. In the **Graph variables** box, **Select** the columns of numbers in the table. In the **Row labels** box, **Select** the column with the categories.



INTERPRET

If the two variables are unrelated, then the patterns exhibited in the bar charts should be approximately the same. If some relationship exists, then some bar charts will differ from others.

The graphs tell us the same story as did the table. The shapes of the bar charts for occupations 2 and 3 (white-collar and professional) are very similar. Both differ considerably from the bar chart for occupation 1 (blue-collar).

Comparing Two or More Sets of Nominal Data

We can interpret the results of the cross-classification table of the bar charts in a different way. In Example 2.4, we can consider the three occupations as defining three different populations. If differences exist between the columns of the frequency distributions (or between the bar charts), then we can conclude that differences exist among the three populations. Alternatively, we can consider the readership of the four newspapers as four different populations. If differences exist among the frequencies or the bar charts, then we conclude that there are differences between the four populations.

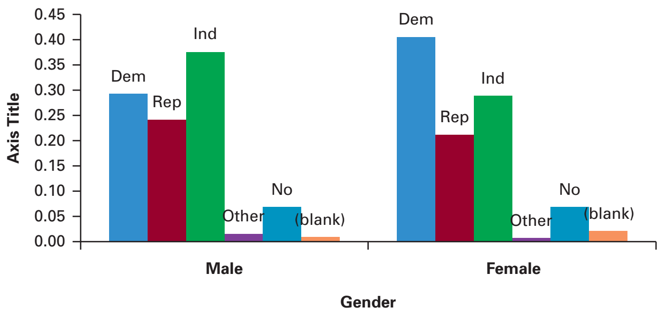
Do Male and Female American Voters Differ in Their Party Affiliation?

DATA
ANES2008*

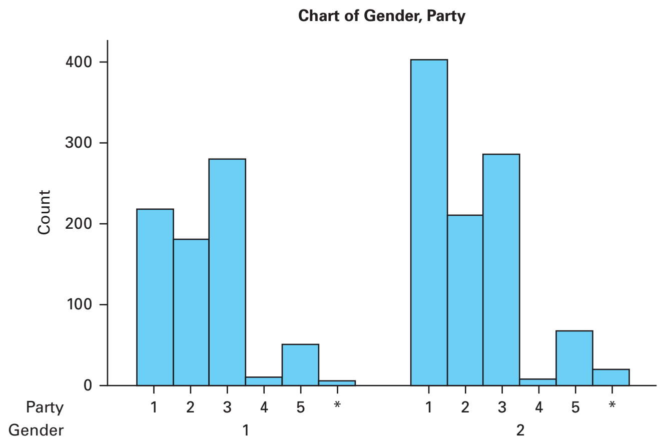
Using the technique introduced above, we produced the bar charts below.

©AP Photo/David Smith

EXCEL



MINITAB



INTERPRET

As you can see, there are substantial differences between the bar charts for men and women. We can conclude that gender and party affiliation are related. However, we can also conclude that differences in party affiliation exist between American male and female voters: Specifically, men tend to identify themselves as independents, whereas women support the Democratic party.

Historically, women tend to be Democrats, and men lean toward the Republican party. However, in this survey, both genders support the Democrats over the Republicans, which explains the results of the 2008 election.

Data Formats

There are several ways to store the data to be used in this section to produce a table or a bar or pie chart.

1. The data are in two columns. The first column represents the categories of the first nominal variable, and the second column stores the categories for the second variable. Each row represents one observation of the two variables. The number of observations in each column must be the same. Excel and Minitab can produce a cross-classification table from these data. (To use Excel's PivotTable, there also must be a third variable representing the observation number.) This is the way the data for Example 2.4 were stored.

2. The data are stored in two or more columns, with each column representing the same variable in a different sample or population. For example, the variable may be the type of undergraduate degree of applicants to an MBA program, and there may be five universities we wish to compare. To produce a cross-classification table, we would have to count the number of observations of each category (undergraduate degree) in each column.

3. The table representing counts in a cross-classification table may have already been created.

We complete this section with the factors that identify the use of the techniques introduced here.

Factors That Identify When to Use a Cross-Classification Table

1. **Objective**: Describe the relationship between two variables and compare two or more sets of data.
2. **Data type**: Nominal

EXERCISES

2.38 Xr02-38 Has the educational level of adults changed over the past 15 years? To help answer this question, the Bureau of Labor Statistics compiled the following table; it lists the number (1,000) of adults 25 years of age and older who are employed. Use a graphical technique to present these figures. Briefly describe what the chart tells you.

Educational level	1995	1999	2003	2007
Less than high school	12,021	12,110	12,646	12,408
High school	36,746	35,335	33,792	32,634
Some college	30,908	30,401	30,338	30,389
College graduate	31,176	33,651	35,454	37,321

Source: Statistical Abstract of the United States, 2009, Table 572.

2.39 Xr02-39 How do governments spend the tax dollars they collect, and has this changed over the past 15 years? The following table displays the amounts spent by the federal, state, and local governments on consumption expenditures and gross investments. Consumption expenditures are services (such as education). Gross investments ($billions) consist of expenditures on fixed assets (such as roads, bridges, and highways). Use a graphical technique to present these figures. Have the ways governments spend money changed over the previous 15 years?

Level of Government and Type	1990	1995	2000	2004
Federal national defense				
Consumption	308.1	297.3	321.5	477.5
Gross	65.9	51.4	48.8	70.4
Federal nondefense				
Consumption	111.7	143.2	177.8	227.0
Gross	22.6	27.3	30.7	35.0
State and local				
Consumption	544.6	696.1	917.8	1,099.7
Gross	127.2	154.0	225.0	274.3

Source: Statistical Abstract of the United States, 2006, Table 419.

2.40 Xr02-15* The table below displays the energy consumption patterns of Australia and New Zealand. The figures measure the heat content in metric tons (1,000 kilograms) of oil equivalent. Use a graphical technique to display the differences between the sources of energy for the two countries.

Energy Sources	Australia	New Zealand
Nonrenewable		
Coal & coal products	55,385	1,281
Oil	33,185	6,275
Natural Gas	20,350	5,324
Nuclear	0	0
Renewable		
Hydroelectric	1,388	1,848
Solid Biomass	4,741	805
Other (Liquid biomass, geothermal, solar, wind, and tide, wave, & ocean)	347	2,761
Total	115,396	18,294

Source: International Energy Association.

The following exercises require a computer and software.

2.41 Xr02-41 The average loss from a robbery in the United States in 2004 was $1,308 (*Source:* U.S. Federal Bureau of Investigation). Suppose that a government agency wanted to know whether the type of robbery differed between 1990, 1995, 2000, and 2006. A random sample of robbery reports was taken from each of these years, and the types were recorded using the codes below. Determine whether there are differences in the types of robbery over the 16-year span. (Adapted from *Statistical Abstract of the United States*, 2009, Table 308.)

1. Street or highway
2. Commercial house
3. Gas station
4. Convenience store
5. Residence
6. Bank
7. Other

2.42 Xr02-42 The associate dean of a business school was looking for ways to improve the quality of the applicants to its MBA program. In particular, she wanted to know whether the undergraduate degree of applicants differed among her school and the three nearby universities with MBA programs. She sampled 100 applicants of her program and an equal number from each of the other universities. She recorded their undergraduate degrees (1 = BA, 2 = BEng, 3 = BBA, 4 = other) as well the university (codes 1, 2, 3, and 4). Use a tabular technique to determine whether the undergraduate degree and the university each person applied to appear to be related.

2.43 Xr02-43 Is there brand loyalty among car owners in their purchases of gasoline? To help answer the question, a random sample of car owners was asked to record the brand of gasoline in their last two purchases (1 = Exxon, 2 = Amoco, 3 = Texaco, 4 = Other). Use a tabular technique to formulate your answer.

2.44 Xr02-44 The costs of smoking for individuals, companies for whom they work, and society in general is in the many billions of dollars. In an effort to reduce smoking, various government and non-government organizations have undertaken information campaigns about the dangers of smoking. Most of these have been directed at young people. This raises the question: Are you more likely to smoke if your parents smoke? To shed light on the issue, a sample of 20- to 40-year-old people were asked whether they smoked and whether their parents smoked. The results are stored the following way:

Column 1: 1 = do not smoke, 2 = smoke
Column 2: 1 = neither parent smoked,
2 = father smoked, 3 = mother smoked,
4 = both parents smoked

Use a tabular technique to produce the information you need.

2.45 Xr02-45 In 2007, 3,882,000 men and 3,196,000 women were unemployed at some time during the year (*Source*: U.S. Bureau of Labor Statistics). A statistics practitioner wanted to investigate the reasons for unemployment and whether the reasons differed by gender. A random sample of people 16 years of age and older was drawn. The reasons given for their status are:

1. Lost job
2. Left job
3. Reentrants
4. New entrants

Determine whether there are differences between unemployed men and women in terms of the reasons for unemployment. (*Source:* Adapted from *Statistical Abstract of the United States*, 2009 Table 604.)

2.46 Xr02-46 In 2004, the total number of prescriptions sold in the United States was 3,274,000,000 (*Source*: National Association of Drug Store Chains). The sales manager of a chain of drugstores wanted to determine whether changes were made in the way the prescriptions were filled. A survey of prescriptions was undertaken in 1995, 2000, and 2007. The year and type of each prescription were recorded using the codes below. Determine whether there are differences between the years. (*Source:* Adapted from the *Statistical Abstract of the United States*, 2009, Table 151.)

1. Traditional chain store
2. Independent drugstore
3. Mass merchant
4. Supermarket
5. Mail order

2.47 Xr02-33* Refer to Exercise 2.33. Also recorded was the gender of the respondents. Use a graphical technique to determine whether the choice of light beers differs between genders.

CHAPTER SUMMARY

Descriptive statistical methods are used to summarize data sets so that we can extract the relevant information. In this chapter, we presented graphical techniques for nominal data.

Bar charts, pie charts, and frequency distributions are employed to summarize single sets of nominal data. Because of the restrictions applied to this type of data, all that we can show is the frequency and proportion of each category.

To describe the relationship between two nominal variables, we produce cross classification tables and bar charts.

IMPORTANT TERMS

- Variable 13
- Values 13
- Data 13
- Datum 13
- Interval 14
- Quantitative 14
- Numerical 14
- Nominal 14
- Qualitative 14
- Categorical 14
- Ordinal 14
- Frequency distribution 18
- Relative frequency distribution 18
- Bar chart 18
- Pie chart 18
- Univariate 32
- Bivariate 32
- Cross-classification table 32
- Cross-tabulation table 32

COMPUTER OUTPUT AND INSTRUCTIONS

Graphical Technique	Excel	Minitab
Bar chart	22	23
Pie chart	22	23

CHAPTER EXERCISES

The following exercises require a computer and software.

2.48 Xr02-48 A sample of 200 people who had purchased food at the concession stand at Yankee Stadium was asked to rate the quality of the food. The responses are:

a. Poor
b. Fair
c. Good
d. Very good
e. Excellent

Draw a graph that describes the data. What does the graph tell you?

2.49 Xr02-49 There are several ways to teach applied statistics. The most popular approaches are:

a. Emphasize manual calculations
b. Use a computer combined with manual calculations
c. Use a computer exclusively with no manual calculations

A survey of 100 statistics instructors asked each one to report his or her approach. Use a graphical method to extract the most useful information about the teaching approaches.

2.50 Xr02-50 Which Internet search engines are the most popular? A survey undertaken by the *Financial Post* (May 14, 2004) asked random samples of Americans and Canadians that question. The responses were:

1. Google
2. Microsoft (MSN)
3. Yahoo
4. Other

Use a graphical technique that compares the proportions of Americans' and Canadians' use of search engines

2.51 Xr02-51 The Wilfrid Laurier University bookstore conducts annual surveys of its customers. One question asks respondents to rate the prices of textbooks. The wording is, "The bookstore's prices of textbooks are reasonable." The responses are:

1. Strongly disagree
2. Disagree
3. Neither agree nor disagree
4. Agree
5. Strongly agree

The responses for a group of 115 students were recorded. Graphically summarize these data and report your findings.

2.52 Xr02-52 The Red Lobster restaurant chain conducts regular surveys of its customers to monitor the performance of individual restaurants. One question asks customers to rate the overall quality of their last visit. The listed responses are poor (1), fair (2), good (3), very good (4), and excellent (5). The survey also asks respondents whether their children accompanied them to the restaurant (1 = yes, 2 = no). Graphically depict these data and describe your findings.

2.53 Xr02-53 Many countries are lowering taxes on corporations in an effort to make their countries more attractive for investment. In the next table, we list the marginal effective corporate tax rates among Organization for Economic Co-Operation and Development (OECD) countries. Develop a graph that depicts these figures. Briefly describe your results.

Country	Manufacturers	Services	Aggregate
Australia	27.7	26.6	26.7
Austria	21.6	19.5	19.9
Belgium	−6.0	−4.1	−0.5
Canada	20.0	29.2	25.2
Czech Republic	1.0	7.8	8.4
Denmark	16.5	12.7	13.4
Finland	22.4	22.9	22.8
France	33.0	31.7	31.9
Germany	30.8	29.4	29.7
Greece	18.0	13.2	13.8
Hungary	12.9	12.0	12.2
Iceland	19.5	17.6	17.9
Ireland	12.7	11.7	12.0
Italy	24.6	28.6	27.8
Japan	35.2	30.4	31.3
Korea	32.8	31.0	31.5
Luxembourg	24.1	20.3	20.6
Mexico	17.1	12.1	13.1
Netherlands	18.3	15.0	15.5
New Zealand	27.1	25.4	25.7
Norway	25.8	23.2	23.5
Poland	14.4	15.0	14.9
Portugal	14.8	16.1	15.9
Slovak	13.3	11.7	12.0
Spain	27.2	25.2	25.5
Sweden	19.3	17.5	17.8
Switzerland	14.8	15.0	14.9
Turkey	22.7	20.2	20.8
United Kingdom	22.7	27.8	26.9
United States	32.7	39.9	36.9

2.54 Xr02-54* A survey of the business school graduates undertaken by a university placement office asked, among other questions, the area in which each person was employed. The areas of employment are:

a. Accounting
b. Finance
c. General management
d. Marketing/Sales
e. Other

Additional questions were asked and the responses were recorded in the following way.

Column	Variable
1	Identification number
2	Area
3	Gender (1 = female, 2 = male)
4	Job satisfaction (4 = very, 3 = quite, 2 = little, 1 = none)

The placement office wants to know the following:

a. Do female and male graduates differ in their areas of employment? If so, how?
b. Are area of employment and job satisfaction related?

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3

GRAPHICAL DESCRIPTIVE TECHNIQUES II

Were Oil Companies Gouging Customers 2000–2009?

DATA
Xm03-00

The price of oil has been increasing for several reasons. First, oil is a finite resource; the world will eventually run out. In January 2009, the world was consuming more than 100 million barrels per day—more than 36 billion barrels per year. The total proven world reserves of oil are 1,348.5 billion barrels. At today's consumption levels, the proven reserves will be exhausted in 37 years. (It should be noted, however, that in 2005 the proven reserves of oil amounted to 1,349.4 billion barrels, indicating that new oil discoveries are offsetting increasing usage.) Second, China's and India's industries are rapidly increasing and require ever-increasing amounts of oil. Third, over the last 10 years, hurricanes have threatened the oil rigs in the Gulf of Mexico.

The result of the price increases in oil is reflected in the price of gasoline. In January 2000, the average retail price of gasoline in the United States was $1.301 per U.S. gallon (one U.S.

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On page 78 you will find our answer

gallon equals 3.79 liters) and the price of oil (West Texas intermediate crude) was $27.18 per barrel (one barrel equals 42 U.S. gallons). (*Sources:* U.S Department of Energy.) Over the next 10 years, the price of both oil and gasoline substantially increased. Many drivers complained that the oil companies were guilty of price gouging; that is, they believed that when the price of oil increased, the price of gas also increased, but when the price of oil decreased, the decrease in the price of gasoline seemed to lag behind. To determine whether this perception is accurate, we determined the monthly figures for both commodities. Were oil and gas prices related?

Introduction

Chapter 2 introduced graphical techniques used to summarize and present nominal data. In this chapter, we do the same for interval data. Section 3.1 presents techniques to describe a set of interval data, Section 3.2 introduces time series and the method used to present time series data, and Section 3.3 describes the technique we use to describe the relationship between two interval variables. We complete this chapter with a discussion of how to properly use graphical methods in Section 3.4.

3.1 Graphical Techniques to Describe a Set of Interval Data

In this section, we introduce several graphical methods that are used when the data are interval. The most important of these graphical methods is the histogram. As you will see, the histogram not only is a powerful graphical technique used to summarize interval data but also is used to help explain an important aspect of probability (see Chapter 8).

APPLICATIONS in MARKETING

© AP Photo/Paul Sakuma

Pricing

Traditionally, marketing has been defined in terms of the four P's: product, price, promotion, and place. *Marketing management* is the functional area of business that focuses on the development of a product, together with its pricing, promotion, and distribution. Decisions are made in these four areas with a view to satisfying the wants and needs of consumers while also satisfying the firm's objective.

The pricing decision must be addressed both for a new product, and, from time to time, for an existing product. Anyone buying a product such as a personal computer has been confronted with a wide variety of prices, accompanied by a correspondingly wide variety of features. From a vendor's standpoint, establishing the appropriate price and corresponding set of attributes for a product is complicated and must be done in the context of the overall marketing plan for the product.

EXAMPLE 3.1

DATA
Xm03-01

Analysis of Long-Distance Telephone Bills

Following deregulation of telephone service, several new companies were created to compete in the business of providing long-distance telephone service. In almost all cases, these companies competed on price because the service each offered is similar. Pricing a service or product in the face of stiff competition is very difficult. Factors to be considered include supply, demand, price elasticity, and the actions of competitors. Long-distance packages may employ per minute charges, a flat monthly rate, or some combination of the two. Determining the appropriate rate structure is facilitated by acquiring information about the behaviors of customers, especially the size of monthly long-distance bills.

As part of a larger study, a long-distance company wanted to acquire information about the monthly bills of new subscribers in the first month after signing with the company. The company's marketing manager conducted a survey of 200 new residential subscribers and recorded the first month's bills. These data are listed here. The general manager planned to present his findings to senior executives. What information can be extracted from these data?

Long-Distance Telephone Bills

42.19	39.21	75.71	8.37	1.62	28.77	35.32	13.9	114.67	15.3
38.45	48.54	88.62	7.18	91.1	9.12	117.69	9.22	27.57	75.49
29.23	93.31	99.5	11.07	10.88	118.75	106.84	109.94	64.78	68.69
89.35	104.88	85	1.47	30.62	0	8.4	10.7	45.81	35
118.04	30.61	0	26.4	100.05	13.95	90.04	0	56.04	9.12
110.46	22.57	8.41	13.26	26.97	14.34	3.85	11.27	20.39	18.49
0	63.7	70.48	21.13	15.43	79.52	91.56	72.02	31.77	84.12
72.88	104.84	92.88	95.03	29.25	2.72	10.13	7.74	94.67	13.68
83.05	6.45	3.2	29.04	1.88	9.63	5.72	5.04	44.32	20.84
95.73	16.47	115.5	5.42	16.44	21.34	33.69	33.4	3.69	100.04
103.15	89.5	2.42	77.21	109.08	104.4	115.78	6.95	19.34	112.94
94.52	13.36	1.08	72.47	2.45	2.88	0.98	6.48	13.54	20.12
26.84	44.16	76.69	0	21.97	65.9	19.45	11.64	18.89	53.21
93.93	92.97	13.62	5.64	17.12	20.55	0	83.26	1.57	15.3
90.26	99.56	88.51	6.48	19.7	3.43	27.21	15.42	0	49.24
72.78	92.62	55.99	6.95	6.93	10.44	89.27	24.49	5.2	9.44
101.36	78.89	12.24	19.6	10.05	21.36	14.49	89.13	2.8	2.67
104.8	87.71	119.63	8.11	99.03	24.42	92.17	111.14	5.1	4.69
74.01	93.57	23.31	9.01	29.24	95.52	21	92.64	3.03	41.38
56.01	0	11.05	84.77	15.21	6.72	106.59	53.9	9.16	45.77

SOLUTION

Little information can be developed just by casually reading through the 200 observations. The manager can probably see that most of the bills are under \$100, but that is likely to be the extent of the information garnered from browsing through the data. If he examines the data more carefully, he may discover that the smallest bill is \$0 and the largest is \$119.63. He has now developed some information. However, his presentation to senior executives will be most unimpressive if no other information is produced. For example, someone is likely to ask how the numbers are distributed between 0 and 119.63. Are there many small bills and few large bills? What is the "typical" bill? Are the bills somewhat similar or do they vary considerably?

To help answer these questions and others like them, the marketing manager can construct a frequency distribution from which a histogram can be drawn. In the previous section a frequency distribution was created by counting the number of times each category of the nominal variable occurred. We create a frequency distribution for interval data by counting the number of observations that fall into each of a series of intervals, called **classes**, that cover the complete range of observations. We discuss how to decide the number of classes and the upper and lower limits of the intervals later. We have chosen eight classes defined in such a way that each observation falls into one and only one class. These classes are defined as follows:

Classes

Amounts that are less than or equal to 15

Amounts that are more than 15 but less than or equal to 30

Amounts that are more than 30 but less than or equal to 45

Amounts that are more than 45 but less than or equal to 60

Amounts that are more than 60 but less than or equal to 75

Amounts that are more than 75 but less than or equal to 90

Amounts that are more than 90 but less than or equal to 105

Amounts that are more than 105 but less than or equal to 120

Notice that the intervals do not overlap, so there is no uncertainty about which interval to assign to any observation. Moreover, because the smallest number is 0 and the largest is 119.63, every observation will be assigned to a class. Finally, the intervals are equally wide. Although this is not essential, it makes the task of reading and interpreting the graph easier.

To create the frequency distribution manually, we count the number of observations that fall into each interval. Table 3.1 presents the frequency distribution.

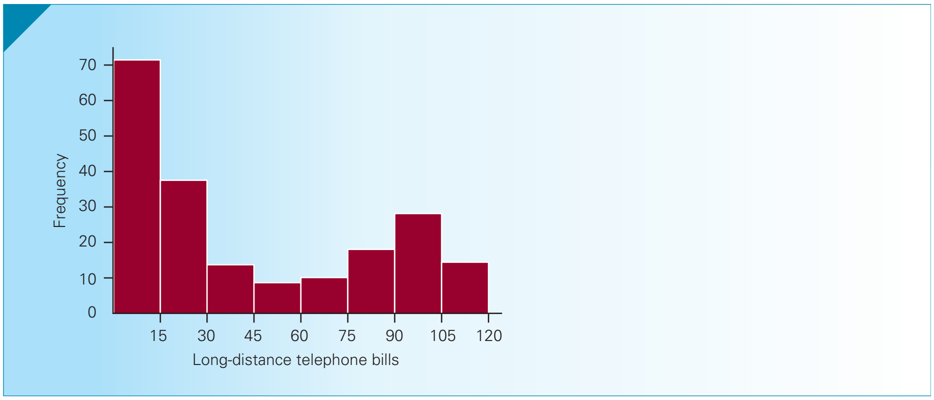
TABLE **3.1** **Frequency Distribution of the Long-Distance Bills in Example 3.1**

CLASS LIMITS	FREQUENCY
0 to 15*	71
15 to 30	37
30 to 45	13
45 to 60	9
60 to 75	10
75 to 90	18
90 to 105	28
105 to 120	14
Total	200

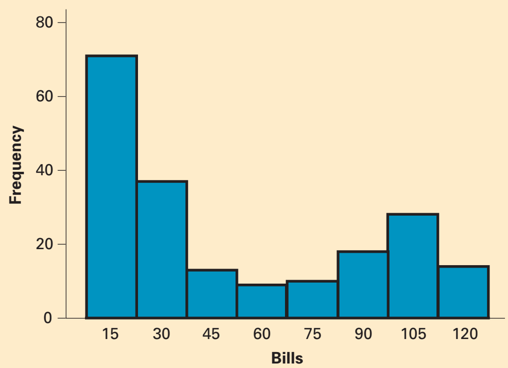
*Classes contain observations greater than their lower limits (except for the first class) and less than or equal to their upper limits.

Although the frequency distribution provides information about how the numbers are distributed, the information is more easily understood and imparted by drawing a picture or graph. The graph is called a **histogram**. A histogram is created by drawing rectangles whose bases are the intervals and whose heights are the frequencies. Figure 3.1 exhibits the histogram that was drawn by hand.

FIGURE **3.1** Histogram for Example 3.1



EXCEL



INSTRUCTIONS

1. Type or import the data into one column. (Open Xm03-01.) In another column, type the upper limits of the class intervals. Excel calls them *bins*. (You can put any name in the first row; we typed "Bills.")
2. Click **Data**, **Data Analysis**, and **Histogram.** If Data Analysis does not appear in the menu box, see our Keller's website, Appendix A1.
3. Specify the **Input Range** (A1:A201) and the **Bin Range** (B1:B9). Click **Chart Output**. Click **Labels** if the first row contains names.
4. To remove the gaps, place the cursor over one of the rectangles and click the right button of the mouse. Click (with the left button) **Format Data Series . . . .** move the pointer to **Gap Width** and use the slider to change the number from 150 to 0.

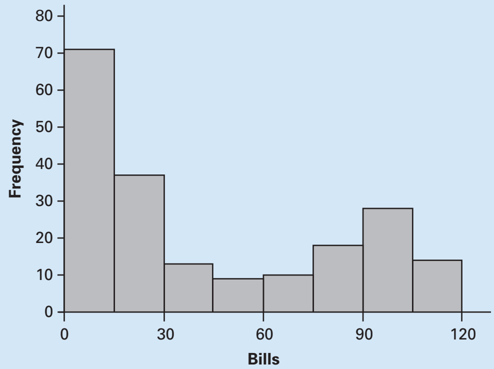
Except for the first class, Excel counts the number of observations in each class that are greater than the lower limit and less than or equal to the upper limit.

Note that the numbers along the horizontal axis represent the upper limits of each class although they appear to be placed in the centers. If you wish, you can replace these numbers with the actual midpoints by making changes to the frequency distribution in cells A1:B14 (change 15 to 7.5, 30 to 22.5, . . . , and 120 to 112.5).

You can also convert the histogram to list relative frequencies instead of frequencies. To do so, change the frequencies to relative frequencies by dividing each frequency by 200; that is, replace 71 by .355, 37 by .185, . . . , and 14 by .07.

If you have difficulty with this technique, turn to the website Appendix A2 or A3, which provides step-by-step instructions for Excel and provides troubleshooting tips.

MINITAB



Note that Minitab counts the number of observations in each class that are strictly less than their upper limits.

INSTRUCTIONS

1. Type or import the data into one column. (Open Xm03-01.)
2. Click **Graph**, **Histogram . . .** , and **Simple**.
3. Type or use the **Select** button to specify the name of the variable in the **Graph Variables** box (Bills). Click **Data View**.
4. Click **Data Display** and **Bars**. Minitab will create a histogram using its own choices of class intervals.
5. To choose your own classes, double-click the horizontal axis. Click **Binning**.
6. Under **Interval Type**, choose **Cutpoint**. Under **Interval Definition**, choose **Midpoint/Cutpoint positions** and type in your choices (0 15 30 45 60 75 90 105 120) to produce the histogram shown here.

INTERPRET

The histogram gives us a clear view of the way the bills are distributed. About half the monthly bills are small ($0 to $30), a few bills are in the middle range ($30 to $75), and a relatively large number of long-distance bills are at the high end of the range. It would appear from this sample of first-month long-distance bills that the company's customers are split unevenly between light and heavy users of long-distance telephone service. If the company assumes that this pattern will continue, it must address a number of pricing issues. For example, customers who incurred large monthly bills may be targets of competitors who offer flat rates for 15-minute or 30-minute calls. The company needs to know more about these customers. With the additional information, the marketing manager may suggest altering the company's pricing.

Determining the Number of Class Intervals

The number of class intervals we select depends entirely on the number of observations in the data set. The more observations we have, the larger the number of class intervals we need to use to draw a useful histogram. Table 3.2 provides guidelines on choosing the number of classes. In Example 3.1, we had 200 observations. The table tells us to use 7, 8, 9, or 10 classes.

TABLE **3.2** Approximate Number of Classes in Histograms

NUMBER OF OBSERVATIONS	NUMBER OF CLASSES
Less than 50	5–7
50–200	7–9
200–500	9–10
500–1,000	10–11
1,000–5,000	11–13
5,000–50,000	13–17
More than 50,000	17–20

An alternative to the guidelines listed in Table 3.2 is to use Sturges's formula, which recommends that the number of class intervals be determined by the following:

$$\text{Number of class intervals} = 1 + 3.3 \log(n)$$

For example, if n = 50 Sturges's formula becomes

$$\text{Number of class intervals} = 1 + 3.3 \log(50) = 1 + 3.3(1.7) = 6.6$$

which we round to 7.

Class Interval Widths We determine the approximate width of the classes by subtracting the smallest observation from the largest and dividing the difference by the number of classes. Thus,

$$\text{Class width} = \frac{\textit{Largest Observation} - \textit{Smallest Observation}}{\textit{Number of Classes}}$$

In Example 3.1, we calculated

$$\text{Class width} = \frac{119.63 - 0}{8} = 14.95$$

We often round the result to some convenient value. We then define our class limits by selecting a lower limit for the first class from which all other limits are determined. The only condition we apply is that the first class interval must contain the smallest observation. In Example 3.1, we rounded the class width to 15 and set the lower limit of the first class to 0. Thus, the first class is defined as "Amounts that are greater than or equal to 0 but less than or equal to 15." (Minitab users should remember that the classes are defined as the number of observations that are *strictly less* than their upper limits.)

Table 3.2 and Sturges's formula are guidelines only. It is more important to choose classes that are easy to interpret. For example, suppose that we have recorded the marks on an exam of the 100 students registered in the course where the highest mark is 94 and the lowest is 48. Table 3.2 suggests that we use 7, 8, or 9 classes, and Sturges's formula computes the approximate number of classes as

$$\text{Number of class intervals} = 1 + 3.3 \log(100) = 1 + 3.3(2) = 7.6$$

which we round to 8. Thus,

$$\text{Class width} = \frac{94 - 48}{8} = 5.75$$

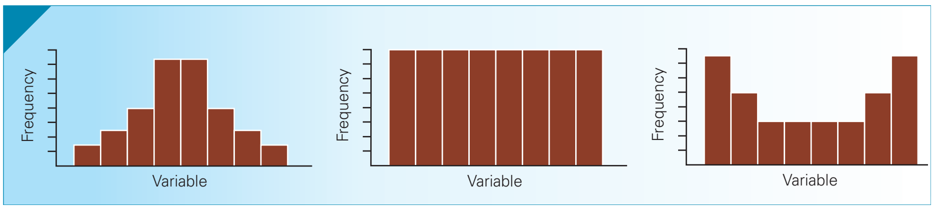
which we would round to 6. We could then produce a histogram whose upper limits of the class intervals are 50, 56, 62, . . . , 98. Because of the rounding and the way in which we defined the class limits, the number of classes is 9. However, a histogram that is easier to interpret would be produced using classes whose widths are 5; that is, the upper limits would be 50, 55, 60, . . . , 95. The number of classes in this case would be 10.

Shapes of Histograms

The purpose of drawing histograms, like that of all other statistical techniques, is to acquire information. Once we have the information, we frequently need to describe what we've learned to others. We describe the shape of histograms on the basis of the following characteristics.

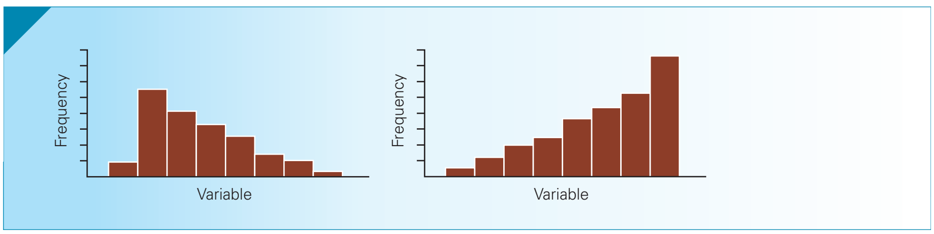
Symmetry A histogram is said to be **symmetric** if, when we draw a vertical line down the center of the histogram, the two sides are identical in shape and size. Figure 3.2 depicts three symmetric histograms.

FIGURE **3.2** Three Symmetric Histograms



Skewness A skewed histogram is one with a long tail extending to either the right or the left. The former is called **positively skewed**, and the latter is called **negatively skewed**. Figure 3.3 shows examples of both. Incomes of employees in large firms tend to be positively skewed because there is a large number of relatively low-paid workers and a small number of well-paid executives. The time taken by students to write exams is frequently negatively skewed because few students hand in their exams early; most prefer to reread their papers and hand them in near the end of the scheduled test period.

FIGURE **3.3** Positively and Negatively Skewed Histograms



Number of Modal Classes As we discuss in Chapter 4, a *mode* is the observation that occurs with the greatest frequency. A **modal class** is the class with the largest number of observations. A **unimodal histogram** is one with a single peak. The histogram in

Figure 3.4 is unimodal. A **bimodal histogram** is one with two peaks, not necessarily equal in height. Bimodal histograms often indicate that two different distributions are present. (See Example 3.4.) Figure 3.5 depicts bimodal histograms.

FIGURE **3.4** A Unimodal Histogram

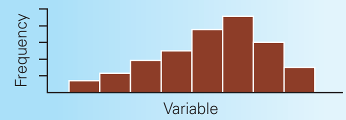
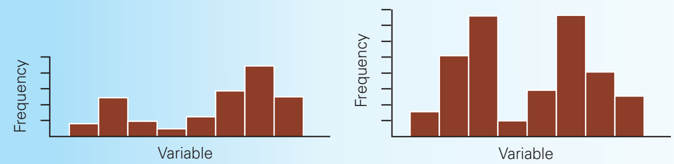
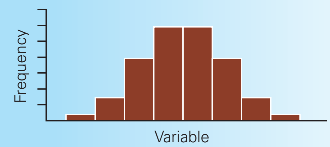


FIGURE **3.5** Bimodal Histograms



Bell Shape A special type of symmetric unimodal histogram is one that is bell shaped. In Chapter 8 we will explain why this type of histogram is important. Figure 3.6 exhibits a bell-shaped histogram.

FIGURE **3.6** Bell-Shaped Histogram



Now that we know what to look for, let's examine some examples of histograms and see what we can discover.

APPLICATIONS in **FINANCE**

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Stock and Bond Valuation

A basic understanding of how financial assets, such as stocks and bonds, are valued is critical to good financial management. Understanding the basics of valuation is necessary for capital budgeting and capital structure decisions. Moreover, understanding the basics of valuing investments such as stocks and bonds is at the heart of the huge and growing discipline known as *investment management*.

A financial manager must be familiar with the main characteristics of the capital markets where long-term financial assets such as stocks and bonds trade. A well-functioning capital market provides managers with useful information concerning the appropriate prices and rates of return that are required for a variety of financial securities with differing levels of risk. Statistical methods can be used to analyze capital markets and summarize their characteristics, such as the shape of the distribution of stock or bond returns.

APPLICATIONS in FINANCE

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Return on Investment

The return on an investment is calculated by dividing the gain (or loss) by the value of the investment. For example, a $100 investment that is worth $106 after 1 year has a 6% rate of return. A $100 investment that loses $20 has a −20% rate of return. For many investments, including individual stocks and stock portfolios (combinations of various stocks), the rate of return is a variable. In other words, the investor does not know in advance what the rate of return will be. It could be a positive number, in which case the investor makes money—or negative, and the investor loses money.

Investors are torn between two goals. The first is to maximize the rate of return on investment. The second goal is to reduce risk. If we draw a histogram of the returns for a certain investment, the location of the center of the histogram gives us some information about the return one might expect from that investment. The spread or variation of the histogram provides us with guidance about the risk. If there is little variation, an investor can be quite confident in predicting what his or her rate of return will be. If there is a great deal of variation, the return becomes much less predictable and thus riskier. Minimizing the risk becomes an important goal for investors and financial analysts.

EXAMPLE 3.2 Comparing Returns on Two Investments

DATA Xm03-02

Suppose that you are facing a decision about where to invest that small fortune that remains after you have deducted the anticipated expenses for the next year from the earnings from your summer job. A friend has suggested two types of investment, and to help make the decision you acquire some rates of return from each type. You would like to know what you can expect by way of the return on your investment, as well as other types of information, such as whether the rates are spread out over a wide range (making the investment risky) or are grouped tightly together (indicating relatively low risk). Do the data indicate that it is possible that you can do extremely well with little likelihood of a large loss? Is it likely that you could lose money (negative rate of return)?

The returns for the two types of investments are listed here. Draw histograms for each set of returns and report on your findings. Which investment would you choose and why?

Returns on Investment A

30.00	6.93	13.77	−8.55
−2.13	−13.24	22.42	−5.29
4.30	−18.95	34.40	−7.04
25.00	9.43	49.87	−12.11
12.89	1.21	22.92	12.89
−20.24	31.76	20.95	63.00
1.20	11.07	43.71	−19.27
−2.59	8.47	−12.83	−9.22
33.00	36.08	0.52	−17.00
14.26	−21.95	61.00	17.30
−15.83	10.33	−11.96	52.00
0.63	12.68	1.94	
38.00	13.09	28.45	

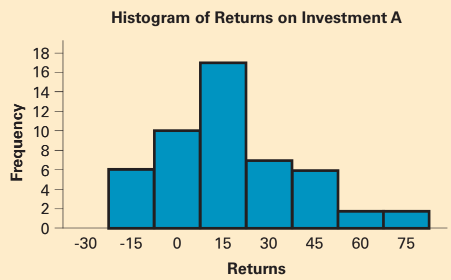
Returns on Investment B

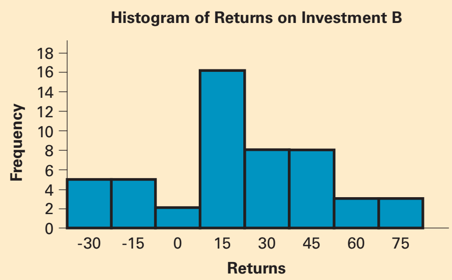
30.33	−34.75	30.31	24.3
−30.37	54.19	6.06	−10.01
−5.61	44.00	14.73	35.24
29.00	−20.23	36.13	40.7
−26.01	4.16	1.53	22.18
0.46	10.03	17.61	3.24
2.07	10.51	1.2	25.1
29.44	39.04	9.94	−24.24
11	24.76	−33.39	−38.47
−25.93	15.28	58.67	13.44
8.29	34.21	0.25	68.00
61.00	52.00	5.23	
−20.44	−32.17	66	

SOLUTION

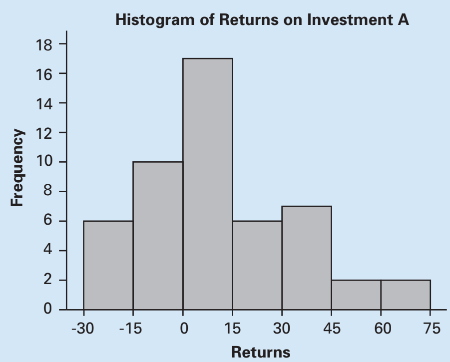
We draw the histograms of the returns on the two investments. We'll use Excel and Minitab to do the work.

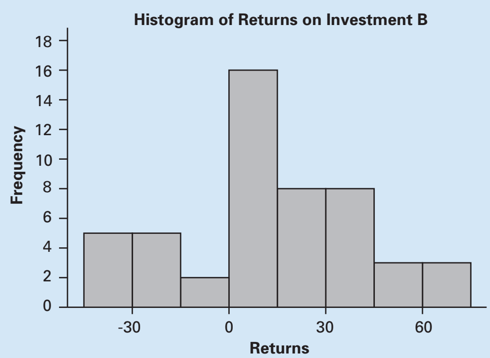
EXCEL





MINITAB





INTERPRET

Comparing the two histograms, we can extract the following information:

1. The center of the histogram of the returns of investment A is slightly lower than that for investment B.
2. The spread of returns for investment A is considerably less than that for investment B.
3. Both histograms are slightly positively skewed.

These findings suggest that investment A is superior. Although the returns for A are slightly less than those for B, the wider spread for B makes it unappealing to most investors. Both investments allow for the possibility of a relatively large return.

The interpretation of the histograms is somewhat subjective. Other viewers may not concur with our conclusion. In such cases, numerical techniques provide the detail and precision lacking in most graphs. We will redo this example in Chapter 4 to illustrate how numerical techniques compare to graphical ones.

EXAMPLE 3.3 Business Statistics Marks

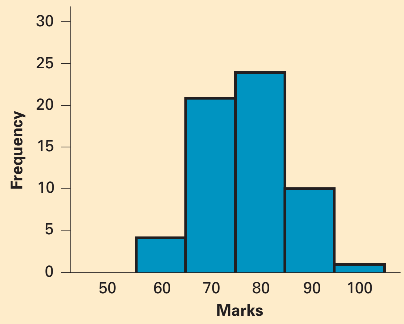
DATA Xm03-03*

A student enrolled in a business program is attending the first class of the required statistics course. The student is somewhat apprehensive because he believes the myth that the course is difficult. To alleviate his anxiety, the student asks the professor about last year's marks. The professor obliges and provides a list of the final marks, which is composed of term work plus the final exam. Draw a histogram and describe the result, based on the following marks:

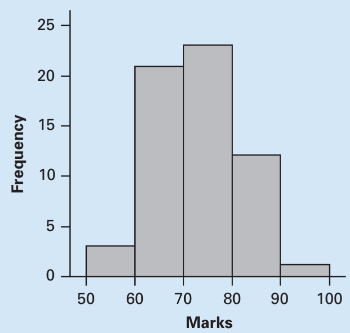
65	81	72	59
71	53	85	66
66	70	72	71
79	76	77	68
65	73	64	72
82	73	77	75
80	85	89	74
86	83	87	77
67	80	78	69
64	67	79	60
62	78	59	92
74	68	63	69
67	67	84	69
72	62	74	73
68	83	74	65

SOLUTION

EXCEL



MINITAB



INTERPRET

The histogram is unimodal and approximately symmetric. There are no marks below 50, with the great majority of marks between 60 and 90. The modal class is 70 to 80, and the center of the distribution is approximately 75.

EXAMPLE 3.4

DATA
Xm03-04*

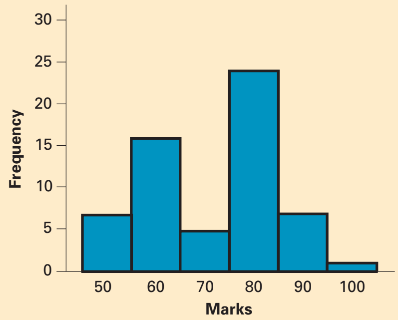
Mathematical Statistics Marks

Suppose the student in Example 3.3 obtained a list of last year's marks in a mathematical statistics course. This course emphasizes derivations and proofs of theorems. Use the accompanying data to draw a histogram and compare it to the one produced in Example 3.3. What does this histogram tell you?

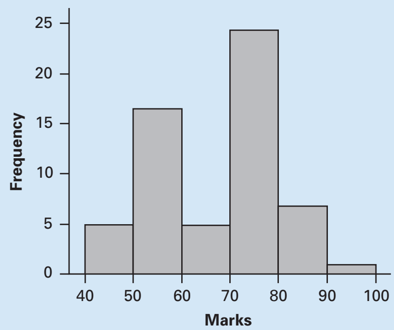
77	67	53	54
74	82	75	44
75	55	76	54
75	73	59	60
67	92	82	50
72	75	82	52
81	75	70	47
76	52	71	46
79	72	75	50
73	78	74	51
59	83	53	44
83	81	49	52
77	73	56	53
74	72	61	56
78	71	61	53

SOLUTION

EXCEL



MINITAB



INTERPRET

The histogram is bimodal. The larger modal class is composed of the marks in the 70s. The smaller modal class includes the marks that are in the 50s. There appear to be few marks in the 60s. This histogram suggests that there are two groups of students. Because of the emphasis on mathematics in the course, one may conclude that those who performed poorly in the course are weaker mathematically than those who performed well. The histograms in this example and in Example 3.3 suggest that the courses are quite different from one another and have a completely different distribution of marks.

Stem-and-Leaf Display

One of the drawbacks of the histogram is that we lose potentially useful information by classifying the observations. In Example 3.1, we learned that there are 71 observations that fall between 0 and 15. By classifying the observations we did acquire useful information. However, the histogram focuses our attention on the frequency of each class and by doing so sacrifices whatever information was contained in the actual observations. A statistician named John Tukey introduced the **stem-and-leaf display**, which is a method that to some extent overcomes this loss.

The first step in developing a stem-and-leaf display is to split each observation into two parts, a stem and a leaf. There are several different ways of doing this. For example, the number 12.3 can be split so that the stem is 12 and the leaf is 3. In this definition the stem consists of the digits to the left of the decimal and the leaf is the digit to the right of the decimal. Another method can define the stem as 1 and the leaf as 2 (ignoring the 3). In this definition the stem is the number of tens and the leaf is the number of ones. We'll use this definition to create a stem-and-leaf display for Example 3.1.

The first observation is 42.19. Thus, the stem is 4 and the leaf is 2. The second observation is 38.45, which has a stem of 3 and a leaf of 8. We continue converting each number in this way. The stem-and-leaf display consists of listing the stems 0, 1, 2, . . . , 11. After each stem, we list that stem's leaves, usually in ascending order. Figure 3.7 depicts the manually created stem-and-leaf display.

FIGURE **3.7** Stem-and-Leaf Display for Example 3.1

Stem	Leaf
0	0000000001111122222223333345555556666666778888999999
1	000001111233333334455555667889999
2	0000111112344666778999
3	001335589
4	124445589
5	33566
6	3458
7	022224556789
8	334457889999
9	00112222233344555999
10	001344446699
11	0124557889

As you can see the stem-and-leaf display is similar to a histogram turned on its side. The length of each line represents the frequency in the class interval defined by the stems. The advantage of the stem-and-leaf display over the histogram is that we can see the actual observations.

EXCEL

	A	B	C	D	E	F	G
1	**Stem & Leaf Display**						
2							
3	**Stems**	**Leaves**					
4	0	->00000000011111222222333334555555666666677888899999999					
5	1	->000001111233333334455555667889999					
6	2	->00001111123446667789999					
7	3	->001335589					
8	4	->12445589					
9	5	->33566					
10	6	->3458					
11	7	->022224556789					
12	8	->334457889999					
13	9	->00112222233344555999					
14	10	->001344446699					
15	11	->0124557889					
16							
17							

INSTRUCTIONS

1. Type or import the data into one column. (Open Xm03-01.)
2. Click **Add-ins, Data Analysis Plus**, and **Stem-and-Leaf Display**.
3. Specify the **Input Range** (A1:A201). Click one of the values of **Increment** (the increment is the difference between stems) (10).

MINITAB

```
Stem-and-Leaf Display: Bills

Stem-and-leaf of Bills  N  = 200
Leaf Unit = 1.0

 52   0   00000000011111222222333334555555666666677888899999999
 85   1   000001111233333334455555667889999
(23)  2   00001111123446667789999
 92   3   001335589
 83   4   12445589
 75   5   33566
 70   6   3458
 66   7   022224556789
 54   8   334457889999
 42   9   00112222233344555999
 22   10  001344446699
 10   11  0124557889
```

The numbers in the left column are called **depths**. Each depth counts the number of observations that are on its line or beyond. For example, the second depth is 85, which means that there are 85 observations that are less than 20. The third depth is displayed in parentheses, which indicates that the third interval contains the observation that falls in the middle of all the observations, a statistic we call the *median* (to be presented in Chapter 4). For this interval, the depth tells us the frequency of the interval; that is, 23 observations are greater than or equal to 20 but less than 30. The fourth depth is 92, which tells us that 92 observations are greater than or equal to 30. Notice that for classes below the median, the depth reports the number of observations that are less than the upper limit of that class. For classes that are above the median, the depth reports the number of observations that are greater than or equal to the lower limit of that class.

INSTRUCTIONS

1. Type or import the data into one column. (Open Xm03-01.)
2. Click **Graph** and **Stem-and-Leaf. . . .**
3. Type or use the **Select** button to specify the variable in the **Variables** box (Bills). Type the increment in the **Increment** box (10).

Ogive

The frequency distribution lists the number of observations that fall into each class interval. We can also create a **relative frequency distribution** by dividing the frequencies by the number of observations. Table 3.3 displays the relative frequency distribution for Example 3.1.

TABLE **3.3** Relative Frequency Distribution for Example 3.1

CLASS LIMITS	RELATIVE FREQUENCY
0 to 15	71/200 = .355
15 to 30	37/200 = .185
30 to 45	13/200 = .065
45 to 60	9/200 = .045
60 to 75	10/200 = .050
75 to 90	18/200 = .090
90 to 105	28/200 = .140
105 to 120	14/200 = .070
Total	200/200 = 1.0

As you can see, the relative frequency distribution highlights the proportion of the observations that fall into each class. In some situations, we may wish to highlight the proportion of observations that lie below each of the class limits. In such cases, we create the **cumulative relative frequency distribution**. Table 3.4 displays this type of distribution for Example 3.1.

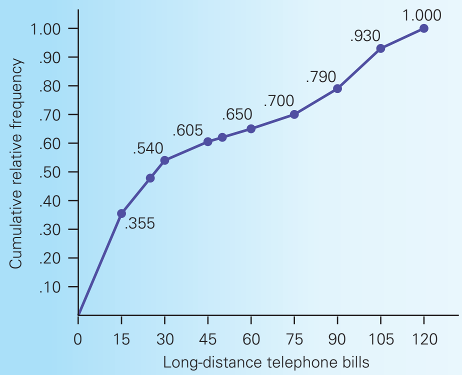
From Table 3.4, you can see that, for example, 54% of the bills were less than or equal to $30 and that 79% of the bills were less than or equal to $90.

Another way of presenting this information is the **ogive**, which is a graphical representation of the cumulative relative frequencies. Figure 3.8 is the manually drawn ogive for Example 3.1.

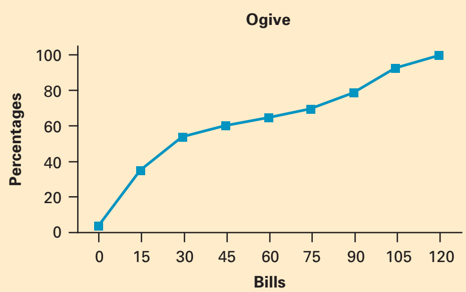
TABLE **3.4** Cumulative Relative Frequency Distribution for Example 3.1

CLASS LIMITS	RELATIVE FREQUENCY	CUMULATIVE RELATIVE FREQUENCY
0 to 15	71/200 = .355	71/200 = .355
15 to 30	37/200 = .185	108/200 = .540
30 to 45	13/200 = .065	121/200 = .605
45 to 60	9/200 = .045	130/200 = .650
60 to 75	10/200 = .05	140/200 = .700
75 to 90	18/200 = .09	158/200 = .790
90 to 105	28/200 = .14	186/200 = .930
105 to 120	14/200 = .07	200/200 = 1.00

FIGURE **3.8** Ogive for Example 3.1



EXCEL



INSTRUCTIONS

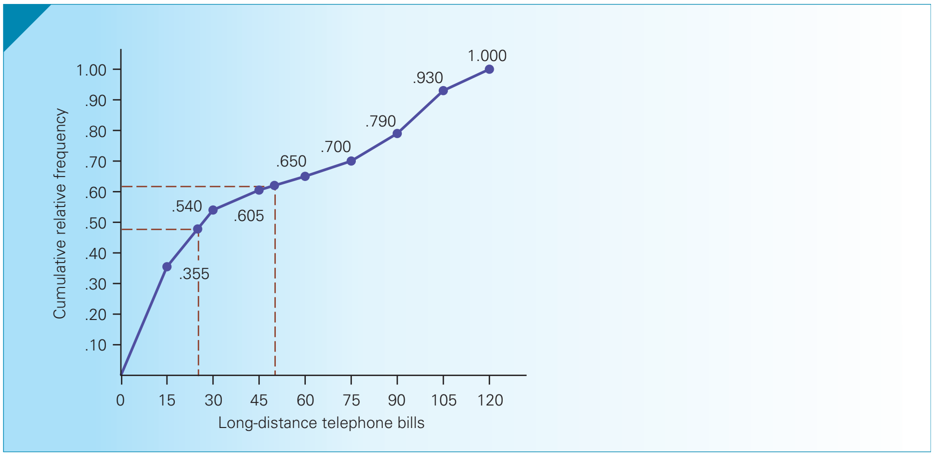
Follow instructions to create a histogram. Make the first bin's upper limit a number that is slightly smaller than the smallest number in the data set. Move the cursor to **Chart Output** and click. Do the same for **Cumulative Percentage**. Remove the "More" category. Click on any of the rectangles and click Delete. Change the **Scale**, if necessary. (Right-click the vertical or horizontal axis, click **Format Axis . . .** , and change the **Maximum** value of **Y** equal to 1.0.)

MINITAB

Minitab does not draw ogives.

We can use the ogive to estimate the cumulative relative frequencies of other values. For example, we estimate that about 62% of the bills lie below $50 and that about 48% lie below $25. (See Figure 3.9.)

FIGURE **3.9** Ogive with Estimated Relative Frequencies for Example 3.1



Here is a summary of this section's techniques.

Factors That Identify When to Use a Histogram, Ogive, or Stem-and-Leaf Display

1. Objective: Describe a single set of data
2. Data type: Interval

EXERCISES

3.1 How many classes should a histogram contain if the number of observations is 250?

3.2 Determine the number of classes of a histogram for 700 observations.

3.3 A data set consists of 125 observations that range between 37 and 188.
a. What is an appropriate number of classes to have in the histogram?
b. What class intervals would you suggest?

3.4 A statistics practitioner would like to draw a histogram of 62 observations that range from 5.2 to 6.1.
a. What is an appropriate number of class intervals?
b. Define the upper limits of the classes you would use.

3.5 Xr03-05 The number of items rejected daily by a manufacturer because of defects was recorded for the past 30 days. The results are as follows.

4	9	13	7	5	8	12	15	5	7	3
8	15	17	19	6	4	10	8	22	16	9
5	3	9	19	14	13	18	7			

a. Construct a histogram.
b. Construct an ogive.
c. Describe the shape of the histogram.

3.6 Xr03-06 The final exam in a third-year organizational behavior course requires students to write several essay-style answers. The numbers of pages for a sample of 25 exams were recorded. These data are shown here.

5	8	9	3	12	8	5	7	3	8	9	5	2
7	12	9	6	3	8	7	10	9	12	7	3	

a. Draw a histogram.
b. Draw an ogive.
c. Describe what you've learned from the answers to parts (a) and (b).

3.7 Xr03-07 A large investment firm on Wall Street wants to review the distribution of ages of its stockbrokers. The firm believes that this information can be useful in developing plans to recruit new brokers. The ages of a sample of 40 brokers are shown here.

46	28	51	34	29	40	38	33	41	52
53	40	50	33	36	41	25	38	37	41
36	50	46	33	61	48	32	28	30	49
41	37	26	39	35	39	46	26	31	35

a. Draw a stem-and-leaf display.
b. Draw a histogram.
c. Draw an ogive.
d. Describe what you have learned.

3.8 Xr03-08 The numbers of weekly sales calls by a sample of 30 telemarketers are listed here. Draw a histogram of these data and describe it.

14	8	6	12	21	4	9	3	25	17
9	5	8	18	16	3	17	19	10	15
5	20	17	14	19	7	10	15	10	8

3.9 Xr03-09 The amount of time (in seconds) needed to complete a critical task on an assembly line was measured for a sample of 50 assemblies. These data are as follows:

30.3	34.5	31.1	30.9	33.7
31.9	33.1	31.1	30.0	32.7
34.4	30.1	34.6	31.6	32.4
32.8	31.0	30.2	30.2	32.8
31.1	30.7	33.1	34.4	31.0
32.2	30.9	32.1	34.2	30.7
30.7	30.7	30.6	30.2	33.4
36.8	30.2	31.5	30.1	35.7
30.5	30.6	30.2	31.4	30.7
30.6	37.9	30.3	34.1	30.4

a. Draw a stem-and-leaf display.
b. Draw a histogram.
c. Describe the histogram.

3.10 Xr03-10 A survey of individuals in a mall asked 60 people how many stores they will enter during this visit to the mall. The responses are listed here.

3	2	4	3	3	9
2	4	3	6	2	2
8	7	6	4	5	1
5	2	3	1	1	7
3	4	1	1	4	8
0	2	5	4	4	4
6	2	2	5	3	8
4	3	1	6	9	1
4	4	1	0	4	6
5	5	5	1	4	3

a. Draw a histogram.
b. Draw an ogive.
c. Describe your findings.

3.11 Xr03-11 A survey asked 50 baseball fans to report the number of games they attended last year. The results are listed here. Use an appropriate graphical technique to present these data and describe what you have learned.

5	15	14	7	8
16	26	6	15	23
11	15	6	4	7
8	19	16	9	9
8	7	10	5	8
8	6	6	21	10
5	24	5	28	9
11	20	24	5	13
14	9	25	10	24
10	18	22	12	17

3.12 Xr03-12 To help determine the need for more golf courses, a survey was undertaken. A sample of 75 self-declared golfers was asked how many rounds of golf they played last year. These data are as follows:

18	26	16	35	30
15	18	15	18	29
25	30	35	14	20
18	24	21	25	18
29	23	15	19	27
28	9	17	28	25
23	20	24	28	36
20	30	26	12	31
13	26	22	30	29
26	17	32	36	24
29	18	38	31	36
24	30	20	13	23
3	28	5	14	24
13	18	10	14	16
28	19	10	42	22

a. Draw a histogram.
b. Draw a stem-and-leaf display.
c. Draw an ogive.
d. Describe what you have learned.

The following exercises require a computer and statistical software.

3.13 Xr03-13 The annual incomes for a sample of 200 first-year accountants were recorded. Summarize these data using a graphical method. Describe your results.

3.14 Xr03-14 The real estate board in a suburb of Los Angeles wanted to investigate the distribution of the prices (in $ thousands) of homes sold during the past year.

a Draw a histogram.
b. Draw an ogive.
c. Draw a stem-and-leaf display (if your software allows it).
d. Describe what you have learned.

3.15 Xr03-15 The number of customers entering a bank in the first hour of operation for each of the last 200 days was recorded. Use a graphical technique to extract information. Describe your findings.

3.16 Xr03-16 The lengths of time (in minutes) to serve 420 customers at a local restaurant were recorded.

a. How many bins should a histogram of these data contain?
b. Draw a histogram using the number of bins specified in part (a).
c. Is the histogram symmetric or skewed?
d. Is the histogram bell shaped?

3.17 Xr03-17 The marks of 320 students on an economics midterm test were recorded. Use a graphical technique to summarize these data. What does the graph tell you?

3.18 Xr03-18 The lengths (in inches) of 150 newborn babies were recorded. Use whichever graphical technique you judge suitable to describe these data. What have you learned from the graph?

3.19 Xr03-19 The number of copies made by an office copier was recorded for each of the past 75 days. Graph the data using a suitable technique. Describe what the graph tells you.

3.20 Xr03-20 Each of a sample of 240 tomatoes grown with a new type of fertilizer was weighed (in ounces) and recorded. Draw a histogram and describe your findings.

3.21 Xr03-21 The volume of water used by each of a sample of 350 households was measured (in gallons) and recorded. Use a suitable graphical statistical method to summarize the data. What does the graph tell you?

3.22 Xr03-22 The number of books shipped out daily by Amazon.com was recorded for 100 days. Draw a histogram and describe your findings.

APPLICATIONS in BANKING

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Credit Scorecards

Credit scorecards are used by banks and financial institutions to determine whether applicants will receive loans. The scorecard is the product of a statistical technique that converts questions about income, residence, and other variables into a score. The higher the score, the higher the probability that the applicant will repay. The scorecard is a formula produced by a statistical technique called *logistic regression*, which is available as an appendix on the Keller's website. For example, a scorecard may score age categories in the following way:

Less than 25	20 points
25 to 39	24
40 to 55	30
Over 55	38

Other variables would be scored similarly. The sum for all variables would be the applicant's score. A cutoff score would be used to predict those who will repay and those who will default. Because no scorecard is perfect, it is possible to make two types of error: granting credit to those who will default and not lending money to those who would have repaid.

(Continued)

Exercises

3.23 Xr03-23 A small bank that had not yet used a scorecard wanted to determine whether a scorecard would be advantageous. The bank manager took a random sample of 300 loans that were granted and scored each on a scorecard borrowed from a similar bank. This scorecard is based on the responses supplied by the applicants to questions such as age, marital status, and household income. The cutoff is 650, which means that those scoring below are predicted to default and those scoring above are predicted to repay. Two hundred twenty of the loans were repaid, the rest were not. The scores of those who repaid and the scores of those who defaulted were recorded.

a. Use a graphical technique to present the scores of those who repaid.
b. Use a graphical technique to present the scores of those who defaulted.
c. What have you learned about the scorecard?

3.24 Xr03-24 Refer to Exercise 3.23. The bank decided to try another scorecard, this one based not on the responses of the applicants but on credit bureau reports, which list problems such as late payments and previous defaults. The scores using the new scorecard of those who repaid and the scores of those who did not repay were recorded. The cutoff score is 650.

a. Use a graphical technique to present the scores of those who repaid.
b. Use a graphical technique to present the scores of those who defaulted.
c. What have you learned about the scorecard?
d. Compare the results of this exercise with those of Exercise 3.23. Which scorecard appears to be better?

General Social Survey Exercises

3.25 The GSS asked respondents to specify their highest year of school completed (EDUC).

a. Is this type of data interval, ordinal, or nominal?
b. GSS2008* Use a graphical technique to present these data for the 2008 survey.
c. Briefly describe your results.

3.26 GSS2008* Graphically display the results of the GSS 2008 question, On average days how many hours do you spend watching television (TVHOURS)? Briefly describe what you have discovered.

3.27 GSS2008* Employ a graphical technique to present the ages (AGE) of the respondents in the 2008 survey. Describe your results.

3.28 GSS2008* The survey in 2008 asked "If working, full- or part-time, how many hours did you work last week at all jobs (HRS)?" Summarize these data with a graphical technique.

3.2 Describing Time-Series Data

Besides classifying data by type, we can also classify them according to whether the observations are measured at the same time or whether they represent measurements at successive points in time. The former are called **cross-sectional data**, and the latter **time-series data**.

The techniques described in Section 3.1 are applied to cross-sectional data. All the data for Example 3.1 were probably determined within the same day. We can probably say the same thing for Examples 3.2 to 3.4.

To give another example, consider a real estate consultant who feels that the selling price of a house is a function of its size, age, and lot size. To estimate the specific form of the function, she samples, say, 100 homes recently sold and records the price, size, age, and lot size for each home. These data are cross-sectional: They all are observations at the same point in time. The real estate consultant is also working on a separate project to forecast the monthly housing starts in the northeastern United States over the next year. To do so, she collects the monthly housing starts in this region for each of the past 5 years. These 60 values (housing starts) represent time-series data because they are observations taken over time.

Note that the original data may be interval or nominal. All the illustrations above deal with interval data. A time series can also list the frequencies and relative frequencies of a nominal variable over a number of time periods. For example, a brand-preference survey asks consumers to identify their favorite brand. These data are nominal. If we repeat the survey once a month for several years, the proportion of consumers who prefer a certain company's product each month would constitute a time series.

Line Chart

Time-series data are often graphically depicted on a **line chart**, which is a plot of the variable over time. It is created by plotting the value of the variable on the vertical axis and the time periods on the horizontal axis.

The chapter-opening example addresses the issue of the relationship between the price of gasoline and the price of oil. We will introduce the technique we need to answer the question in Section 3.3. Another question arises: Is the recent price of gasoline high compared to the past prices?

EXAMPLE 3.5

DATA
Xm03-05

Price of Gasoline

We recorded the monthly average retail price of gasoline (in cents per gallon) since January 1976. Some of these data are displayed below. Draw a line chart to describe these data and briefly describe the results.

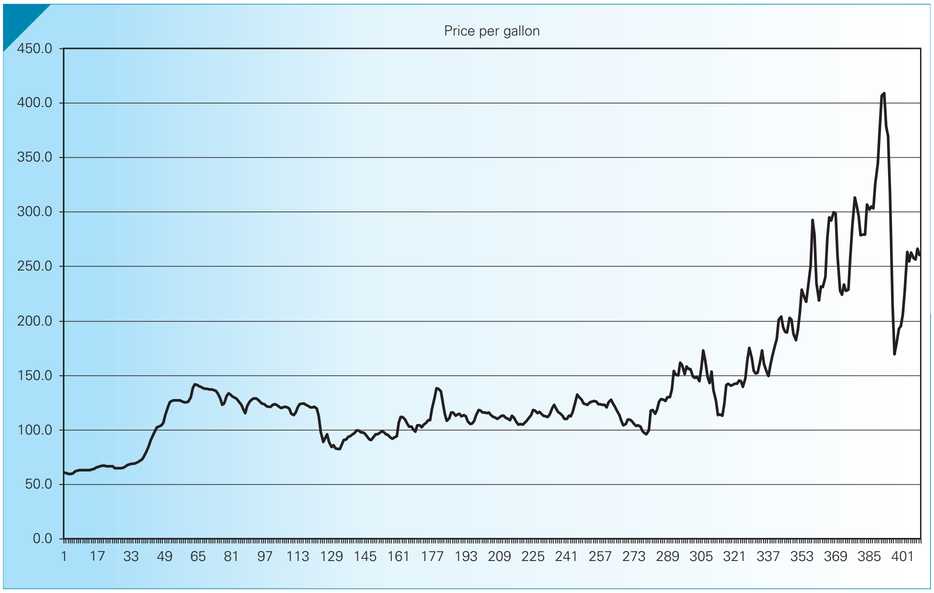
Year	Month	Price per gallon
1976	1	60.5
1976	2	60.0
1976	3	59.4
1976	4	59.2
1976	5	60.0
1976	6	61.6
1976	7	62.3
1976	8	62.8
1976	9	63.0
1976	10	62.9
1976	11	62.9
1976	12	62.6
2009	1	178.7
2009	2	192.8
2009	3	194.9
2009	4	205.6

2009	5	226.5
2009	6	263.1
2009	7	254.3
2009	8	262.7
2009	9	257.4
2009	10	256.1
2009	11	266.0
2009	12	260.0

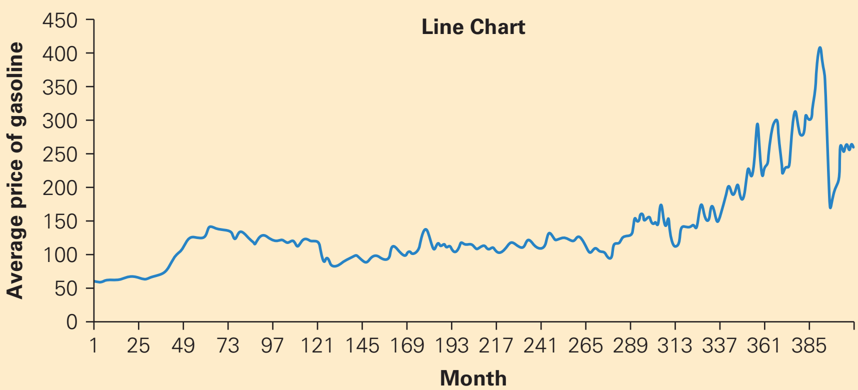
SOLUTION

Here are the line charts produced manually, and by Excel and Minitab.

FIGURE **3.10** Line Chart for Example 3.5



EXCEL

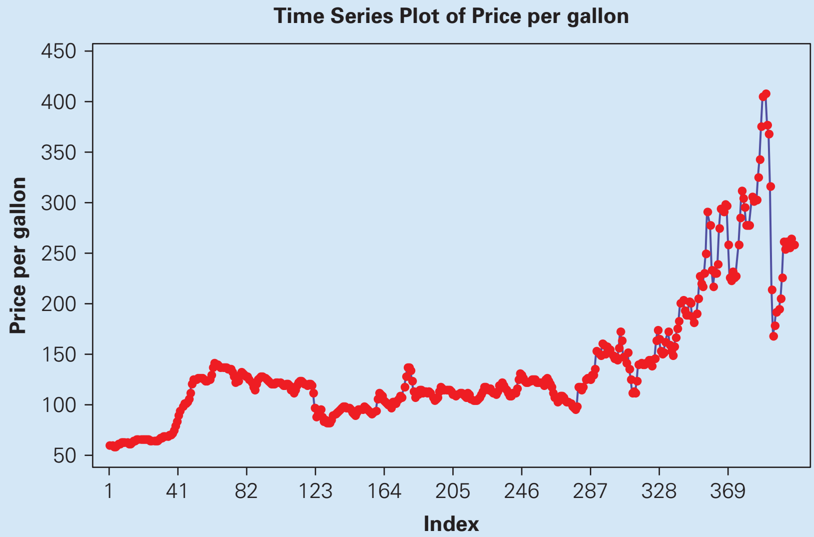


INSTRUCTIONS

1. Type or import the data into one column. (Open Xm03-05.)
2. Highlight the column of data. Click **Insert, Line,** and the first **2-D Line**. Click **Chart Tools** and **Layout** to make whatever changes you wish.

You can draw two or more line charts (for two or more variables) by highlighting all columns of data you wish to graph.

MINITAB



INSTRUCTIONS

1. Type or import the data into one column. (Open Xm03-05.)
2. Click **Graph** and **Time Series Plot . . . .** Click **Simple.**
3. In the **series** box type or use the **Select** button to specify the variable (Price). Click **Time/Scale**.
4. Click the **Time** tab, and under **Time Scale** click **Index**.

INTERPRET

The price of gasoline rose from about \$.60 to more than a dollar in the late 1970s (months 1 to 49), fluctuated between \$.90 and \$1.50 until 2000 (months 49 to 289), then generally rose with large fluctuations (months 289 to 380), then declined sharply before rallying in the last 10 months.

APPLICATIONS in **ECONOMICS**

Measuring Inflation: Consumer Price Index*

Inflation is the increase in the prices for goods and services. In most countries, inflation is measured using the Consumer Price Index (CPI). The Consumer Price Index works with a basket of some 300 goods and services in the United States (and a similar number in other countries), including such diverse items as food, housing, clothing, transportation, health, and recreation. The basket is defined for the "typical" or "average" middle-income family, and the set of items and their weights are revised periodically (every 10 years in the United States and every 7 years in Canada).

Prices for each item in this basket are computed on a monthly basis and the CPI is computed from these prices. Here is how it works. We start by setting a period of time as the base. In the United States the base is the years 1982–1984. Suppose that the basket of goods and services cost $1,000 during this period. Thus, the base is $1,000, and the CPI is set at 100. Suppose that in the next month (January 1985) the price increases to $1,010. The CPI for January 1985 is calculated in the following way:

$$\text{CPI(January 1985)} = \frac{1{,}010}{1{,}000} \times 100 = 101$$

If the price increases to $1,050 in the next month, the CPI is

$$\text{CPI(February 1985)} = \frac{1{,}050}{1{,}000} \times 100 = 105$$

The CPI, despite never really being intended to serve as the official measure of inflation, has come to be interpreted in this way by the general public. Pension-plan payments, old-age Social Security, and some labor contracts are automatically linked to the CPI and automatically indexed (so it is claimed) to the level of inflation. Despite its flaws, the CPI is used in numerous applications. One application involves adjusting prices by removing the effect of inflation, making it possible to track the "real" changes in a time series of prices.

In Example 3.5, the figures shown are the actual prices measured in what are called *current* dollars. To remove the effect of inflation, we divide the monthly prices by the CPI for that month and multiply by 100. These prices are then measured in *constant* 1982–1984 dollars. This makes it easier to see what has happened to the prices of the goods and services of interest.

We created two data sets to help you calculate prices in constant 1982–1984 dollars. File Ch03:\\CPI-Annual and Ch03:\\CPI-Monthly list the values of the CPI where 1982–1984 is set at 100 for annual values and monthly values, respectively.

*Keller's website Appendix Index Numbers, located at www.cengage.com/bstatistics/keller, describes index numbers and how they are calculated.

EXAMPLE 3.6 Price of Gasoline in 1982–1984 Constant Dollars

DATA
Xm03-06

Remove the effect of inflation in Example 3.5 to determine whether gasoline prices are higher than they have been in the past.

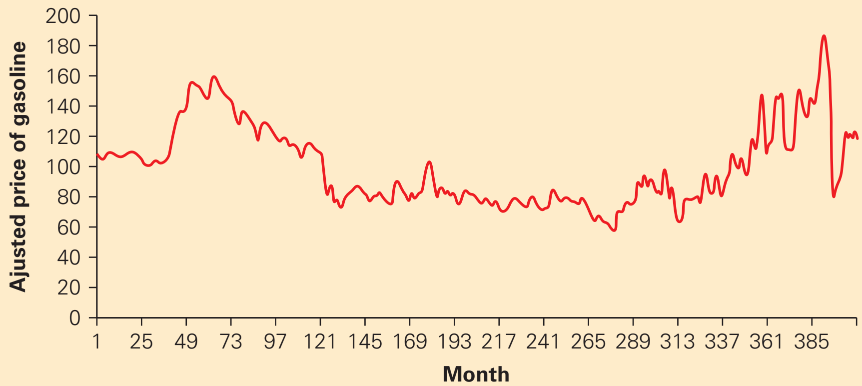
SOLUTION

Here are the 1976 and 2009 average monthly prices of gasoline, the CPI, and the adjusted prices.

The adjusted figures for all months were used in the line chart produced by Excel. Minitab's chart is similar.

Year	Month	Price per gallon	CPI	Adjusted price
1976	1	60.5	55.8	108.4
1976	2	60.0	55.9	107.3
1976	3	59.4	56.0	106.1
1976	4	59.2	56.1	105.5
1976	5	60.0	56.4	106.4
1976	6	61.6	56.7	108.6
1976	7	62.3	57.0	109.3
1976	8	62.8	57.3	109.6
1976	9	63.0	57.6	109.4
1976	10	62.9	57.9	108.6
1976	11	62.9	58.1	108.3
1976	12	62.6	58.4	107.2
2009	1	178.7	212.17	84.2
2009	2	192.8	213.01	90.5
2009	3	194.9	212.71	91.6
2009	4	205.6	212.67	96.7
2009	5	226.5	212.88	106.4
2009	6	263.1	214.46	122.7
2009	7	254.3	214.47	118.6
2009	8	262.7	215.43	121.9
2009	9	257.4	215.79	119.3
2009	10	256.1	216.39	118.4
2009	11	266.0	217.25	122.4
2009	12	260.0	217.54	119.5

EXCEL



INTERPRET

Using constant 1982–1984 dollars, we can see that the average price of a gallon of gasoline hit its peak in the middle of 2008 (month 390). From there it dropped rapidly and in late 2009 was about equal to the adjusted price in 1976.

There are two more factors to consider in judging whether the price of gasoline is high. The first is distance traveled and the second is fuel consumption. Exercise 3.41 deals with this issue.

Exercises

3.29 Xr03-29 The fees television broadcasters pay to cover the summer Olympic Games has become the largest source of revenue for the host country. Below we list the year, city, and revenue in millions of U.S. dollars paid by television broadcasters around the world. Draw a chart to describe these prices paid by the networks.

Year	City	Broadcast Revenue
1960	Rome	1.2
1964	Tokyo	1.6
1968	Mexico City	9.8
1972	Munich	17.8
1976	Montreal	34.9
1980	Moscow	88.0
1984	Los Angeles	266.9
1988	Seoul	402.6
1992	Barcelona	636.1
1996	Atlanta	898.3
2000	Sydney	1331.6
2004	Athens	1494.0
2008	Beijing	1737.0

Source: Bloomberg News.

3.30 Xr03-30 The number of females enlisted in the United States Army from 1971 to 2007 are listed here. Draw a line chart, and describe what the chart tells you.

Year	Females enlisted	Year	Females enlisted
1971	11.8	1990	71.2
1972	12.3	1991	67.8
1973	16.5	1992	61.7
1974	26.3	1993	60.2
1975	37.7	1994	59.0
1976	43.8	1995	57.3
1977	46.1	1996	59.0
1978	50.5	1997	62.4
1979	55.2	1998	61.4
1980	61.7	1999	61.5
1981	65.3	2000	62.9
1982	64.1	2001	63.4
1983	66.5	2002	63.2
1984	67.1	2003	63.5
1985	68.4	2004	61.0
1986	69.7	2005	57.9
1987	71.6	2006	58.5
1988	72.0	2007	58.8
1989	74.3		

(Continued)

Source: Statistical Abstract of the United States, 2009, Table 494.

3.31 Xr03-31 The United States spends more money on health care than any other country. To gauge how fast costs are increasing, the following table was produced, listing the total health-care expenditures in the United States annually for 1981 to 2006 (costs are in $billions).

a. Graphically present these data.
b. Use the data in CPI-Annual to remove the effect of inflation. Graph the results and describe your findings.

Year	Health expenditures	Year	Health expenditures
1981	294	1994	962
1982	331	1995	1017
1983	365	1996	1069
1984	402	1997	1125
1985	440	1998	1191
1986	472	1999	1265
1987	513	2000	1353
1988	574	2001	1470
1989	639	2002	1603
1990	714	2003	1732
1991	782	2004	1852
1992	849	2005	1973
1993	913	2006	2106

Source: Statistical Abstract of the United States, 2009, Table 124.

3.32 Xr03-32 The number of earned degrees (thousands) for males and females is listed below for the years 1987 to 2006. Graph both sets of data. What do the graphs tell you?

Year	Female	Male
1987	941	882
1988	954	881
1989	986	887
1990	1035	905
1991	1097	928
1992	1147	961
1993	1182	985
1994	1211	995
1995	1223	995
1996	1255	993
1997	1290	998
1998	1304	994
1999	1330	993
2000	1369	1016
2001	1391	1025
2002	1441	1053
2003	1517	1104
2004	1603	1152
2005	1666	1185
2006	1725	1211

Source: U.S. National Center for Education Statistics, *Statistical Abstract of the United States, 2009*, Table 288.

3.33 Xr03-33 The number of property crimes (burglary, larceny, theft, car theft) (in thousands) for the years 1992 to 2006 are listed next. Draw a line chart and interpret the results.

Year	Crimes	Year	Crimes
1992	12506	2000	10183
1993	12219	2001	10437
1994	12132	2002	10455
1995	12064	2003	10443
1996	11805	2004	10319
1997	11558	2005	10175
1998	10952	2006	9984
1999	10208		

Source: U.S. Federal Bureau of Investigation *Statistical Abstract of the United States, 2009*, Table 295.

3.34 Xr03-34 Refer to Exercise 3.33. Another way of measuring the number of property crimes is to calculate the number of crimes per 100,000 of population. This allows us to remove the effect of the increasing population. Graph these data and interpret your findings.

Year	Crimes	Year	Crimes
1992	4868	2000	3606
1993	4695	2001	3658
1994	4605	2002	3628
1995	4526	2003	3589
1996	4378	2004	3515
1997	4235	2005	3434
1998	3966	2006	3337
1999	3655		

3.35 Xr03-35 The gross national product (GNP) is the sum total of the economic output of a the citizens (nationals) of a country. It is an important measure of the wealth of a country. The following table lists the year and the GNP in billions of current dollars for the United States.

a. Graph the GNP. What have you learned?
b. Use the data in CPI-Annual to compute the per capita GNP in constant 1982–1984 dollars. Graph the results and describe your findings.

Year	GNP	Year	GNP
1980	2822	1995	7444
1981	3160	1996	7870
1982	3290	1997	8356
1983	3572	1998	8811
1984	3967	1999	9381
1985	4244	2000	9989
1986	4478	2001	10338
1987	4754	2002	10691
1988	5124	2003	11211
1989	5508	2004	11959
1990	5835	2005	12736
1991	6022	2006	13471
1992	6371	2007	14193
1993	6699	2008	14583
1994	7109		

Source: U.S. Bureau of Economic Activity.

3.36 Xr03-36 The average daily U.S. oil consumption and production (thousands of barrels) is shown for the years 1973 to 2007. Use a graphical technique to describe these figures. What does the graph tell you?

Year	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982
Consumption	17,318	16,655	16,323	17,460	18,443	18,857	18,527	17,060	16,061	15,301
Production	9,209	8,776	8,376	8,132	8,245	8,706	8,551	8,597	8,572	8,649
Year	**1983**	**1984**	**1985**	**1986**	**1987**	**1988**	**1989**	**1990**	**1991**	**1992**
Consumption	15,228	15,722	15,726	16,277	16,666	17,284	17,327	16,988	16,710	17,031
Production	8,689	8,879	8,972	8,683	8,349	8,140	7,615	7,356	7,418	7,172
Year	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**	**2000**	**2001**	**2002**
Consumption	17,328	17,721	17,730	18,308	18,618	18,913	19,515	19,699	19,647	19,758
Production	6,847	6,662	6,561	6,465	6,452	6,253	5,882	5,822	5,801	5,746
Year	**2003**	**2004**	**2005**	**2006**	**2007**					
Consumption	20,034	20,731	20,799	20,800	20,680					
Production	5,682	5,419	5,179	5,102	5,065					

Source: U.S. Department of Energy: Monthly Energy Review.

3.37 Xr03-37 Has housing been a hedge against inflation in the last 20 years? To answer this question, we produced the following table, which lists the average selling price of one-family homes in all of the United States, the Northeast, Midwest, South, and West for the years 1988 to 2007, as well as the annual CPI. For the entire country and for each area, use a graphical technique to determine whether housing prices stayed ahead of inflation.

Year	All	Northeast	Midwest	South	West	CPI
1988	89,300	143,000	68,400	82,200	124,900	118.3
1989	94,600	147,700	73,100	85,600	138,400	124.0
1990	97,300	146,200	76,700	86,300	141,200	130.7
1991	102,700	149,300	81,000	89,800	147,400	136.2
1992	105,500	149,000	84,600	92,900	143,300	140.3
1993	109,100	149,300	87,600	95,800	144,400	144.5
1994	113,500	149,300	90,900	97,200	151,900	148.2
1995	117,000	146,500	96,500	99,200	153,600	152.4
1996	122,600	147,800	102,800	105,000	160,200	156.9
1997	129,000	152,400	108,900	111,300	169,000	160.5
1998	136,000	157,100	116,300	118,000	179,500	163.0
1999	141,200	160,700	121,600	122,100	189,400	166.6
2000	147,300	161,200	125,600	130,300	199,200	172.2
2001	156,600	169,400	132,300	139,600	211,700	177.1
2002	167,600	190,100	138,300	149,700	234,300	179.9
2003	180,200	220,300	143,700	159,700	254,700	184.0
2004	195,200	254,400	151,500	171,800	289,100	188.9
2005	219,000	281,600	168,300	181,100	340,300	195.3
2006	221,900	280,300	164,800	183,700	350,500	201.6
2007	217,900	288,100	161,400	178,800	342,500	207.3

Source: *Statistical Abstract of the United States, 2009*, Table 935.

3.38 Xr03-38 How has the size of government changed? To help answer this question, we recorded the U.S. federal budget receipts and outlays (billions of current dollars) for the years 1980 to 2007.

a. Use a graphical technique to describe the receipts and outlays of the annual U.S. federal government budgets since 1980.

b. Calculate the difference between receipts and outlays. If the difference is positive the result is a surplus; if the difference is negative the result is a deficit. Graph the surplus/deficit variable and describe the results.

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989
Receipts	517.1	599.3	617.8	600.6	666.5	734.1	769.2	854.4	909.3	991.2
Outlays	590.9	678.2	745.7	808.4	851.9	946.4	990.4	1,004.1	1,064.5	1,143.6
Year	**1990**	**1991**	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**
Receipts	1,032.0	1,055.0	1,091.3	1,154.4	1,258.6	1,351.8	1,453.1	1,579.3	1,721.8	1,827.5
Outlays	1,253.2	1,324.4	1,381.7	1,409.5	1,461.9	1,515.8	1,560.5	1,601.3	1,652.6	1,701.9
Year	**2000**	**2001**	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**		
Receipts	2,025.2	1,991.2	1,853.2	1,782.3	1,880.1	2,153.9	2,407.3	2,568.2		
Outlays	1,789.1	1,863.9	2,011.0	2,159.9	2,293.0	2,472.2	2,655.4	2,730.2		

Source: *Statistical Abstract of the United States, 2009*, Table 451.

3.39 Refer to Exercise 3.38. Another way of judging the size of budget surplus/deficits is to calculate the deficit as a percentage of GNP. Use the data in Exercises 3.35 and 3.38 to calculate this variable and use a graphical technique to display the results.

3.40 Repeat Exercise 3.39 using the CPI-Annual file to convert all amounts to constant 1982–1984 dollars. Draw a line chart to show these data.

3.41 Xr03-41 Refer to Example 3.5. The following table lists the average gasoline consumption in miles per gallon (MPG) and the average distance (thousands of miles) driven by cars in each of the years 1980 to 2006. (The file contains the average price for each year, the annual CPI, fuel consumption and distance (thousands).) For each year calculate the inflation-adjusted cost per year of driving. Use a graphical technique to present the results.

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989
MPG	13.3	13.6	14.1	14.2	14.5	14.6	14.7	15.1	15.6	15.9
Distance	9.5	9.5	9.6	9.8	10.0	10.0	10.1	10.5	10.7	10.9
Year	**1990**	**1991**	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**
MPG	16.4	16.9	16.9	16.7	16.7	16.8	16.9	17.0	16.9	17.7
Distance	11.1	11.3	11.6	11.6	11.7	11.8	11.8	12.1	12.2	12.2
Year	**2000**	**2001**	**2002**	**2003**	**2004**	**2005**	**2006**			
MPG	16.9	17.1	16.9	17.0	17.1	17.2	17.1			
Distance	12.2	11.9	12.2	12.2	12.2	12.1	12.4			

Source: Statistical Abstract of the United States, 2009, Tables 1061 and 1062.

The following exercises require a computer and software.

3.42 Xr03-42 The monthly value of U.S. exports to Canada (in $millions) and imports from Canada from 1985 to 2009 were recorded. (*Source*: Federal Reserve Economic Data.)

a. Draw a line chart of U.S. exports to Canada.
b. Draw a line chart of U.S. imports from Canada.
c. Calculate the trade balance and draw a line chart.
d. What do all the charts reveal?

3.43 Xr03-43 The monthly value of U.S. exports to Japan (in $ millions) and imports from Japan from 1985 to 2009 were recorded. (*Source*: Federal Reserve Economic Data.)

a. Draw a line chart of U.S. exports to Japan.
b. Draw a line chart of U.S. imports from Japan.
c. Calculate the trade balance and draw a line chart.
d. What do all the charts reveal?

3.44 Xr03-44 The value of the Canadian dollar in U.S. dollars was recorded monthly for the period 1971 to 2009. Draw a graph of these figures and interpret your findings.

3.45 Xr03-45 The value of the Japanese yen in U.S. dollars was recorded monthly for the period 1971 to 2009. Draw a graph of these figures and interpret your findings.

3.46 Xr03-46 The Dow Jones Industrial Average was recorded monthly for the years 1950 to 2009. Use a graph to describe these numbers. (*Source: Wall Street Journal.*)

3.47 Refer to Exercise 3.46. Use the CPI-monthly file to measure the Dow Jones Industrial Average in 1982–1984 constant dollars. What have you learned?

3.3 Describing the Relationship between Two Interval Variables

Statistics practitioners frequently need to know how two interval variables are related. For example, financial analysts need to understand how the returns of individual stocks are related to the returns of the entire market. Marketing managers need to understand the relationship between sales and advertising. Economists develop statistical techniques to describe the relationship between such variables as unemployment rates and inflation. The technique is called a **scatter diagram**.

To draw a scatter diagram, we need data for two variables. In applications where one variable depends to some degree on the other variable, we label the dependent variable Y and the other, called the *independent variable*, X. For example, an individual's income depends somewhat on the number of years of education. Accordingly, we identify income as the dependent variable and label it Y, and we identify years of education as the independent variable and label it X. In other cases where no dependency is evident, we label the variables arbitrarily.

Example 3.7 Analyzing the Relationship between Price and Size of Houses

DATA Xm03-07

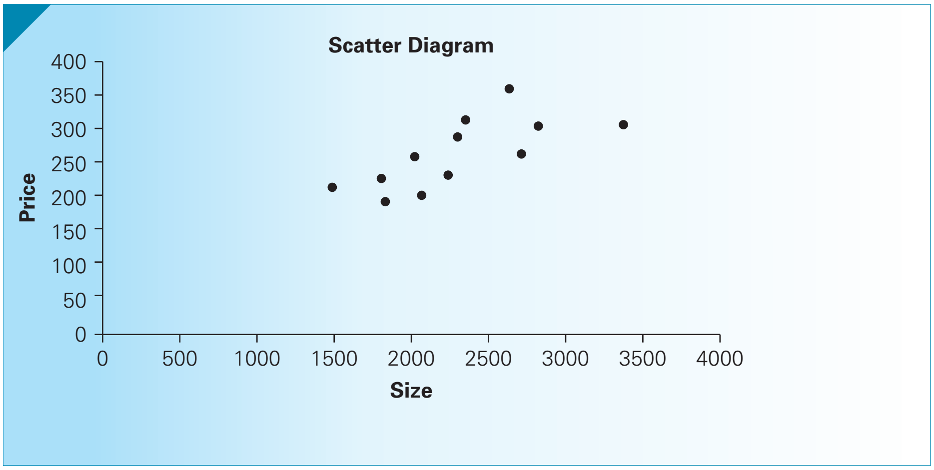
A real estate agent wanted to know to what extent the selling price of a home is related to its size. To acquire this information, he took a sample of 12 homes that had recently sold, recording the price in thousands of dollars and the size in square feet. These data are listed in the accompanying table. Use a graphical technique to describe the relationship between size and price.

Size (ft^2)	Price ($1,000)
2,354	315
1,807	229
2,637	355
2,024	261
2,241	234
1,489	216
3,377	308
2,825	306
2,302	289
2,068	204
2,715	265
1,833	195

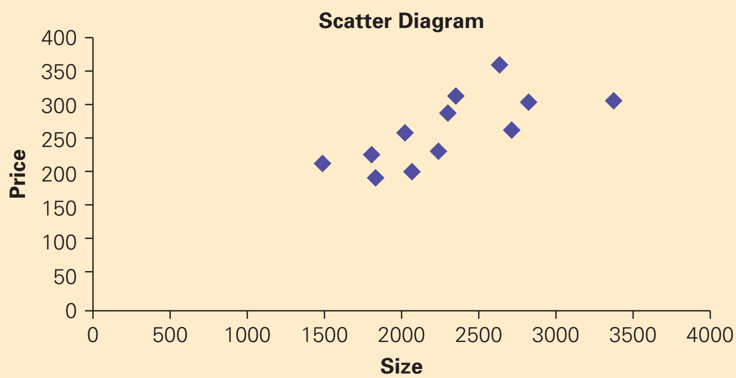
SOLUTION

Using the guideline just stated, we label the price of the house Y (dependent variable) and the size X (independent variable). Figure 3.11 depicts the scatter diagram.

FIGURE **3.11** Scatter Diagram for Example 3.7



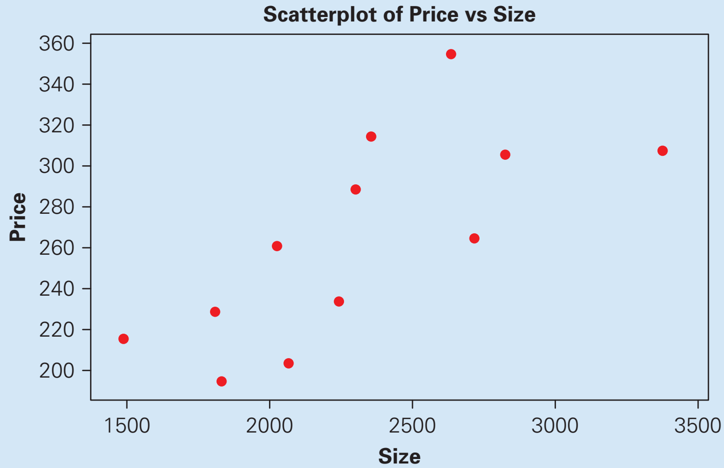
EXCEL



INSTRUCTIONS

1. Type or import the data into two adjacent columns. Store variable X in the first column and variable Y in the next column. (Open Xm03-07.)
2. Click **Insert** and **Scatter**.
3. To make cosmetic changes, click **Chart Tools** and **Layout**. (We chose to add titles and remove the gridlines.) If you wish to change the scale, click **Axes, Primary Horizontal Axis** or **Primary Vertical Axis, More Primary Horizontal** or **Vertical Axis Options . . . ,** and make the changes you want.

MINITAB



INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm03-07.)
2. Click **Graph** and **Scatterplot. . . .**
3. Click Simple.
4. Type or use the **Select** button to specify the variable to appear on the Y-axis (Price) and the X-axis (Size).

INTERPRET

The scatter diagram reveals that, in general, the greater the size of the house, the greater the price. However, there are other variables that determine price. Further analysis may reveal what these other variables are.

Patterns of Scatter Diagrams

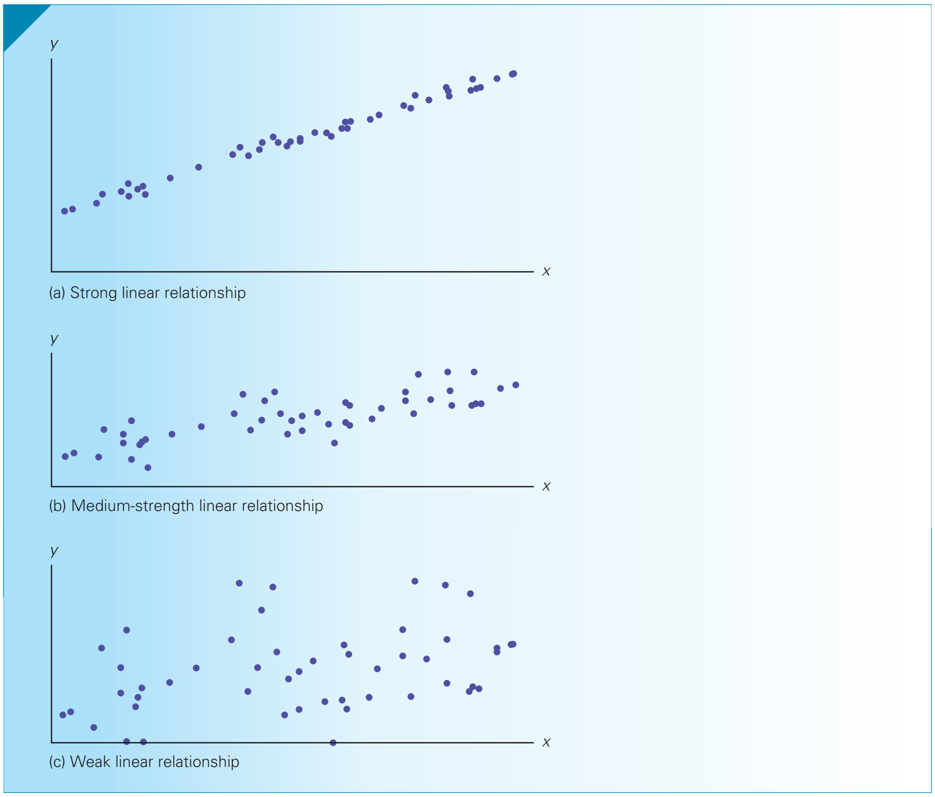
As was the case with histograms, we frequently need to describe verbally how two variables are related. The two most important characteristics are the strength and direction of the linear relationship.

Linearity

To determine the strength of the linear relationship, we draw a straight line through the points in such a way that the line represents the relationship. If most of the points fall close to the line, we say that there is a **linear relationship**. If most of the points appear to be scattered randomly with only a semblance of a straight line, there is no, or at best, a weak linear relationship. Figure 3.12 depicts several scatter diagrams that exhibit various levels of linearity.

In drawing the line freehand, we would attempt to draw it so that it passes through the middle of the data. Unfortunately, different people drawing a straight line through the same set of data will produce somewhat different lines. Fortunately, statisticians have produced an objective way to draw the straight line. The method is called the *least squares method*, and it will be presented in Chapter 4 and employed in Chapters 16, 17, and 18.

FIGURE **3.12** Scatter Diagrams Depicting Linearity



Note that there may well be some other type of relationship, such as a quadratic or exponential one.

Direction

In general, if one variable increases when the other does, we say that there is a **positive linear relationship**. When the two variables tend to move in opposite directions, we describe the nature of their association as a **negative linear relationship**. (The terms *positive* and *negative* will be explained in Chapter 4.) See Figure 3.13 for examples of scatter diagrams depicting a positive linear relationship, a negative linear relationship, no relationship, and a nonlinear relationship.

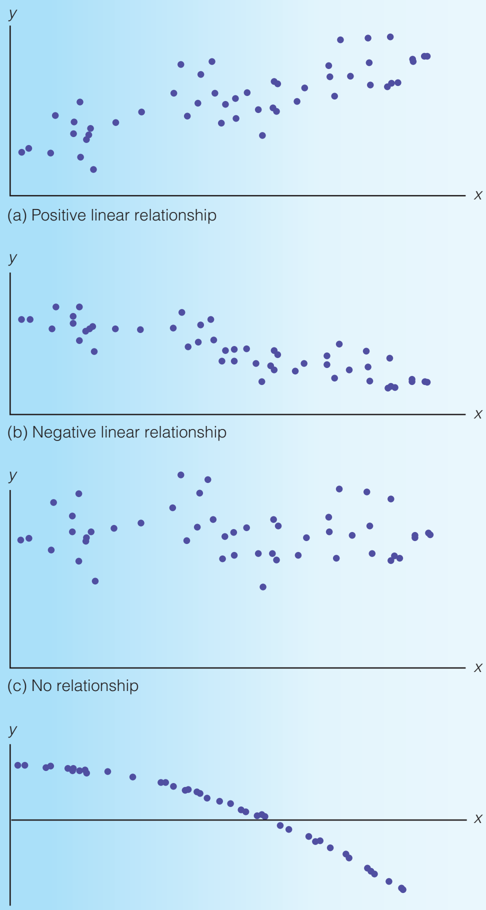
Interpreting a Strong Linear Relationship

In interpreting the results of a scatter diagram it is important to understand that if two variables are linearly related it does not mean that one is causing the other. In fact, we can never conclude that one variable causes another variable. We can express this more eloquently as

Correlation is not causation.

Now that we know what to look for, we can answer the chapter-opening example.

FIGURE **3.13** Scatter Diagrams Describing Direction



(a) Positive linear relationship

(b) Negative linear relationship

(c) No relationship

(d) Nonlinear relationship

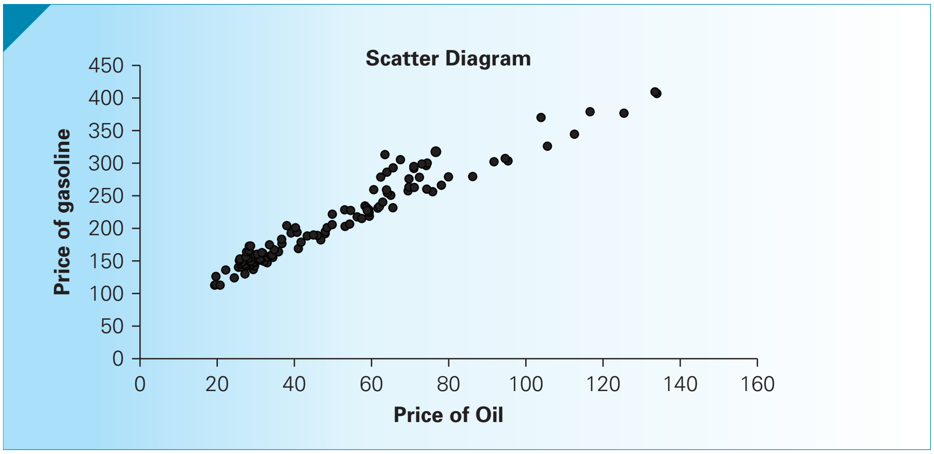
Were Oil Companies Gouging Customers 2000–2009: Solution

© Comstock Images/Jupiterimages

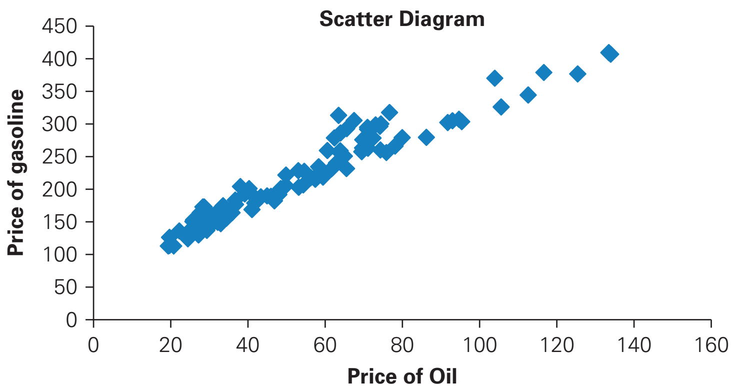
To determine whether drivers' perceptions that oil companies were gouging consumers, we need to determine whether and to what extent the two variables are related. The appropriate statistical technique is the scatter diagram.

We label the price of gasoline Y and the price of oil X. Figure 3.14 displays the scatter diagram.

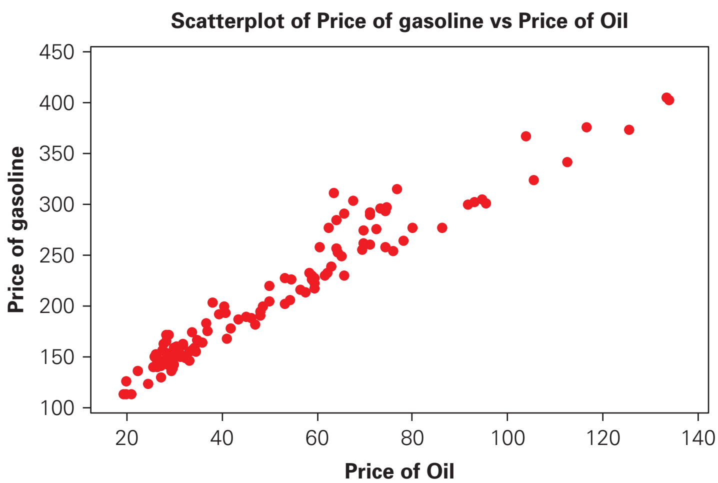
FIGURE **3.14** Scatter Diagram for Chapter-Opening Example



EXCEL



MINITAB



(Continued)

INTERPRET

The scatter diagram reveals that the two prices are strongly related linearly. When the price of oil was below $40, the relationship between the two was stronger than when the price of oil exceeded $40.

We close this section by reviewing the factors that identify the use of the scatter diagram.

Factors That Identify When to Use a Scatter Diagram

1. **Objective**: Describe the relationship between two variables
2. **Data type**: Interval

EXERCISES

3.48 Xr03-48 Between 2002 and 2005, there was a decrease in movie attendance. There are several reasons for this decline. One reason may be the increase in DVD sales. The percentage of U.S. homes with DVD players and the movie attendance (billions) in the United States for the years 2000 to 2005 are shown next. Use a graphical technique to describe the relationship between these variables.

Year	2000	2001	2002	2003	2004	2005
DVD percentage	12	23	37	42	59	74
Movie attendance	1.41	1.49	1.63	1.58	1.53	1.40

Sources: Northern Technology & Telecom Research and Motion Picture Association.

3.49 Xr03-49 Because inflation reduces the purchasing power of the dollar, investors seek investments that will provide higher returns when inflation is higher. It is frequently stated that common stocks provide just such a hedge against inflation. The annual percentage rates of return on common stock and annual inflation rates for a recent 10-year period are listed here.

Year	1	2	3	4	5	6	7	8	9	10
Returns	25	8	6	11	21	−15	12	−1	33	0
Inflation	4.4	4.2	4.1	4.0	5.2	5.0	3.8	2.1	1.7	0.2

a. Use a graphical technique to depict the relationship between the two variables.
b. Does it appear that the returns on common stocks and inflation are linearly related?

3.50 Xr03-50 In a university where calculus is a prerequisite for the statistics course, a sample of 15 students was drawn. The marks for calculus and statistics were recorded for each student. The data are as follows:

Calculus	65	58	93	68	74	81	58	85
Statistics	74	72	84	71	68	85	63	73
Calculus	88	75	63	79	80	54	72	
Statistics	79	65	62	71	74	68	73	

a. Draw a scatter diagram of the data.
b. What does the graph tell you about the relationship between the marks in calculus and statistics?

3.51 Xr03-51 The cost of repairing cars involved in accidents is one reason that insurance premiums are so high. In an experiment, 10 cars were driven into a wall. The speeds were varied between 2 and 20 mph. The costs of repair were estimated and are listed here. Draw an appropriate graph to analyze the relationship between the two variables. What does the graph tell you?

Speed	2	4	6	8	10	12
Cost of Repair ($)	88	124	358	519	699	816
Speed	14	16	18	20		
Cost of Repair ($)	905	1,521	1,888	2,201		

3.52 Xr03-52 The growing interest in and use of the Internet have forced many companies into considering ways to sell their products on the Web. Therefore, it is of interest to these companies to determine who is using the Web. A statistics practitioner undertook a study to determine how education and Internet use are connected. She took a random sample of 15 adults (20 years of age and older) and asked each to report the years of education they had completed and the number of hours of Internet use in the previous week. These data follow.

a. Employ a suitable graph to depict the data.
b. Does it appear that there is a linear relationship between the two variables? If so, describe it.

Education	11	11	8	13	17	11	11	11
Internet use	10	5	0	14	24	0	15	12
Education	19	13	15	9	15	15	11	
Internet use	20	10	5	8	12	15	0	

3.53 Xr03-53 A statistics professor formed the opinion that students who handed in quiz and exams early outperformed students who handed in their papers later. To develop data to decide whether her opinion is valid, she recorded the amount of time (in minutes) taken by students to submit their midterm tests (time limit 90 minutes) and the subsequent mark for a sample of 12 students.

Time	90	73	86	85	80	87	90	78	84	71	72	88
Mark	68	65	58	94	76	91	62	81	75	83	85	74

The following exercises require a computer and software.

3.54 Xr03-54 In an attempt to determine the factors that affect the amount of energy used, 200 households were analyzed. The number of occupants and the amount of electricity used were measured for each household.

a. Draw a graph of the data.
b. What have you learned from the graph?

3.55 Xr03-55 Many downhill skiers eagerly look forward to the winter months and fresh snowfalls. However, winter also entails cold days. How does the temperature affect skiers' desire? To answer this question, a local ski resort recorded the temperature for 50 randomly selected days and the number of lift tickets they sold. Use a graphical technique to describe the data and interpret your results.

3.56 Xr03-56 One general belief held by observers of the business world is that taller men earn more money than shorter men. In a University of Pittsburgh study, 250 MBA graduates, all about 30 years old, were polled and asked to report their height (in inches) and their annual income (to the nearest $1,000).

a. Draw a scatter diagram of the data.
b. What have you learned from the scatter diagram?

3.57 Xr03-57 Do chief executive officers (CEOs) of publicly traded companies earn their compensation? Every year the *National Post's Business* magazine attempts to answer the question by reporting the CEO's annual compensation ($1,000), the profit (or loss) ($1,000), and the three-year share return (%) for the top 50 Canadian companies. Use a graphical technique to answer the question.

3.58 Xr03-58 Are younger workers less likely to stay with their jobs? To help answer this question, a random sample of workers was selected. All were asked to report their ages and how many months they had been employed with their current employers. Use a graphical technique to summarize these data. (Adapted from *Statistical Abstract of the United States, 2006*, Table 599.)

3.59 Xr03-59 A very large contribution to profits for a movie theater is the sales of popcorn, soft drinks, and candy. A movie theater manager speculated that the longer the time between showings of a movie, the greater the sales of concession items. To acquire more information, the manager conducted an experiment. For a month he varied the amount of time between movie showings and calculated the sales. Use a graphical technique to help the manager determine whether longer time gaps produces higher concession stand sales.

3.60 Xr03-60 An analyst employed at a commodities trading firm wanted to explore the relationship between prices of grains and livestock. Theoretically, the prices should move in the same direction because, as the price of livestock increases, more livestock are bred, resulting in a greater demand for grains to feed them. The analyst recorded the monthly grains and livestock subindexes for 1971 to 2008. (Subindexes are based on the prices of several similar commodities. For example, the livestock subindex represents the prices of cattle and hogs.) Using a graphical technique, describe the relationship between the two subindexes and report your findings. (*Source:* Bridge Commodity Research Bureau.)

3.61 Xr03-61 It is generally believed that higher interest rates result in less employment because companies are more reluctant to borrow to expand their business. To determine whether there is a relationship between bank prime rate and unemployment, an economist collected the monthly prime bank rate and the monthly unemployment rate for the years 1950 to 2009. Use a graphical technique to supply your answer. (*Source:* Bridge Commodity Research Bureau.)

American National Election Survey Exercises

3.62 ANES2004* Do younger people have more education (EDUC) than older people (AGE)? Use the American National Election Survey from 2004 and a graphical technique to help answer the question.

In the 2008 survey American adults were asked to report the amount of time (in minutes) that each person spent in an average day watching, reading, or listening about news in four different media. They are

Internet (TIME1)
Television (TIME2)
Printed newspaper (TIME3)
Radio (TIME4)

3.63 ANES2008* Use a graphical technique to determine whether people who spend more time reading news on the Internet also devote more time to watching news on television.

3.64 ANES2008* Analyze the relationship between the amount time reading news on the Internet and reading news in a printed newspaper. Does it appear that they are linearly related?

3.65 ANES2008* Refer to Exercise 3.64. Study the scatter diagram. Does it appear that something is wrong with the data? If so, how do you correct the problem and determine whether a linear relationship exists?

3.66 ANES2008* Graphically describe the relationship between the amount of time watching news on television and listening to news on the radio. Are the two linearly related?

3.67 ANES2008* Do younger people spend more time reading news on the Internet than older people? Use a graphical technique to help answer the question.

Genral Social Survey Exercises

3.68 GSS2008* Do more educated people tend to marry people with more education? Draw a scatter diagram of EDUC and SPEDUC to answer the question.

3.69 GSS2008* Do the children of more educated men (PAEDUC) have more education (EDUC)? Produce a graph that helps answer the question.

3.70 GSS2008* Is there a positive linear relationship between the amount of education of mothers (MAEDUC) and their children (EDUC)? Draw a scatter diagram to answer the question.

3.71 GSS2008* If one member of a married couple works more hours (HRS) does his or her spouse work less hours (SPHRS)? Draw a graph to produce the information you need.

3.4 Art and Science of Graphical Presentations

In this chapter and in Chapter 2, we introduced a number of graphical techniques. The emphasis was on how to construct each one manually and how to command the computer to draw them. In this section, we discuss how to use graphical techniques effectively. We introduce the concept of **graphical excellence**, which is a term we apply to techniques that are informative and concise and that impart information clearly to their viewers. Additionally, we discuss an equally important concept: graphical integrity and its enemy graphical deception.

Graphical Excellence

Graphical excellence is achieved when the following characteristics apply.

1. **The graph presents large data sets concisely and coherently.** Graphical techniques were created to summarize and describe large data sets. Small data sets are easily summarized with a table. One or two numbers can best be presented in a sentence.

2. **The ideas and concepts the statistics practitioner wants to deliver are clearly understood by the viewer.** The chart is designed to describe what would otherwise be described in words. An excellent chart is one that can replace a thousand words and still be clearly comprehended by its readers.

3. **The graph encourages the viewer to compare two or more variables.** Graphs displaying only one variable provide very little information. Graphs are often best used to depict relationships between two or more variables or to explain how and why the observed results occurred.

4. **The display induces the viewer to address the substance of the data and not the form of the graph.** The form of the graph is supposed to help present the substance. If the form replaces the substance, the chart is not performing its function.

5. **There is no distortion of what the data reveal.** You cannot make statistical techniques say whatever you like. A knowledgeable reader will easily see through distortions and deception. We will endeavor to make you a knowledgeable reader by describing graphical deception later in this section.

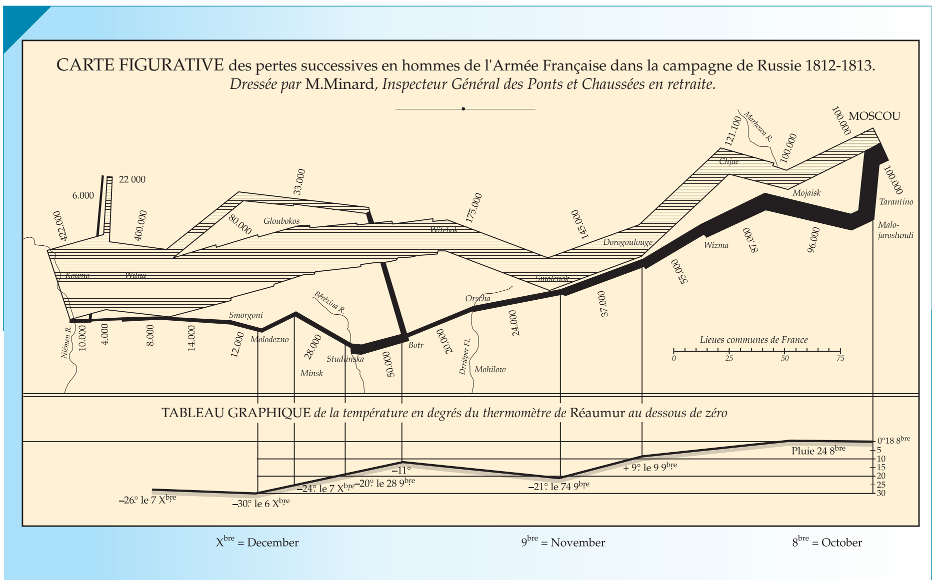
Edward Tufte, professor of statistics at Yale University, summarized graphical excellence this way:

1. Graphical excellence is the well-designed presentation of interesting data—a matter of substance, of statistics, and of design.
2. Graphical excellence is that which gives the viewer the greatest number of ideas in the shortest time with the least ink in the smallest space.
3. Graphical excellence is nearly always multivariate.
4. And graphical excellence requires telling the truth about the data.

Now let's examine the chart that has been acclaimed the best chart ever drawn.

Figure 3.15 depicts Minard's graph. The striped band is a time series depicting the size of the army at various places on the map, which is also part of the chart. When

FIGURE **3.15** Chart Depicting Napoleon's Invasion and Retreat from Russia in 1812



Source : Edward Tufte, *The Visual Display of Quantitative Information* (Cheshire, CT: Graphics Press, 1983), p. 41.

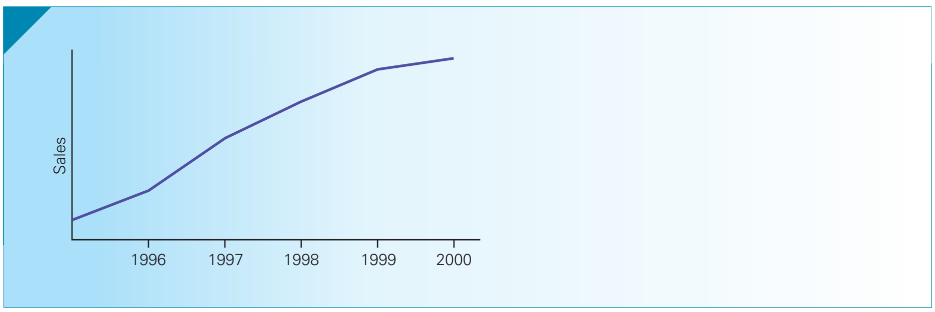
Napoleon invaded Russia by crossing the Niemen River on June 21, 1812, there were 422,000 soldiers. By the time the army reached Moscow, the number had dwindled to 100,000. At that point, the army started its retreat. The black band represents the army in retreat. At the bottom of the chart, we see the dates starting with October 1813. Just above the dates, Minard drew another time series, this one showing the temperature. It was bitterly cold during the fall, and many soldiers died of exposure. As you can see, the temperature dipped to –30 on December 6. The chart is effective because it depicts five variables clearly and succinctly.

Graphical Deception

The use of graphs and charts is pervasive in newspapers, magazines, business and economic reports, and seminars, in large part because of the increasing availability of computers and software that allow the storage, retrieval, manipulation, and summary of large masses of raw data. It is therefore more important than ever to be able to evaluate critically the information presented by means of graphical techniques. In the final analysis, graphical techniques merely create a visual impression, which is easy to distort. In fact, distortion is so easy and commonplace that in 1992 the Canadian Institute of Chartered Accountants found it necessary to begin setting guidelines for financial graphics, after a study of hundreds of the annual reports of major corporations found that 8% contained at least one misleading graph that covered up bad results. Although the heading for this section mentions deception, it is quite possible for an inexperienced person inadvertently to create distorted impressions with graphs. In any event, you should be aware of possible methods of graphical deception. This section illustrates a few of them.

The first thing to watch for is a graph without a scale on one axis. The line chart of a firm's sales in Figure 3.16 might represent a growth rate of 100% or 1% over the 5 years depicted, depending on the vertical scale. It is best simply to ignore such graphs.

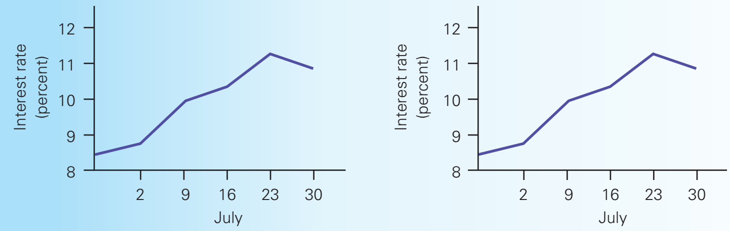
FIGURE **3.16** Graph without Scale



A second trap to avoid is being influenced by a graph's caption. Your impression of the trend in interest rates might be different, depending on whether you read a newspaper carrying caption (a) or caption (b) in Figure 3.17.

Perspective is often distorted if only absolute changes in value, rather than percentage changes, are reported. A $1 drop in the price of your $2 stock is relatively more distressing than a $1 drop in the price of your $100 stock. On January 9, 1986, newspapers throughout North America displayed graphs similar to the one shown in Figure 3.18 and reported that the stock market, as measured by the Dow Jones Industrial Average (DJIA), had suffered its worst 1-day loss ever on the previous day.

FIGURE **3.17** Graphs with Different Captions



(a) Interest rates have finally begun to turn downward.

(b) Last week provided temporary relief from the upward trend in interest rates.

The loss was 39 points, exceeding even the loss of Black Tuesday: October 28, 1929. While the loss was indeed a large one, many news reports failed to mention that the 1986 level of the DJIA was much higher than the 1929 level. A better perspective on the situation could be gained by noticing that the loss on January 8, 1986, represented a 2.5% decline, whereas the decline in 1929 was 12.8%. As a point of interest, we note that the stock market was 12% higher within 2 months of this historic drop and 40% higher 1 year later. The largest one-day percentage drop in the DJIA is 24.4% (December 12, 1914).

FIGURE **3.18** Graph Showing Drop in the DJIA

We now turn to some rather subtle methods of creating distorted impressions with graphs. Consider the graph in Figure 3.19, which depicts the growth in a firm's quarterly sales during the past year, from $100 million to $110 million. This 10% growth in quarterly sales can be made to appear more dramatic by stretching the vertical axis—a technique that involves changing the scale on the vertical axis so that a given dollar amount is represented by a greater height than before. As a result, the rise in sales appears to be greater because the slope of the graph is visually (but not numerically) steeper. The expanded scale is usually accommodated by employing a break in the vertical axis, as in Figure 3.20(a), or by truncating the vertical axis, as in Figure 3.20(b), so that the vertical scale begins at a point greater than zero. The effect of making slopes appear steeper can also be created by shrinking the horizontal axis, in which case points on the horizontal axis are moved closer together.

FIGURE 3.19 Graph Showing Growth in Quarterly Sales 1

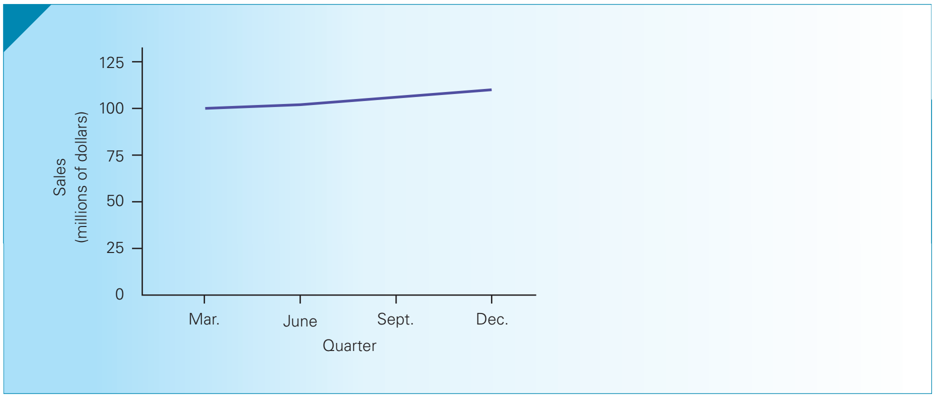
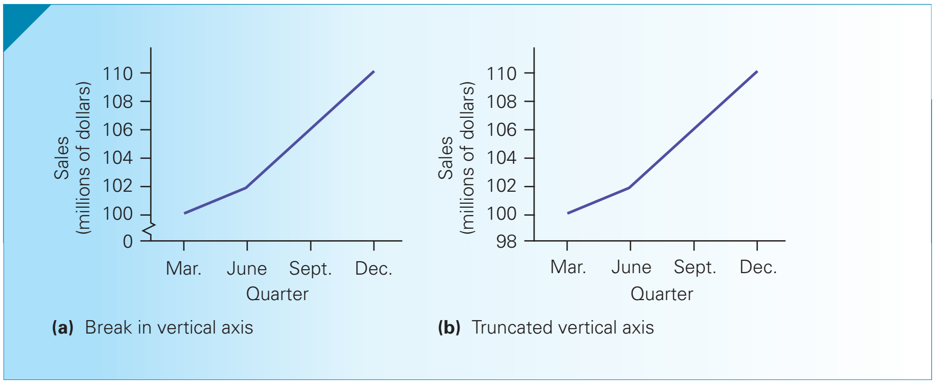
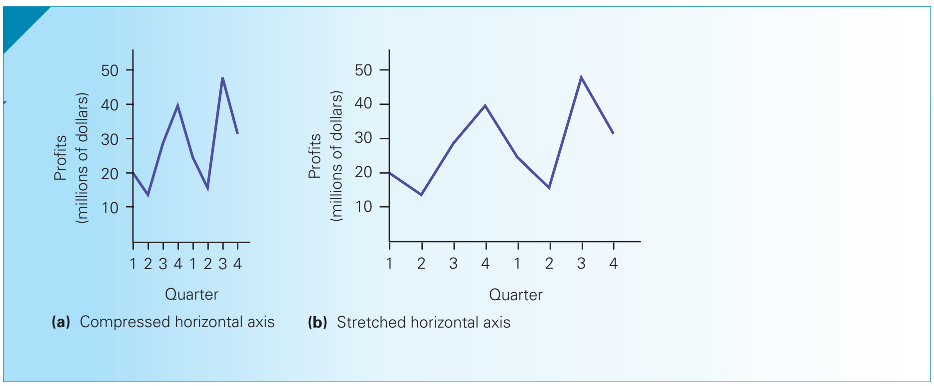


FIGURE 3.20 Graph Showing Growth in Quarterly Sales 2



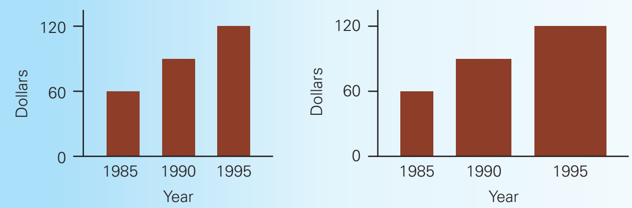
Just the opposite effect is obtained by stretching the horizontal axis; that is, spreading out the points on the horizontal axis to increase the distance between them so that slopes and trends will appear to be less steep. The graph of a firm's profits presented in Figure 3.21(a) shows considerable swings, both upward and downward in the profits from one quarter to the next. However, the firm could convey the impression of reasonable stability in profits from quarter to quarter by stretching the horizontal axis, as shown in Figure 3.21(b).

FIGURE 3.21 Graph Showing Considerable Swings or Relative Stability



Similar illusions can be created with bar charts by stretching or shrinking the vertical or horizontal axis. Another popular method of creating distorted impressions with bar charts is to construct the bars so that their widths are proportional to their heights. The bar chart in Figure 3.22(a) correctly depicts the average weekly amount spent on food by Canadian families during three particular years. This chart correctly uses bars of equal width so that both the height and the area of each bar are proportional to the expenditures they represent. The growth in food expenditures is exaggerated in Figure 3.22(b), in which the widths of the bars increase with their heights. A quick glance at this bar chart might leave the viewer with the mistaken impression that food expenditures increased fourfold over the decade, because the 1995 bar is four times the size of the 1985 bar.

FIGURE **3.22** Correct and Distorted Bar Charts

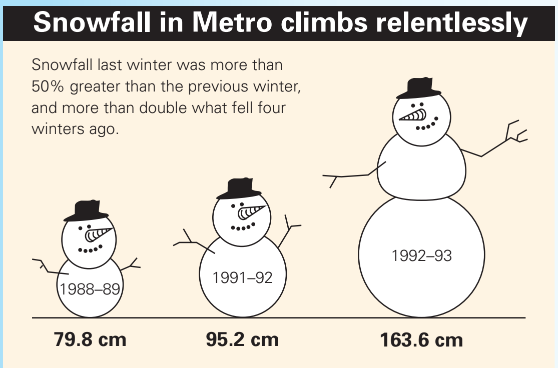


(a) Correct bar chart

(b) Increasing bar widths to create distortion

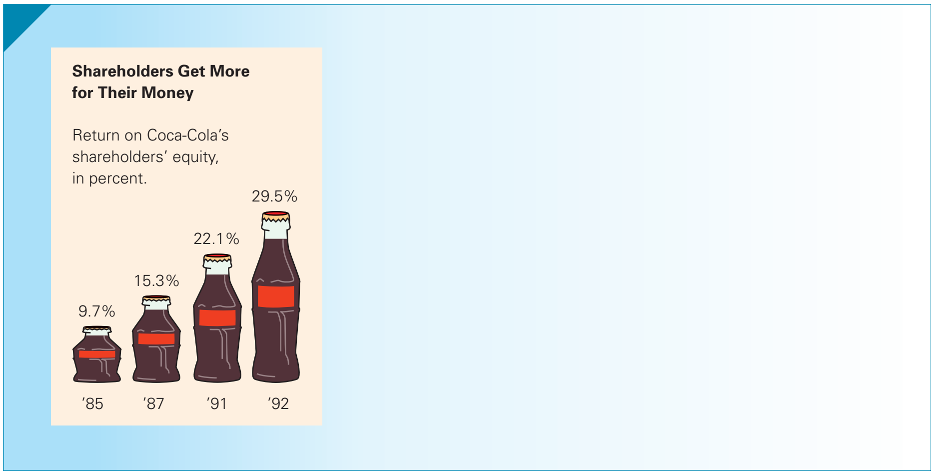
You should be on the lookout for size distortions, particularly in pictograms, which replace the bars with pictures of objects (such as bags of money, people, or animals) to enhance the visual appeal. Figure 3.23 displays the misuse of a pictogram—the snowman grows in width as well as height. The proper use of a pictogram is shown in Figure 3.24, which effectively uses pictures of Coca-Cola bottles.

FIGURE **3.23** Misuse of Pictogram



The preceding examples of creating a distorted impression using graphs are not exhaustive, but they include some of the more popular methods. They should also serve to make the point that graphical techniques are used to create a visual impression, and

FIGURE **3.24** Correct Pictogram



the impression you obtain may be a distorted one unless you examine the graph with care. You are less likely to be misled if you focus your attention on the numerical values that the graph represents. Begin by carefully noting the scales on both axes; graphs with unmarked axes should be ignored completely.

EXERCISES

3.72 Xr03-72 A computer company has diversified its operations into financial services, construction, manufacturing, and hotels. In a recent annual report, the following tables were provided. Create charts to present these data so that the differences between last year and the previous year are clear. (*Note:* It may be necessary to draw the charts manually.)

	Sales (Millions of Dollars) by Region	
Region	**Last Year**	**Previous Year**
United States	67.3	40.4
Canada	20.9	18.9
Europe	37.9	35.5
Australasia	26.2	10.3
Total	152.2	105.1

	Sales (Millions of Dollars) by Division	
Division	**Last Year**	**Previous Year**
Customer service	54.6	43.8
Library systems	49.3	30.5
Construction/property management	17.5	7.7
Manufacturing and distribution	15.4	8.9
Financial systems	9.4	10.9
Hotels and clubs	5.9	3.4

3.73 Xr03-73 The following table lists the number (in thousands) of violent crimes and property crimes committed annually in 1985 to 2006 (the last year data were available).

a. Draw a chart that displays both sets of data.
b. Does it appear that crime rates are decreasing? Explain.
c. Is there another variable that should be included to show the trends in crime rates?

Year	Violent crimes	Property crimes
1985	1,328	11,103
1986	1,489	11,723
1987	1,484	12,025
1988	1,566	12,357

1989	1,646	12,605
1990	1,820	12,655
1991	1,912	12,961
1992	1,932	12,506
1993	1,926	12,219
1994	1,858	12,132
1995	1,799	12,064
1996	1,689	11,805
1997	1,636	11,558
1998	1,534	10,952
1999	1,426	10,208
2000	1,425	10,183
2001	1,439	10,437
2002	1,424	10,455
2003	1,384	10,443
2004	1,360	10,319
2005	1,391	10,175
2006	1,418	9,984

Source: Statistical Abstract of the United States, 2006, Table 293; and *2009,* Table 293.

3.74 Xr03-74 Refer to Exercise 3.73. We've added the United States population.

a. Incorporate this variable into your charts to show crime rate trends.
b. Summarize your findings.
c. Can you think of another demographic variable that may explain crime rate trends?

3.75 Xr03-75 Refer to Exercises 3.73 and 3.74. We've included the number of Americans aged 15 to 24.

a. What is the significance of adding the populations aged 15 to 24?
b. Include these data in your analysis. What have you discovered?

3.76 Xr03-76 To determine premiums for automobile insurance, companies must have an understanding of the variables that affect whether a driver will have an accident. The age of the driver may top the list of variables. The following table lists the number of drivers in the United States, the number of fatal accidents, and the number of total accidents in each age group in 2002.

a. Calculate the accident rate (per driver) and the fatal accident rate (per 1,000 drivers) for each age group.
b. Graphically depict the relationship between the ages of drivers, their accident rates, and their fatal accident rates (per 1,000 drivers).
c. Briefly describe what you have learned.

Age Group	Number of Drivers (1,000s)	Number of Accidents (1,000s)	Number of Fatal Accidents
Under 20	9,508	3,543	6,118
20–24	16,768	2,901	5,907
25–34	33,734	7,061	10,288
35–44	41,040	6,665	10,309
45–54	38,711	5,136	8,274
55–64	25,609	2,775	5,322
65–74	15,812	1,498	2,793
Over 74	12,118	1,121	3,689
Total	193,300	30,700	52,700

Source: National Safety Council.

3.77 Xr03-77 During 2002 in the state of Florida, a total of 365,474 drivers were involved in car accidents. The accompanying table breaks down this number by the age group of the driver and whether the driver was injured or killed. (There were actually 371,877 accidents, but the driver's age was not recorded in 6,413 of these.)

a. Calculate the injury rate (per 100 accidents) and the death rate (per accident) for each age group.
b. Graphically depict the relationship between the ages of drivers, their injury rate (per 100 accidents), and their death rate.
c. Briefly describe what you have learned from these graphs.
d. What is the difference between the information extracted from Exercise 3.9 and this one?

Age Group	Number of Accidents	Drivers Injured	Drivers Killed
20 or less	52,313	21,762	217
21–24	38,449	16,016	185
25–34	78,703	31,503	324
35–44	76,152	30,542	389
45–54	54,699	22,638	260
55–64	31,985	13,210	167
65–74	18,896	7,892	133
75–84	11,526	5,106	138
85 or more	2,751	1,223	65
Total	365,474	149,892	1,878

Source: Florida Department of Highway Safety and Motor Vehicles.

3.78 Xr03-78 The accompanying table lists the average test scores in the Scholastic Assessment Test (SAT) for the years 1967, 1970, 1975, 1980, 1985, 1990, 1995, and 1997 to 2007.

Year	Verbal All	Verbal Male	Verbal Female	Math All	Math Male	Math Female
1967	543	540	545	516	535	595
1970	537	536	538	512	531	493
1975	512	515	509	498	518	479
1980	502	506	498	492	515	473
1985	509	514	503	500	522	480
1990	500	505	496	501	521	483
1995	504	505	502	506	525	490
1997	505	507	503	511	530	494
1998	505	509	502	512	531	496
1999	505	509	502	511	531	495
2000	505	507	504	514	533	498
2001	506	509	502	514	533	498
2002	504	507	502	516	534	500
2003	507	512	503	519	537	503
2004	508	512	504	518	537	501
2005	508	513	505	520	538	504
2006	503	505	502	518	536	502
2007	502	504	502	515	533	499

Source: Statistical Abstract of the United States, 2003, Table 264; *2006*, Table 252; *2009*, Table 258.

Draw a chart for each of the following.

a. You wish to show that both verbal and mathematics test scores for all students have not changed much over the years.
b. The exact opposite of part (a).
c. You want to claim that there are no differences between genders.
d. You want to "prove" that differences between genders exist.

3.79 Xr03-79 The monthly unemployment rate in one state for the past 12 months is listed here.

a. Draw a bar chart of these data with 6.0% as the lowest point on the vertical axis.
b. Draw a bar chart of these data with 0.0% as the lowest point on the vertical axis.
c. Discuss the impression given by the two charts.
d. Which chart would you use? Explain.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Rate	7.5	7.6	7.5	7.3	7.2	7.1	7.0	6.7	6.4	6.5	6.3	6.0

3.80 Xr03-80 The accompanying table lists the federal minimum wage from 1955 to 2007. The actual and adjusted minimum wages (in constant 1996 dollars) are listed.

a. Suppose you wish to show that the federal minimum wage has grown rapidly over the years. Draw an appropriate chart.
b. Draw a chart to display the actual changes in the federal minimum wage.

Year	Current Dollars	Constant 1996 Dollars	Year	Current Dollars	Constant 1996 Dollars
1955	0.75	4.39	1982	3.35	5.78
1956	1.00	5.77	1983	3.35	5.28
1957	1.00	5.58	1984	3.35	5.06
1958	1.00	5.43	1985	3.35	4.88
1959	1.00	5.39	1986	3.35	4.80
1960	1.00	5.30	1987	3.35	4.63
1961	1.15	6.03	1988	3.35	4.44
1962	1.15	5.97	1989	3.35	4.24
1963	1.25	6.41	1990	3.80	4.56
1964	1.25	6.33	1991	4.25	4.90
1965	1.25	6.23	1992	4.25	4.75
1966	1.25	6.05	1993	4.25	4.61
1967	1.40	6.58	1994	4.25	4.50

1968	1.60	7.21	1995	4.25	4.38
1969	1.60	6.84	1996	4.75	4.75
1970	1.60	6.47	1997	5.15	5.03
1971	1.60	6.20	1998	5.15	4.96
1972	1.60	6.01	1999	5.15	4.85
1973	1.60	5.65	2000	5.15	4.69
1974	2.00	6.37	2001	5.15	4.56
1975	2.10	6.12	2002	5.15	4.49
1976	2.30	6.34	2003	5.15	4.39
1977	2.30	5.95	2004	5.15	4.28
1978	2.65	6.38	2005	5.15	4.14
1979	2.90	6.27	2006	5.15	4.04
1980	3.10	5.90	2007	5.85	4.41
1981	3.35	5.78			

Source: U.S. Department of Labor.

3.81 Xr03-81 The following table shows school enrollment (in thousands) for public and private schools for the years 1965 to 2005.

a. Draw charts that allow you to claim that enrollment in private schools is "skyrocketing."
b. Draw charts that "prove" public school enrollment is stagnant.

Year	Public_K_8	Private_K_8	Public_9_12	Private_9_12	College_Public	College_Private
1965	30,563	4,900	11,610	1,400	3,970	1,951
1966	31,145	4,800	11,894	1,400	4,349	2,041
1967	31,641	4,600	12,250	1,400	4,816	2,096
1968	32,226	4,400	12,718	1,400	5,431	2,082
1969	32,513	4,200	13,037	1,300	5,897	2,108
1970	32,558	4,052	13,336	1,311	6,428	2,153
1971	32,318	3,900	13,753	1,300	6,804	2,144
1972	31,879	3,700	13,848	1,300	7,071	2,144
1973	31,401	3,700	14,044	1,300	7,420	2,183
1974	30,971	3,700	14,103	1,300	7,989	2,235
1975	30,515	3,700	14,304	1,300	8,835	2,350
1976	29,997	3,825	14,314	1,342	8,653	2,359
1977	29,375	3,797	14,203	1,343	8,847	2,439
1978	28,463	3,732	14,088	1,353	8,786	2,474
1979	28,034	3,700	13,616	1,300	9,037	2,533
1980	27,647	3,992	13,231	1,339	9,457	2,640
1981	27,280	4,100	12,764	1,400	9,647	2,725
1982	27,161	4,200	12,405	1,400	9,696	2,730
1983	26,981	4,315	12,271	1,400	9,683	2,782
1984	26,905	4,300	12,304	1,400	9,477	2,765
1985	27,034	4,195	12,388	1,362	9,479	2,768
1986	27,420	4,116	12,333	1,336	9,714	2,790
1987	27,933	4,232	12,076	1,247	9,973	2,793
1988	28,501	4,036	11,687	1,206	10,161	2,894
1989	29,152	4,035	11,390	1,163	10,578	2,961
1990	29,878	4,084	11,338	1,150	10,845	2,974
1991	30,506	4,518	11,541	1,163	11,310	3,049
1992	31,088	4,528	11,735	1,148	11,385	3,102
1993	31,504	4,536	11,961	1,132	11,189	3,116
1994	31,898	4,624	12,213	1,162	11,134	3,145
1995	32,341	4,721	12,500	1,197	11,092	3,169
1996	32,764	4,720	12,847	1,213	11,121	3,247

1997	33,073	4,726	13,054	1,218	11,196	3,306
1998	33,346	4,748	13,193	1,240	11,138	3,369
1999	33,488	4,765	13,369	1,254	11,309	3,482
2000	33,688	4,878	13,515	1,292	11,753	3,560
2001	33,938	4,993	13,734	1,326	12,233	3,695
2002	34,116	4,886	14,067	1,334	12,752	3,860
2003	34,202	4,761	14,338	1,338	12,857	4,043
2004	34,178	4,731	14,617	1,356	12,980	4,292
2005	34,205	4,699	14,909	1,374	13,022	4,466

Source: Statistical Abstract of the United States, 2009, Table 211.

3.82 Xr03-82 The following table lists the percentage of single and married women in the United States who had jobs outside the home during the period 1970 to 2007.

a. Construct a chart that shows that the percentage of married women who are working outside the home has not changed much in the past 47 years.

b. Use a chart to show that the percentage of single women in the workforce has increased "dramatically."

Year	Single	Married	Year	Single	Married
1970	56.8	40.5	1989	68.0	57.8
1971	56.4	40.6	1990	66.7	58.4
1972	57.5	41.2	1991	66.2	58.5
1973	58.6	42.3	1992	66.2	59.3
1974	59.5	43.3	1993	66.2	59.4
1975	59.8	44.3	1994	66.7	60.7
1976	61.0	45.3	1995	66.8	61.0
1977	62.1	46.4	1996	67.1	61.2
1978	63.7	47.8	1997	67.9	61.6
1979	64.6	49.0	1998	68.5	61.2
1980	64.4	49.8	1999	68.7	61.2
1981	64.5	50.5	2000	68.9	61.1
1982	65.1	51.1	2001	68.1	61.2
1983	65.0	51.8	2002	67.4	61.0
1984	65.6	52.8	2003	66.2	61.0
1985	66.6	53.8	2004	65.9	60.5
1986	67.2	54.9	2005	66.0	60.7
1987	67.4	55.9	2006	65.7	61.0
1988	67.7	56.7	2007	65.3	61.0

Source: Statistical Abstract of the United States, 2009, Table 286.

CHAPTER SUMMARY

Histograms are used to describe a single set of interval data. Statistics practitioners examine several aspects of the shapes of histograms. These are symmetry, number of modes, and its resemblance to a bell shape.

We described the difference between time-series data and cross-sectional data. Time series are graphed by line charts.

To analyze the relationship between two interval variables, we draw a scatter diagram. We look for the direction and strength of the linear relationship.

IMPORTANT TERMS

COMPUTER OUTPUT AND INSTRUCTIONS

Graphical Technique	Excel	Minitab
Histogram	47	48
Stem-and-leaf display	58	58
Ogive	60	61
Line chart	66	67
Scatter diagram	75	76

Chapter Exercises

The following exercises require a computer and software.

3.83 Xr03-83 Gold and other precious metals have traditionally been considered a hedge against inflation. If this is true, we would expect that a fund made up of precious metals (gold, silver, platinum, and others) would have a strong positive relationship with the inflation rate. To see whether this is true, a statistics practitioner collected the monthly CPI and the monthly precious metals subindex, which is based on the prices of gold, silver, platinum, etc., for the years 1975 to 2008. These figures were used to calculate the monthly inflation rate and the monthly return on the precious metals subindex. Use a graphical technique to determine the nature of the relationship between the inflation rate and the return on the subindex. What does the graph tell you? (*Source:* U.S. Treasury and Bridge Commodity Research Bureau.)

3.84 Xr03-84 The monthly values of one Australian dollar measured in American dollars since 1971 were recorded. Draw a graph that shows how the exchange rate has varied over the 38-year period. (*Source*: Federal Reserve Economic Data.)

3.85 Xr03-85 Studies of twins may reveal more about the "nature or nurture" debate. The issue being debated is whether nature or the environment has more of an effect on individual traits such as intelligence. Suppose that a sample of identical twins was selected and their IQs measured. Use a suitable graphical technique to depict the data, and describe what it tells you about the relationship between the IQs of identical twins.

3.86 Xr03-86 An economist wanted to determine whether a relationship existed between interest rates and currencies (measured in U.S. dollars). He recorded the monthly interest rate and the currency indexes for the years 1982 to 2008. Graph the data and describe the results. (*Source:* Bridge Commodity Research Bureau.)

3.87 Xr03-87 One hundred students who had reported that they use their computers for at least 20 hours per week were asked to keep track of the number of crashes their computers incurred during a 12-week period. Using an appropriate statistical method, summarize the data. Describe your findings.

3.88 Xr03-88 In Chapters 16, 17, and 18, we introduce regression analysis, which addresses the relationships among variables. One of the first applications of regression analysis was to analyze the relationship between the heights of fathers and sons. Suppose that a sample of 80 fathers and sons was drawn. The heights of the fathers and of the adult sons were measured.

a. Draw a scatter diagram of the data. Draw a straight line that describes the relationship.
b. What is the direction of the line?
c. Does it appear that there is a linear relationship between the two variables?

3.89 Xr03-89 When the Dow Jones Industrial Averages index increases, it usually means that the economy is growing, which in turn usually means that the unemployment rate is low. A statistics professor pointed out that in numerous periods (including when this edition was being written), the stock market had been booming while the rest of the economy was performing poorly. To learn more about the issue, the monthly closing DJIA and the monthly unemployment rates were recorded for the years 1950 to 2009. Draw a graph of the data and report your results. (*Source:* Federal Reserve Economic Data and the *Wall Street Journal*.)

3.90 Xr03-90 The monthly values of one British pound measured in American dollars since 1987 were recorded. Produce a graph that shows how the exchange rate has varied over the past 23 years. (*Source*: Federal Reserve Economic Data.)

3.91 Xr03-91 Do better golfers play faster than poorer ones? To determine whether a relationship exists, a sample of 125 foursomes was selected. Their total scores and the amount of time taken to complete the 18 holes were recorded. Graphically depict the data, and describe what they tell you about the relationship between score and time.

3.92 Xr03-92 The value of monthly U.S. exports to Mexico and imports from Mexico (in $ millions) since 1985 were recorded. (*Source*: Federal Reserve Economic Data.)

a. Draw a chart that depicts exports.
b. Draw a chart that exhibits imports.
c. Compute the trade balance and graph these data.
d. What do these charts tell you?

3.93 Xr03-93 An increasing number of consumers prefer to use debit cards in place of cash or credit cards. To analyze the relationship between the amounts of purchases made with debit and credit cards, 240 people were interviewed and asked to report the amount of money spent on purchases using debit cards and the amount spent using credit cards during the last month. Draw a graph of the data and summarize your findings.

3.94 Xr03-94 Most publicly traded companies have boards of directors. The rate of pay varies considerably. A survey was undertaken by the *Globe and Mail* (February 19, 2001) wherein 100 companies were surveyed and asked to report how much their directors were paid annually. Use a graphical technique to present these data.

3.95 Xr03-95 Refer to Exercise 3.94. In addition to reporting the annual payment per director, the survey recorded the number of meetings last year. Use a graphical technique to summarize and present these data.

3.96 Xr03-96 Is airline travel becoming safer? To help answer this question, a student recorded the number of fatal accidents and the number of deaths that occurred in the years 1986 to 2007 for scheduled airlines. Use a graphical method to answer the question. (*Source: Statistical Abstract of the United States, 2009*, Table 1036.)

3.97 Xr03-97 Most car-rental companies keep their cars for about a year and then sell them to used car dealerships. Suppose one company decided to sell the used cars themselves. Because most used car buyers make their decision on what to buy and how much to spend based on the car's odometer reading, this would be an important issue for the car-rental company. To develop information about the mileage shown on the company's rental cars, the general manager took a random sample of 658 customers and recorded the average number of miles driven per day. Use a graphical technique to display these data.

3.98 Xr03-98 Several years ago, the Barnes Exhibit toured major cities all over the world, with millions of people flocking to see it. Dr. Albert Barnes was a wealthy art collector who accumulated a large number of impressionist masterpieces; the total exceeds 800 paintings. When Dr. Barnes died in 1951, he stated in his will that his collection was not to be allowed to tour. However, because of the deterioration of the exhibit's home near Philadelphia, a judge ruled that the collection could go on tour to raise enough money to renovate the building. Because of the size and value of the collection, it was predicted (correctly) that in each city a large number of people would come to view the paintings. Because space was limited, most galleries had to sell tickets that were valid at one time (much like a play). In this way, they were able to control the number of visitors at any one time. To judge how many people to let in at any time, it was necessary to know the length of time people would spend at the exhibit; longer times would dictate smaller audiences; shorter times would allow for the sale of more tickets. The manager of a gallery that will host the exhibit realized her facility can comfortably and safely hold about

250 people at any one time. Although the demand will vary throughout the day and from weekday to weekend, she believes that the demand will not drop below 500 at any time. To help make a decision about how many tickets to sell, she acquired the amount of time a sample of 400 people spent at the exhibit from another city. What ticket procedure should the museum management institute?

The following exercises are based on data sets that include additional data referenced in previously presented examples and exercises.

3.99 Xm03-03* Xm03-04* Examples 3.3 and 3.4 listed the final marks in the business statistics course and the mathematical statistics course. The professor also provided the final marks in the first-year required calculus course. Graphically describe the relationship between calculus and statistics marks. What information were you able to extract?

3.100 Xm03-03* Xm03-04* In addition to the previously discussed data in Examples 3.3 and 3.4, the professor listed the midterm mark. Conduct an analysis of the relationship between final exam mark and midterm mark in each course. What does this analysis tell you?

3.101 Xr02-54* Two other questions were asked in Exercise 2.54:

Number of weeks job searching?
Salary ($ thousands)?

The placement office wants the following:

a. Graphically describe salary.
b. Is salary related to the number of weeks needed to land the job?

CASE 3.1 The Question of Global Warming

DATA
C03-01a
C03-01b

In the last part of the 20th century, scientists developed the theory that the planet was warming and that the primary cause was the increasing amounts of atmospheric carbon dioxide (CO_2), which are the product of burning oil, natural gas, and coal (fossil fuels). Although many climatologists believe in the so-called greenhouse effect, many others do not subscribe to this theory. There are three critical questions that need to be answered in order to resolve the issue.

1. Is Earth actually warming? To answer this question, we need accurate temperature measurements over a large number of years. But how do we measure the temperature before the invention of accurate thermometers? Moreover, how do we go about measuring Earth's temperature even with accurate thermometers?
2. If the planet is warming, is there a human cause or is it natural fluctuation? Earth's temperature has increased and decreased many times in its long history. We've had higher temperatures, and we've had lower temperatures, including various ice ages. In fact, a period called the "Little Ice Age" ended around the middle to the end of the 19th century. Then the temperature rose until about 1940, at which point it decreased until 1975. In fact, an April 28, 1975, *Newsweek* article discussed the possibility of global cooling, which seemed to be the consensus among scientists.
3. If the planet is warming, is CO_2 the cause? There are greenhouse gases in the atmosphere, without which Earth would be considerably colder. These gases include methane, water vapor, and carbon dioxide. All occur naturally in nature. Carbon dioxide is vital to our life on Earth because it is necessary for growing plants. The amount of CO_2 produced by fossil fuels is a relatively small proportion of all the CO_2 in the atmosphere.

The generally accepted procedure is to record monthly temperature anomalies. To do so, we calculate the average for each month over many years. We then calculate any deviations between the latest month's temperature reading and its average. A positive anomaly would represent a month's temperature that is above the average. A negative anomaly indicates a month where the temperature is less than the average. One key question is how we measure the temperature.

Although there are many different sources of data, we have chosen to provide you with one, the National Climatic Data Center (NCDC), which is affiliated with the National Oceanic and

(Case 3.1 continued)

Atmospheric Administration (NOAA). (Other sources tend to agree with the NCDC's data.) C03-01a stores the monthly temperature anomalies from 1880 to 2009.

The best measures of CO_2 levels in the atmosphere come from the Mauna Loa Observatory in Hawaii, which has measured this variable since December 1958. However, attempts to estimate CO_2 levels prior to 1958 are as controversial as the methods used to estimate temperatures. These techniques include taking ice-core samples from the arctic and measuring the amount of carbon dioxide trapped in the ice from which estimates of atmospheric CO_2 are produced. To avoid this controversy, we will use the Mauna Loa Observatory numbers only. These data are stored in file C03-01b. (Note that some of the original data are missing and were replaced by interpolated values.)

1. Use whichever techniques you wish to determine whether there is global warming.
2. Use a graphical technique to determine whether there is a relationship between temperature anomalies and CO_2 levels.

CASE 3.2 Economic Freedom and Prosperity

DATA
C03-02a
C03-02b

Adam Smith published *The Wealth of Nations* in 1776. In that book he argued that when institutions protect the liberty of individuals, greater prosperity results for all. Since 1995, the *Wall Street Journal* and the Heritage Foundation, a think tank in Washington, D.C., have produced the Index of Economic Freedom for all countries in the world. The index is based on a subjective score for 10 freedoms: business freedom, trade freedom, fiscal freedom, government size, monetary freedom, investment freedom, financial freedom, property rights, freedom from corruption, and labor freedom. We downloaded the scores for the years 1995 to 2009 and stored them in C03-02a. From the *CIA Factbook*, we determined the gross domestic product (GDP), measured in terms purchasing power parity (PPP), which makes it possible to compare the GDP for all countries. The GDP PPP figures for 2008 (the latest year available) are stored in C03-02b. Use the 2009 Freedom Index scores, the GDP PPP figures, and a graphical technique to see how freedom and prosperity are related.

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4

NUMERICAL DESCRIPTIVE TECHNIQUES

The Cost of One More Win in Major League Baseball

© AP Photo/Charles Krupa

DATA
Xm04-00

In the era of free agency, professional sports teams must compete for the services of the best players. It is generally believed that only teams whose salaries place them in the top quarter have a chance of winning the championship. Efforts have been made to provide balance by establishing salary caps or some form of equalization. To examine the problem, we gathered data from the 2009 baseball season. For each team in major league baseball, we recorded the number of wins and the team payroll.

To make informed decisions, we need to know how the number of wins and the team payroll are related. After the statistical technique is presented, we return to this problem and solve it.

Introduction

In Chapters 2 and 3, we presented several graphical techniques that describe data. In this chapter we introduce numerical descriptive techniques that allow the statistics practitioner to be more precise in describing various characteristics of a sample or population. These techniques are critical to the development of statistical inference.

As we pointed out in Chapter 2, arithmetic calculations can be applied to interval data only. Consequently, most of the techniques introduced here may be used only to numerically describe interval data. However, some of the techniques can be used for ordinal data, and one of the techniques can be employed for nominal data.

When we introduced the histogram, we commented that there are several bits of information that we look for. The first is the location of the center of the data. In Section 4.1, we will present **measures of central location**. Another important characteristic that we seek from a histogram is the spread of the data. The spread will be measured more precisely by measures of variability, which we present in Section 4.2. Section 4.3 introduces measures of relative standing and another graphical technique, the box plot.

In Section 3.3, we introduced the scatter diagram, which is a graphical method that we use to analyze the relationship between two interval variables. The numerical counterparts to the scatter diagram are called *measures of linear relationship*, and they are presented in Section 4.4.

Sections 4.5 and 4.6 feature statistical applications in baseball and finance, respectively. In Section 4.7, we compare the information provided by graphical and numerical techniques. Finally, we complete this chapter by providing guidelines on how to explore data and retrieve information.

Sample Statistic or Population Parameter

Recall the terms introduced in Chapter 1: population, sample, parameter, and statistic. A parameter is a descriptive measurement about a population, and a statistic is a descriptive measurement about a sample. In this chapter, we introduce a dozen descriptive measurements. For each one, we describe how to calculate both the population parameter and the sample statistic. However, in most realistic applications, populations are very large—in fact, virtually infinite. The formulas describing the calculation of parameters are not practical and are seldom used. They are provided here primarily to teach the concept and the notation. In Chapter 7, we introduce probability distributions, which describe populations. At that time we show how parameters are calculated from probability distributions. In general, small data sets of the type we feature in this book are samples.

4.1 Measures of Central Location

Arithmetic Mean

There are three different measures that we use to describe the center of a set of data. The first is the best known, the *arithmetic mean*, which we'll refer to simply as the **mean**. Students may be more familiar with its other name, the *average.* The mean is computed by summing the observations and dividing by the number of observations.

We label the observations in a sample $x_1, x_2, \ldots, x_n$, where x_1 is the first observation, x_2 is the second, and so on until x_n, where n is the sample size. As a result, the sample mean is denoted $\bar{x}$. In a population, the number of observations is labeled N and the population mean is denoted by μ (Greek letter *mu*).

Mean

Population mean: $$\mu = \frac{\sum_{i=1}^{N} x_i}{N}$$

Sample mean: $$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

EXAMPLE 4.1 Mean Time Spent on the Internet

A sample of 10 adults was asked to report the number of hours they spent on the Internet the previous month. The results are listed here. Manually calculate the sample mean.

0	7	12	5	33	14	8	0	9	22

SOLUTION

Using our notation, we have $x_1 = 0$, $x_2 = 7, \ldots, x_{10} = 22$, and $n = 10$. The sample mean is

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{0 + 7 + 12 + 5 + 33 + 14 + 8 + 0 + 9 + 22}{10} = \frac{110}{10} = 11.0$$

EXAMPLE 4.2 Mean Long-Distance Telephone Bill

DATA
Xm03-01

Refer to Example 3.1. Find the mean long-distance telephone bill.

SOLUTION

To calculate the mean, we add the observations and divide the sum by the size of the sample. Thus,

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{42.19 + 38.45 + \cdots + 45.77}{200} = \frac{8717.52}{200} = 43.59$$

Using the Computer

There are several ways to command Excel and Minitab to compute the mean. If we simply want to compute the mean and no other statistics, we can proceed as follows.

EXCEL

INSTRUCTIONS

Type or import the data into one or more columns. (Open Xm03-01.) Type into any empty cell

=**AVERAGE**([Input range])

For Example 4.2, we would type into any cell

=**AVERAGE**(A1:A201)

The active cell would store the mean as 43.5876.

MINITAB

INSTRUCTIONS

1. Type or import the data into one column. (Open Xm03-01.)
2. Click **Calc** and **Column Statistics . . .** . Specify **Mean** in the **Statistic** box. Type or use the **Select** button to specify the **Input variable** and click **OK**. The sample mean is outputted in the session window as 43.5876.

Median

The second most popular measure of central location is the *median*.

Median

The **median** is calculated by placing all the observations in order (ascending or descending). The observation that falls in the middle is the median. The sample and population medians are computed in the same way.

When there is an even number of observations, the median is determined by averaging the two observations in the middle.

EXAMPLE 4.3

Median Time Spent on Internet

Find the median for the data in Example 4.1.

SOLUTION

When placed in ascending order, the data appear as follows:

0 0 5 7 8 9 12 14 22 33

The median is the average of the fifth and sixth observations (the middle two), which are 8 and 9, respectively. Thus, the median is 8.5.

EXAMPLE 4.4 Median Long-Distance Telephone Bill

DATA
Xm03-01

Find the median of the 200 observations in Example 3.1.

SOLUTION

All the observations were placed in order. We observed that the 100th and 101st observations are 26.84 and 26.97, respectively. Thus, the median is the average of these two numbers:

$$\text{Median} = \frac{26.84 + 26.97}{2} = 26.905$$

EXCEL

INSTRUCTIONS

To calculate the median, substitute **MEDIAN** in place of **AVERAGE** in the instructions for the mean (page 100). The median is reported as 26.905.

MINITAB

INSTRUCTIONS

Follow the instructions for the mean to compute the mean except click **Median** instead of **Mean**. The median is outputted as 26.905 in the session window.

INTERPRET

Half the observations are below 26.905, and half the observations are above 26.905.

Mode

The third and last measure of central location that we present here is the *mode*.

Mode

The **mode** is defined as the observation (or observations) that occurs with the greatest frequency. Both the statistic and parameter are computed in the same way.

For populations and large samples, it is preferable to report the **modal class**, which we defined in Chapter 2.

There are several problems with using the mode as a measure of central location. First, in a small sample it may not be a very good measure. Second, it may not be unique.

EXAMPLE 4.5 Mode Time Spent on Internet

Find the mode for the data in Example 4.1.

SOLUTION

All observations except 0 occur once. There are two 0s. Thus, the mode is 0. As you can see, this is a poor measure of central location. It is nowhere near the center of the data. Compare this with the mean 11.0 and median 8.5 and you can appreciate that in this example the mean and median are superior measures.

EXAMPLE 4.6 Mode of Long-Distance Bill

DATA Xm03-01

Determine the mode for Example 3.1.

SOLUTION

An examination of the 200 observations reveals that, except for 0, it appears that each number is unique. However, there are 8 zeroes, which indicates that the mode is 0.

EXCEL

INSTRUCTIONS

To compute the mode, substitute **MODE** in place of **AVERAGE** in the previous instructions. Note that if there is more than one mode, Excel prints only the smallest one, without indicating whether there are other modes. In this example, Excel reports that the mode is 0.

MINITAB

Follow the instructions to compute the mean except click **Mode** instead of **Mean**. The mode is outputted as 0 in the session window. (See page 20.)

Excel and Minitab: Printing All the Measures of Central Location plus Other Statistics Both Excel and Minitab can produce the measures of central location plus a variety of others that we will introduce in later sections.

EXCEL

Excel Output for Examples 4.2, 4.4, and 4.6

	A	B
1	*Bills*	
2		
3	Mean	43.59
4	Standard Error	2.76
5	Median	26.91
6	Mode	0
7	Standard Deviation	38.97
8	Sample Variance	1518.64
9	Kurtosis	-1.29
10	Skewness	0.54
11	Range	119.63
12	Minimum	0
13	Maximum	119.63
14	Sum	8717.5
15	Count	200

Excel reports the mean, median, and mode as the same values we obtained previously. Most of the other statistics will be discussed later.

INSTRUCTIONS

1. Type or import the data into one column. (Open Xm03-01.)
2. Click **Data, Data Analysis**, and **Descriptive Statistics.**
3. Specify the **Input Range** (A1:A201) and click **Summary statistics**.

MINITAB

Minitab Output for Examples 4.2, 4.4, and 4.6

Descriptive Statistics: Bills

Variable	Mean	Median	Mode	N for Mode
Bills	43.59	26.91	0	8

INSTRUCTIONS

1. Type or import the data into one column. (Open Xm03-01.)
2. Click **Stat, Basic Statistics,** and **Display Descriptive Statistics . . . .**
3. Type or use **Select** to identify the name of the variable or column (Bills). Click **Statistics . . .** to add or delete particular statistics.

Mean, Median, Mode: Which Is Best?

With three measures from which to choose, which one should we use? There are several factors to consider when making our choice of measure of central location. The mean is generally our first selection. However, there are several circumstances when the median is better. The mode is seldom the best measure of central location. One advantage the median holds is that it is not as sensitive to extreme values as is the mean.

To illustrate, consider the data in Example 4.1. The mean was 11.0, and the median was 8.5. Now suppose that the respondent who reported 33 hours actually reported 133 hours (obviously an Internet addict). The mean becomes

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{0 + 7 + 12 + 5 + 133 + 14 + 8 + 0 + 22}{10} = \frac{210}{10} = 21.0$$

This value is exceeded by only 2 of the 10 observations in the sample, making this statistic a poor measure of *central* location. The median stays the same. When there is a relatively small number of extreme observations (either very small or very large, but not both), the median usually produces a better measure of the center of the data.

To see another advantage of the median over the mean, suppose you and your classmates have written a statistics test and the instructor is returning the graded tests. What piece of information is most important to you? The answer, of course, is *your* mark. What is the next important bit of information? The answer is how well you performed relative to the class. Most students ask their instructor for the class mean. This is the wrong statistic to request. You want the *median* because it divides the class into two halves. This information allows you to identify which half of the class your mark falls into. The median provides this information; the mean does not. Nevertheless, the mean can also be useful in this scenario. If there are several sections of the course, the section means can be compared to determine whose class performed best (or worst).

Measures of Central Location for Ordinal and Nominal Data

When the data are interval, we can use any of the three measures of central location. However, for ordinal and nominal data, the calculation of the mean is not valid. Because the calculation of the median begins by placing the data in order, this statistic is appropriate for ordinal data. The mode, which is determined by counting the frequency of each observation, is appropriate for nominal data. However, nominal data do not have a "center," so we cannot interpret the mode of nominal data in that way. It is generally pointless to compute the mode of nominal data.

APPLICATIONS in FINANCE


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Geometric Mean

The arithmetic mean is the single most popular and useful measure of central location. We noted certain situations where the median is a better measure of central location. However, there is another circumstance where neither the mean nor the median is the best measure. When the variable is a growth rate or rate of change, such as the value of an investment over periods of time, we need another measure. This will become apparent from the following illustration.

Suppose you make a 2-year investment of \$1,000, and it grows by 100% to \$2,000 during the first year. During the second year, however, the investment suffers a 50% loss, from \$2,000 back to \$1,000. The rates of return for years 1 and 2 are $R_1 = 100\%$ and $R_2 = -50\%$, respectively. The arithmetic mean (and the median) is computed as

$$\bar{R} = \frac{R_1 + R_2}{2} = \frac{100 + (-50)}{2} = 25\%$$

But this figure is misleading. Because there was no change in the value of the investment from the beginning to the end of the 2-year period, the "average" compounded rate of return is 0%. As you will see, this is the value of the *geometric mean.*

Let R_i denote the rate of return (in decimal form) in period i ($i = 1, 2, \ldots, n$). The **geometric mean** R_g of the returns $R_1, R_2, \ldots, R_n$ is defined such that

$$(1 + R_g)^n = (1 + R_1)(1 + R_2)\cdots(1 + R_n)$$

Solving for R_g, we produce the following formula:

$$R_g = \sqrt[n]{(1 + R_1)(1 + R_2)\cdots(1 + R_n)} - 1$$

The geometric mean of our investment illustration is

$$R_g = \sqrt[n]{(1 + R_1)(1 + R_2)\cdots(1 + R_n)} - 1 = \sqrt[2]{(1 + 1)(1 + [-.50])} - 1 = 1 - 1 = 0$$

The geometric mean is therefore 0%. This is the single "average" return that allows us to compute the value of the investment at the end of the investment period from the beginning value. Thus, using the formula for compound interest with the rate = 0%, we find

$$\text{Value at the end of the investment period} = 1{,}000(1 + R_g)^2 = 1{,}000(1 + 0)^2 = 1{,}000$$

The geometric mean is used whenever we wish to find the "average" growth rate, or rate of change, in a variable *over time*. However, the arithmetic mean of n returns (or growth rates) is the appropriate mean to calculate if you wish to estimate the mean rate of return (or growth rate) for any *single* period in the future; that is, in the illustration above if we wanted to estimate the rate of return in year 3, we would use the arithmetic mean of the two annual rates of return, which we found to be 25%.

EXCEL

INSTRUCTIONS

1. Type or import the values of $1 + R_i$ into a column.
2. Follow the instructions to produce the mean (page 100) except substitute **GEOMEAN** in place of **AVERAGE**.
3. To determine the geometric mean, subtract 1 from the number produced.

MINITAB

Minitab does not compute the geometric mean.

Here is a summary of the numerical techniques introduced in this section and when to use them.

Factors That Identify When to Compute the Mean

1. **Objective**: Describe a single set of data
2. **Type of data**: Interval
3. **Descriptive measurement**: Central location

Factors That Identify When to Compute the Median

1. **Objective**: Describe a single set of data
2. **Type of data**: Ordinal or interval (with extreme observations)
3. **Descriptive measurement**: Central location

Factors That Identify When to Compute the Mode

1. **Objective**: Describe a single set of data
2. **Type of data**: Nominal, ordinal, interval

Factors That Identify When to Compute the Geometric Mean

1. **Objective**: Describe a single set of data
2. **Type of data**: Interval; growth rates

EXERCISES

4.1 A sample of 12 people was asked how much change they had in their pockets and wallets. The responses (in cents) are

52	25	15	0	104	44
60	30	33	81	40	5

Determine the mean, median, and mode for these data.

4.2 The number of sick days due to colds and flu last year was recorded by a sample of 15 adults. The data are

5	7	0	3	15	6	5	9
3	8	10	5	2	0	12	

Compute the mean, median, and mode.

4.3 A random sample of 12 joggers was asked to keep track and report the number of miles they ran last week. The responses are

5.5	7.2	1.6	22.0	8.7	2.8
5.3	3.4	12.5	18.6	8.3	6.6

a. Compute the three statistics that measure central location.
b. Briefly describe what each statistic tells you.

4.4 The midterm test for a statistics course has a time limit of 1 hour. However, like most statistics exams this one was quite easy. To assess how easy, the professor recorded the amount of time taken by a sample of nine students to hand in their test papers. The times (rounded to the nearest minute) are

33	29	45	60	42	19	52	38	36

a. Compute the mean, median, and mode.
b. What have you learned from the three statistics calculated in part (a)?

4.5 The professors at Wilfrid Laurier University are required to submit their final exams to the registrar's office 10 days before the end of the semester. The exam coordinator sampled 20 professors and recorded the number of days before the final exam that each submitted his or her exam. The results are

14	8	3	2	6	4	9	13	10	12
7	4	9	13	15	8	11	12	4	0

a. Compute the mean, median, and mode.
b. Briefly describe what each statistic tells you.

4.6 Compute the geometric mean of the following rates of return.

.25 −.10 .50

4.7 What is the geometric mean of the following rates of return?

.50 .30 −.50 −.25

4.8 The following returns were realized on an investment over a 5-year period.

Year	1	2	3	4	5
Rate of Return	.10	.22	.06	−.05	.20

a. Compute the mean and median of the returns.
b. Compute the geometric mean.
c. Which one of the three statistics computed in parts (a) and (b) best describes the return over the 5-year period? Explain.

4.9 An investment you made 5 years ago has realized the following rates of return.

Year	1	2	3	4	5
Rate of Return	−.15	−.20	.15	−.08	.50

a. Compute the mean and median of the rates of return.
b. Compute the geometric mean.
c. Which one of the three statistics computed in parts (a) and (b) best describes the return over the 5-year period? Explain.

4.10 An investment of $1,000 you made 4 years ago was worth $1,200 after the first year, $1,200 after the second year, $1,500 after the third year, and $2,000 today.
a. Compute the annual rates of return.
b. Compute the mean and median of the rates of return.
c. Compute the geometric mean.
d. Discuss whether the mean, median, or geometric mean is the best measure of the performance of the investment.

4.11 Suppose that you bought a stock 6 years ago at $12.The stock's price at the end of each year is shown here.

Year	1	2	3	4	5	6
Price	10	14	15	22	30	25

a. Compute the rate of return for each year.
b. Compute the mean and median of the rates of return.
c. Compute the geometric mean of the rates of return.
d. Explain why the best statistic to use to describe what happened to the price of the stock over the 6-year period is the geometric mean.

The following exercises require the use of a computer and software.

4.12 Xr04-12 The starting salaries of a sample of 125 recent MBA graduates are recorded.
a. Determine the mean and median of these data.
b. What do these two statistics tell you about the starting salaries of MBA graduates?

4.13 Xr04-13 To determine whether changing the color of its invoices would improve the speed of payment, a company selected 200 customers at random and sent their invoices on blue paper. The number of days until the bills were paid was recorded. Calculate the mean and median of these data. Report what you have discovered.

4.14 Xr04-14 A survey undertaken by the U.S. Bureau of Labor Statistics, Annual Consumer Expenditure, asks American adults to report the amount of money spent on reading material in 2006. (*Source:* Adapted from *Statistical Abstract of the United States*, *2009*, Table 664.)
a. Compute the mean and median of the sample.
b. What do the statistics computed in part (a) tell you about the reading materials expenditures?

4.15 Xr04-15 A survey of 225 workers in Los Angeles and 190 workers in New York asked each to report the average amount of time spent commuting to work. (*Source:* Adapted from *Statistical Abstract of the United States*, *2009*, Table 1060.)
a. Compute the mean and median of the commuting times for workers in Los Angeles.
b. Repeat part (a) for New York workers.
c. Summarize your findings.

4.16 Xr04-16 In the United States, banks and financial institutions often require buyers of houses to pay fees in order to arrange mortgages. In a survey conducted by the U.S. Federal Housing Finance Board, 350 buyers of new houses who received a mortgage from a bank were asked to report the amount of fees (fees include commissions, discounts, and points) they paid as a percentage of the whole mortgage. (*Source:* Adapted from *Statistical Abstract of the United States*, *2009*, Table 1153.)
a. Compute the mean and median.
b. Interpret the statistics you computed.

4.17 Xr04-17 In an effort to slow drivers, traffic engineers painted a solid line 3 feet from the curb over the entire length of a road and filled the space with diagonal lines. The lines made the road look narrower. A sample of car speeds was taken after the lines were drawn.

a. Compute the mean, median, and mode of these data.
b. Briefly describe the information you acquired from each statistic calculated in part (a).

4.18 Xr04-18 How much do Americans spend on various food groups? Two hundred American families were surveyed and asked to report the amount of money spent annually on fruits and vegetables. Compute the mean and median of these data and interpret the results. (*Source:* Adapted from *Statistical Abstract of the United States*, *2009*, Table 662.)

4.2 Measures of Variability

The statistics introduced in Section 4.1 serve to provide information about the central location of the data. However, as we have already discussed in Chapter 2, there are other characteristics of data that are of interest to practitioners of statistics. One such characteristic is the spread or variability of the data. In this section, we introduce four **measures of variability**. We begin with the simplest.

Range

Range

Range = Largest observation − Smallest observation

The advantage of the **range** is its simplicity. The disadvantage is also its simplicity. Because the range is calculated from only two observations, it tells us nothing about the other observations. Consider the following two sets of data.

Set 1:	4	4	4	4	4	50
Set 2:	4	8	15	24	39	50

The range of both sets is 46. The two sets of data are completely different, yet their ranges are the same. To measure variability, we need other statistics that incorporate all the data and not just two observations.

Variance

The **variance** and its related measure, the **standard deviation**, are arguably the most important statistics. They are used to measure variability, but, as you will discover, they play a vital role in almost all statistical inference procedures.

Variance

Population variance: $$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}$$

Sample variance:* $$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n - 1}$$

The population variance is represented by σ^2(Greek letter *sigma* squared).

Examine the formula for the sample variance s^2. It may appear to be illogical that in calculating s^2 we divide by $n - 1$ rather than by n.* However, we do so for the following reason. Population parameters in practical settings are seldom known. One objective of statistical inference is to estimate the parameter from the statistic. For example, we estimate the population mean μ from the sample mean $\bar{x}$. Although it is not obviously logical, the statistic created by dividing $\sum(x_i - \bar{x})^2$ by $n - 1$ is a better estimator than the one created by dividing by n. We will discuss this issue in greater detail in Section 10.1.

To compute the sample variance s^2, we begin by calculating the sample mean $\bar{x}$. Next we compute the difference (also call the **deviation**) between each observation and the mean. We square the deviations and sum. Finally, we divide the sum of squared deviations by $n - 1$.

We'll illustrate with a simple example. Suppose that we have the following observations of the numbers of hours five students spent studying statistics last week:

8 4 9 11 3

The mean is

$$\bar{x} = \frac{8 + 4 + 9 + 11 + 3}{5} = \frac{35}{5} = 7$$

For each observation, we determine its deviation from the mean. The deviation is squared, and the sum of squares is determined as shown in Table 4.1.

TABLE **4.1** Calculation of Sample Variance

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
8	$(8 - 7) = 1$	$(1)^2 = 1$
4	$(4 - 7) = -3$	$(-3)^2 = 9$
9	$(9 - 7) = 2$	$(2)^2 = 4$
11	$(11 - 7) = 4$	$(4)^2 = 16$
3	$(3 - 7) = -4$	$(-4)^2 = 16$
	$\sum_{l=1}^{5}(x_i - \bar{x}) = 0$	$\sum_{l=1}^{5}(x_i - \bar{x})^2 = 46$

The sample variance is

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n - 1} = \frac{46}{5 - 1} = 11.5$$

The calculation of this statistic raises several questions. Why do we square the deviations before averaging? If you examine the deviations, you will see that some of the

*Technically, the variance of the sample is calculated by dividing the sum of squared deviations by n. The statistic computed by dividing the sum of squared deviations by $n - 1$ is called the *sample variance corrected for the mean*. Because this statistic is used extensively, we will shorten its name to *sample variance*.

deviations are positive and some are negative. When you add them together, the sum is 0. This will always be the case because the sum of the positive deviations will always equal the sum of the negative deviations. Consequently, we square the deviations to avoid the "canceling effect."

Is it possible to avoid the canceling effect without squaring? We could average the *absolute* value of the deviations. In fact, such a statistic has already been invented. It is called the **mean absolute deviation** or MAD. However, this statistic has limited utility and is seldom calculated.

What is the unit of measurement of the variance? Because we squared the deviations, we also squared the units. In this illustration the units were hours (of study). Thus, the sample variance is 11.5 hours^2.

EXAMPLE 4.7

Summer Jobs

The following are the number of summer jobs a sample of six students applied for. Find the mean and variance of these data.

17 15 23 7 9 13

SOLUTION

The mean of the six observations is

$$\bar{x} = \frac{17 + 15 + 23 + 7 + 9 + 13}{6} = \frac{84}{6} = 14 \text{ jobs}$$

The sample variance is

$$\begin{aligned} s^2 &= \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1} \\ &= \frac{(17-14)^2 + (15-14)^2 + (23-14)^2 + (7-14)^2 + (9-14)^2 + (13-14)^2}{6-1} \\ &= \frac{9 + 1 + 81 + 49 + 25 + 1}{5} = \frac{166}{5} = 33.2 \text{ jobs}^2 \end{aligned}$$

(Optional) Shortcut Method for Variance The calculations for larger data sets are quite time-consuming. The following shortcut for the sample variance may help lighten the load.

Shortcut for Sample Variance

$$s^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}\right]$$

To illustrate, we'll do Example 4.7 again.

$$\sum_{I=1}^{n} x_i^2 = 17^2 + 15^2 + 23^2 + 7^2 + 9^2 + 13^2 = 1{,}342$$

$$\sum_{i=1}^{n} x_i = 17 + 15 + 23 + 7 + 9 + 13 = 84$$

$$\left(\sum_{i=1}^{n} x_i\right)^2 = 84^2 = 7{,}056$$

$$s^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}\right] = \frac{1}{6-1}\left[1342 - \frac{7056}{6}\right] = 33.2 \text{ jobs}^2$$

Notice that we produced the same exact answer.

EXCEL

INSTRUCTIONS

Follow the instructions to compute the mean (page 100) except type VAR instead of AVERAGE.

MINITAB

1. Type or import data into one column.
2. Click **Stat, Basic Statistics, Display Descriptive Statistics . . .** , and select the variable.
3. Click **Statistics** and **Variance**.

Interpreting the Variance

We calculated the variance in Example 4.7 to be 33.2 jobs^2. What does this statistic tell us? Unfortunately, the variance provides us with only a rough idea about the amount of variation in the data. However, this statistic is useful when comparing two or more sets of data of the same type of variable. If the variance of one data set is larger than that of a second data set, we interpret that to mean that the observations in the first set display more variation than the observations in the second set.

The problem of interpretation is caused by the way the variance is computed. Because we squared the deviations from the mean, the unit attached to the variance is the square of the unit attached to the original observations. In other words, in Example 4.7 the unit of the data is jobs; the unit of the variance is jobs squared. This contributes to the problem of interpretation. We resolve this difficulty by calculating another related measure of variability.

Standard Deviation

Standard Deviation

Population standard deviation: $\sigma = \sqrt{\sigma^2}$

Sample standard deviation: $s = \sqrt{s^2}$

The standard deviation is simply the positive square root of the variance. Thus, in Example 4.7, the sample standard deviation is

$$s = \sqrt{s^2} = \sqrt{33.2} = 5.76 \text{ jobs}$$

Notice that the unit associated with the standard deviation is the unit of the original data set.

EXAMPLE 4.8 Comparing the Consistency of Two Types of Golf Clubs

DATA Xm04-08

Consistency is the hallmark of a good golfer. Golf equipment manufacturers are constantly seeking ways to improve their products. Suppose that a recent innovation is designed to improve the consistency of its users. As a test, a golfer was asked to hit 150 shots using a 7 iron, 75 of which were hit with his current club and 75 with the new innovative 7 iron. The distances were measured and recorded. Which 7 iron is more consistent?

SOLUTION

To gauge the consistency, we must determine the standard deviations. (We could also compute the variances, but as we just pointed out, the standard deviation is easier to interpret.) We can get Excel and Minitab to print the sample standard deviations. Alternatively, we can calculate all the descriptive statistics, a course of action we recommend because we often need several statistics. The printouts for both 7 irons are shown here.

EXCEL

	A	B	C	D	E
1	*Current*			*Innovation*	
2					
3	Mean	150.55		Mean	150.15
4	Standard Error	0.67		Standard Error	0.36
5	Median	151		Median	150
6	Mode	150		Mode	149
7	Standard Deviation	5.79		Standard Deviation	3.09
8	Sample Variance	33.55		Sample Variance	9.56
9	Kurtosis	0.13		Kurtosis	-0.89
10	Skewness	-0.43		Skewness	0.18
11	Range	28		Range	12
12	Minimum	134		Minimum	144
13	Maximum	162		Maximum	156
14	Sum	11291		Sum	11261
15	Count	75		Count	75

MINITAB

Descriptive Statistics: Current, Innovation

Variable	N	N*	Mean	StDev	Variance	Minimum	Q1	Median	Q3
Current	75	0	150.55	5.79	33.55	134.00	148.00	151.00	155.00
Innovation	75	0	150.15	3.09	9.56	144.00	148.00	150.00	152.00

Variable	Maximum
Current	162.00
Innovation	156.00

INTERPRET

The standard deviation of the distances of the current 7 iron is 5.79 yards whereas that of the innovative 7 iron is 3.09 yards. Based on this sample, the innovative club is more consistent. Because the mean distances are similar it would appear that the new club is indeed superior.

Interpreting the Standard Deviation Knowing the mean and standard deviation allows the statistics practitioner to extract useful bits of information. The information depends on the shape of the histogram. If the histogram is bell shaped, we can use the **Empirical Rule**.

Empirical Rule

1. Approximately 68% of all observations fall within one standard deviation of the mean.
2. Approximately 95% of all observations fall within two standard deviations of the mean.
3. Approximately 99.7% of all observations fall within three standard deviations of the mean.

EXAMPLE 4.9

Using the Empirical Rule to Interpret Standard Deviation

After an analysis of the returns on an investment, a statistics practitioner discovered that the histogram is bell shaped and that the mean and standard deviation are 10% and 8%, respectively. What can you say about the way the returns are distributed?

SOLUTION

Because the histogram is bell shaped, we can apply the Empirical Rule:

1. Approximately 68% of the returns lie between 2% (the mean minus one standard deviation $= 10 - 8$) and 18% (the mean plus one standard deviation $= 10 + 8$).

2. Approximately 95% of the returns lie between −6% [the mean minus two standard deviations = 10 − 2(8)] and 26% [the mean plus two standard deviations = 10 + 2(8)].
3. Approximately 99.7% of the returns lie between −14% [the mean minus three standard deviations = 10 − 3(8)] and 34% [the mean plus three standard deviations = 10 − 3(8)].

A more general interpretation of the standard deviation is derived from *Chebysheff's Theorem*, which applies to all shapes of histograms.

Chebysheff's Theorem

The proportion of observations in any sample or population that lie within k standard deviations of the mean is at least

$$1 - \frac{1}{k^2} \quad \text{for} \quad k > 1$$

When $k = 2$, **Chebysheff's Theorem** states that at least three-quarters (75%) of all observations lie within two standard deviations of the mean. With $k = 3$, Chebysheff's Theorem states that at least eight-ninths (88.9%) of all observations lie within three standard deviations of the mean.

Note that the Empirical Rule provides approximate proportions, whereas Chebysheff's Theorem provides lower bounds on the proportions contained in the intervals.

EXAMPLE 4.10 Using Chebysheff's Theorem to Interpret Standard Deviation

The annual salaries of the employees of a chain of computer stores produced a positively **skewed** histogram. The mean and standard deviation are \$28,000 and \$3,000, respectively. What can you say about the salaries at this chain?

SOLUTION

Because the histogram is not bell shaped, we cannot use the Empirical Rule. We must employ Chebysheff's Theorem instead.

The intervals created by adding and subtracting two and three standard deviations to and from the mean are as follows:

1. At least 75% of the salaries lie between \$22,000 [the mean minus two standard deviations = 28,000 − 2(3,000)] and \$34,000 [the mean plus two standard deviations = 28,000 + 2(3,000)].
2. At least 88.9% of the salaries lie between \$19,000 [the mean minus three standard deviations = 28,000 − 3(3,000)] and \$37,000 [the mean plus three standard deviations = 28,000 + 3(3,000)].

Coefficient of Variation

Is a standard deviation of 10 a large number indicating great variability or a small number indicating little variability? The answer depends somewhat on the magnitude of the observations in the data set. If the observations are in the millions, then a standard deviation of 10 will probably be considered a small number. On the other hand, if the observations are less than 50, then the standard deviation of 10 would be seen as a large number. This logic lies behind yet another measure of variability, the *coefficient of variation*.

Coefficient of Variation

The **coefficient of variation** of a set of observations is the standard deviation of the observations divided by their mean:

$$\text{Population coefficient of variation: CV} = \frac{\sigma}{\mu}$$

$$\text{Sample coefficient of variation: cv} = \frac{s}{\bar{x}}$$

Measures of Variability for Ordinal and Nominal Data

The measures of variability introduced in this section can be used only for interval data. The next section will feature a measure that can be used to describe the variability of ordinal data. There are no measures of variability for nominal data.

Approximating the Mean and Variance from Grouped Data

The statistical methods presented in this chapter are used to compute descriptive statistics from data. However, in some circumstances, the statistics practitioner does not have the raw data but instead has a frequency distribution. This is often the case when data are supplied by government organizations. In Appendix Approximating Means and Variances for Grouped Data on Keller's website we provide the formulas used to approximate the sample mean and variance.

We complete this section by reviewing the factors that identify the use of measures of variability.

Factors That Identify When to Compute the Range, Variance, Standard Deviation, and Coefficient of Variation

1. **Objective**: Describe a single set of data
2. **Type of Data**: Interval
3. **Descriptive measurement**: Variability

EXERCISES

4.19 Calculate the variance of the following data.

9 3 7 4 1 7 5 4

4.20 Calculate the variance of the following data.

4 5 3 6 5 6 5 6

4.21 Determine the variance and standard deviation of the following sample.

12 6 22 31 23 13 15 17 21

4.22 Find the variance and standard deviation of the following sample.

0 −5 −3 6 4 −4 1 −5 0 3

4.23 Examine the three samples listed here. Without performing any calculations, indicate which sample has the largest amount of variation and which sample has the smallest amount of variation. Explain how you produced your answer.

a.	17	29	12	16	11
b.	22	18	23	20	17
c.	24	37	6	39	29

4.24 Refer to Exercise 4.23. Calculate the variance for each part. Was your answer in Exercise 4.23 correct?

4.25 A friend calculates a variance and reports that it is –25.0. How do you know that he has made a serious calculation error?

4.26 Create a sample of five numbers whose mean is 6 and whose standard deviation is 0.

4.27 A set of data whose histogram is bell shaped yields a mean and standard deviation of 50 and 4, respectively. Approximately what proportion of observations

a. are between 46 and 54?
b. are between 42 and 58?
c. are between 38 and 62?

4.28 Refer to Exercise 4.27. Approximately what proportion of observations

a. are less than 46?
b. are less than 58?
c. are greater than 54?

4.29 A set of data whose histogram is extremely skewed yields a mean and standard deviation of 70 and 12, respectively. What is the minimum proportion of observations that

a. are between 46 and 94?
b. are between 34 and 106?

4.30 A statistics practitioner determined that the mean and standard deviation of a data set were 120 and 30, respectively. What can you say about the proportions of observations that lie between each of the following intervals?

a. 90 and 150
b. 60 and 180
c. 30 and 210

The following exercises require a computer and software.

4.31 Xr04-31 There has been much media coverage of the high cost of medicinal drugs in the United States. One concern is the large variation from pharmacy to pharmacy. To investigate, a consumer advocacy group took a random sample of 100 pharmacies around the country and recorded the price (in dollars per 100 pills) of Prozac. Compute the range, variance, and standard deviation of the prices. Discuss what these statistics tell you.

4.32 Xr04-32 Many traffic experts argue that the most important factor in accidents is not the average speed of cars but the amount of variation. Suppose that the speeds of a sample of 200 cars were taken over a stretch of highway that has seen numerous accidents. Compute the variance and standard deviation of the speeds, and interpret the results.

4.33 Xr04-33 Three men were trying to make the football team as punters. The coach had each of them punt the ball 50 times, and the distances were recorded.

a. Compute the variance and standard deviation for each punter.
b. What do these statistics tell you about the punters?

4.34 Xr04-34 Variance is often used to measure quality in production-line products. Suppose that a sample of steel rods that are supposed to be exactly 100 cm long is taken. The length of each is determined, and the results are recorded. Calculate the variance and the standard deviation. Briefly describe what these statistics tell you.

4.35 Xr04-35 To learn more about the size of withdrawals at a banking machine, the proprietor took a sample of 75 withdrawals and recorded the amounts. Determine the mean and standard deviation of these data, and describe what these two statistics tell you about the withdrawal amounts.

4.36 Xr04-36 Everyone is familiar with waiting lines or queues. For example, people wait in line at a supermarket to go through the checkout counter. There are two factors that determine how long the queue becomes. One is the speed of service. The other is the number of arrivals at the checkout counter. The mean number of arrivals is an important number, but so is the standard deviation. Suppose that a consultant for the supermarket counts the number of arrivals per hour during a sample of 150 hours.

a. Compute the standard deviation of the number of arrivals.
b. Assuming that the histogram is bell shaped, interpret the standard deviation.

American National Election Survey Exercises

4.37 ANES2008* The ANES in 2008 asked respondents to state their ages stored as AGE.

a. Calculate the mean, variance, and standard deviation.
b. Draw a histogram.
c. Use the Empirical Rule, if applicable, or Chebysheff's Theorem to interpret the mean and standard deviation.

4.38 ANES2008* Respondents were asked to report the number of minutes spent watching news on television during a typical day (TIME2).

a. Calculate the mean and standard deviation.
b. Draw a histogram.
c. Use the Empirical Rule, if applicable, or Chebysheff's Theorem to interpret the mean and standard deviation.

General Social Survey Exercise

4.39 GSS2008* One of the questions in the 2008 General Social Survey was, If you were born outside the United States, at what age did you permanently move to the United States (AGECMEUS)?

a. Calculate the mean, variance, and standard deviation.
b. Draw a histogram.
c. Use the Empirical Rule, if applicable, or Chebysheff's Theorem to interpret the mean and standard deviation.

4.3 Measures of Relative Standing and Box Plots

Measures of relative standing are designed to provide information about the position of particular values relative to the entire data set. We've already presented one measure of relative standing, the median, which is also a measure of central location. Recall that the median divides the data set into halves, allowing the statistics practitioner to determine which half of the data set each observation lies in. The statistics we're about to introduce will give you much more detailed information.

Percentile

The Pth **percentile** is the value for which P percent are less than that value and (100–P)% are greater than that value.

The scores and the percentiles of the Scholastic Achievement Test (SAT) and the Graduate Management Admission Test (GMAT), as well as various other admissions tests, are reported to students taking them. Suppose for example, that your SAT score is reported to be at the 60th percentile. This means that 60% of all the other marks are below yours and 40% are above it. You now know exactly where you stand relative to the population of SAT scores.

We have special names for the 25th, 50th, and 75th percentiles. Because these three statistics divide the set of data into quarters, these measures of relative standing are also called **quartiles**. The *first* or *lower quartile* is labeled Q_1. It is equal to the 25th percentile. The *second quartile*, Q_2, is equal to the 50th percentile, which is also the median. The *third* or *upper quartile*, Q_3, is equal to the 75th percentile. Incidentally, many people confuse the terms *quartile* and *quarter*. A common error is to state that someone is in the lower *quartile* of a group when they actually mean that someone is in the lower *quarter* of a group.

Besides quartiles, we can also convert percentiles into quintiles and deciles. *Quintiles* divide the data into fifths, and *deciles* divide the data into tenths.

Locating Percentiles

The following formula allows us to approximate the location of any percentile.

Location of a Percentile

$$L_P = (n + 1)\frac{P}{100}$$

where L_P is the location of the Pth percentile.

EXAMPLE 4.11

Percentiles of Time Spent on Internet

Calculate the 25th, 50th, and 75th percentiles (first, second, and third quartiles) of the data in Example 4.1.

SOLUTION

Placing the 10 observations in ascending order we get

0 0 5 7 8 9 12 14 22 33

The location of the 25th percentile is

$$L_{25} = (10 + 1)\frac{25}{100} = (11)(.25) = 2.75$$

The 25th percentile is three-quarters of the distance between the second (which is 0) and the third (which is 5) observations. Three-quarters of the distance is

$$(.75)(5 - 0) = 3.75$$

Because the second observation is 0, the 25th percentile is $0 + 3.75 = 3.75$.

To locate the 50th percentile, we substitute $P = 50$ into the formula and produce

$$L_{50} = (10 + 1)\frac{50}{100} = (11)(.5) = 5.5$$

which means that the 50th percentile is halfway between the fifth and sixth observations. The fifth and sixth observations are 8 and 9, respectively. The 50th percentile is 8.5. This is the median calculated in Example 4.3.

The 75th percentile's location is

$$L_{75} = (10 + 1)\frac{75}{100} = (11)(.75) = 8.25$$

Thus, it is located one-quarter of the distance between the eighth and ninth observations, which are 14 and 22, respectively. One-quarter of the distance is

$$(.25)(22 - 14) = 2$$

which means that the 75th percentile is

$$14 + 2 = 16$$

EXAMPLE 4.12 Quartiles of Long-Distance Telephone Bills

DATA
Xm03-01

Determine the quartiles for Example 3.1.

SOLUTION

EXCEL

	A	B
1	*Bills*	
2		
3	Mean	43.59
4	Standard Error	2.76
5	Median	26.91
6	Mode	0
7	Standard Deviation	38.97
8	Sample Variance	1518.64
9	Kurtosis	-1.29
10	Skewness	0.54
11	Range	119.63
12	Minimum	0
13	Maximum	119.63
14	Sum	8717.52
15	Count	200
16	Largest(50)	85
17	Smallest(50)	9.22

INSTRUCTIONS

Follow the instructions for **Descriptive Statistics** (page 103). In the dialog box, click **Kth Largest** and type in the integer closest to $n/4$. Repeat for **Kth Smallest**, typing in the integer closest to $n/4$.

Excel approximates the third and first percentiles in the following way. The **Largest(50)** is 85, which is the number such that 150 numbers are below it and 49 numbers are above it. The **Smallest(50)** is 9.22, which is the number such that 49 numbers are below it and 150 numbers are above it. The median is 26.91, a statistic we discussed in Example 4.4.

MINITAB

Descriptive Statistics: Bills

Variable	Mean	StDev	Variance	Minimum	Q1	Median	Q3	Maximum
Bills	43.59	38.97	1518.64	0.00	9.28	26.91	84.94	119.63

Minitab outputs the first and third quartiles as Q1 (9.28) and Q3 (84.94), respectively. (See page 103.)

We can often get an idea of the shape of the histogram from the quartiles. For example, if the first and second quartiles are closer to each other than are the second and third quartiles, then the histogram is positively skewed. If the first and second quartiles are farther apart than the second and third quartiles, then the histogram is negatively skewed. If the difference between the first and second quartiles is approximately equal to the difference between the second and third quartiles, then the histogram is approximately symmetric. The box plot described subsequently is particularly useful in this regard.

Interquartile Range

The quartiles can be used to create another measure of variability, the **interquartile range**, which is defined as follows.

Interquartile Range

$$\text{Interquartile range} = Q_3 - Q_1$$

The interquartile range measures the spread of the middle 50% of the observations. Large values of this statistic mean that the first and third quartiles are far apart, indicating a high level of variability.

EXAMPLE 4.13

Interquartile Range of Long-Distance Telephone Bills

DATA Xm03-01

Determine the interquartile range for Example 3.1.

SOLUTION

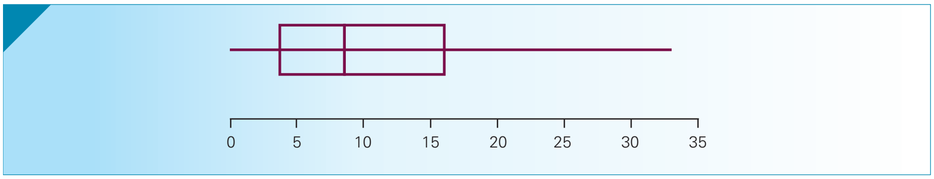
Using Excel's approximations of the first and third quartiles, we find

$$\text{Interquartile range} = Q_3 - Q_1 = 85 - 9.22 = 75.78$$

Box Plots

Now that we have introduced quartiles we can present one more graphical technique, the **box plot**. This technique graphs five statistics: the minimum and maximum observations, and the first, second, and third quartiles. It also depicts other features of a set of data. Figure 4.1 exhibits the box plot of the data in Example 4.1.

FIGURE **4.1** Box Plot for Example 4.1



The three vertical lines of the box are the first, second, and third quartiles. The lines extending to the left and right are called *whiskers*. Any points that lie outside the whiskers are called *outliers*. The whiskers extend outward to the smaller of 1.5 times the interquartile range or to the most extreme point that is not an outlier.

Outliers **Outliers** are unusually large or small observations. Because an outlier is considerably removed from the main body of the data set, its validity is suspect. Consequently, outliers should be checked to determine that they are not the result of an error in recording their values. Outliers can also represent unusual observations that should be investigated. For example, if a salesperson's performance is an outlier on the high end of the distribution, the company could profit by determining what sets that salesperson apart from the others.

EXAMPLE 4.14

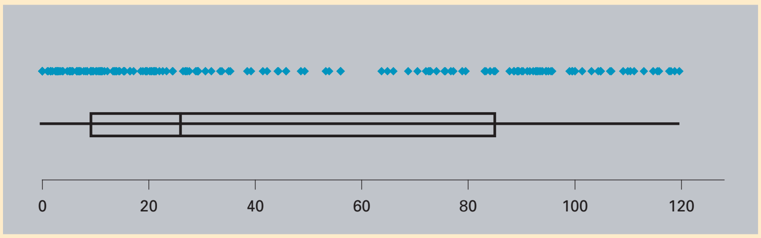
Box Plot of Long-Distance Telephone Bills

DATA
Xm03-01

Draw the box plot for Example 3.1.

SOLUTION

EXCEL



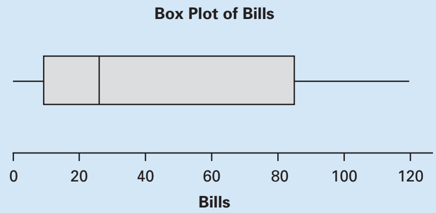
INSTRUCTIONS

1. Type or import the data into one column or two or more adjacent columns. (Open Xm03-01.)
2. Click **Add-Ins, Data Analysis Plus**, and **Box Plot**.
3. Specify the **Input Range** (A1:A201).

A box plot will be created for each column of data that you have specified or highlighted.

Notice that the quartiles produced in the **Box Plot** are not exactly the same as those produced by **Descriptive Statistics**. The **Box Plot** command uses a slightly different method than the **Descriptive Methods** command.

MINITAB



INSTRUCTIONS

1. Type or import the data into one column or more columns. (Open Xm03-01.)
2. Click **Graph** and **Box Plot . . . .**
3. Click **Simple** if there is only one column of data or **Multiple Y's** if there are two or more columns.
4. Type or **Select** the variable or variables in the **Graph variables** box (Bills).
5. The box plot will be drawn so that the values will appear on the vertical axis. To turn the box plot on its side click **Scale . . .** , **Axes and Ticks**, and **Transpose value and category scales**.

INTERPRET

The smallest value is 0, and the largest is 119.63. The first, second, and third quartiles are 9.275, 26.905, and 84.9425, respectively. The interquartile range is 75.6675. One and one-half times the interquartile range is $1.5 \times 75.6675 = 113.5013$. Outliers are defined as any observations that are less than $9.275 - 113.5013 = -104.226$ and any observations that are larger than $84.9425 + 113.5013 = 198.4438$. The whisker to the left extends only to 0, which is the smallest observation that is not an outlier. The whisker to the right extends to 119.63, which is the largest observation that is not an outlier. There are no outliers.

The box plot is particularly useful when comparing two or more data sets.

EXAMPLE 4.15 Comparing Service Times of Fast-Food Restaurants' Drive-Throughs

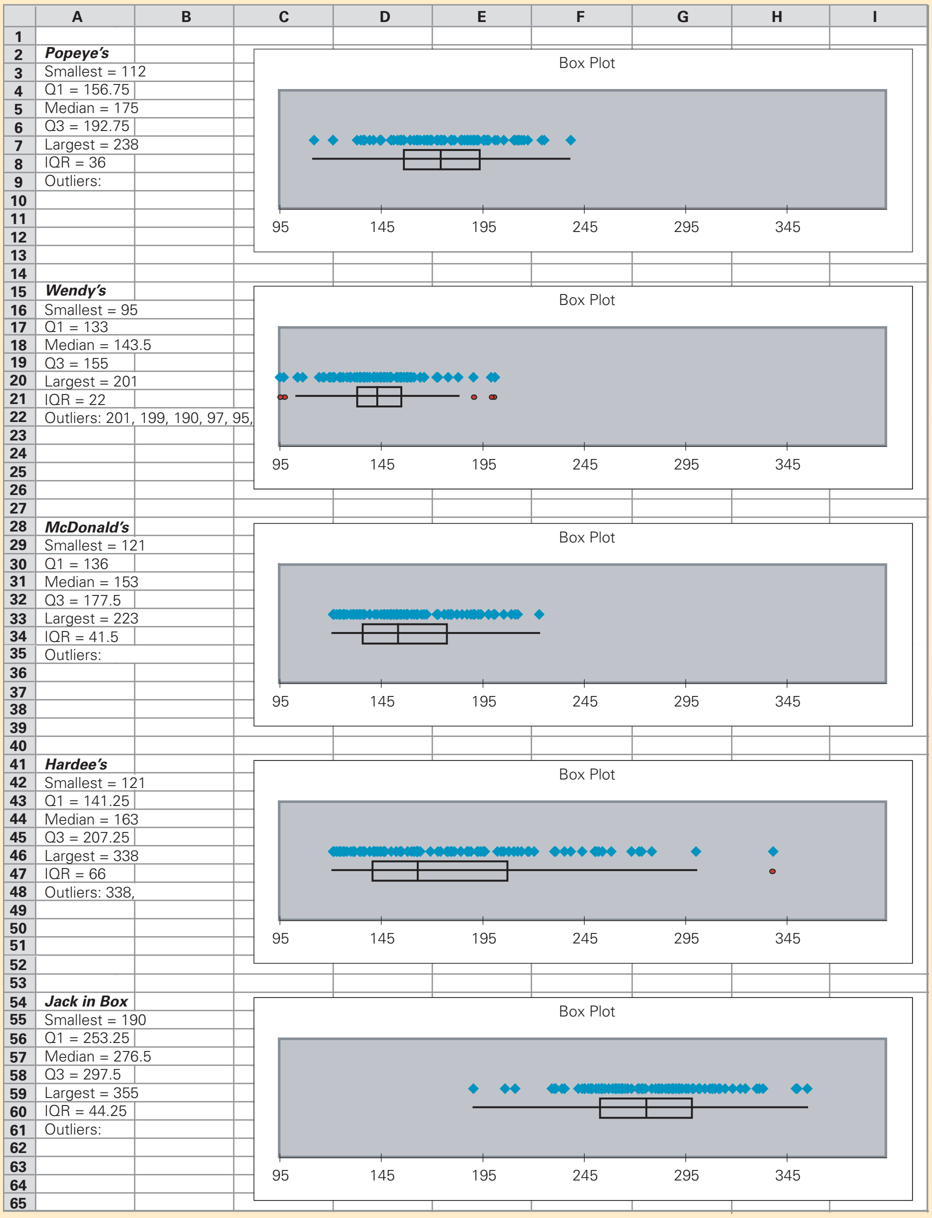
DATA
Xm04-15

A large number of fast-food restaurants with drive-through windows offer drivers and their passengers the advantages of quick service. To measure how good the service is, an organization called QSR planned a study in which the amount of time taken by a sample of drive-through customers at each of five restaurants was recorded. Compare the five sets of data using a box plot and interpret the results.

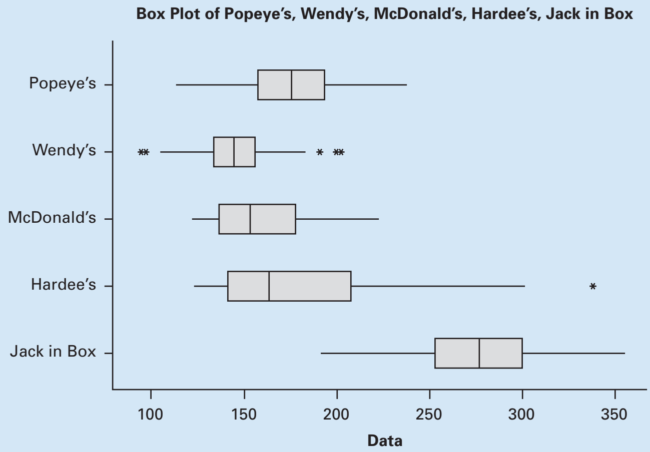
SOLUTION

We use the computer and our software to produce the box plots.

EXCEL


Popeye's
Smallest = 112
Q1 = 156.75
Median = 175
Q3 = 192.75
Largest = 238
IQR = 36
Outliers:
Box Plot
95 145 195 245 295 345
Wendy's
Smallest = 95
Q1 = 133
Median = 143.5
Q3 = 155
Largest = 201
IQR = 22
Outliers: 201, 199, 190, 97, 95,
Box Plot
95 145 195 245 295 345
McDonald's
Smallest = 121
Q1 = 136
Median = 153
Q3 = 177.5
Largest = 223
IQR = 41.5
Outliers:
Box Plot
95 145 195 245 295 345
Hardee's
Smallest = 121
Q1 = 141.25
Median = 163
Q3 = 207.25
Largest = 338
IQR = 66
Outliers: 338,
Box Plot
95 145 195 245 295 345
Jack in Box
Smallest = 190
Q1 = 253.25
Median = 276.5
Q3 = 297.5
Largest = 355
IQR = 44.25
Outliers:
Box Plot
95 145 195 245 295 345

MINITAB



INTERPRET

Wendy's times appear to be the lowest and most consistent. The service times for Hardee's display considerably more variability. The slowest service times are provided by Jack in the Box. The service times for Popeye's, Wendy's, and Jack in the Box seem to be symmetric. However, the times for McDonald's and Hardee's are positively skewed.

Measures of Relative Standing and Variability for Ordinal Data

Because the measures of relative standing are computed by ordering the data, these statistics are appropriate for ordinal as well as for interval data. Furthermore, because the interquartile range is calculated by taking the difference between the upper and lower quartiles, it too can be employed to measure the variability of ordinal data.

Here are the factors that tell us when to use the techniques presented in this section.

Factors That Identify When to Compute Percentiles and Quartiles

1. **Objective**: Describe a single set of data
2. **Type of data**: Interval or ordinal
3. **Descriptive measurement**: Relative standing

Factors That Identify When to Compute the Interquartile Range

1. **Objective**: Describe a single set of data
2. **Type of data**: Interval or ordinal
3. **Descriptive measurement**: Variability

EXERCISES

4.40 Calculate the first, second, and third quartiles of the following sample.

5 8 2 9 5 3 7 4 2 7 4 10 4 3 5

4.41 Find the third and eighth deciles (30th and 80th percentiles) of the following data set.

26	23	29	31	24
22	15	31	30	20

4.42 Find the first and second quintiles (20th and 40th percentiles) of the data shown here.

52	61	88	43	64
71	39	73	51	60

4.43 Determine the first, second, and third quartiles of the following data.

10.5	14.7	15.3	17.7	15.9	12.2	10.0
14.1	13.9	18.5	13.9	15.1	14.7	

4.44 Calculate the 3rd and 6th deciles of the accompanying data.

7	18	12	17	29	18	4	27	30	2
4	10	21	5	8					

4.45 Refer to Exercise 4.43. Determine the interquartile range.

4.46 Refer to Exercise 4.40. Determine the interquartile range.

4.47 Compute the interquartile range from the following data.

5 8 14 6 21 11 9 10 18 2

4.48 Draw the box plot of the following set of data.

9	28	15	21	12	22	29	
20	23	31	11	19	24	16	13

The following exercises require a computer and software.

4.49 Xr04-49 Many automotive experts believe that speed limits on highways are too low. One particular expert has stated that he thinks that most drivers drive at speeds that they consider safe. He suggested that the "correct" speed limit should be set at the 85th percentile. Suppose that a random sample of 400 speeds on a highway where the limit is 60 mph was recorded. Find the "correct" speed limit.

4.50 Xr04-50 Accountemps, a company that supplies temporary workers, sponsored a survey of 100 executives. Each was asked to report the number of minutes they spend screening each job resume they receive.
a. Compute the quartiles.
b. What information did you derive from the quartiles? What does this suggest about writing your resume?

4.51 Xr04-51 How much do pets cost? A random sample of dog and cat owners was asked to compute the amounts of money spent on their pets (exclusive of pet food). Draw a box plot for each data set and describe your findings.

4.52 Xr04-52 The Travel Industry Association of America sponsored a poll that asked a random sample of people how much they spent in preparation for pleasure travel. Determine the quartiles and describe what they tell you.

4.53 Xr04-53 The career-counseling center at a university wanted to learn more about the starting salaries of the university's graduates. They asked each graduate to report the highest salary offer received. The survey also asked each graduate to report the degree and starting salary (column 1 = BA, column 2 = BSc, column 3 = BBA, column 4 = other). Draw box plots to compare the four groups of starting salaries. Report your findings.

4.54 Xr04-54 A random sample of Boston Marathon runners was drawn and the times to complete the race were recorded.
a. Draw the box plot.
b. What are the quartiles?

c. Identify outliers.
d. What information does the box plot deliver?

4.55 Xr04-55 Do golfers who are members of private courses play faster than players on a public course? The amount of time taken for a sample of private-course and public-course golfers was recorded.
a. Draw box plots for each sample.
b. What do the box plots tell you?

4.56 Xr04-56 For many restaurants, the amount of time customers linger over coffee and dessert negatively affect profits. To learn more about this variable, a sample of 200 restaurant groups was observed, and the amount of time customers spent in the restaurant was recorded.
a. Calculate the quartiles of these data.
b. What do these statistics tell you about the amount of time spent in this restaurant?

4.57 Xr04-57 In the United States, taxpayers are allowed to deduct mortgage interest from their incomes before calculating the amount of income tax they are required to pay. In 2005, the Internal Revenue Service sampled 500 tax returns that had a mortgage-interest deduction. Calculate the quartiles and describe what they tell you. (Adapted from *Statistical Abstract of the United States, 2009*, Table 471.)

American National Election Survey Exercises

4.58 ANES2008* In the 2008 survey, people were asked to indicate the amount of time they spent in a typical day receiving news about the election on the Internet (TIME1) and on television (TIME2). Compare the two amounts of time by drawing box plots (using the same scale) and describe what the graphs tell you. (*Excel users:* You must have adjacent columns. We recommend that you copy the two columns into adjacent columns in a separate spreadsheet.)

4.59 ANES2008* Draw a box plot of the ages (AGE) of respondents in the 2008 survey.

4.60 ANES2008* Draw a box plot of the education level of both married spouses (EDUC and SPEDUC). Describe your findings.

General Social Survey Exercises

4.61 GSS2008* Draw a box plot of the ages (AGE) of respondents from the 2008 survey. Briefly describe the graph.

4.62 GSS2008* Produce a box plot of the amount of television watched (TVHOURS). State what the graph tells you.

4.4 Measures of Linear Relationship

In Chapter 3, we introduced the scatter diagram, a graphical technique that describes the relationship between two interval variables. At that time, we pointed out that we were particularly interested in the direction and strength of the linear relationship. We now present three numerical measures of linear relationship that provide this information: *covariance*, *coefficient of correlation*, and *coefficient of determination*. Later in this section we discuss another related numerical technique, the *least squares line*.

Covariance

As we did in Chapter 3, we label one variable X and the other Y.

Covariance

$$\text{Population covariance: } \sigma_{xy} = \frac{\sum_{i=1}^{N}(x_i - \mu_x)(y_i - \mu_y)}{N}$$

$$\text{Sample covariance: } s_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n - 1}$$

The denominator in the calculation of the sample **covariance** is $n - 1$, not the more logical n for the same reason we divide by $n - 1$ to calculate the sample variance (see page 109). If you plan to compute the sample covariance manually, here is a shortcut calculation.

Shortcut for Sample Covariance

$$s_{xy} = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i y_i - \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n}\right]$$

To illustrate how covariance measures the linear relationship, examine the following three sets of data.

Set 1

x_i	y_i	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$
2	13	−3	−7	21
6	20	1	0	0
7	27	2	7	14
	$\bar{x} = 5$	$\bar{y} = 20$		$s_{xy} = 35/2 = 17.5$

Set 2

x_i	y_i	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$
2	27	−3	7	−21
6	20	1	0	0
7	13	2	−7	−14
	$\bar{x} = 5$	$\bar{y} = 20$		$s_{xy} = -35/2 = -17.5$

Set 3

x_i	y_i	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$
2	20	−3	0	0
6	27	1	7	7
7	13	2	−7	−14
	$\bar{x} = 5$	$\bar{y} = 20$		$s_{xy} = -7/2 = -3.5$

Notice that the values of x are the same in all three sets and that the values of y are also the same. The only difference is the *order* of the values of y.

In set 1, as x increases so does y. When x is larger than its mean, y is at least as large as its mean. Thus $(x_i - \bar{x})$ and $(y_i - \bar{y})$ have the same sign or 0. Their product is also positive or 0. Consequently, the covariance is a positive number. Generally, when two variables move in the same direction (both increase or both decrease), the covariance will be a large positive number.

If you examine set 2, you will discover that as x increases, y decreases. When x is larger than its mean, y is less than or equal to its mean. As a result when $(x_i - \bar{x})$ is positive, $(y_i - \bar{y})$ is negative or 0. Their products are either negative or 0. It follows that the covariance is a negative number. In general, when two variables move in opposite directions, the covariance is a large negative number.

In set 3, as x increases, y does not exhibit any particular direction. One of the products $(x_i - \bar{x})(y_i - \bar{y})$ is 0, one is positive, and one is negative. The resulting covariance is a small number. In general, when there is no particular pattern, the covariance is a small number.

We would like to extract two pieces of information. The first is the sign of the covariance, which tells us the nature of the relationship. The second is the magnitude, which describes the strength of the association. Unfortunately, the magnitude may be difficult to judge. For example, if you're told that the covariance between two variables is 500, does this mean that there is a strong linear relationship? The answer is that it is impossible to judge without additional statistics. Fortunately, we can improve on the information provided by this statistic by creating another one.

Coefficient of Correlation

The **coefficient of correlation** is defined as the covariance divided by the standard deviations of the variables.

Coefficient of Correlation

$$\text{Population coefficient of correlation: } \rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

$$\text{Sample coefficient of correlation: } r = \frac{s_{xy}}{s_x s_y}$$

The population parameter is denoted by the Greek letter *rho*.

The advantage that the coefficient of correlation has over the covariance is that the former has a set lower and upper limit. The limits are -1 and $+1$, respectively—that is,

$$-1 \le r \le +1 \qquad \text{and} \qquad -1 \le \rho \le +1$$

When the coefficient of correlation equals –1, there is a negative linear relationship and the scatter diagram exhibits a straight line. When the coefficient of correlation equals +1, there is a perfect positive relationship. When the coefficient of correlation equals 0, there is no linear relationship. All other values of correlation are judged in relation to these three values. The drawback to the coefficient of correlation is that—except for the three values −1, 0, and +1—we cannot interpret the correlation. For example, suppose that we calculated the coefficient of correlation to be −.4. What does this tell us? It tells us two things. The minus sign tells us the relationship is negative and because .4 is closer to 0 than to 1, we judge that the linear relationship is weak. In many applications, we need a better interpretation than the "linear relationship is weak." Fortunately, there is yet another measure of the strength of a linear relationship, which gives us more information. It is the *coefficient of determination*, which we introduce later in this section.

EXAMPLE 4.16 Calculating the Coefficient of Correlation

Calculate the coefficient of correlation for the three sets of data on pages 126–127.

SOLUTION

Because we've already calculated the covariances we need to compute only the standard deviations of X and Y.

$$\bar{x} = \frac{2 + 6 + 7}{3} = 5.0$$

$$\bar{y} = \frac{13 + 20 + 27}{3} = 20.0$$

$$s_x^2 = \frac{(2 - 5)^2 + (6 - 5)^2 + (7 - 5)^2}{3 - 1} = \frac{9 + 1 + 4}{2} = 7.0$$

$$s_y^2 = \frac{(13 - 20)^2 + (20 - 20)^2 + (27 - 20)^2}{3 - 1} = \frac{49 + 0 + 49}{2} = 49.0$$

The standard deviations are

$$s_x = \sqrt{7.0} = 2.65$$

$$s_y = \sqrt{49.0} = 7.00$$

The coefficients of correlation are:

Set 1: $$r = \frac{s_{xy}}{s_x s_y} = \frac{17.5}{(2.65)(7.0)} = .943$$

Set 2: $$r = \frac{s_{xy}}{s_x s_y} = \frac{-17.5}{(2.65)(7.0)} = -.943$$

Set 3: $$r = \frac{s_{xy}}{s_x s_y} = \frac{-3.5}{(2.65)(7.0)} = -.189$$

It is now easier to see the strength of the linear relationship between X and Y.

Comparing the Scatter Diagram, Covariance, and Coefficient of Correlation

The scatter diagram depicts relationships graphically; the covariance and the coefficient of correlation describe the linear relationship numerically. Figures 4.2, 4.3, and 4.4 depict three scatter diagrams. To show how the graphical and numerical techniques compare, we calculated the covariance and the coefficient of correlation for each. (The data are stored in files Fig04-02, Fig04-03, and Fig04-04.) As you can see, Figure 4.2 depicts a strong positive relationship between the two variables. The covariance is 36.87, and the coefficient of correlation is .9641. The variables in Figure 4.3 produced a relatively strong negative linear relationship; the covariance and coefficient of correlation are −34.18 and −.8791, respectively. The covariance and coefficient of correlation for the data in Figure 4.4 are 2.07 and .1206, respectively. There is no apparent linear relationship in this figure.

FIGURE **4.2** Strong Positive Linear Relationship

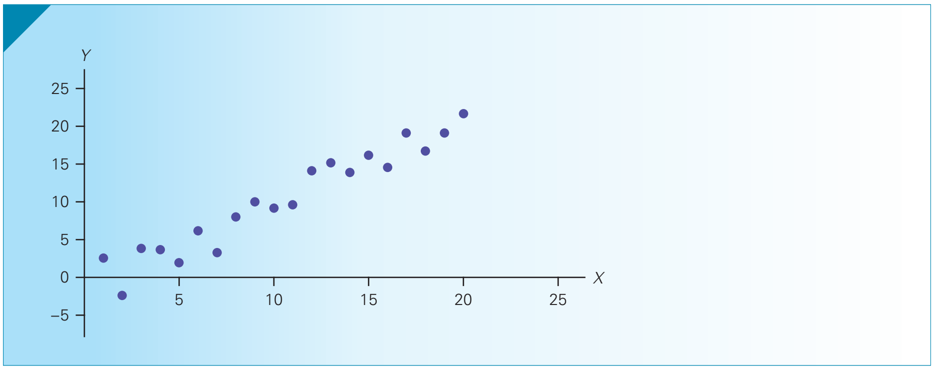


FIGURE **4.3** Strong Negative Linear Relationship

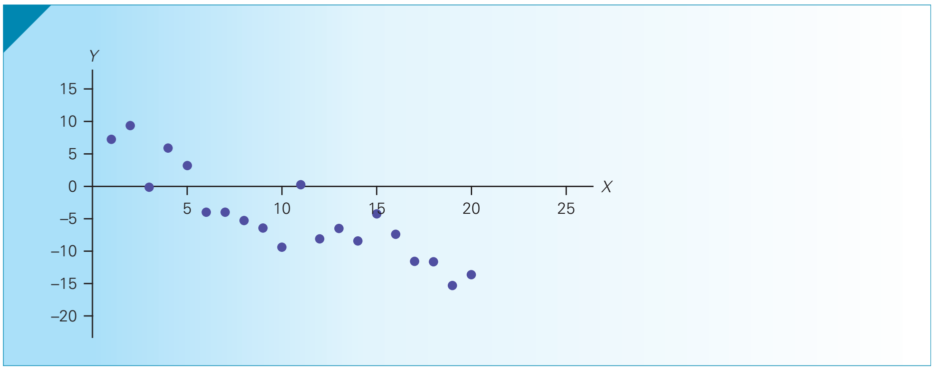
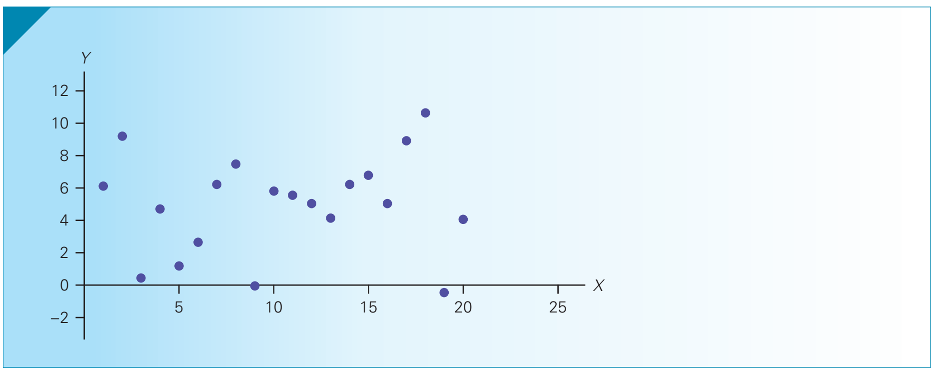


FIGURE **4.4** No Linear Relationship



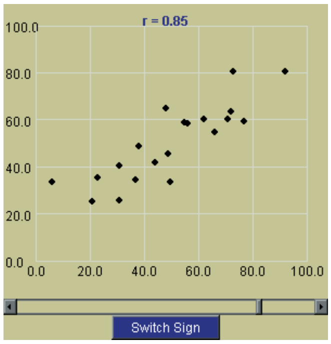
SEEING STATISTICS

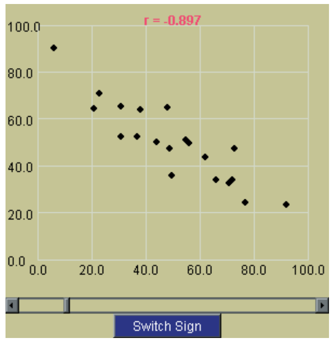
applet 1 Scatter Diagrams and Correlation

In Section 1.3, we introduced applets as a method to show students of applied statistics how statistical techniques work and gain insights into the underlying principles. The applets are stored on Keller's website that accompanies this book. See the README file for instructions on how to use them.

Instructions for Applet 1

Use your mouse to move the slider in the graph. As you move the slider, observe how the coefficient of correlation changes as the points become more "organized" in the scatter diagram. If you click **Switch sign**, you can see the difference between positive and negative coefficients. The following figures displays the applet for two values of *r*.





Applet Exercises

1.1 Drag the slider to the right until the correlation coefficient *r* is 1.0. Describe the pattern of the data points.

1.2 Drag the slider to the left until the correlation coefficient *r* is -1.0. Describe the pattern of the data points. In what way does it differ from the case where $r = 1.0$?

1.3 Drag the slider toward the center until the correlation coefficient *r* is 0 (approximately). Describe the pattern of the data points. Is there a pattern? Or do the points appear to be scattered randomly?

1.4 Drag the slider until the correlation coefficient *r* is .5 (approximately). Can you detect a pattern? Now click on the **Switch Sign** button to change the correlation coefficient *r* to $-.5$. How does the pattern change when the sign switches? Switch back and forth several times so you can see the changes.

SEEING STATISTICS

applet 2 Scatter Patterns and Correlation

This applet allows you to place points on a graph and see the resulting value of the coefficient of correlation.

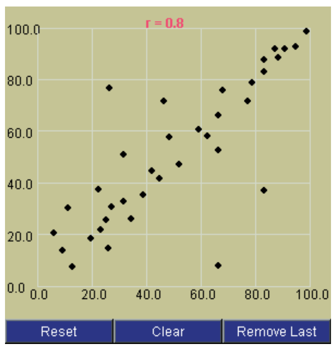
Instructions

Click on the graph to place a point. As you add points, the correlation coefficient is recalculated. Click to add points in various patterns to see how the correlation does (or does not) reflect those patterns. Click on the **Reset** button to clear all points. The figure shown here depicts a scatter diagram and its coefficient of correlation.

Applet Exercises

2.1 Create a scatter diagram where *r* is approximately 0. Describe how you did it.

2.2 Create a scatter diagram where *r* is approximately 1. Describe how this was done.



2.3 Plot points such that r is approximately .5. How would you describe the resulting scatter diagram?

2.4 Plot the points on a scatter diagram where r is approximately 1. Now add one more point, decreasing r by as much as possible. What does this tell you about extreme points?

2.5 Repeat Applet Exercise 2.4, adding two points. How close to $r = 0$ did you get?

Least Squares Method

When we presented the scatter diagram in Section 3.3, we pointed out that we were interested in measuring the strength and direction of the linear relationship. Both can be more easily judged by drawing a straight line through the data. However, if different people draw a line through the same data set, it is likely that each person's line will differ from all the others. Moreover, we often need to know the equation of the line. Consequently, we need an objective method of producing a straight line. Such a method has been developed; it is called the **least squares method**.

The least squares method produces a straight line drawn through the points so that the sum of squared deviations between the points and the line is minimized. The line is represented by the equation:

$$\hat{y} = b_0 + b_1 x$$

where b_0 is the y-intercept (where the line intercepts the y-axis), and b_1 is the slope (defined as rise/run), and $\hat{y}$ (y *hat*) is the value of y determined by the line. The coefficients b_0 and b_1 are derived using calculus so that we minimize the sum of squared deviations:

$$\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

Least Squares Line Coefficients

$$b_1 = \frac{s_{xy}}{s_x^2}$$

$$b_0 = \bar{y} - b_1\bar{x}$$

APPLICATIONS in **ACCOUNTING**

Breakeven Analysis

Breakeven analysis is an extremely important business tool, one that you will likely encounter repeatedly in your course of studies. It can used to determine how much sales volume your business needs to start making a profit.

In Section 3.1 (page 44) we briefly introduced the four P's of marketing and illustrated the problem of pricing with Example 3.1. Breakeven analysis is especially useful when managers are attempting to determine the appropriate price for the company's products and services.

A company's profit can be calculated simply as

Profit = (Price per unit – variable cost per unit) × (Number of units sold) – Fixed costs

The breakeven point is the number of units sold such that the profit is 0. Thus, the breakeven point is calculated as

Number of units sold = Fixed cost / (Price – Variable cost)

Managers can use the formula to help determine the price that will produce a profit. However, to do so requires knowledge of the fixed and variable costs. For example, suppose that a bakery sells only loaves of bread. The bread sells for $1.20, the variable cost is $0.40, and the fixed annual costs are $10,000. The breakeven point is

Number of units sold = 10,000/ (1.20 – 0.40) = 12,500

The bakery must sell more than 12,500 loaves per year to make a profit.

In the next application box, we discuss fixed and variable costs.

APPLICATIONS in ACCOUNTING

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Fixed and Variable Costs

Fixed costs are costs that must be paid whether or not any units are produced. These costs are "fixed" over a specified period of time or range of production. Variable costs are costs that vary directly with the number of products produced. For the previous bakery example, the fixed costs would include rent and maintenance of the shop, wages paid to employees, advertising costs, telephone, and any other costs that are not related to the number of loaves baked. The variable cost is primarily the cost of ingredients, which rises in relation to the number of loaves baked.

Some expenses are mixed. For the bakery example, one such cost is the cost of electricity. Electricity is needed for lights, which is considered a fixed cost, but also for the ovens and other equipment, which are variable costs.

There are several ways to break the mixed costs into fixed and variable components. One such method is the least squares line; that is, we express the total costs of some component as

$$y = b_0 + b_1x$$

where y = total mixed cost, b_0 = fixed cost, b_1 = variable cost, and x is the number of units.

EXAMPLE 4.17 Estimating Fixed and Variable Costs

DATA
Xm04-17

A tool and die maker operates out of a small shop making specialized tools. He is considering increasing the size of his business and needs to know more about his costs. One such cost is electricity, which he needs to operate his machines and lights. (Some jobs require that he turn on extra bright lights to illuminate his work.) He keeps track of his daily electricity costs and the number of tools that he made that day. These data are listed next. Determine the fixed and variable electricity costs.

Day	1	2	3	4	5	6	7	8	9	10
Number of tools	7	3	2	5	8	11	5	15	3	6
Electricity cost	23.80	11.89	15.98	26.11	31.79	39.93	12.27	40.06	21.38	18.65

SOLUTION

The dependent variable is the daily cost of electricity, and the independent variable is the number of tools. To calculate the coefficients of the least squares line and other statistics (calculated below), we need the sum of X, Y, XY, X^2, and Y^2.

Day	X	Y	XY	X^2	Y^2
1	7	23.80	166.60	49	566.44
2	3	11.89	35.67	9	141.37
3	2	15.98	31.96	4	255.36
4	5	26.11	130.55	25	681.73
5	8	31.79	254.32	64	1010.60
6	11	39.93	439.23	121	1594.40
7	5	12.27	61.35	25	150.55
8	15	40.06	600.90	225	1604.80
9	3	21.38	64.14	9	457.10
10	6	18.65	111.90	36	347.82
Total	65	241.86	1896.62	567	6810.20

Covariance:

$$s_{xy} = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i y_i - \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n}\right] = \frac{1}{10-1}\left[1896.62 - \frac{(65)(241.86)}{10}\right] = 36.06$$

Variance of X:

$$s_x^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}\right] = \frac{1}{10-1}\left[567 - \frac{(65)^2}{10}\right] = 16.06$$

Sample means

$$\bar{x} = \frac{\sum x_I}{n} = \frac{65}{10} = 6.5$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{241.86}{10} = 24.19$$

The coefficients of the least squares line are

Slope

$$b_1 = \frac{s_{xy}}{s_x^2} = \frac{36.06}{16.06} = 2.25$$

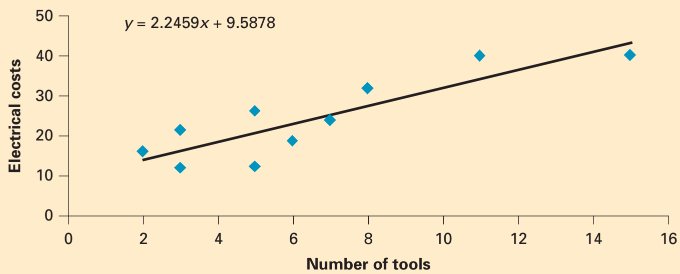
y-intercept:

$$b_0 = \bar{y} - b_1\bar{x} = 24.19 - (2.25)(6.5) = 9.57$$

The least squares line is

$$\hat{y} = 9.57 + 2.25x$$

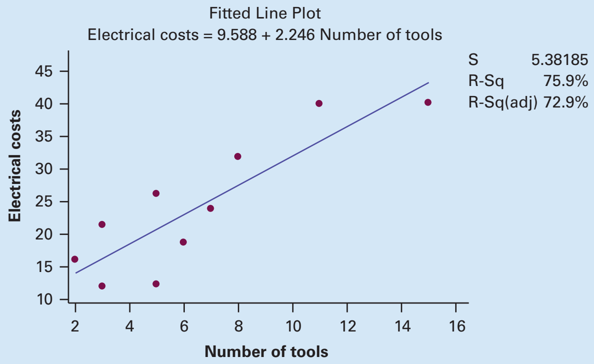
EXCEL



INSTRUCTIONS

1. Type or import the data into two columns where the first column stores the values of X and the second stores Y. (Open Xm04-17.) Highlight the columns containing the variables. Follow the instructions to draw a scatter diagram (page 75).
2. In the **Chart Tools** and **Layout** menu, click **Trendline** and **Linear Trendline.**
3. Click **Trendline** and **More Trendline Options . . . .** Click **Display Equation on Chart.**

MINITAB



INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm04-17.)
2. Click **Stat, Regression**, and **Fitted Line Plot**.
3. Specify the **Response [Y]** (Electrical cost) and the **Predictor [X]** (Number of tools) variables. Specify **Linear.**

INTERPRET

The slope is defined as rise/run, which means that it is the change in y (rise) for a one-unit increase in x (run). Put less mathematically, the slope measures the *marginal* rate of change in the dependent variable. The marginal rate of change refers to the effect of increasing the independent variable by one additional unit. In this example, the slope is 2.25, which means that in this sample, for each one-unit increase in the number of tools, the marginal increase in the electricity cost is \$2.25. Thus, the estimated variable cost is \$2.25 per tool.

The y-intercept is 9.57; that is, the line strikes the y-axis at 9.57. This is simply the value of $\hat{y}$ when $x = 0$. However, when $x = 0$, we are producing no tools and hence the estimated fixed cost of electricity is \$9.57 per day.

Because the costs are estimates based on a straight line, we often need to know how well the line fits the data.

EXAMPLE 4.18 Measuring the Strength of the Linear Relationship

DATA
Xm04-17

Calculate the coefficient of correlation for Example 4.17.

SOLUTION

To calculate the coefficient of correlation, we need the covariance and the standard deviations of both variables. The covariance and the variance of X were calculated in Example 4.17. The covariance is

$$s_{xy} = 36.06$$

and the variance of X is

$$s_x^2 = 16.06$$

Standard deviation of X is

$$s_x = \sqrt{s_x^2} = \sqrt{16.06} = 4.01$$

All we need is the standard deviation of Y.

$$s_y^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} y_i^2 - \frac{\left(\sum_{i=1}^{n} y_i\right)^2}{n}\right] = \frac{1}{10-1}\left[6810.20 - \frac{(241.86)^2}{10}\right] = 106.73$$

$$s_y = \sqrt{s_y^2} = \sqrt{106.73} = 10.33$$

The coefficient of correlation is

$$r = \frac{s_{XY}}{s_x s_y} = \frac{36.06}{(4.01)(10.33)} = .8705$$

EXCEL

As with the other statistics introduced in this chapter, there is more than one way to calculate the coefficient of correlation and the covariance. Here are the instructions for both.

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm04-17.) Type the following into any empty cell.

= **CORREL**([Input range of one variable], [Input range of second variable])

In this example, we would enter

= **CORREL**(B1:B11, C1:C11)

To calculate the covariance, replace **CORREL** with **COVAR.**

Another method, which is also useful if you have more than two variables and would like to compute the coefficient of correlation or the covariance for each pair of variables, is to produce the correlation matrix and the variance–covariance matrix. We do the correlation matrix first.

	A	B	C
1		*Number of tools*	*Electrical costs*
2	Number of tools	1	
3	Electrical costs	0.8711	1

INSTRUCTIONS

1. Type or import the data into adjacent columns. (Open Xm04-17.)
2. Click **Data, Data Analysis**, and **Correlation**.
3. Specify the **Input Range (B1:C11).**

The coefficient of correlation between number of tools and electrical costs is .8711 (slightly different from the manually calculated value). (The two 1s on the diagonal of the matrix are the coefficients of number of tools and number of tools, and electrical costs and electrical costs, telling you the obvious.)

Incidentally, the formula for the population parameter ρ (Greek letter *rho*) and for the sample statistic r produce exactly the same value.

The variance–covariance matrix is shown next.

	A	B	C
1		*Number of tools*	*Electrical costs*
2	Number of tools	14.45	
3	Electrical costs	32.45	96.06

INSTRUCTIONS

1. Type or import the data into adjacent columns. (Open Xm04-17.)
2. Click **Data, Data Analysis**, and **Covariance**.
3. Specify the **Input Range (B1:C11).**

Unfortunately, Excel computes the population parameters. In other words, the variance of the number of tools is $\sigma_x^2 = 14.45$, the variance of the electrical costs is $\sigma_y^2 = 96.06$, and the covariance is $\sigma_{xy} = 32.45$. You can convert these parameters to statistics by multiplying each by $n/(n-1)$.

	D	E	F
1		*Number of tools*	*Electrical costs*
2	Number of tools	16.06	
3	Electrical costs	36.06	106.73

MINITAB

Correlations: Number of tools, Electrical costs

Pearson correlation of Number of tools and Electrical costs = 0.871

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm04-17.)
2. Click **Calc, Basic Statistics** and **Correlation . . .** .
3. In the **Variables** box, type **Select** the variables (**Number of Tools, Electrical Costs**).

Covariances: Number of tools, Electrical costs

	Number of tools	Electrical costs
Number of tools	16.0556	
Electrical costs	36.0589	106.7301

INSTRUCTIONS

Click **Covariance . . .** instead of **Correlation . . .** in step 2 above.

INTERPRET

The coefficient of correlation is .8711, which tells us that there is a positive linear relationship between the number of tools and the electricity cost. The coefficient of correlation tells us that the linear relationship is quite strong and thus the estimates of the fixed and variable costs should be good.

Coefficient of Determination

When we introduced the coefficient of correlation (page 128), we pointed out that except for -1, 0, and $+1$ we cannot precisely interpret its meaning. We can judge the coefficient of correlation in relation to its proximity to only -1, 0, and $+1$. Fortunately, we have another measure that can be precisely interpreted. It is the coefficient of determination, which is calculated by squaring the coefficient of correlation. For this reason, we denote it R^2 .

The coefficient of determination measures the amount of variation in the dependent variable that is explained by the variation in the independent variable. For example, if the coefficient of correlation is -1 or $+1$, a scatter diagram would display all the points lining up in a straight line. The coefficient of determination is 1, which we interpret to mean that 100% of the variation in the dependent variable Y is explained by the variation in the independent variable X. If the coefficient of correlation is 0, then there is no linear relationship between the two variables, $R^2 = 0$, and none of the variation in Y is explained by the variation in X. In Example 4.18, the coefficient of correlation was calculated to be $r = .8711$. Thus, the coefficient of determination is

$$r^2 = (.8711)^2 = .7588$$

This tells us that 75.88% of the variation in electrical costs is explained by the number of tools. The remaining 24.12% is unexplained.

Using the Computer

EXCEL

You can use Excel to calculate the coefficient of correlation and then square the result. Alternatively, use Excel to draw the least squares line. After doing so, click **Trendline, Trendline Options**, and **Display R-squared value on chart.**

MINITAB

Minitab automatically prints the coefficient of determination.

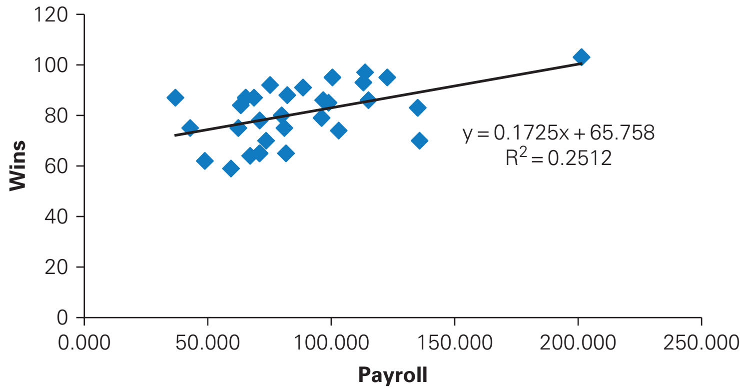
The concept of explained variation is an extremely important one in statistics. We return to this idea repeatedly in Chapters 13, 14, 16, 17, and 18. In Chapter 16, we explain why we interpret the coefficient of determination in the way that we do.

Cost of One More Win: Solution

To determine the cost of an additional win, we must describe the relationship between two variables. To do so, we use the least squares method to produce a straight line through the data. Because we believe that the number of games a baseball team wins depends to some extent on its team payroll, we label Wins as the dependent variable and Payroll as the independent variable. Because of rounding problems, we expressed the payroll in the number of millions of dollars.

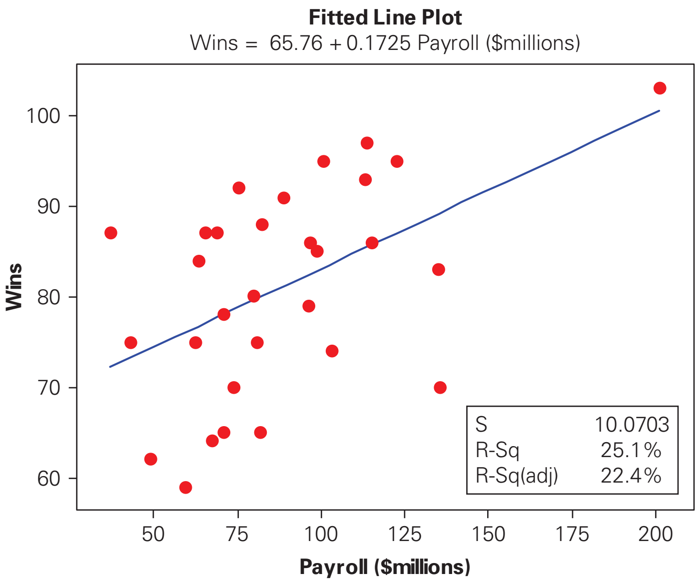
© AP Photo/Charles Krupa

EXCEL



As you can see, Excel outputs the least squares line and the coefficient of determination.

MINITAB



INTERPRET

The least squares line is

$$\hat{y} = 65.758 + .1725\,x$$

The slope is equal to .1725, which is the marginal rate of change in games won for each one-unit increase in payroll. Because payroll is measured in millions of dollars, we estimate that for each $1 million increase in the payroll, the number of games won increases on average by .1725. Thus, to win one more game requires on average an additional expenditure of an incredible $5,797,101 (calculated as 1 million/.1725).

Besides analyzing the least squares line, we should determine the strength of the linear relationship. The coefficient of determination is .2512, which means that the variation in the team's payroll explains 25.12% of the variation in the team's number of games won. This suggests that some teams win a small number of games with large payrolls, whereas others win a large number of games with small payrolls. In the next section, we will return to this issue and examine why some teams perform better than predicted by the least squares line.

Interpreting Correlation

Because of its importance, we remind you about the correct interpretation of the analysis of the relationship between two interval variables that we discussed in Chapter 3. In other words, if two variables are linearly related, it does not mean that X causes Y. It may mean that another variable causes both X and Y or that Y causes X. Remember

Correlation is not Causation

We complete this section with a review of when to use the techniques introduced in this section.

Factors That Identify When to Compute Covariance, Coefficient of Correlation, Coefficient of Determination, and Least Squares Line

1. **Objective**: Describe the relationship between two variables
2. **Type of data**: Interval

EXERCISES

4.63 The covariance of two variables has been calculated to be –150. What does the statistic tell you about the relationship between the two variables?

4.64 Refer to Exercise 4.63. You've now learned that the two sample standard deviations are 16 and 12.

a. Calculate the coefficient of correlation. What does this statistic tell you about the relationship between the two variables?

b. Calculate the coefficient of determination and describe what this says about the relationship between the two variables.

4.65 Xr04-65 A retailer wanted to estimate the monthly fixed and variable selling expenses. As a first step, she collected data from the past 8 months. The total selling expenses ($1,000) and the total sales ($1,000) were recorded and listed below.

Total Sales	Selling Expenses
20	14
40	16
60	18
50	17
50	18
55	18
60	18
70	20

a. Compute the covariance, the coefficient of correlation, and the coefficient of determination and describe what these statistics tell you.
b. Determine the least squares line and use it to produce the estimates the retailer wants.

4.66 Xr04-66 Are the marks one receives in a course related to the amount of time spent studying the subject? To analyze this mysterious possibility, a student took a random sample of 10 students who had enrolled in an accounting class last semester. He asked each to report his or her mark in the course and the total number of hours spent studying accounting. These data are listed here.

Marks	77	63	79	86	51	78	83	90	65	47
Time spent studying	40	42	37	47	25	44	41	48	35	28

a. Calculate the covariance.
b. Calculate the coefficient of correlation.
c. Calculate the coefficient of determination.
d. Determine the least squares line.
e. What do the statistics calculated above tell you about the relationship between marks and study time?

4.67 Xr04-67 Students who apply to MBA programs must take the Graduate Management Admission Test (GMAT). University admissions committees use the GMAT score as one of the critical indicators of how well a student is likely to perform in the MBA program. However, the GMAT may not be a very strong indicator for all MBA programs. Suppose that an MBA program designed for middle managers who wish to upgrade their skills was launched 3 years ago. To judge how well the GMAT score predicts MBA performance, a sample of 12 graduates was taken. Their grade point averages in the MBA program (values from 0 to 12) and their GMAT score (values range from 200 to 800) are listed here. Compute the covariance, the coefficient of correlation, and the coefficient of determination. Interpret your findings.

GMAT and GPA Scores for 12 MBA Students

GMAT	599	689	584	631	594	643
MBA GPA	9.6	8.8	7.4	10.0	7.8	9.2
GMAT	656	594	710	611	593	683
MBA GPA	9.6	8.4	11.2	7.6	8.8	8.0

The following exercises require a computer and software.

4.68 Xr04-68 The unemployment rate is an important measure of a country's economic health. The unemployment rate measures the percentage of people who are looking for work and who are without jobs. Another way of measuring this economic variable is to calculate the employment rate, which is the percentage of adults who are employed. Here are the unemployment rates and employment rates of 19 countries. Calculate the coefficient of determination and describe what you have learned.

Country	Unemployment Rate	Employment Rate
Australia	6.7	70.7
Austria	3.6	74.8
Belgium	6.6	59.9
Canada	7.2	72.0
Denmark	4.3	77.0
Finland	9.1	68.1
France	8.6	63.2
Germany	7.9	69.0
Hungary	5.8	55.4
Ireland	3.8	67.3
Japan	5.0	74.3
Netherlands	2.4	65.4
New Zealand	5.3	62.3
Poland	18.2	53.5
Portugal	4.1	72.2
Spain	13.0	57.5
Sweden	5.1	73.0
United Kingdom	5.0	72.2
United States	4.8	73.1

(*Source:* National Post Business.)

4.69 Xr04-69 All Canadians have government-funded health insurance, which pays for any medical care they require. However, when traveling out of the country, Canadians usually acquire supplementary health insurance to cover the difference between the costs incurred for emergency treatment and what the government program pays. In the United States, this cost differential can be prohibitive. Until recently, private insurance companies (such as BlueCross BlueShield) charged everyone the same weekly rate, regardless of age. However, because of rising costs and the realization that older people frequently incur greater medical emergency expenses, insurers had to change their premium plans. They decided to offer rates that depend on the age of the customer. To help determine the new rates, one insurance company gathered data concerning the age and mean daily medical expenses of a random sample of 1,348 Canadians during the previous 12-month period.

a. Calculate the coefficient of determination.
b. What does the statistic calculated in part (a) tell you?
c. Determine the least squares line.
d. Interpret the coefficients.
e. What rate plan would you suggest?

4.70 Xr04-70 A real estate developer of single-family dwellings across the country is in the process of developing plans for the next several years. An analyst for the company believes that interest rates are likely to increase but remain at low levels. To help make decisions about the number of homes to build, the developer acquired the monthly bank prime rate and the number of new single-family homes sold monthly (thousands) from 1963 to 2009. (*Source:* Federal Reserve Statistics and U.S. Census Bureau.)

Calculate the coefficient of determination. Explain what this statistic tells you about the relationship between the prime bank rate and the number of single-family homes sold.

4.71 Xr04-71 When the price of crude oil increases, do oil companies drill more oil wells? To determine the strength and nature of the relationship, an economist recorded the price of a barrel of domestic crude oil (West Texas crude) and the number of exploratory oil wells drilled for each month from 1973 to 2009. Analyze the data and explain what you have discovered. (*Source:* U.S. Department of Energy.)

4.72 Xr04-72 One way of measuring the extent of unemployment is through the help wanted index, which measures the number of want ads in the nation's newspapers. The higher the index, the greater the demand for workers. Another measure is the unemployment rate among insured workers. An economist wanted to know whether these two variables are related and, if so, how. He acquired the help wanted index and unemployment rates for each month between 1951 and 2006 (last year available). Determine the strength and direction of the relationship. (*Source*: U.S. Department of Labor Statistics.)

4.73 Xr04-73 A manufacturing firm produces its products in batches using sophisticated machines and equipment. The general manager wanted to investigate the relationship between direct labor costs and the number of units produced per batch. He recorded the data from the last 30 batches. Determine the fixed and variable labor costs.

4.74 Xr04-74 A manufacturer has recorded its cost of electricity and the total number of hours of machine time for each of 52 weeks. Estimate the fixed and variable electricity costs.

4.75 Xr04-75 The chapter-opening example showed that there is a linear relationship between a baseball team's payroll and the number of wins. This raises the question, are success on the field and attendance related? If the answer is no, then profit-driven owners may not be inclined to spend money to improve their teams. The statistics practitioner recorded the number of wins and the average home attendance for the 2009 baseball season.
a. Calculate whichever parameters you wish to help guide baseball owners.
b. Estimate the marginal number of tickets sold for each additional game won.

4.76 Xr04-76 Refer to Exercise 4.75. The practitioner also recorded the average away attendance for each team. Visiting teams take a share of the gate, so every owner should be interested in this analysis.
a. Are visiting team attendance figures related to number of wins?
b. Estimate the marginal number of tickets sold for each additional game won.

4.77 Xr04-77 The number of wins and payrolls for the each team in the National Basketball Association (NBA) in the 2008–2009 season were recorded.
a. Determine the marginal cost of one more win.
b. Calculate the coefficient of determination and describe what this number tells you.

4.78 Xr04-78 The number of wins and payrolls for each team in the National Football League (NFL) in the 2009–2010 season were recorded.
a. Determine the marginal cost of one more win.
b. Calculate the coefficient of determination and describe what this number tells you.

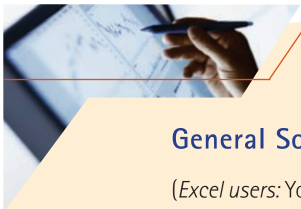
4.79 Xr04-79 The number of wins and payrolls for each team in the National Hockey League (NHL) in the 2008–2009 season were recorded.
a. Determine the marginal cost of one more win.
b. Calculate the coefficient of determination and describe what this number tells you.

4.80 Xr04-80 We recorded the home and away attendance for the NBA for the 2008–2009 season.
a. Analyze the relationship between the number of wins and home attendance.
b. Perform a similar analysis for away attendance.

4.81 Xr04-81 Refer to Exercise 4.77. The relatively weak relationship between the number of wins and home attendance may be explained by the size of the arena each team plays in. The ratio of home attendance to the arena's capacity was calculated. Is percent of capacity more strongly related to the number of wins than average home attendance? Explain.

4.82 Xr04-82 Analyze the relationship between the number of wins and home and away attendance in the National Football League in the 2009–2010 season.

4.83 Xr04-83 Repeat Exercise 4.81 for the NFL.



General Social Survey Exercises

(*Excel users:* You must have adjacent columns. We recommend that you copy the two columns into adjacent columns in a separate spreadsheet.)

4.84 GSS2008* Do more educated people watch less television?

a. To answer this question use the least squares method to determine how education (EDUC) affects the amount of time spent watching television (TVHOURS).

b. Measure the strength of the linear relationship using an appropriate statistic and explain what the statistic tells you.

4.85 GSS2006* Using the 2006 survey, determine whether the number of years of education (EDUC) of the respondent is linearly related to the number of years of education of his or her father (PAEDUC).

American National Election Survey Exercise

4.86 ANES2008* Determine whether the age (AGE) of the respondent and the amount of time he or she watches television news in a typical week (TIME2) are linearly related.

4.5 (Optional) Applications in Professional Sports: Baseball

In the chapter-opening example, we provided the payrolls and the number of wins from the 2009 season. We discovered that there is a weak positive linear relationship between number of wins and payroll. The strength of the linear relationship tells us that some teams with large payrolls are not successful on the field, whereas some teams with small payrolls win a large number of games. It would appear that although the amount of money teams spend is a factor, another factor is *how* teams spend their money. In this section, we will analyze the eight seasons between 2002 and 2009 to see how small-payroll teams succeed.

Professional sports in North America is a multibillion-dollar business. The cost of a new franchise in baseball, football, basketball, and hockey is often in the hundreds of millions of dollars. Although some teams are financially successful during losing seasons, success on the field is often related to financial success. (Exercises 4.75 and 4.76 reveal that there is a financial incentive to win more games.)

It is obvious that winning teams have better players. But how does a team get better players? Teams acquire new players in three ways:

1. They can draft players from high school and college.
2. They can sign free agents on their team or on other teams.
3. They can trade with other teams.

Drafting Players

Every year, high school and university players are drafted by major league baseball teams. The order of the draft is in reverse order of the winning percentage the previous season. Teams that rank low in the standings rank high in the draft. A team that drafts and signs a player owns the rights to that player for his first 7 years in the minor leagues and his first 6 years in the major leagues. The decision on whom to draft and in what order is made by the general manager and a group of scouts who travel the country watching high school and college games. Young players are often invited to a camp where variables such as running speed, home run power, and, for pitchers, velocity are measured. They are often judged by whether a young man "looks" like a player. That is, taller, more athletic men are judged to be better than shorter, heavier ones.

Free Agency

For the first 3 years in the major leagues, the team can pay a player the minimum, which in 2009 was $400,000 per year. After 3 years, the player is eligible for arbitration. A successful player can usually increase his salary from $2 million to $3 million through arbitration. After 6 years, the player can become a free agent and can sign with any major league team. The top free agents can make well in excess of $10 million per year in a multiyear contract.

Trading

Teams will often trade with each other hoping that the players they acquire will help them more than the players they traded away. Many trades produce little improvement in both teams. However, in the history of baseball, there have been some very one-sided trades that resulted in great improvement in one team and the weakening of the other.

As you can see from the solved chapter-opening example "Cost of One More Win," there is a great variation in team payrolls. In 2009, the New York Yankees spent $201 million, while the Florida Marlins spent $37 million (the amounts listed are payrolls at the beginning of the season). To a very large extent, the ability to finance expensive teams is a function of the city in which the team plays. For example, teams in New York, Los Angeles, Atlanta, and Arlington, Texas, are in large markets. Tampa Bay, Oakland, and Minnesota are small-market teams. Large-market teams can afford higher salaries because of the higher gate receipts and higher fees for local television. This means that small-market teams cannot compete for the services of the top free agency players, and thus are more likely to be less successful than large-market teams.

The question arises, can small-market teams be successful on the field and, if so, how? The answer lies in how players are assessed for the draft and for trades. The decisions about whom to draft, whom to trade for, and whom to give in return are made by the team's general manager with the assistance of his assistants and the team's scouts. Because scouts are usually former major league and minor league players who were trained by other former minor league and major league players, they tend to generally agree on the value of the players in the draft. Similarly, teams making trades often trade players of "equal" value. As a result, most teams evaluate players in the same way, yielding little differences in a player's worth. This raises the question, how can a team get the edge on other teams? The answer lies in statistics.

You won't be surprised to learn that the two most important variables in determining the number of wins are the number of runs the team scores and the number of runs the team allows. The number of runs allowed is a function of the quality of the team's pitchers and, to a lesser extent, the defense. Most major league teams evaluate pitchers on the velocity of their fastball. Velocities in the 90 to 100 mile per hour range get the scouts' attention. High school and college pitchers with fastball speeds in the 80s are seldom drafted in the early rounds even when they appear to allow fewer runs by opposing teams.

Scouts also seek out high school and college players with high batting averages and who hit home runs in high school and college.

The only way that small-budget teams can succeed is for them to evaluate players more accurately. In practice, this means that they need to judge players differently from the other teams. In the following analysis, we concentrate on the number of runs a team scores and the statistics that are related to this variable.

If the scouts are correct in their method of evaluating young players, the variables that would be most strongly related to the number of runs a team scores are batting average (BA) and the number of home runs (HR). (A player's batting average is computed by calculating the number of times the player hits divided by the number of at bats less bases on balls.) The coefficients of correlation for seasons 2002 to 2009 are listed here.

	Year							
Coefficients of Correlation	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**
Number of runs & batting average	.828	.889	.803	.780	.672	.762	.680	.748
Number of runs & home runs	.682	.747	.765	.713	.559	.536	.617	.744

Are there better statistics? In other words, are there other team statistics that correlate more highly with the number of runs a team scores? There are two candidates. The first is the teams' on-base average (OBA); the second is the slugging percentage (SLG). The OBA is the number of hits plus bases on balls plus being hit by the pitcher divided by the number of at bats. The SLG is calculated by dividing the total number of bases (single = 1, double = 2, triple = 3, and home run = 4) by the number of at bats minus bases on balls and hit by pitcher. The coefficients of correlation are listed here.

	Year							
Coefficients of Correlation	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**
Number of runs and on-base average	.835	.916	.875	.783	.800	.875	.834	.851
Number of runs and slugging percentage	.913	.951	.929	.790	.854	.885	.903	.911

As you can see, for all eight seasons the OBA had a higher correlation with runs than did the BA.

Comparing the coefficients of correlation of runs with HR and SLG, we can see that in all eight seasons SLG was more strongly correlated than was HR.

As we've pointed out previously, we cannot definitively conclude a causal relationship from these statistics. However, because most decisions are based on the BA and HR, these statistics suggest that general managers should place much greater weight on batters' ability to get on base instead of simply reading the batting averages.

The Oakland Athletics (and a Statistics) Success Story*

From 2002 to 2006, no team was as successful as the Oakland Athletics in converting a small payroll into a large number of wins. In 2002, Oakland's payroll was $40 million and the team won 103 games. In the same season, the New York Yankees spent $126 million and won the same number of games. In 2003, Oakland won 96 games, second to

*The Oakland success story is described in the book Moneyball : *The Art of Winning an Unfair Game* by Michael Lewis, New York London: W.W. Norton

New York's 101 games. Oakland's payroll was $50 million, whereas New York's was $153 million. In 2004, Oakland won 91 games with a payroll of $59 million, and the Yankees won 101 games with a payroll of $184 million. Oakland won 88 games in 2005, and the Yankees won 95 games. Payrolls were Oakland $55 million, Yankees $208 million. In 2006, the team payrolls were Oakland $62 million, Yankees $199 million. Team wins were Oakland 93, Yankees 97.

The Athletics owe their success to general managers who were willing to rethink how teams win. The first of these general managers was Sandy Alderson, who was hired by Oakland in 1993. He was a complete outsider with no baseball experience. This was an unusual hire in an organization in which managers and general managers are either former players or individuals who worked their way up the organization after years of service in a variety of jobs. At the time, Oakland was a high-payroll team that was successful on the field. It was in the World Series in 1988, 1989, and 1990, and had the highest payroll in 1991. The team was owned by Walter A. Haas, Jr., who was willing to spend whatever was necessary to win. When he died in 1995, the new owners decided that the payroll was too large and limited it. This forced Alderson to rethink strategy.

Sandy Alderson was a lawyer and a former marine. Because he was an outsider, he approached the job in a completely different way. He examined each aspect of the game and, among other things, concluded that before three outs everything was possible, but after three outs nothing was possible. This led him to the conclusion that the way to score runs is to minimize each player's probability of making an out. Rather than judge a player by his batting average, which is the way every other general manger assessed players, it would make more sense to judge the player on his on-base average. The on-base average (explained previously) is the probability of *not* making an out. Thus was born something quite rare in baseball—a new idea.

Alderson's replacement is Billy Beane, who continued and extended Alderson's thinking, including hiring a Harvard graduate to help manage the statistics.

In the previous edition of this book, we asked the question, why don't other teams do the same? The answer in 2010 is that other teams have. In the last three years, Oakland has not been anywhere nearly as successful as it was in the previous five (winning about 75 games each year). Apparently, Oakland's approach to evaluating players has influenced other teams. However, there is still a weak linear relationship between team success and team payroll. This means that other variables affect how many games a team wins besides what the team pays its players. Perhaps some clever general manager will find these variables. If he or she does, it may be best not to publicize the discovery in another book.

4.6 (Optional) Applications in Finance: Market Model

In the Applications in Finance box in Chapter 3 (page 52), we introduced the terms *return on investment* and *risk*. We described two goals of investing. The first is to maximize the expected or mean return and the second is to minimize the risk. Financial analysts use a variety of statistical techniques to achieve these goals. Most investors are risk averse, which means that for them minimizing risk is of paramount importance. In Section 4.2, we pointed out that variance and standard deviation are used to measure the risk associated with investments.

APPLICATIONS in **FINANCE**

Stock Market Indexes

Stock markets such as the New York Stock Exchange (NYSE), NASDAQ, Toronto Stock Exchange (TSE), and many others around the world calculate indexes to provide information about the prices of stocks on their exchanges. A stock market index is composed of a number of stocks that more or less represent the entire market. For example, the Dow Jones Industrial Average (DJIA) is the average price of a group of 30 NYSE stocks of large publicly traded companies. The Standard and Poor's 500 Index (S&P) is the average price of 500 NYSE stocks. These indexes represent their stock exchanges and give readers a quick view of how well the exchange is doing as well as the economy of the country as a whole. The NASDAQ 100 is the average price of the 100 largest nonfinancial companies on the NASDAQ exchange. The S&P/TSX Composite Index is composed of the largest companies on the TSE.

In this section, we describe one of the most important applications of the use of a least squares line. It is the well-known and often applied *market model.* This model assumes that the rate of return on a stock is linearly related to the rate of return on the stock market index. The return on the index is calculated in the same way the return on a single stock is computed. For example, if the index at the end of last year was 10,000 and the value at the end of this year is 11,000, then the market index annual return is 10%. The return on the stock is the dependent variable Y, and the return on the index is the independent variable X.

We use the least squares line to represent the linear relationship between X and Y. The coefficient b_1 is called the stock's *beta coefficient*, which measures how sensitive the stock's rate of return is to changes in the level of the overall market. For example, if b_1 is greater than 1, then the stock's rate of return is more sensitive to changes in the level of the overall market than the average stock. To illustrate, suppose that $b_1 = 2$. Then a 1% increase in the index results in an average increase of 2% in the stock's return. A 1% decrease in the index produces an average 2% decrease in the stock's return. Thus, a stock with a beta coefficient greater than 1 will tend to be more volatile than the market.

EXAMPLE 4.19 Market Model for Research in Motion

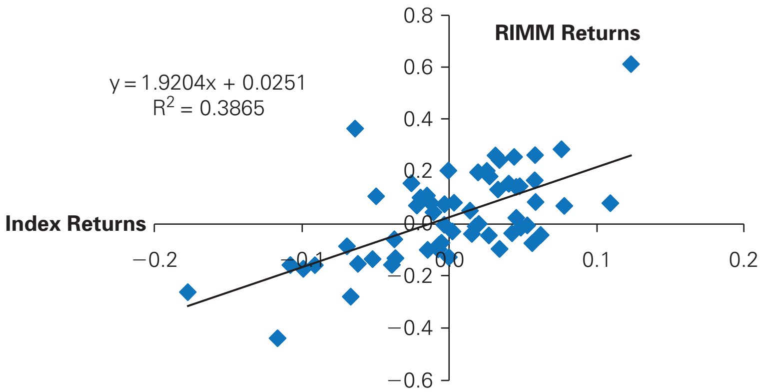
DATA Xm04-19

The monthly rates of return for Research in Motion, maker of the BlackBerry (symbol RIMM), and the NASDAQ index (a measure of the overall NASDAQ stock market) were recorded for each month between January 2005 and December 2009. Some of these data are shown below. Estimate the market model and analyze the results.

Month-Year	Index	RIMM
Jan-05	−0.05196	−0.13506
Feb-05	−0.00518	−0.07239
Mar-05	−0.02558	0.15563
Apr-05	−0.03880	−0.15705
May-05	0.07627	0.28598
Jun-05	−0.00544	−0.10902
Jul-09	0.07818	0.06893
Aug-09	0.01545	−0.03856
Sep-09	0.05642	−0.07432
Oct-09	−0.03643	−0.13160
Nov-09	0.04865	−0.01430
Dec-09	0.05808	0.16670

SOLUTION

Excel's scatter diagram and least squares line are shown below. (Minitab produces a similar result.) We added the equation and the coefficient of determination to the scatter diagram.



We note that the slope coefficient for RIMM is 1.9204. We interpret this to mean that for each 1% increase in the NASDAQ index return in this sample, the average increase in RIMM's return is 1.9204%. Because b_1 is greater than 1, we conclude that the return on investing in Research in Motion is more volatile and therefore riskier than the entire NASDAQ market.

Systematic and Firm-Specific Risk

The slope coefficient b_1 is a measure of the stock's *market-related* (or *systematic*) *risk* because it measures the volatility of the stock price that is related to the overall market volatility. The slope coefficient only informs us about the nature of the relationship between the two sets of returns. It tells us nothing about the *strength* of the linear relationship.

The coefficient of determination measures the proportion of the total risk that is market related. In this case, we see that 38.65% of RIMM's total risk is market related. That is, 38.65% of the variation in RIMM's returns is explained by the variation in the NASDAQ index's returns. The remaining 61.35% is the proportion of the risk that is associated with events specific to RIMM rather than the market. Financial analysts (and most everyone else) call this the *firm-specific* (or *nonsystematic) risk*. The firm-specific risk is attributable to variables and events not included in the market model, such as the effectiveness of RIMM's sales force and managers. This is the part of the risk that can be "diversified away" by creating a portfolio of stocks (as will be discussed in Section 7.3). We cannot, however, diversify away the part of the risk that is market related.

When a portfolio has been created, we can estimate its beta by averaging the betas of the stocks that compose the portfolio. If an investor believes that the market is likely to rise, then a portfolio with a beta coefficient greater than 1 is desirable. Risk-averse investors or ones who believe that the market will fall will seek out portfolios with betas less than 1.

EXERCISES

The following exercises require the use of a computer and software.

4.87 Xr04-87 We have recorded the monthly returns for the S&P 500 index and the following six stocks listed on the New York Stock Exchange for the period January 2005 to December 2009.

AT&T
Aetna
Cigna
Coca-Cola
Disney
Ford
McDonald's

Calculate the beta coefficient for each stock and briefly describe what it means. (*Excel users:* To use the scatter diagram to compute the beta coefficient, the data must be stored in two adjacent columns. The first must contain the returns on the index, and the second stores the returns for whichever stock whose coefficient you wish to calculate.)

4.88 Xm04-88 Monthly returns for the Toronto Stock Exchange index and the following stocks on the Toronto Stock Exchange were recorded for the years 2005 to 2009.

Barrick Gold
Bell Canada Enterprises (BCE)
Bank of Montreal (BMO)
Enbridge
Fortis
Methanex
Research in Motion (RIM)
Telus
Trans Canada Pipeline

Calculate the beta coefficient for each stock and discuss what you have learned about each stock.

4.89 X04-89 We calculated the returns on the NASDAQ index and the following stocks on the NASDAQ exchange for the period January 2005 to December 2009.

Amazon
Amgen
Apple
Cisco Systems
Google
Intel
Microsoft
Oracle
Research in Motion

Calculate the beta coefficient for each stock and briefly describe what it means.

4.7 COMPARING GRAPHICAL AND NUMERICAL TECHNIQUES

As we mentioned before, graphical techniques are useful in producing a quick picture of the data. For example, you learn something about the location, spread, and shape of a set of interval data when you examine its histogram. Numerical techniques provide the same approximate information. We have measures of central location, measures of variability, and measures of relative standing that do what the histogram does. The scatter diagram graphically describes the relationship between two interval variables, but so do the numerical measures covariance, coefficient of correlation, coefficient of determination, and least squares line. Why then do we need to learn both categories of techniques? The answer is that they differ in the information each provides. We illustrate the difference between graphical and numerical methods by redoing four examples we used to illustrate graphical techniques in Chapter 3.

EXAMPLE 3.2 Comparing Returns on Two Investments

In Example 3.2, we wanted to judge which investment appeared to be better. As we discussed in the Applications in Finance: Return on Investment (page 52), we judge investments in terms of the return we can expect and its risk. We drew histograms and attempted to interpret them. The centers of the histograms provided us with information about the expected return and their spreads gauged the risk. However, the histograms were not clear. Fortunately, we can use numerical measures. The mean and median provide us with information about the return we can expect, and the variance or standard deviation tell us about the risk associated with each investment.

Here are the descriptive statistics produced by Excel. Minitab's are similar. (We combined the output into one worksheet.)

Microsoft Excel Output for Example 3.2

	A	B	C	D	E
1	*Return A*			*Return B*	
2					
3	Mean	10.95		Mean	12.76
4	Standard Error	3.10		Standard Error	3.97
5	Median	9.88		Median	10.76
6	Mode	12.89		Mode	#N/A
7	Standard Deviation	21.89		Standard Deviation	28.05
8	Sample Variance	479.35		Sample Variance	786.62
9	Kurtosis	-0.32		Kurtosis	-0.62
10	Skewness	0.54		Skewness	0.01
11	Range	84.95		Range	106.47
12	Minimum	-21.95		Minimum	-38.47
13	Maximum	63		Maximum	68
14	Sum	547.27		Sum	638.01
15	Count	50		Count	50

We can now see that investment B has a larger mean and median but that investment A has a smaller variance and standard deviation. If an investor were interested in low-risk investments, then he or she would choose investment A. If you reexamine the histograms from Example 3.2 (page 53), you will see that the precision provided by the numerical techniques (mean, median, and standard deviation) provides more useful information than did the histograms.

EXAMPLES 3.3 AND 3.4

Business Statistics Marks; Mathematical Statistical Marks

In these examples we wanted to see what differences existed between the marks in the two statistics classes. Here are the descriptive statistics. (We combined the two printouts in one worksheet.)

Microsoft Excel Output for Examples 3.3 and 3.4

	A	B	C	D	E
1	*Marks (Example 3.3)*			*Marks (Example 3.4)*	
2					
3	Mean	72.67		Mean	66.40
4	Standard Error	1.07		Standard Error	1.610
5	Median	72		Median	71.5
6	Mode	67		Mode	75
7	Standard Deviation	8.29		Standard Deviation	12.470
8	Sample Variance	68.77		Sample Variance	155.498
9	Kurtosis	-0.36		Kurtosis	-1.241
10	Skewness	0.16		Skewness	-0.217
11	Range	39		Range	48
12	Minimum	53		Minimum	44
13	Maximum	92		Maximum	92
14	Sum	4360		Sum	3984
15	Count	60		Count	60
16	Largest(15)	79		Largest(15)	76
17	Smallest(15)	67		Smallest(15)	53

The statistics tell us that the mean and median of the marks in the business statistics course (Example 3.3) are higher than in the mathematical statistics course (Example 3.4). We found that the histogram of the mathematical statistics marks was bimodal, which we interpreted to mean that this type of approach created differences between students. The unimodal histogram of the business statistics marks informed us that this approach eliminated those differences.

Chapter 3 Opening Example

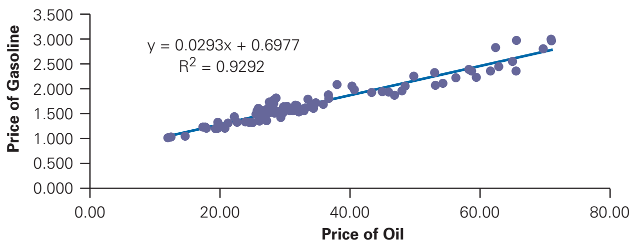
In this example, we wanted to know whether the prices of gasoline and oil were related. The scatter diagram did reveal a strong positive linear relationship. We can improve on the quality of this information by computing the coefficient of correlation and drawing the least squares line.

Excel Output for Chapter 3 Opening Example: Coefficient of Correlation

	A	B	C
1		*Oil*	*Gasoline*
2	Oil	1	
3	Gasoline	0.8574	1

The coefficient of correlation seems to confirm what we learned from the scatter diagram: There is a moderately strong positive linear relationship between the two variables.

Excel Output for Chapter 3 Opening Example: Least Squares Line



The slope coefficient tells us that for each dollar increase in the price of a barrel of oil, the price of a (U.S.) gallon of gasoline increases an average of 2.9 cents. However, because there are 42 gallons per barrel, we would expect a dollar increase in a barrel of oil to yield a 2.4† cents per gallon increase (calculated as $1.00/42). It does appear that the oil companies are taking some small advantage by adding an extra half-cent per gallon. The coefficient of determination is .929, which indicates that 92.9% of the variation in gasoline prices is explained by the variation in oil prices.

†This is a simplification. In fact a barrel of oil yields a variety of other profitable products. See Exercise 2.14.

EXERCISES

The following exercises require a computer and statistical software.

4.90 Xr03-23 Refer to Exercise 3.23
a. Calculate the mean, median, and standard deviation of the scores of those who repaid and of those who defaulted.
b. Do these statistics produce more useful information than the histograms?

4.91 Xr03-24 Refer to Exercise 3.24.
a. Draw box plots of the scores of those who repaid and of those who defaulted.
b. Compare the information gleaned from the histograms to that contained in the box plots. Which are better?

4.92 Xr03-50 Calculate the coefficient of determination for Exercise 3.50. Is this more informative than the scatter diagram?

4.93 Xr03-51 Refer to Exercise 3.51. Compute the coefficients of the least squares line and compare your results with the scatter diagram.

4.94 Xr03-56 Compute the coefficient of determination and the least squares line for Exercise 3.56. Compare this information with that developed by the scatter diagram alone.

4.95 Xr03-59 Refer to Exercise 3.59. Calculate the coefficient of determination and the least squares line. Is this more informative than the scatter diagram?

4.96 Xm03-07 a. Calculate the coefficients of the least squares line for the data in Example 3.7.
b. Interpret the coefficients.
c. Is this information more useful than the information extracted from the scatter diagram?

4.97 Xr04-53 In Exercise 4.53, you drew box plots. Draw histograms instead and compare the results.

4.98 Xr04-55 Refer to Exercise 4.55. Draw histograms of the data. What have you learned?

4.8 GENERAL GUIDELINES FOR EXPLORING DATA

The purpose of applying graphical and numerical techniques is to describe and summarize data. Statisticians usually apply graphical techniques as a first step because we need to know the shape of the distribution. The shape of the distribution helps answer the following questions:

1. Where is the approximate center of the distribution?
2. Are the observations close to one another, or are they widely dispersed?
3. Is the distribution unimodal, bimodal, or multimodal? If there is more than one mode, where are the peaks, and where are the valleys?
4. Is the distribution symmetric? If not, is it skewed? If symmetric, is it bell shaped?

Histograms and box plots provide most of the answers. We can frequently make several inferences about the nature of the data from the shape. For example, we can assess the relative risk of investments by noting their spreads. We can attempt to improve the teaching of a course by examining whether the distribution of final grades is bimodal or skewed.

The shape can also provide some guidance on which numerical techniques to use. As we noted in this chapter, the central location of highly skewed data may be more appropriately measured by the median. We may also choose to use the interquartile range instead of the standard deviation to describe the spread of skewed data.

When we have an understanding of the structure of the data, we may do additional analysis. For example, we often want to determine how one variable, or several variables, affects another. Scatter diagrams, covariance, and the coefficient of correlation are useful techniques for detecting relationships between variables. A number of techniques to be introduced later in this book will help uncover the nature of these associations.

CHAPTER SUMMARY

This chapter extended our discussion of descriptive statistics, which deals with methods of summarizing and presenting the essential information contained in a set of data. After constructing a frequency distribution to obtain a general idea about the distribution of a data set, we can use numerical measures to describe the central location and variability of interval data. Three popular measures of central location, or averages, are the mean, the median, and the mode. Taken by themselves, these measures provide an inadequate description of the data because they say nothing about the extent to which the data vary. Information regarding the variability of interval data is conveyed by such numerical measures as the range, variance, and standard deviation.

For the special case in which a sample of measurements has a mound-shaped distribution, the Empirical Rule provides a good approximation of the percentages of measurements that fall within one, two, and three standard deviations of the mean. Chebysheff's Theorem applies to all sets of data no matter the shape of the histogram.

Measures of relative standing that were presented in this chapter are percentiles and quartiles. The box plot graphically depicts these measures as well as several others. The linear relationship between two interval variables is measured by the covariance, the coefficient of correlation, the coefficient of determination, and the least squares line.

IMPORTANT TERMS

Measures of central location 98
Mean 98
Median 100
Mode 101
Modal class 102
Geometric mean 105
Measures of variability 108
Range 108
Variance 108
Standard deviation 108
Deviation 109
Mean absolute deviation 110
Empirical Rule 113
Chebysheff's Theorem 114
Skewed 114
Coefficient of variation 115
Percentiles 117
Quartiles 118
Interquartile range 120
Box plots 120
Outlier 121
Covariance 127
Coefficient of correlation 128
Least squares method 132

SYMBOLS

Symbol	Pronounced	Represents
μ	mu	Population mean
σ^2	sigma squared	Population variance
σ	sigma	Population standard deviation
ρ	rho	Population coefficient of correlation
$\sum$	Sum of	Summation

Symbol	Pronounced	Represents
$\sum_{i=1}^{n} x_i$	Sum of x_i from 1 to n	Summation of n numbers
$\hat{y}$	y hat	Fitted or calculated value of y
b_0	b zero	y-Intercept
b_1	b one	Slope coefficient

FORMULAS

Population mean

$$\mu = \frac{\sum_{i=1}^{N} x_i}{N}$$

Sample mean

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Range

Largest observation – Smallest observation

Population variance

$$\sigma^2 = \frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N}$$

Sample variance

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}$$

Population standard deviation

$$\sigma = \sqrt{\sigma^2}$$

Sample standard deviation

$$s = \sqrt{s^2}$$

Population covariance

$$\sigma_{xy} = \frac{\sum_{i=1}^{N} (x_i - \mu_x)(y_i - \mu_y)}{N}$$

Sample covariance

$$s_{xy} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{n - 1}$$

Population coefficient of correlation

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

Sample coefficient of correlation

$$r = \frac{s_{xy}}{s_x s_y}$$

Coefficient of determination

$$R^2 = r^2$$

Slope coefficient

$$b_1 = \frac{s_{xy}}{s_x^2}$$

y-intercept

$$b_0 = \bar{y} - b_1 \bar{x}$$

COMPUTER OUTPUT AND INSTRUCTIONS

Technique	Excel	Minitab
Mean	100	100
Median	101	101
Mode	102	102
Variance	111	111
Standard deviation	112	113
Descriptive statistics	119	120
Box plot	121	122
Least squares line	135	136

(Continued)

Technique	Excel	Minitab
Covariance	137	138
Correlation	137	138
Coefficient of determination	139	139

CHAPTER EXERCISES

4.99 Xr04-99* Osteoporosis is a condition in which bone density decreases, often resulting in broken bones. Bone density usually peaks at age 30 and decreases thereafter. To understand more about the condition, researchers recruited a random sample of women aged 50 and older. Each woman's bone density loss was recorded.

a. Compute the mean and median of these data.
b. Compute the standard deviation of the bone density losses.
c. Describe what you have learned from the statistics.

4.100 Xr04-100* The temperature in December in Buffalo, New York, is often below 40 degrees Fahrenheit (4 degrees Celsius). Not surprisingly, when the National Football League Buffalo Bills play at home in December, hot coffee is a popular item at the concession stand. The concession manager would like to acquire more information so that he can manage inventories more efficiently. The number of cups of coffee sold during 50 games played in December in Buffalo were recorded.

a. Determine the mean and median.
b. Determine the variance and standard deviation.
c. Draw a box plot.
d. Briefly describe what you have learned from your statistical analysis.

4.101 Refer to Exercise 4.99. In addition to the bone density losses, the ages of the women were also recorded. Compute the coefficient of determination and describe what this statistic tells you.

4.102 Refer to Exercise 4.100. Suppose that in addition to recording the coffee sales, the manager also recorded the average temperature (measured in degrees Fahrenheit) during the game. These data together with the number of cups of coffee sold were recorded.

a. Compute the coefficient of determination.
b. Determine the coefficients of the least squares line.
c. What have you learned from the statistics calculated in parts (a) and (b) about the relationship between the number of cups of coffee sold and the temperature?
d. Discuss the information obtained here and in Exercise 4.100. Which is more useful to the manager?

4.103 Xr04-103* Chris Golfnut loves the game of golf. Chris also loves statistics. Combining both passions, Chris records a sample of 100 scores.

a. What statistics should Chris compute to describe the scores?
b. Calculate the mean and standard deviation of the scores.
c. Briefly describe what the statistics computed in part (b) divulge.

4.104 Xr04-104* The Internet is growing rapidly with an increasing number of regular users. However, among people older than 50, Internet use is still relatively low. To learn more about this issue, a sample of 250 men and women older than 50 who had used the Internet at least once were selected. The number of hours on the Internet during the past month was recorded.

a. Calculate the mean and median.
b. Calculate the variance and standard deviation.
c. Draw a box plot.
d. Briefly describe what you have learned from the statistics you calculated.

4.105 Refer to Exercise 4.103. For each score, Chris also recorded the number of putts as well as his scores. Conduct an analysis of both sets of data. What conclusions can be achieved from the statistics?

4.106 Refer to Exercise 4.104. In addition to Internet use, the numbers of years of education were recorded.

a. Compute the coefficient of determination.
b. Determine the coefficients of the least squares line.
c. Describe what these statistics tell you about the relationship between Internet use and education.
d. Discuss the information obtained here and in Exercise 4.104.

4.107 Xr04-107* A sample was drawn of one-acre plots of land planted with corn. The crop yields were recorded. Calculate the descriptive statistics you judge to be useful. Interpret these statistics.

4.108 Refer to Exercise 4.107. For each plot, the amounts of rainfall were also recorded.
a. Compute the coefficient of determination.
b. Determine the coefficients of the least squares line.
c. Describe what these statistics tell you about the relationship between crop yield and rainfall.
d. Discuss the information obtained here and in Exercise 4.107.

4.109 Refer to Exercise 4.107. For each plot, the amounts of fertilizer were recorded.
a. Compute the coefficient of determination.
b. Determine the coefficients of the least squares line.
c. Describe what these statistics tell you about the relationship between crop yield and the amount of fertilizer.
d. Discuss the information obtained here and in Exercise 4.107.

4.110 Xr04-110 Increasing tuition has resulted in some students being saddled with large debts at graduation. To examine this issue, a random sample of recent graduates was asked to report whether they had student loans, and, if so, how much was the debt at graduation.
a. Compute all three measures of central location.
b. What do these statistics reveal about student loan debt at graduation?

CASE 4.1 Return to the Global Warming Question

DATA
C04-01a
C04-01b

Now that we have presented techniques that allow us to conduct more precise analyses we'll return to Case 3.1. Recall that there are two issues in this discussion. First, is there global warming; second, if so, is carbon dioxide the cause? The only tools available at the end of Chapter 3 were graphical techniques including line charts and scatter diagrams. You are now invited to apply the more precise techniques in this chapter to answer the same questions.

Here are the data sets you can work with.

C04-01a: Column 1: Months numbered 1 to 1559
Column 2: Temperature anomalies produced by the National Climatic Data Center
C04-01b: Column 1: Monthly carbon dioxide levels measured by the Mauna Loa Observatory
Column 2: Temperature anomalies produced by the National Climatic Data Center

a. Use the least squares method to estimate average monthly changes in temperature anomalies.
b. Calculate the least squares line and the coefficient of correlation between CO_2 levels and temperature anomalies and describe your findings.

CASE 4.2 Another Return to the Global Warming Question

DATA
C04-02a
C04-02b
C04-02c
C04-02d

Did you conclude in Case 4.1 that Earth has warmed since 1880 and that there is some linear relationship between CO_2 and temperature anomalies? If so, here is another look at the same data. C04-02a lists the temperature anomalies from 1880 to 1940, C04-02b lists the data from 1941 to 1975, C04-02c stores temperature anomalies from 1976 to 1997, and C04-02d contains the data from 1998 to 2009. For each set of data, calculate the least squares line and the coefficient of determination. Report your findings.

CASE 4.3 The Effect of the Players' Strike in the 2004–05 Hockey Season

DATA
C04-03a
C04-03b

The 2004–2005 hockey season was canceled because of a players' strike. The key issue in the labor dispute was a "salary cap." The team owners wanted a salary cap to cut their costs. The owners of small-market teams wanted the cap to help their teams become competitive. Of course, caps on salaries would lower the salaries of most players; as a result, the players association fought against it. The team owners prevailed, and the collective bargaining agreement specified a salary cap of $39 million and a floor of $21.5 million for the 2005–2006 season.

Conduct an analysis of the 2003–2004 season (C04-02a) and the 2005–2006 season (C04-02b). For each season:

a. Estimate how much on average a team needs to spend to win one more game.
b. Measure the strength of the linear relationship.
c. Discuss the differences between the two seasons.

CASE 4.4 Quebec Referendum Vote: Was There Electoral Fraud?*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DATA
C04-04

Since the 1960s, Quebecois (citizens of the province of Quebec) have been debating whether to separate from Canada and form an independent nation. A referendum was held on October 30, 1995, in which the people of Quebec voted not to separate. The vote was extremely close with the "no" side winning by only 52,448 votes. A large number of no votes was cast by the non-Francophone (non-French speaking) people of Quebec, who make up about 20% of the population and who very much want to remain Canadians. The remaining 80% are Francophones, a majority of whom voted "yes."

After the votes were counted, it became clear that the tallied vote was much closer than it should have been. Supporters of the no side charged that poll scrutineers, all of whom were appointed by the proseparatist provincial government, rejected a disproportionate number of ballots in ridings (electoral districts) where the percentage of yes votes was low and where there are large numbers of Allophone (people whose first language is neither English nor French) and Anglophone (English-speaking) residents. (Electoral laws require the rejection of ballots that do not appear to be properly marked.) They were outraged that in a strong democracy like Canada, votes would be rigged much as they are in many non-democratic countries around the world.

If, in ridings where there was a low percentage of "yes" votes, there was a high percentage of rejected ballots, this would be evidence of electoral fraud. Moreover, if, in ridings where there were large percentages of Allophone or Anglophone voters (or both), there were high percentages of rejected ballots, this too would constitute evidence of fraud on the part of the scrutineers and possibly the government.

To determine the veracity of the charges, the following variables were recorded for each riding.

Percentage of rejected ballots in referendum

Percentage of "yes" votes

Percentage of Allophones

Percentage of Anglophones

Conduct a statistical analysis of these data to determine whether there are indications that electoral fraud took place.

*This case is based on "Voting Irregularities in the 1995 Referendum on Quebec Sovereignty" by Jason Cawley and Paul Sommers, *Chance*, Vol. 9, No. 4, Fall, 1996. We are grateful to Dr. Paul Sommers, Middlebury College, for his assistance in writing this case.

APPENDIX 4 Review of Descriptive Techniques

Here is a list of the statistical techniques introduced in Chapters 2, 3, and 4. This is followed by a flowchart designed to help you select the most appropriate method to use to address any problem requiring a descriptive method.

To provide practice in identifying the correct descriptive method to use we have created a number of review exercises. These are in Keller's website Appendix Descriptive Techniques Review Exercises.

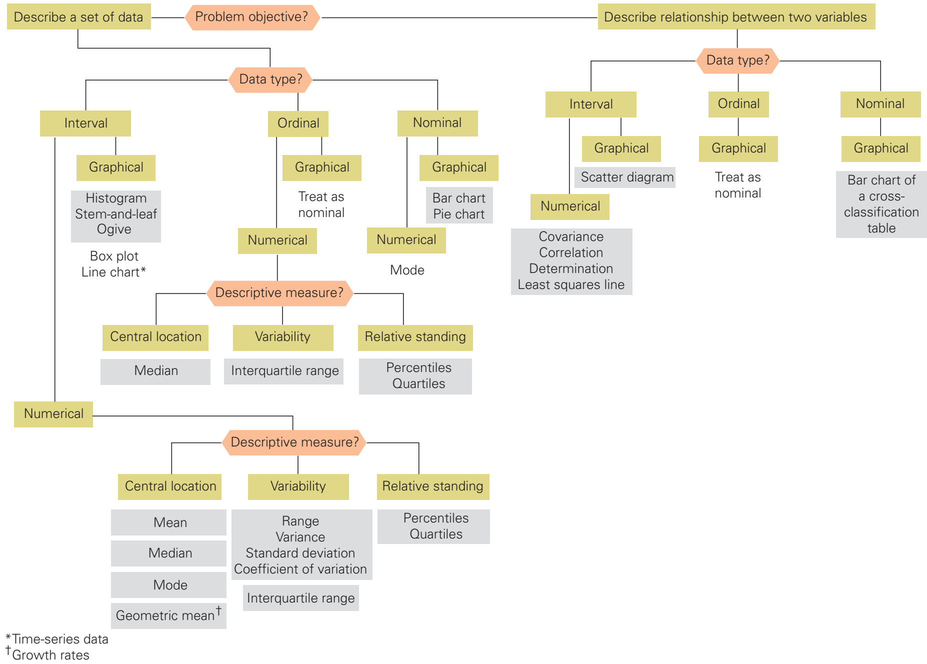
Graphical Techniques

Histogram
Stem-and-leaf display
Ogive
Bar chart
Pie chart
Scatter diagram
Line chart (time series)
Box plot

Numerical Techniques

Measures of Central Location
Mean
Median
Mode
Geometric mean (growth rates)
Measures of Variability
Range
Variance
Standard deviation
Coefficient of variation
Interquartile range
Measures of Relative Standing
Percentiles
Quartiles
Measures of Linear Relationship
Covariance
Coefficient of correlation
Coefficient of determination
Least squares line

Flowchart: Graphical and Numerical Techniques



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5

DATA COLLECTION AND SAMPLING

Sampling and the Census

Courtesy, US Census Bureau

The census, which is conducted every 10 years in the United States, serves an important function. It is the basis for deciding how many congressional representatives and how many votes in the electoral college each state will have. Businesses often use the information derived from the census to help make decisions about products, advertising, and plant locations.

One of the problems with the census is the issue of undercounting, which occurs when some people are not included. For example, the 1990 census reported that 12.05% of adults were African American; the true value was 12.41%. To address undercounting, the Census Bureau adjusts the numbers it gets from the census. The adjustment is based on another survey. The mechanism is called the Accuracy and Coverage Evaluation. Using sampling methods described in this chapter, the Census Bureau is able to adjust the numbers in American subgroups. For example, the Bureau may discover that the number of Hispanics has been undercounted or that the number of people living in California has not been accurately counted.

Later in this chapter we'll discuss how the sampling is conducted and how the adjustments are made.

INTRODUCTION

In Chapter 1, we briefly introduced the concept of statistical inference—the process of inferring information about a population from a sample. Because information about populations can usually be described by parameters, the statistical technique used generally deals with drawing inferences about population parameters from sample statistics. (Recall that a parameter is a measurement about a population, and a statistic is a measurement about a sample.)

Working within the covers of a statistics textbook, we can assume that population parameters are known. In real life, however, calculating parameters is virtually impossible because populations tend to be very large. As a result, most population parameters are not only unknown but also unknowable. The problem that motivates the subject of statistical inference is that we often need information about the value of parameters in order to make decisions. For example, to make decisions about whether to expand a line of clothing, we may need to know the mean annual expenditure on clothing by North American adults. Because the size of this population is approximately 200 million, determining the mean is prohibitive. However, if we are willing to accept less than 100% accuracy, we can use statistical inference to obtain an estimate. Rather than investigating the entire population, we select a sample of people, determine the annual expenditures on clothing in this group, and calculate the sample mean. Although the probability that the sample mean will equal the population mean is very small, we would expect them to be close. For many decisions, we need to know how close. We postpone that discussion until Chapters 10 and 11. In this chapter, we will discuss the basic concepts and techniques of sampling itself. But first we take a look at various sources for collecting data.

5.1 METHODS OF COLLECTING DATA

Most of this book addresses the problem of converting data into information. The question arises, where do data come from? The answer is that a large number of methods produce data. Before we proceed however, we'll remind you of the definition of data introduced in Section 2.1. Data are the observed values of a variable; that is, we define a variable or variables that are of interest to us and then proceed to collect observations of those variables.

Direct Observation

The simplest method of obtaining data is by direct observation. When data are gathered in this way, they are said to be **observational**. For example, suppose that a researcher for a pharmaceutical company wants to determine whether aspirin actually reduces the incidence of heart attacks. Observational data may be gathered by selecting a sample of men and women and asking each whether he or she has taken aspirin regularly over the past 2 years. Each person would be asked whether he or she had suffered a heart attack over the same period. The proportions reporting heart attacks would be compared and a statistical technique that is introduced in Chapter 13 would be used to determine whether aspirin is effective in reducing the likelihood of heart attacks. There are many drawbacks to this method. One of the most critical is that it is difficult to produce useful information in this way. For example, if the statistics practitioner concludes that people who take aspirin suffer fewer heart attacks, can we conclude that aspirin is effective? It may be that people who take aspirin tend to be more health conscious, and health-conscious people tend to have fewer heart attacks. The one advantage to direct observation is that it is relatively inexpensive.

Experiments

A more expensive but better way to produce data is through experiments. Data produced in this manner are called **experimental**. In the aspirin illustration, a statistics practitioner can randomly select men and women. The sample would be divided into two groups. One group would take aspirin regularly, and the other would not. After 2 years, the statistics practitioner would determine the proportion of people in each group who had suffered heart attacks, and statistical methods again would be used to determine whether aspirin works. If we find that the aspirin group suffered fewer heart attacks, then we may more confidently conclude that taking aspirin regularly is a healthy decision.

Surveys

One of the most familiar methods of collecting data is the **survey**, which solicits information from people concerning such things as their income, family size, and opinions on various issues. We're all familiar, for example, with opinion polls that accompany each political election. The Gallup Poll and the Harris Survey are two well-known surveys of public opinion whose results are often reported by the media. But the majority of surveys are conducted for private use. Private surveys are used extensively by market researchers to determine the preferences and attitudes of consumers and voters. The results can be used for a variety of purposes, from helping to determine the target market for an advertising campaign to modifying a candidate's platform in an election campaign. As an illustration, consider a television network that has hired a market research firm to provide the network with a profile of owners of luxury automobiles, including what they watch on television and at what times. The network could then use this information to develop a package of recommended time slots for Cadillac commercials, including costs, which it would present to General Motors. It is quite likely that many students reading this book will one day be marketing executives who will "live and die" by such market research data.

An important aspect of surveys is the **response rate**. The response rate is the proportion of all people who were selected who complete the survey. As we discuss in the next section, a low response rate can destroy the validity of any conclusion resulting from the statistical analysis. Statistics practitioners need to ensure that data are reliable.

Personal Interview Many researchers feel that the best way to survey people is by means of a personal interview, which involves an interviewer soliciting information from a respondent by asking prepared questions. A personal interview has the advantage of having a higher expected response rate than other methods of data collection. In addition, there will probably be fewer incorrect responses resulting from respondents misunderstanding some questions because the interviewer can clarify misunderstandings when asked to. But the interviewer must also be careful not to say too much for fear of biasing the response. To avoid introducing such biases, as well as to reap the potential benefits of a personal interview, the interviewer must be well trained in proper interviewing techniques and well informed on the purpose of the study. The main disadvantage of personal interviews is that they are expensive, especially when travel is involved.

Telephone Interview A telephone interview is usually less expensive, but it is also less personal and has a lower expected response rate. Unless the issue is of interest, many people will refuse to respond to telephone surveys. This problem is exacerbated by telemarketers trying to sell something.

Self-Administered Survey A third popular method of data collection is the self-administered questionnaire, which is usually mailed to a sample of people. This is an inexpensive method of conducting a survey and is therefore attractive when the number of people to be surveyed is large. But self-administered questionnaires usually have a low response rate and may have a relatively high number of incorrect responses due to respondents misunderstanding some questions.

Questionnaire Design Whether a questionnaire is self-administered or completed by an interviewer, it must be well designed. Proper questionnaire design takes knowledge, experience, time, and money. Some basic points to consider regarding questionnaire design follow.

1. First and foremost, the questionnaire should be kept as short as possible to encourage respondents to complete it. Most people are unwilling to spend much time filling out a questionnaire.
2. The questions themselves should also be short, as well as simply and clearly worded, to enable respondents to answer quickly, correctly, and without ambiguity. Even familiar terms such as "*unemployed*" and "*family*" must be defined carefully because several interpretations are possible.
3. Questionnaires often begin with simple demographic questions to help respondents get started and become comfortable quickly.
4. Dichotomous questions (questions with only two possible responses such as "yes" and "no" and multiple-choice questions) are useful and popular because of their simplicity, but they also have possible shortcomings. For example, a respondent's choice of yes or no to a question may depend on certain assumptions not stated in the question. In the case of a multiple-choice question, a respondent may feel that none of the choices offered is suitable.
5. Open-ended questions provide an opportunity for respondents to express opinions more fully, but they are time consuming and more difficult to tabulate and analyze.
6. Avoid using leading questions, such as "Wouldn't you agree that the statistics exam was too difficult?" These types of questions tend to lead the respondent to a particular answer.

7. Time permitting, it is useful to pretest a questionnaire on a small number of people in order to uncover potential problems such as ambiguous wording.

8. Finally, when preparing the questions, think about how you intend to tabulate and analyze the responses. First, determine whether you are soliciting values (i.e., responses) for an interval variable or a nominal variable. Then consider which type of statistical techniques—descriptive or inferential—you intend to apply to the data to be collected, and note the requirements of the specific techniques to be used. Thinking about these questions will help ensure that the questionnaire is designed to collect the data you need.

Whatever method is used to collect primary data, we need to know something about sampling, the subject of the next section.

EXERCISES

5.1 Briefly describe the difference between observational and experimental data.

5.2 A soft drink manufacturer has been supplying its cola drink in bottles to grocery stores and in cans to small convenience stores. The company is analyzing sales of this cola drink to determine which type of packaging is preferred by consumers.
- a. Is this study observational or experimental? Explain your answer.
- b. Outline a better method for determining whether a store will be supplied with cola in bottles or in cans so that future sales data will be more helpful in assessing the preferred type of packaging.

5.3 a. Briefly describe how you might design a study to investigate the relationship between smoking and lung cancer.
- b. Is your study in part (a) observational or experimental? Explain why.

5.4 a. List three methods of conducting a survey of people.
- b. Give an important advantage and disadvantage of each of the methods listed in part (a).

5.5 List five important points to consider when designing a questionnaire.

5.2 SAMPLING

The chief motive for examining a sample rather than a population is cost. Statistical inference permits us to draw conclusions about a population parameter based on a sample that is quite small in comparison to the size of the population. For example, television executives want to know the proportion of television viewers who watch a network's programs. Because 100 million people may be watching television in the United States on a given evening, determining the actual proportion of the population that is watching certain programs is impractical and prohibitively expensive. The Nielsen ratings provide approximations of the desired information by observing what is watched by a sample of 5,000 television viewers. The proportion of households watching a particular program can be calculated for the households in the Nielsen sample. This sample proportion is then used as an **estimate** of the proportion of all households (the population proportion) that watched the program.

Another illustration of sampling can be taken from the field of quality management. To ensure that a production process is operating properly, the operations manager needs to know what proportion of items being produced is defective. If the quality technician must destroy the item to determine whether it is defective, then there is no alternative to sampling: A complete inspection of the product population would destroy the entire output of the production process.

We know that the sample proportion of television viewers or of defective items is probably not exactly equal to the population proportion we want to estimate. Nonetheless, the sample statistic can come quite close to the parameter it is designed to estimate if the **target population** (the population about which we want to draw inferences) and the **sampled population** (the actual population from which the sample has been taken) are the same. In practice, these may not be the same. One of statistics' most famous failures illustrates this phenomenon.

The *Literary Digest* was a popular magazine of the 1920s and 1930s that had correctly predicted the outcomes of several presidential elections. In 1936, the *Digest* predicted that the Republican candidate, Alfred Landon, would defeat the Democratic incumbent, Franklin D. Roosevelt, by a 3 to 2 margin. But in that election, Roosevelt defeated Landon in a landslide victory, garnering the support of 62% of the electorate. The source of this blunder was the sampling procedure, and there were two distinct mistakes.* First, the *Digest* sent out 10 million sample ballots to prospective voters. However, most of the names of these people were taken from the *Digest*'s subscription list and from telephone directories. Subscribers to the magazine and people who owned telephones tended to be wealthier than average and such people then, as today, tended to vote Republican. In addition, only 2.3 million ballots were returned resulting in a self-selected sample.

Self-selected samples are almost always biased because the individuals who participate in them are more keenly interested in the issue than are the other members of the population. You often find similar surveys conducted today when radio and television stations ask people to call and give their opinion on an issue of interest. Again, only listeners who are concerned about the topic and have enough patience to get through to the station will be included in the sample. Hence, the sampled population is composed entirely of people who are interested in the issue, whereas the target population is made up of all the people within the listening radius of the radio station. As a result, the conclusions drawn from such surveys are frequently wrong.

An excellent example of this phenomenon occurred on ABC's *Nightline* in 1984. Viewers were given a 900 telephone number (cost: 50 cents) and asked to phone in their responses to the question of whether the United Nations should continue to be located in the United States. More than 186,000 people called, with 67% responding "no." At the same time, a (more scientific) market research poll of 500 people revealed that 72% wanted the United Nations to remain in the United States. In general, because the true value of the parameter being estimated is never known, these surveys give the impression of providing useful information. In fact, the results of such surveys are likely to be no more accurate than the results of the 1936 *Literary Digest* poll or *Nightline*'s phone-in show. Statisticians have coined two terms to describe these polls: SLOP (self-selected opinion poll) and *Oy vey* (from the Yiddish lament), both of which convey the contempt that statisticians have for such data-gathering processes.

* Many statisticians ascribe the *Literary Digest*'s statistical debacle to the wrong causes. For an understanding of what really happened, read Maurice C. Bryson, "The Literary Digest Poll: Making of a Statistical Myth" *American Statistician* 30(4) (November 1976): 184–185.

EXERCISES

5.6 For each of the following sampling plans, indicate why the target population and the sampled population are not the same.

a. To determine the opinions and attitudes of customers who regularly shop at a particular mall, a surveyor stands outside a large department store in the mall and randomly selects people to participate in the survey.

b. A library wants to estimate the proportion of its books that have been damaged. The librarians decide to select one book per shelf as a sample by measuring 12 inches from the left edge of each shelf and selecting the book in that location.

c. Political surveyors visit 200 residences during one afternoon to ask eligible voters present in the house at the time whom they intend to vote for.

5.7 a. Describe why the *Literary Digest* poll of 1936 has become infamous.

b. What caused this poll to be so wrong?

5.8 a. What is meant by *self-selected sample*?

b. Give an example of a recent poll that involved a self-selected sample.

c. Why are self-selected samples not desirable?

5.9 A regular feature in a newspaper asks readers to respond via e-mail to a survey that requires a yes or no response. In the following day's newspaper, the percentage of yes and no responses are reported. Discuss why we should ignore these statistics.

5.10 Suppose your statistics professor distributes a questionnaire about the course. One of the questions asks, "Would you recommend this course to a friend?" Can the professor use the results to infer something about all statistics courses? Explain.

5.3 Sampling Plans

Our objective in this section is to introduce three different sampling plans: simple random sampling, stratified random sampling, and cluster sampling. We begin our presentation with the most basic design.

Simple Random Sampling

Simple Random Sample

A **simple random sample** is a sample selected in such a way that every possible sample with the same number of observations is equally likely to be chosen.

One way to conduct a simple random sample is to assign a number to each element in the population, write these numbers on individual slips of paper, toss them into a hat, and draw the required number of slips (the sample size, n) from the hat. This is the kind of procedure that occurs in raffles, when all the ticket stubs go into a large rotating drum from which the winners are selected.

Sometimes the elements of the population are already numbered. For example, virtually all adults have Social Security numbers (in the United States) or Social Insurance numbers (in Canada); all employees of large corporations have employee numbers; many people have driver's license numbers, medical plan numbers, student numbers, and so on. In such cases, choosing which sampling procedure to use is simply a matter of deciding how to select from among these numbers.

In other cases, the existing form of numbering has built-in flaws that make it inappropriate as a source of samples. Not everyone has a phone number, for example, so the telephone book does not list all the people in a given area. Many households have two (or more) adults but only one phone listing. Couples often list the phone number under the man's name, so telephone listings are likely to be disproportionately male. Some people do not have phones, some have unlisted phone numbers, and some have more than one phone; these differences mean that each element of the population does not have an equal probability of being selected.

After each element of the chosen population has been assigned a unique number, sample numbers can be selected at random. A random number table can be used to select these sample numbers. (See, for example, *CRC Standard Management Tables*, W. H. Beyer, ed., Boca Raton FL: CRC Press.) Alternatively, we can use Excel to perform this function.

EXAMPLE 5.1

Random Sample of Income Tax Returns

A government income tax auditor has been given responsibility for 1,000 tax returns. A computer is used to check the arithmetic of each return. However, to determine whether the returns have been completed honestly, the auditor must check each entry and confirm its veracity. Because it takes, on average, 1 hour to completely audit a return and she has only 1 week to complete the task, the auditor has decided to randomly select 40 returns. The returns are numbered from 1 to 1,000. Use a computer random-number generator to select the sample for the auditor.

SOLUTION

We generated 50 numbers between 1 and 1,000 even though we needed only 40 numbers. We did so because it is likely that there will be some duplicates. We will use the first 40 unique random numbers to select our sample. The following numbers were generated by Excel. The instructions for both Excel and Minitab are provided here. [Notice that the 24th and 36th (counting down the columns) numbers generated were the same—467.]

Computer-Generated Random Numbers

383	246	372	952	75
101	46	356	54	199
597	33	911	706	65
900	165	467	817	359
885	220	427	973	488
959	18	304	467	512
15	286	976	301	374
408	344	807	751	986
864	554	992	352	41
139	358	257	776	231

EXCEL

INSTRUCTIONS

1. Click **Data, Data Analysis**, and **Random Number Generation**.
2. Specify the **Number of Variables** (1) and the **Number of Random Numbers** (50).
3. Select **Uniform Distribution.**
4. Specify the range of the uniform distribution (**Parameters**) (0 and 1).
5. Click **OK**. Column A will fill with 50 numbers that range between 0 and 1.

6. Multiply column A by 1,000 and store the products in column B.
7. Make cell C1 active, and click f_x, **Math & Trig**, **ROUNDUP**, and **OK**.
8. Specify the first number to be rounded (**B1**).
9. Type the **number of digits** (decimal places) (**0**). Click **OK**.
10. Complete column C.

The first five steps command Excel to generate 50 uniformly distributed random numbers between 0 and 1 to be stored in column A. Steps 6 through 10 convert these random numbers to integers between 1 and 1,000. Each tax return has the same probability (1/1,000 = .001) of being selected. Thus, each member of the population is equally likely to be included in the sample.

MINITAB

INSTRUCTIONS

1. Click **Calc**, **Random Data**, and **Integer . . .**.
2. Type the number of random numbers you wish (**50**).
3. Specify where the numbers are to be stored (**C1**).
4. Specify the **Minimum value** (**1**).
5. Specify the **Maximum value** (**1000**). Click **OK**.

INTERPRET

The auditor would examine the tax returns selected by the computer. She would pick returns numbered 383, 101, 597, . . . , 352, 776, and 75 (the first 40 unique numbers). Each of these returns would be audited to determine whether it is fraudulent. If the objective is to audit these 40 returns, no statistical procedure would be employed. However, if the objective is to estimate the proportion of all 1,000 returns that are dishonest, then she would use one of the inferential techniques presented later in this book.

Stratified Random Sampling

In making inferences about a population, we attempt to extract as much information as possible from a sample. The basic sampling plan, simple random sampling, often accomplishes this goal at low cost. Other methods, however, can be used to increase the amount of information about the population. One such procedure is *stratified random sampling*.

Stratified Random Sample

A **stratified random sample** is obtained by separating the population into mutually exclusive sets, or strata, and then drawing simple random samples from each stratum.

Examples of criteria for separating a population into strata (and of the strata themselves) follow.

1. Gender
    - male
    - female
2. Age
    - under 20
    - 20–30
    - 31–40
    - 41–50
    - 51–60
    - over 60
3. Occupation
    - professional
    - clerical
    - blue-collar
    - other
4. Household income
    - under $25,000
    - $25,000–$39,999
    - $40,000–$60,000
    - over $60,000

To illustrate, suppose a public opinion survey is to be conducted to determine how many people favor a tax increase. A stratified random sample could be obtained by selecting a random sample of people from each of the four income groups we just described. We usually stratify in a way that enables us to obtain particular kinds of information. In this example, we would like to know whether people in the different income categories differ in their opinions about the proposed tax increase, because the tax increase will affect the strata differently. We avoid stratifying when there is no connection between the survey and the strata. For example, little purpose is served in trying to determine whether people within religious strata have divergent opinions about the tax increase.

One advantage of stratification is that, besides acquiring information about the entire population, we can also make inferences within each stratum or compare strata. For instance, we can estimate what proportion of the lowest income group favors the tax increase, or we can compare the highest and lowest income groups to determine whether they differ in their support of the tax increase.

Any stratification must be done in such a way that the strata are mutually exclusive: Each member of the population must be assigned to exactly one stratum. After the population has been stratified in this way, we can use simple random sampling to generate the complete sample. There are several ways to do this. For example, we can draw random samples from each of the four income groups according to their proportions in the population. Thus, if in the population the relative frequencies of the four groups are as listed here, our sample will be stratified in the same proportions. If a total sample of 1,000 is to be drawn, then we will randomly select 250 from stratum 1, 400 from stratum 2, 300 from stratum 3, and 50 from stratum 4.

Stratum	Income Categories ($)	Population Proportions (%)
1	Less than 25,000	25
2	25,000–39,999	40
3	40,000–60,000	30
4	More than 60,000	5

The problem with this approach, however, is that if we want to make inferences about the last stratum, a sample of 50 may be too small to produce useful information. In such cases, we usually increase the sample size of the smallest stratum to ensure that the sample data provide enough information for our purposes. An adjustment must then be made before we attempt to draw inferences about the entire population. The required procedure is beyond the level of this book. We recommend that anyone planning such a survey consult an expert statistician or a reference book on the subject. Better still, become an expert statistician yourself by taking additional statistics courses.

Cluster Sampling

Cluster Sample

A **cluster sample** is a simple random sample of groups or clusters of elements.

Cluster sampling is particularly useful when it is difficult or costly to develop a complete list of the population members (making it difficult and costly to generate a simple random sample). It is also useful whenever the population elements are widely dispersed geographically. For example, suppose we wanted to estimate the average annual household income in a large city. To use simple random sampling, we would need a complete list of households in the city from which to sample. To use stratified random sampling, we would need the list of households, and we would also need to have each household categorized by some other variable (such as age of household head) in order to develop the strata. A less-expensive alternative would be to let each block within the city represent a cluster. A sample of clusters could then be randomly selected, and every household within these clusters could be questioned to determine income. By reducing the distances the surveyor must cover to gather data, cluster sampling reduces the cost.

But cluster sampling also increases sampling error (see Section 5.4) because households belonging to the same cluster are likely to be similar in many respects, including household income. This can be partially offset by using some of the cost savings to choose a larger sample than would be used for a simple random sample.

Sample Size

Whichever type of sampling plan you select, you still have to decide what size sample to use. Determining the appropriate sample size will be addressed in detail in Chapters 10 and 12. Until then, we can rely on our intuition, which tells us that the larger the sample size is, the more accurate we can expect the sample estimates to be.

Sampling and the Census

Courtesy, US Census Bureau

To adjust for undercounting, the Census Bureau conducts cluster sampling. The clusters are geographic blocks. For the year 2000 census, the bureau randomly sampled 11,800 blocks, which contained 314,000 housing units. Each unit was intensively revisited to ensure that all residents were counted. From the results of this survey, the Census Bureau estimated the number of people missed by the first census in various subgroups, defined by several variables including gender, race, and age. Because of the importance of determining state populations, adjustments were made to state totals. For example, by comparing the results of the census and of the sampling, the Bureau determined that the undercount in the state of Texas was 1.7087%. The official census produced a state population of 20,851,820. Taking 1.7087% of this total produced an adjustment of 356,295. Using this method changed the population of the state of Texas to 21,208,115.

It should be noted that this process is contentious. The controversy concerns the way in which subgroups are defined. Changing the definition alters the undercounts, making this statistical technique subject to politicking.

EXERCISES

5.11 A statistics practitioner would like to conduct a survey to ask people their views on a proposed new shopping mall in their community. According to the latest census, there are 500 households in the community. The statistician has numbered each household (from 1 to 500), and she would like to randomly select 25 of these households to participate in the study. Use Excel or Minitab to generate the sample.

5.12 A safety expert wants to determine the proportion of cars in his state with worn tire treads. The state license plate contains six digits. Use Excel or Minitab to generate a sample of 20 cars to be examined.

5.13 A large university campus has 60,000 students. The president of the students' association wants to conduct a survey of the students to determine their views on an increase in the student activity fee. She would like to acquire information about all the students but would also like to compare the school of business, the faculty of arts and sciences, and the graduate school. Describe a sampling plan that accomplishes these goals.

5.14 A telemarketing firm has recorded the households that have purchased one or more of the company's products. These number in the millions. The firm would like to conduct a survey of purchasers to acquire information about their attitude concerning the timing of the telephone calls. The president of the company would like to know the views of all purchasers but would also like to compare the attitudes of people in the West, South, North, and East. Describe a suitable sampling plan.

5.15 The operations manager of a large plant with four departments wants to estimate the person-hours lost per month from accidents. Describe a sampling plan that would be suitable for estimating the plantwide loss and for comparing departments.

5.16 A statistics practitioner wants to estimate the mean age of children in his city. Unfortunately, he does not have a complete list of households. Describe a sampling plan that would be suitable for his purposes.

5.4 SAMPLING AND NONSAMPLING ERRORS

Two major types of error can arise when a sample of observations is taken from a population: *sampling error* and *nonsampling error*. Anyone reviewing the results of sample surveys and studies, as well as statistics practitioners conducting surveys and applying statistical techniques, should understand the sources of these errors.

Sampling Error

Sampling error refers to differences between the sample and the population that exists only because of the observations that happened to be selected for the sample. Sampling error is an error that we expect to occur when we make a statement about a population that is based only on the observations contained in a sample taken from the population.

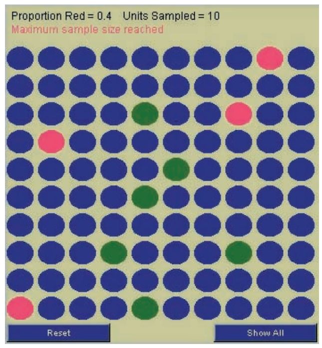
To illustrate, suppose that we wish to determine the mean annual income of North American blue-collar workers. To determine this parameter we would have to ask each North American blue-collar worker what his or her income is and then calculate the mean of all the responses. Because the size of this population is several million, the task is both expensive and impractical. We can use statistical inference to estimate the mean income μ of the population if we are willing to accept less than 100% accuracy. We record the incomes of a sample of the workers and find the mean $\bar{x}$ of this sample of incomes. This sample mean is an estimate, of the desired, population mean. But the value of the sample mean will deviate from the population mean simply by chance because the value of the sample mean depends on which incomes just happened to be selected for the sample. The difference between the true (unknown) value of the population mean and its estimate, the sample mean, is the sampling error. The size of this deviation may be large simply because of bad luck—bad luck that a particularly unrepresentative sample happened to be selected. The only way we can reduce the expected size of this error is to take a larger sample.

Given a fixed sample size, the best we can do is to state the probability that the sampling error is less than a certain amount (as we will discuss in Chapter 10). It is common today for such a statement to accompany the results of an opinion poll. If an opinion poll states that, based on sample results, the incumbent candidate for mayor has the support of 54% of eligible voters in an upcoming election, the statement may be accompanied by the following explanatory note: "This percentage is correct to within three percentage points, 19 times out of 20." This statement means that we estimate that the actual level of support for the candidate is between 51% and 57%, and that in the long run this type of procedure is correct 95% of the time.

SEEING STATISTICS

applet 3 Sampling

When you select this applet, you will see 100 circles. Imagine that each of the circles represents a household. You want to estimate the proportion of households having high-speed Internet access (DSL, cable modem, etc.). You may collect data from a sample of 10 households by clicking on a household's circle. If the circle turns red, the household has high-speed Internet access. If the circle turns green, the household does not have high-speed access. After collecting your sample and obtaining your estimate, click on the **Show All** button to see information for all the households. How well did your sample estimate the true proportion? Click the **Reset** button to try again. (*Note*: This page uses a randomly determined base proportion each time it is loaded or reloaded.)



Applet Exercises

3.1 Run the applet 25 times. How many times did the sample proportion equal the population proportion?

3.2 Run the applet 20 times. For each simulation, record the sample proportion of homes with high-speed Internet access as well as the population proportion. Compute the average sampling error.

Nonsampling Error

Nonsampling error is more serious than sampling error because taking a larger sample won't diminish the size, or the possibility of occurrence, of this error. Even a census can (and probably will) contain nonsampling errors. **Nonsampling errors** result from mistakes made in the acquisition of data or from the sample observations being selected improperly.

1. *Errors in data acquisition*. This type of error arises from the recording of incorrect responses. Incorrect responses may be the result of incorrect measurements being

taken because of faulty equipment, mistakes made during transcription from primary sources, inaccurate recording of data because terms were misinterpreted, or inaccurate responses were given to questions concerning sensitive issues such as sexual activity or possible tax evasion.

2. *Nonresponse error*: **Nonresponse error** refers to error (or **bias**) introduced when responses are not obtained from some members of the sample. When this happens, the sample observations that are collected may not be representative of the target population, resulting in biased results (as was discussed in Section 5.2). Nonresponse can occur for a number of reasons. An interviewer may be unable to contact a person listed in the sample, or the sampled person may refuse to respond for some reason. In either case, responses are not obtained from a sampled person, and bias is introduced. The problem of nonresponse is even greater when self-administered questionnaires are used rather than an interviewer, who can attempt to reduce the nonresponse rate by means of callbacks. As noted previously, the *Literary Digest* fiasco was largely the result of a high nonresponse rate, resulting in a biased, self-selected sample.

3. *Selection bias*. **Selection bias** occurs when the sampling plan is such that some members of the target population cannot possibly be selected for inclusion in the sample. Together with nonresponse error, selection bias played a role in the *Literary Digest* poll being so wrong, as voters without telephones or without a subscription to *Literary Digest* were excluded from possible inclusion in the sample taken.

EXERCISES

5.17 a. Explain the difference between sampling error and nonsampling error.
b. Which type of error in part (a) is more serious? Why?

5.18 Briefly describe three types of nonsampling error.

5.19 Is it possible for a sample to yield better results than a census? Explain.

CHAPTER SUMMARY

Because most populations are very large, it is extremely costly and impractical to investigate each member of the population to determine the values of the parameters. As a practical alternative, we take a sample from the population and use the sample statistics to draw inferences about the parameters. Care must be taken to ensure that the **sampled population** is the same as the **target population**.

We can choose from among several different sampling plans, including **simple random sampling**, **stratified random sampling**, and **cluster sampling**. Whatever sampling plan is used, it is important to realize that both **sampling error** and **nonsampling error** will occur and to understand what the sources of these errors are.

IMPORTANT TERMS

Observational 162
Experimental 163
Survey 163
Response rate 163
Estimate 165
Target population 166
Sampled population 166
Self-selected sample 166
Simple random sample 167
Stratified random sample 169
Cluster sample 171
Sampling error 172
Nonsampling error 173
Nonresponse error (bias) 174
Selection bias 174

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6

PROBABILITY

Auditing Tax Returns

Government auditors routinely check tax returns to determine whether calculation errors were made. They also attempt to detect fraudulent returns. There are several methods that dishonest taxpayers use to evade income tax. One method is not to declare various sources of income. Auditors have several detection methods, including spending patterns. Another form of tax fraud is to invent deductions that are not real. After analyzing the returns of thousands of self-employed taxpayers, an auditor has determined that 45% of fraudulent returns contain two suspicious deductions, 28% contain one suspicious deduction, and the rest no suspicious deductions. Among honest returns the rates are 11% for two deductions, 18% for one deduction, and 71% for no deductions. The auditor believes that 5% of the returns of self-employed individuals contain significant fraud. The auditor has just received a tax return for a self-employed individual that contains one suspicious expense deduction. What is the probability that this tax return contains significant fraud?

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See page 202 for the answer.

INTRODUCTION

In Chapters 2, 3, and 4, we introduced graphical and numerical descriptive methods. Although the methods are useful on their own, we are particularly interested in developing statistical inference. As we pointed out in Chapter 1, statistical inference is the process by which we acquire information about populations from samples. A critical component of inference is *probability* because it provides the link between the population and the sample.

Our primary objective in this and the following two chapters is to develop the probability-based tools that are at the basis of statistical inference. However, probability can also play a critical role in decision making, a subject we explore in Chapter 22.

6.1 ASSIGNING PROBABILITY TO EVENTS

To introduce probability, we must first define a *random experiment*.

Random Experiment

A **random experiment** is an action or process that leads to one of several possible outcomes.

Here are six illustrations of random experiments and their outcomes.

Illustration 1.	Experiment:	Flip a coin.
	Outcomes:	Heads and tails
Illustration 2.	Experiment:	Record marks on a statistics test (out of 100).
	Outcomes:	Numbers between 0 and 100
Illustration 3.	Experiment:	Record grade on a statistics test.
	Outcomes:	A, B, C, D, and F
Illustration 4.	Experiment:	Record student evaluations of a course.
	Outcomes:	Poor, fair, good, very good, and excellent
Illustration 5.	Experiment:	Measure the time to assemble a computer.
	Outcomes:	Number whose smallest possible value is 0 seconds with no predefined upper limit
Illustration 6.	Experiment:	Record the party that a voter will vote for in an upcoming election.
	Outcomes:	Party A, Party B, . . .

The first step in assigning probabilities is to produce a list of the outcomes. The listed outcomes must be **exhaustive**, which means that all possible outcomes must be included. In addition, the outcomes must be **mutually exclusive**, which means that no two outcomes can occur at the same time.

To illustrate the concept of exhaustive outcomes consider this list of the outcomes of the toss of a die:

1 2 3 4 5

This list is not exhaustive, because we have omitted 6.

The concept of mutual exclusiveness can be seen by listing the following outcomes in illustration 2:

0–50 50–60 60–70 70–80 80–100

If these intervals include both the lower and upper limits, then these outcomes are not mutually exclusive because two outcomes can occur for any student. For example, if a student receives a mark of 70, both the third and fourth outcomes occur.

Note that we could produce more than one list of exhaustive and mutually exclusive outcomes. For example, here is another list of outcomes for illustration 3:

Pass and fail

A list of exhaustive and mutually exclusive outcomes is called a *sample space* and is denoted by S. The outcomes are denoted by $O_1, O_2, \ldots, O_k$.

Sample Space

A **sample space** of a random experiment is a list of all possible outcomes of the experiment. The outcomes must be exhaustive and mutually exclusive.

Using set notation, we represent the sample space and its outcomes as

$$S = \{O_1, O_2, \ldots, O_k\}$$

Once a sample space has been prepared we begin the task of assigning probabilities to the outcomes. There are three ways to assign probability to outcomes. However it is done, there are two rules governing probabilities as stated in the next box.

Requirements of Probabilities

Given a sample space $S = \{O_1, O_2, \ldots, O_k\}$, the probabilities assigned to the outcomes must satisfy two requirements.

1. The probability of any outcome must lie between 0 and 1; that is,

$$0 \leq P(O_i) \leq 1 \quad \text{for each } i$$

[Note: $P(O_i)$ is the notation we use to represent the probability of outcome i.]

2. The sum of the probabilities of all the outcomes in a sample space must be 1. That is,

$$\sum_{i=1}^{k} P(O_i) = 1$$

Three Approaches to Assigning Probabilities

The **classical approach** is used by mathematicians to help determine probability associated with games of chance. For example, the classical approach specifies that the probabilities of heads and tails in the flip of a balanced coin are equal to each other.

Because the sum of the probabilities must be 1, the probability of heads and the probability of tails are both 50%. Similarly, the six possible outcomes of the toss of a balanced die have the same probability; each is assigned a probability of 1/6. In some experiments, it is necessary to develop mathematical ways to count the number of outcomes. For example, to determine the probability of winning a lottery, we need to determine the number of possible combinations. For details on how to count events, see Keller's website Appendix Counting Formulas.

The **relative frequency approach** defines probability as the long-run relative frequency with which an outcome occurs. For example, suppose that we know that of the last 1,000 students who took the statistics course you're now taking, 200 received a grade of A. The relative frequency of A's is then 200/1000 or 20%. This figure represents an estimate of the probability of obtaining a grade of A in the course. It is only an estimate because the relative frequency approach defines probability as the "long-run" relative frequency. One thousand students do not constitute the long run. The larger the number of students whose grades we have observed, the better the estimate becomes. In theory, we would have to observe an infinite number of grades to determine the exact probability.

When it is not reasonable to use the classical approach and there is no history of the outcomes, we have no alternative but to employ the **subjective approach**. In the subjective approach, we define probability as the degree of belief that we hold in the occurrence of an event. An excellent example is derived from the field of investment. An investor would like to know the probability that a particular stock will increase in value. Using the subjective approach, the investor would analyze a number of factors associated with the stock and the stock market in general and, using his or her judgment, assign a probability to the outcomes of interest.

Defining Events

An individual outcome of a sample space is called a *simple event*. All other events are composed of the simple events in a sample space.

Event

An **event** is a collection or set of one or more simple events in a sample space.

In illustration 2, we can define the event, achieve a grade of A, as the set of numbers that lie between 80 and 100, inclusive. Using set notation, we have

$$A = \{80, 81, 82, \ldots, 99, \ 100\}$$

Similarly,

$$F = \{0, 1, 2, \ldots, 48, 49\}$$

Probability of Events

We can now define the probability of any event.

Probability of an Event

The probability of an event is the sum of the probabilities of the simple events that constitute the event.

For example, suppose that in illustration 3, we employed the relative frequency approach to assign probabilities to the simple events as follows:

$P(A) = .20$

$P(B) = .30$

$P(C) = .25$

$P(D) = .15$

$P(F) = .10$

The probability of the event, pass the course, is

$$P(\text{Pass the course}) = P(A) + P(B) + P(C) + P(D) = .20 + .30 + .25 + .15 = .90$$

Interpreting Probability

No matter what method was used to assign probability, we interpret it using the relative frequency approach for an infinite number of experiments. For example, an investor may have used the subjective approach to determine that there is a 65% probability that a particular stock's price will increase over the next month. However, we interpret the 65% figure to mean that if we had an infinite number of stocks with exactly the same economic and market characteristics as the one the investor will buy, 65% of them will increase in price over the next month. Similarly, we can determine that the probability of throwing a 5 with a balanced die is 1/6. We may have used the classical approach to determine this probability. However, we interpret the number as the proportion of times that a 5 is observed on a balanced die thrown an infinite number of times.

This relative frequency approach is useful to interpret probability statements such as those heard from weather forecasters or scientists. You will also discover that this is the way we link the population and the sample in statistical inference.

EXERCISES

6.1 The weather forecaster reports that the probability of rain tomorrow is 10%.
a. Which approach was used to arrive at this number?
b. How do you interpret the probability?

6.2 A sportscaster states that he believes that the probability that the New York Yankees will win the World Series this year is 25%.
a. Which method was used to assign that probability?
b. How would you interpret the probability?

6.3 A quiz contains a multiple-choice question with five possible answers, only one of which is correct. A student plans to guess the answer because he knows absolutely nothing about the subject.
a. Produce the sample space for each question.
b. Assign probabilities to the simple events in the sample space you produced.
c. Which approach did you use to answer part (b)?
d. Interpret the probabilities you assigned in part (b).

6.4 An investor tells you that in her estimation there is a 60% probability that the Dow Jones Industrial Averages index will increase tomorrow.
a. Which approach was used to produce this figure?
b. Interpret the 60% probability.

6.5 The sample space of the toss of a fair die is

$$S = \{1, 2, 3, 4, 5, 6\}$$

If the die is balanced each simple event has the same probability. Find the probability of the following events.
a. An even number
b. A number less than or equal to 4
c. A number greater than or equal to 5

6.6 Four candidates are running for mayor. The four candidates are Adams, Brown, Collins, and Dalton. Determine the sample space of the results of the election.

6.7 Refer to Exercise 6.6. Employing the subjective approach a political scientist has assigned the following probabilities:

$P(\text{Adams wins}) = .42$

$P(\text{Brown wins}) = .09$

$P(\text{Collins wins}) = .27$

$P(\text{Dalton wins}) = .22$

Determine the probabilities of the following events.
a. Adams loses.
b. Either Brown or Dalton wins.
c. Adams, Brown, or Collins wins.

6.8 The manager of a computer store has kept track of the number of computers sold per day. On the basis of this information, the manager produced the following list of the number of daily sales.

Number of Computers Sold	Probability
0	.08
1	.17
2	.26
3	.21
4	.18
5	.10

a. If we define the experiment as observing the number of computers sold tomorrow, determine the sample space.
b. Use set notation to define the event, sell more than three computers.
c. What is the probability of selling five computers?
d. What is the probability of selling two, three, or four computers?
e. What is the probability of selling six computers?

6.9 Three contractors (call them contractors 1, 2, and 3) bid on a project to build a new bridge. What is the sample space?

6.10 Refer to Exercise 6.9. Suppose that you believe that contractor 1 is twice as likely to win as contractor 3 and that contractor 2 is three times as likely to win as contactor 3. What are the probabilities of winning for each contractor?

6.11 Shoppers can pay for their purchases with cash, a credit card, or a debit card. Suppose that the proprietor of a shop determines that 60% of her customers use a credit card, 30% pay with cash, and the rest use a debit card.
a. Determine the sample space for this experiment.
b. Assign probabilities to the simple events.
c. Which method did you use in part (b)?

6.12 Refer to Exercise 6.11.
a. What is the probability that a customer does not use a credit card?
b. What is the probability that a customer pays in cash or with a credit card?

6.13 A survey asks adults to report their marital status. The sample space is $S = \{\text{single, married, divorced, widowed}\}$. Use set notation to represent the event the adult is not married.

6.14 Refer to Exercise 6.13. Suppose that in the city in which the survey is conducted, 50% of adults are married, 15% are single, 25% are divorced, and 10% are widowed.
a. Assign probabilities to each simple event in the sample space.
b. Which approach did you use in part (a)?

6.15 Refer to Exercises 6.13 and 6.14. Find the probability of each of the following events.
a. The adult is single.
b. The adult is not divorced
c. The adult is either widowed or divorced.

6.2 Joint, Marginal, and Conditional Probability

In the previous section, we described how to produce a sample space and assign probabilities to the simple events in the sample space. Although this method of determining probability is useful, we need to develop more sophisticated methods. In this section, we discuss how to calculate the probability of more complicated events from the probability of related events. Here is an illustration of the process.

The sample space for the toss of a die is

$$S = \{1, 2, 3, 4, 5, 6\}$$

If the die is balanced, the probability of each simple event is 1/6. In most parlor games and casinos, players toss two dice. To determine playing and wagering strategies, players need to compute the probabilities of various totals of the two dice. For example, the probability of tossing a total of 3 with two dice is 2/36. This probability was derived by creating combinations of the simple events. There are several different types of combinations. One of the most important types is the *intersection* of two events.

Intersection

Intersection of Events A and B

The **intersection** of events A and B is the event that occurs when both A and B occur. It is denoted as

$$A \text{ and } B$$

The probability of the intersection is called the **joint probability**.

For example, one way to toss a 3 with two dice is to toss a 1 on the first die *and* a 2 on the second die, which is the intersection of two simple events. Incidentally, to compute the probability of a total of 3, we need to combine this intersection with another intersection, namely, a 2 on the first die and a 1 on the second die. This type of combination is called a *union* of two events, and it will be described later in this section. Here is another illustration.

APPLICATIONS in FINANCE

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Mutual funds

A mutual fund is a pool of investments made on behalf of people who share similar objectives. In most cases, a professional manager who has been educated in finance and statistics manages the fund. He or she makes decisions to buy and sell individual stocks and bonds in accordance with a specified investment philosophy. For example, there are funds that concentrate on other publicly traded mutual fund companies. Other mutual funds specialize in Internet stocks (so-called dot-coms), whereas others buy stocks of biotechnology firms. Surprisingly, most mutual funds do not outperform the market; that is, the increase in the net asset value (NAV) of the mutual fund is often less than the increase in the value of stock indexes that represent their stock markets. One reason for this is the management expense ratio (MER) which is a measure of the costs charged to the fund by the manager to cover expenses, including the salary and bonus of the managers. The MERs for most funds range from .5% to more than 4%. The ultimate success of the fund depends on the skill and knowledge of the fund manager. This raises the question, which managers do best?

EXAMPLE 6.1

Determinants of Success among Mutual Fund Managers—Part 1*

Why are some mutual fund managers more successful than others? One possible factor is the university where the manager earned his or her master of business administration (MBA). Suppose that a potential investor examined the relationship between how well the mutual fund performs and where the fund manager earned his or her MBA. After the analysis, Table 6.1, a table of joint probabilities, was developed. Analyze these probabilities and interpret the results.

TABLE **6.1** Determinants of Success among Mutual Fund Managers, Part 1*

	MUTUAL FUND OUTPERFORMS MARKET	MUTUAL FUND DOES NOT OUTPERFORM MARKET
Top-20 MBA program	.11	.29
Not top-20 MBA program	.06	.54

Table 6.1 tells us that the joint probability that a mutual fund outperforms the market *and* that its manager graduated from a top-20 MBA program is .11; that is, 11% of all mutual funds outperform the market and their managers graduated from a top-20 MBA program. The other three joint probabilities are defined similarly:

The probability that a mutual fund outperforms the market and its manager did not graduate from a top-20 MBA program is .06.

The probability that a mutual fund does not outperform the market and its manager graduated from a top-20 MBA program is .29.

The probability that a mutual fund does not outperform the market and its manager did not graduate from a top-20 MBA program is .54.

To help make our task easier, we'll use notation to represent the events. Let

A_1 = Fund manager graduated from a top-20 MBA program

A_2 = Fund manager did not graduate from a top-20 MBA program

B_1 = Fund outperforms the market

B_2 = Fund does not outperform the market

Thus,

$$P(A_1 \text{ and } B_1) = .11$$
$$P(A_2 \text{ and } B_1) = .06$$
$$P(A_1 \text{ and } B_2) = .29$$
$$P(A_2 \text{ and } B_2) = .54$$

*This example is adapted from "Are Some Mutual Fund Managers Better than Others? Cross-Sectional Patterns in Behavior and Performance" by Judith Chevalier and Glenn Ellison, Working paper 5852, National Bureau of Economic Research.

Marginal Probability

The joint probabilities in Table 6.1 allow us to compute various probabilities. **Marginal probabilities**, computed by adding across rows or down columns, are so named because they are calculated in the margins of the table.

Adding across the first row produces

$$P(A_1 \text{ and } B_1) + P(A_1 \text{ and } B_2) = .11 + .29 = .40$$

Notice that both intersections state that the manager graduated from a top-20 MBA program (represented by A_1). Thus, when randomly selecting mutual funds, the probability that its manager graduated from a top-20 MBA program is .40. Expressed as relative frequency, 40% of all mutual fund managers graduated from a top-20 MBA program.

Adding across the second row:

$$P(A_2 \text{ and } B_1) + P(A_2 \text{ and } B_2) = .06 + .54 = .60$$

This probability tells us that 60% of all mutual fund managers did not graduate from a top-20 MBA program (represented by A_2). Notice that the probability that a mutual fund manager graduated from a top-20 MBA program and the probability that the manager did not graduate from a top-20 MBA program add to 1.

Adding down the columns produces the following marginal probabilities.

Column 1: $P(A_1 \text{ and } B_1) + P(A_2 \text{ and } B_1) = .11 + .06 = .17$

Column 2: $P(A_1 \text{ and } B_2) + P(A_2 \text{ and } B_2) = .29 + .54 = .83$

These marginal probabilities tell us that 17% of all mutual funds outperform the market and that 83% of mutual funds do not outperform the market.

Table 6.2 lists all the joint and marginal probabilities.

TABLE **6.2** Joint and Marginal Probabilities

	MUTUAL FUND OUTPERFORMS MARKET	MUTUAL FUND DOES NOT OUTPERFORM MARKET	TOTALS
Top-20 MBA program	$P(A_1 \text{ and } B_1) = .11$	$P(A_1 \text{ and } B_2) = .29$	$P(A_1) = .40$
Not top-20 MBA program	$P(A_2 \text{ and } B_1) = .06$	$P(A_2 \text{ and } B_2) = .54$	$P(A_2) = .60$
Totals	$P(B_1) = .17$	$P(B_2) = .83$	1.00

Conditional Probability

We frequently need to know how two events are related. In particular, we would like to know the probability of one event given the occurrence of another related event. For example, we would certainly like to know the probability that a fund managed by a graduate of a top-20 MBA program will outperform the market. Such a probability will allow us to make an informed decision about where to invest our money. This probability is called a **conditional probability** because we want to know the probability that a fund will outperform the market *given* the condition that the manager graduated from a top-20 MBA program. The conditional probability that we seek is represented by

$$P(B_1|A_1)$$

where the "|" represents the word *given*. Here is how we compute this conditional probability.

The marginal probability that a manager graduated from a top-20 MBA program is .40, which is made up of two joint probabilities. They are (1) the probability that the mutual fund outperforms the market and the manager graduated from a top-20 MBA program [P(A_1 and B_1)] and (2) the probability that the fund does not outperform the market and the manager graduated from a top-20 MBA program [P(A_1 and B_2)]. Their joint probabilities are .11 and .29, respectively. We can interpret these numbers in the following way. On average, for every 100 mutual funds, 40 will be managed by a graduate of a top-20 MBA program. Of these 40 managers, on average 11 of them will manage a mutual fund that will outperform the market. Thus, the conditional probability is 11/40 = .275. Notice that this ratio is the same as the ratio of the joint probability to the marginal probability .11/.40. All conditional probabilities can be computed this way.

Conditional Probability

The probability of event A given event B is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

The probability of event B given event A is

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

EXAMPLE 6.2

Determinants of Success among Mutual Fund Managers—Part 2

Suppose that in Example 6.1 we select one mutual fund at random and discover that it did not outperform the market. What is the probability that a graduate of a top-20 MBA program manages it?

SOLUTION

We wish to find a conditional probability. The condition is that the fund did not outperform the market (event B_2), and the event whose probability we seek is that the fund is managed by a graduate of a top-20 MBA program (event A_1). Thus, we want to compute the following probability:

$$P(A_1|B_2)$$

Using the conditional probability formula, we find

$$P(A_1|B_2) = \frac{P(A_1 \text{ and } B_2)}{P(B_2)} = \frac{.29}{.83} = .349$$

Thus, 34.9% of all mutual funds that do not outperform the market are managed by top-20 MBA program graduates.

The calculation of conditional probabilities raises the question of whether the two events, the fund outperformed the market and the manager graduated from a top-20 MBA program, are related, a subject we tackle next.

Independence

One of the objectives of calculating conditional probability is to determine whether two events are related. In particular, we would like to know whether they are **independent events**.

Independent Events

Two events A and B are said to be independent if

$$P(A|B) = P(A)$$

or

$$P(B|A) = P(B)$$

Put another way, two events are independent if the probability of one event is not affected by the occurrence of the other event.

EXAMPLE 6.3

Determinants of Success among Mutual Fund Managers—Part 3

Determine whether the event that the manager graduated from a top-20 MBA program and the event the fund outperforms the market are independent events.

SOLUTION

We wish to determine whether A_1 and B_1 are independent. To do so, we must calculate the probability of A_1 given B_1; that is,

$$P(A_1 | B_1) = \frac{P(A_1 \text{ and } B_1)}{P(B_1)} = \frac{.11}{.17} = .647$$

The marginal probability that a manager graduated from a top-20 MBA program is

$$P(A_1) = .40$$

Since the two probabilities are not equal, we conclude that the two events are dependent.

Incidentally, we could have made the decision by calculating $P(B_1|A_1) = .275$ and observing that it is not equal to $P(B_1) = .17$.

Note that there are three other combinations of events in this problem. They are (A_1 and B_2), (A_2 and B_1), (A_2 and B_2) [ignoring mutually exclusive combinations (A_1 and A_2) and (B_1 and B_2), which are dependent]. In each combination, the two events are dependent. In this type of problem, where there are only four combinations, if one combination is dependent, then all four will be dependent. Similarly, if one combination is independent, then all four will be independent. This rule does not apply to any other situation.

Union

Another event that is the combination of other events is the *union*.

Union of Events A and B

The **union** of events A and B is the event that occurs when either A or B or both occur. It is denoted as

$$A \text{ or } B$$

EXAMPLE 6.4

Determinants of Success among Mutual Fund Managers—Part 4

Determine the probability that a randomly selected fund outperforms the market or the manager graduated from a top-20 MBA program.

SOLUTION

We want to compute the probability of the union of two events

$$P(A_1 \text{ or } B_1)$$

The union A_1 or B_1 consists of three events; That is, the union occurs whenever any of the following joint events occurs:

1. Fund outperforms the market and the manager graduated from a top-20 MBA program
2. Fund outperforms the market and the manager did not graduate from a top-20 MBA program
3. Fund does not outperform the market and the manager graduated from a top-20 MBA program

Their probabilities are

$$P(A_1 \text{ and } B_1) = .11$$
$$P(A_2 \text{ and } B_1) = .06$$
$$P(A_1 \text{ and } B_2) = .29$$

Thus, the probability of the union—the fund outperforms the market or the manager graduated from a top-20 MBA program—is the sum of the three probabilities; That is,

$$P(A_1 \text{ or } B_1) = P(A_1 \text{ and } B_1) + P(A_2 \text{ and } B_1) + P(A_1 \text{ and } B_2) = .11 + .06 + .29 = .46$$

Notice that there is another way to produce this probability. Of the four probabilities in Table 6.1, the only one representing an event that is not part of the union is the probability of the event the fund does not outperform the market and the manager did not graduate from a top-20 MBA program. That probability is

$$P(A_2 \text{ and } B_2) = .54$$

which is the probability that the union *does not* occur. Thus, the probability of the union is

$$P(A_1 \text{ or } B_1) = 1 - P(A_2 \text{ and } B_2) = 1 - .54 = .46.$$

Thus, we determined that 46% of mutual funds either outperform the market or are managed by a top-20 MBA program graduate or have both characteristics.

EXERCISES

6.16 Given the following table of joint probabilities, calculate the marginal probabilities.

	A_1	A_2	A_3
B_1	.1	.3	.2
B_2	.2	.1	.1

6.17 Calculate the marginal probabilities from the following table of joint probabilities.

	A_1	A_2
B_1	.4	.3
B_2	.2	.1

6.18 Refer to Exercise 6.17.
a. Determine $P(A_1|B_1)$.
b. Determine $P(A_2|B_1)$.
c. Did your answers to parts (a) and (b) sum to 1? Is this a coincidence? Explain.

6.19 Refer to Exercise 6.17. Calculate the following probabilities.
a. $P(A_1|B_2)$
b. $P(B_2|A_1)$
c. Did you expect the answers to parts (a) and (b) to be reciprocals? In other words, did you expect that $P(A_1|B_2) = 1/P(B_2|A_1)$? Why is this impossible (unless both probabilities are 1)?

6.20 Are the events in Exercise 6.17 independent? Explain.

6.21 Refer to Exercise 6.17. Compute the following.
a. P(A_1 or B_1)
b. P(A_1 or B_2)
c. P(A_1 or A_2)

6.22 Suppose that you have been given the following joint probabilities. Are the events independent? Explain.

	A_1	A_2
B_1	.20	.60
B_2	.05	.15

6.23 Determine whether the events are independent from the following joint probabilities.

	A_1	A_2
B_1	.20	.15
B_2	.60	.05

6.24 Suppose we have the following joint probabilities.

	A_1	A_2	A_3
B_1	.15	.20	.10
B_2	.25	.25	.05

Compute the marginal probabilities.

6.25 Refer to Exercise 6.24.
a. Compute $P(A_2|B_2)$.
b. Compute $P(B_2|A_2)$.
c. Compute $P(B_1|A_2)$.

6.26 Refer to Exercise 6.24.
a. Compute P(A_1 or A_2).
b. Compute P(A_2 or B_2).
c. Compute P(A_3 or B_1).

6.27 Discrimination in the workplace is illegal, and companies that discriminate are often sued. The female instructors at a large university recently lodged a complaint about the most recent round of promotions from assistant professor to associate professor. An analysis of the relationship between gender and promotion produced the following joint probabilities.

	Promoted	Not Promoted
Female	.03	.12
Male	.17	.68

a. What is the rate of promotion among female assistant professors?
b. What is the rate of promotion among male assistant professors?
c. Is it reasonable to accuse the university of gender bias?

6.28 A department store analyzed its most recent sales and determined the relationship between the way the customer paid for the item and the price category of the item. The joint probabilities in the following table were calculated.

	Cash	Credit Card	Debit Card
Less than $20	.09	.03	.04
$20–$100	.05	.21	.18
More than $100	.03	.23	.14

a. What proportion of purchases was paid by debit card?
b. Find the probability that a credit card purchase was more than $100.
c. Determine the proportion of purchases made by credit card or by debit card.

6.29 The following table lists the probabilities of unemployed females and males and their educational attainment.

	Female	Male
Less than high school	.077	.110
High school graduate	.154	.201
Some college or university—no degree	.141	.129
College or university graduate	.092	.096

(*Source: Statistical Abstract of the United States, 2009*, Table 607.)

a. If one unemployed person is selected at random, what is the probability that he or she did not finish high school?
b. If an unemployed female is selected at random, what is the probability that she has a college or university degree?
c. If an unemployed high school graduate is selected at random, what is the probability that he is a male?

6.30 The costs of medical care in North America are increasing faster than inflation, and with the baby boom generation soon to need health care, it becomes imperative that countries find ways to reduce both costs and demand. The following table lists the joint probabilities associated with smoking and lung disease among 60- to 65-year-old men.

	He is a smoker	He is a nonsmoker
He has lung disease	.12	.03
He does not have lung disease	.19	.66

One 60- to 65-year-old man is selected at random. What is the probability of the following events?
a. He is a smoker.
b. He does not have lung disease.
c. He has lung disease given that he is a smoker.
d. He has lung disease given that he does not smoke.

6.31 Refer to Exercise 6.30. Are smoking and lung disease among 60- to 65-year-old men related?

6.32 The method of instruction in college and university applied statistics courses is changing. Historically, most courses were taught with an emphasis on manual calculation. The alternative is to employ a computer and a software package to perform the calculations. An analysis of applied statistics courses investigated whether the instructor's educational background is primarily mathematics (or statistics) or some other field. The result of this analysis is the accompanying table of joint probabilities.

	Statistics Course Emphasizes Manual Calculations	Statistics Course Employs Computer and Software
Mathematics or statistics education	.23	.36
Other education	.11	.30

a. What is the probability that a randomly selected applied statistics course instructor whose education was in statistics emphasizes manual calculations?
b. What proportion of applied statistics courses employ a computer and software?
c. Are the educational background of the instructor and the way his or her course is taught independent?

6.33 A restaurant chain routinely surveys its customers. Among other questions, the survey asks each customer whether he or she would return and to rate the quality of food. Summarizing hundreds of thousands of questionnaires produced this table of joint probabilities.

Rating	Customer Will Return	Customer Will Not Return
Poor	.02	.10
Fair	.08	.09
Good	.35	.14
Excellent	.20	.02

a. What proportion of customers say that they will return and rate the restaurant's food as good?
b. What proportion of customers who say that they will return rate the restaurant's food as good?
c. What proportion of customers who rate the restaurant's food as good say that they will return?
d. Discuss the differences in your answers to parts (a), (b), and (c).

6.34 To determine whether drinking alcoholic beverages has an effect on the bacteria that cause ulcers, researchers developed the following table of joint probabilities.

Number of Alcoholic Drinks per Day	Ulcer	No Ulcer
None	.01	.22
One	.03	.19
Two	.03	.32
More than two	.04	.16

a. What proportion of people have ulcers?
b. What is the probability that a teetotaler (no alcoholic beverages) develops an ulcer?
c. What is the probability that someone who has an ulcer does not drink alcohol?
d. What is the probability that someone who has an ulcer drinks alcohol?

6.35 An analysis of fired or laid-off workers, their age, and the reasons for their departure produced the following table of joint probabilities.

	Age Category			
Reason for job loss	20–24	25–54	55–64	65 and older
Plant or company closed or moved	.015	.320	.089	.029
Insufficient work	.014	.180	.034	.011
Position or shift abolished	.006	.214	.071	.016

(*Source: Statistical Abstract of the United States, 2009*, Table 593.)

a. What is the probability that a 25- to 54-year-old employee was laid off or fired because of insufficient work?
b. What proportion of laid-off or fired workers is age 65 and older?
c. What is the probability that a laid-off or fired worker because the plant or company closed is 65 or older?

6.36 Many critics of television claim that there is too much violence and that it has a negative effect on society. There may also be a negative effect on advertisers. To examine this issue, researchers developed two versions of a cops-and-robbers made-for-television movie. One version depicted several violent crimes, and the other removed these scenes. In the middle of the movie, one 60-second commercial was shown advertising a new product and brand name. At the end of the movie, viewers were asked to name the brand. After observing the results, the researchers produced the following table of joint probabilities.

	Watch Violent Movie	Watch Nonviolent Movie
Remember the brand name	.15	.18
Do not remember the brand name	.35	.32

a. What proportion of viewers remember the brand name?
b. What proportion of viewers who watch the violent movie remember the brand name?
c. Does watching a violent movie affect whether the viewer will remember the brand name? Explain.

6.37 Is there a relationship between the male hormone testosterone and criminal behavior? To answer this question, medical researchers measured the testosterone level of penitentiary inmates and recorded whether they were convicted of murder. After analyzing the results, the researchers produced the following table of joint probabilities.

Testosterone Level	Murderer	Other Felon
Above average	.27	.24
Below average	.21	.28

a. What proportion of murderers have above-average testosterone levels?
b. Are levels of testosterone and the crime committed independent? Explain.

6.38 The issue of health care coverage in the United States is becoming a critical issue in American politics. A large-scale study was undertaken to determine who is and is not covered. From this study, the following table of joint probabilities was produced.

Age Category	Has Health Insurance	Does Not Have Health Insurance
25–34	.167	.085
35–44	.209	.061
45–54	.225	.049
55–64	.177	.026

(*Source*: U.S. Department of Health and Human Services.)

If one person is selected at random, find the following probabilities.
a. P(Person has health insurance)
b. P(Person 55–64 has no health insurance)
c. P(Person without health insurance is between 25 and 34 years old)

6.39 Violent crime in many American schools is an unfortunate fact of life. An analysis of schools and violent crime yielded the table of joint probabilities shown next.

Level	Violent Crime Committed This Year	No Violent Crime Committed
Primary	.393	.191
Middle	.176	.010
High School	.134	.007
Combined	.074	.015

(*Source: Statistical Abstract of the United States, 2009*, Table 237.)

If one school is randomly selected find the following probabilities.

a. Probability of at least one incident of violent crime during the year in a primary school
b. Probability of no violent crime during the year

6.40 Refer to Exercise 6.39. A similar analysis produced these joint probabilities.

Enrollment	Violent Crime Committed This Year	No Violent Crime Committed
Less than 300	.159	.091
300 to 499	.221	.065
500 to 999	.289	.063
1,000 or more	.108	.004

(*Source: Statistical Abstract of the United States, 2009*, Table 237.)

a. What is the probability that a school with an enrollment of less than 300 had at least one violent crime during the year?
b. What is the probability that a school that has at least one violent crime had an enrollment of less than 300?

6.41 A firm has classified its customers in two ways: (1) according to whether the account is overdue and (2) whether the account is new (less than 12 months) or old. An analysis of the firm's records provided the input for the following table of joint probabilities.

	Overdue	Not Overdue
New	.06	.13
Old	.52	.29

One account is randomly selected.

a. If the account is overdue, what is the probability that it is new?
b. If the account is new, what is the probability that it is overdue?
c. Is the age of the account related to whether it is overdue? Explain.

6.42 How are the size of a firm (measured in terms of the number of employees) and the type of firm related? To help answer the question, an analyst referred to the U.S. Census and developed the following table of joint probabilities.

Number of Employees	Industry		
	Construction	Manufacturing	Retail
Fewer than 20	.464	.147	.237
20 to 99	.039	.049	.035
100 or more	.005	.019	.005

(*Source: Statistical Abstract of the United States, 2009*, Table 737.)

If one firm is selected at random, find the probability of the following events.

a. The firm employs fewer than 20 employees.
b. The firm is in the retail industry.
c. A firm in the construction industry employs between 20 and 99 workers.

6.43 Credit scorecards are used by financial institutions to help decide to whom loans should be granted (see the Applications in Banking: Credit Scorecards summary on page 63). An analysis of the records of one bank produced the following probabilities.

Loan Performance	Score	
	Under 400	400 or More
Fully repaid	.19	.64
Defaulted	.13	.04

a. What proportion of loans are fully repaid?
b. What proportion of loans given to scorers of less than 400 fully repay?
c. What proportion of loans given to scorers of 400 or more fully repay?
d. Are score and whether the loan is fully repaid independent? Explain.

6.44 A retail outlet wanted to know whether its weekly advertisement in the daily newspaper works. To acquire this critical information, the store manager surveyed the people who entered the store and determined whether each individual saw the ad and whether a purchase was made. From the information developed, the manager produced the following table of joint probabilities. Are the ads effective? Explain.

	Purchase	No Purchase
See ad	.18	.42
Do not see ad	.12	.28

6.45 To gauge the relationship between education and unemployment, an economist turned to the U.S. Census from which the following table of joint probabilities was produced.

Education	Employed	Unemployed
Not a high school graduate	.091	.008
High school graduate	.282	.014

(Continued)

Some college, no degree	.166	.007
Associate's degree	.095	.003
Bachelor's degree	.213	.004
Advanced degree	.115	.002

(*Source: Statistical Abstract of the United States, 2009*, Table 223.)

a. What is the probability that a high school graduate is unemployed?
b. Determine the probability that a randomly selected individual is employed.
c. Find the probability that an unemployed person possesses an advanced degree.
d. What is the probability that a randomly selected person did not finish high school?

6.46 The decision about where to build a new plant is a major one for most companies. One factor that is often considered is the education level of the location's residents. Census information may be useful in this regard. After analyzing a recent census, a company produced the following joint probabilities.

	Region			
Education	**Northeast**	**Midwest**	**South**	**West**
Not a high school graduate	.024	.024	.059	.036
High school graduate	.063	.078	.117	.059
Some college, no degree	.023	.039	.061	.045
Associate's degree	.015	.021	.030	.020
Bachelor's degree	.038	.040	.065	.046
Advanced degree	.024	.020	.032	.023

(*Source: Statistical Abstract of the United States, 2009*, Table 223.)

a. Determine the probability that a person living in the West has a bachelor's degree.
b. Find the probability that a high school graduate lives in the Northeast.
c. What is the probability that a person selected at random lives in the South?
d. What is the probability that a person selected at random does not live in the South?

6.3 Probability Rules and Trees

In Section 6.2, we introduced intersection and union and described how to determine the probability of the intersection and the union of two events. In this section, we present other methods of determining these probabilities. We introduce three rules that enable us to calculate the probability of more complex events from the probability of simpler events.

Complement Rule

The **complement** of event A is the event that occurs when event A does not occur. The complement of event A is denoted by A^C. The **complement rule** defined here derives from the fact that the probability of an event and the probability of the event's complement must sum to 1.

Complement Rule

$$P(A^C) = 1 - P(A)$$

for any event A.

We will demonstrate the use of this rule after we introduce the next rule.

Multiplication Rule

The **multiplication rule** is used to calculate the joint probability of two events. It is based on the formula for conditional probability supplied in the previous section; that is, from the following formula

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

we derive the multiplication rule simply by multiplying both sides by $P(B)$.

Multiplication Rule

The joint probability of any two events A and B is

$$P(A \text{ and } B) = P(B)P(A|B)$$

or, altering the notation,

$$P(A \text{ and } B) = P(A)P(B|A)$$

If A and B are independent events, $P(A|B) = P(A)$ and $P(B|A) = P(B)$. It follows that the joint probability of two independent events is simply the product of the probabilities of the two events. We can express this as a special form of the multiplication rule.

Multiplication Rule for Independent Events

The joint probability of any two independent events A and B is

$$P(A \text{ and } B) = P(A)P(B)$$

EXAMPLE 6.5*

Selecting Two Students without Replacement

A graduate statistics course has seven male and three female students. The professor wants to select two students at random to help her conduct a research project. What is the probability that the two students chosen are female?

SOLUTION

Let A represent the event that the first student chosen is female and B represent the event that the second student chosen is also female. We want the joint probability $P(A \text{ and } B)$. Consequently, we apply the multiplication rule:

$$P(A \text{ and } B) = P(A)P(B|A)$$

Because there are 3 female students in a class of 10, the probability that the first student chosen is female is

$$P(A) = 3/10$$

*This example can be solved using the Hypergeometric distribution, which is described in the Keller's website Appendix Hypergeometric Distribution.

After the first student is chosen, there are only nine students left. Given that the first student chosen was female, there are only two female students left. It follows that

$$P(B|A) = 2/9$$

Thus, the joint probability is

$$P(A \text{ and } B) = P(A)P(B|A) = \left(\frac{3}{10}\right)\left(\frac{2}{9}\right) = \frac{6}{90} = .067$$

EXAMPLE 6.6

Selecting Two Students with Replacement

Refer to Example 6.5. The professor who teaches the course is suffering from the flu and will be unavailable for two classes. The professor's replacement will teach the next two classes. His style is to select one student at random and pick on him or her to answer questions during that class. What is the probability that the two students chosen are female?

SOLUTION

The form of the question is the same as in Example 6.5: We wish to compute the probability of choosing two female students. However, the experiment is slightly different. It is now possible to choose the *same* student in each of the two classes taught by the replacement. Thus, A and B are independent events, and we apply the multiplication rule for independent events:

$$P(A \text{ and } B) = P(A)P(B)$$

The probability of choosing a female student in each of the two classes is the same; that is,

$$P(A) = 3/10 \text{ and } P(B) = 3/10$$

Hence,

$$P(A \text{ and } B) = P(A)P(B) = \left(\frac{3}{10}\right)\left(\frac{3}{10}\right) = \frac{9}{100} = .09$$

Addition Rule

The **addition rule** enables us to calculate the probability of the union of two events.

Addition Rule

The probability that event A, or event B, or both occur is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

If you're like most students, you're wondering why we subtract the joint probability from the sum of the probabilities of A and B. To understand why this is necessary, examine Table 6.2 (page 183), which we have reproduced here as Table 6.3.

TABLE **6.3** Joint and Marginal Probabilities

	B_1	B_2	TOTALS
A_1	$P(A_1 \text{ and } B_1) = .11$	$P(A_1 \text{ and } B_2) = .29$	$P(A_1) = .40$
A_2	$P(A_2 \text{ and } B_1) = .06$	$P(A_2 \text{ and } B_2) = .54$	$P(A_2) = .60$
Totals	$P(B_1) = .17$	$P(B_2) = .83$	1.00

This table summarizes how the marginal probabilities were computed. For example, the marginal probability of A_1 and the marginal probability of B_1 were calculated as

$$P(A_1) = P(A_1 \text{ and } B_1) + P(A_1 \text{ and } B_2) = .11 + .29 = .40$$

$$P(B_1) = P(A_1 \text{ and } B_1) + P(A_2 \text{ and } B_1) = .11 + .06 = .17$$

If we now attempt to calculate the probability of the union of A_1 and B_1 by summing their probabilities, we find

$$P(A_1) + P(B_1) = .11 + .29 + .11 + .06$$

Notice that we added the joint probability of A_1 and B_1 (which is .11) twice. To correct the double counting, we subtract the joint probability from the sum of the probabilities of $A_1$and B_1. Thus,

$$\begin{aligned} P(A_1 \text{ or } B_1) &= P(A_1) + P(B_1) - P(A_1 \text{ and } B_1) \\ &= [.11 + .29] + [.11 + .06] - .11 \\ &= .40 + .17 - .11 = .46 \end{aligned}$$

This is the probability of the union of A_1 and B_1, which we calculated in Example 6.4 (page 186).

As was the case with the multiplication rule, there is a special form of the addition rule. When two events are mutually exclusive (which means that the two events cannot occur together), their joint probability is 0.

Addition Rule for Mutually Exclusive Events

The probability of the union of two mutually exclusive events A and B is

$$P(A \text{ or } B) = P(A) + P(B)$$

EXAMPLE 6.7

Applying the Addition Rule

In a large city, two newspapers are published, the *Sun* and the *Post*. The circulation departments report that 22% of the city's households have a subscription to the *Sun* and 35% subscribe to the *Post*. A survey reveals that 6% of all households subscribe to both newspapers. What proportion of the city's households subscribe to either newspaper?

SOLUTION

We can express this question as, what is the probability of selecting a household at random that subscribes to the *Sun*, the *Post*, or both? Another way of asking the question is, what is the probability that a randomly selected household subscribes to *at least one* of the newspapers? It is now clear that we seek the probability of the union, and we must apply the addition rule. Let A = the household subscribes to the *Sun* and B = the household subscribes to the *Post*. We perform the following calculation:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = .22 + .35 - .06 = .51$$

The probability that a randomly selected household subscribes to either newspaper is .51. Expressed as relative frequency, 51% of the city's households subscribe to either newspaper.

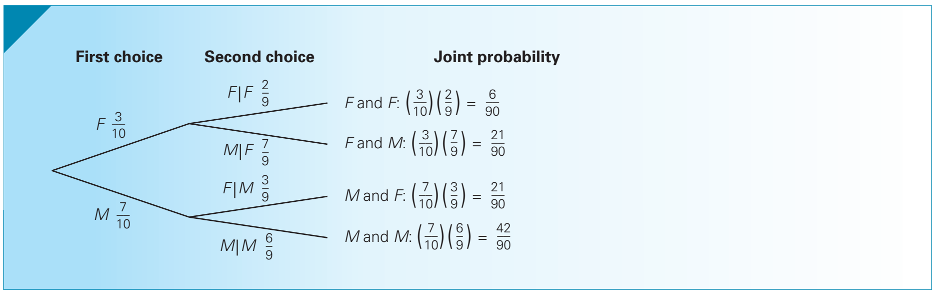
Probability Trees

An effective and simpler method of applying the probability rules is the probability tree, wherein the events in an experiment are represented by lines. The resulting figure resembles a tree, hence the name. We will illustrate the probability tree with several examples, including two that we addressed using the probability rules alone.

In Example 6.5, we wanted to find the probability of choosing two female students, where the two choices had to be different. The tree diagram in Figure 6.1 describes this experiment. Notice that the first two branches represent the two possibilities, female and male students, on the first choice. The second set of branches represents the two possibilities on the second choice. The probabilities of female and male student chosen first are 3/10 and 7/10, respectively. The probabilities for the second set of branches are conditional probabilities based on the choice of the first student selected.

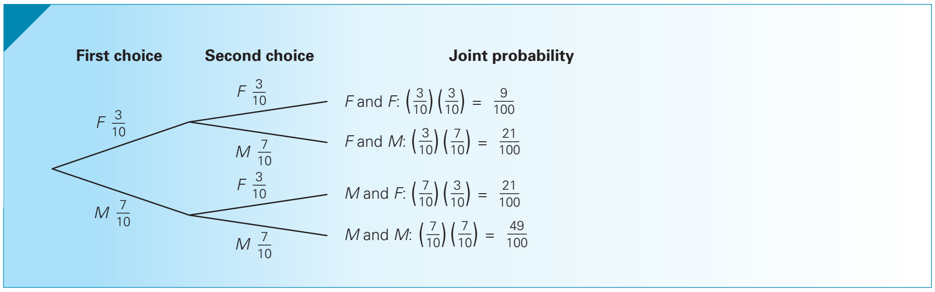
We calculate the joint probabilities by multiplying the probabilities on the linked branches. Thus, the probability of choosing two female students is $P(F \text{ and } F) = (3/10)(2/9) = 6/90$. The remaining joint probabilities are computed similarly.

FIGURE **6.1** **Probability Tree for Example 6.5**



In Example 6.6, the experiment was similar to that of Example 6.5. However, the student selected on the first choice was returned to the pool of students and was eligible to be chosen again. Thus, the probabilities on the second set of branches remain the same as the probabilities on the first set, and the probability tree is drawn with these changes, as shown in Figure 6.2.

FIGURE **6.2** Probability Tree for Example 6.6



The advantage of a probability tree on this type of problem is that it restrains its users from making the wrong calculation. Once the tree is drawn and the probabilities of the branches inserted, virtually the only allowable calculation is the multiplication of the probabilities of linked branches. An easy check on those calculations is available. The joint probabilities at the ends of the branches must sum to 1 because all possible events are listed. In both figures, notice that the joint probabilities do indeed sum to 1.

The special form of the addition rule for mutually exclusive events can be applied to the joint probabilities. In both probability trees, we can compute the probability that one student chosen is female and one is male simply by adding the joint probabilities. For the tree in Example 6.5, we have

$$P(F \text{ and } M) + P(M \text{ and } F) = 21/90 + 21/90 = 42/90$$

In the probability tree in Example 6.6, we find

$$P(F \text{ and } M) + P(M \text{ and } F) = 21/100 + 21/100 = 42/100$$

EXAMPLE 6.8

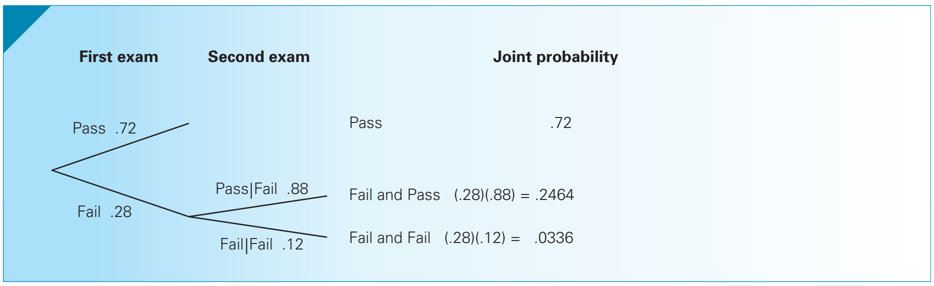
Probability of Passing the Bar Exam

Students who graduate from law schools must still pass a bar exam before becoming lawyers. Suppose that in a particular jurisdiction the pass rate for first-time test takers is 72%. Candidates who fail the first exam may take it again several months later. Of those who fail their first test, 88% pass their second attempt. Find the probability that a randomly selected law school graduate becomes a lawyer. Assume that candidates cannot take the exam more than twice.

SOLUTION

The probability tree in Figure 6.3 is employed to describe the experiment. Note that we use the complement rule to determine the probability of failing each exam.

FIGURE **6.3** Probability Tree for Example 6.8



We apply the multiplication rule to calculate P(Fail and Pass), which we find to be .2464. We then apply the addition rule for mutually exclusive events to find the probability of passing the first or second exam:

$$P(\text{Pass [on first exam]}) + P(\text{Fail [on first exam] and Pass [on second exam]})$$
$$= .72 + .2464 = .9664$$

Thus, 96.64% of applicants become lawyers by passing the first or second exam.

EXERCISES

6.47 Given the following probabilities, compute all joint probabilities.

$P(A) = .9$ $\quad P(A^C) = .1$

$P(B|A) = .4$ $\quad P(B|A^C) = .7$

6.48 Determine all joint probabilities from the following.

$P(A) = .8$ $\quad P(A^C) = .2$

$P(B|A) = .4$ $\quad P(B|A^C) = .7$

6.49 Draw a probability tree to compute the joint probabilities from the following probabilities.

$P(A) = .5$ $\quad P(A^C) = .2$

$P(B|A) = .4$ $\quad P(B|A^C) = .7$

6.50 Given the following probabilities, draw a probability tree to compute the joint probabilities.

$P(A) = .8$ $\quad P(A^C) = .2$

$P(B|A) = .3$ $\quad P(B|A^C) = .3$

6.51 Given the following probabilities, find the joint probability $P(A \text{ and } B)$.

$P(A) = .7$ $\quad P(B|A) = .3$

6.52 Approximately 10% of people are left-handed. If two people are selected at random, what is the probability of the following events?
a. Both are right-handed.
b. Both are left-handed.
c. One is right-handed and the other is left-handed.
d. At least one is right-handed.

6.53 Refer to Exercise 6.52. Suppose that three people are selected at random.
a. Draw a probability tree to depict the experiment.
b. If we use the notation RRR to describe the selection of three right-handed people, what are the descriptions of the remaining seven events? (Use L for left-hander.)
c. How many of the events yield no right-handers, one right-hander, two right-handers, three right-handers?
d. Find the probability of no right-handers, one right-hander, two right-handers, three right-handers.

6.54 Suppose there are 100 students in your accounting class, 10 of whom are left-handed. Two students are selected at random.
a. Draw a probability tree and insert the probabilities for each branch.

What is the probability of the following events?
b. Both are right-handed.
c. Both are left-handed.
d. One is right-handed and the other is left-handed.
e. At least one is right-handed

6.55 Refer to Exercise 6.54. Suppose that three people are selected at random.
a. Draw a probability tree and insert the probabilities of each branch.
b. What is the probability of no right-handers, one right-hander, two right-handers, three right-handers?

6.56 An aerospace company has submitted bids on two separate federal government defense contracts. The company president believes that there is a 40% probability of winning the first contract. If they win the first contract, the probability of winning the second is 70%. However, if they lose the first contract, the president thinks that the probability of winning the second contract decreases to 50%.
a. What is the probability that they win both contracts?
b. What is the probability that they lose both contracts?
c. What is the probability that they win only one contract?

6.57 A telemarketer calls people and tries to sell them a subscription to a daily newspaper. On 20% of her calls, there is no answer or the line is busy. She sells subscriptions to 5% of the remaining calls. For what proportion of calls does she make a sale?

6.58 A foreman for an injection-molding firm admits that on 10% of his shifts, he forgets to shut off the injection machine on his line. This causes the machine to overheat, increasing the probability from 2% to 20% that a defective molding will be produced during the early morning run. What proportion of moldings from the early morning run is defective?

6.59 A study undertaken by the Miami-Dade Supervisor of Elections in 2002 revealed that 44% of registered voters are Democrats, 37% are Republicans, and 19% are others. If two registered voters are selected at random, what is the probability that both of them have the same party affiliation? (*Source: Miami Herald*, April 11, 2002.)

6.60 In early 2001, the U.S. Census Bureau started releasing the results of the 2000 census. Among many other pieces of information, the bureau recorded the race or ethnicity of the residents of every county in every state. From these results, the bureau calculated a "diversity index" that measures the probability that two people chosen at random are of different races or ethnicities. Suppose that the census determined that in a county in Wisconsin 80% of its residents are white, 15% are black, and 5% are Asian. Calculate the diversity index for this county.

6.61 A survey of middle-aged men reveals that 28% of them are balding at the crown of their heads. Moreover, it is known that such men have an 18% probability of suffering a heart attack in the next 10 years. Men who are not balding in this way have an 11% probability of a heart attack. Find the probability that a middle-aged man will suffer a heart attack sometime in the next 10 years.

6.62 The chartered financial analyst (CFA) is a designation earned after a candidate has taken three annual exams (CFA I, II, and III). The exams are taken in early June. Candidates who pass an exam are eligible to take the exam for the next level in the following year. The pass rates for levels I, II, and III are .57, .73, and .85, respectively. Suppose that 3,000 candidates take the level I exam, 2,500 take the level II exam, and 2,000 take the level III exam. Suppose that one student is selected at random. What is the probability that he or she has passed the exam? (*Source*: Institute of Financial Analysts.)

6.63 The Nickels restaurant chain regularly conducts surveys of its customers. Respondents are asked to assess food quality, service, and price. The responses are

Excellent   Good   Fair

Surveyed customers are also asked whether they would come back. After analyzing the responses, an expert in probability determined that 87% of customers say that they will return. Of those who so indicate, 57% rate the restaurant as excellent, 36% rate it as good, and the remainder rate it as fair. Of those who say that they won't return, the probabilities are 14%, 32%, and 54%, respectively. What proportion of customers rate the restaurant as good?

6.64 Researchers at the University of Pennsylvania School of Medicine have determined that children under 2 years old who sleep with the lights on have a 36% chance of becoming myopic before they are 16. Children who sleep in darkness have a 21% probability of becoming myopic. A survey indicates that 28% of children under 2 sleep with some light on. Find the probability that a child under 16 is myopic.

6.65 All printed circuit boards (PCBs) that are manufactured at a certain plant are inspected. An analysis of the company's records indicates that 22% of all PCBs are flawed in some way. Of those that are flawed, 84% are reparable and the rest must be discarded. If a newly produced PCB is randomly selected, what is the probability that it does not have to be discarded?

6.66 A financial analyst has determined that there is a 22% probability that a mutual fund will outperform the market over a 1-year period provided that it outperformed the market the previous year. If only 15% of mutual funds outperform the market during any year, what is the probability that a mutual fund will outperform the market 2 years in a row?

6.67 An investor believes that on a day when the Dow Jones Industrial Average (DJIA) increases, the probability that the NASDAQ also increases is 77%. If the investor believes that there is a 60% probability that the DJIA will increase tomorrow, what is the probability that the NASDAQ will increase as well?

6.68 The controls of an airplane have several backup systems or redundancies so that if one fails the plane will continue to operate. Suppose that the mechanism that controls the flaps has two backups. If the probability that the main control fails is .0001 and the probability that each backup will fail is .01, what is the probability that all three fail to operate?

6.69 According to TNS Intersearch, 69% of wireless web users use it primarily for receiving and sending e-mail. Suppose that three wireless web users are selected at random. What is the probability that all of them use it primarily for e-mail?

6.70 A financial analyst estimates that the probability that the economy will experience a recession in the next 12 months is 25%. She also believes that if the economy encounters a recession, the probability that her mutual fund will increase in value is 20%. If there is no recession, the probability that the mutual fund will increase in value is 75%. Find the probability that the mutual fund's value will increase.

6.4 BAYES'S LAW

Conditional probability is often used to gauge the relationship between two events. In many of the examples and exercises you've already encountered, conditional probability measures the probability that an event occurs given that a possible cause of the event has occurred. In Example 6.2, we calculated the probability that a mutual fund outperforms the market (the effect) given that the fund manager graduated from a top-20 MBA program (the possible cause). There are situations, however, where we witness a particular event and we need to compute the probability of one of its possible causes. **Bayes's Law** is the technique we use.

EXAMPLE 6.9 Should an MBA Applicant Take a Preparatory Course?

The Graduate Management Admission Test (GMAT) is a requirement for all applicants of MBA programs. A variety of preparatory courses are designed to help applicants improve their GMAT scores, which range from 200 to 800. Suppose that a survey of MBA students reveals that among GMAT scorers above 650, 52% took a preparatory course; whereas among GMAT scorers of less than 650 only 23% took a preparatory course. An applicant to an MBA program has determined that he needs a score of more than 650 to get into a certain MBA program, but he feels that his probability of getting that high a score is quite low—10%. He is considering taking a preparatory course that costs $500. He is willing to do so only if his probability of achieving 650 or more doubles. What should he do?

SOLUTION

The easiest way to address this problem is to draw a tree diagram. The following notation will be used:

A = GMAT score is 650 or more

A^C = GMAT score less than 650

B = Took preparatory course

B^C = Did not take preparatory course

The probability of scoring 650 or more is

$$P(A) = .10$$

The complement rule gives us

$$P(A^C) = 1 - .10 = .90$$

Conditional probabilities are

$$P(B|A) = .52$$

and

$$P(B|A^C) = .23$$

Again using the complement rule, we find the following conditional probabilities:

$$P(B^C|A) = 1 - .52 = .48$$

and

$$P(B^C|A^C) = 1 - .23 = .77$$

We would like to determine the probability that he would achieve a GMAT score of 650 or more given that he took the preparatory course; that is, we need to compute

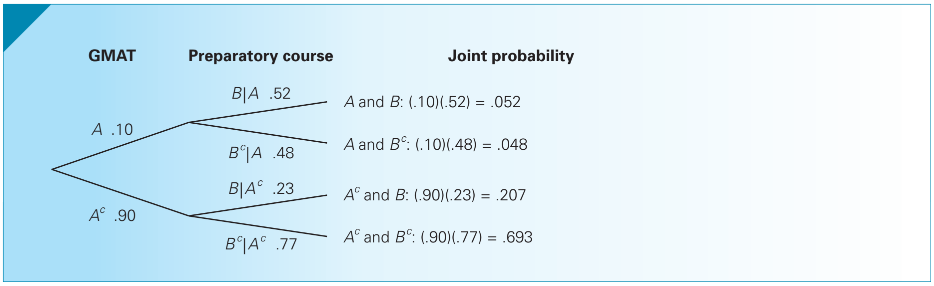
$$P(A|B)$$

Using the definition of conditional probability (page 184), we have

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Neither the numerator nor the denominator is known. The probability tree (Figure 6.4) will provide us with the probabilities.

FIGURE **6.4** Probability Tree for Example 6.9



As you can see,

$$P(A \text{ and } B) = (.10)(.52) = .052$$
$$P(A^C \text{ and } B) = (.90)(.23) = .207$$

and

$$P(B) = P(A \text{ and } B) + P(A^C \text{ and } B) = .052 + .207 = .259$$

Thus,

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{.052}{.259} = .201$$

The probability of scoring 650 or more on the GMAT doubles when the preparatory course is taken.

Thomas Bayes first employed the calculation of conditional probability as shown in Example 6.9 during the 18th century. Accordingly, it is called Bayes's Law.

The probabilities $P(A)$ and $P(A^C)$ are called **prior probabilities** because they are determined *prior* to the decision about taking the preparatory course. The conditional probabilities are called **likelihood probabilities** for reasons that are beyond the mathematics in this book. Finally, the conditional probability $P(A|B)$ and similar conditional probabilities $P(A^C|B)$, $P(A|B^C)$, and $P(A^C|B^C)$ are called **posterior probabilities** or **revised probabilities** because the prior probabilities are revised *after* the decision about taking the preparatory course.

You may be wondering why we did not get $P(A|B)$ directly. In other words, why not survey people who took the preparatory course and ask whether they received a score of 650 or more? The answer is that using the likelihood probabilities and using Bayes's Law allows individuals to set their own prior probabilities, which can then be revised. For example, another MBA applicant may assess her probability of scoring 650 or more as .40. Inputting the new prior probabilities produces the following probabilities:

$$P(A \text{ and } B) = (.40)(.52) = .208$$
$$P(A^C \text{ and } B) = (.60)(.23) = .138$$
$$P(B) = P(A \text{ and } B) + P(A^C \text{and } B) = .208 + .138 = .346$$
$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{.208}{.346} = .601$$

The probability of achieving a GMAT score of 650 or more increases by a more modest 50% (from .40 to .601).

Bayes's Law Formula (Optional)

Bayes's Law can be expressed as a formula for those who prefer an algebraic approach rather than a probability tree. We use the following notation.

The event B is the given event and the events

$$A_1, A_2, \ldots, A_k$$

are the events for which prior probabilities are known; that is,

$$P(A_1), P(A_2), \ldots, P(A_k)$$

are the prior probabilities.

The likelihood probabilities are

$$P(B|A_1), P(B|A_2), \ldots, P(B|A_k)$$

and

$$P(A_1|B), P(A_2|B), \ldots, P(A_k|B)$$

are the posterior probabilities, which represent the probabilities we seek.

Bayes's Law Formula

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_k)P(B|A_k)}$$

To illustrate the use of the formula, we'll redo Example 6.9. We begin by defining the events.

A_1 = GMAT score is 650 or more
A_2 = GMAT score less than 650
B = Take preparatory course

The probabilities are

$$P(A_1) = .10$$

The complement rule gives us

$$P(A_2) = 1 - .10 = .90$$

Conditional probabilities are

$$P(B|A_1) = .52$$

and

$$P(B|A_2) = .23$$

Substituting the prior and likelihood probabilities into the Bayes's Law formula yields the following:

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)} = \frac{(.10)(.52)}{(.10)(.52) + (.90)(.23)}$$

$$= \frac{.052}{.052 + .207} = \frac{.052}{.259} = .201$$

As you can see, the calculation of the Bayes's Law formula produces the same results as the probability tree.

Auditing Tax Returns: Solution

© Gary Buss/Taxi/Getty Images

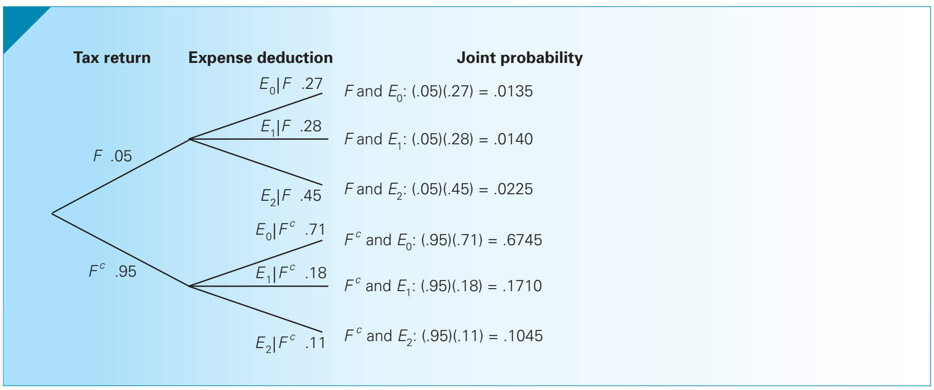
We need to revise the prior probability that this return contains significant fraud. The tree shown in Figure 6.5 details the calculation.

F = Tax return is fraudulent
F^C = Tax return is honest
E_0 = Tax return contains no expense deductions
E_1 = Tax return contains one expense deduction
E_2 = tax return contains two expense deductions

$$P(E_1) = P(F \text{ and } E_1) + P(F^C \text{ and } E_1) = .0140 + .1710 = .1850$$
$$P(F|E_1) = P(F \text{ and } E_1)/P(E_1) = .0140/.1850 = .0757$$

The probability that this return is fraudulent is .0757.

FIGURE **6.5** Probability Tree for Auditing Tax Returns



Applications in Medicine and Medical Insurance (Optional)

Physicians routinely perform medical tests, called *screenings*, on their patients. Screening tests are conducted for all patients in a particular age and gender group, regardless of their symptoms. For example, men in their 50s are advised to take a prostate-specific antigen (PSA) test to determine whether there is evidence of prostate cancer. Women undergo a Pap test for cervical cancer. Unfortunately, few of these tests are 100% accurate. Most can produce *false-positive* and *false-negative* results. A **false-positive** result is one in which the patient does not have the disease, but the test shows positive. A **false-negative** result is one in which the patient does have the disease, but the test produces a negative result. The consequences of each test are serious and costly. A false-negative test results in not detecting a disease in a patient, therefore postponing treatment, perhaps indefinitely. A false-positive test leads to apprehension and fear for the patient. In most cases, the patient is required to undergo further testing such as a biopsy. The unnecessary follow-up procedure can pose medical risks.

False-positive test results have financial repercussions. The cost of the follow-up procedure, for example, is usually far more expensive than the screening test. Medical insurance companies as well as government-funded plans are all adversely affected by false-positive test results. Compounding the problem is that physicians and patients are incapable of properly interpreting the results. A correct analysis can save both lives and money.

Bayes's Law is the vehicle we use to determine the true probabilities associated with screening tests. Applying the complement rule to the false-positive and false-negative rates produces the conditional probabilities that represent correct conclusions. Prior probabilities are usually derived by looking at the overall proportion of people with the diseases. In some cases, the prior probabilities may themselves have been revised because of heredity or demographic variables such as age or race. Bayes's Law allows us to revise the prior probability after the test result is positive or negative.

Example 6.10 is based on the actual false-positive and false-negative rates. Note however, that different sources provide somewhat different probabilities. The differences may be the result of the way positive and negative results are defined or the way technicians conduct the tests. Students who are affected by the diseases described in the example and exercises should seek clarification from their physicians.

EXAMPLE 6.10

Probability of Prostate Cancer

Prostate cancer is the most common form of cancer found in men. The probability of developing prostate cancer over a lifetime is 16%. (This figure may be higher since many prostate cancers go undetected.) Many physicians routinely perform a PSA test, particularly for men over age 50. PSA is a protein produced only by the prostate gland and thus is fairly easy to detect. Normally, men have PSA levels between 0 and 4 mg/ml. Readings above 4 may be considered high and potentially indicative of cancer. However, PSA levels tend to rise with age even among men who are cancer free. Studies have shown that the test is not very accurate. In fact, the probability of having an elevated PSA level given that the man does not have cancer (false positive) is .135. If the man does have cancer, the probability of a normal PSA level (false negative) is almost .300. (This figure may vary by age and by the definition of *high* PSA level.) If a physician concludes that the PSA is high, a biopsy is performed. Besides the concerns and health needs of the men, there are also financial costs. The cost of the blood test is low (approximately $50). However, the cost of the biopsy is considerably higher (approximately $1,000). A false-positive PSA test will lead to an unnecessary biopsy. Because the PSA test is so inaccurate, some private and public medical plans do not pay for it. Suppose you are a manager in a medical insurance company and must decide on guidelines for whom should be routinely screened for prostate cancer. An analysis of prostate cancer incidence and age produces the following table of probabilities. (The probability of a man under 40 developing prostate cancer is less than .0001, or small enough to treat as 0.)

Age	Probability of Developing Prostate Cancer
40–49	.010
50–59	.022
60–69	.046
70 and older	.079

Assume that a man in each of the age categories undergoes a PSA test with a positive result. Calculate the probability that each man actually has prostate cancer and the probability that he does not. Perform a cost–benefit analysis to determine the cost per cancer detected.

SOLUTION

As we did in Example 6.9 and the chapter-opening example, we'll draw a probability tree (Figure 6.6). The notation is

C = Has prostate cancer

C^C = Does not have prostate cancer

PT = Positive test result

NT = Negative test result

Starting with a man between 40 and 50 years old, we have the following probabilities

Prior

$P(C) = .010$

$P(C^C) = 1 - .010 = .990$

Likelihood probabilities

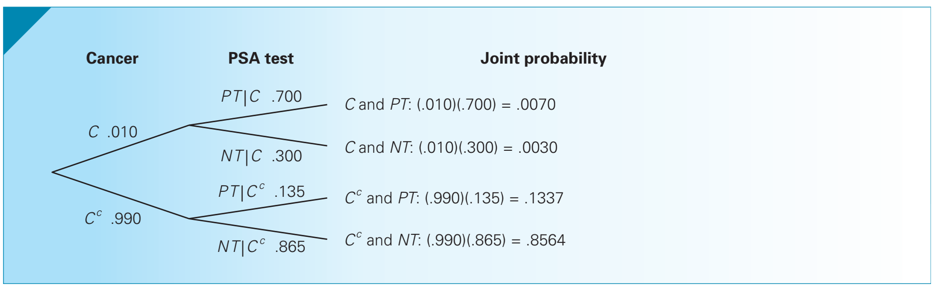
False negative: $P(NT|C) = .300$

True positive: $P(\text{PT}|C) = 1 - .300 = .700$

False positive: $P(\text{PT}|C^C) = .135$

True negative: $P(NT|C^C) = 1 - .135 = .865$

FIGURE **6.6** Probability Tree for Example 6.10



The tree allows you to determine the probability of obtaining a positive test result. It is

$$P(PT) = P(C \text{ and } PT) + P(C^C \text{ and } PT) = .0070 + .1337 = .1407$$

We can now compute the probability that the man has prostate cancer given a positive test result:

$$P(C|PT) = \frac{P(C \text{ and } PT)}{P(PT)} = \frac{.0070}{.1407} = .0498$$

The probability that he does not have prostate cancer is

$$P(C^C|PT) = 1 - P(C|PT) = 1 - .0498 = .9502$$

We can repeat the process for the other age categories. Here are the results.

	Probabilities Given a Positive PSA Test	
Age	Has Prostate Cancer	Does Not Have Prostate Cancer
40–49	.0498	.9502
50–59	.1045	.8955
60–69	.2000	.8000
70 and older	.3078	.6922

The following table lists the proportion of each age category wherein the PSA test is positive $[P(PT)]$

Age	Proportion of Tests That Are Positive	Number of Biopsies Performed per Million	Number of Cancers Detected	Number of Biopsies per Cancer Detected
40–49	.1407	140,700	.0498(140,700) = 7,007	20.10
50–59	.1474	147,400	.1045(147,400) = 15,403	9.57
60–79	.1610	161,000	.2000(161,000) = 32,200	5.00
70 and older	.1796	179,600	.3078(179,600) = 55,281	3.25

If we assume a cost of $1,000 per biopsy, the cost per cancer detected is $20,100 for 40 to 50, $9,570 for 50 to 60, $5,000 for 60 to 70, and $3,250 for over 70.

We have created an Excel spreadsheet to help you perform the calculations in Example 6.10. Open the **Excel Workbooks** folder and select **Medical screening**. There are three cells that you may alter. In cell B5, enter a new prior probability for prostate cancer. Its complement will be calculated in cell B15. In cells D6 and D15, type new values for the false-negative and false-positive rates, respectively. Excel will do the rest. We will use this spreadsheet to demonstrate some terminology standard in medical testing.

Terminology We will illustrate the terms using the probabilities calculated for the 40 to 50 age category.

The false-negative rate is .300. Its complement is the likelihood probability $P(PT|C)$, called the *sensitivity*. It is equal to $1 - .300 = .700$. Among men with prostate cancer, this is the proportion of men who will get a positive test result.

The complement of the false-positive rate (.135) is $P(NT|C^C)$, which is called the *specificity*. This likelihood probability is $1 - .135 = .865$

The posterior probability that someone has prostate cancer given a positive test result $[P(C|PT) = .0498]$ is called the *positive predictive value*. Using Bayes's Law, we can compute the other three posterior probabilities.

The probability that the patient does not have prostate cancer given a positive test result is

$$P(C^C|PT) = .9502$$

The probability that the patient has prostate cancer given a negative test result is

$$P(C|NT) = .0035$$

The probability that the patient does not have prostate cancer given a negative test result:

$$P(C^C|NT) = .9965$$

This revised probability is called the *negative predictive value*.

Developing an Understanding of Probability Concepts

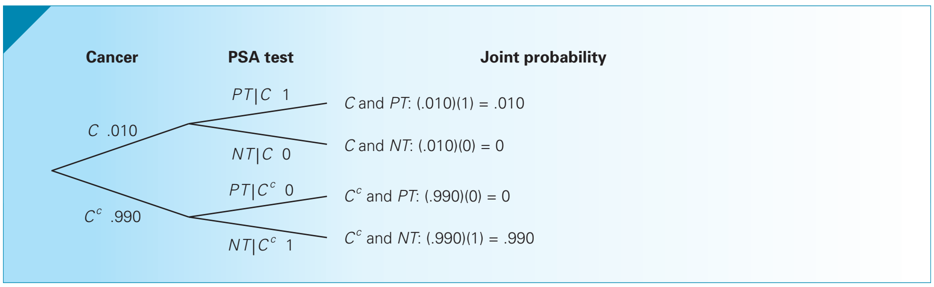
If you review the computations made previously, you'll realize that the prior probabilities are as important as the probabilities associated with the test results (the likelihood probabilities) in determining the posterior probabilities. The following table shows the prior probabilities and the revised probabilities.

Age	Prior Probabilities for Prostate Cancer	Posterior Probabilities Given a Positive PSA Test
40–49	.010	.0498
50–59	.022	.1045
60–69	.046	.2000
70 and older	.079	.3078

As you can see, if the prior probability is low, then unless the screening test is quite accurate, the revised probability will still be quite low.

To see the effects of different likelihood probabilities, suppose the PSA test is a perfect predictor. In other words, the false-positive and false-negative rates are 0. Figure 6.7 displays the probability tree.

FIGURE **6.7** Probability Tree for Example 6.10 with a Perfect Predictor Test



We find

$$P(PT) = P(C \text{ and } PT) + P(C^C \text{ and } PT) = .01 + 0 = .01$$

$$P(C|PT) = \frac{P(C \text{ and } PT)}{P(PT)} = \frac{.01}{.01} = 1.00$$

Now we calculate the probability of prostate cancer when the test is negative.

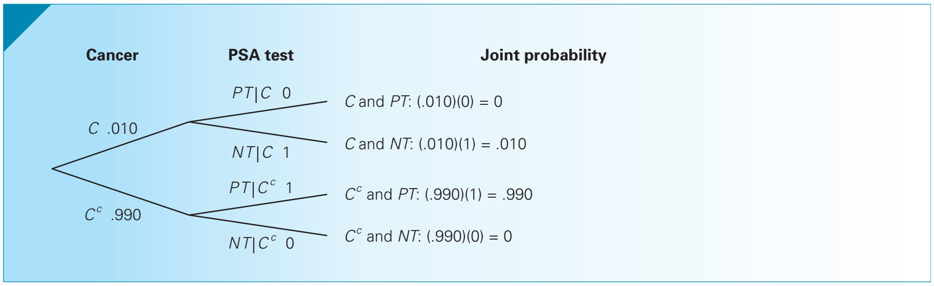
$$P(NT) = P(C \text{ and } NT) + P(C^C \text{ and } NT) = 0 + .99 = .99$$

$$P(C|NT) = \frac{P(C \text{ and } NT)}{P(NT)} = \frac{0}{.99} = 0$$

Thus, if the test is a perfect predictor and a man has a positive test, then as expected the probability that he has prostate cancer is 1.0. The probability that he does not have cancer when the test is negative is 0.

Now suppose that the test is always wrong; that is, the false-positive and false-negative rates are 100%. The probability tree is shown in Figure 6.8.

FIGURE 6.8 Probability Tree for Example 6.10 with a Test That Is Always Wrong



$$P(PT) = P(C \text{ and } PT) + P(C^C \text{ and } PT) = 0 + .99 = .99$$

$$P(C|PT) = \frac{P(C \text{ and } PT) \quad 0}{P(PT) \quad .99} = 0$$

and

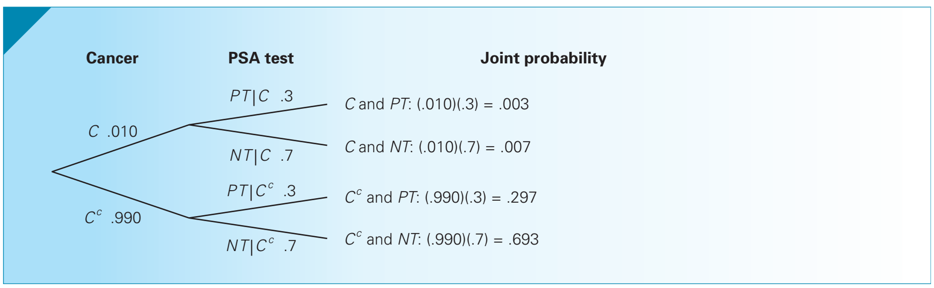
$$P(NT) = P(C \text{ and } NT) + P(C^C \text{ and } NT) = .01 + 0 = .01$$

$$P(C|NT) = \frac{P(C \text{ and } NT)}{P(NT)} = \frac{.01}{.01} = 1.00$$

Notice we have another perfect predictor except that it is reversed. The probability of prostate cancer given a positive test result is 0, but the probability becomes 1.00 when the test is negative.

Finally we consider the situation when the set of likelihood probabilities are the same. Figure 6.9 depicts the probability tree for a 40- to 50-year-old male and the probability of a positive test is (say) .3 and a the probability of a negative test is .7.

FIGURE 6.9 Probability Tree for Example 6.10 with Identical Likelihood Probabilities



$$P(PT) = P(C \text{ and } PT) + P(C^C \text{ and } PT) = .003 + .297 = .300$$

$$P(C|PT) = \frac{P(C \text{ and } PT)}{P(PT)} = \frac{.003}{.300} = .01$$

and

$$P(NT) = P(C \text{ and } NT) + P(C^C \text{ and } NT) = .007 + .693 = .700$$

$$P(C|NT) = \frac{P(C \text{ and } NT)}{P(NT)} = .007 + .700 = .01$$

As you can see, the posterior and prior probabilities are the same. That is, the PSA test does not change the prior probabilities. Obviously, the test is useless.

We could have used any probability for the false-positive and false-negative rates, including .5. If we had used .5, then one way of performing this PSA test is to flip a fair coin. One side would be interpreted as positive and the other side as negative. It is clear that such a test has no predictive power.

The exercises and Case 6.4 offer the probabilities for several other screening tests.

EXERCISES

6.71 Refer to Exercise 6.47. Determine $P(A|B)$.

6.72 Refer to Exercise 6.48. Find the following.

a. $P(A|B)$
b. $P(A^C|B)$
c. $P(A|B^C)$
d. $P(A^C|B^C)$

6.73 Refer to Example 6.9. An MBA applicant believes that the probability of scoring more than 650 on the GMAT without the preparatory course is .95. What is the probability of attaining that level after taking the preparatory course?

6.74 Refer to Exercise 6.58. The plant manager randomly selects a molding from the early morning run and discovers it is defective. What is the probability that the foreman forgot to shut off the machine the previous night?

6.75 The U.S. National Highway Traffic Safety Administration gathers data concerning the causes of highway crashes where at least one fatality has occurred. The following probabilities were determined from the 1998 annual study (BAC is blood-alcohol content). (*Source: Statistical Abstract of the United States, 2000*, Table 1042.)

$P(BAC = 0 \mid \text{Crash with fatality}) = .616$

$P(BAC \text{ is between .01 and .09} \mid \text{Crash with fatality}) = .300$

$P(BAC \text{ is greater than .09} \mid \text{Crash with fatality}) = .084$

Over a certain stretch of highway during a 1-year period, suppose the probability of being involved in a crash that results in at least one fatality is .01. It has been estimated that 12% of the drivers on this highway drive while their BAC is greater than .09. Determine the probability of a crash with at least one fatality if a driver drives while legally intoxicated (BAC greater than .09).

6.76 Refer to Exercise 6.62. A randomly selected candidate who took a CFA exam tells you that he has passed the exam. What is the probability that he took the CFA I exam?

6.77 Bad gums may mean a bad heart. Researchers discovered that 85% of people who have suffered a heart attack had periodontal disease, an inflammation of the gums. Only 29% of healthy people have this disease. Suppose that in a certain community heart attacks are quite rare, occurring with only 10% probability. If someone has periodontal disease, what is the probability that he or she will have a heart attack?

6.78 Refer to Exercise 6.77. If 40% of the people in a community will have a heart attack, what is the probability that a person with periodontal disease will have a heart attack?

6.79 Data from the Office on Smoking and Health, Centers for Disease Control and Prevention, indicate that 40% of adults who did not finish high school, 34% of high school graduates, 24% of adults who completed some college, and 14% of college graduates smoke. Suppose that one individual is selected at random, and it is discovered that the individual smokes. What is the probability that the individual is a college graduate? Use the probabilities in Exercise 6.45 to calculate the probability that the individual is a college graduate.

6.80 Three airlines serve a small town in Ohio. Airline A has 50% of all the scheduled flights, airline B has 30%, and airline C has the remaining 20%. Their on-time rates are 80%, 65%, and 40%, respectively. A plane has just left on time. What is the probability that it was airline A?

6.81 Your favorite team is in the final playoffs. You have assigned a probability of 60% that it will win the championship. Past records indicate that when teams win the championship, they win the first game of the series 70% of the time. When they lose the series, they win the first game 25% of the time. The first game is over; your team has lost. What is the probability that it will win the series?

The following exercises are based on the Applications in Medical Screening and Medical Insurance subsection.

6.82 Transplant operations have become routine. One common transplant operation is for kidneys. The most dangerous aspect of the procedure is the possibility that the body may reject the new organ. Several new drugs are available for such circumstances, and the earlier the drug is administered, the higher the probability of averting rejection. The *New England Journal of Medicine* recently reported the development of a new urine test to detect early warning signs that the body is rejecting a transplanted kidney. However, like most other tests, the new test is not perfect. When the test is conducted on someone whose kidney will be rejected, approximately one out of five tests will be negative (i.e., the test is wrong). When the test is conducted on a person whose kidney will not be rejected, 8% will show a positive test result (i.e., another incorrect result). Physicians know that in about 35% of kidney transplants the body tries to reject the organ. Suppose that the test was performed and the test is positive (indicating early warning of rejection). What is the probability that the body is attempting to reject the kidney?

6.83 The Rapid Test is used to determine whether someone has HIV (the virus that causes AIDS). The false-positive and false-negative rates are .027 and .080, respectively. A physician has just received the Rapid Test report that his patient tested positive. Before receiving the result, the physician assigned his patient to the low-risk group (defined on the basis of several variables) with only a 0.5% probability of having HIV. What is the probability that the patient actually has HIV?

6.84 What are the sensitivity, specificity, positive predictive value, and negative predictive value in the previous exercise?

6.85 The Pap smear is the standard test for cervical cancer. The false-positive rate is .636; the false-negative rate is .180. Family history and age are factors that must be considered when assigning a probability of cervical cancer. Suppose that, after obtaining a medical history, a physician determines that 2% of women of his patient's age and with similar family histories have cervical cancer. Determine the effects a positive and a negative Pap smear test have on the probability that the patient has cervical cancer.

6.5 Identifying the Correct Method

As we've previously pointed out, the emphasis in this book will be on identifying the correct statistical technique to use. In Chapters 2 and 4, we showed how to summarize data by first identifying the appropriate method to use. Although it is difficult to offer strict rules on which probability method to use, we can still provide some general guidelines.

In the examples and exercises in this text (and most other introductory statistics books), the key issue is whether joint probabilities are provided or are required.

Joint Probabilities Are Given

In Section 6.2, we addressed problems where the joint probabilities were given. In these problems, we can compute marginal probabilities by adding across rows and down columns. We can use the joint and marginal probabilities to compute conditional probabilities, for which a formula is available. This allows us to determine whether the events described by the table are independent or dependent.

We can also apply the addition rule to compute the probability that either of two events occur.

Joint Probabilities Are Required

The previous section introduced three probability rules and probability trees. We need to apply some or all of these rules in circumstances where one or more joint probabilities are required. We apply the multiplication rule (either by formula or through a probability tree) to calculate the probability of intersections. In some problems, we're interested in adding these joint probabilities. We're actually applying the addition rule for mutually exclusive events here. We also frequently use the complement rule. In addition, we can also calculate new conditional probabilities using Bayes's Law.

CHAPTER SUMMARY

The first step in assigning probability is to create an **exhaustive** and **mutually exclusive** list of outcomes. The second step is to use the **classical**, **relative frequency**, or **subjective approach** and assign probability to the outcomes. A variety of methods are available to compute the probability of other events. These methods include **probability rules** and **trees**.

An important application of these rules is **Bayes's Law**, which allows us to compute conditional probabilities from other forms of probability.

IMPORTANT TERMS

Random experiment 176
Exhaustive 176
Mutually exclusive 176
Sample space 177
Classical approach 177
Relative frequency approach 178
Subjective approach 178
Event 178
Intersection 181
Joint probability 181
Marginal probability 183
Conditional probability 183
Independent events 185
Union 186
Complement 191
Complement rule 191
Multiplication rule 191
Addition rule 193
Bayes's Law 199
Prior probability 200
Likelihood probability 200
Posterior probability 200
False-positive 203
False-negative 203

FORMULAS

Conditional probability

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Complement rule

$$P(A^C) = 1 - P(A)$$

Multiplication rule

$$P(A \text{ and } B) = P(A|B)P(B)$$

Addition rule

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

CHAPTER EXERCISES

6.86 The following table lists the joint probabilities of achieving grades of A and not achieving A's in two MBA courses.

	Achieve a Grade of A in Marketing	Does Not Achieve a Grade of A in Marketing
Achieve a grade of A in statistics	.053	.130
Does not achieve a grade of A in statistics	.237	.580

a. What is the probability that a student achieves a grade of A in marketing?
b. What is the probability that a student achieves a grade of A in marketing, given that he or she does not achieve a grade of A in statistics?
c. Are achieving grades of A in marketing and statistics independent events? Explain.

6.87 A construction company has bid on two contracts. The probability of winning contract A is .3. If the company wins contract A, then the probability of winning contract B is .4. If the company loses contract A, then the probability of winning contract B decreases to .2. Find the probability of the following events.
a. Winning both contracts
b. Winning exactly one contract
c. Winning at least one contract

6.88 Laser surgery to fix shortsightedness is becoming more popular. However, for some people, a second procedure is necessary. The following table lists the joint probabilities of needing a second procedure and whether the patient has a corrective lens with a factor (diopter) of minus 8 or less.

	Vision Corrective Factor of More Than Minus 8	Vision Corrective Factor of Minus 8 or Less
First procedure is successful	.66	.15
Second procedure is required	.05	.14

a. Find the probability that a second procedure is required.
b. Determine the probability that someone whose corrective lens factor is minus 8 or less does not require a second procedure.
c. Are the events independent? Explain your answer.

6.89 The effect of an antidepressant drug varies from person to person. Suppose that the drug is effective on 80% of women and 65% of men. It is known that 66% of the people who take the drug are women. What is the probability that the drug is effective?

6.90 Refer to Exercise 6.89. Suppose that you are told that the drug is effective. What is the probability that the drug taker is a man?

6.91 In a four-cylinder engine there are four spark plugs. If any one of them malfunctions, the car will idle roughly and power will be lost. Suppose that for a certain brand of spark plugs the probability that a spark plug will function properly after 5,000 miles is .90. Assuming that the spark plugs operate independently, what is the probability that the car will idle roughly after 5,000 miles?

6.92 A telemarketer sells magazine subscriptions over the telephone. The probability of a busy signal or no answer is 65%. If the telemarketer does make contact, the probability of 0, 1, 2, or 3 magazine subscriptions is .5, .25, .20, and .05, respectively. Find the probability that in one call she sells no magazines.

6.93 A statistics professor believes that there is a relationship between the number of missed classes and the grade on his midterm test. After examining his records, he produced the following table of joint probabilities.

	Student Fails the Test	Student Passes the Test
Student misses fewer than 5 classes	.02	.86
Student misses 5 or more classes	.09	.03

a. What is the pass rate on the midterm test?
b. What proportion of students who miss five or more classes passes the midterm test?
c. What proportion of students who miss fewer than five classes passes the midterm test?
d. Are the events independent?

6.94 In Canada, criminals are entitled to parole after serving only one-third of their sentence. Virtually all prisoners, with several exceptions including murderers, are released after serving two-thirds of their sentence. The government has proposed a new law that would create a special category of inmates based on whether they had committed crimes involving violence or drugs. Such criminals would be subject to additional detention if the Correction Services judges them highly likely to reoffend. Currently, 27% of prisoners who are released commit another crime within 2 years of release. Among those who have reoffended, 41% would have been detained under the new law, whereas 31% of those who have not reoffended would have been detained.

a. What is the probability that a prisoner who would have been detained under the new law does commit another crime within 2 years?
b. What is the probability that a prisoner who would not have been detained under the new law does commit another crime within 2 years?

6.95 Casino Windsor conducts surveys to determine the opinions of its customers. Among other questions, respondents are asked to give their opinion about "Your overall impression of Casino Windsor." The responses are

Excellent Good Average Poor

In addition, the gender of the respondent is noted. After analyzing the results, the following table of joint probabilities was produced.

Rating	Women	Men
Excellent	.27	.22
Good	.14	.10
Average	.06	.12
Poor	.03	.06

a. What proportion of customers rate Casino Windsor as excellent?
b. Determine the probability that a male customer rates Casino Windsor as excellent.
c. Find the probability that a customer who rates Casino Windsor as excellent is a man.
d. Are gender and rating independent? Explain your answer.

6.96 A customer-service supervisor regularly conducts a survey of customer satisfaction. The results of the latest survey indicate that 8% of customers were not satisfied with the service they received at their last visit to the store. Of those who are not satisfied, only 22% return to the store within a year. Of those who are satisfied, 64% return within a year. A customer has just entered the store. In response to your question, he informs you that it is less than 1 year since his last visit to the store. What is the probability that he was satisfied with the service he received?

6.97 How does level of affluence affect health care? To address one dimension of the problem, a group of heart attack victims was selected. Each was categorized as a low-, medium-, or high-income earner. Each was also categorized as having survived or died. A demographer notes that in our society 21% fall into the low-income group, 49% are in the medium-income group, and 30% are in the high-income group. Furthermore, an analysis of heart attack victims reveals that 12% of low-income people, 9% of medium-income people, and 7% of high-income people die of heart attacks. Find the probability that a survivor of a heart attack is in the low-income group.

6.98 A statistics professor and his wife are planning to take a 2-week vacation in Hawaii, but they can't decide whether to spend 1 week on each of the islands of Maui and Oahu, 2 weeks on Maui, or 2 weeks on Oahu. Placing their faith in random chance, they insert two Maui brochures in one envelope, two Oahu brochures in a second envelope, and one brochure from each island in a third envelope. The professor's wife will select one envelope at random, and their vacation schedule will be based on the brochures of the islands so selected. After his wife randomly selects an envelope, the professor removes one brochure from the envelope (without looking at the second brochure) and observes that it is a Maui brochure. What is the probability that the other brochure in the envelope is a Maui brochure? (Proceed with caution: The problem is more difficult than it appears.)

6.99 The owner of an appliance store is interested in the relationship between the price at which an item is sold (regular or sale price) and the customer's decision on whether to purchase an extended warranty. After analyzing her records, she produced the following joint probabilities.

	Purchased Extended Warranty	Did Not Purchase Extended Warranty
Regular price	.21	.57
Sale price	.14	.08

a. What is the probability that a customer who bought an item at the regular price purchased the extended warranty?
b. What proportion of customers buy an extended warranty?
c. Are the events independent? Explain.

6.100 Researchers have developed statistical models based on financial ratios that predict whether a company will go bankrupt over the next 12 months. In a test of one such model, the model correctly predicted the bankruptcy of 85% of firms that did in fact fail, and it correctly predicted nonbankruptcy for 74% of firms that did not fail. Suppose that we expect 8% of the firms in a particular city to fail over the next year. Suppose that the model predicts bankruptcy for a firm that you own. What is the probability that your firm will fail within the next 12 months?

6.101 A union's executive conducted a survey of its members to determine what the membership felt were the important issues to be resolved during upcoming negotiations with management. The results indicate that 74% of members felt that job security was an important issue, whereas 65% identified pension benefits as an important issue. Of those who felt that pension benefits were important, 60% also felt that job security was an important issue. One member is selected at random.

a. What is the probability that he or she felt that both job security and pension benefits were important?

b. What is the probability that the member felt that at least one of these two issues was important?

6.102 In a class on probability, a statistics professor flips two balanced coins. Both fall to the floor and roll under his desk. A student in the first row informs the professor that he can see both coins. He reports that at least one of them shows tails. What is the probability that the other coin is also tails? (Beware the obvious.)

6.103 Refer to Exercise 6.102. Suppose the student informs the professor that he can see only one coin and it shows tails. What is the probability that the other coin is also tails?

CASE 6.1 Let's Make a Deal

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A number of years ago, there was a popular television game show called *Let's Make a Deal*. The host, Monty Hall, would randomly select contestants from the audience and, as the title suggests, he would make deals for prizes. Contestants would be given relatively modest prizes and would then be offered the opportunity to risk those prizes to win better ones.

Suppose that you are a contestant on this show. Monty has just given you a free trip touring toxic waste sites around the country. He now offers you a trade: Give up the trip in exchange for a gamble. On the stage are three curtains, A, B, and C. Behind one of them is a brand new car worth $20,000. Behind the other two curtains, the stage is empty. You decide to gamble and select curtain A. In an attempt to make things more interesting, Monty then exposes an empty stage by opening curtain C (he knows there is nothing behind curtain C). He then offers you the free trip again if you quit now or, if you like, he will propose another deal (i.e., you can keep your choice of curtain A or perhaps switch to curtain B). What do you do?

To help you answer that question, try first answering these questions.

1. Before Monty shows you what's behind curtain C, what is the probability that the car is behind curtain A? What is the probability that the car is behind curtain B?

2. After Monty shows you what's behind curtain C, what is the probability that the car is behind curtain A? What is the probability that the car is behind curtain B?

CASE 6.2 To Bunt or Not to Bunt, That Is the Question

© AlBehrmanAP

No sport generates as many statistics as baseball. Reporters, managers, and fans argue and discuss strategies on the basis of these statistics. An article in *Chance* ("A Statistician Reads the Sports Page," Hal S. Stern, Vol. 1, Winter 1997) offers baseball lovers another opportunity to analyze numbers associated with the game. Table 1 lists the probabilities of scoring at least one run in situations that are defined by the number of outs and the bases occupied. For example, the probability of scoring at least one run when there are no outs and a man

(Case 6.4 continued)

Mother's Age	False-Positive Rate	False-Negative Rate
Under 30	.04	.376
30–34	.082	.290
35–37	.178	.269
Over 38	.343	.029

The probability that a baby has Down syndrome is primarily a function of the mother's age. The probabilities are listed here.

Age	Probability of Down Syndrome
25	1/1300
30	1/900
35	1/350
40	1/100
45	1/25
49	1/12

a. For each of the ages 25, 30, 35, 40, 45, and 49 determine the probability of Down syndrome if the maternity serum screening produces a positive result.

b. Repeat for a negative result.

CASE 6.5 Probability That at Least Two People in the Same Room Have the Same Birthday


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Suppose that there are two people in a room. The probability that they share the same birthday (date, not necessarily year) is 1/365, and the probability that they have different birthdays is 364/365. To illustrate, suppose that you're in a room with one other person and that your birthday is July 1. The probability that the other person does not have the same birthday is 364/365 because there are 364 days in the year that are not July 1. If a third person now enters the room, the probability that he or she has a different birthday from the first two people in the room is 363/365. Thus, the probability that three people in a room having different birthdays is (364/365)(363/365). You can continue this process for any number of people.

Find the number of people in a room so that there is about a 50% probability that at least two have the same birthday.

Hint1: Calculate the probability that they don't have the same birthday.

Hint 2: Excel users can employ the **product** function to calculate joint probabilities.

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7

RANDOM VARIABLES AND DISCRETE PROBABILITY DISTRIBUTIONS

Investing to Maximize Returns and Minimize Risk

DATA Xm07-00

An investor has $100,000 to invest in the stock market. She is interested in developing a stock portfolio made up of stocks on the New York Stock Exchange (NYSE), the Toronto Stock Exchange (TSX), and the NASDAQ. The stocks are Coca Cola and Disney (NYSE), Barrick Gold (TSX), and Amazon (NASDAQ). However, she doesn't know how much to invest in each one. She wants to maximize her return, but she would also like to minimize the risk. She has computed the monthly returns for all four stocks during a 60-month period (January 2005 to December 2009). After some consideration, she narrowed her choices down to the following three. What should she do?

1. $25,000 in each stock
2. Coca Cola: $10,000, Disney: $20,000, Barrick Gold: $30,000, Amazon: $40,000
3. Coca Cola: $10,000, Disney: $50,000, Barrick Gold: $30,000, Amazon: $10,000

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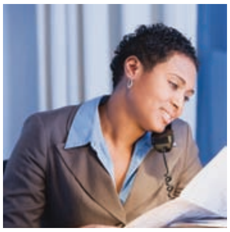

We will provide our answer after we've developed the necessary tools in Section 7.3.

(Case 6.2 continued)

is on first base is .39. If the bases are loaded with one out, then the probability of scoring any runs is .67.

TABLE 1 Probability of Scoring Any Runs

Bases Occupied	0 Outs	1 Out	2 Outs
Bases empty	.26	.16	.07
First base	.39	.26	.13
Second base	.57	.42	.24
Third base	.72	.55	.28
First base and second base	.59	.45	.24
First base and third base	.76	.61	.37
Second base and third base	.83	.74	.37
Bases loaded	.81	.67	.43

(Probabilities are based on results from the American League during the 1989 season. The results for the National League are also shown in the article and are similar.)

Table 1 allows us to determine the best strategy in a variety of circumstances. This case will concentrate on the strategy of the sacrifice bunt. The purpose of the sacrifice bunt is to sacrifice the batter to move base runners to the next base. It can be employed when there are fewer than two outs and men on base. Ignoring the suicide squeeze, any of four outcomes can occur:

1. The bunt is successful. The runner (or runners) advances one base, and the batter is out.
2. The batter is out but fails to advance the runner.
3. The batter bunts into a double play.
4. The batter is safe (hit or error), and the runner advances.

Suppose that you are an American League manager. The game is tied in the middle innings of a game, and there is a runner on first base with no one out. Given the following probabilities of the four outcomes of a bunt for the batter at the plate, should you signal the batter to sacrifice bunt?

$P(\text{Outcome } 1) = .75$
$P(\text{Outcome } 2) = .10$
$P(\text{Outcome } 3) = .10$
$P(\text{Outcome } 4) = .05$

Assume for simplicity that after the hit or error in outcome 4, there will be men on first and second base and no one out.

CASE 6.3 Should He Attempt to Steal a Base?

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Refer to Case 6.2. Another baseball strategy is to attempt to steal second base. Historically the probability of a successful steal of second base is approximately 68%. The probability of being thrown out is 32%. (We'll ignore the relatively rare event wherein the catcher throws the ball into center field allowing the base runner to advance to third base.) Suppose there is a runner on first base. For each of the possible number of outs (0, 1, or 2), determine whether it is advisable to have the runner attempt to steal second base.

CASE 6.4 Maternal Serum Screening Test for Down Syndrome

© AP Photo/Javier Galeano

Pregnant women are screened for a birth defect called Down syndrome. Down syndrome babies are mentally and physically challenged. Some mothers choose to abort the fetus when they are certain that their baby will be born with the syndrome. The most common screening is maternal serum screening, a blood test that looks for markers in the blood to indicate whether the birth defect may occur. The false-positive and false-negative rates vary according to the age of the mother.

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Introduction

In this chapter, we extend the concepts and techniques of probability introduced in Chapter 6. We present random variables and probability distributions, which are essential in the development of statistical inference.

Here is a brief glimpse into the wonderful world of statistical inference. Suppose that you flip a coin 100 times and count the number of heads. The objective is to determine whether we can infer from the count that the coin is not balanced. It is reasonable to believe that observing a large number of heads (say, 90) or a small number (say, 15) would be a statistical indication of an unbalanced coin. However, where do we draw the line? At 75 or 65 or 55? Without knowing the probability of the frequency of the number of heads from a balanced coin, we cannot draw any conclusions from the sample of 100 coin flips.

The concepts and techniques of probability introduced in this chapter will allow us to calculate the probability we seek. As a first step, we introduce random variables and probability distributions.

7.1 Random Variables and Probability Distributions

Consider an experiment where we flip two balanced coins and observe the results. We can represent the events as

Heads on the first coin and heads on the second coin

Heads on the first coin and tails on the second coin

Tails on the first coin and heads on the second coin

Tails on the first coin and tails on the second coin

However, we can list the events in a different way. Instead of defining the events by describing the outcome of each coin, we can count the number of heads (or, if we wish, the number of tails). Thus, the events are now

2 heads

1 heads

1 heads

0 heads

The number of heads is called the **random variable**. We often label the random variable X, and we're interested in the probability of each value of X. Thus, in this illustration, the values of X are 0, 1, and 2.

Here is another example. In many parlor games as well as in the game of craps played in casinos, the player tosses two dice. One way of listing the events is to describe the number on the first die and the number on the second die as follows.

1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

However, in almost all games, the player is primarily interested in the total. Accordingly, we can list the totals of the two dice instead of the individual numbers.

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

If we define the random variable X as the total of the two dice, then X can equal 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12.

Random Variable

A **random variable** is a function or rule that assigns a number to each outcome of an experiment.

In some experiments the outcomes are numbers. For example, when we observe the return on an investment or measure the amount of time to assemble a computer, the experiment produces events that are numbers. Simply stated, the value of a random variable is a numerical event.

There are two types of random variables, discrete and continuous. A **discrete random variable** is one that can take on a countable number of values. For example, if we define X as the number of heads observed in an experiment that flips a coin 10 times, then the values of X are 0, 1, 2, . . . , 10. The variable X can assume a total of 11 values. Obviously, we counted the number of values; hence, X is discrete.

A **continuous random variable** is one whose values are uncountable. An excellent example of a continuous random variable is the amount of time to complete a task. For example, let X = time to write a statistics exam in a university where the time limit is 3 hours and students cannot leave before 30 minutes. The smallest value of X is 30 minutes. If we attempt to count the number of values that X can take on, we need to identify the next value. Is it 30.1 minutes? 30.01 minutes? 30.001 minutes? None of these is the second possible value of X because there exist numbers larger than 30 and smaller than 30.001. It becomes clear that we cannot identify the second, or third, or any other values of X (except for the largest value 180 minutes). Thus, we cannot count the number of values, and X is continuous.

A **probability distribution** is a table, formula, or graph that describes the values of a random variable and the probability associated with these values. We will address discrete probability distributions in the rest of this chapter and cover continuous distributions in Chapter 8.

As we noted above, an uppercase letter will represent the *name* of the random variable, usually X. Its lowercase counterpart will represent the value of the random variable. Thus, we represent the probability that the random variable X will equal x as

$$P(X = x)$$

or more simply

$$P(x)$$

Discrete Probability Distributions

The probabilities of the values of a discrete random variable may be derived by means of probability tools such as tree diagrams or by applying one of the definitions of probability. However, two fundamental requirements apply as stated in the box.

Requirements for a Distribution of a Discrete Random Variable

1. $0 \le P(x) \le 1 \quad$ for all x
2. $\sum_{\text{all } x} P(x) = 1$

where the random variable can assume values x and $P(x)$ is the probability that the random variable is equal to x.

These requirements are equivalent to the rules of probability provided in Chapter 6. To illustrate, consider the following example.

EXAMPLE 7.1

Probability Distribution of Persons per Household

The *Statistical Abstract of the United States* is published annually. It contains a wide variety of information based on the census as well as other sources. The objective is to provide information about a variety of different aspects of the lives of the country's residents. One of the questions asks households to report the number of persons living in the household. The following table summarizes the data. Develop the probability distribution of the random variable defined as the number of persons per household.

Number of Persons	Number of Households (Millions)
1	31.1
2	38.6
3	18.8
4	16.2
5	7.2
6	2.7
7 or more	1.4
Total	116.0

Source: Statistical Abstract of the United States, 2009, Table 61.

SOLUTION

The probability of each value of X, the number of persons per household is computed as the relative frequency. We divide the frequency for each value of X by the total number of households, producing the following probability distribution.

x	$P(x)$
1	31.1/116.0 = .268
2	38.6/116.0 = .333
3	18.8/116.0 = .162
4	16.2/116.0 = .140
5	7.2/116.0 = .062
6	2.7/116.0 = .023
7 or more	1.4/116.0 = .012
Total	1.000

As you can see, the requirements are satisfied. Each probability lies between 0 and 1, and the total is 1.

We interpret the probabilities in the same way we did in Chapter 6. For example, if we select one household at random, the probability that it has three persons is

$$P(3) = .162$$

We can also apply the addition rule for mutually exclusive events. (The values of X are mutually exclusive; a household can have 1, 2, 3, 4, 5, 6, or 7 or more persons.) The probability that a randomly selected household has four or more persons is

$$\begin{aligned} P(X \geq 4) &= P(4) + P(5) + P(6) + P(7 \text{ or more}) \\ &= .140 + .062 + .023 + .012 = .237 \end{aligned}$$

In Example 7.1, we calculated the probabilities using census information about the entire population. The next example illustrates the use of the techniques introduced in Chapter 6 to develop a probability distribution.

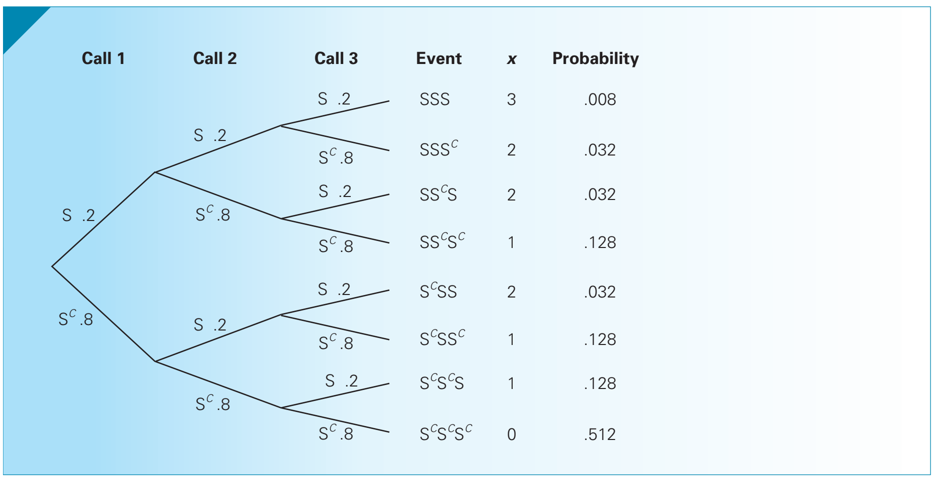
EXAMPLE 7.2 Probability Distribution of the Number of Sales

A mutual fund salesperson has arranged to call on three people tomorrow. Based on past experience, the salesperson knows there is a 20% chance of closing a sale on each call. Determine the probability distribution of the number of sales the salesperson will make.

SOLUTION

We can use the probability rules and trees introduced in Section 6.3. Figure 7.1 displays the probability tree for this example. Let X = the number of sales.

FIGURE **7.1**



The tree exhibits each of the eight possible outcomes and their probabilities. We see that there is one outcome that represents no sales, and its probability is $P(0) = .512$. There are three outcomes representing one sale, each with probability .128, so we add these probabilities. Thus,

$$P(1) = .128 + .128 + .128 = 3(.128) = .384$$

The probability of two sales is computed similarly:

$$P(X) = 3(.032) = .096$$

There is one outcome where there are three sales:

$$P(3) = .008$$

The probability distribution of X is listed in Table 7.1.

TABLE **7.1** Probability Distribution of the Number of Sales in Example 7.2

x	$P(x)$
0	.512
1	.384
2	.096
3	.008

Probability Distributions and Populations

The importance of probability distributions derives from their use as representatives of populations. In Example 7.1, the distribution provided us with information about the population of numbers of persons per household. In Example 7.2, the population was the number of sales made in three calls by the salesperson. And as we noted before, statistical inference deals with inference about populations.

Describing the Population/Probability Distribution

In Chapter 4, we showed how to calculate the mean, variance, and standard deviation of a population. The formulas we provided were based on knowing the value of the random variable for each member of the population. For example, if we want to know the mean and variance of annual income of all North American blue-collar workers, we would record each of their incomes and use the formulas introduced in Chapter 4:

$$\mu = \frac{\sum_{i=1}^{N} X_i}{N}$$

$$\sigma^2 = \frac{\sum_{i=1}^{N} (X_i - \mu)^2}{N}$$

where X_1 is the income of the first blue-collar worker, X_2 is the second worker's income, and so on. It is likely that N equals several million. As you can appreciate, these formulas are seldom used in practical applications because populations are so large. It is unlikely that we would be able to record all the incomes in the population of North American blue-collar workers. However, probability distributions often represent populations. Rather than record each of the many observations in a population, we list the values and their associated probabilities as we did in deriving the probability distribution of the number of persons per household in Example 7.1 and the number of successes in three calls by the mutual fund salesperson. These can be used to compute the mean and variance of the population.

The population mean is the weighted average of all of its values. The weights are the probabilities. This parameter is also called the **expected value** of X and is represented by $E(X)$.

Population Mean

$$E(X) = \mu = \sum_{\text{all } x} xP(x)$$

The population variance is calculated similarly. It is the weighted average of the squared deviations from the mean.

Population Variance

$$V(X) = \sigma^2 = \sum_{\text{all } x} (x - \mu)^2 P(x)$$

There is a shortcut calculation that simplifies the calculations for the population variance. This formula is not an approximation; it will yield the same value as the formula above.

Shortcut Calculation for Population Variance

$$V(X) = \sigma^2 = \sum_{\text{all } x} x^2 P(x) - \mu^2$$

The standard deviation is defined as in Chapter 4.

Population Standard Deviation

$$\sigma = \sqrt{\sigma^2}$$

EXAMPLE 7.3

Describing the Population of the Number of Persons per Household

Find the mean, variance, and standard deviation for the population of the number of persons per household Example 7.1.

SOLUTION

For this example, we will assume that the last category is exactly seven persons. The mean of X is

$$\begin{aligned} E(X) = \mu = \sum_{\text{all } x} xP(x) &= 1P(1) + 2P(2) + 3P(3) + 4P(4) + 5P(5) + 6P(6) + 7P(7) \\ &= 1(.268) + 2(.333) + 3(.162) + 4(.140) + 5(.062) + 6(.023) + 7(.012) \\ &= 2.512 \end{aligned}$$

Notice that the random variable can assume integer values only, yet the mean is 2.513.

The variance of X is

$$\begin{aligned} V(X) = \sigma^2 = \sum_{\text{all } x} (x - \mu)^2 P(x) \\ &= (1 - 2.512)^2(.268) + (2 - 2.512)^2(.333) + (3 - 2.512)^2(.162) \\ &\quad + (4 - 2.512)^2(.140) + (5 - 2.512)^2(.062) + (6 - 2.512)^2(.023) \\ &\quad + (7 - 2.512)^2(.012) \\ &= 1.954 \end{aligned}$$

To demonstrate the shortcut method, we'll use it to recompute the variance:

$$\begin{aligned} \sum_{\text{all } x} x^2 P(x) &= 1^2(.268) + 2^2(.333) + 3^2(.162) + 4^2(.140) + 5^2(.062) \\ &\quad + 6^2(.023) + 7^2(.012) = 8.264 \end{aligned}$$

and

$$\mu = 2.512$$

Thus,

$$\sigma^2 = \sum_{\text{all } x} x^2 P(x) - \mu^2 = 8.264 - (2.512)^2 = 1.954$$

The standard deviation is

$$\sigma = \sqrt{\sigma^2} = \sqrt{1.954} = 1.398$$

These parameters tell us that the mean and standard deviation of the number of persons per household are 2.512 and 1.398, respectively.

Laws of Expected Value and Variance

As you will discover, we often create new variables that are functions of other random variables. The formulas given in the next two boxes allow us to quickly determine the expected value and variance of these new variables. In the notation used here, X is the random variable and c is a constant.

Laws of Expected Value

1. $E(c) = c$
2. $E(X + c) = E(X) + c$
3. $E(cX) = cE(X)$

Laws of Variance

1. $V(c) = 0$
2. $V(X + c) = V(X)$
3. $V(cX) = c^2V(X)$

EXAMPLE 7.4 Describing the Population of Monthly Profits

The monthly sales at a computer store have a mean of $25,000 and a standard deviation of $4,000. Profits are calculated by multiplying sales by 30% and subtracting fixed costs of $6,000. Find the mean and standard deviation of monthly profits.

SOLUTION

We can describe the relationship between profits and sales by the following equation:

$$\text{Profit} = .30(\text{Sales}) - 6{,}000$$

The expected or mean profit is

$$E(\text{Profit}) = E[.30(\text{Sales}) - 6{,}000]$$

Applying the second law of expected value, we produce

$$E(\text{Profit}) = E[.30(\text{Sales})] - 6{,}000$$

Applying law 3 yields

$$E(\text{Profit}) = .30E(\text{Sales}) - 6{,}000 = .30(25{,}000) - 6{,}000 = 1{,}500$$

Thus, the mean monthly profit is $1,500.

The variance is

$$V(\text{Profit}) = V[.30(\text{Sales}) - 6{,}000]$$

The second law of variance states that

$$V(\text{Profit}) = V[.30(\text{Sales})]$$

and law 3 yields

$$V(\text{Profit}) = (.30)^2V(\text{Sales}) = .09(4{,}000)^2 = 1{,}440{,}000$$

Thus, the standard deviation of monthly profits is

$$\sigma_{\text{Profit}} = \sqrt{1{,}440{,}000} = \$1{,}200$$

EXERCISES

7.1 The number of accidents that occur on a busy stretch of highway is a random variable.
a. What are the possible values of this random variable?
b. Are the values countable? Explain.
c. Is there a finite number of values? Explain.
d. Is the random variable discrete or continuous? Explain.

7.2 The distance a car travels on a tank of gasoline is a random variable.
a. What are the possible values of this random variable?
b. Are the values countable? Explain.
c. Is there a finite number of values? Explain.
d. Is the random variable discrete or continuous? Explain.

7.3 The amount of money students earn on their summer jobs is a random variable.
a. What are the possible values of this random variable?
b. Are the values countable? Explain.
c. Is there a finite number of values? Explain.
d. Is the random variable discrete or continuous? Explain.

7.4 The mark on a statistics exam that consists of 100 multiple-choice questions is a random variable.
a. What are the possible values of this random variable?
b. Are the values countable? Explain.
c. Is there a finite number of values? Explain.
d. Is the random variable discrete or continuous? Explain.

7.5 Determine whether each of the following is a valid probability distribution.

a.

x	0	1	2	3
P(x)	.1	.3	.4	.1

b.

x	5	−6	10	0
P(x)	.01	.01	.01	.97

c.

x	14	12	−7	13
P(x)	.25	.46	.04	.24

7.6 Let X be the random variable designating the number of spots that turn up when a balanced die is rolled. What is the probability distribution of X?

7.7 In a recent census the number of color televisions per household was recorded

Number of color televisions	0	1	2	3	4	5
Number of households (thousands)	1,218	32,379	37,961	19,387	7,714	2,842

a. Develop the probability distribution of X, the number of color televisions per household.
b. Determine the following probabilities.

$P(X \le 2)$
$P(X > 2)$
$P(X \ge 4)$

7.8 Using historical records, the personnel manager of a plant has determined the probability distribution of X, the number of employees absent per day. It is

x	0	1	2	3	4	5	6	7
P(x)	.005	.025	.310	.340	.220	.080	.019	.001

a. Find the following probabilities.

$P(2 \le X \le 5)$
$P(X > 5)$
$P(X < 4)$

b. Calculate the mean of the population.
c. Calculate the standard deviation of the population.

7.9 Second-year business students at many universities are required to take 10 one-semester courses. The number of courses that result in a grade of A is a discrete random variable. Suppose that each value of this random variable has the same probability. Determine the probability distribution.

7.10 The random variable X has the following probability distribution.

x	−3	2	6	8
P(x)	.2	.3	.4	.1

Find the following probabilities.
a. $P(X > 0)$
b. $P(X \ge 1)$
c. $P(X \ge 2)$
d. $P(2 \le X \le 5)$

7.11 An Internet pharmacy advertises that it will deliver the over-the-counter products that customers purchase in 3 to 6 days. The manager of the company wanted to be more precise in its advertising. Accordingly, she recorded the number of days it took to deliver to customers. From the data, the following probability distribution was developed.

Number of days	0	1	2	3	4	5	6	7	8
Probability	0	0	.01	.04	.28	.42	.21	.02	.02

a. What is the probability that a delivery will be made within the advertised 3- to 6-day period?

b. What is the probability that a delivery will be late?
c. What is the probability that a delivery will be early?

7.12 A gambler believes that a strategy called "doubling up" is an effective way to gamble. The method requires the gambler to double the stake after each loss. Thus, if the initial bet is $1, after losing he will double the bet until he wins. After a win, he resorts back to a $1 bet. The result is that he will net $1 for every win. The problem however, is that he will eventually run out of money or bump up against the table limit. Suppose that for a certain game the probability of winning is .5 and that losing six in a row will result in bankrupting the gambler. Find the probability of losing six times in a row.

7.13 The probability that a university graduate will be offered no jobs within a month of graduation is estimated to be 5%. The probability of receiving one, two, and three job offers has similarly been estimated to be 43%, 31%, and 21%, respectively. Determine the following probabilities.
a. A graduate is offered fewer than two jobs.
b. A graduate is offered more than one job.

7.14 Use a probability tree to compute the probability of the following events when flipping two fair coins.
a. Heads on the first coin and heads on the second coin
b. Heads on the first coin and tails on the second coin
c. Tails on the first coin and heads on the second coin
d. Tails on the first coin and tails on the second coin

7.15 Refer to Exercise 7.14. Find the following probabilities.
a. No heads
b. One head
c. Two heads
d. At least one head

7.16 Draw a probability tree to describe the flipping of three fair coins.

7.17 Refer to Exercise 7.16. Find the following probabilities.
a. Two heads
b. One head
c. At least one head
d. At least two heads

7.18 The random variable X has the following distribution.

x	−2	5	7	8
$P(x)$	.59	.15	.25	.01

a. Find the mean and variance for the probability distribution below.
b. Determine the probability distribution of Y where $Y = 5X$.
c. Use the probability distribution in part (b) to compute the mean and variance of Y.
d. Use the laws of expected value and variance to find the expected value and variance of Y from the parameters of X.

7.19 We are given the following probability distribution.

x	0	1	2	3
$P(x)$	.4	.3	.2	.1

a. Calculate the mean, variance, and standard deviation.
b. Suppose that $Y = 3X + 2$. For each value of X, determine the value of Y. What is the probability distribution of Y?
c. Calculate the mean, variance, and standard deviation from the probability distribution of Y.
d. Use the laws of expected value and variance to calculate the mean, variance, and standard deviation of Y from the mean, variance, and standard deviation of X. Compare your answers in parts (c) and (d). Are they the same (except for rounding)?

7.20 The number of pizzas delivered to university students each month is a random variable with the following probability distribution.

x	0	1	2	3
$P(X)$	.1	.3	.4	.2

a. Find the probability that a student has received delivery of two or more pizzas this month.
b. Determine the mean and variance of the number of pizzas delivered to students each month.

7.21 Refer to Exercise 7.20. If the pizzeria makes a profit of $3 per pizza, determine the mean and variance of the profits per student.

7.22 After watching a number of children playing games at a video arcade, a statistics practitioner estimated the following probability distribution of X, the number of games per visit.

x	1	2	3	4	5	6	7
$P(x)$	.05	.15	.15	.25	.20	.10	.10

a. What is the probability that a child will play more than four games?
b. What is the probability that a child will play at least two games?

7.23 Refer to Exercise 7.22. Determine the mean and variance of the number of games played.

7.24 Refer to Exercise 7.23. Suppose that each game costs the player 25 cents. Use the laws of expected value and variance to determine the expected value and variance of the amount of money the arcade takes in.

7.25 Refer to Exercise 7.22.

a. Determine the probability distribution of the amount of money the arcade takes in per child.
b. Use the probability distribution to calculate the mean and variance of the amount of money the arcade takes in.
c. Compare the answers in part (b) with those of Exercise 7.24. Are they identical (except for rounding errors)?

7.26 A survey of Amazon.com shoppers reveals the following probability distribution of the number of books purchased per hit.

x	0	1	2	3	4	5	6	7
$P(x)$	.35	.25	.20	.08	.06	.03	.02	.01

a. What is the probability that an Amazon.com visitor will buy four books?
b. What is the probability that an Amazon.com visitor will buy eight books?
c. What is the probability that an Amazon.com visitor will not buy any books?
d. What is the probability that an Amazon.com visitor will buy at least one book?

7.27 A university librarian produced the following probability distribution of the number of times a student walks into the library over the period of a semester.

x	0	5	10	15	20	25	30	40	50	75	100
$P(x)$	.22	.29	.12	.09	.08	.05	.04	.04	.03	.03	.01

Find the following probabilities.

a. $P(X \geq 20)$
b. $P(X = 60)$
c. $P(X > 50)$
d. $P(X > 100)$

7.28 After analyzing the frequency with which cross-country skiers participate in their sport, a sportswriter created the following probability distribution for X = number of times per year cross-country skiers ski.

x	0	1	2	3	4	5	6	7	8
$P(x)$	.04	.09	.19	.21	.16	.12	.08	.06	.05

Find the following.

a. $P(3)$
b. $P(X \geq 5)$
c. $P(5 \leq X \leq 7)$

7.29 The natural remedy echinacea is reputed to boost the immune system, which will reduce the number of flu and colds. A 6-month study was undertaken to determine whether the remedy works. From this study, the following probability distribution of the number of respiratory infections per year (X) for echinacea users was produced.

x	0	1	2	3	4
$P(x)$	.45	.31	.17	.06	.01

Find the following probabilities.

a. An echinacea user has more than one infection per year.
b. An echinacea user has no infections per year.
c. An echinacea user has between one and three (inclusive) infections per year.

7.30 A shopping mall estimates the probability distribution of the number of stores mall customers actually enter, as shown in the table.

x	0	1	2	3	4	5	6
$P(x)$	.04	.19	.22	.28	.12	.09	.06

Find the mean and standard deviation of the number of stores entered.

7.31 Refer to Exercise 7.30. Suppose that, on average, customers spend 10 minutes in each store they enter. Find the mean and standard deviation of the total amount of time customers spend in stores.

7.32 When parking a car in a downtown parking lot, drivers pay according to the number of hours or parts thereof. The probability distribution of the number of hours cars are parked has been estimated as follows.

x	1	2	3	4	5	6	7	8
$P(x)$	.24	.18	.13	.10	.07	.04	.04	.20

Find the mean and standard deviation of the number of hours cars are parked in the lot.

7.33 Refer to Exercise 7.32. The cost of parking is \$2.50 per hour. Calculate the mean and standard deviation of the amount of revenue each car generates.

7.34 You have been given the choice of receiving \$500 in cash or receiving a gold coin that has a face value of \$100. However, the actual value of the gold coin depends on its gold content. You are told that the coin has a 40% probability of being worth \$400, a 30% probability of being worth \$900, and a 30% probability of being worth its face value. Basing your decision on expected value, should you choose the coin?

7.35 The manager of a bookstore recorded the number of customers who arrive at a checkout counter every 5 minutes from which the following distribution was calculated. Calculate the mean and standard deviation of the random variable.

x	0	1	2	3	4
$P(x)$	.10	.20	.25	.25	.20

7.36 The owner of a small firm has just purchased a personal computer, which she expects will serve her for the next 2 years. The owner has been told that she "must" buy a surge suppressor to provide protection for her new hardware against possible surges or variations in the electrical current, which have the capacity to damage the computer. The amount of damage to the computer depends on the strength of the surge. It has been estimated that there is a 1% chance of incurring \$400 damage, a 2% chance of incurring \$200 damage, and 10% chance of \$100 damage. An inexpensive suppressor, which would provide protection for only one surge can be purchased. How much should the owner be willing to pay if she makes decisions on the basis of expected value?

7.37 It cost one dollar to buy a lottery ticket, which has five prizes. The prizes and the probability that a player wins the prize are listed here. Calculate the expected value of the payoff.

Prize (\$)	1 million	200,000	50,000
Probability	1/10 million	1/1 million	1/500,000

Prize (\$)	10,000	1,000
Probability	1/50,000	1/10,000

7.38 After an analysis of incoming faxes the manager of an accounting firm determined the probability distribution of the number of pages per facsimile as follows:

x	1	2	3	4	5	6	7
$P(x)$	.05	.12	.20	.30	.15	.10	.08

Compute the mean and variance of the number of pages per fax.

7.39 Refer to Exercise 7.38. Further analysis by the manager revealed that the cost of processing each page of a fax is \$.25. Determine the mean and variance of the cost per fax.

7.40 To examine the effectiveness of its four annual advertising promotions, a mail-order company has sent a questionnaire to each of its customers, asking how many of the previous year's promotions prompted orders that would not otherwise have been made. The table lists the probabilities that were derived from the questionnaire, where X is the random variable representing the number of promotions that prompted orders. If we assume that overall customer behavior next year will be the same as last year, what is the expected number of promotions that each customer will take advantage of next year by ordering goods that otherwise would not be purchased?

x	0	1	2	3	4
$P(x)$	.10	.25	.40	.20	.05

7.41 Refer to Exercise 7.40. A previous analysis of historical records found that the mean value of orders for promotional goods is \$20, with the company earning a gross profit of 20% on each order. Calculate the expected value of the profit contribution next year.

7.42 Refer to Exercises 7.40 and 7.41. The fixed cost of conducting the four promotions is estimated to be \$15,000, with a variable cost of \$3.00 per customer for mailing and handling costs. How large a customer base does the company need to cover the cost of promotions?

7.2 Bivariate Distributions

Thus far, we have dealt with the distribution of a *single* variable. However, there are circumstances where we need to know about the relationship between two variables. Recall that we have addressed this problem statistically in Chapter 3 by drawing the scatter diagram and in Chapter 4 by calculating the covariance and the coefficient of correlation. In this section, we present the **bivariate distribution**, which provides probabilities of combinations of two variables. Incidentally, when we need to distinguish between the bivariate distributions and the distributions of one variable, we'll refer to the latter as *univariate* distributions.

The joint probability that two variables will assume the values x and y is denoted $P(x, y)$. A bivariate (or joint) probability distribution of X and Y is a table or formula that lists the joint probabilities for all pairs of values of x and y. As was the case with univariate distributions, the joint probability must satisfy two requirements.

Requirements for a Discrete Bivariate Distribution

1. $0 \leq P(x, y) \leq 1$ for all pairs of values (x, y)
2. $\sum_{\text{all } x} \sum_{\text{all } y} P(x, y) = 1$

EXAMPLE 7.5

Bivariate Distribution of the Number of House Sales

Xavier and Yvette are real estate agents. Let X denote the number of houses that Xavier will sell in a month and let Y denote the number of houses Yvette will sell in a month. An analysis of their past monthly performances has the following joint probabilities.

Bivariate Probability Distribution

		X		
		0	1	2
	0	.12	.42	.06
Y	1	.21	.06	.03
	2	.07	.02	.01

We interpret these joint probabilities in the same way we did in Chapter 6. For example, the probability that Xavier sells 0 houses and Yvette sells 1 house in the month is $P(0, 1) = .21$.

Marginal Probabilities

As we did in Chapter 6, we can calculate the marginal probabilities by summing across rows or down columns.

Marginal Probability Distribution of X in Example 7.5

$$P(X = 0) = P(0, 0) + P(0, 1) + P(0, 2) = .12 + .21 + .07 = .4$$
$$P(X = 1) = P(1, 0) + P(1, 1) + P(1, 2) = .42 + .06 + .02 = .5$$
$$P(X = 2) = P(2, 0) + P(2, 1) + P(2, 2) = .06 + .03 + .01 = .1$$

The marginal probability distribution of X is

x	$P(x)$
0	.4
1	.5
2	.1

Marginal Probability Distribution of Y in Example 7.5

$$P(Y = 0) = P(0, 0) + P(1, 0) + P(2, 0) = .12 + .42 + .06 = .6$$
$$P(Y = 1) = P(0, 1) + P(1, 1) + P(2, 1) = .21 + .06 + .03 = .3$$
$$P(Y = 2) = P(0, 2) + P(1, 2) + P(2, 2) = .07 + .02 + .01 = .1$$

The marginal probability distribution of Y is

y	$P(y)$
0	.6
1	.3
2	.1

Notice that both marginal probability distributions meet the requirements; the probabilities are between 0 and 1, and they add to 1.

Describing the Bivariate Distribution

As we did with the univariate distribution, we often describe the bivariate distribution by computing the mean, variance, and standard deviation of each variable. We do so by utilizing the marginal probabilities.

Expected Value, Variance, and Standard Deviation of X in Example 7.5

$$E(X) = \mu_X = \sum xP(x) = 0(.4) + 1(.5) + 2(.1) = .7$$

$$V(X) = \sigma_X^2 = \sum (x - \mu_X)^2 P(x) = (0 - .7)^2(.4) + (1 - .7)^2(.5) + (2 - .7)^2(.1) = .41$$

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{.41} = .64$$

Expected Value, Variance, and Standard Deviation of Y in Example 7.5

$$E(Y) = \mu_Y = \sum yP(y) = 0(.6) + 1(.3) + 2(.1) = .5$$

$$V(Y) = \sigma_Y^2 = \sum (y - \mu_Y)^2 P(y) = (0 - .5)^2(.6) + (1 - .5)^2(.3) + (2 - .5)^2(.1) = .45$$

$$\sigma_Y = \sqrt{\sigma_Y^2} = \sqrt{.45} = .67$$

There are two more parameters we can and need to compute. Both deal with the relationship between the two variables. They are the covariance and the coefficient of correlation. Recall that both were introduced in Chapter 4, where the formulas were based on the assumption that we knew each of the N observations of the population. In this chapter, we compute parameters like the covariance and the coefficient of correlation from the bivariate distribution.

Covariance

The covariance of two discrete variables is defined as

$$\text{COV}(X,Y) = \sigma_{xy} = \sum_{\text{all } x} \sum_{\text{all } y} (x - \mu_X)(y - \mu_Y)P(x, y)$$

Notice that we multiply the deviations from the mean for both X and Y and then multiply by the joint probability.

The calculations are simplified by the following shortcut method.

Shortcut Calculation for Covariance

$$\text{COV}(X,Y) = \sigma_{xy} = \sum_{\text{all } x} \sum_{\text{all } y} xyP(x, y) - \mu_X\mu_Y$$

The coefficient of correlation is calculated in the same way as in Chapter 4.

Coefficient of Correlation

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

EXAMPLE 7.6 Describing the Bivariate Distribution

Compute the covariance and the coefficient of correlation between the numbers of houses sold by the two agents in Example 7.5.

SOLUTION

We start by computing the covariance.

$$\begin{aligned}\sigma_{xy} &= \sum_{\text{all } x} \sum_{\text{all } y} (x - \mu_X)(y - \mu_Y)P(x, y)\\ &= (0 - .7)(0 - .5)(.12) + (1 - .7)(0 - .5)(.42) + (2 - .7)(0 - .5)(.06)\\ &\quad + (0 - .7)(1 - .5)(.21) + (1 - .7)(1 - .5)(.06) + (2 - .7)(1 - .5)(03)\\ &\quad + (0 - .7)(2 - .5)(.07) + (1 - .7)(2 - .5)(.02) + (2 - .7)(2 - .5)(.01)\\ &= -.15\end{aligned}$$

As we did with the shortcut method for the variance, we'll recalculate the covariance using its shortcut method.

$$\begin{aligned}\sum_{\text{all } x} \sum_{\text{all } y} xyP(x, y) &= (0)(0)(.12) + (1)(0)(.42) + (2)(0)(.06)\\ &\quad + (0)(1)(.21) + (1)(1)(.06) + (2)(1)(.03)\\ &\quad + (0)(2)(.07) + (1)(2)(.02) + (2)(2)(.01)\\ &= .2\end{aligned}$$

Using the expected values computed above we find

$$\sigma_{xy} = \sum_{\text{all } x} \sum_{\text{all } y} xyP(x, y) - \mu_X\mu_Y = .2 - (.7)(.5) = -.15$$

We also computed the standard deviations above. Thus, the coefficient of correlation is

$$\rho = \frac{\sigma_{xy}}{\sigma_X \sigma_Y} = \frac{-.15}{(.64)(.67)} = -.35$$

There is a weak negative relationship between the two variables: the number of houses Xavier will sell in a month (X) and the number of houses Yvette will sell in a month (Y).

Sum of Two Variables

The bivariate distribution allows us to develop the probability distribution of any combination of the two variables. Of particular interest to us is the sum of two variables. The analysis of this type of distribution leads to an important statistical application in finance, which we present in the next section.

To see how to develop the probability distribution of the sum of two variables from their bivariate distribution, return to Example 7.5. The sum of the two variables X and Y is the total number of houses sold per month. The possible values of $X + Y$ are 0, 1, 2, 3, and 4. The probability that $X + Y = 2$, for example, is obtained by summing the joint probabilities of all pairs of values of X and Y that sum to 2:

$$P(X + Y = 2) = P(0,2) + P(1,1) + P(2,0) = .07 + .06 + .06 = .19$$

We calculate the probabilities of the other values of $X + Y$ similarly, producing the following table.

Probability Distribution of $X + Y$ in Example 7.5

$x + y$	0	1	2	3	4
$P(x + y)$	.12	.63	.19	.05	.01

We can compute the expected value, variance, and standard deviation of $X + Y$ in the usual way.

$$E(X + Y) = 0(.12) + 1(.63) + 2(.19) + 3(.05) + 4(.01) = 1.2$$

$$\begin{aligned} V(X + Y) = \sigma^2_{X+Y} &= (0 - 1.2)^2(.12) + (1 - 1.2)^2(.63) + (2 - 1.2)^2(.19) \\ &\quad + (3 - 1.2)^2(.05) + (4 - 1.2)^2(.01) \\ &= .56 \end{aligned}$$

$$\sigma_{X+Y} = \sqrt{.56} = .75$$

We can derive a number of laws that enable us to compute the expected value and variance of the sum of two variables.

Laws of Expected Value and Variance of the Sum of Two Variables

1. $E(X + Y) = E(X) + E(Y)$
2. $V(X + Y) = V(X) + V(Y) + 2\text{COV}(X, Y)$

If X and Y are independent, $\text{COV}(X, Y) = 0$ and thus $V(X + Y) = V(X) + V(Y)$

EXAMPLE 7.7 Describing the Population of the Total Number of House Sales

Use the rules of expected value and variance of the sum of two variables to calculate the mean and variance of the total number of houses sold per month in Example 7.5.

SOLUTION

Using law 1 we compute the expected value of $X + Y$:

$$E(X + Y) = E(X) + E(Y) = .7 + .5 = 1.2$$

which is the same value we produced directly from the probability distribution of $X + Y$.

We apply law 3 to determine the variance:

$$V(X + Y) = V(X) + V(Y) + 2\text{COV}(X,Y) = .41 + .45 + 2(-.15) = .56$$

This is the same value we obtained from the probability distribution of $X + Y$.

We will encounter several applications where we need the laws of expected value and variance for the sum of two variables. Additionally, we will demonstrate an important application in operations management where we need the formulas for the expected value and variance of the sum of more than two variables. See Exercises 7.57–7.60.

EXERCISES

7.43 The following table lists the bivariate distribution of X and Y.

	x	
y	1	2
1	.5	.1
2	.1	.3

a. Find the marginal probability distribution of X.
b. Find the marginal probability distribution of Y.
c. Compute the mean and variance of X.
d. Compute the mean and variance of Y.

7.44 Refer to Exercise 7.43. Compute the covariance and the coefficient of correlation.

7.45 Refer to Exercise 7.43. Use the laws of expected value and variance of the sum of two variables to compute the mean and variance of $X + Y$.

7.46 Refer to Exercise 7.43.
a. Determine the distribution of $X + Y$.
b. Determine the mean and variance of $X + Y$.
c. Does your answer to part (b) equal the answer to Exercise 7.45?

7.47 The bivariate distribution of X and Y is described here.

	x	
y	1	2
1	.28	.42
2	.12	.18

a. Find the marginal probability distribution of X.
b. Find the marginal probability distribution of Y.
c. Compute the mean and variance of X.
d. Compute the mean and variance of Y.

7.48 Refer to Exercise 7.47. Compute the covariance and the coefficient of correlation.

7.49 Refer to Exercise 7.47. Use the laws of expected value and variance of the sum of two variables to compute the mean and variance of $X + Y$.

7.50 Refer to Exercise 7.47.
a. Determine the distribution of $X + Y$.
b. Determine the mean and variance of $X + Y$.
c. Does your answer to part (b) equal the answer to Exercise 7.49?

7.51 The joint probability distribution of X and Y is shown in the following table.

	x		
y	**1**	**2**	**3**
1	.42	.12	.06
2	.28	.08	.04

a. Determine the marginal distributions of X and Y.
b. Compute the covariance and coefficient of correlation between X and Y.
c. Develop the probability distribution of $X + Y$.

7.52 The following distributions of X and of Y have been developed. If X and Y are independent, determine the joint probability distribution of X and Y.

x	0	1	2
***p*(*x*)**	.6	.3	.1

y	1	2
***p*(*y*)**	.7	.3

7.53 The distributions of X and of Y are described here. If X and Y are independent, determine the joint probability distribution of X and Y.

x	0	1
***P*(*x*)**	.2	.8

y	1	2	3
***P*(*y*)**	.2	.4	.4

7.54 After analyzing several months of sales data, the owner of an appliance store produced the following joint probability distribution of the number of refrigerators and stoves sold daily.

	Refrigerators		
Stoves	**0**	**1**	**2**
0	.08	.14	.12
1	.09	.17	.13
2	.05	.18	.04

a. Find the marginal probability distribution of the number of refrigerators sold daily.
b. Find the marginal probability distribution of the number of stoves sold daily.
c. Compute the mean and variance of the number of refrigerators sold daily.
d. Compute the mean and variance of the number of stoves sold daily.
e. Compute the covariance and the coefficient of correlation.

7.55 Canadians who visit the United States often buy liquor and cigarettes, which are much cheaper in the United States. However, there are limitations. Canadians visiting in the United States for more than 2 days are allowed to bring into Canada one bottle of liquor and one carton of cigarettes. A Canada customs agent has produced the following joint probability distribution of the number of bottles of liquor and the number of cartons of cigarettes imported by Canadians who have visited the United States for 2 or more days.

	Bottles of Liquor	
Cartons of Cigarettes	**0**	**1**
0	.63	.18
1	.09	.10

a. Find the marginal probability distribution of the number of bottles imported.
b. Find the marginal probability distribution of the number of cigarette cartons imported.
c. Compute the mean and variance of the number of bottles imported.
d. Compute the mean and variance of the number of cigarette cartons imported.
e. Compute the covariance and the coefficient of correlation.

7.56 Refer to Exercise 7.54. Find the following conditional probabilities.
a. P(1 refrigerator | 0 stoves)
b. P(0 stoves | 1 refrigerator)
c. P(2 refrigerators | 2 stoves)

APPLICATIONS in OPERATIONS MANAGEMENT

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PERT/CPM

The Project Evaluation and Review Technique (**PERT**) and the Critical Path Method (**CPM**) are related management-science techniques that help operations managers control the activities and the amount of time it takes to complete a project. Both techniques are based on the order in which the activities must be performed. For example, in building a house the excavation of the foundation must precede the pouring of the foundation, which in turn precedes the framing. A **path**

(Continued)

is defined as a sequence of related activities that leads from the starting point to the completion of a project. In most projects, there are several paths with differing amounts of time needed for their completion. The longest path is called the **critical path** because any delay in the activities along this path will result in a delay in the completion of the project. In some versions of PERT/CPM, the activity completion times are fixed and the chief task of the operations manager is to determine the critical path. In other versions, each activity's completion time is considered to be a random variable, where the mean and variance can be estimated. By extending the laws of expected value and variance for the sum of two variables to more than two variables, we produce the following, where $X_1, X_2, \ldots, X_k$ are the times for the completion of activities $1, 2, \ldots, k$, respectively. These times are independent random variables.

Laws of Expected Value and Variance for the Sum of More than Two Independent Variables

1. $E(X_1 + X_2 + \cdots + X_k) = E(X_1) + E(X_2) + \cdots + E(X_k)$
2. $V(X_1 + X_2 + \cdots + X_k) = V(X_1) + V(X_2) + \cdots + V(X_k)$

Using these laws, we can then produce the expected value and variance for the complete project. Exercises 7.57–7.60 address this problem.

7.57 There are four activities along the critical path for a project. The expected values and variances of the completion times of the activities are listed here. Determine the expected value and variance of the completion time of the project.

Activity	Expected Completion Time (Days)	Variance
1	18	8
2	12	5
3	27	6
4	8	2

7.58 The operations manager of a large plant wishes to overhaul a machine. After conducting a PERT/CPM analysis he has developed the following critical path.

1. Disassemble machine
2. Determine parts that need replacing
3. Find needed parts in inventory
4. Reassemble machine
5. Test machine

He has estimated the mean (in minutes) and variances of the completion times as follows.

Activity	Mean	Variance
1	35	8
2	20	5
3	20	4
4	50	12
5	20	2

Determine the mean and variance of the completion time of the project.

7.59 In preparing to launch a new product, a marketing manager has determined the critical path for her department. The activities and the mean and variance of the completion time for each activity along the critical path are shown in the accompanying table. Determine the mean and variance of the completion time of the project.

Activity	Expected Completion Time (Days)	Variance
Develop survey questionnaire	8	2
Pretest the questionnaire	14	5
Revise the questionnaire	5	1
Hire survey company	3	1
Conduct survey	30	8
Analyze data	30	10
Prepare report	10	3

7.60 A professor of business statistics is about to begin work on a new research project. Because his time is quite limited, he has developed a PERT/CPM critical path, which consists of the following activities:

1. Conduct a search for relevant research articles.
2. Write a proposal for a research grant.
3. Perform the analysis.
4. Write the article and send to journal.
5. Wait for reviews.
6. Revise on the basis of the reviews and resubmit.

The mean (in days) and variance of the completion times are as follows

Activity	Mean	Variance
1	10	9
2	3	0
3	30	100
4	5	1
5	100	400
6	20	64

Compute the mean and variance of the completion time of the entire project.

7.3 (OPTIONAL) APPLICATIONS IN FINANCE: PORTFOLIO DIVERSIFICATION AND ASSET ALLOCATION

In this section we introduce an important application in finance that is based on the previous section.

In Chapter 3 (page 51), we described how the variance or standard deviation can be used to measure the risk associated with an investment. Most investors tend to be risk averse, which means that they prefer to have lower risk associated with their investments. One of the ways in which financial analysts lower the risk that is associated with the stock market is through **diversification**. This strategy was first mathematically developed by Harry Markowitz in 1952. His model paved the way for the development of modern portfolio theory (MPT), which is the concept underlying mutual funds (see page 181).

To illustrate the basics of portfolio diversification, consider an investor who forms a portfolio, consisting of only two stocks, by investing \$4,000 in one stock and \$6,000 in a second stock. Suppose that the results after 1 year are as listed here. (We've previously defined return on investment. See Applications in Finance: Return on Investment on page 52.)

One-Year Results

Stock	Initial Investment (\$)	Value of Investment After One Year (\$)	Rate of Return on Investment
1	4,000	5,000	$R_1 = .25$ (25%)
2	6,000	5,400	$R_2 = -.10$ (-10%)
Total	10,000	10,400	$R_p = .04$ (4%)

Another way of calculating the portfolio return R_p is to compute the weighted average of the individual stock returns R_1 and R_2, where the weights w_1 and w_2 are the proportions of the initial \$10,000 invested in stocks 1 and 2, respectively. In this illustration, $w_1 = .4$ and $w_2 = .6$. (Note that w_1 and w_2 must always sum to 1 because the two stocks constitute the entire portfolio.) The weighted average of the two returns is

$$\begin{aligned} R_p &= w_1R_1 + w_2R_2 \\ &= (.4)(.25) + (.6)(-.10) = .04 \end{aligned}$$

This is how portfolio returns are calculated. However, when the initial investments are made, the investor does not know what the returns will be. In fact, the returns are random variables. We are interested in determining the expected value and variance of the portfolio. The formulas in the box were derived from the laws of expected value and variance introduced in the two previous sections.

Mean and Variance of a Portfolio of Two Stocks

$$\begin{aligned} E(R_p) &= w_1E(R_1) + w_2E(R_2) \\ V(R_p) &= w_1^2V(R_1) + w_2^2V(R_2) + 2w_1w_2\text{COV}(R_1,R_2) \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2 \end{aligned}$$

where w_1 and w_2 are the proportions or weights of investments 1 and 2, $E(R_1)$ and $E(R_2)$ are their expected values, σ_1 and σ_2 are their standard deviations, $\text{COV}(R_1,R_2)$ is the covariance, and ρ is the coefficient of correlation.

(Recall that $\rho = \dfrac{\text{COV}(R_1,R_2)}{\sigma_1\sigma_2}$, which means that $\text{COV}(R_1,R_2) = \rho\sigma_1\sigma_2$.)

EXAMPLE 7.8 Describing the Population of the Returns on a Portfolio

An investor has decided to form a portfolio by putting 25% of his money into McDonald's stock and 75% into Cisco Systems stock. The investor assumes that the expected returns will be 8% and 15%, respectively, and that the standard deviations will be 12% and 22%, respectively.

a. Find the expected return on the portfolio.
b. Compute the standard deviation of the returns on the portfolio assuming that
   i. the two stocks' returns are perfectly positively correlated.
   ii. the coefficient of correlation is .5.
   iii. the two stocks' returns are uncorrelated.

SOLUTION

a. The expected values of the two stocks are

$$E(R_1) = .08 \quad \text{and} \quad E(R_2) = .15$$

The weights are $w_1 = .25$ and $w_2 = .75$.
Thus,

$$E(R_p) = w_1E(R_1) + w_2E(R_2) = .25(.08) + .75(.15) = .1325$$

b. The standard deviations are

$$\sigma_1 = .12 \quad \text{and} \quad \sigma_2 = .22$$

Thus,

$$\begin{aligned} V(R_p) &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2 \\ &= (.25^2)(.12^2) + (.75^2)(.22^2) + 2(.25)(.75)\,\rho\,(.12)(.22) \\ &= .0281 + .0099\rho \end{aligned}$$

When $\rho = 1$

$$V(R_p) = .0281 + .0099(1) = .0380$$

$$\text{Standard deviation} = \sqrt{V(R_p)} = \sqrt{.0380} = .1949$$

When $\rho = .5$

$$V(R_p) = .0281 + .0099(.5) = .0331$$

$$\text{Standard deviation} = \sqrt{V(R_p)} = \sqrt{.0331} = .1819$$

When $\rho = 0$

$$V(R_p) = .0281 + .0099(0) = .0281$$

$$\text{Standard deviation} = \sqrt{V(R_p)} = \sqrt{.0281} = .1676$$

Notice that the variance and standard deviation of the portfolio returns decrease as the coefficient of correlation decreases.

Portfolio Diversification in Practice

The formulas introduced in this section require that we know the expected values, variances, and covariance (or coefficient of correlation) of the investments we're interested in. The question arises, how do we determine these parameters? (Incidentally, this question is rarely addressed in finance textbooks!) The most common procedure is to estimate the parameters from historical data, using sample statistics.

Portfolios with More Than Two Stocks

We can extend the formulas that describe the mean and variance of the returns of a portfolio of two stocks to a portfolio of any number of stocks.

Mean and Variance of a Portfolio of *k* Stocks

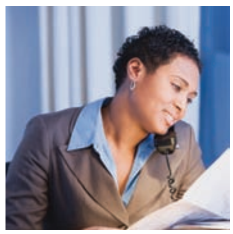
$$E(R_p) = \sum_{i=1}^{k} w_i E(R_i)$$

$$V(R_p) = \sum_{i=1}^{k} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{k}\sum_{j=i+1}^{k} w_i w_j \text{COV}(R_i, R_j)$$

Where R_i is the return of the ith stock, w_i is the proportion of the portfolio invested in stock i, and k is the number of stocks in the portfolio.

When k is greater than 2, the calculations can be tedious and time consuming. For example, when $k = 3$, we need to know the values of the three weights, three expected values, three variances, and three covariances. When $k = 4$, there are four expected values, four variances, and six covariances. [The number of covariances required in general is $k(k - 1)/2$.] To assist you, we have created an Excel worksheet to perform the computations when $k = 2$, 3, or 4. To demonstrate, we'll return to the problem described in this chapter's introduction.

Investing to Maximize Returns and Minimize Risk: Solution



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Because of the large number of calculations, we will solve this problem using only Excel. From the file, we compute the means of each stock's returns.

Excel Means

	A	B	C	D
1	0.00881	0.00562	0.01253	0.02834

Next we compute the variance–covariance matrix. (The commands are the same as those described in Chapter 4: Simply include all the columns of the returns of the investments you wish to include in the portfolio.)

Excel Variance-Covariance Matrix

	A	B	C	D	E
1		Coca Cola	Disney	Barrick	Amazon
2	Coca Cola	0.00235			
3	Disney	0.00141	0.00434		
4	Barrick	0.00184	–0.00058	0.01174	
5	Amazon	0.00167	0.00182	–0.00170	0.02020

Notice that the variances of the returns are listed on the diagonal. Thus, for example, the variance of the 60 monthly returns of Barrick Gold is .01174. The covariances appear below the diagonal. The covariance between the returns of Coca Cola and Disney is .00141.

The means and the variance–covariance matrix are copied to the spreadsheet using the commands described here. The weights are typed producing the accompanying output.

Excel Worksheet: Portfolio Diversification—Plan 1

	A	B	C	D	E	F
1	**Portfolio of 4 Stocks**					
2			**Coca Cola**	**Disney**	**Barrick**	**Amazon**
3	**Variance-Covariance Matrix**	**Coca Cola**	0.00235			
4		**Disney**	0.00141	0.00434		
5		**Barrick**	0.00184	–0.00058	0.01174	
6		**Amazon**	0.00167	0.00182	–0.00170	0.02020
7						
8	**Expected Returns**		0.00881	0.00562	0.01253	0.02834
9						
10	**Weights**		0.25000	0.25000	0.25000	0.25000
11						
12	**Portfolio Return**					
13	**Expected Value**	**0.01382**				
14	**Variance**	**0.00297**				
15	**Standard Deviation**	**0.05452**				

The expected return on the portfolio is .01382, and the variance is .00297.

INSTRUCTIONS

1. Open the file containing the returns. In this example, open file Ch7:\ Xm07-00
2. Compute the means of the columns containing the returns of the stocks in the portfolio.
3. Using the commands described in Chapter 4 (page 137) compute the variance–covariance matrix.
4. Open the **Portfolio Diversification** workbook. Use the tab to select the **4 Stocks** worksheet. Do not change any cells that appear in bold print. Do not save any worksheets.
5. Copy the means into cells C8 to F8. (Use **Copy**, **Paste Special** with **Values and number formats**.)
6. Copy the variance–covariance matrix (including row and column labels) into columns B, C, D, E, and F.
7. Type the weights into cells C10 to F10

The mean, variance, and standard deviation of the portfolio will be printed. Use similar commands for 2 stock and 3 stock portfolios.

The results for plan 2 are

	A	B
12	**Portfolio Return**	
13	**Expected Value**	**0.01710**
14	**Variance**	**0.00460**
15	**Standard Deviation**	**0.06783**

Plan 3

	A	B
12	**Portfolio Return**	
13	**Expected Value**	**0.01028**
14	**Variance**	**0.00256**
15	**Standard Deviation**	**0.05059**

Plan 3 has the smallest expected value and the smallest variance. Plan 2 has the largest expected value and the largest variance. Plan 1's expected value and variance are in the middle. If the investor is like most investors, she would select Plan 3 because of its lower risk. Other, more daring investors may choose plan 2 to take advantage of its higher expected value.

In this example, we showed how to compute the expected return, variance, and standard deviation from a sample of returns on the investments for any combination of weights. (We illustrated the process with three sets of weights.) It is possible to determine the "optimal" weights that minimize risk for a given expected value or maximize expected return for a given standard deviation. This is an extremely important function of financial analysts and investment advisors. Solutions can be determined using a management-science technique called *linear programming*, a subject taught by most schools of business and faculties of management.

EXERCISES

7.61 Describe what happens to the expected value and standard deviation of the portfolio returns when the coefficient of correlation decreases.

7.62 A portfolio is composed of two stocks. The proportion of each stock, their expected values, and standard deviations are listed next.

Stock	1	2
Proportion of portfolio	.30	.70
Mean	.12	.25
Standard deviation	.02	.15

For each of the following coefficients of correlation calculate the expected value and standard deviation of the portfolio.

a. $\rho = .5$
b. $\rho = .2$
c. $\rho = 0$

7.63 An investor is given the following information about the returns on two stocks.

Stock	1	2
Mean	.09	.13
Standard deviation	.15	.21

a. If he is most interested in maximizing his returns, which stock should he choose?
b. If he is most interested in minimizing his risk, which stock should he choose?

7.64 Refer to Exercise 7.63. Compute the expected value and standard deviation of the portfolio composed of 60% stock 1 and 40% stock 2. The coefficient of correlation is .4.

7.65 Refer to Exercise 7.63. Compute the expected value and standard deviation of the portfolio composed of 30% stock 1 and 70% stock 2.

The following exercises require the use of a computer.

Xr07-66 *The monthly returns for the following stocks on the New York Stock Exchange were recorded.*

AT&T, Aetna, Cigna, Coca-Cola, Disney, Ford, and McDonald's

The next seven exercises are based on this set of data.

7.66 a. Calculate the mean and variance of the monthly return for each stock.
b. Determine the variance–covariance matrix.

7.67 Select the two stocks with the largest means and construct a portfolio consisting of equal amounts of both. Determine the expected value and standard deviation of the portfolio.

7.68 Select the two stocks with the smallest variances and construct a portfolio consisting of equal amounts of both. Determine the expected value and standard deviation of the portfolio.

7.69 Describe the results of Exercises 7.66 to 7.68.

7.70 An investor wants to develop a portfolio composed of shares of AT&T, Coca-Cola, Ford, and Disney. Calculate the expected value and standard deviation of the returns for a portfolio with equal proportions of all three stocks.

7.71 Suppose you want a portfolio composed of AT&T, Cigna, Disney, and Ford. Find the expected value and standard deviation of the returns for the following portfolio.

AT&T	30%
Cigna	20%
Disney	40%
Ford	10%

7.72 Repeat Exercise 7.71 using the following proportions. Compare your results with those of Exercise 7.71.

AT&T	30%
Cigna	10%
Disney	40%
Ford	20%

The following seven exercises are directed at Canadian students.

Xr07-73 *The monthly returns for the following stocks on the Toronto Stock Exchange were recorded:*

Barrick Gold, Bell Canada Enterprises (BCE),Bank of Montreal (BMO), Enbridge, Fortis, Methanex, Research in Motion, Telus, and Trans Canada Pipeline

The next seven exercises are based on this set of data.

7.73 a. Calculate the mean and variance of the monthly return for each stock.
b. Determine the correlation matrix.

7.74 Select the two stocks with the largest means and construct a portfolio consisting of equal amounts of both. Determine the expected value and standard deviation of the portfolio.

7.75 Select the two stocks with the smallest variances and construct a portfolio consisting of equal amounts of both. Determine the expected value and standard deviation of the portfolio.

7.76 Describe the results of Exercises 7.73 to 7.75.

7.77 An investor wants to develop a portfolio composed of shares of Bank of Montreal, Enbridge, and Fortis. Calculate the expected value and standard deviation of the returns for a portfolio with the following proportions.

Bank of Montreal	20%
Enbridge	30%
Fortis	50%

7.78 Suppose you want a portfolio composed of Barrick Gold, Bell Canada Enterprises, Telus, and TransCanada Ltd. Find the expected value and standard deviation of the returns for the following portfolio.

Barrick Gold	50%
Bell Canada Enterprises	25%
Telus	15%
TransCanada	10%

7.79 Repeat Exercise 7.78 using the following proportions. Compare your results with those of Exercise 7.78.

Barrick Gold	20%
Bell Canada Enterprises	40%
Telus	20%
TransCanada	20%

Xr07-80 *The monthly returns for the following stocks on the NASDAQ Stock Exchange were recorded:*

Amazon, Amgen, Apple, Cisco Systems, Google, Intel, Microsoft, Oracle, and Research in Motion

The next four exercises are based on this set of data.

7.80 a. Calculate the mean and variance of the monthly return for each stock.
b. Determine which four stocks you would include in your portfolio if you wanted a large expected value.
c. Determine which four stocks you would include in your portfolio if you wanted a small variance.

7.81 Suppose you want a portfolio composed of Cisco Systems, Intel, Microsoft, and Research in Motion. Find the expected value and standard deviation of the returns for the following portfolio.

Cisco Systems	30%
Intel	15%
Microsoft	25%
Research in Motion	30%

7.82 An investor wants to acquire a portfolio composed of Cisco Systems, Intel, Microsoft, and Research in Motion. Moreover, he wants the expected value to be at least 1%. Try several sets of proportions (remember, they must add to 1.0) to see if you can find the portfolio with the smallest variance.

7.83 Refer to Exercise 7.81.
a. Compute the expected value and variance of the portfolio described next.

Cisco Systems	26.59%
Intel	2.49%
Microsoft	54.74%
Research in Motion	16.19%

b. Can you do better? In other words, can you find a portfolio whose expected value is greater than or equal to 1% and whose variance is less than the one you calculated in part (a)? (*Hint:* Don't spend too much time at this. You won't be able to do better.)
c. If you want to learn how we produced the portfolio above, take a course that teaches linear and nonlinear programming.

7.4 Binomial Distribution

Now that we've introduced probability distributions in general, we need to introduce several specific probability distributions. In this section, we present the *binomial distribution*.

The binomial distribution is the result of a *binomial experiment*, which has the following properties.

Binomial Experiment

1. The **binomial experiment** consists of a fixed number of trials. We represent the number of trials by n.
2. Each trial has two possible outcomes. We label one outcome a *success*, and the other a *failure*.
3. The probability of success is p. The probability of failure is $1 - p$.
4. The trials are independent, which means that the outcome of one trial does not affect the outcomes of any other trials.

If properties 2, 3, and 4 are satisfied, we say that each trial is a **Bernoulli process**. Adding property 1 yields the binomial experiment. The random variable of a binomial experiment is defined as the number of successes in the n trials. It is called the **binomial random variable**. Here are several examples of binomial experiments.

1. Flip a coin 10 times. The two outcomes per trial are heads and tails. The terms *success* and *failure* are arbitrary. We can label either outcome success. However, generally, we call success anything we're looking for. For example, if we were betting on heads, we would label heads a success. If the coin is fair, the probability of heads is 50%. Thus, $p = .5$. Finally, we can see that the trials are independent because the outcome of one coin flip cannot possibly affect the outcomes of other flips.

2. Draw five cards out of a shuffled deck. We can label as success whatever card we seek. For example, if we wish to know the probability of receiving five clubs, a club is labeled a success. On the first draw, the probability of a club is 13/52 = .25. However, if we draw a second card without replacing the first card and shuffling, the trials are not independent. To see why, suppose that the first draw is a club. If we draw again without replacement the probability of drawing a second club is 12/51, which is not .25. In this experiment, the trials are *not* independent.* Hence, this is not a binomial experiment. However, if we replace the card and shuffle before drawing again, the experiment is binomial. Note that in most card games, we do not replace the card, and as a result the experiment is not binomial.

3. A political survey asks 1,500 voters who they intend to vote for in an approaching election. In most elections in the United States, there are only two candidates, the Republican and Democratic nominees. Thus, we have two outcomes per trial. The trials are independent because the choice of one voter does not affect the choice of other voters. In Canada, and in other countries with parliamentary systems of government, there are usually several candidates in the race. However, we can label a vote for our favored candidate (or the party that is paying us to do the survey) a success and all the others are failures.

As you will discover, the third example is a very common application of statistical inference. The actual value of p is unknown, and the job of the statistics practitioner is to estimate its value. By understanding the probability distribution that uses p, we will be able to develop the statistical tools to estimate p.

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*The hypergeometric distribution described in Keller's website Appendix of the same name is used to calculate probabilities in such cases.

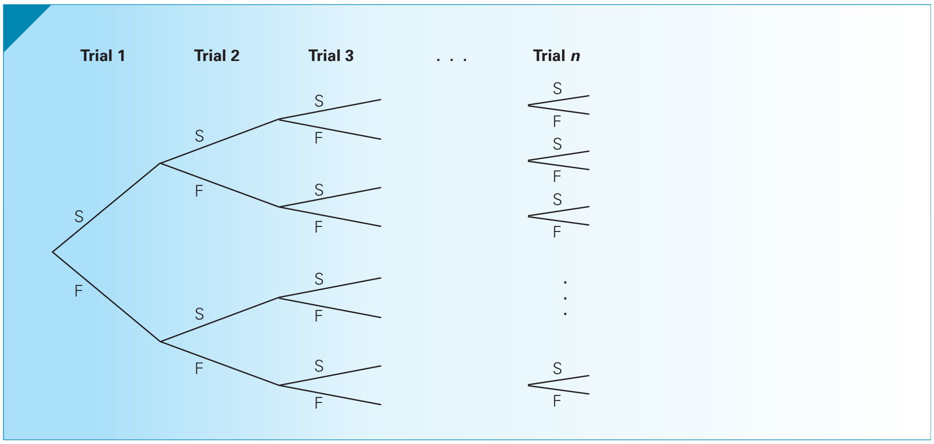
Binomial Random Variable

The binomial random variable is the number of successes in the experiment's n trials. It can take on values 0, 1, 2, . . . , n. Thus, the random variable is discrete. To proceed, we must be capable of calculating the probability associated with each value.

Using a probability tree, we draw a series of branches as depicted in Figure 7.2. The stages represent the outcomes for each of the n trials. At each stage, there are two branches representing success and failure. To calculate the probability that there are X successes in n trials, we note that for each success in the sequence we must multiply by p. And if there are X successes, there must be $n - X$ failures. For each failure in the sequence, we multiply by $1 - p$. Thus, the probability for each sequence of branches that represent x successes and $n - x$ failures has probability

$$p^x(1 - p)^{n-x}$$

FIGURE **7.2** Probability Tree for a Binomial Experiment



There are a number of branches that yield x successes and $n - x$ failures. For example, there are two ways to produce exactly one success and one failure in two trials: SF and FS. To count the number of branch sequences that produce x successes and $n - x$ failures, we use the combinatorial formula

$$C_x^n = \frac{n!}{x!(n - x)!}$$

where $n! = n(n - 1)\,(n - 2) \ldots (2)(1)$. For example, $3! = 3(2)(1) = 6$. Incidentally, although it may not appear to be logical $0! = 1$.

Pulling together the two components of the probability distribution yields the following.

Binomial Probability Distribution

The probability of x successes in a binomial experiment with n trials and probability of success $= p$ is

$$P(x) = \frac{n!}{x!(n - x)!}p^x(1 - p)^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n$$

EXAMPLE 7.9 Pat Statsdud and the Statistics Quiz

Pat Statsdud is a student taking a statistics course. Unfortunately, Pat is not a good student. Pat does not read the textbook before class, does not do homework, and regularly misses class. Pat intends to rely on luck to pass the next quiz. The quiz consists of 10 multiple-choice questions. Each question has five possible answers, only one of which is correct. Pat plans to guess the answer to each question.

a. What is the probability that Pat gets no answers correct?

b. What is the probability that Pat gets two answers correct?

SOLUTION

The experiment consists of 10 identical trials, each with two possible outcomes and where success is defined as a correct answer. Because Pat intends to guess, the probability of success is 1/5 or .2. Finally, the trials are independent because the outcome of any of the questions does not affect the outcomes of any other questions. These four properties tell us that the experiment is binomial with $n = 10$ and $p = .2$.

a. From

$$P(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

we produce the probability of no successes by letting $n = 10$, $p = .2$, and $x = 0$. Hence,

$$P(0) = \frac{10!}{0!(10-0)!}(.2)^0(1-.2)^{10-0}$$

The combinatorial part of the formula is $\frac{10!}{0!10!}$, which is 1. This is the number of ways to get 0 correct and 10 incorrect. Obviously, there is only one way to produce $X = 0$. And because $(.2)^0 = 1$,

$$P(X = 0) = 1(1)(.8)^{10} = .1074$$

b. The probability of two correct answers is computed similarly by substituting $n = 10$, $p = .2$, and $x = 2$:

$$P(x) = \frac{n!}{x!(n-x)!}\,p^x(1-p)^{n-x}$$

$$\begin{aligned} P(0) &= \frac{10!}{2!(10-2)!}\,(.2)^2(1-.2)^{10-2} \\ &= \frac{(10)(9)(8)(7)(6)(5)(4)(3)(2)(1)}{(2)(1)(8)(7)(6)(5)(4)(3)(2)(1)}\,(.04)(.1678) \\ &= 45(.006712) \\ &= .3020 \end{aligned}$$

In this calculation, we discovered that there are 45 ways to get exactly two correct and eight incorrect answers, and that each such outcome has probability .006712. Multiplying the two numbers produces a probability of .3020.

Cumulative Probability

The formula of the binomial distribution allows us to determine the probability that X equals individual values. In Example 7.9, the values of interest were 0 and 2. There are many circumstances where we wish to find the probability that a random variable is less than or equal to a value; that is, we want to determine $P(X \leq x)$, where x is that value. Such a probability is called a **cumulative probability**.

EXAMPLE 7.10

Will Pat Fail the Quiz?

Find the probability that Pat fails the quiz. A mark is considered a failure if it is less than 50%.

SOLUTION

In this quiz, a mark of less than 5 is a failure. Because the marks must be integers, a mark of 4 or less is a failure. We wish to determine $P(X \leq 4)$. So,

$$P(X \leq 4) = P(0) + P(1) + P(2) + P(3) + P(4)$$

From Example 7.9, we know $P(0) = .1074$ and $P(2) = .3020$. Using the binomial formula, we find $P(1) = .2684$, $P(3) = .2013$, and $P(4) = .0881$. Thus

$$P(X \leq 4) = .1074 + .2684 + .3020 + .2013 + .0881 = .9672$$

There is a 96.72% probability that Pat will fail the quiz by guessing the answer for each question.

Binomial Table

There is another way to determine binomial probabilities. Table 1 in Appendix B provides cumulative binomial probabilities for selected values of n and p. We can use this table to answer the question in Example 7.10, where we need $P(X \leq 4)$. Refer to Table 1, find $n = 10$, and in that table find $p = .20$. The values in that column are $P(X \leq x)$ for $x = 0, 1, 2, \ldots, 10$, which are shown in Table 7.2.

TABLE **7.2** Cumulative Binomial Probabilities with $n = 10$ and $p = .2$

x	$P(X \leq x)$
0	.1074
1	.3758
2	.6778
3	.8791
4	.9672
5	.9936
6	.9991
7	.9999
8	1.000
9	1.000
10	1.000

The first cumulative probability is $P(X \leq 0)$, which is $P(0) = .1074$. The probability we need for Example 7.10 is $P(X \leq 4) = .9672$, which is the same value we obtained manually.

We can use the table and the complement rule to determine probabilities of the type $P(X \geq x)$. For example, to find the probability that Pat will pass the quiz, we note that

$$P(X \leq 4) + P(X \geq 5) = 1$$

Thus,

$$P(X \geq 5) = 1 - P(X \leq 4) = 1 - .9672 = .0328$$

Using Table 1 to Find the Binomial Probability $P(X \geq x)$

$$P(X \geq x) = 1 - P(X \leq [x - 1])$$

The table is also useful in determining the probability of an individual value of X. For example, to find the probability that Pat will get exactly two right answers we note that

$$P(X \leq 2) = P(0) + P(1) + P(2)$$

and

$$P(X \leq 1) = P(0) + P(1)$$

The difference between these two cumulative probabilities is $p(2)$. Thus,

$$P(2) = P(X \leq 2) - P(X \leq 1) = .6778 - .3758 = .3020$$

Using Table 1 to Find the Binomial Probability $P(X = x)$

$$P(x) = P(X \leq x) - P(X \leq [x - 1])$$

Using the Computer

EXCEL

INSTRUCTIONS

Type the following into any empty cell:

=**BINOMDIST**([x], [n], [p], [True] or [False])

Typing "True" calculates a cumulative probability and typing "False" computes the probability of an individual value of X. For Example 7.9(a), type

=**BINOMDIST**(0, 10, .2, False)

For Example 7.10, enter

=**BINOMDIST**(4, 10, .2, True)

MINITAB

INSTRUCTIONS

This is the first of seven probability distributions for which we provide instructions. All work in the same way. Click **Calc**, **Probability Distributions**, and the specific distribution whose probability you wish to compute. In this case, select **Binomial . . . .** Check either **Probability** or **Cumulative probability**. If you wish to make a probability statement about one value of x, specify **Input constant** and type the value of x.

If you wish to make probability statements about several values of x from the same binomial distribution, type the values of x into a column before checking **Calc**. Choose **Input column** and type the name of the column. Finally, enter the components of the distribution. For the binomial, enter the **Number of trials** n and the **Event Probability** p.

For the other six distributions, we list the distribution (here it is **Binomial**) and the components only (for this distribution it is n and p).

Mean and Variance of a Binomial Distribution

Statisticians have developed general formulas for the mean, variance, and standard deviation of a binomial random variable. They are

$$\mu = np$$
$$\sigma^2 = np(1 - p)$$
$$\sigma = \sqrt{np(1 - p)}$$

EXAMPLE 7.11

Pat Statsdud Has Been Cloned!

Suppose that a professor has a class full of students like Pat (a nightmare!). What is the mean mark? What is the standard deviation?

SOLUTION

The mean mark for a class of Pat Statsduds is

$$\mu = np = 10(.2) = 2$$

The standard deviation is

$$\sigma = \sqrt{np(1 - p)} = \sqrt{10(.2)(1 - .2)} = 1.26$$

EXERCISES

7.84 Given a binomial random variable with $n = 10$ and $p = .3$, use the formula to find the following probabilities.
a. $P(X = 3)$
b. $P(X = 5)$
c. $P(X = 8)$

7.85 Repeat Exercise 7.84 using Table 1 in Appendix B.

7.86 Repeat Exercise 7.84 using Excel or Minitab.

7.87 Given a binomial random variable with $n = 6$ and $p = .2$, use the formula to find the following probabilities.
a. $P(X = 2)$
b. $P(X = 3)$
c. $P(X = 5)$

7.88 Repeat Exercise 7.87 using Table 1 in Appendix B.

7.89 Repeat Exercise 7.87 using Excel or Minitab.

7.90 Suppose X is a binomial random variable with $n = 25$ and $p = .7$. Use Table 1 to find the following.
a. $P(X = 18)$
b. $P(X = 15)$
c. $P(X \leq 20)$
d. $P(X \geq 16)$

7.91 Repeat Exercise 7.90 using Excel or Minitab.

7.92 A sign on the gas pumps of a chain of gasoline stations encourages customers to have their oil checked with the claim that one out of four cars needs to have oil added. If this is true, what is the probability of the following events?
a. One out of the next four cars needs oil
b. Two out of the next eight cars need oil
c. Three out of the next 12 cars need oil

7.93 The leading brand of dishwasher detergent has a 30% market share. A sample of 25 dishwasher detergent customers was taken. What is the probability that 10 or fewer customers chose the leading brand?

7.94 A certain type of tomato seed germinates 90% of the time. A backyard farmer planted 25 seeds.
a. What is the probability that exactly 20 germinate?
b. What is the probability that 20 or more germinate?
c. What is the probability that 24 or fewer germinate?
d. What is the expected number of seeds that germinate?

7.95 According to the American Academy of Cosmetic Dentistry, 75% of adults believe that an unattractive smile hurts career success. Suppose that 25 adults are randomly selected. What is the probability that 15 or more of them would agree with the claim?

7.96 A student majoring in accounting is trying to decide on the number of firms to which he should apply. Given his work experience and grades, he can expect to receive a job offer from 70% of the firms to which he applies. The student decides to apply to only four firms. What is the probability that he receives no job offers?

7.97 In the United States, voters who are neither Democrat nor Republican are called Independents. It is believed that 10% of all voters are Independents. A survey asked 25 people to identify themselves as Democrat, Republican, or Independent.
a. What is the probability that none of the people are Independent?
b. What is the probability that fewer than five people are Independent?
c. What is the probability that more than two people are Independent?

7.98 Most dial-up Internet service providers (ISPs) attempt to provide a large enough service so that customers seldom encounter a busy signal. Suppose that the customers of one ISP encounter busy signals 8% of the time. During the week, a customer of this ISP called 25 times. What is the probability that she did not encounter any busy signals?

7.99 Major software manufacturers offer a help line that allows customers to call and receive assistance in solving their problems. However, because of the volume of calls, customers frequently are put on hold. One software manufacturer claims that only 20% of callers are put on hold. Suppose that 100 customers call. What is the probability that more than 25 of them are put on hold?

7.100 A statistics practitioner working for major league baseball determined the probability that the hitter will be out on ground balls is .75. In a game where there are 20 ground balls, find the probability that all of them were outs.

The following exercises are best solved with a computer.

7.101 The probability of winning a game of craps (a dice-throwing game played in casinos) is 244/495.
a. What is the probability of winning 5 or more times in 10 games?
b. What is the expected number of wins in 100 games?

7.102 In the game of blackjack as played in casinos in Las Vegas, Atlantic City, and Niagara Falls, as well as in many other cities, the dealer has the advantage. Most players do not play very well. As a result, the probability that the average player wins a hand is about 45%. Find the probability that an average player wins.
a. Twice in 5 hands
b. Ten or more times in 25 hands

7.103 Several books teach blackjack players the "basic strategy," which increases the probability of winning any hand to 50%. Repeat Exercise 7.102, assuming the player plays the basic strategy.

7.104 The best way of winning at blackjack is to "case the deck," which involves counting 10s, non-10s, and aces. For card counters, the probability of winning a hand may increase to 52%. Repeat Exercise 7.102 for a card counter.

7.105 In the game of roulette, a steel ball is rolled onto a wheel that contains 18 red, 18 black, and 2 green slots. If the ball is rolled 25 times, find the probabilities of the following events.
a. The ball falls into the green slots two or more times.
b. The ball does not fall into any green slots.
c. The ball falls into black slots 15 or more times.
d. The ball falls into red slots 10 or fewer times.

7.106 According to a Gallup Poll conducted March 5–7, 2001, 52% of American adults think that protecting the environment should be given priority over developing U.S. energy supplies. Thirty-six percent think that developing energy supplies is more important, and 6% believe the two are equally important. The rest had no opinion. Suppose that a sample of 100 American adults is quizzed on the subject. What is the probability of the following events?

a. Fifty or more think that protecting the environment should be given priority.
b. Thirty or fewer think that developing energy supplies is more important.
c. Five or fewer have no opinion.

7.107 In a *Bon Appetit* poll, 38% of people said that chocolate was their favorite flavor of ice cream. A sample of 20 people was asked to name their favorite flavor of ice cream. What is the probability that half or more of them prefer chocolate?

7.108 The statistics practitioner in Exercise 7.100 also determined that if a batter hits a line drive, the probability of an out is .23. Determine the following probabilities.

a. In a game with 10 line drives, at least 5 are outs.
b. In a game with 25 line drives, there are 5 outs or less.

7.109 According to the last census, 45% of working women held full-time jobs in 2002. If a random sample of 50 working women is drawn, what is the probability that 19 or more hold full-time jobs?

7.5 Poisson Distribution

Another useful discrete probability distribution is the **Poisson distribution**, named after its French creator. Like the binomial random variable, the **Poisson random variable** is the number of occurrences of events, which we'll continue to call *successes*. The difference between the two random variables is that a binomial random variable is the number of successes in a set number of trials, whereas a Poisson random variable is the number of successes in an interval of time or specific region of space. Here are several examples of Poisson random variables.

1. The number of cars arriving at a service station in 1 hour. (The interval of time is 1 hour.)
2. The number of flaws in a bolt of cloth. (The specific region is a bolt of cloth.)
3. The number of accidents in 1 day on a particular stretch of highway. (The interval is defined by both time, 1 day, and space, the particular stretch of highway.)

The Poisson experiment is described in the box.

Poisson Experiment

A **Poisson experiment** is characterized by the following properties:

1. The number of successes that occur in any interval is independent of the number of successes that occur in any other interval.
2. The probability of a success in an interval is the same for all equal-size intervals.
3. The probability of a success in an interval is proportional to the size of the interval.
4. The probability of more than one success in an interval approaches 0 as the interval becomes smaller.

Poisson Random Variable

The **Poisson random variable** is the number of successes that occur in a period of time or an interval of space in a Poisson experiment.

There are several ways to derive the probability distribution of a Poisson random variable. However, all are beyond the mathematical level of this book. We simply provide the formula and illustrate how it is used.

Poisson Probability Distribution

The probability that a Poisson random variable assumes a value of x in a specific interval is

$$P(x) = \frac{e^{-\mu}\mu^{x}}{x!} \quad \text{for } x = 0,1,2,\ldots$$

where μ is the mean number of successes in the interval or region and e is the base of the natural logarithm (approximately 2.71828). Incidentally, the variance of a Poisson random variable is equal to its mean; that is, $\sigma^2 = \mu$.

EXAMPLE 7.12 Probability of the Number of Typographical Errors in Textbooks

A statistics instructor has observed that the number of typographical errors in new editions of textbooks varies considerably from book to book. After some analysis, he concludes that the number of errors is Poisson distributed with a mean of 1.5 per 100 pages. The instructor randomly selects 100 pages of a new book. What is the probability that there are no typographical errors?

SOLUTION

We want to determine the probability that a Poisson random variable with a mean of 1.5 is equal to 0. Using the formula

$$P(x) = \frac{e^{-\mu}\mu^{x}}{x!}$$

and substituting $x = 0$ and $\mu = 1.5$, we get

$$P(0) = \frac{e^{-1.5}1.5^{0}}{0!} = \frac{(2.71828)^{-1.5}(1)}{1} = .2231$$

The probability that in the 100 pages selected there are no errors is .2231.

Notice that in Example 7.12 we wanted to find the probability of 0 typographical errors in 100 pages given a mean of 1.5 typos in 100 pages. The next example illustrates how we calculate the probability of events where the intervals or regions do not match.

EXAMPLE 7.13

Probability of the Number of Typographical Errors in 400 Pages

Refer to Example 7.12. Suppose that the instructor has just received a copy of a new statistics book. He notices that there are 400 pages.

a. What is the probability that there are no typos?
b. What is the probability that there are five or fewer typos?

SOLUTION

The specific region that we're interested in is 400 pages. To calculate Poisson probabilities associated with this region, we must determine the mean number of typos per 400 pages. Because the mean is specified as 1.5 per 100 pages, we multiply this figure by 4 to convert to 400 pages. Thus, $\mu = 6$ typos per 400 pages.

a. The probability of no typos is

$$P(0) = \frac{e^{-6}6^0}{0!} = \frac{(2.71828)^{-6}(1)}{1} = .002479$$

b. We want to determine the probability that a Poisson random variable with a mean of 6 is 5 or less; that is, we want to calculate

$$P(X \le 5) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$$

To produce this probability, we need to compute the six probabilities in the summation.

$$P(0) = .002479$$

$$P(1) = \frac{e^{-\mu}\mu^x}{x!} = \frac{e^{-6}6^1}{1!} = \frac{(2.71828)^{-6}(6)}{1} = .01487$$

$$P(2) = \frac{e^{-\mu}\mu^x}{x!} = \frac{e^{-6}6^2}{2!} = \frac{(2.71828)^{-6}(36)}{2} = .04462$$

$$P(3) = \frac{e^{-\mu}\mu^x}{x!} = \frac{e^{-6}6^3}{3!} = \frac{(2.71828)^{-6}(216)}{6} = .08924$$

$$P(4) = \frac{e^{-\mu}\mu^x}{x!} = \frac{e^{-6}6^4}{4!} = \frac{(2.71828)^{-6}(1296)}{24} = .1339$$

$$P(5) = \frac{e^{-\mu}\mu^x}{x!} = \frac{e^{-6}6^5}{5!} = \frac{(2.71828)^{-6}(7776)}{120} = .1606$$

Thus,

$$P(X \le 5) = .002479 + .01487 + .04462 + .08924 + .1339 + .1606$$
$$= .4457$$

The probability of observing 5 or fewer typos in this book is .4457.

Poisson Table

As was the case with the binomial distribution, a table is available that makes it easier to compute Poisson probabilities of individual values of x as well as cumulative and related probabilities.

Table 2 in Appendix B provides cumulative Poisson probabilities for selected values of μ. This table makes it easy to find cumulative probabilities like those in Example 7.13, part (b), where we found $P(X \leq 5)$.

To do so, find $\mu = 6$ in Table 2. The values in that column are $P(X \leq x)$ for $x = 0, 1, 2, \ldots, 18$ which are shown in Table 7.3.

TABLE **7.3** Cumulative Poisson Probabilities for $\mu = 6$

x	$P(X \leq x)$
0	.0025
1	.0174
2	.0620
3	.1512
4	.2851
5	.4457
6	.6063
7	.7440
8	.8472
9	.9161
10	.9574
11	.9799
12	.9912
13	.9964
14	.9986
15	.9995
16	.9998
17	.9999
18	1.0000

Theoretically, a Poisson random variable has no upper limit. The table provides cumulative probabilities until the sum is 1.0000 (using four decimal places).

The first cumulative probability is $P(X \leq 0)$, which is $P(0) = .0025$. The probability we need for Example 7.13, part (b), is $P(X \leq 5) = .4457$, which is the same value we obtained manually.

Like Table 1 for binomial probabilities, Table 2 can be used to determine probabilities of the type $P(X \geq x)$. For example, to find the probability that in Example 7.13 there are 6 or more typos, we note that $P(X \leq 5) + P(X \geq 6) = 1$. Thus,

$$P(X \geq 6) = 1 - P(X \leq 5) = 1 - .4457 = .5543$$

Using Table 2 to Find the Poisson Probability $P(X \geq x)$

$$P(X \geq x) = 1 - P(X \leq [x - 1])$$

We can also use the table to determine the probability of one individual value of X. For example, to find the probability that the book contains exactly 10 typos, we note that

$$P(X \leq 10) = P(0) + P(1) + \cdots + P(9) + P(10)$$

and

$$P(X \leq 9) = P(0) + P(1) + \cdots + P(9)$$

The difference between these two cumulative probabilities is $P(10)$. Thus,

$$P(10) = P(X \leq 10) - P(X \leq 9) = .9574 - .9161 = .0413$$

Using Table 2 to Find the Poisson Probability $P(X = x)$

$$P(x) = P(X \leq x) - P(X \leq [x - 1])$$

Using the Computer

EXCEL

INSTRUCTIONS

Type the following into any empty cell:

= **POISSON**([x], [μ], [True] or [False])

We calculate the probability in Example 7.12 by typing

= **POISSON**(0, 1.5, False)

For Example 7.13, we type

= **POISSON**(5, 6, True)

MINITAB

INSTRUCTIONS

Click **Calc**, **Probability Distributions**, and **Poisson . . .** and type the mean.

EXERCISES

7.110 Given a Poisson random variable with $\mu = 2$, use the formula to find the following probabilities.
a. $P(X = 0)$
b. $P(X = 3)$
c. $P(X = 5)$

7.111 Given that X is a Poisson random variable with $\mu = .5$, use the formula to determine the following probabilities.
a. $P(X = 0)$
b. $P(X = 1)$
c. $P(X = 2)$

7.112 The number of accidents that occur at a busy intersection is Poisson distributed with a mean of 3.5 per week. Find the probability of the following events.

a. No accidents in one week
b. Five or more accidents in one week
c. One accident today

7.113 Snowfalls occur randomly and independently over the course of winter in a Minnesota city. The average is one snowfall every 3 days.

a. What is the probability of five snowfalls in 2 weeks?
b. Find the probability of a snowfall today.

7.114 The number of students who seek assistance with their statistics assignments is Poisson distributed with a mean of two per day.

a. What is the probability that no students seek assistance tomorrow?
b. Find the probability that 10 students seek assistance in a week.

7.115 Hits on a personal website occur quite infrequently. They occur randomly and independently with an average of five per week.

a. Find the probability that the site gets 10 or more hits in a week.
b. Determine the probability that the site gets 20 or more hits in 2 weeks.

7.116 In older cities across North America, infrastructure is deteriorating, including water lines that supply homes and businesses. A report to the Toronto city council stated that there are on average 30 water line breaks per 100 kilometers per year in the city of Toronto. Outside of Toronto, the average number of breaks is 15 per 100 kilometers per year.

a. Find the probability that in a stretch of 100 kilometers in Toronto there are 35 or more breaks next year.
b. Find the probability that there are 12 or fewer breaks in a stretch of 100 kilometers outside of Toronto next year.

7.117 The number of bank robberies that occur in a large North American city is Poisson distributed with a mean of 1.8 per day. Find the probabilities of the following events.

a. Three or more bank robberies in a day
b. Between 10 and 15 (inclusive) robberies during a 5-day period

7.118 Flaws in a carpet tend to occur randomly and independently at a rate of one every 200 square feet. What is the probability that a carpet that is 8 feet by 10 feet contains no flaws?

7.119 Complaints about an Internet brokerage firm occur at a rate of five per day. The number of complaints appears to be Poisson distributed.

a. Find the probability that the firm receives 10 or more complaints in a day.
b. Find the probability that the firm receives 25 or more complaints in a 5-day period.

APPLICATIONS in OPERATIONS MANAGEMENT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Waiting Lines

Everyone is familiar with waiting lines. We wait in line at banks, groceries, and fast-food restaurants. There are also waiting lines in firms where trucks wait to load and unload and on assembly lines where stations wait for new parts. Management scientists have developed mathematical models that allow managers to determine the operating characteristics of waiting lines. Some of the operating characteristics are

The probability that there are no units in the system
The average number of units in the waiting line
The average time a unit spends in the waiting line
The probability that an arriving unit must wait for service

The Poisson probability distribution is used extensively in waiting-line (also called *queuing*) models. Many models assume that the arrival of units for service is Poisson distributed with a specific value of μ. In the next chapter, we will discuss the operating characteristics of waiting lines.

Exercises 7.120–7.122 require the calculation of the probability of a number of arrivals.

7.120 The number of trucks crossing at the Ambassador Bridge connecting Detroit, Michigan, and Windsor, Ontario, is Poisson distributed with a mean of 1.5 per minute.

a. What is the probability that in any 1-minute time span two or more trucks will cross the bridge?

b. What is the probability that fewer than four trucks will cross the bridge over the next 4 minutes?

7.121 Cars arriving for gasoline at a particular gas station follow a Poisson distribution with a mean of 5 per hour.

a. Determine the probability that over the next hour only one car will arrive.

b. Compute the probability that in the next 3 hours more than 20 cars will arrive.

7.122 The number of users of an automatic banking machine is Poisson distributed. The mean number of users per 5-minute interval is 1.5. Find the probability of the following events.

a. No users in the next 5 minutes

b. Five or fewer users in the next 15 minutes

c. Three or more users in the next 10 minutes

Chapter Summary

There are two types of random variables. A **discrete random variable** is one whose values are countable. A **continuous random variable** can assume an uncountable number of values. In this chapter, we discussed discrete random variables and their **probability distributions**. We defined the **expected value**, **variance**, and **standard deviation** of a population represented by a discrete probability distribution. Also introduced in this chapter were **bivariate discrete distributions** on which an important application in finance was based. Finally, the two most important discrete distributions—the **binomial** and the **Poisson**—were presented.

Important Terms

Random variable 218
Discrete random variable 219
Continuous random variable 219
Probability distribution 219
Expected value 223
Bivariate distribution 229
PERT (Project Evaluation and Review Technique) 235
CPM (Critical Path Method) 235
Path 235
Critical path 236
Diversification 237
Binomial experiment 244
Bernoulli process 244
Binomial random variable 244
Binomial probability distribution 245
Cumulative probability 247
Poisson distribution 251
Poisson random variable 251
Poisson experiment 251

SYMBOLS

Symbol	Pronounced	Represents
$\sum_{\text{all } x} x$	Sum of x for all values of x	Summation
C_x^n	n choose x	Number of combinations
$n!$	n factorial	$n(n-1)(n-2)\cdots(3)(2)(1)$
e		2.71828 . . .

FORMULAS

Expected value (mean)

$$E(X) = \mu = \sum_{\text{all } x} xP(x)$$

Variance

$$V(x) = \sigma^2 = \sum_{\text{all } x} (x-\mu)^2 P(x)$$

Standard deviation

$$\sigma = \sqrt{\sigma^2}$$

Covariance

$$\text{COV}(X, Y) = \sigma_{xy} = \sum (x-\mu_x)(y-\mu_y)P(x, y)$$

Coefficient of Correlation

$$\rho = \frac{\text{COV}(X, Y)}{\sigma_x \sigma_y} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

Laws of expected value

1. $E(c) = c$
2. $E(X + c) = E(X) + c$
3. $E(cX) = cE(X)$

Laws of variance

1. $V(c) = 0$
2. $V(X + c) = V(X)$
3. $V(cX) = c^2 V(X)$

Laws of expected value and variance of the sum of two variables

1. $E(X + Y) = E(X) + E(Y)$
2. $V(X + Y) = V(X) + V(Y) + 2\text{COV}(X, Y)$

Laws of expected value and variance for the sum of k variables, where $k \geq 2$

1. $E(X_1 + X_2 + \cdots + X_k) = E(X_1) + E(X_2) + \cdots + E(X_k)$
2. $V(X_1 + X_2 + \cdots + X_k) = V(X_1) + V(X_2) + \cdots + V(X_k)$

   if the variables are independent

Mean and variance of a portfolio of two stocks

$$E(Rp) = w_1 E(R_1) + w_2 E(R_2)$$

$$\begin{aligned} V(R_p) &= w_1^2 V(R_1) + w_2^2 V(R_2) \\ &\quad + 2w_1 w_2 \text{COV}(R_1, R_2) \\ &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \rho \sigma_1 \sigma_2 \end{aligned}$$

Mean and variance of a portfolio of k stocks

$$E(R_p) = \sum_{i=1}^{k} w_i E(R_i)$$

$$V(R_p) = \sum_{i=1}^{k} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{k} \sum_{j=i+1}^{k} w_i w_j \text{COV}(R_i, R_j)$$

Binomial probability

$$P(X = x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$\mu = np$$

$$\sigma^2 = np(1-p)$$

$$\sigma = \sqrt{np(1-p)}$$

Poisson probability

$$P(X = x) = \frac{e^{-\mu}\mu^x}{x!}$$

COMPUTER INSTRUCTIONS

Probability Distribution	Excel	Minitab
Binomial	248	249
Poisson	255	255

CHAPTER EXERCISES

7.123 In 2000, Northwest Airlines boasted that 77.4% of its flights were on time. If we select five Northwest flights at random, what is the probability that all five are on time? (*Source*: Department of Transportation.)

7.124 The final exam in a one-term statistics course is taken in the December exam period. Students who are sick or have other legitimate reasons for missing the exam are allowed to write a deferred exam scheduled for the first week in January. A statistics professor has observed that only 2% of all students legitimately miss the December final exam. Suppose that the professor has 40 students registered this term.

a. How many students can the professor expect to miss the December exam?
b. What is the probability that the professor will not have to create a deferred exam?

7.125 The number of magazine subscriptions per household is represented by the following probability distribution.

Magazine subscriptions per household	0	1	2	3	4
Probability	.48	.35	.08	.05	.04

a. Calculate the mean number of magazine subscriptions per household.
b. Find the standard deviation.

7.126 The number of arrivals at a car wash is Poisson distributed with a mean of eight per hour.

a. What is the probability that 10 cars will arrive in the next hour?
b. What is the probability that more than 5 cars will arrive in the next hour?
c. What is the probability that fewer than 12 cars will arrive in the next hour?

7.127 The percentage of customers who enter a restaurant and ask to be seated in a smoking section is 15%. Suppose that 100 people enter the restaurant.

a. What is the expected number of people who request a smoking table?
b. What is the standard deviation of the number of requests for a smoking table?
c. What is the probability that 20 or more people request a smoking table?

7.128 Lotteries are an important income source for various governments around the world. However, the availability of lotteries and other forms of gambling have created a social problem: gambling addicts. A critic of government-controlled gambling contends that 30% of people who regularly buy lottery tickets are gambling addicts. If we randomly select 10 people among those who report that they regularly buy lottery tickets, what is the probability that more than 5 of them are addicts?

7.129 The distribution of the number of home runs in soft-ball games is shown here.

Number of home runs	0	1	2	3	4	5
Probability	.05	.16	.41	.27	.07	.04

a. Calculate the mean number of home runs.
b. Find the standard deviation.

7.130 An auditor is preparing for a physical count of inventory as a means of verifying its value. Items counted are reconciled with a list prepared by the storeroom supervisor. In one particular firm, 20% of the items counted cannot be reconciled without reviewing invoices. The auditor selects 10 items. Find the probability that 6 or more items cannot be reconciled.

7.131 Shutouts in the National Hockey League occur randomly and independently at a rate of 1 every 20 games. Calculate the probability of the following events.

a. 2 shutouts in the next 10 games
b. 25 shutouts in 400 games
c. a shutout in to.night's game

7.132 Most Miami Beach restaurants offer "early-bird" specials. These are lower-priced meals that are available only from 4 to 6 P.M. However, not all customers who arrive between 4 and 6 P.M. order the special. In fact, only 70% do.

a. Find the probability that of 80 customers between 4 and 6 P.M., more than 65 order the special.
b. What is the expected number of customers who order the special?
c. What is the standard deviation?

7.133 According to climatologists, the long-term average for Atlantic storms is 9.6 per season (June 1 to November 30), with 6 becoming hurricanes and 2.3 becoming intense hurricanes. Find the probability of the following events.

a. Ten or more Atlantic storms
b. Five or fewer hurricanes
c. Three or more intense hurricanes

7.134 Researchers at the University of Pennsylvania School of Medicine theorized that children under 2 years old who sleep in rooms with the light on have a 40% probability of becoming myopic by age 16. Suppose that researchers found 25 children who slept with the light on before they were 2.

a. What is the probability that 10 of them will become myopic before age 16?
b. What is the probability that fewer than 5 of them will become myopic before age 16?
c. What is the probability that more than 15 of them will become myopic before age 16?

7.135 A pharmaceutical researcher working on a cure for baldness noticed that middle-aged men who are balding at the crown of their head have a 45% probability of suffering a heart attack over the next decade. In a sample of 100 middle-age balding men, what are the following probabilities?

a. More than 50 will suffer a heart attack in the next decade.
b. Fewer than 44 will suffer a heart attack in the next decade.
c. Exactly 45 will suffer a heart attack in the next decade.

7.136 Advertising researchers have developed a theory that states that commercials that appear in violent television shows are less likely to be remembered and will thus be less effective. After examining samples of viewers who watch violent and nonviolent programs and asking them a series of five questions about the commercials, the researchers produced the following probability distributions of the number of correct answers.

Viewers of violent shows

x	0	1	2	3	4	5
$P(x)$	.36	.22	.20	.09	.08	.05

Viewers of nonviolent shows

x	0	1	2	3	4	5
$P(x)$	.15	.18	.23	.26	.10	.08

a. Calculate the mean and standard deviation of the number of correct answers among viewers of violent television programs.
b. Calculate the mean and standard deviation of the number of correct answers among viewers of nonviolent television programs.

7.137 According to the U.S. census, one-third of all businesses are owned by women. If we select 25 businesses at random, what is the probability that 10 or more of them are owned by women?

7.138 It is recommended that women age 40 and older have a mammogram annually. A recent report indicated that if a woman has annual mammograms over a 10-year period, there is a 60% probability that there will be at least one false-positive result. (A false-positive mammogram test result is one that indicates the presence of cancer when, in fact, there is no cancer.) If the annual test results are independent, what is the probability that in any one year a mammogram will produce a false-positive result? (*Hint:* Find the value of p such that the probability that a binomial random variable with $n = 10$ is greater than or equal to 1 is .60.)

7.139 In a recent election, the mayor received 60% of the vote. Last week, a survey was undertaken that asked 100 people whether they would vote for the mayor. Assuming that her popularity has not changed, what is the probability that more than 50 people in the sample would vote for the mayor?

7.140 When Earth traveled through the storm of meteorites trailing the comet Tempel-Tuttle on November 17, 1998, the storm was 1,000 times as intense as the average meteor storm. Before the comet arrived, telecommunication companies worried about the potential damage that might be inflicted on the approximately 650 satellites in orbit. It was estimated that each satellite had a 1% chance of being hit, causing damage to the satellite's electronic system. One company had five satellites in orbit at the time. Determine the probability distribution of the number of the company's satellites that would be damaged.

CASE 7.1 To Bunt or Not to Bunt, That Is the Question—Part 2

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In Case 6.2, we presented the probabilities of scoring at least one run and asked you to determine whether the manager should signal for the batter to sacrifice bunt. The decision was made on the basis of comparing the probability of scoring at least one run when the manager signaled for the bunt and when he signaled the batter to swing away. Another factor that should be incorporated into the decision is the *number* of runs the manager expects his team to score. In the same article referred to in Case 6.2, the author also computed the expected number of runs scored for each situation. Table 1 lists the expected number of runs in situations that are defined by the number of outs and the bases occupied.

TABLE 1 Expected Number of Runs Scored

Bases Occupied	0 Out	1 Out	2 Outs
Bases empty	.49	.27	.10
First base	.85	.52	.23
Second base	1.06	.69	.34
Third base	1.21	.82	.38
First base and second base	1.46	1.00	.48
First base and third base	1.65	1.10	.51
Second base and third base	1.94	1.50	.62
Bases loaded	2.31	1.62	.82

Assume that the manager wishes to score as many runs as possible. Using the same probabilities of the four outcomes of a bunt listed in Case 6.2, determine whether the manager should signal the batter to sacrifice bunt.

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8 CONTINUOUS PROBABILITY DISTRIBUTIONS

Minimum GMAT Score to Enter Executive MBA Program

A university has just approved a new Executive MBA Program. The new director believes that to maintain the prestigious image of the business school, the new program must be seen as having high standards. Accordingly, the Faculty Council decides that one of the entrance requirements will be that applicants must score in the top 1% of Graduate Management Admission Test (GMAT) scores. The director knows that GMAT scores are normally distributed with a mean of 490 and a standard deviation of 61. The only thing she doesn't know is what the minimum GMAT score for admission should be.

After introducing the normal distribution, we will return to this question and answer it.

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See page 281.

INTRODUCTION

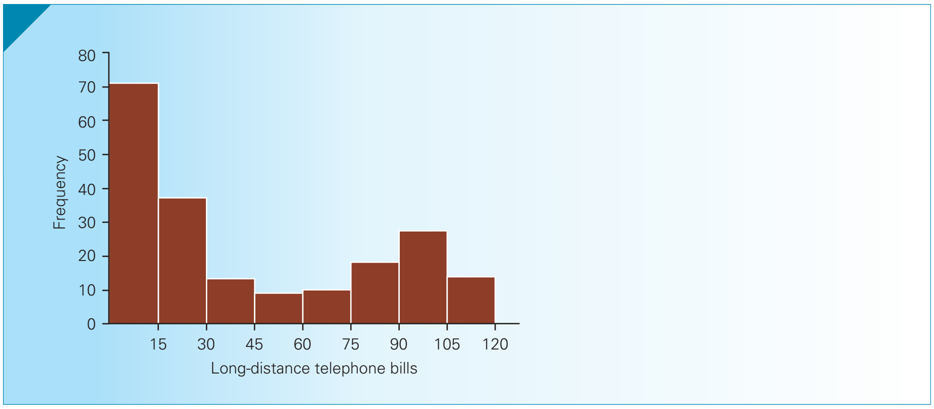
This chapter completes our presentation of probability by introducing continuous random variables and their distributions. In Chapter 7, we introduced discrete probability distributions that are employed to calculate the probability associated with discrete random variables. In Section 7.4, we introduced the binomial distribution, which allows us to determine the probability that the random variable equals a particular value (the number of successes). In this way we connected the population represented by the probability distribution with a sample of nominal data. In this chapter, we introduce continuous probability distributions, which are used to calculate the probability associated with an interval variable. By doing so, we develop the link between a population and a sample of interval data.

Section 8.1 introduces probability density functions and uses the uniform density function to demonstrate how probability is calculated. In Section 8.2, we focus on the normal distribution, one of the most important distributions because of its role in the development of statistical inference. Section 8.3 introduces the exponential distribution, a distribution that has proven to be useful in various management-science applications. Finally, in Section 8.4 we introduce three additional continuous distributions. They will be used in statistical inference throughout the book.

8.1 PROBABILITY DENSITY FUNCTIONS

A continuous random variable is one that can assume an uncountable number of values. Because this type of random variable is so different from a discrete variable, we need to treat it completely differently. First, we cannot list the possible values because there is an infinite number of them. Second, because there is an infinite number of values, the probability of each individual value is virtually 0. Consequently, we can determine the probability of only a range of values. To illustrate how this is done, consider the histogram we created for the long-distance telephone bills (Example 3.1), which is depicted in Figure 8.1.

FIGURE **8.1** Histogram for Example 3.1



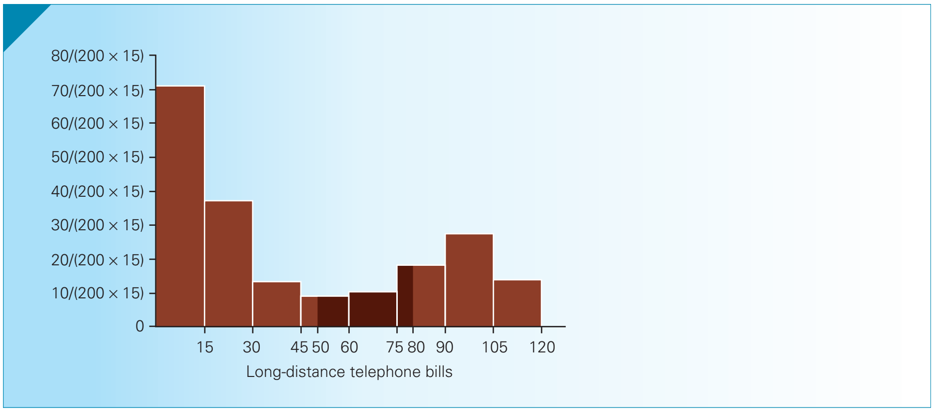
We found, for example, that the relative frequency of the interval 15 to 30 was 37/200. Using the relative frequency approach, we estimate that the probability that a randomly selected long-distance bill will fall between \$15 and \$30 is 37/200 = .185. We can similarly estimate the probabilities of the other intervals in the histogram.

Interval	Relative Frequency
$0 \leq X \leq 15$	71/200
$15 < X \leq 30$	37/200
$30 < X \leq 45$	13/200
$45 < X \leq 60$	9/200
$60 < X \leq 75$	10/200
$75 < X \leq 90$	18/200
$90 < X \leq 105$	28/200
$105 < X \leq 120$	14/200

Notice that the sum of the probabilities equals 1. To proceed, we set the values along the vertical axis so that the *area* in all the rectangles together adds to 1. We accomplish this by dividing each relative frequency by the width of the interval, which is 15. The result is a rectangle over each interval whose *area* equals the probability that the random variable will fall into that interval.

To determine probabilities of ranges other than the ones created when we drew the histogram, we apply the same approach. For example, the probability that a long-distance bill will fall between \$50 and \$80 is equal to the area between 50 and 80 as shown in Figure 8.2.

FIGURE **8.2** Histogram for Example 3.1: Relative Frequencies Divided by Interval Width



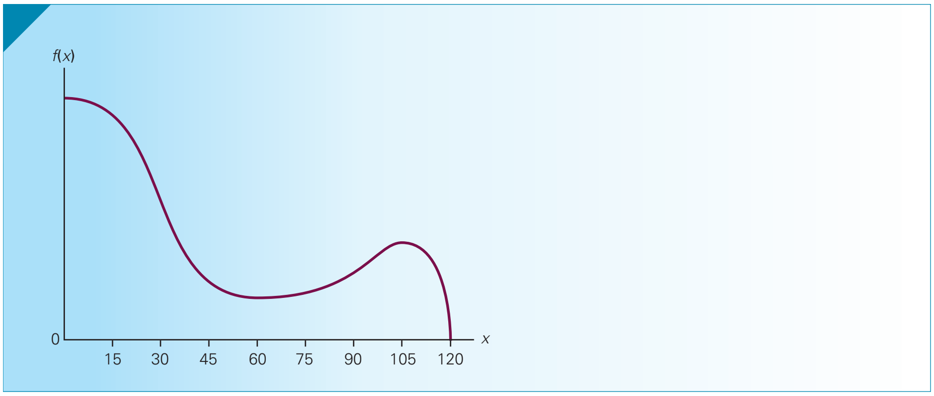
The areas in each shaded rectangle are calculated and added together as follows:

Interval	Height of Rectangle	Base Multiplied by Height
$50 < X \leq 60$	$9/(200 \times 15) = .00300$	$(60 - 50) \times .00300 = .030$
$60 < X \leq 75$	$10/(200 \times 15) = .00333$	$(75 - 60) \times .00333 = .050$
$75 < X \leq 80$	$18/(200 \times 15) = .00600$	$(80 - 75) \times .00600 = .030$
		Total = .110

We estimate that the probability that a randomly selected long-distance bill falls between \$50 and \$80 is .11.

If the histogram is drawn with a large number of small intervals, we can smooth the edges of the rectangles to produce a smooth curve as shown in Figure 8.3. In many cases, it is possible to determine a function $f(x)$ that approximates the curve. The function is called a **probability density function**. Its requirements are stated in the box.

FIGURE 8.3 Density Function for Example 3.1



Requirements for a Probability Density Function

The following requirements apply to a probability density function $f(x)$ whose range is $a \leq x \leq b$.

1. $f(x) \geq 0$ for all x between a and b
2. The total area under the curve between a and b is 1.0

Integral calculus* can often be used to calculate the area under a curve. Fortunately, the probabilities corresponding to continuous probability distributions that we deal with do not require this mathematical tool. The distributions will be either simple or too complex for calculus. Let's start with the simplest continuous distribution.

Uniform Distribution

To illustrate how we find the area under the curve that describes a probability density function, consider the **uniform probability distribution** also called the **rectangular probability distribution**.

Uniform Probability Density Function

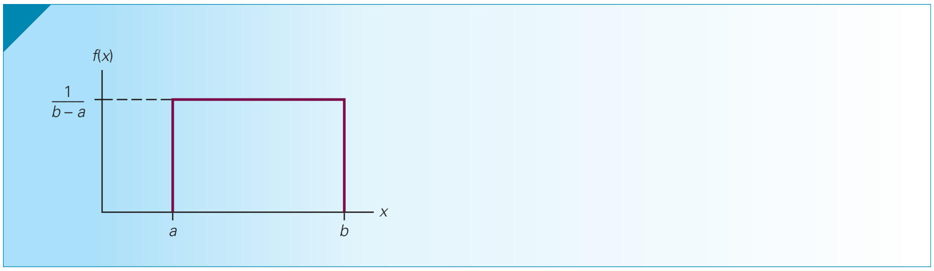
The uniform distribution is described by the function

$$f(x) = \frac{1}{b - a} \quad \text{where } a \leq x \leq b$$

The function is graphed in Figure 8.4. You can see why the distribution is called *rectangular*.

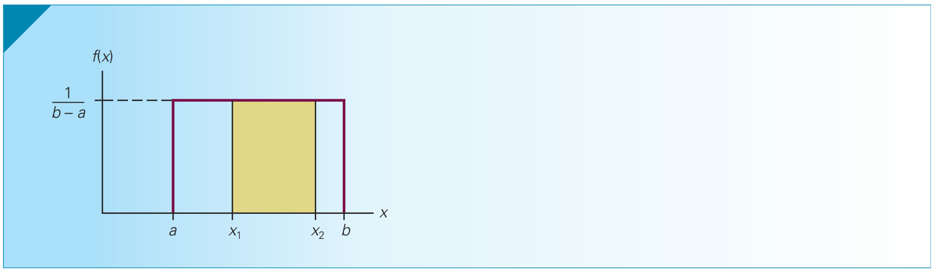
*Keller's website Appendix Continuous Probability Distributions: Calculus Approach demonstrates how to use integral calculus to determine probabilities and parameters for continuous random variables.

FIGURE **8.4** Uniform Distribution



To calculate the probability of any interval, simply find the area under the curve. For example, to find the probability that X falls between x_1 and x_2 determine the area in the rectangle whose base is $x_2 - x_1$ and whose height is $1/(b - a)$. Figure 8.5 depicts the area we wish to find. As you can see, it is a rectangle and the area of a rectangle is found by multiplying the base times the height.

FIGURE **8.5** $P(x_1 < X < x_2)$



Thus,

$$P(x_1 < X < x_2) = \text{Base} \times \text{Height} = (x_2 - x_1) \times \frac{1}{b - a}$$

EXAMPLE 8.1

Uniformly Distributed Gasoline Sales

The amount of gasoline sold daily at a service station is uniformly distributed with a minimum of 2,000 gallons and a maximum of 5,000 gallons.

a. Find the probability that daily sales will fall between 2,500 and 3,000 gallons.
b. What is the probability that the service station will sell at least 4,000 gallons?
c. What is the probability that the station will sell exactly 2,500 gallons?

SOLUTION

The probability density function is

$$f(x) = \frac{1}{5000 - 2000} = \frac{1}{3000} \qquad 2000 \le x \le 5000$$

a. The probability that X falls between 2,500 and 3,000 is the area under the curve between 2,500 and 3,000 as depicted in Figure 8.6a. The area of a rectangle is the base times the height. Thus,

$$P(2{,}500 \le X \le 3{,}000) = (3{,}000 - 2{,}500) \times \left(\frac{1}{3{,}000}\right) = .1667$$

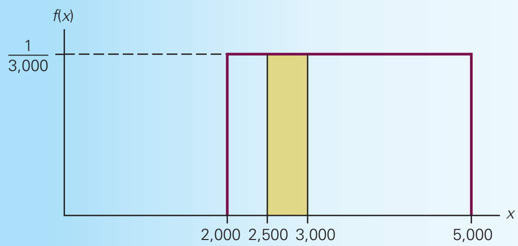
b. $P(X \ge 4{,}000) = (5{,}000 - 4{,}000) \times \left(\frac{1}{3{,}000}\right) = .3333$ [See Figure 8.6b.]

c. $P(X = 2{,}500) = 0$

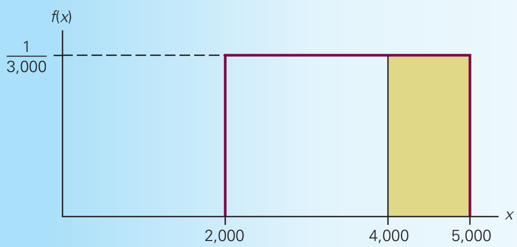
Because there is an uncountable infinite number of values of X, the probability of each individual value is zero. Moreover, as you can see from Figure 8.6c, the area of a line is 0.

Because the probability that a continuous random variable equals any individual value is 0, there is no difference between $P(2{,}500 \le X \le 3{,}000)$ and $P(2{,}500 < X < 3{,}000)$. Of course, we cannot say the same thing about discrete random variables.

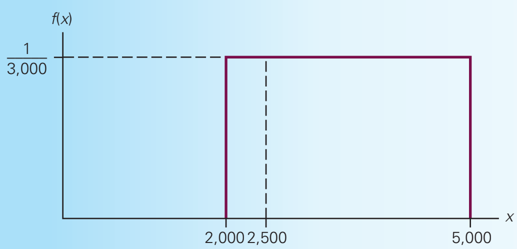
FIGURE **8.6** Density Functions for Example 8.1



(a) $P(2{,}500 < X < 3{,}000)$



(b) $P(4{,}000 < X < 5{,}000)$



(c) $P(X = 2{,}500)$

Using a Continuous Distribution to Approximate a Discrete Distribution

In our definition of discrete and continuous random variables, we distinguish between them by noting whether the number of possible values is countable or uncountable. However, in practice, we frequently use a continuous distribution to approximate a discrete one when the number of values the variable can assume is countable but large. For example, the number of possible values of weekly income is countable. The values of weekly income expressed in dollars are 0, .01, .02, ... Although there is no set upper limit, we can easily identify (and thus count) all the possible values. Consequently, weekly income is a discrete random variable. However, because it can assume such a large number of values, we prefer to employ a continuous probability distribution to determine the probability associated with such variables. In the next section, we introduce the normal distribution, which is often used to describe discrete random variables that can assume a large number of values.

EXERCISES

8.1 Refer to Example 3.2. From the histogram for investment A, estimate the following probabilities.
a. $P(X > 45)$
b. $P(10 < X < 40)$
c. $P(X < 25)$
d. $P(35 < X < 65)$

8.2 Refer to Example 3.2. Estimate the following from the histogram of the returns on investment B.
a. $P(X > 45)$
b. $P(10 < X < 40)$
c. $P(X < 25)$
d. $P(35 < X < 65)$

8.3 Refer to Example 3.3. From the histogram of the marks, estimate the following probabilities.
a. $P(55 < X < 80)$
b. $P(X > 65)$
c. $P(X < 85)$
d. $P(75 < X < 85)$

8.4 A random variable is uniformly distributed between 5 and 25.
a. Draw the density function.
b. Find $P(X > 25)$.
c. Find $P(10 < X < 15)$.
d. Find $P(5.0 < X < 5.1)$.

8.5 A uniformly distributed random variable has minimum and maximum values of 20 and 60, respectively.
a. Draw the density function.
b. Determine $P(35 < X < 45)$.
c. Draw the density function including the calculation of the probability in part (b).

8.6 The amount of time it takes for a student to complete a statistics quiz is uniformly distributed between 30 and 60 minutes. One student is selected at random. Find the probability of the following events.
a. The student requires more than 55 minutes to complete the quiz.
b. The student completes the quiz in a time between 30 and 40 minutes.
c. The student completes the quiz in exactly 37.23 minutes.

8.7 Refer to Exercise 8.6. The professor wants to reward (with bonus marks) students who are in the lowest quarter of completion times. What completion time should he use for the cutoff for awarding bonus marks?

8.8 Refer to Exercise 8.6. The professor would like to track (and possibly help) students who are in the top 10% of completion times. What completion time should he use?

8.9 The weekly output of a steel mill is a uniformly distributed random variable that lies between 110 and 175 metric tons.
a. Compute the probability that the steel mill will produce more than 150 metric tons next week.
b. Determine the probability that the steel mill will produce between 120 and 160 metric tons next week.

8.10 Refer to Exercise 8.9. The operations manager labels any week that is in the bottom 20% of production a "bad week." How many metric tons should be used to define a bad week?

8.11 A random variable has the following density function.

$$f(x) = 1 - .5x \quad 0 < x < 2$$

a. Graph the density function.
b. Verify that $f(x)$ is a density function.

c. Find $P(X > 1)$.
d. Find $P(X < .5)$.
e. Find $P(X = 1.5)$.

8.12 The following function is the density function for the random variable X:

$$f(x) = \frac{x-1}{8} \quad 1 < x < 5$$

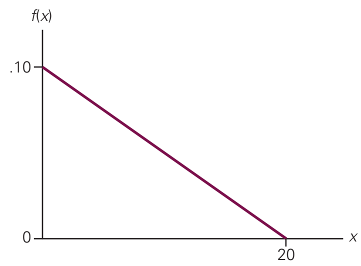
a. Graph the density function.
b. Find the probability that X lies between 2 and 4.
c. What is the probability that X is less than 3?

8.13 The following density function describes the random variable X.

$$f(x) = \begin{cases} \dfrac{x}{25} & 0 < x < 5 \\ \dfrac{10-x}{25} & 5 < x < 10 \end{cases}$$

a. Graph the density function.
b. Find the probability that X lies between 1 and 3.
c. What is the probability that X lies between 4 and 8?
d. Compute the probability that X is less than 7.
e. Find the probability that X is greater than 3.

8.14 The following is a graph of a density function.



a. Determine the density function.
b. Find the probability that X is greater than 10.
c. Find the probability that X lies between 6 and 12.

8.2 Normal Distribution

The **normal distribution** is the most important of all probability distributions because of its crucial role in statistical inference.

Normal Density Function

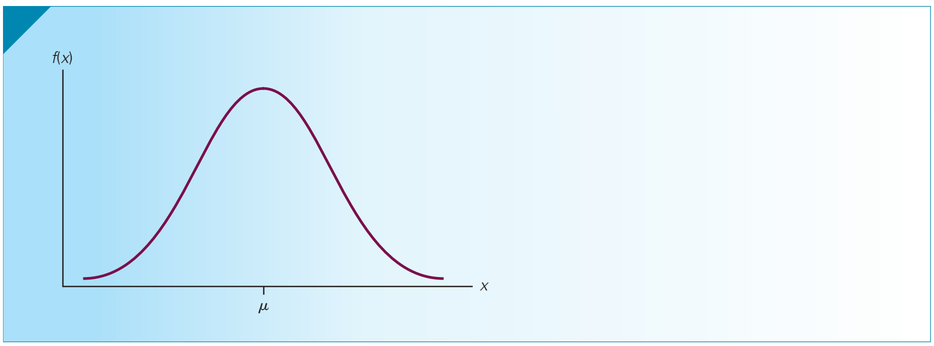
The probability density function of a **normal random variable** is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$

where $e = 2.71828\ldots$ and $\pi = 3.14159\ldots$

Figure 8.7 depicts a normal distribution. Notice that the curve is symmetric about its mean and the random variable ranges between $-\infty$ and $+\infty$.

FIGURE **8.7** Normal Distribution



The normal distribution is described by two parameters, the mean μ and the standard deviation σ. In Figure 8.8, we demonstrate the effect of changing the value of μ. Obviously, increasing μ shifts the curve to the right and decreasing μ shifts it to the left.

FIGURE **8.8** Normal Distributions with the Same Variance but Different Means

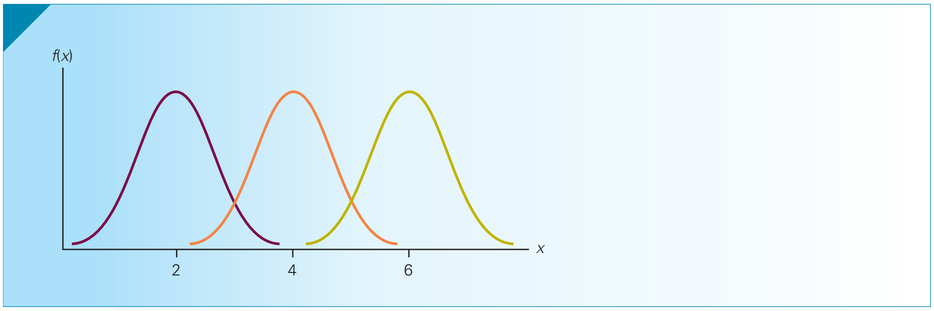
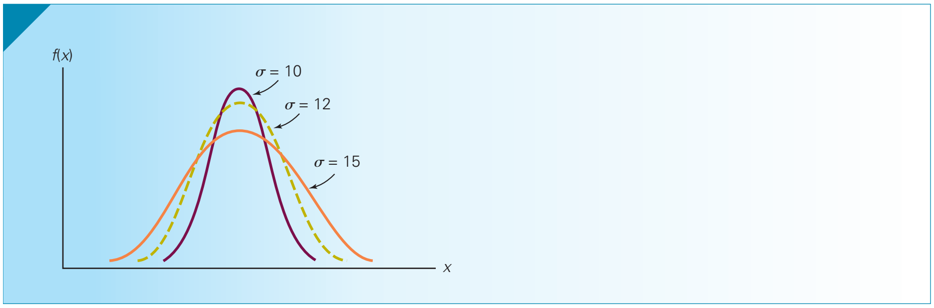


Figure 8.9 describes the effect of σ. Larger values of σ widen the curve and smaller ones narrow it.

FIGURE **8.9** Normal Distributions with the Same Means but Different Variances



SEEING STATISTICS

applet 4 Normal Distribution Parameters

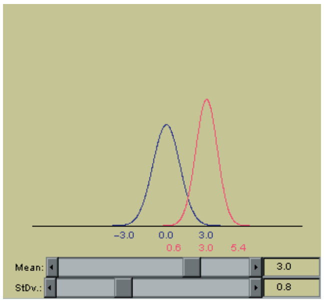
This applet can be used to see the effect of changing the values of the mean and standard deviation of a normal distribution.

Move the top slider left or right to decrease or increase the mean of the distribution. Notice that when you change the value of the mean, the shape stays the same; only the location changes. Move the second slider to change the standard deviation. The shape of the bell curve is changed when you increase or decrease the standard deviation.

Applet Exercises

4.1 Move the slider bar for the standard deviation so that the standard deviation of the red distribution is greater than 1. What does this do to the spread of the normal distribution? Does it squeeze it or stretch it?

4.2 Move the slider bar for the standard deviation so that the standard deviation of the red distribution is less than 1. What does this do to the spread of the normal distribution? Does it squeeze it or stretch it?



(Continued)

4.3 Move both the mean and standard deviation sliders so that the red distribution is different from the blue distribution. What would you need to subtract from the red values to slide the red distribution back (forward) so that the centers of the red and blue distributions would overlap? By what would you need to divide the red values to squeeze or stretch the red distribution so that it would have the same spread as the blue distribution?

Calculating Normal Probabilities

To calculate the probability that a normal random variable falls into any interval, we must compute the area in the interval under the curve. Unfortunately, the function is not as simple as the uniform precluding the use of simple mathematics or even integral calculus. Instead we will resort to using a probability table similar to Tables 1 and 2 in Appendix B, which are used to calculate binomial and Poisson probabilities, respectively. Recall that to determine binomial probabilities from Table 1 we needed probabilities for selected values of n and p. Similarly, to find Poisson probabilities we needed probabilities for each value of μ that we chose to include in Table 2. It would appear then that we will need a separate table for normal probabilities for a selected set of values of μ and σ. Fortunately, this won't be necessary. Instead, we reduce the number of tables needed to one by standardizing the random variable. We standardize a random variable by subtracting its mean and dividing by its standard deviation. When the variable is normal, the transformed variable is called a **standard normal random variable** and denoted by Z; that is,

$$Z = \frac{X - \mu}{\sigma}$$

The probability statement about X is transformed by this formula into a statement about Z. To illustrate how we proceed, consider the following example.

EXAMPLE 8.2 Normally Distributed Gasoline Sales

Suppose that the daily demand for regular gasoline at another gas station is normally distributed with a mean of 1,000 gallons and a standard deviation of 100 gallons. The station manager has just opened the station for business and notes that there is exactly 1,100 gallons of regular gasoline in storage. The next delivery is scheduled later today at the close of business. The manager would like to know the probability that he will have enough regular gasoline to satisfy today's demands.

SOLUTION

The amount of gasoline on hand will be sufficient to satisfy demand if the demand is less than the supply. We label the demand for regular gasoline as X, and we want to find the probability

$$P(X \le 1{,}100)$$

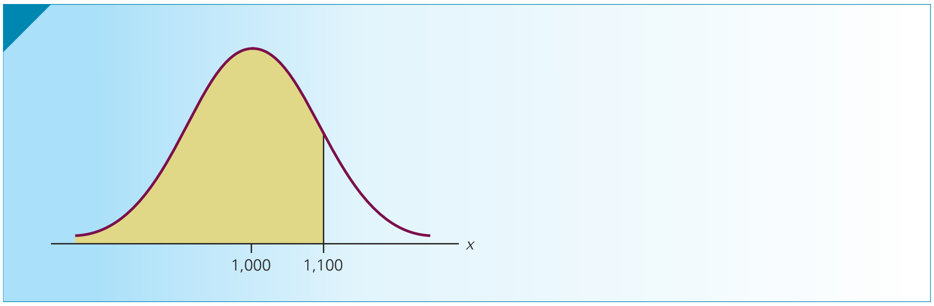
Note that because X is a continuous random variable, we can also express the probability as

$$P(X < 1{,}100)$$

because the area for $X = 1{,}100$ is 0.

Figure 8.10 describes a normal curve with mean of 1,000 and standard deviation of 100, and the area we want to find.

FIGURE **8.10** $P(X < 1{,}100)$

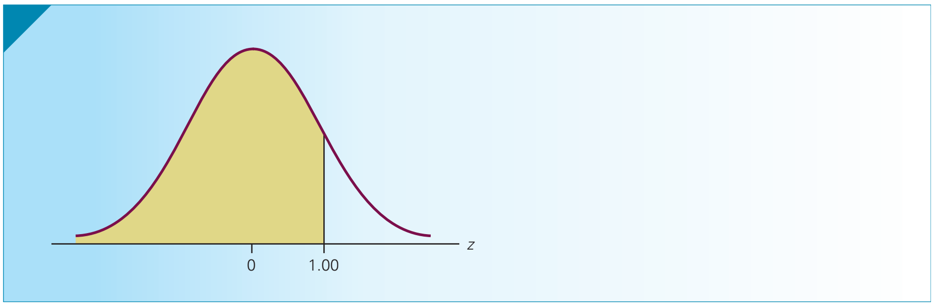


The first step is to standardize X. However, if we perform any operations on X, we must perform the same operations on 1,100. Thus,

$$P(X < 1{,}100) = P\left(\frac{X - \mu}{\sigma} < \frac{1{,}100 - 1{,}000}{100}\right) = P(Z < 1.00)$$

Figure 8.11 describes the transformation that has taken place. Notice that the variable X was transformed into Z, and 1,100 was transformed into 1.00. However, the area has not changed. In other words, the probability that we wish to compute $P(X < 1{,}100)$ is identical to $P(Z < 1.00)$.

FIGURE **8.11** $P(Z < 1.00)$



The values of Z specify the location of the corresponding value of X. A value of $Z = 1$ corresponds to a value of X that is 1 standard deviation above the mean. Notice as well that the mean of Z, which is 0, corresponds to the mean of X.

If we know the mean and standard deviation of a normally distributed random variable, we can always transform the probability statement about X into a probability statement about Z. Consequently, we need only one table, Table 3 in Appendix B, the standard normal probability table, which is reproduced here as Table 8.1.*

*In previous editions we have used another table, which lists $P(0 < Z < z)$. Supplementary Appendix Determining Normal Probabilities using $P(0 < Z < z)$ provides instructions and examples using this table.

TABLE **8.1** Normal Probabilities (Table 3 in Appendix B)

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

This table is similar to the ones we used for the binomial and Poisson distributions; that is, this table lists cumulative probabilities

$$P(Z < z)$$

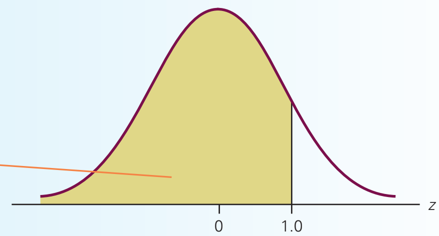
for values of z ranging from -3.09 to $+ 3.09$

To use the table, we simply find the value of z and read the probability. For example, the probability $P(Z < 2.00)$ is found by finding 2.0 in the left margin and under the heading .00 finding .9772. The probability $P(Z < 2.01)$ is found in the same row but under the heading .01. It is .9778.

Returning to Example 8.2, the probability we seek is found in Table 8.1 by finding 1.0 in the left margin. The number to its right under the heading .00 is .8413. See Figure 8.12.

FIGURE **8.12** $P(Z < 1.00)$

z	.00	.01	.02
0.8	.7881	.7910	.7939
0.9	.8159	.8186	.8212
1.0	.8413	.8438	.8461
1.1	.8643	.8665	.8686
1.2	.8849	.8869	.8888



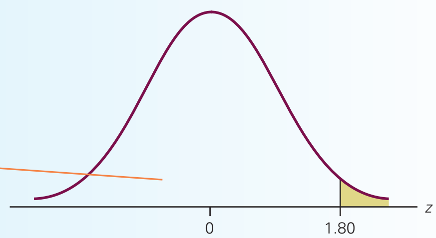
As was the case with Tables 1 and 2, we can also determine the probability that the standard normal random variable is greater than some value of z. For example, we find the probability that Z is greater than 1.80 by determining the probability that Z is less than 1.80 and subtracting that value from 1. Applying the complement rule, we get

$$P(Z > 1.80) = 1 - P(Z < 1.80) = 1 - .9641 = .0359$$

See Figure 8.13.

FIGURE **8.13** $P(Z > 1.80)$

z	.00	.01	.02
1.6	.9452	.9463	.9474
1.7	.9554	.9564	.9573
1.8	.9641	.9649	.9656
1.9	.9713	.9719	.9726
2.0	.9772	.9778	.9783



We can also easily determine the probability that a standard normal random variable lies between two values of z. For example, we find the probability

$$P(-0.71 < Z < 0.92)$$

by finding the two cumulative probabilities and calculating their difference; that is,

$$P(Z < -0.71) = .2389$$

and

$$P(Z < 0.92) = .8212$$

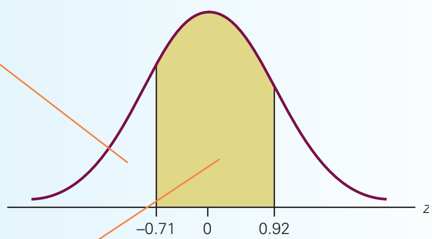
Hence,

$$P(-0.71 < Z < 0.92) = P(Z < 92) - P(Z < -0.71) = .8212 - .2389 = .5823$$

Figure 8.14 depicts this calculation.

FIGURE **8.14** $P(-0.71 < Z < 0.92)$

z	.00	.01	.02
−0.8	.2119	.2090	.2061
−0.7	.2420	.2389	.2358
−0.6	.2743	.2709	.2676
−0.5	.3085	.3050	.3015
−0.4	.3446	.3409	.3372
−0.3	.3821	.3783	.3745
−0.2	.4207	.4168	.4129
−0.1	.4602	.4562	.4522
−0.0	.5000	.4960	.4920
0.0	.5000	.5040	.5080
0.1	.5398	.5438	.5478
0.2	.5793	.5832	.5871
0.3	.6179	.6217	.6255
0.4	.6554	.6591	.6628
0.5	.6915	.6950	.6985
0.6	.7257	.7291	.7324
0.7	.7580	.7611	.7642
0.8	.7881	.7910	.7939
0.9	.8159	.8186	.8212
1.0	.8413	.8438	.8461



Notice that the largest value of z in the table is 3.09, and that $P(Z < 3.09) = .9990$. This means that

$$P(Z > 3.09) = 1 - .9990 = .0010$$

However, because the table lists no values beyond 3.09, we approximate any area beyond 3.10 as 0. In other words,

$$P(Z > 3.10) = P(Z < -3.10) \approx 0$$

Recall that in Tables 1 and 2 we were able to use the table to find the probability that X is *equal* to some value of x, but we won't do the same with the normal table. Remember that the normal random variable is continuous and the probability that a continuous random variable is equal to any single value is 0.

SEEING STATISTICS

applet 5 Normal Distribution Areas

This applet can be used to show the calculation of the probability of any interval for any values of μ and σ. Click or drag anywhere in the graph to move the nearest end to that point. Adjust the ends to correspond to either z-scores or actual scores. The area under the normal curve between the two endpoints is highlighted in red. The size of this area corresponds to the probability of obtaining a score between the two endpoints. You can change the mean and standard deviation of the actual scores by changing the numbers in the text boxes. After changing a number, press the **Enter** or **Return** key to update the graph. When this page first loads, the mean and standard deviation correspond to a mean of 50 and a standard deviation of 10.

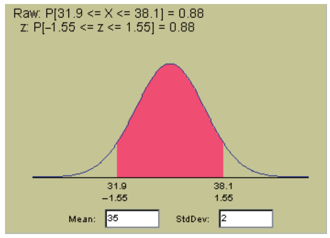
Applet Exercises

The graph is initially set with mean 50 and standard deviation 10. Change it so that it represents the distribution of IQs, which are normally distributed with a mean of 100 and a standard deviation of 16.

5.1 About what proportion of people have IQ scores equal to or less than 116?

5.2 About what proportion of people have IQ scores between 100 and 116?

5.3 About what proportion have IQ scores greater than 120?

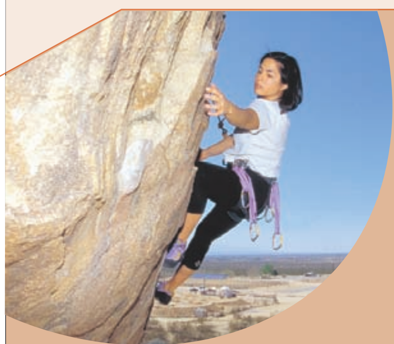


5.4 About what proportion of the scores are within one standard deviation of the mean?

5.5 About what proportion of the scores are within two standard deviations of the mean?

5.6 About what proportion of the scores are within three standard deviations of the mean?

APPLICATIONS in FINANCE



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Measuring Risk

In previous chapters, we discussed several probability and statistical applications in finance where we wanted to measure and perhaps reduce the risk associated with investments. In Example 3.2, we drew histograms to gauge the spread of the histogram of the returns on two investments. We repeated this example in Chapter 4, where we computed the standard deviation and variance as numerical measures of risk. In Section 7.3, we developed an important application in finance in which we emphasized reducing the variance of the returns on a portfolio. However, we have not demonstrated why risk is measured by the variance and standard deviation. The following example corrects this deficiency.

EXAMPLE 8.3 Probability of a Negative Return on Investment

Consider an investment whose return is normally distributed with a mean of 10% and a standard deviation of 5%.

a. Determine the probability of losing money.
b. Find the probability of losing money when the standard deviation is equal to 10%.

SOLUTION

a. The investment loses money when the return is negative. Thus, we wish to determine

$$P(X < 0)$$

The first step is to standardize both X and 0 in the probability statement:

$$P(X < 0) = P\left(\frac{X - \mu}{\sigma} < \frac{0 - 10}{5}\right) = P(Z < -2.00) = .0228$$

Therefore, the probability of losing money is .0228.

b. If we increase the standard deviation to 10%, the probability of suffering a loss becomes

$$P(X < 0) = P\left(\frac{X - \mu}{\sigma} < \frac{0 - 10}{10}\right) = P(Z < -1.00) = .1587$$

As you can see, increasing the standard deviation increases the probability of losing money. Note that increasing the standard deviation will also increase the probability that the return will exceed some relatively large amount. However, because investors tend to be risk averse, we emphasize the increased probability of negative returns when discussing the effect of increasing the standard deviation.

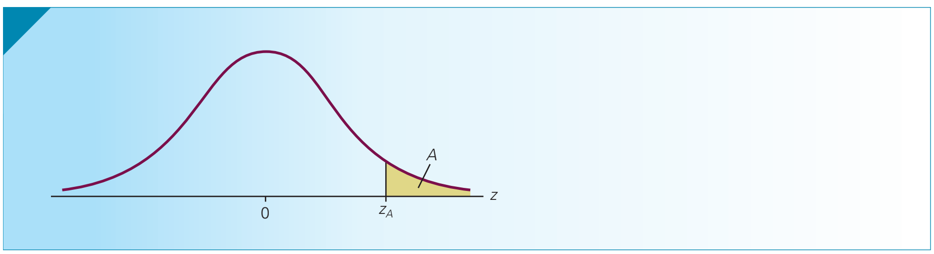
Finding Values of Z

There is a family of problems that require us to determine the value of Z given a probability. We use the notation Z_A to represent the value of z such that the area to its right under the standard normal curve is A; that is, Z_A is a value of a standard normal random variable such that

$$P(Z > Z_A) = A$$

Figure 8.15 depicts this notation.

FIGURE **8.15** $P(Z > Z_A) = A$

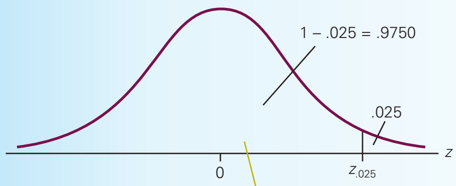


To find Z_A for any value of A requires us to use the standard normal table backward. As you saw in Example 8.2, to find a probability about Z, we must find the value of z in the table and determine the probability associated with it. To use the table backward, we need to specify a probability and then determine the z-value associated with it.

We'll demonstrate by finding $Z_{.025}$. Figure 8.16 depicts the standard normal curve and $Z_{.025}$. Because of the format of the standard normal table, we begin by determining the area *less than* $Z_{.025}$, which is $1 - .025 = .9750$. (Notice that we expressed this probability with four decimal places to make it easier for you to see what you need to do.) We now search through the probability part of the table looking for .9750. When we locate it, we see that the z-value associated with it is 1.96.

Thus, $Z_{.025} = 1.96$, which means that $P(Z > 1.96) = .025$.

FIGURE **8.16** $Z_{.025}$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990

EXAMPLE 8.4

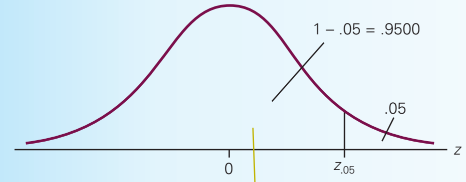
Finding $Z_{.05}$

Find the value of a standard normal random variable such that the probability that the random variable is greater than it is 5%.

SOLUTION

We wish to determine $Z_{.05}$. Figure 8.17 depicts the normal curve and $Z_{.05}$. If .05 is the area in the tail, then the probability less than $Z_{.05}$ must be $1 - .05 = .9500$. To find $Z_{.05}$ we search the table looking for the probability .9500. We don't find this probability, but we find two values that are equally close: .9495 and .9505. The Z-values associated with these probabilities are 1.64 and 1.65, respectively. The average is taken as $Z_{.05}$. Thus, $Z_{.05} = 1.645$.

FIGURE **8.17** $Z_{.05}$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990

EXAMPLE 8.5

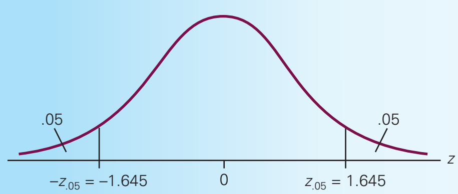
Finding $-Z_{.05}$

Find the value of a standard normal random variable such that the probability that the random variable is less than it is 5%.

SOLUTION

Because the standard normal curve is symmetric about 0, we wish to find $-Z_{.05}$. In Example 8.4 we found $Z_{.05} = 1.645$. Thus, $-Z_{.05} = -1.645$. See Figure 8.18.

FIGURE **8.18** $-Z_{.05}$



Minimum GMAT Score to Enter Executive MBA Program: Solution

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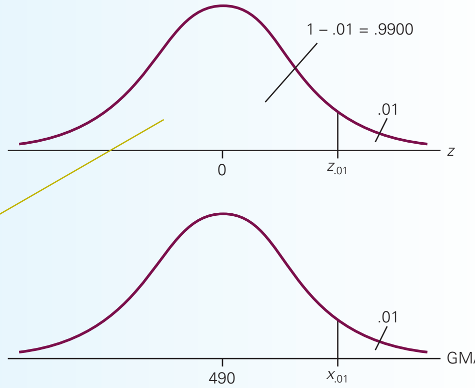
Figure 8.19 depicts the distribution of GMAT scores. We've labeled the minimum score needed to enter the new MBA program $X_{.01}$ such that

$$P(X > X_{.01}) = .01$$

FIGURE 8.19 Minimum GMAT Score

z	.00	.01	.02	.03	.04
1.0	.8413	.8438	.8461	.8485	.8508
1.1	.8643	.8665	.8686	.8708	.8729
1.2	.8849	.8869	.8888	.8907	.8925
1.3	.9032	.9049	.9066	.9082	.9099
1.4	.9192	.9207	.9222	.9236	.9251
1.5	.9332	.9345	.9357	.9370	.9382
1.6	.9452	.9463	.9474	.9484	.9495
1.7	.9554	.9564	.9573	.9582	.9591
1.8	.9641	.9649	.9656	.9664	.9671
1.9	.9713	.9719	.9726	.9732	.9738
2.0	.9772	.9778	.9783	.9788	.9793
2.1	.9821	.9826	.9830	.9834	.9838
2.2	.9861	.9864	.9868	.9871	.9875
2.3	.9893	.9896	.9898	.9901	.9904
2.4	.9918	.9920	.9922	.9925	.9927
2.5	.9938	.9940	.9941	.9943	.9945
2.6	.9953	.9955	.9956	.9957	.9959
2.7	.9965	.9966	.9967	.9968	.9969
2.8	.9974	.9975	.9976	.9977	.9977
2.9	.9981	.9982	.9982	.9983	.9984
3.0	.9987	.9987	.9987	.9988	.9988



Above the normal curve, we depict the standard normal curve and $Z_{.01}$. We can determine the value of $Z_{.01}$ as we did in Example 8.4. In the standard normal table, we find $1 - .01 = .9900$ (its closest value in the table is .9901) and the Z-value 2.33. Thus, the standardized value of $X_{.01}$ is $Z_{.01} = 2.33$. To find $X_{.01}$, we must unstandardize $Z_{.01}$. We do so by solving for $X_{.01}$ in the equation

$$Z_{.01} = \frac{X_{.01} - \mu}{\sigma}$$

Substituting $Z_{.01} = 2.33$, $\mu = 490$, and $\sigma = 61$, we find

$$2.33 = \frac{X_{.01} - 490}{61}$$

Solving, we get

$$X_{.01} = 2.33(61) + 490 = 632.13$$

Rounding up (GMAT scores are integers), we find that the minimum GMAT score to enter the Executive MBA Program is 633.

Z_A and Percentiles

In Chapter 4, we introduced percentiles, which are measures of relative standing. The values of Z_A are the $100(1 - A)$th percentiles of a standard normal random variable. For example, $Z_{.05} = 1.645$, which means that 1.645 is the 95th percentile: 95% of all values of Z are below it, and 5% are above it. We interpret other values of Z_A similarly.

Using the Computer

EXCEL

INSTRUCTIONS

We can use Excel to compute probabilities as well as values of X and Z. To compute cumulative normal probabilities $P(X < x)$, type (in any cell)

= **NORMDIST**($[X]$, $[\mu]$, $[\sigma]$,True)

(Typing "True" yields a cumulative probability. Typing "False" will produce the value of the normal density function, a number with little meaning.)

If you type 0 for μ and 1 for σ, you will obtain standard normal probabilities. Alternatively, type

NORMSDIST instead of NORMDIST and enter the value of z.

In Example 8.2 we found $P(X < 1{,}100) = P(Z < 1.00) = .8413$. To instruct Excel to calculate this probability, we enter

= NORMDIST(1100, 1000, 100, True)

or

= NORMSDIST(1.00)

To calculate a value for Z_A, type

= **NORMSINV**($[1 - A]$)

In Example 8.4, we would type

= **NORMSINV**(.95)

and produce 1.6449. We calculated $Z_{.05} = 1.645$.

To calculate a value of x given the probability $P(X > x) = A$, enter

= **NORMINV**($1 - A$, μ, σ)

The chapter-opening example would be solved by typing

= **NORMINV**(.99, 490, 61)

which yields 632.

MINITAB

INSTRUCTIONS

We can use Minitab to compute probabilities as well as values of X and Z.

Check **Calc**, **Probability Distributions**, and **Normal . . .** and either **Cumulative probability** [to determine $P(X < x)$] or **Inverse cumulative probability** to find the value of x. Specify the **Mean** and **Standard deviation**.

APPLICATIONS in OPERATIONS MANAGEMENT

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Inventory Management

Every organization maintains some inventory, which is defined as a stock of items. For example, grocery stores hold inventories of almost all the products they sell. When the total number of products drops to a specified level, the manager arranges for the delivery of more products. An automobile repair shop keeps an inventory of a large number of replacement parts. A school keeps stock of items that it uses regularly, including chalk, pens, envelopes, file folders, and paper clips. There are costs associated with inventories.

These include the cost of capital, losses (theft and obsolescence), and warehouse space, as well as maintenance and record keeping. Management scientists have developed many models to help determine the optimum inventory level that balances the cost of inventory with the cost of shortages and the cost of making many small orders. Several of these models are deterministic—that is, they assume that the demand for the product is constant. However, in most realistic situations, the demand is a random variable. One commonly applied probabilistic model assumes that the demand during lead time is a normally distributed random variable. *Lead time* is defined as the amount of time between when the order is placed and when it is delivered.

The quantity ordered is usually calculated by attempting to minimize the total costs, including the cost of ordering and the cost of maintaining inventory. (This topic is discussed in most management-science courses.) Another critical decision involves the *reorder point*, which is the level of inventory at which an order is issued to its supplier. If the reorder point is too low, the company will run out of product, suffering the loss of sales and potentially customers who will go to a competitor. If the reorder point is too high, the company will be carrying too much inventory, which costs money to buy and store. In some companies, inventory has a tendency to walk out the back door or become obsolete. As a result, managers create a *safety stock*, which is the extra amount of inventory to reduce the times when the company has a shortage. They do so by setting a service level, which is the probability that the company will not experience a shortage. The method used to determine the reorder point is be demonstrated with Example 8.6.

EXAMPLE 8.6 Determining the Reorder Point

During the spring, the demand for electric fans at a large home-improvement store is quite strong. The company tracks inventory using a computer system so that it knows how many fans are in the inventory at any time. The policy is to order a new shipment of 250 fans when the inventory level falls to the reorder point, which is 150. However, this policy has resulted in frequent shortages and thus lost sales because both lead time and demand are highly variable. The manager would like to reduce the incidence of shortages so that only 5% of orders will arrive after inventory drops to 0 (resulting in a shortage). This policy is expressed as a 95% service level. From previous periods, the company has determined that demand during lead time is normally distributed with a mean of 200 and a standard deviation of 50. Find the reorder point.

SOLUTION

The reorder point is set so that the probability that demand during lead time exceeds this quantity is 5%. Figure 8.20 depicts demand during lead time and the reorder point. As we did in the solution to the chapter-opening example, we find the standard normal value such that the area to its right is .05. The standardized value of the reorder point is $Z_{.05} = 1.645$. To find the reorder point (ROP), we must unstandardize $Z_{.05}$.

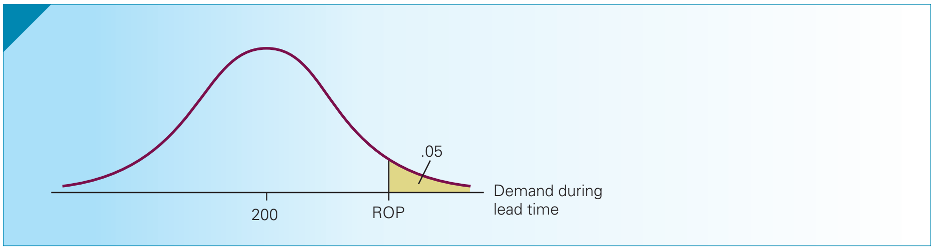
$$Z_{.05} = \frac{\text{ROP} - \mu}{\sigma}$$

$$1.645 = \frac{\text{ROP} - 200}{50}$$

$$\text{ROP} = 50(1.645) + 200 = 282.25$$

which we round up to 283. The policy is to order a new batch of fans when there are 283 fans left in inventory.

FIGURE 8.20 Distribution of Demand During Lead Time



EXERCISES

In Exercises 8.15 to 8.30, find the probabilities.

8.15 $P(Z < 1.50)$

8.16 $P(Z < 1.51)$

8.17 $P(Z < 1.55)$

8.18 $P(Z < -1.59)$

8.19 $P(Z < -1.60)$

8.20 $P(Z < -2.30)$

8.21 $P(-1.40 < Z < .60)$

8.22 $P(Z > -1.44)$

8.23 $P(Z < 2.03)$

8.24 $P(Z > 1.67)$

8.25 $P(Z < 2.84)$

8.26 $P(1.14 < Z < 2.43)$

8.27 $P(-0.91 < Z < -0.33)$

8.28 $P(Z > 3.09)$

8.29 $P(Z > 0)$

8.30 $P(Z > 4.0)$

8.31 Find $z_{.02}$.

8.32 Find $z_{.045}$.

8.33 Find $z_{.20}$.

8.34 X is normally distributed with mean 100 and standard deviation 20. What is the probability that X is greater than 145?

8.35 X is normally distributed with mean 250 and standard deviation 40. What value of X does only the top 15% exceed?

8.36 X is normally distributed with mean 1,000 and standard deviation 250. What is the probability that X lies between 800 and 1,100?

8.37 X is normally distributed with mean 50 and standard deviation 8. What value of X is such that only 8% of values are below it?

8.38 The long-distance calls made by the employees of a company are normally distributed with a mean of 6.3 minutes and a standard deviation of 2.2 minutes. Find the probability that a call
a. lasts between 5 and 10 minutes.
b. lasts more than 7 minutes.
c. lasts less than 4 minutes.

8.39 Refer to Exercise 8.38. How long do the longest 10% of calls last?

8.40 The lifetimes of lightbulbs that are advertised to last for 5,000 hours are normally distributed with a mean of 5,100 hours and a standard deviation of 200 hours. What is the probability that a bulb lasts longer than the advertised figure?

8.41 Refer to Exercise 8.40. If we wanted to be sure that 98% of all bulbs last longer than the advertised figure, what figure should be advertised?

8.42 Travelbyus is an Internet-based travel agency wherein customers can see videos of the cities they plan to visit. The number of hits daily is a normally distributed random variable with a mean of 10,000 and a standard deviation of 2,400.
a. What is the probability of getting more than 12,000 hits?
b. What is the probability of getting fewer than 9,000 hits?

8.43 Refer to Exercise 8.42. Some Internet sites have bandwidths that are not sufficient to handle all their traffic, often causing their systems to crash. Bandwidth can be measured by the number of hits a system can handle. How large a bandwidth should Travelbyus have in order to handle 99.9% of daily traffic?

8.44 A new gas–electric hybrid car has recently hit the market. The distance traveled on 1 gallon of fuel is normally distributed with a mean of 65 miles and a standard deviation of 4 miles. Find the probability of the following events.
a. The car travels more than 70 miles per gallon.
b. The car travels less than 60 miles per gallon.
c. The car travels between 55 and 70 miles per gallon.

8.45 The top-selling Red and Voss tire is rated 70,000 miles, which means nothing. In fact, the distance the tires can run until they wear out is a normally distributed random variable with a mean of 82,000 miles and a standard deviation of 6,400 miles.
a. What is the probability that a tire wears out before 70,000 miles?
b. What is the probability that a tire lasts more than 100,000 miles?

8.46 The heights of children 2 years old are normally distributed with a mean of 32 inches and a standard deviation of 1.5 inches. Pediatricians regularly measure the heights of toddlers to determine whether there is a problem. There may be a problem when a child is in the top or bottom 5% of heights. Determine the heights of 2-year-old children that could be a problem.

8.47 Refer to Exercise 8.46. Find the probability of these events.
a. A 2-year-old child is taller than 36 inches.
b. A 2-year-old child is shorter than 34 inches.
c. A 2-year-old child is between 30 and 33 inches tall.

8.48 University and college students average 7.2 hours of sleep per night, with a standard deviation of 40 minutes. If the amount of sleep is normally distributed, what proportion of university and college students sleep for more than 8 hours?

8.49 Refer to Exercise 8.48. Find the amount of sleep that is exceeded by only 25% of students.

8.50 The amount of time devoted to studying statistics each week by students who achieve a grade of A in the course is a normally distributed random variable with a mean of 7.5 hours and a standard deviation of 2.1 hours.
a. What proportion of A students study for more than 10 hours per week?
b. Find the probability that an A student spends between 7 and 9 hours studying.
c. What proportion of A students spend fewer than 3 hours studying?
d. What is the amount of time below which only 5% of all A students spend studying?

8.51 The number of pages printed before replacing the cartridge in a laser printer is normally distributed with a mean of 11,500 pages and a standard deviation of 800 pages. A new cartridge has just been installed.
a. What is the probability that the printer produces more than 12,000 pages before this cartridge must be replaced?
b. What is the probability that the printer produces fewer than 10,000 pages?

8.52 Refer to Exercise 8.51. The manufacturer wants to provide guidelines to potential customers advising them of the minimum number of pages they can expect from each cartridge. How many pages should it advertise if the company wants to be correct 99% of the time?

8.53 Battery manufacturers compete on the basis of the amount of time their products last in cameras and toys. A manufacturer of alkaline batteries has observed that its batteries last for an average of 26 hours when used in a toy racing car. The amount of time is normally distributed with a standard deviation of 2.5 hours.

a. What is the probability that the battery lasts between 24 and 28 hours?
b. What is the probability that the battery lasts longer than 28 hours?
c. What is the probability that the battery lasts less than 24 hours?

8.54 Because of the relatively high interest rates, most consumers attempt to pay off their credit card bills promptly. However, this is not always possible. An analysis of the amount of interest paid monthly by a bank's Visa cardholders reveals that the amount is normally distributed with a mean of \$27 and a standard deviation of \$7.
a. What proportion of the bank's Visa cardholders pay more than \$30 in interest?
b. What proportion of the bank's Visa cardholders pay more than \$40 in interest?
c. What proportion of the bank's Visa cardholders pay less than \$15 in interest?
d. What interest payment is exceeded by only 20% of the bank's Visa cardholders?

8.55 It is said that sufferers of a cold virus experience symptoms for 7 days. However, the amount of time is actually a normally distributed random variable whose mean is 7.5 days and whose standard deviation is 1.2 days.
a. What proportion of cold sufferers experience fewer than 4 days of symptoms?
b. What proportion of cold sufferers experience symptoms for between 7 and 10 days?

8.56 How much money does a typical family of four spend at a McDonald's restaurant per visit? The amount is a normally distributed random variable with a mean of \$16.40 and a standard deviation of \$2.75.
a. Find the probability that a family of four spends less than \$10.
b. What is the amount below which only 10% of families of four spend at McDonald's?

8.57 The final marks in a statistics course are normally distributed with a mean of 70 and a standard deviation of 10. The professor must convert all marks to letter grades. She decides that she wants 10% A's, 30% B's, 40% C's, 15% D's, and 5% F's. Determine the cutoffs for each letter grade.

8.58 Mensa is an organization whose members possess IQs that are in the top 2% of the population. It is known that IQs are normally distributed with a mean of 100 and a standard deviation of 16. Find the minimum IQ needed to be a Mensa member.

8.59 According to the 2001 Canadian census, university-educated Canadians earned a mean income of \$61,823. The standard deviation is \$17,301. If incomes are normally distributed, what is the probability that a randomly selected university-educated Canadian earns more than \$70,000?

8.60 The census referred to in the previous exercise also reported that college-educated Canadians earn on average \$41,825. Suppose that incomes are normally distributed with a standard deviation of \$13,444. Find the probability that a randomly selected college-educated Canadian earns less than \$45,000.

8.61 The lifetimes of televisions produced by the Hishobi Company are normally distributed with a mean of 75 months and a standard deviation of 8 months. If the manufacturer wants to have to replace only 1% of its televisions, what should its warranty be?

8.62 According to the *Statistical Abstract of the United States, 2000* (Table 764), the mean family net worth of families whose head is between 35 and 44 years old is approximately \$99,700. If family net worth is normally distributed with a standard deviation of \$30,000, find the probability that a randomly selected family whose head is between 35 and 44 years old has a net worth greater than \$150,000.

8.63 A retailer of computing products sells a variety of computer-related products. One of his most popular products is an HP laser printer. The average weekly demand is 200. Lead time for a new order from the manufacturer to arrive is 1 week. If the demand for printers were constant, the retailer would reorder when there were exactly 200 printers in inventory. However, the demand is a random variable. An analysis of previous weeks reveals that the weekly demand standard deviation is 30. The retailer knows that if a customer wants to buy an HP laser printer but he has none available, he will lose that sale plus possibly additional sales. He wants the probability of running short in any week to be no more than 6%. How many HP laser printers should he have in stock when he reorders from the manufacturer?

8.64 The demand for a daily newspaper at a newsstand at a busy intersection is known to be normally distributed with a mean of 150 and a standard deviation of 25. How many newspapers should the newsstand operator order to ensure that he runs short on no more than 20% of days?

8.65 Every day a bakery prepares its famous marble rye. A statistically savvy customer determined that daily demand is normally distributed with a mean of 850 and a standard deviation of 90. How many loaves should the bakery make if it wants the probability of running short on any day to be no more than 30%?

8.66 Refer to Exercise 8.65. Any marble ryes that are unsold at the end of the day are marked down and sold for half-price. How many loaves should the bakery prepare so that the proportion of days that result in unsold loaves is no more than 60%?

APPLICATIONS in OPERATIONS MANAGEMENT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PERT/CPM

In the Applications in Operations Management box on page 235, we introduced PERT/CPM. The purpose of this powerful management-science procedure is to determine the critical path of a project. The expected value and variance of the completion time of the project are based on the expected values and variances of the completion times of the activities on the critical path. Once we have the expected value and variance of the completion time of the project, we can use these figures to determine the probability that the project will be completed by a certain date. Statisticians have established that the completion time of the project is approximately normally distributed, enabling us to compute the needed probabilities.

8.67 Refer to Exercise 7.57. Find the probability that the project will take more than 60 days to complete.

8.68 The mean and variance of the time to complete the project in Exercise 7.58 was 145 minutes and 31 minutes^2. What is the probability that it will take less than 2.5 hours to overhaul the machine?

8.69 The annual rate of return on a mutual fund is normally distributed with a mean of 14% and a standard deviation of 18%.
a. What is the probability that the fund returns more than 25% next year?
b. What is the probability that the fund loses money next year?

8.70 In Exercise 7.64, we discovered that the expected return is .1060 and the standard deviation is .1456. Working with the assumption that returns are normally distributed, determine the probability of the following events.
a. The portfolio loses money.
b. The return on the portfolio is greater than 20%.

8.3 (Optional) Exponential Distribution

Another important continuous distribution is the **exponential distribution**.

Exponential Probability Density Function

A random variable X is exponentially distributed if its probability density function is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

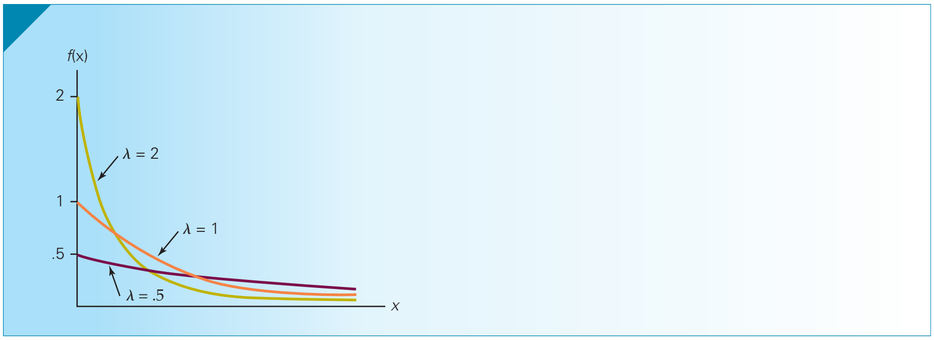
where $e = 2.71828\ldots$ and λ is the parameter of the distribution.

Statisticians have shown that the mean and standard deviation of an exponential random variable are equal to each other:

$$\mu = \sigma = 1/\lambda$$

Recall that the normal distribution is a two-parameter distribution. The distribution is completely specified once the values of the two parameters μ and σ are known. In contrast, the exponential distribution is a one-parameter distribution. The distribution is completely specified once the value of the parameter λ is known. Figure 8.21 depicts three exponential distributions, corresponding to three different values of the parameter λ. Notice that for any exponential density function $f(x)$, $f(0) = \lambda$ and $f(x)$ approaches 0 as x approaches infinity.

FIGURE **8.21** Exponential Distributions



The exponential density function is easier to work with than the normal. As a result, we can develop formulas for the calculation of the probability of any range of values. Using integral calculus, we can determine the following probability statements.

Probability Associated with an Exponential Random Variable

If X is an exponential random variable,

$$P(X > x) = e^{-\lambda x}$$

$$P(X < x) = 1 - e^{-\lambda x}$$

$$P(x_1 < X < x_2) = P(X < x_2) - P(X < x_1) = e^{-\lambda x_1} - e^{-\lambda x_2}$$

The value of $e^{-\lambda x}$ can be obtained with the aid of a calculator.

EXAMPLE 8.7

Lifetimes of Alkaline Batteries

The lifetime of an alkaline battery (measured in hours) is exponentially distributed with $\lambda = .05$.

a. What is the mean and standard deviation of the battery's lifetime?

b. Find the probability that a battery will last between 10 and 15 hours.

c. What is the probability that a battery will last for more than 20 hours?

SOLUTION

a. The mean and standard deviation are equal to $1/\lambda$. Thus,

$$\mu = \sigma = 1/\lambda = 1/.05 = 20 \text{ hours}$$

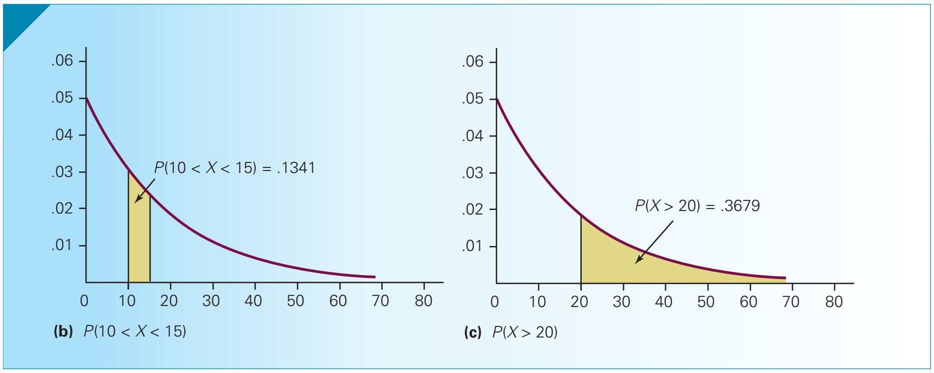
b. Let X denote the lifetime of a battery. The required probability is

$$\begin{aligned} P(10 < X < 15) &= e^{-.05(10)} - e^{-.05(15)} \\ &= e^{-.5} - e^{-.75} \\ &= .6065 - .4724 \\ &= .1341 \end{aligned}$$

c. $$\begin{aligned} P(X > 20) &= e^{-.05(20)} \\ &= e^{-1} \\ &= .3679 \end{aligned}$$

Figure 8.22 depicts these probabilities.

FIGURE 8.22 Probabilities for Example 8.7



Using the Computer

EXCEL

INSTRUCTIONS

Type (in any cell)

= **EXPONDIST** ([X],[λ], True)

To produce the answer for Example 8.7c we would find $P(X < 20)$ and subtract it from 1.

To find $P(X < 20)$, type

= **EXPONDIST**(20, .05, True)

which outputs .6321 and hence $P(X > 20) = 1 - .6321 = .3679$, which is exactly the number we produced manually.

MINITAB

INSTRUCTIONS

Click **Calc**, **Probability Distributions**, and **Exponential . . .** and specify **Cumulative probability**. In the **Scale** box, type the mean, which is $1/\lambda$. In the **Threshold** box, type 0.

APPLICATIONS in OPERATIONS MANAGEMENT

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Waiting Lines

In Section 7.5, we described waiting-line models and how the Poisson distribution is used to calculate the probabilities of the number of arrivals per time period. To calculate the operating characteristics of waiting lines, management scientists often assume that the times to complete a service are exponentially distributed. In this application, the parameter λ is the service rate, which is defined as the mean number of service completions per time period. For example, if service times are exponentially distributed with $\lambda = 5$/hour, this tells us that the service rate is 5 units per hour or 5 per 60 minutes. Recall that the mean of an exponential distribution is $\mu = 1/\lambda$. In this case, the service facility can complete a service in an average of 12 minutes. This was calculated as

$$\mu = \frac{1}{\lambda} = \frac{1}{5/\text{hr,}} = \frac{1}{5/60 \text{ minutes}} = \frac{60 \text{ minutes}}{5} = 12 \text{ minutes.}$$

We can use this distribution to make a variety of probability statements.

EXAMPLE 8.8 Supermarket Checkout Counter

A checkout counter at a supermarket completes the process according to an exponential distribution with a service rate of 6 per hour. A customer arrives at the checkout counter. Find the probability of the following events.

a. The service is completed in fewer than 5 minutes

b. The customer leaves the checkout counter more than 10 minutes after arriving

c. The service is completed in a time between 5 and 8 minutes

SOLUTION

One way to solve this problem is to convert the service rate so that the time period is 1 minute. (Alternatively, we can solve by converting the probability statements so that the time periods are measured in fractions of an hour.) Let the service rate $= \lambda = .1$/minute.

a. $P(X < 5) = 1 - e^{-\lambda x} = 1 - e^{-.1(5)} = 1 - e^{-.5} = 1 - .6065 = .3935$

b. $P(X > 10) = e^{-\lambda x} = e^{-.1(10)} = e^{-1} = .3679$

c. $P(5 < X < 8) = e^{-.1(5)} - e^{-.1(8)} = e^{-.5} - e^{-.8} = .6065 - .4493 = .1572$

EXERCISES

8.71 The random variable X is exponentially distributed with $\lambda = 3$. Sketch the graph of the distribution of X by plotting and connecting the points representing $f(x)$ for $x = 0, .5, 1, 1.5$, and 2.

8.72 X is an exponential random variable with $\lambda = .25$. Sketch the graph of the distribution of X by plotting and connecting the points representing $f(x)$ for $x = 0, 2, 4, 6, 8, 10, 15, 20$.

8.73 Let X be an exponential random variable with $\lambda = .5$. Find the following probabilities.
a. $P(X > 1)$
b. $P(X > .4)$
c. $P(X < .5)$
d. $P(X < 2)$

8.74 X is an exponential random variable with $\lambda = .3$. Find the following probabilities.
a. $P(X > 2)$
b. $P(X < 4)$
c. $P(1 < X < 2)$
d. $P(X = 3)$

8.75 The production of a complex chemical needed for anticancer drugs is exponentially distributed with $\lambda = 6$ kilograms per hour. What is the probability that the production process requires more than 15 minutes to produce the next kilogram of drugs?

8.76 The time between breakdowns of aging machines is known to be exponentially distributed with a mean of 25 hours. The machine has just been repaired. Determine the probability that the next breakdown occurs more than 50 hours from now.

8.77 When trucks arrive at the Ambassador Bridge, each truck must be checked by customs agents. The times are exponentially distributed with a service rate of 10 per hour. What is the probability that a truck requires more than 15 minutes to be checked?

8.78 A bank wishing to increase its customer base advertises that it has the fastest service and that virtually all of its customers are served in less than 10 minutes. A management scientist has studied the service times and concluded that service times are exponentially distributed with a mean of 5 minutes. Determine what the bank means when it claims "virtually all" its customers are served in under 10 minutes.

8.79 Toll booths on the New York State Thruway are often congested because of the large number of cars waiting to pay. A consultant working for the state concluded that if service times are measured from the time a car stops in line until it leaves, service times are exponentially distributed with a mean of 2.7 minutes. What proportion of cars can get through the toll booth in less than 3 minutes?

8.80 The manager of a gas station has observed that the times required by drivers to fill their car's tank and pay are quite variable. In fact, the times are exponentially distributed with a mean of 7.5 minutes. What is the probability that a car can complete the transaction in less than 5 minutes?

8.81 Because automatic banking machine (ABM) customers can perform a number of transactions, the times to complete them can be quite variable. A banking consultant has noted that the times are exponentially distributed with a mean of 125 seconds. What proportion of the ABM customers take more than 3 minutes to do their banking?

8.82 The manager of a supermarket tracked the amount of time needed for customers to be served by the cashier. After checking with his statistics professor, he concluded that the checkout times are exponentially distributed with a mean of 6 minutes. What proportion of customers require more than 10 minutes to check out?

8.4 OTHER CONTINUOUS DISTRIBUTIONS

In this section, we introduce three more continuous distributions that are used extensively in statistical inference.

Student *t* Distribution

The Student t distribution was first derived by William S. Gosset in 1908. (Gosset published his findings under the pseudonym "Student" and used the letter t to represent the random variable, hence the **Student *t* distribution**—also called the *Student's t* *distribution*.) It is very commonly used in statistical inference, and we will employ it in Chapters 12, 13, 14, 16, 17, and 18.

Student t Density Function

The density function of the Student t distribution is as follows:

$$f(t) = \frac{\Gamma[(\nu + 1)/2]}{\sqrt{\nu\pi}\Gamma(\nu/2)}\left[1 + \frac{t^2}{\nu}\right]^{-(\nu+1)/2}$$

where ν (Greek letter *nu*) is the parameter of the Student t distribution called the **degrees of freedom**, $\pi = 3.14159$ (approximately), and Γ is the gamma function (its definition is not needed here).

The mean and variance of a Student t random variable are

$$E(t) = 0$$

and

$$V(t) = \frac{\nu}{\nu - 2} \quad \text{for } \nu > 2$$

Figure 8.23 depicts the Student t distribution. As you can see, it is similar to the standard normal distribution. Both are symmetrical about 0. (Both random variables have a mean of 0.) We describe the Student t distribution as mound shaped, whereas the normal distribution is bell shaped.

FIGURE **8.23** Student t Distribution

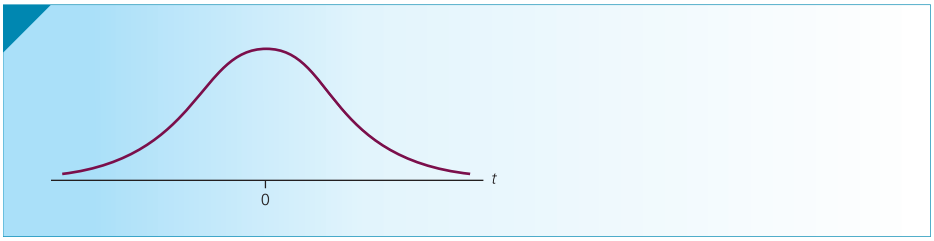


Figure 8.24 shows both a Student t and the standard normal distributions. The former is more widely spread out than the latter. [The variance of a standard normal random variable is 1, whereas the variance of a Student t random variable is $\nu/(\nu - 2)$, which is greater than 1 for all ν.]

FIGURE **8.24** Student t and Normal Distributions

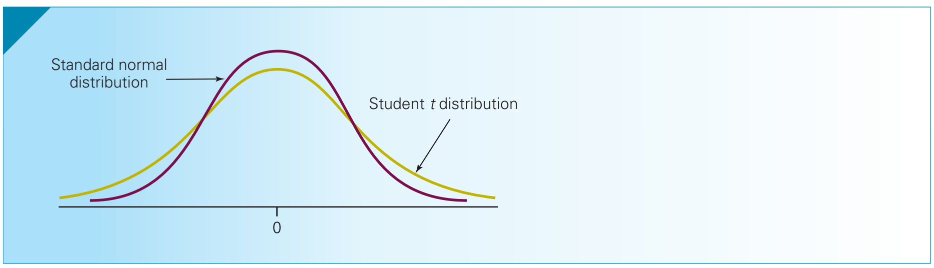
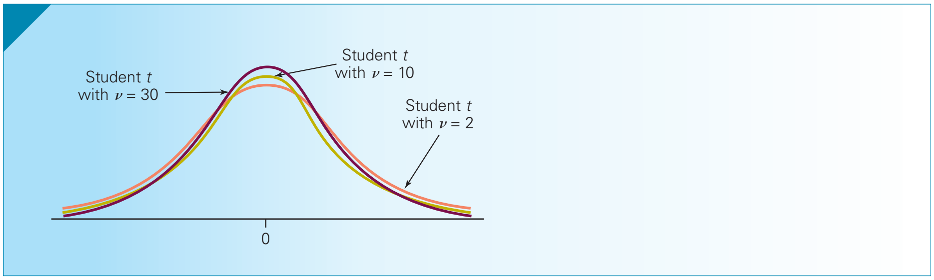


Figure 8.25 depicts Student t distributions with several different degrees of freedom. Notice that for larger degrees of freedom the Student t distribution's dispersion is smaller. For example, when $\nu = 10$, $V(t)=1.25$; when $\nu = 50$, $V(t)=1.042$; and when $\nu = 200$, $V(t)=1.010$. As ν grows larger, the Student t distribution approaches the standard normal distribution.

FIGURE **8.25** Student t Distribution with ν = 2, 10, and 30



Student *t* Probabilities For each value of ν (the number of degrees of freedom), there is a different Student t distribution. If we wanted to calculate probabilities of the Student t random variable manually as we did for the normal random variable, then we would need a different table for each ν, which is not practical. Alternatively, we can use Microsoft Excel or Minitab. The instructions are given later in this section.

Determining Student *t* Values As you will discover later in this book, the Student t distribution is used extensively in statistical inference. And for inferential methods, we often need to find values of the random variable. To determine values of a normal random variable, we used Table 3 backward. Finding values of a Student t random variable is considerably easier. Table 4 in Appendix B (reproduced here as Table 8.2) lists values of $t_{A,\nu}$, which are the values of a Student t random variable with ν degrees of freedom such that

$$P(t > t_{A,\nu}) = A$$

Figure 8.26 depicts this notation.

FIGURE **8.26** Student t Distribution with t_A

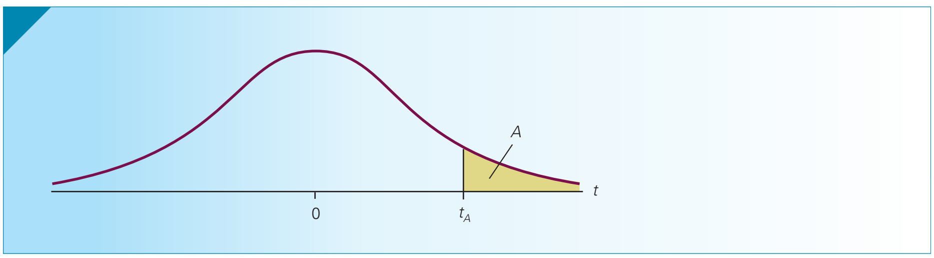


TABLE **8.2** Critical Values of t

ν	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$	ν	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$
1	3.078	6.314	12.71	31.82	63.66	29	1.311	1.699	2.045	2.462	2.756
2	1.886	2.920	4.303	6.965	9.925	30	1.310	1.697	2.042	2.457	2.750
3	1.638	2.353	3.182	4.541	5.841	35	1.306	1.690	2.030	2.438	2.724
4	1.533	2.132	2.776	3.747	4.604	40	1.303	1.684	2.021	2.423	2.704
5	1.476	2.015	2.571	3.365	4.032	45	1.301	1.679	2.014	2.412	2.690
6	1.440	1.943	2.447	3.143	3.707	50	1.299	1.676	2.009	2.403	2.678
7	1.415	1.895	2.365	2.998	3.499	55	1.297	1.673	2.004	2.396	2.668
8	1.397	1.860	2.306	2.896	3.355	60	1.296	1.671	2.000	2.390	2.660
9	1.383	1.833	2.262	2.821	3.250	65	1.295	1.669	1.997	2.385	2.654
10	1.372	1.812	2.228	2.764	3.169	70	1.294	1.667	1.994	2.381	2.648
11	1.363	1.796	2.201	2.718	3.106	75	1.293	1.665	1.992	2.377	2.643
12	1.356	1.782	2.179	2.681	3.055	80	1.292	1.664	1.990	2.374	2.639
13	1.350	1.771	2.160	2.650	3.012	85	1.292	1.663	1.988	2.371	2.635
14	1.345	1.761	2.145	2.624	2.977	90	1.291	1.662	1.987	2.368	2.632
15	1.341	1.753	2.131	2.602	2.947	95	1.291	1.661	1.985	2.366	2.629
16	1.337	1.746	2.120	2.583	2.921	100	1.290	1.660	1.984	2.364	2.626
17	1.333	1.740	2.110	2.567	2.898	110	1.289	1.659	1.982	2.361	2.621
18	1.330	1.734	2.101	2.552	2.878	120	1.289	1.658	1.980	2.358	2.617
19	1.328	1.729	2.093	2.539	2.861	130	1.288	1.657	1.978	2.355	2.614
20	1.325	1.725	2.086	2.528	2.845	140	1.288	1.656	1.977	2.353	2.611
21	1.323	1.721	2.080	2.518	2.831	150	1.287	1.655	1.976	2.351	2.609
22	1.321	1.717	2.074	2.508	2.819	160	1.287	1.654	1.975	2.350	2.607
23	1.319	1.714	2.069	2.500	2.807	170	1.287	1.654	1.974	2.348	2.605
24	1.318	1.711	2.064	2.492	2.797	180	1.286	1.653	1.973	2.347	2.603
25	1.316	1.708	2.060	2.485	2.787	190	1.286	1.653	1.973	2.346	2.602
26	1.315	1.706	2.056	2.479	2.779	200	1.286	1.653	1.972	2.345	2.601
27	1.314	1.703	2.052	2.473	2.771	∞	1.282	1.645	1.960	2.326	2.576
28	1.313	1.701	2.048	2.467	2.763						

Observe that $t_{A,\nu}$ is provided for degrees of freedom ranging from 1 to 200 and ∞. To read this table, simply identify the degrees of freedom and find that value or the closest number to it if it is not listed. Then locate the column representing the t_A value you wish. For example, if we want the value of t with 10 degrees of freedom such that the area under the Student t curve is .05, we locate 10 in the first column and move across this row until we locate the number under the heading $t_{.05}$. From Table 8.3, we find

$$t_{.05,10} = 1.812$$

If the number of degrees of freedom is not shown, find its closest value. For example, suppose we wanted to find $t_{.025,32}$. Because 32 degrees of freedom is not listed, we find the closest number of degrees of freedom, which is 30 and use $t_{.025,30} = 2.042$ as an approximation.

TABLE **8.3** Finding $t_{.05,10}$

DEGREES OF FREEDOM	$t_{.10}$	$t_{.05}$	$t_{.025}$	$t_{.01}$	$t_{.005}$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055

Because the Student t distribution is symmetric about 0, the value of t such that the area to its *left* is A is $-t_{A,\nu}$. For example, the value of t with 10 degrees of freedom such that the area to its left is .05 is

$$-t_{.05,10} = -1.812$$

Notice the last row in the Student t table. The number of degrees of freedom is infinite, and the t values are identical (except for the number of decimal places) to the values of z. For example,

$$t_{.10,\infty} = 1.282$$
$$t_{.05,\infty} = 1.645$$
$$t_{.025,\infty} = 1.960$$
$$t_{.01,\infty} = 2.326$$
$$t_{.005,\infty} = 2.576$$

In the previous section, we showed (or showed how we determine) that

$$z_{.10} = 1.28$$
$$z_{.05} = 1.645$$
$$z_{.025} = 1.96$$
$$z_{.01} = 2.23$$
$$z_{.005} = 2.575$$

Using the Computer

EXCEL

INSTRUCTIONS

To compute Student t probabilities, type

$$= \textbf{TDIST}([x], [\nu], [\text{Tails}])$$

where x must be positive, ν is the number of degrees of freedom, and "Tails" is 1 or 2. Typing 1 for "Tails" produces the area to the right of x. Typing 2 for "Tails" produces the area to the right of x plus the area to the left of $-x$. For example,

$$= \textbf{TDIST}(2, 50, 1) = .02547$$

and

$$= \textbf{TDIST}(2, 50, 2) = .05095$$

To determine t_A, type

$$= \textbf{TINV}([2A], [\nu])$$

For example, to find $t_{.05,200}$ enter

$$= \textbf{TINV}(.10, 200)$$

yielding 1.6525.

MINITAB

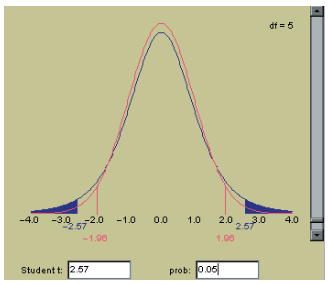
INSTRUCTIONS

Click **Calc**, **Probability Distributions**, and **t . . .** and type the **Degrees of freedom**.

SEEING STATISTICS

applet 6 Student *t* Distribution

The Student *t* distribution applet allows you to see for yourself the shape of the distribution, how the degrees of freedom change the shape, and its resemblance to the standard normal curve. The first graph shows the comparison of the normal distribution (red curve) to the Student *t* distribution (blue curve). Use the right slider to change the degrees of freedom for the *t* distribution. Use the text boxes to change either the value of *t* or the two-tail probability. Remember to press the **Return** key in the text box to record the change.



The second graph is the same as the one above except the comparison to the normal distribution has been removed. This graph is a little easier to use to find critical values of *t* or to find the probability of specific values of *t*.

Applet Exercises

The following exercises refer to Graph 1.

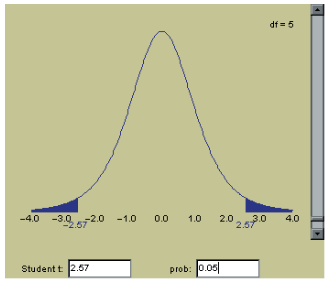
6.1 Set the degrees of freedom equal to 2. For values (on the horizontal axis) near 0, which curve is higher? The higher curve is more likely to have observations in that region.

6.2 Again for $df = 2$, for values around either $+4$ or -4, which curve is higher? In other words, which distribution is more likely to have extreme values—the normal (red) or Student t (blue) distribution?

The following exercises refer to Graph 2.

6.3 As you use the scrollbar to increase (slowly) the degrees of freedom, what happens to the value of $t_{.025}$ and $-t_{.025}$?

6.4 When the degrees of freedom = 100, is there still a small difference between the critical values of $t_{.025}$ and $z_{.025}$? How large do you think the degrees of freedom would have to be before the two sets of critical values were identical?



Chi-Squared Distribution The density function of another very useful random variable is exhibited next.

Chi-Squared Density Function

The chi-squared density function is

$$f(\chi^2) = \frac{1}{\Gamma(\nu/2)}\frac{1}{2^{\nu/2}}(\chi^2)^{(\nu/2)-1}e^{-\chi^2/2} \quad \chi^2 > 0$$

The parameter ν is the number of degrees of freedom, which like the degrees of freedom of the Student *t* distribution affects the shape.

Figure 8.27 depicts a **chi-squared distribution**. As you can see, it is positively skewed ranging between 0 and ∞. Like that of the Student *t* distribution, its shape depends on its number of degrees of freedom. The effect of increasing the degrees of freedom is seen in Figure 8.28.

FIGURE **8.27** **Chi-Squared Distribution**

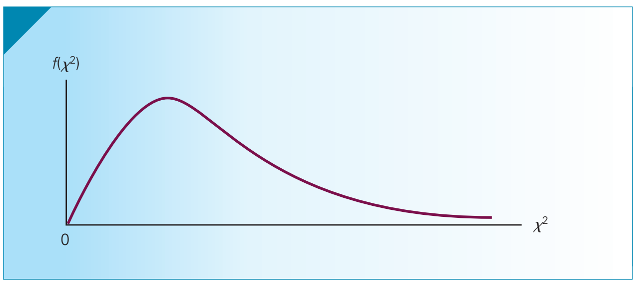
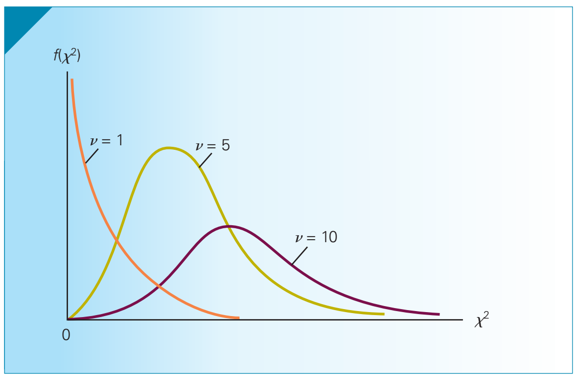


FIGURE **8.28** **Chi-Squared Distribution with ν = 1, 5, and 10**



The mean and variance of a chi-squared random variable are

$$E(\chi^2) = \nu$$

and

$$V(\chi^2) = 2\nu$$

Determining Chi-Squared Values The value of χ^2 with ν degrees of freedom such that the area to its right under the chi-squared curve is equal to A is denoted $\chi^2_{A,\nu}$. We cannot use $-\chi^2_{A,\nu}$ to represent the point such that the area to its *left* is A (as we did with the standard normal and Student t values) because χ^2 is always greater than 0. To represent left-tail critical values, we note that if the area to the left of a point is A, the area to its right must be $1 - A$ because the entire area under the chi-squared curve (as well as all continuous distributions) must equal 1. Thus, $\chi^2_{1-A,\nu}$ denotes the point such that the area to its left is A. See Figure 8.29.

FIGURE **8.29** χ^2_A and χ^2_{1-A}

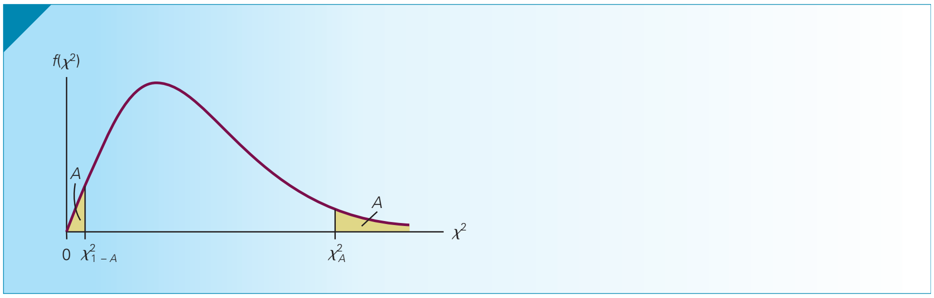


Table 5 in Appendix B (reproduced here as Table 8.4) lists critical values of the chi-squared distribution for degrees of freedom equal to 1 to 30, 40, 50, 60, 70, 80, 90, and 100. For example, to find the point in a chi-squared distribution with 8 degrees of freedom such that the area to its right is .05, locate 8 degrees of freedom in the left column and $\chi^2_{.050}$ across the top. The intersection of the row and column contains the number we seek as shown in Table 8.5; that is,

$$\chi^2_{.050,8} = 15.5$$

To find the point in the same distribution such that the area to its *left* is .05, find the point such that the area to its *right* is .95. Locate $\chi^2_{.950}$ across the top row and 8 degrees of freedom down the left column (also shown in Table 8.5). You should see that

$$\chi^2_{.950,8} = 2.73$$

TABLE **8.4** Critical Values of χ^2

ν	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000039	0.000157	0.000982	0.00393	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.103	0.211	4.61	5.99	7.38	9.21	10.6
3	0.072	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.3	12.8
4	0.207	0.297	0.484	0.711	1.06	7.78	9.49	11.1	13.3	14.9
5	0.412	0.554	0.831	1.15	1.61	9.24	11.1	12.8	15.1	16.7
6	0.676	0.872	1.24	1.64	2.20	10.6	12.6	14.4	16.8	18.5
7	0.989	1.24	1.69	2.17	2.83	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.09	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.86	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.12	11.65	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.85	12.44	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.28	11.59	13.24	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	10.98	12.34	14.04	30.8	33.9	36.8	40.3	42.8
23	9.26	10.20	11.69	13.09	14.85	32.0	35.2	38.1	41.6	44.2
24	9.89	10.86	12.40	13.85	15.66	33.2	36.4	39.4	43.0	45.6
25	10.52	11.52	13.12	14.61	16.47	34.4	37.7	40.6	44.3	46.9
26	11.16	12.20	13.84	15.38	17.29	35.6	38.9	41.9	45.6	48.3
27	11.81	12.88	14.57	16.15	18.11	36.7	40.1	43.2	47.0	49.6
28	12.46	13.56	15.31	16.93	18.94	37.9	41.3	44.5	48.3	51.0
29	13.12	14.26	16.05	17.71	19.77	39.1	42.6	45.7	49.6	52.3
30	13.79	14.95	16.79	18.49	20.60	40.3	43.8	47.0	50.9	53.7
40	20.71	22.16	24.43	26.51	29.05	51.8	55.8	59.3	63.7	66.8
50	27.99	29.71	32.36	34.76	37.69	63.2	67.5	71.4	76.2	79.5
60	35.53	37.48	40.48	43.19	46.46	74.4	79.1	83.3	88.4	92.0
70	43.28	45.44	48.76	51.74	55.33	85.5	90.5	95.0	100	104
80	51.17	53.54	57.15	60.39	64.28	96.6	102	107	112	116
90	59.20	61.75	65.65	69.13	73.29	108	113	118	124	128
100	67.33	70.06	74.22	77.93	82.36	118	124	130	136	140

TABLE **8.5** Critical Values of $\chi^2_{.05,8}$ and $\chi^2_{.950,8}$

DEGREES OF FREEDOM	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000039	0.000157	0.000982	0.00393	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.103	0.211	4.61	5.99	7.38	9.21	10.6
3	0.072	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.3	12.8
4	0.207	0.297	0.484	0.711	1.06	7.78	9.49	11.1	13.3	14.9
5	0.412	0.554	0.831	1.15	1.61	9.24	11.1	12.8	15.1	16.7
6	0.676	0.872	1.24	1.64	2.20	10.6	12.6	14.4	16.8	18.5
7	0.989	1.24	1.69	2.17	2.83	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	17.3	19.7	21.9	24.7	26.8

For values of degrees of freedom greater than 100, the chi-squared distribution can be approximated by a normal distribution with $\mu = \nu$ and $\sigma = \sqrt{2\nu}$.

Using the Computer

EXCEL

INSTRUCTIONS

To calculate $P(\chi^2 > x)$, type into any cell

= **CHIDIST**([x], [ν])

For example, **CHIDIST**(6.25, 3) = .100.

To determine $\chi_{A,\nu}$, type

= **CHIINV**([A], [ν])

For example, = **CHIINV**(.10, 3) = 6.25

MINITAB

INSTRUCTIONS

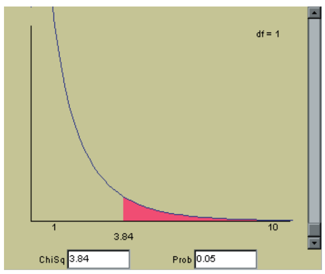
Click **Calc**, **Probability Distributions**, and **Chi-square . . . .** Specify the **Degrees of freedom**.

SEEING STATISTICS

applet 7 Chi-Squared Distribution

Like the Student *t* applet, this applet allows you to see how the degrees of freedom affect the shape of the chi-squared distribution. Additionally, you can use the applet to determine probabilities and values of the chi-squared random variable.

Use the right slider to change the degrees of freedom. Use the text boxes to change either the value of ChiSq or the probability. Remember to press the **Return** key in the text box to record the change.



Applet Exercises

7.1 What happens to the shape of the chi-squared distribution as the degrees of freedom increase?

7.2 Describe what happens to $\chi^2_{.05}$ when the degrees of freedom increase.

7.3 Describe what happens to $\chi^2_{.95}$ when the degrees of freedom increase.

F Distribution

The density function of the ***F* distribution** is given in the box.

_F_ Density Function

$$f(F) = \frac{\Gamma\left(\frac{\nu_1 + \nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{F^{\frac{\nu_1 - 2}{2}}}{\left(1 + \frac{\nu_1 F}{\nu_2}\right)^{\frac{\nu_1 + \nu_2}{2}}} \quad F > 0$$

where F ranges from 0 to ∞ and ν_1 and ν_2 are the parameters of the distribution called degrees of freedom. For reasons that are clearer in Chapter 13, we call ν_1 the *numerator degrees of freedom* and ν_2 the *denominator degrees of freedom.*

The mean and variance of an F random variable are

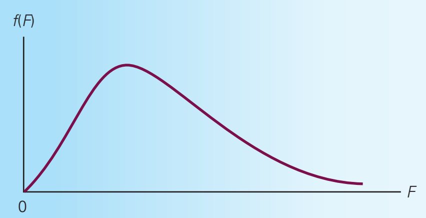
$$E(F) = \frac{\nu_2}{\nu_2 - 2} \quad \nu_2 > 2$$

and

$$V(F) = \frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)} \quad \nu_2 > 4$$

Notice that the mean depends only on the denominator degrees of freedom and that for large ν_2 the mean of the F distribution is approximately 1. Figure 8.30 describes the density function when it is graphed. As you can see, the F distribution is positively skewed. Its actual shape depends on the two numbers of degrees of freedom.

FIGURE **8.30** *F* Distribution



Determining Values of *F* We define F_{A,ν_1,ν_2} as the value of F with ν_1 and ν_2 degrees of freedom such that the area to its right under the curve is A; that is,

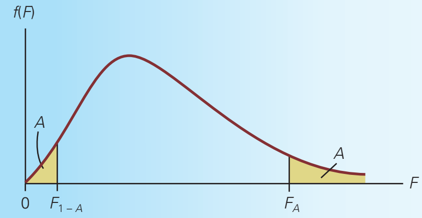
$$P(F > F_{A,\nu_1,\nu_2}) = A$$

TABLE **8.6** Critical Values of F_A for $A = .05$

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10
1	161	199	216	225	230	234	237	239	241	242
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16
35	4.12	3.27	2.87	2.64	2.49	2.37	2.29	2.22	2.16	2.11
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08
45	4.06	3.20	2.81	2.58	2.42	2.31	2.22	2.15	2.10	2.05
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91
140	3.91	3.06	2.67	2.44	2.28	2.16	2.08	2.01	1.95	1.90
160	3.90	3.05	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89
180	3.89	3.05	2.65	2.42	2.26	2.15	2.06	1.99	1.93	1.88
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88
∞	3.84	3.00	2.61	2.37	2.21	2.10	2.01	1.94	1.88	1.83

Because the F random variable like the chi-squared can equal only positive values, we define F_{1-A,ν_1,ν_2} as the value such that the area to its left is A. Figure 8.31 depicts this notation. Table 6 in Appendix B provides values of F_{A,ν_1,ν_2} for $A = .05, .025, .01$, and $.005$. Part of Table 6 is reproduced here as Table 8.6.

FIGURE **8.31** F_{1-A} and F_A



Values of F_{1-A,ν_1,ν_2} are unavailable. However, we do not need them because we can determine F_{1-A,ν_1,ν_2} from F_{A,ν_1,ν_2}. Statisticians can show that

$$F_{1-A,\nu_1,\nu_2} = \frac{1}{F_{A,\nu_2,\nu_1}}$$

To determine any critical value, find the numerator degrees of freedom ν_1 across the top of Table 6 and the denominator degrees of freedom ν_2 down the left column. The intersection of the row and column contains the number we seek. To illustrate, suppose that we want to find $F_{.05,5,7}$. Table 8.7 shows how this point is found. Locate the numerator degrees of freedom, 5, across the top and the denominator degrees of freedom, 7, down the left column. The intersection is 3.97. Thus, $F_{.05,5,7} = 3.97$.

TABLE **8.7** $F_{.05,5,7}$

Denominator Degrees of Freedom

ν_2 \ ν_1	NUMERATOR DEGREES OF FREEDOM								
	1	2	3	4	5	6	7	8	9
1	161	199	216	225	230	234	237	239	241
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.1
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.5	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02

Note that the order in which the degrees of freedom appear is important. To find $F_{.05,7,5}$ (numerator degrees of freedom = 7, and denominator degrees of freedom = 5), we locate 7 across the top and 5 down the side. The intersection is $F_{.05,7,5} = 4.88$.

Suppose that we want to determine the point in an F distribution with $\nu_1 = 4$ and $\nu_2 = 8$ such that the area to its right is .95. Thus,

$$F_{.95,4,8} = \frac{1}{F_{.05,8,4}} = \frac{1}{6.04} = .166$$

Using the Computer

EXCEL

INSTRUCTIONS

For probabilities, type

$$= \mathbf{FDIST}([X], [\nu_1], [\nu_2])$$

For example, $= \mathbf{FDIST}(3.97, 5, 7) = .05$.

To determine F_{A,ν_1,ν_2}, type

$$= \mathbf{FINV}([A], [\nu_1], [\nu_2])$$

For example, $= \mathbf{FINV}(.05, 5, 7) = 3.97$.

MINITAB

INSTRUCTIONS

Click **Calc**, **Probability Distributions**, and ***F*** **. . .** . Specify the **Numerator degrees of freedom** and the **Denominator degrees of freedom**.

SEEING STATISTICS

applet 8 *F* Distribution

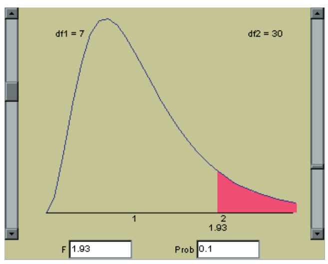
The graph shows the *F* distribution. Use the left and right sliders to change the numerator and denominator degrees of freedom, respectively. Use the text boxes to change either the value of *F* or the probability. Remember to press the **Return** key in the text box to record the change.

Applet Exercises

8.1 Set the numerator degrees of freedom equal to 1. What happens to the shape of the *F* distribution as the denominator degrees of freedom increase?

8.2 Set the numerator degrees of freedom equal to 10. What happens to the shape of the *F* distribution as the denominator degrees of freedom increase?

8.3 Describe what happens to $F_{.05}$ when either the numerator or the denominator degrees of freedom increase.



8.4 Describe what happens to $F_{.95}$ when either the numerator or the denominator degrees of freedom increase.

EXERCISES

Some of the following exercises require the use of a computer and software.

8.83 Use the t table (Table 4) to find the following values of t.
a. $t_{.10,15}$ b. $t_{.10,23}$ c. $t_{.025,83}$ d. $t_{.05,195}$

8.84 Use the t table (Table 4) to find the following values of t.
a. $t_{.005,33}$ b. $t_{.10,600}$ c. $t_{.05,4}$ d. $t_{.01,20}$

8.85 Use a computer to find the following values of t.
a. $t_{.10,15}$ b. $t_{.10,23}$ c. $t_{.025,83}$ d. $t_{.05,195}$

8.86 Use a computer to find the following values of t.
a. $t_{.05,143}$ b. $t_{.01,12}$ c. $t_{.025,\infty}$ d. $t_{.05,100}$

8.87 Use a computer to find the following probabilities.
a. $P(t_{64} > 2.12)$ b. $P(t_{27} > 1.90)$
c. $P(t_{159} > 1.33)$ d. $P(t_{550} > 1.85)$

8.88 Use a computer to find the following probabilities.
a. $P(t_{141} > .94)$ b. $P(t_{421} > 2.00)$
c. $P(t_{1000} > 1.96)$ d. $P(t_{82} > 1.96)$

8.89 Use the χ^2 table (Table 5) to find the following values of χ^2.
a. $\chi^2_{.10,5}$ b. $\chi^2_{.01,100}$ c. $\chi^2_{.95,18}$ d. $\chi^2_{.99,60}$

8.90 Use the χ^2 table (Table 5) to find the following values of χ^2.
a. $\chi^2_{.90,26}$ b. $\chi^2_{.01,30}$ c. $\chi^2_{.10,1}$ d. $\chi^2_{.99,80}$

8.91 Use a computer to find the following values of χ^2.
a. $\chi^2_{.25,66}$ b. $\chi^2_{.40,100}$ c. $\chi^2_{.50,17}$ d. $\chi^2_{.10,17}$

8.92 Use a computer to find the following values of χ^2.
a. $\chi^2_{.99,55}$ b. $\chi^2_{.05,800}$ c. $\chi^2_{.99,43}$ d. $\chi^2_{.10,233}$

8.93 Use a computer to find the following probabilities.
a. $P(\chi^2_{73} > 80)$ b. $P(\chi^2_{200} > 125)$
c. $P(\chi^2_{88} > 60)$ d. $P(\chi^2_{1000} > 450)$

8.94 Use a computer to find the following probabilities.
a. $P(\chi^2_{250} > 250)$ b. $P(\chi^2_{36} > 25)$
c. $P(\chi^2_{600} > 500)$ d. $P(\chi^2_{120} > 100)$

8.95 Use the F table (Table 6) to find the following values of F.
a. $F_{.05,3,7}$ b. $F_{.05,7,3}$ c. $F_{.025,5,20}$ d. $F_{.01,12,60}$

8.96 Use the F table (Table 6) to find the following values of F.
a. $F_{.025,8,22}$ b. $F_{.05,20,30}$
c. $F_{.01,9,18}$ d. $F_{.025,24,10}$

8.97 Use a computer to find the following values of F.
a. $F_{.05,70,70}$ b. $F_{.01,45,100}$
c. $F_{.025,36,50}$ d. $F_{.05,500,500}$

8.98 Use a computer to find the following values of F.
a. $F_{.01,100,150}$ b. $F_{.05,25,125}$
c. $F_{.01,11,33}$ d. $F_{.05,300,800}$

8.99 Use a computer to find the following probabilities.
a. $P(F_{7,20} > 2.5)$ b. $P(F_{18,63} > 1.4)$
c. $P(F_{34,62} > 1.8)$ d. $P(F_{200,400} > 1.1)$

8.100 Use a computer to find the following probabilities.
a. $P(F_{600,800} > 1.1)$ b. $P(F_{35,100} > 1.3)$
c. $P(F_{66,148} > 2.1)$ d. $P(F_{17,37} > 2.8)$

CHAPTER SUMMARY

This chapter dealt with **continuous random variables** and their distributions. Because a continuous random variable can assume an infinite number of values, the probability that the random variable equals any single value is 0. Consequently, we address the problem of computing the probability of a range of values. We showed that the probability of any interval is the area in the interval under the curve representing the **density function**.

We introduced the most important distribution in statistics and showed how to compute the probability that a **normal random variable** falls into any interval. Additionally, we demonstrated how to use the normal table backward to find values of a normal random variable given a probability. Next we introduced the **exponential distribution**, a distribution that is particularly useful in several management-science applications. Finally, we presented three more continuous random variables and their probability density functions. The **Student *t***, **chi-squared**, and ***F* distributions** will be used extensively in statistical inference.

IMPORTANT TERMS

Probability density function 265
Uniform probability distribution 266
Rectangular probability distribution 266
Normal distribution 270
Normal random variable 270
Standard normal random variable 272
Exponential distribution 287
Student *t* distribution 291
Degrees of freedom 292
Chi-squared distribution 297
F distribution 301

SYMBOLS

Symbol	Pronounced	Represents
π	pi	3.14159 . . .
z_A	*z*-sub-*A* or z-*A*	Value of Z such that area to its right is A
ν	nu	Degrees of freedom
t_A	*t*-sub-*A* or t-*A*	Value of t such that area to its right is A
χ^2_A	chi-squared-sub-*A* or chi-squared-*A*	Value of chi-squared such that area to its right is A
F_A	F-sub-*A* or *F*-*A*	Value of F such that area to its right is A
ν_1	nu-sub-one or nu-one	Numerator degrees of freedom
ν_2	nu-sub-two or nu-two	Denominator degrees of freedom

COMPUTER OUTPUT AND INSTRUCTIONS

Probability/Random Variable	Excel	Minitab
Normal probability	282	282
Normal random variable	282	282
Exponential probability	289	290
Exponential random variable	289	290
Student *t* probability	296	296
Student *t* random variable	296	296
Chi-squared probability	300	300
Chi-squared random variable	300	300
F probability	304	304
F random variable	304	304

9

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SAMPLING DISTRIBUTIONS

Salaries of a Business School's Graduates

Deans and other faculty members in professional schools often monitor how well the graduates of their programs fare in the job market. Information about the types of jobs and their salaries may provide useful information about the success of a program.

In the advertisements for a large university, the dean of the School of Business claims that the average salary of the school's graduates one year after graduation is $800 per week, with a standard deviation of $100. A second-year student in the business school who has just completed his statistics course would like to check whether the claim about the mean is correct. He does a survey of 25 people who graduated one year earlier and determines their weekly salary. He discovers the sample mean to be $750. To interpret his finding, he needs to calculate the probability that a sample of 25 graduates would have a mean of $750 or less when the population mean is $800 and the standard deviation is $100. After calculating the probability, he needs to draw some conclusion.

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See page 317 for the answer.

Introduction

This chapter introduces the *sampling distribution*, a fundamental element in statistical inference. We remind you that statistical inference is the process of converting data into information. Here are the parts of the process we have thus far discussed:

1. Parameters describe populations.
2. Parameters are almost always unknown.
3. We take a random sample of a population to obtain the necessary data.
4. We calculate one or more statistics from the data.

For example, to estimate a population mean, we compute the sample mean. Although there is very little chance that the sample mean and the population mean are identical, we would expect them to be quite close. However, for the purposes of statistical inference, we need to be able to measure *how* close. The sampling distribution provides this service. It plays a crucial role in the process because the measure of proximity it provides is the key to statistical inference.

9.1 Sampling Distribution of the Mean

A **sampling distribution** is created by, as the name suggests, sampling. There are two ways to create a sampling distribution. The first is to actually draw samples of the same size from a population, calculate the statistic of interest, and then use descriptive techniques to learn more about the sampling distribution. The second method relies on the rules of probability and the laws of expected value and variance to derive the sampling distribution. We'll demonstrate the latter approach by developing the sampling distribution of the mean of two dice.

Sampling Distribution of the Mean of Two Dice

The population is created by throwing a fair die infinitely many times, with the random variable X indicating the number of spots showing on any one throw. The probability distribution of the random variable X is as follows:

x	1	2	3	4	5	6
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6

The population is infinitely large because we can throw the die infinitely many times (or at least imagine doing so). From the definitions of expected value and variance presented in Section 7.1, we calculate the population mean, variance, and standard deviation.

Population mean:

$$\begin{aligned}\mu &= \sum xP(x) \\ &= 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6) \\ &= 3.5\end{aligned}$$

Population variance:

$$\sigma^2 = \sum(x - \mu)^2 P(x)$$

$$= (1 - 3.5)^2(1/6) + (2 - 3.5)^2(1/6) + (3 - 3.5)^2(1/6) + (4 - 3.5)^2(1/6)$$
$$+ (5 - 3.5)^2(1/6) + (6 - 3.5)^2(1/6)$$
$$= 2.92$$

Population standard deviation:

$$\sigma = \sqrt{\sigma^2} = \sqrt{2.92} = 1.71$$

The sampling distribution is created by drawing samples of size 2 from the population. In other words, we toss two dice. Figure 9.1 depicts this process in which we compute the mean for each sample. Because the value of the sample mean varies randomly from sample to sample, we can regard $\overline{X}$ as a new random variable created by sampling. Table 9.1 lists all the possible samples and their corresponding values of $\bar{x}$.

FIGURE **9.1** Drawing Samples of Size 2 from a Population

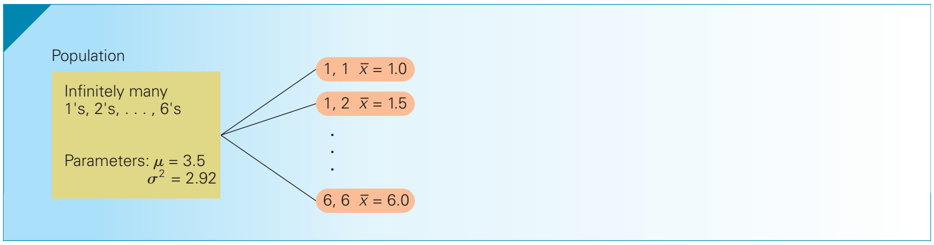


TABLE **9.1** All Samples of Size 2 and Their Means

SAMPLE	$\bar{x}$	SAMPLE	$\bar{x}$	SAMPLE	$\bar{x}$
1, 1	1.0	3, 1	2.0	5, 1	3.0
1, 2	1.5	3, 2	2.5	5, 2	3.5
1, 3	2.0	3, 3	3.0	5, 3	4.0
1, 4	2.5	3, 4	3.5	5, 4	4.5
1, 5	3.0	3, 5	4.0	5, 5	5.0
1, 6	3.5	3, 6	4.5	5, 6	5.5
2, 1	1.5	4, 1	2.5	6, 1	3.5
2, 2	2.0	4, 2	3.0	6, 2	4.0
2, 3	2.5	4, 3	3.5	6, 3	4.5
2, 4	3.0	4, 4	4.0	6, 4	5.0
2, 5	3.5	4, 5	4.5	6, 5	5.5
2, 6	4.0	4, 6	5.0	6, 6	6.0

There are 36 different possible samples of size 2; because each sample is equally likely, the probability of any one sample being selected is 1/36. However, $\bar{x}$ can assume only 11 different possible values: 1.0, 1.5, 2.0, . . . , 6.0, with certain values of $\bar{x}$ occurring more frequently than others. The value $\bar{x} = 1.0$ occurs only once, so its probability is 1/36. The value $\bar{x} = 1.5$ can occur in two ways—(1, 2) and (2, 1)—each having the same probability (1/36). Thus, $P(\bar{x} = 1.5) = 2/36$. The probabilities of the other values of $\bar{x}$

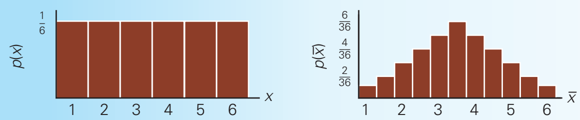
are determined in similar fashion, and the resulting **sampling distribution of the sample mean** is shown in Table 9.2.

TABLE **9.2** Sampling Distribution of $\overline{X}$

$\bar{x}$	$P(\bar{x})$
1.0	1/36
1.5	2/36
2.0	3/36
2.5	4/36
3.0	5/36
3.5	6/36
4.0	5/36
4.5	4/36
5.0	3/36
5.5	2/36
6.0	1/36

The most interesting aspect of the sampling distribution of $\overline{X}$ is how different it is from the distribution of X, as can be seen in Figure 9.2.

FIGURE **9.2** Distributions of X and $\overline{X}$



(a) Distribution of X **(b)** Sampling distribution of $\overline{X}$

We can also compute the mean, variance, and standard deviation of the sampling distribution. Once again using the definitions of expected value and variance, we determine the following parameters of the sampling distribution.

Mean of the sampling distribution of $\overline{X}$:

$$\begin{aligned}\mu_{\bar{x}} &= \sum \bar{x}P(\bar{x}) \\ &= 1.0(1/36) + 1.5(2/36) + \cdots + 6.0(1/36) \\ &= 3.5\end{aligned}$$

Notice that the mean of the sampling distribution of $\overline{X}$ is equal to the mean of the population of the toss of a die computed previously.

Variance of the sampling distribution of $\overline{X}$:

$$\begin{aligned}\sigma^2_{\bar{x}} &= \sum (\bar{x} - \mu_{\bar{x}})^2 P(\bar{x}) \\ &= (1.0 - 3.5)^2(1/36) + (1.5 - 3.5)^2(2/36) + \cdots + (6.0 - 3.5)^2(1/36) \\ &= 1.46\end{aligned}$$

It is no coincidence that the variance of the sampling distribution of $\overline{X}$ is exactly half of the variance of the population of the toss of a die (computed previously as $\sigma^2 = 2.92$).

Standard deviation of the sampling distribution of $\overline{X}$:

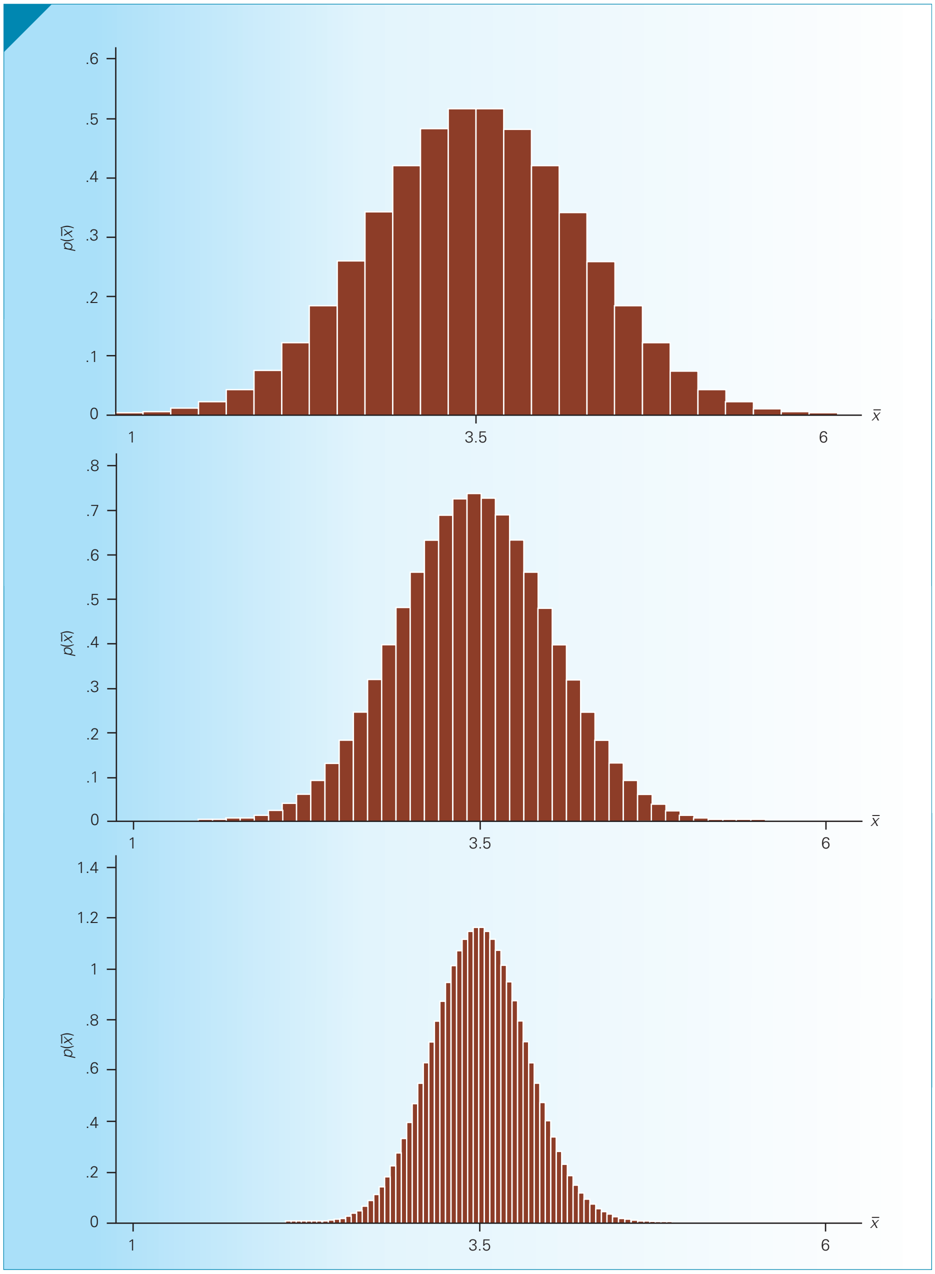
$$\sigma_{\bar{x}} = \sqrt{\sigma^2_{\bar{x}}} = \sqrt{1.46} = 1.21$$

It is important to recognize that the distribution of $\overline{X}$ is different from the distribution of X as depicted in Figure 9.2. However, the two random variables are related. Their means are the same ($\mu_{\overline{x}} = \mu = 3.5$) and their variances are related ($\sigma^2_{\overline{x}} = \sigma^2/2$).

Don't get lost in the terminology and notation. Remember that μ and σ^2 are the parameters of the population of X. To create the sampling distribution of $\overline{X}$, we repeatedly drew samples of size $n = 2$ from the population and calculated $\bar{x}$ for each sample. Thus, we treat $\overline{X}$ as a brand-new random variable, with its own distribution, mean, and variance. The mean is denoted $\mu_{\overline{x}}$, and the variance is denoted $\sigma^2_{\overline{x}}$.

If we now repeat the sampling process with the same population but with other values of n, we produce somewhat different sampling distributions of $\overline{X}$. Figure 9.3 shows the sampling distributions of $\overline{X}$ when $n = 5$, 10, and 25.

FIGURE 9.3 Sampling Distributions of $\overline{X}$ for n = 5, 10, and 25



For each value of n, the mean of the sampling distribution of $\overline{X}$ is the mean of the population from which we're sampling; that is,

$$\mu_{\bar{x}} = \mu = 3.5$$

The variance of the sampling distribution of the sample mean is the variance of the population divided by the sample size:

$$\sigma^2_{\bar{x}} = \frac{\sigma^2}{n}$$

The standard deviation of the sampling distribution is called the **standard error of the mean**; that is,

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

As you can see, the variance of the sampling distribution of $\overline{X}$ is less than the variance of the population we're sampling from all sample sizes. Thus, a randomly selected value of $\overline{X}$ (the mean of the number of spots observed in, say, five throws of the die) is likely to be closer to the mean value of 3.5 than is a randomly selected value of X (the number of spots observed in one throw). Indeed, this is what you would expect, because in five throws of the die you are likely to get some 5s and 6s and some 1s and 2s, which will tend to offset one another in the averaging process and produce a sample mean reasonably close to 3.5. As the number of throws of the die increases, the probability that the sample mean will be close to 3.5 also increases. Thus, we observe in Figure 9.3 that the sampling distribution of $\overline{X}$ becomes narrower (or more concentrated about the mean) as n increases.

Another thing that happens as n gets larger is that the sampling distribution of $\bar{x}$ becomes increasingly bell shaped. This phenomenon is summarized in the **central limit theorem**.

Central Limit Theorem

The sampling distribution of the mean of a random sample drawn from any population is approximately normal for a sufficiently large sample size. The larger the sample size, the more closely the sampling distribution of $\overline{X}$ will resemble a normal distribution.

The accuracy of the approximation alluded to in the central limit theorem depends on the probability distribution of the population and on the sample size. If the population is normal, then $\overline{X}$ is normally distributed for all values of n. If the population is nonnormal, then $\overline{X}$ is approximately normal only for larger values of n. In many practical situations, a sample size of 30 may be sufficiently large to allow us to use the normal distribution as an approximation for the sampling distribution of $\overline{X}$. However, if the population is extremely nonnormal (for example, bimodal and highly skewed distributions), the sampling distribution will also be nonnormal even for moderately large values of n.

Sampling Distribution of the Mean of Any Population We can extend the discoveries we've made to all infinitely large populations. Statisticians have shown that the mean of the sampling distribution is always equal to the mean of the population and that the standard error is equal to $\sigma/\sqrt{n}$ for infinitely large populations. (In Keller's website Appendix Using the Laws of Expected Value and Variance to Derive the Parameters of Sampling Distributions we describe how to mathematically prove that $\mu_{\bar{x}} = \mu$ and $\sigma^2_{\bar{x}} = \sigma^2/n$.) However, if the population is finite the standard error is

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$$

where N is the population size and $\sqrt{\frac{N-n}{N-1}}$ is called the **finite population correction factor**. (The source of the correction factor is provided in Keller's website Appendix Hypergeometric Distribution.) An analysis (see Exercises 9.13 and 9.14) reveals that if the population size is large relative to the sample size, then the finite population correction factor is close to 1 and can be ignored. As a rule of thumb, we will treat any population that is at least 20 times larger than the sample size as large. In practice, most applications involve populations that qualify as large because if the population is small, it may be possible to investigate each member of the population, and in so doing, calculate the parameters precisely. As a consequence, the finite population correction factor is usually omitted.

We can now summarize what we know about the sampling distribution of the sample mean for large populations.

Sampling Distribution of the Sample Mean

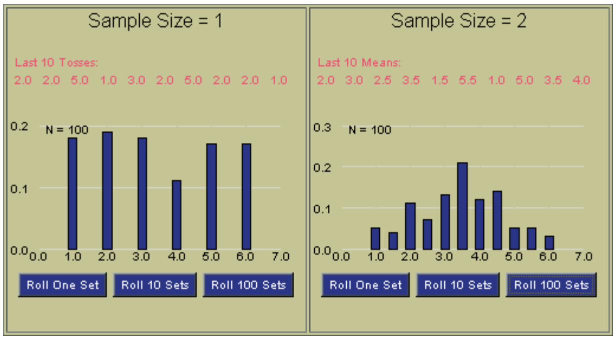
1. $\mu_{\bar{x}} = \mu$
2. $\sigma^2_{\bar{x}} = \sigma^2/n$ and $\sigma_{\bar{x}} = \sigma/\sqrt{n}$
3. If X is normal, then $\bar{X}$ is normal. If X is nonnormal, then $\bar{X}$ is approximately normal for sufficiently large sample sizes. The definition of "sufficiently large" depends on the extent of nonnormality of X.

Creating the Sampling Distribution Empirically

In the previous analysis, we created the sampling distribution of the mean theoretically. We did so by listing all the possible samples of size 2 and their probabilities. (They were all equally likely with probability 1/36.) From this distribution, we produced the sampling distribution. We could also create the distribution empirically by actually tossing two fair dice repeatedly, calculating the sample mean for each sample, counting the number of times each value of $\bar{X}$ occurs, and computing the relative frequencies to estimate the theoretical probabilities. If we toss the two dice a large enough number of times, the relative frequencies and theoretical probabilities (computed previously) will be similar. Try it yourself. Toss two dice 500 times, calculate the mean of the two tosses, count the number of times each sample mean occurs, and construct the histogram representing the sampling distribution. Obviously, this approach is far from ideal because of the excessive amount of time required to toss the dice enough times to make the relative frequencies good approximations for the theoretical probabilities. However, we can use the computer to quickly simulate tossing dice many times.

SEEING STATISTICS

applet 9 Fair Dice 1



This applet has two parts. The first part simulates the tossing of one fair die. You can toss 1 at a time, 10 at a time, or 100 at a time. The histogram of the cumulative results is shown. The second part allows you to simulate tossing 2 dice one set at a time, 10 sets a time, or 100 sets a time. The histogram of the means of the cumulative results is exhibited. To start again, click **Refresh** or **Reload** on the browser menu. The value *N* represents the number of sets. The larger the value of *N*, the closer the histogram approximates the theoretical distribution.

Applet Exercises

Simulate 2,500 tosses of one fair die and 2,500 tosses of two fair dice.

9.1 Does the simulated probability distribution of one die look like the theoretical distribution displayed in Figure 9.2? Discuss the reason for the deviations.

9.2 Does the simulated sampling distribution of the mean of two dice look like the theoretical distribution displayed in Figure 9.2? Discuss the reason for the deviations.

9.3 Do the distribution of one die and the sampling distribution of the mean of two dice have the same or different shapes? How would you characterize the difference?

9.4 Do the centers of the distribution of one die and the sampling distribution of the mean of two dice appear to be about the same?

9.5 Do the spreads of the distribution of one die and the sampling distribution of the mean of two dice appear to be about the same? Which one has the smaller spread?

SEEING STATISTICS

applet 10 Fair Dice 2

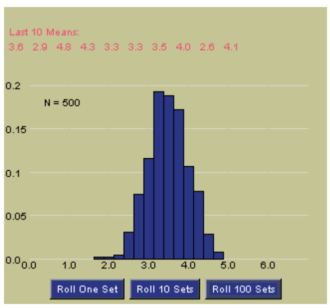
This applet allows you to simulate tossing 12 fair dice and drawing the sampling distribution of the mean. As was the case with the previous applet, you can toss 1 set, 10 sets, or 100 sets. To start again, click **Refresh** or **Reload** on the browser menu.

Applet Exercises

Simulate 2,500 tosses of 12 fair dice.

10.1 Does the simulated sampling distribution of $\overline{X}$ appear to be bell shaped?

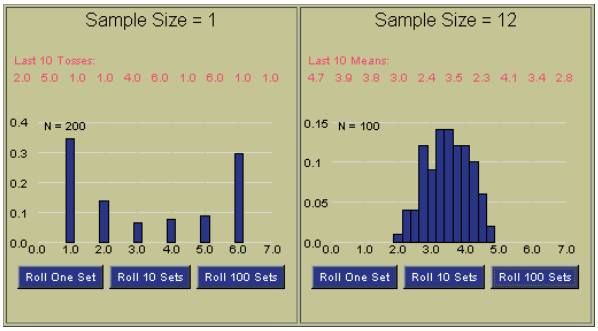
10.2 Does it appear that the simulated sampling distribution of the mean of 12 fair dice is narrower than that of 2 fair dice? Explain why this is so.



SEEING STATISTICS

applet 11 Loaded Dice

This applet has two parts. The first part simulates the tossing of a loaded die. "Loaded" refers to the inequality of the probabilities of the six outcomes. You can toss 1 at a time, 10 at a time, or 100 at a time. The second part allows you to simulate tossing 12 loaded dice 1 set at a time, 10 sets a time, or 100 sets at a time.



Applet Exercises

Simulate 2,500 tosses of one loaded die.

11.1 Estimate the probability of each value of X.

11.2 Use the estimated probabilities to compute the expected value, variance, and standard deviation of X.

Simulate 2,500 tosses of 12 loaded dice.

11.3 Does it appear that the mean of the simulated sampling distribution of $\overline{X}$ is equal to 3.5?

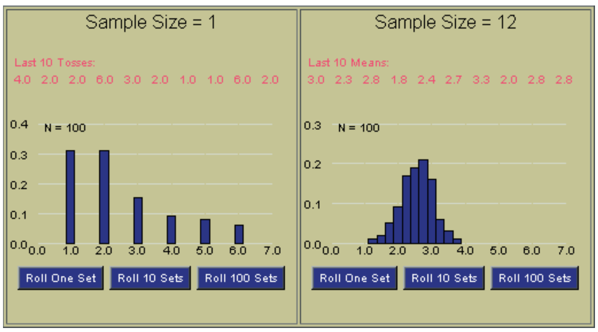
11.4 Does it appear that the standard deviation of the simulated sampling distribution of the mean of 12 loaded dice is greater than that for 12 fair dice? Explain why this is so.

11.5 Does the simulated sampling distribution of the mean of 12 loaded dice appear to be bell shaped? Explain why this is so.

SEEING STATISTICS

applet 12 Skewed Dice

This applet has two parts. The first part simulates the tossing of a skewed die. You can toss it 1 at a time, 10 at a time, or 100 at a time. The second part allows you to simulate tossing 2 dice 1 set at a time, 10 sets a time, or 100 sets at a time



Applet Exercises

Simulate 2,500 tosses of one skewed die.

12.1 Estimate the probability of each value of X.

12.2 Use the estimated probabilities to compute the expected value, variance, and standard deviation of X.

Simulate 2,500 tosses of 12 skewed dice.

12.3. Does it appear that the mean of the simulated sampling distribution of $\overline{X}$ is less than 3.5?

12.4. Does the simulated sampling distribution of the mean of 12 skewed dice appear to be bell shaped? Explain why this is so.

EXAMPLE 9.1 Contents of a 32-Ounce Bottle

The foreman of a bottling plant has observed that the amount of soda in each 32-ounce bottle is actually a normally distributed random variable, with a mean of 32.2 ounces and a standard deviation of .3 ounce.

a. If a customer buys one bottle, what is the probability that the bottle will contain more than 32 ounces?
b. If a customer buys a carton of four bottles, what is the probability that the mean amount of the four bottles will be greater than 32 ounces?

SOLUTION

a. Because the random variable is the amount of soda in one bottle, we want to find $P(X > 32)$, where X is normally distributed, $\mu = 32.2$, and $\sigma = .3$. Hence,

$$\begin{aligned} P(X > 32) &= P\left(\frac{X - \mu}{\sigma} > \frac{32 - 32.2}{.3}\right) \\ &= P(Z > -.67) \\ &= 1 - P(Z < -.67) \\ &= 1 - .2514 = .7486 \end{aligned}$$

b. Now we want to find the probability that the mean amount of four filled bottles exceeds 32 ounces; that is, we want $P(\overline{X} > 32)$. From our previous analysis and from the central limit theorem, we know the following:

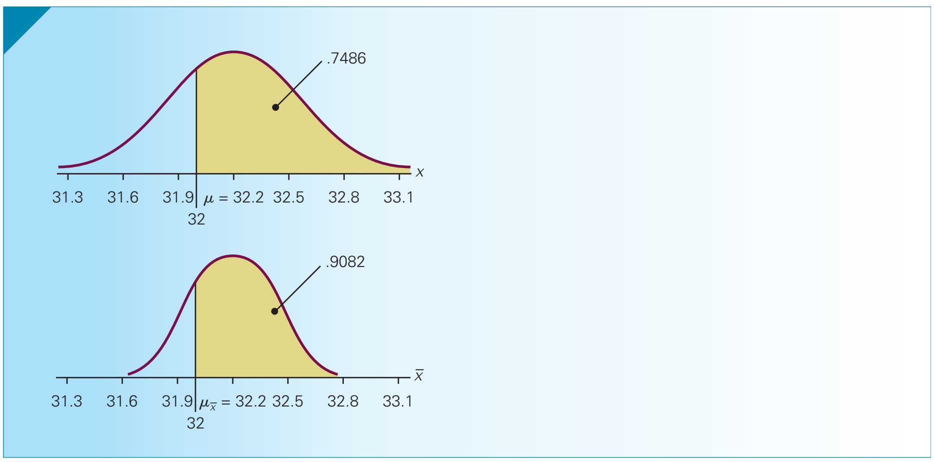
1. $\overline{X}$ is normally distributed.
2. $\mu_{\bar{x}} = \mu = 32.2$
3. $\sigma_{\bar{x}} = \sigma/\sqrt{n} = .3/\sqrt{4} = .15$

Hence,

$$\begin{aligned} P(\overline{X} > 32) &= P\left(\frac{\overline{X} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} > \frac{32 - 32.2}{.15}\right) = P(Z > -1.33) \\ &= 1 - P(Z < -1.33) = 1 - .0918 = .9082 \end{aligned}$$

Figure 9.4 illustrates the distributions used in this example.

FIGURE 9.4 Distribution of X and Sampling Distribution of $\overline{X}$



In Example 9.1(b), we began with the assumption that both μ and σ were known. Then, using the sampling distribution, we made a probability statement about $\bar{X}$. Unfortunately, the values of μ and σ are not usually known, so an analysis such as that in Example 9.1 cannot usually be conducted. However, we can use the sampling distribution to infer something about an unknown value of μ on the basis of a sample mean.

Salaries of a Business School's Graduates: Solution

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We want to find the probability that the sample mean is less than \$750. Thus, we seek

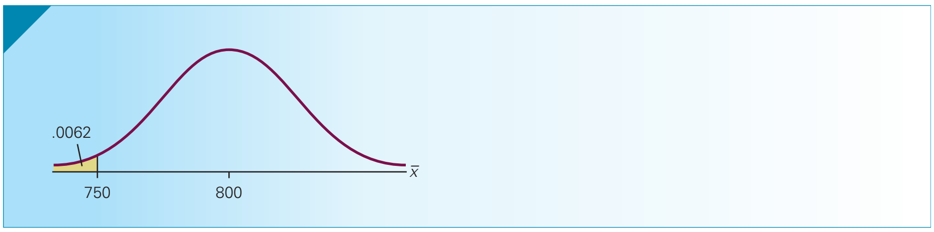
$$P(\bar{X} < 750)$$

The distribution of X, the weekly income, is likely to be positively skewed, but not sufficiently so to make the distribution of $\bar{X}$ nonnormal. As a result, we may assume that $\bar{X}$ is normal with mean $\mu_{\bar{x}} = \mu = 800$ and standard deviation $\sigma_{\bar{x}} = \sigma/\sqrt{n} = 100/\sqrt{25} = 20$. Thus,

$$P(\bar{X} < 750) = P\left(\frac{\bar{X} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} < \frac{750 - 800}{20}\right) = P(Z < -2.5) = .0062$$

Figure 9.5 illustrates the distribution.

FIGURE 9.5 $P(\bar{X} < 750)$



The probability of observing a sample mean as low as \$750 when the population mean is \$800 is extremely small. Because this event is quite unlikely, we would have to conclude that the dean's claim is not justified.

Using the Sampling Distribution for Inference

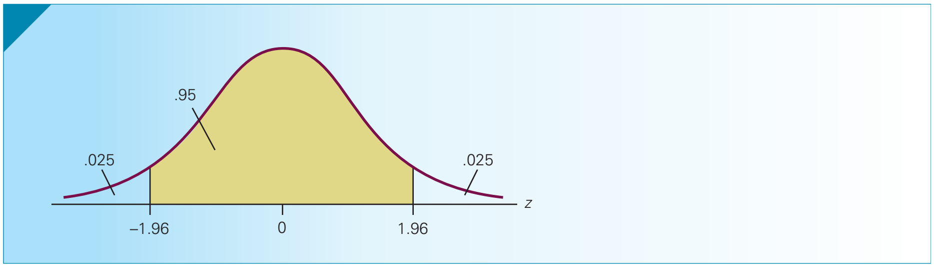
Our conclusion in the chapter-opening example illustrates how the sampling distribution can be used to make inferences about population parameters. The first form of inference is estimation, which we introduce in the next chapter. In preparation for this momentous occasion, we'll present another way of expressing the probability associated with the sampling distribution.

Recall the notation introduced in Section 8.2 (see page 278). We defined z_A to be the value of z such that the area to the right of z_A under the standard normal curve is equal to A. We also showed that $z_{.025} = 1.96$. Because the standard normal distribution is symmetric about 0, the area to the left of -1.96 is also .025. The area between

-1.96 and 1.96 is .95. Figure 9.6 depicts this notation. We can express the notation algebraically as

$$P(-1.96 < Z < 1.96) = .95$$

FIGURE **9.6** $P(-1.96 < Z < 1.96) = .05$



In this section, we established that

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

is standard normally distributed. Substituting this form of Z into the previous probability statement, we produce

$$P\left(-1.96 < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < 1.96\right) = .95$$

With a little algebraic manipulation (multiply all three terms by $\sigma/\sqrt{n}$ and add μ to all three terms), we determine

$$P\left(\mu - 1.96\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + 1.96\frac{\sigma}{\sqrt{n}}\right) = .95$$

Returning to the chapter-opening example where $\mu = 800$, $\sigma = 100$, and $n = 25$, we compute

$$P\left(800 - 1.96\frac{100}{\sqrt{25}} < \overline{X} < 800 + 1.96\frac{100}{\sqrt{25}}\right) = .95$$

Thus, we can say that

$$P(760.8 < \overline{X} < 839.2) = .95$$

This tells us that there is a 95% probability that a sample mean will fall between 760.8 and 839.2. Because the sample mean was computed to be $750, we would have to conclude that the dean's claim is not supported by the statistic.

Changing the probability from .95 to .90 changes the probability statement to

$$P\left(\mu - 1.645\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + 1.645\frac{\sigma}{\sqrt{n}}\right) = .90$$

We can also produce a general form of this statement:

$$P\left(\mu - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

In this formula α (Greek letter *alpha*) is the probability that $\overline{X}$ does not fall into the interval. To apply this formula, all we need do is substitute the values for μ, σ, n, and α. For example, with $\mu = 800$, $\sigma = 100$, $n = 25$, and $\alpha = .01$, we produce

$$P\left(\mu - z_{.005}\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + z_{.005}\frac{\sigma}{\sqrt{n}}\right) = 1 - .01$$

$$P\left(800 - 2.575\frac{100}{\sqrt{25}} < \overline{X} < 800 + 2.575\frac{100}{\sqrt{25}}\right) = .99$$

$$P(748.5 < \overline{X} < 851.5) = .99$$

which is another probability statement about $\overline{X}$. In Section 10.2, we will use a similar type of probability statement to derive the first statistical inference technique.

EXERCISES

9.1 Let X represent the result of the toss of a fair die. Find the following probabilities.
a. $P(X = 1)$
b. $P(X = 6)$

9.2 Let $\overline{X}$ represent the mean of the toss of two fair dice. Use the probabilities listed in Table 9.2 to determine the following probabilities.
a. $P(\overline{X} = 1)$
b. $P(\overline{X} = 6)$

9.3 An experiment consists of tossing five balanced dice. Find the following probabilities. (Determine the exact probabilities as we did in Tables 9.1 and 9.2 for two dice.)
a. $P(\overline{X} = 1)$
b. $P(\overline{X} = 6)$

9.4 Refer to Exercises 9.1 to 9.3. What do the probabilities tell you about the variances of X and $\overline{X}$?

9.5 A normally distributed population has a mean of 40 and a standard deviation of 12. What does the central limit theorem say about the sampling distribution of the mean if samples of size 100 are drawn from this population?

9.6 Refer to Exercise 9.5. Suppose that the population is not normally distributed. Does this change your answer? Explain.

9.7 A sample of $n = 16$ observations is drawn from a normal population with $\mu = 1{,}000$ and $\sigma = 200$. Find the following.
a. $P(\overline{X} > 1{,}050)$
b. $P(\overline{X} < 960)$
c. $P(\overline{X} > 1{,}100)$

9.8 Repeat Exercise 9.7 with $n = 25$.

9.9 Repeat Exercise 9.7 with $n = 100$.

9.10 Given a normal population whose mean is 50 and whose standard deviation is 5, find the probability that a random sample of
a. 4 has a mean between 49 and 52.
b. 16 has a mean between 49 and 52.
c. 25 has a mean between 49 and 52.

9.11 Repeat Exercise 9.10 for a standard deviation of 10.

9.12 Repeat Exercise 9.10 for a standard deviation of 20.

9.13 a. Calculate the finite population correction factor when the population size is $N = 1{,}000$ and the sample size is $n = 100$.
b. Repeat part (a) when $N = 3{,}000$.
c. Repeat part (a) when $N = 5{,}000$.
d. What have you learned about the finite population correction factor when N is large relative to n?

9.14 a. Suppose that the standard deviation of a population with $N = 10{,}000$ members is 500. Determine the standard error of the sampling distribution of the mean when the sample size is 1,000.
b. Repeat part (a) when $n = 500$.
c. Repeat part (a) when $n = 100$.

9.15 The heights of North American women are normally distributed with a mean of 64 inches and a standard deviation of 2 inches.
a. What is the probability that a randomly selected woman is taller than 66 inches?
b. A random sample of four women is selected. What is the probability that the sample mean height is greater than 66 inches?
c. What is the probability that the mean height of a random sample of 100 women is greater than 66 inches?

9.16 Refer to Exercise 9.15. If the population of women's heights is not normally distributed, which, if any, of the questions can you answer? Explain.

9.17 An automatic machine in a manufacturing process is operating properly if the lengths of an important subcomponent are normally distributed with mean = 117 cm and standard deviation = 5.2 cm.

a. Find the probability that one selected subcomponent is longer than 120 cm.
b. Find the probability that if four subcomponents are randomly selected, their mean length exceeds 120 cm.
c. Find the probability that if four subcomponents are randomly selected, all four have lengths that exceed 120 cm.

9.18 The amount of time the university professors devote to their jobs per week is normally distributed with a mean of 52 hours and a standard deviation of 6 hours.

a. What is the probability that a professor works for more than 60 hours per week?
b. Find the probability that the mean amount of work per week for three randomly selected professors is more than 60 hours.
c. Find the probability that if three professors are randomly selected all three work for more than 60 hours per week.

9.19 The number of pizzas consumed per month by university students is normally distributed with a mean of 10 and a standard deviation of 3.

a. What proportion of students consume more than 12 pizzas per month?
b. What is the probability that in a random sample of 25 students more than 275 pizzas are consumed? (*Hint:* What is the mean number of pizzas consumed by the sample of 25 students?)

9.20 The marks on a statistics midterm test are normally distributed with a mean of 78 and a standard deviation of 6.

a. What proportion of the class has a midterm mark of less than 75?
b. What is the probability that a class of 50 has an average midterm mark that is less than 75?

9.21 The amount of time spent by North American adults watching television per day is normally distributed with a mean of 6 hours and a standard deviation of 1.5 hours.

a. What is the probability that a randomly selected North American adult watches television for more than 7 hours per day?
b. What is the probability that the average time watching television by a random sample of five North American adults is more than 7 hours?
c. What is the probability that, in a random sample of five North American adults, all watch television for more than 7 hours per day?

9.22 The manufacturer of cans of salmon that are supposed to have a net weight of 6 ounces tells you that the net weight is actually a normal random variable with a mean of 6.05 ounces and a standard deviation of .18 ounces. Suppose that you draw a random sample of 36 cans.

a. Find the probability that the mean weight of the sample is less than 5.97 ounces.
b. Suppose your random sample of 36 cans of salmon produced a mean weight that is less than 5.97 ounces. Comment on the statement made by the manufacturer.

9.23 The number of customers who enter a supermarket each hour is normally distributed with a mean of 600 and a standard deviation of 200. The supermarket is open 16 hours per day. What is the probability that the total number of customers who enter the supermarket in one day is greater than 10,000? (*Hint:* Calculate the average hourly number of customers necessary to exceed 10,000 in one 16-hour day.)

9.24 The sign on the elevator in the Peters Building, which houses the School of Business and Economics at Wilfrid Laurier University, states, "Maximum Capacity 1,140 kilograms (2,500 pounds) or 16 Persons." A professor of statistics wonders what the probability is that 16 persons would weigh more than 1,140 kilograms. Discuss what the professor needs (besides the ability to perform the calculations) in order to satisfy his curiosity.

9.25 Refer to Exercise 9.24. Suppose that the professor discovers that the weights of people who use the elevator are normally distributed with an average of 75 kilograms and a standard deviation of 10 kilograms. Calculate the probability that the professor seeks.

9.26 The time it takes for a statistics professor to mark his midterm test is normally distributed with a mean of 4.8 minutes and a standard deviation of 1.3 minutes. There are 60 students in the professor's class. What is the probability that he needs more than 5 hours to mark all the midterm tests? (The 60 midterm tests of the students in this year's class can be considered a random sample of the many thousands of midterm tests the professor has marked and will mark.)

9.27 Refer to Exercise 9.26. Does your answer change if you discover that the times needed to mark a midterm test are not normally distributed?

9.28 The restaurant in a large commercial building provides coffee for the building's occupants. The restaurateur has determined that the mean number of cups of coffee consumed in a day by all the occupants is 2.0 with a standard deviation of .6. A new tenant of the building intends to have a total of 125 new employees. What is the probability that the new employees will consume more than 240 cups per day?

9.29 The number of pages produced by a fax machine in a busy office is normally distributed with a mean of 275 and a standard deviation of 75. Determine the probability that in 1 week (5 days) more than 1,500 faxes will be received?

9.2 Sampling Distribution of a Proportion

In Section 7.4, we introduced the binomial distribution whose parameter is p, the probability of success in any trial. In order to compute binomial probabilities, we assumed that p was known. However, in the real world p is unknown, requiring the statistics practitioner to estimate its value from a sample. The estimator of a population proportion of successes is the sample proportion; that is, we count the number of successes in a sample and compute

$$\hat{P} = \frac{X}{n}$$

($\hat{P}$ is read as *p hat*) where X is the number of successes and n is the sample size. When we take a sample of size n, we're actually conducting a binomial experiment; as a result, X is binomially distributed. Thus, the probability of any value of $\hat{P}$ can be calculated from its value of X. For example, suppose that we have a binomial experiment with $n = 10$ and $p = .4$. To find the probability that the sample proportion $\hat{P}$ is less than or equal to .50, we find the probability that X is less than or equal to 5 (because $5/10 = .50$). From Table 1 in Appendix B we find with $n = 10$ and $p = .4$

$$P(\hat{P} \le .50) = P(X \le 5) = .8338$$

We can calculate the probability associated with other values of $\hat{P}$ similarly.

Discrete distributions such as the binomial do not lend themselves easily to the kinds of calculation needed for inference. And inference is the reason we need sampling distributions. Fortunately, we can approximate the binomial distribution by a normal distribution.

What follows is an explanation of how and why the normal distribution can be used to approximate a binomial distribution. Disinterested readers can skip to page 325, where we present the approximate **sampling distribution of a sample proportion**.

(Optional) Normal Approximation to the Binomial Distribution

Recall how we introduced continuous probability distributions in Chapter 8. We developed the density function by converting a histogram so that the total area in the rectangles equaled 1. We can do the same for a binomial distribution. To illustrate, let X be a binomial random variable with $n = 20$ and $p = .5$. We can easily determine the probability of each value of X, where $X = 0, 1, 2, \ldots, 19, 20$. A rectangle representing a value of x is drawn so that its area equals the probability. We accomplish this by letting the height of the rectangle equal the probability and the base of the rectangle equal 1. Thus, the base of each rectangle for x is the interval $x - .5$ to $x + .5$. Figure 9.7 depicts this graph. As you can see, the rectangle representing $x = 10$ is the rectangle whose base is the interval 9.5 to 10.5 and whose height is $P(X = 10) = .1762$.

FIGURE **9.7** Binomial Distribution with $n = 20$ and $p = .5$

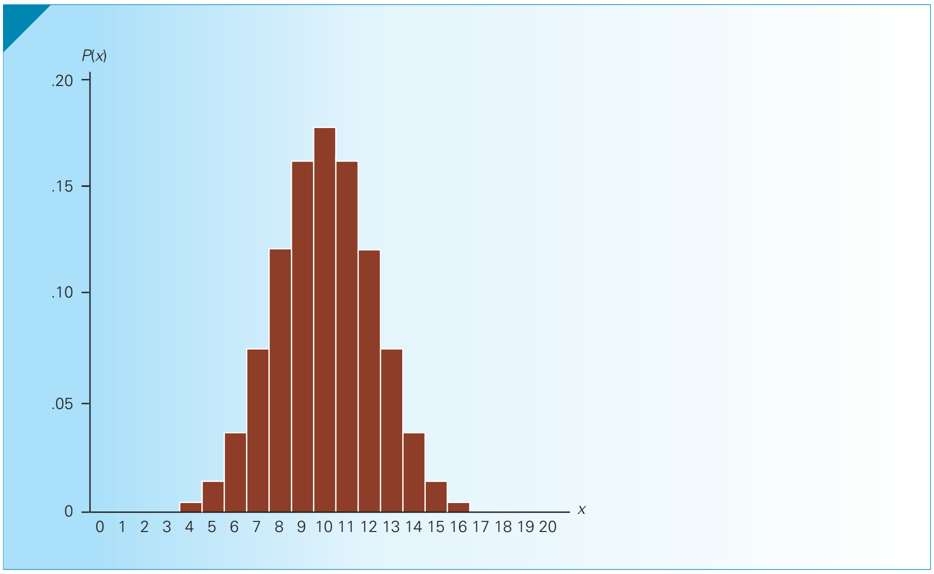
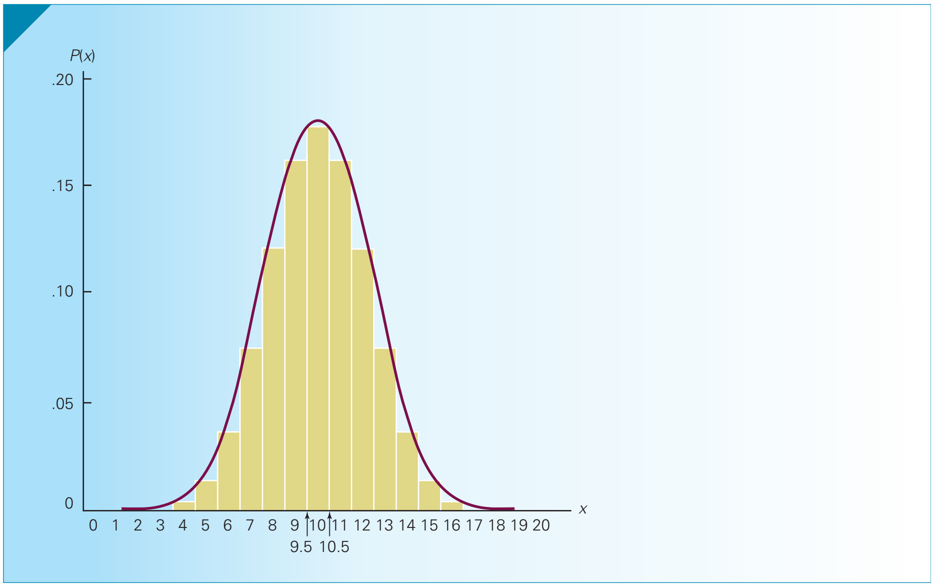


FIGURE **9.8** Binomial Distribution with $n = 20$ and $p = .5$ and Normal Approximation



If we now smooth the ends of the rectangles, we produce a bell-shaped curve as seen in Figure 9.8. Thus, to use the normal approximation, all we need do is find the area under the *normal* curve between 9.5 and 10.5.

To find normal probabilities requires us to first standardize x by subtracting the mean and dividing by the standard deviation. The values for μ and σ are derived from the binomial distribution being approximated. In Section 7.4 we pointed out that

$$\mu = np$$

and

$$\sigma = \sqrt{np(1-p)}$$

For $n = 20$ and $p = .5$, we have

$$\mu = np = 20(.5) = 10$$

and

$$\sigma = \sqrt{np(1-p)} = \sqrt{20(.5)(1-.5)} = 2.24$$

To calculate the probability that $X = 10$ using the normal distribution requires that we find the area under the normal curve between 9.5 and 10.5; that is,

$$P(X = 10) \approx P(9.5 < Y < 10.5)$$

where Y is a normal random variable approximating the binomial random variable X. We standardize Y and use Table 3 of Appendix B to find

$$P(9.5 < Y < 10.5) = P\left(\frac{9.5 - 10}{2.24} < \frac{Y - \mu}{\sigma} < \frac{10.5 - 10}{2.24}\right)$$

$$= P(-.22 < Z < .22) = (Z < .22) - P(Z < -.22)$$

$$= .5871 - .4129 = .1742$$

The actual probability that X equals 10 is

$$P(X = 10) = .1762$$

As you can see, the approximation is quite good.

Notice that to draw a binomial distribution, which is discrete, it was necessary to draw rectangles whose bases were constructed by adding and subtracting .5 to the values of X. The .5 is called the **continuity correction factor**.

The approximation for any other value of X would proceed in the same manner. In general, the binomial probability $P(X = x)$ is approximated by the area under a normal curve between $x - .5$ and $x + .5$. To find the binomial probability $P(X \leq x)$, we calculate the area under the normal curve to the left of $x + .5$. For the same binomial random variable, the probability that its value is less than or equal to 8 is $P(X \leq 8) = .2517$. The normal approximation is

$$P(X \leq 8) \approx P(Y < 8.5) = P\left(\frac{Y - \mu}{\sigma} < \frac{8.5 - 10}{2.24}\right) = P(Z < -.67) = .2514$$

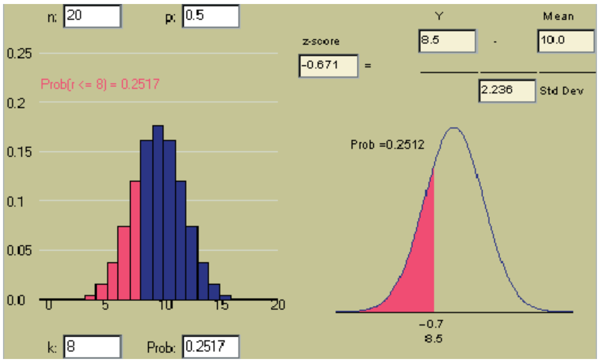
We find the area under the normal curve to the right of $x - .5$ to determine the binomial probability $P(X \geq x)$. To illustrate, the probability that the binomial random variable (with $n = 20$ and $p = .5$) is greater than or equal to 14 is $P(X \geq 14) = .0577$. The normal approximation is

$$P(X \geq 14) \approx P(Y > 13.5) = P\left(\frac{Y - \mu}{\sigma} > \frac{13.5 - 10}{2.24}\right) = P(Z > 1.56) = .0594$$

SEEING STATISTICS

applet 13 Normal Approximation to Binomial Probabilities

This applet shows how well the normal distribution approximates the binomial distribution. Select values for n and p, which will specify a binomial distribution. Then set a value for k. The applet calculates and graphs both the binomial and normal probabilities for $P(X \leq k)$.



Applet Exercises

13.1 Given a binomial distribution with $n = 5$ and $p = .2$, use the applet to compute the actual and normal approximations of the following.

a. $P(X \leq 0)$
b. $P(X \leq 1)$
c. $P(X \leq 2)$
d. $P(X \leq 3)$

Describe how well the normal distribution approximates the binomial when n and p are small.

13.2 Repeat Exercise 13.1 with $p = .5$. Describe how well the normal distribution approximates the binomial when n is small and when p is .5.

13.3 Suppose that X is a binomial random variable with $n = 10$ and $p = .2$. Use the applet to calculate the actual and normal approximations of the following.

a. $P(X \leq 2)$
b. $P(X \leq 3)$
c. $P(X \leq 4)$
d. $P(X \leq 5)$

Describe how well the normal distribution approximates the binomial when $n = 10$ and when p is small.

13.4 Repeat Exercise 13.3 with $p = .5$. Describe how well the normal distribution approximates the binomial when $n = 10$ and when p is .5.

13.5 Describe the effect on the normal approximation to the binomial as n increases.

Omitting the Correction Factor for Continuity

When calculating the probability of *individual* values of X as we did when we computed the probability that X equals 10 above, the correction factor *must* be used. If we don't, we are left with finding the area in a line, which is 0. When computing the probability of a *range* of values of X, we can omit the correction factor. However, the omission of the correction factor will decrease the accuracy of the approximation. For example, if we approximate $P(X \leq 8)$ as we did previously except without the correction factor, we find

$$P(X \leq 8) \approx P(Y < 8) = P\left(\frac{Y - \mu}{\sigma} < \frac{8 - 10}{2.24}\right) = P(Z < -.89) = .1867$$

The absolute size of the error between the actual cumulative binomial probability and its normal approximation is quite small when the values of x are in the tail regions of the distribution. For example, the probability that a binomial random variable with $n = 20$ and $p = .5$ is less than or equal to 3 is

$$P(X \leq 3) = .0013$$

The normal approximation with the correction factor is

$$P(X \leq 3) \approx P(Y < 3.5) = P\left(\frac{Y - \mu}{\sigma} < \frac{3.5 - 10}{2.24}\right) = P(Z < -2.90) = .0019$$

The normal approximation without the correction factor is (using Excel)

$$P(X \le 3) \approx P(Y < 3) = P\left(\frac{Y - \mu}{\sigma} < \frac{3 - 10}{2.24}\right) = P(Z < -3.13) = .0009$$

For larger values of n, the differences between the normal approximation with and without the correction factor are small even for values of X near the center of the distribution. For example, the probability that a binomial random variable with $n = 1000$ and $p = .3$ is less than or equal to 260 is

$P(X \le 260) = .0029$ (using Excel)

The normal approximation with the correction factor is

$$P(X \le 260) \approx P(Y < 260.5) = P\left(\frac{Y - \mu}{\sigma} < \frac{260.5 - 300}{14.49}\right) = P(Z < -2.73) = .0032$$

The normal approximation without the correction factor is

$$P(X \le 260) \approx P(Y < 260) = P\left(\frac{Y - \mu}{\sigma} < \frac{260 - 300}{14.49}\right) = P(Z < -2.76) = .0029$$

As we pointed out, the normal approximation of the binomial distribution is made necessary by the needs of statistical inference. As you will discover, statistical inference generally involves the use of large values of n, and the part of the sampling distribution that is of greatest interest lies in the tail regions. The correction factor was a temporary tool that allowed us to convince you that a binomial distribution can be approximated by a normal distribution. Now that we have done so, we will use the normal approximation of the binomial distribution to approximate the sampling distribution of a sample proportion, and in such applications the correction factor will be omitted.

Approximate Sampling Distribution of a Sample Proportion

Using the laws of expected value and variance (see Keller's website Appendix Using the Laws of Expected Value and Variance to Derive the Parameters of Sampling Distributions), we can determine the mean, variance, and standard deviation of $\hat{P}$. We will summarize what we have learned.

Sampling Distribution of a Sample Proportion

1. $\hat{P}$ is approximately normally distributed provided that np and $n(1 - p)$ are greater than or equal to 5.
2. The expected value: $E(\hat{P}) = p$
3. The variance: $V(\hat{P}) = \sigma^2_{\hat{p}} = \dfrac{p(1 - p)^*}{n}$
4. The standard deviation: $\sigma_{\hat{p}} = \sqrt{p(1 - p)/n}$

(The standard deviation of $\hat{P}$ is called the **standard error of the proportion**.)

*As was the case with the standard error of the mean (page 313), the standard error of a proportion is $\sqrt{p(1 - p)/n}$ when sampling from infinitely large populations. When the population is finite, the standard error of the proportion must include the finite population correction factor, which can be omitted when the population is large relative to the sample size, a very common occurrence in practice.

The sample size requirement is theoretical because, in practice, much larger sample sizes are needed for the normal approximation to be useful.

EXAMPLE 9.2 Political Survey

In the last election, a state representative received 52% of the votes cast. One year after the election, the representative organized a survey that asked a random sample of 300 people whether they would vote for him in the next election. If we assume that his popularity has not changed, what is the probability that more than half of the sample would vote for him?

SOLUTION

The number of respondents who would vote for the representative is a binomial random variable with $n = 300$ and $p = .52$. We want to determine the probability that the sample proportion is greater than 50%. In other words, we want to find $P(\hat{P} > .50)$.

We now know that the sample proportion $\hat{P}$ is approximately normally distributed with mean $p = .52$ and standard deviation $= \sqrt{p(1-p)/n} = \sqrt{(.52)(.48)/300} = .0288$.

Thus, we calculate

$$P(\hat{P} > .50) = P\left(\frac{\hat{P} - p}{\sqrt{p(1-p)/n}} > \frac{.50 - .52}{.0288}\right)$$
$$= P(Z > -.69) = 1 - P(Z < -.69) = 1 - .2451 = .7549$$

If we assume that the level of support remains at 52%, the probability that more than half the sample of 300 people would vote for the representative is .7549.

EXERCISES

Use the normal approximation without the correction factor to find the probabilities in the following exercises.

9.30 a. In a binomial experiment with $n = 300$ and $p = .5$, find the probability that $\hat{P}$ is greater than 60%.
b. Repeat part (a) with $p = .55$.
c. Repeat part (a) with $p = .6$

9.31 a. The probability of success on any trial of a binomial experiment is 25%. Find the probability that the proportion of successes in a sample of 500 is less than 22%.
b. Repeat part (a) with $n = 800$.
c. Repeat part (a) with $n = 1{,}000$.

9.32 Determine the probability that in a sample of 100 the sample proportion is less than .75 if $p = .80$.

9.33 A binomial experiment where $p = .4$ is conducted. Find the probability that in a sample of 60 the proportion of successes exceeds .35.

9.34 The proportion of eligible voters in the next election who will vote for the incumbent is assumed to be 55%. What is the probability that in a random sample of 500 voters less than 49% say they will vote for the incumbent?

9.35 The assembly line that produces an electronic component of a missile system has historically resulted in a 2% defective rate. A random sample of 800 components is drawn. What is the probability that the defective rate is greater than 4%? Suppose that in the random sample the defective rate is 4%. What does that suggest about the defective rate on the assembly line?

9.36 a. The manufacturer of aspirin claims that the proportion of headache sufferers who get relief with just two aspirins is 53%. What is the probability that in a random sample of 400 headache sufferers, less than 50% obtain relief? If 50% of the sample actually obtained relief, what does this suggest about the manufacturer's claim?
b. Repeat part (a) using a sample of 1,000.

9.37 The manager of a restaurant in a commercial building has determined that the proportion of customers who drink tea is 14%. What is the probability that in the next 100 customers at least 10% will be tea drinkers?

9.38 A commercial for a manufacturer of household appliances claims that 3% of all its products require a service call in the first year. A consumer protection association wants to check the claim by surveying 400 households that recently purchased one of the company's appliances. What is the probability that more than 5% require a service call within the first year? What would you say about the commercial's honesty if in a random sample of 400 households 5% report at least one service call?

9.39 The Laurier Company's brand has a market share of 30%. Suppose that 1,000 consumers of the product are asked in a survey which brand they prefer. What is the probability that more than 32% of the respondents say they prefer the Laurier brand?

9.40 A university bookstore claims that 50% of its customers are satisfied with the service and prices.

a. If this claim is true, what is the probability that in a random sample of 600 customers less than 45% are satisfied?

b. Suppose that in a random sample of 600 customers, 270 express satisfaction with the bookstore. What does this tell you about the bookstore's claim?

9.41 A psychologist believes that 80% of male drivers when lost continue to drive hoping to find the location they seek rather than ask directions. To examine this belief, he took a random sample of 350 male drivers and asked each what they did when lost. If the belief is true, determine the probability that less than 75% said they continue driving.

9.42 The Red Lobster restaurant chain regularly surveys its customers. On the basis of these surveys, the management of the chain claims that 75% of its customers rate the food as excellent. A consumer testing service wants to examine the claim by asking 460 customers to rate the food. What is the probability that less than 70% rate the food as excellent?

9.43 An accounting professor claims that no more than one-quarter of undergraduate business students will major in accounting. What is the probability that in a random sample of 1,200 undergraduate business students, 336 or more will major in accounting?

9.44 Refer to Exercise 9.43. A survey of a random sample of 1,200 undergraduate business students indicates that 336 students plan to major in accounting. What does this tell you about the professor's claim?

9.3 Sampling Distribution of the Difference between Two Means

Another sampling distribution that you will soon encounter is that of the **difference between two sample means**. The sampling plan calls for independent random samples drawn from each of two normal populations. The samples are said to be independent if the selection of the members of one sample is independent of the selection of the members of the second sample. We will expand upon this discussion in Chapter 13. We are interested in the sampling distribution of the difference between the two sample means.

In Section 9.1, we introduced the central limit theorem, which states that in repeated sampling from a normal population whose mean is μ and whose standard deviation is σ, the sampling distribution of the sample mean is normal with mean μ and standard deviation $\sigma/\sqrt{n}$. Statisticians have shown that the difference between two independent normal random variables is also normally distributed. Thus, the difference between two sample means $\bar{X}_1 - \bar{X}_2$ is normally distributed if both populations are normal.

Through the use of the laws of expected value and variance we derive the expected value and variance of the **sampling distribution of $\bar{X}_1 - \bar{X}_2$**:

$$\mu_{\bar{x}_1 - \bar{x}_2} = \mu_1 - \mu_2$$

and

$$\sigma^2_{\bar{x}_1 - \bar{x}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Thus, it follows that in repeated independent sampling from two populations with means μ_1 and μ_2 and standard deviations σ_1 and σ_2, respectively, the sampling distribution of $\bar{X}_1 - \bar{X}_2$ is normal with mean

$$\mu_{\bar{x}_1 - \bar{x}_2} = \mu_1 - \mu_2$$

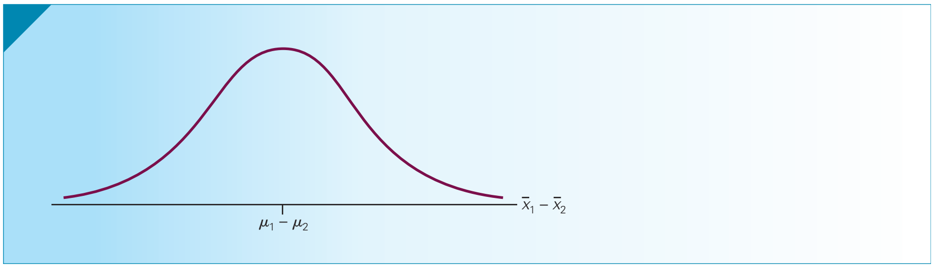
and standard deviation (which is the **standard error of the difference between two means**)

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

If the populations are nonnormal, then the sampling distribution is only approximately normal for large sample sizes. The required sample sizes depend on the extent of nonnormality. However, for most populations, sample sizes of 30 or more are sufficient.

Figure 9.9 depicts the sampling distribution of the difference between two means.

FIGURE **9.9** Sampling Distribution of $\bar{X}_1 - \bar{X}_2$



EXAMPLE 9.3

Starting Salaries of MBAs

Suppose that the starting salaries of MBAs at Wilfrid Laurier University (WLU) are normally distributed, with a mean of \$62,000 and a standard deviation of \$14,500. The starting salaries of MBAs at the University of Western Ontario (UWO) are normally distributed, with a mean of \$60,000 and a standard deviation of \$18,300. If a random sample of 50 WLU MBAs and a random sample of 60 UWO MBAs are selected, what is the probability that the sample mean starting salary of WLU graduates will exceed that of the UWO graduates?

SOLUTION

We want to determine $P(\bar{X}_1 - \bar{X}_2 > 0)$. We know that $\bar{X}_1 - \bar{X}_2$ is normally distributed with mean $\mu_1 - \mu_2 = 62{,}000 - 60{,}000 = 2{,}000$ and standard deviation

$$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{14{,}500^2}{50} + \frac{18{,}300^2}{60}} = 3{,}128$$

We can standardize the variable and refer to Table 3 of Appendix B:

$$P(\bar{X}_1 - \bar{X}_2 > 0) = P\left(\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > \frac{0 - 2{,}000}{3{,}128}\right)$$

$$= P(Z > -.64) = 1 - P(Z < -.64) = 1 - .2611 = .7389$$

There is a .7389 probability that for a sample of size 50 from the WLU graduates and a sample of size 60 from the UWO graduates, the sample mean starting salary of WLU graduates will exceed the sample mean of UWO graduates.

SEEING STATISTICS

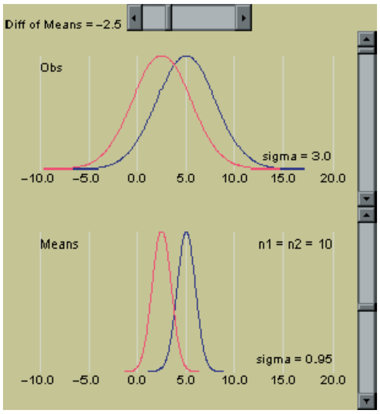
applet 14 Distribution of the Differences between Means

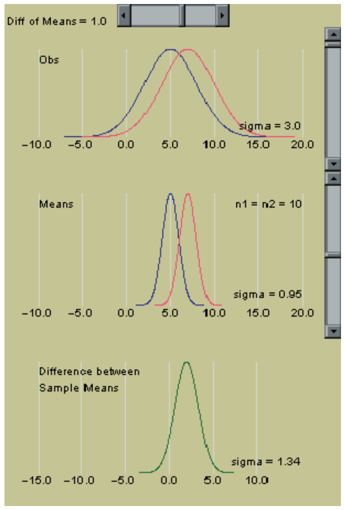
The first part of this applet depicts two graphs. The first graph shows the distribution of the random variable of two populations. Moving the top slider shifts the first distribution left or right. The right slider controls the value of the population standard deviations, which are assumed to be equal. By moving each slider, you can see the relationship between the two populations.

The second graph describes the sampling distribution of the mean of each population in the first graph. Moving the right slider increases or decreases the sample size, which is the same for both samples.

The second part of the applet has three graphs. The first two graphs are identical to the graphs in the first part. The third graph depicts the sampling distribution of the difference between the two sample means from the populations described previously.

Moving the sliders allows you to see the effect on the sampling distribution of $\bar{x}_1 - \bar{x}_2$ of changing the relationship among the two population means, the common population standard deviation, and the sample size.





Applet Exercises

14.1 Describe the effect of changing the difference between population means from − 5.0 to 4.5 on the population random variables, the sampling distribution of $\bar{x}_1$, the sampling distribution of $\bar{x}_2$, and the sampling distribution of $\bar{x}_1 - \bar{x}_2$. Describe what happened.

14.2 Describe the effect of changing the standard deviations from $\sigma_1 = \sigma_2 = 1.1$ to $\sigma_1 = \sigma_2 = 3.0$ on the population random variables, the sampling distribution of $\bar{x}_1$, the sampling distribution of $\bar{x}_2$, and the sampling distribution of $\bar{x}_1 - \bar{x}_2$. Describe what happened.

14.3 Describe the effect of changing the sample sizes from $n_1 = n_2 = 2$ to $n_1 = n_2 = 20$ on the sampling distribution of $\bar{x}_1$, the sampling distribution of $\bar{x}_2$, and the sampling distribution of $\bar{x}_1 - \bar{x}_2$. Describe the effect.

EXERCISES

9.45 Independent random samples of 10 observations each are drawn from normal populations. The parameters of these populations are

Population 1: $\mu = 280, \sigma = 25$
Population 2: $\mu = 270, \sigma = 30$

Find the probability that the mean of sample 1 is greater than the mean of sample 2 by more than 25.

9.46 Repeat Exercise 9.45 with samples of size 50.

9.47 Repeat Exercise 9.45 with samples of size 100.

9.48 Suppose that we have two normal populations with the means and standard deviations listed here. If random samples of size 25 are drawn from each population, what is the probability that the mean of sample 1 is greater than the mean of sample 2?

Population 1: $\mu = 40$, $\sigma = 6$
Population 2: $\mu = 38$, $\sigma = 8$

9.49 Repeat Exercise 9.48 assuming that the standard deviations are 12 and 16, respectively.

9.50 Repeat Exercise 9.48 assuming that the means are 140 and 138, respectively.

9.51 A factory's worker productivity is normally distributed. One worker produces an average of 75 units per day with a standard deviation of 20. Another worker produces at an average rate of 65 per day with a standard deviation of 21. What is the probability that during one week (5 working days), worker 1 will outproduce worker 2?

9.52 A professor of statistics noticed that the marks in his course are normally distributed. He has also noticed that his morning classes average 73%, with a standard deviation of 12% on their final exams. His afternoon classes average 77%, with a standard deviation of 10%. What is the probability that the mean mark of four randomly selected students from a morning class is greater than the average mark of four randomly selected students from an afternoon class?

9.53 The manager of a restaurant believes that waiters and waitresses who introduce themselves by telling customers their names will get larger tips than those who don't. In fact, she claims that the average tip for the former group is 18%, whereas that of the latter is only 15%. If tips are normally distributed with a standard deviation of 3%, what is the probability that in a random sample of 10 tips recorded from waiters and waitresses who introduce themselves and 10 tips from waiters and waitresses who don't, the mean of the former will exceed that of the latter?

9.54 The average North American loses an average of 15 days per year to colds and flu. The natural remedy echinacea reputedly boosts the immune system. One manufacturer of echinacea pills claims that consumers of its product will reduce the number of days lost to colds and flu by one-third. To test the claim, a random sample of 50 people was drawn. Half took echinacea, and the other half took placebos. If we assume that the standard deviation of the number of days lost to colds and flu with and without echinacea is 3 days, find the probability that the mean number of days lost for echinacea users is less than that for nonusers.

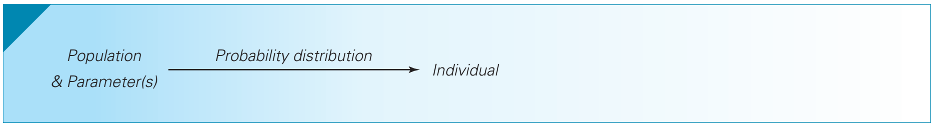
9.4 From Here to Inference

The primary function of the sampling distribution is statistical inference. To see how the sampling distribution contributes to the development of inferential methods, we need to briefly review how we got to this point.

In Chapters 7 and 8, we introduced probability distributions, which allowed us to make probability statements about values of the random variable. A prerequisite of this calculation is knowledge of the distribution and the relevant parameters. In Example 7.9, we needed to know that the probability that Pat Statsdud guesses the correct answer is 20% ($p = .2$) and that the number of correct answers (successes) in 10 questions (trials) is a binomial random variable. We could then compute the probability of any number of successes. In Example 8.3, we needed to know that the return on investment is normally distributed with a mean of 10% and a standard deviation of 5%. These three bits of information allowed us to calculate the probability of various values of the random variable.

Figure 9.10 symbolically represents the use of probability distributions. Simply put, knowledge of the population and its parameter(s) allows us to use the probability distribution to make probability statements about individual members of the population. The direction of the arrows indicates the direction of the flow of information.

FIGURE **9.10** **Probability Distribution**

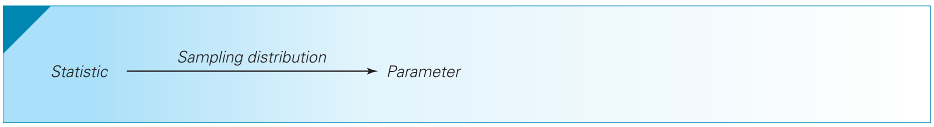


In this chapter, we developed the sampling distribution, wherein knowledge of the parameter(s) and some information about the distribution allow us to make probability statements about a sample statistic. In Example 9.1(b), knowing the population mean and standard deviation and assuming that the population is not extremely nonnormal enabled us to calculate a probability statement about a sample mean. Figure 9.11 describes the application of sampling distributions.

FIGURE **9.11** **Sampling Distribution**

Notice that in applying both probability distributions and sampling distributions, we must know the value of the relevant parameters, a highly unlikely circumstance. In the real world, parameters are almost always unknown because they represent descriptive measurements about extremely large populations. Statistical inference addresses this problem. It does so by reversing the direction of the flow of knowledge in Figure 9.11. In Figure 9.12, we display the character of statistical inference. Starting in Chapter 10, we will assume that most population parameters are unknown. The statistics practitioner will sample from the population and compute the required statistic. The sampling distribution of that statistic will enable us to draw inferences about the parameter.

FIGURE **9.12** **Sampling Distribution in Inference**



You may be surprised to learn that, by and large, that is all we do in the remainder of this book. Why then do we need another 14 chapters? They are necessary because there are many more parameter and sampling distribution combinations that define the inferential procedures to be presented in an introductory statistics course. However, they all work in the same way. If you understand how one procedure is developed, then you will likely understand all of them. Our task in the next two chapters is to ensure that you understand the first inferential method. Your job is identical.

Chapter Summary

The sampling distribution of a statistic is created by repeated sampling from one population. In this chapter, we introduced the sampling distribution of the mean, the proportion, and the difference between two means. We described how these distributions are created theoretically and empirically.

IMPORTANT TERMS

Sampling distribution 308
Sampling distribution of the sample mean 310
Standard error of the mean 312
Central limit theorem 312
Finite population correction factor 313
Sampling distribution of a sample proportion 321
Continuity correction factor 323
Standard error of the proportion 325
Difference between two sample means 327
Sampling distribution of $\bar{X}_1 - \bar{X}_2$ 327
Standard error of the difference between two means 328

SYMBOLS

Symbol	Pronounced	Represents
$\mu_{\bar{x}}$	mu *x* bar	Mean of the sampling distribution of the sample mean
$\sigma^2_{\bar{x}}$	sigma squared *x* bar	Variance of the sampling distribution of the sample mean
$\sigma_{\bar{x}}$	sigma *x* bar	Standard deviation (standard error) of the sampling distribution of the sample mean
α	alpha	Probability
$\hat{P}$	*p* hat	Sample proportion
$\sigma^2_{\hat{p}}$	sigma squared *p* hat	Variance of the sampling distribution of the sample proportion
$\sigma_{\hat{p}}$	sigma *p* hat	Standard deviation (standard error) of the sampling distribution of the sample proportion
$\mu_{\bar{x}_1-\bar{x}_2}$	mu *x* bar 1 minus *x* bar 2	Mean of the sampling distribution of the difference between two sample means
$\sigma^2_{\bar{x}_1-\bar{x}_2}$	sigma squared *x* bar 1 minus *x* bar 2	Variance of the sampling distribution of the difference between two sample means
$\sigma_{\bar{x}_1-\bar{x}_2}$	sigma *x* bar 1 minus *x* bar 2	Standard deviation (standard error) of the sampling distribution of the difference between two sample means

FORMULAS

Expected value of the sample mean

$$E(\bar{X}) = \mu_{\bar{x}} = \mu$$

Variance of the sample mean

$$V(\bar{X}) = \sigma^2_{\bar{x}} = \frac{\sigma^2}{n}$$

Standard error of the sample mean

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

Standardizing the sample mean

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Expected value of the sample proportion

$$E(\hat{P}) = \mu_{\hat{p}} = p$$

Variance of the sample proportion

$$V(\hat{P}) = \sigma^2_{\hat{p}} = \frac{p(1-p)}{n}$$

Standard error of the sample proportion

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$

Standardizing the sample proportion

$$Z = \frac{\hat{P} - p}{\sqrt{p(1-p)/n}}$$

Expected value of the difference between two means

$$E(\overline{X}_1 - \overline{X}_2) = \mu_{\bar{x}_1 - \bar{x}_2} = \mu_1 - \mu_2$$

Variance of the difference between two means

$$V(\overline{X}_1 - \overline{X}_2) = \sigma^2_{\bar{x}_1 - \bar{x}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Standard error of the difference between two means

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Standardizing the difference between two sample means

$$Z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

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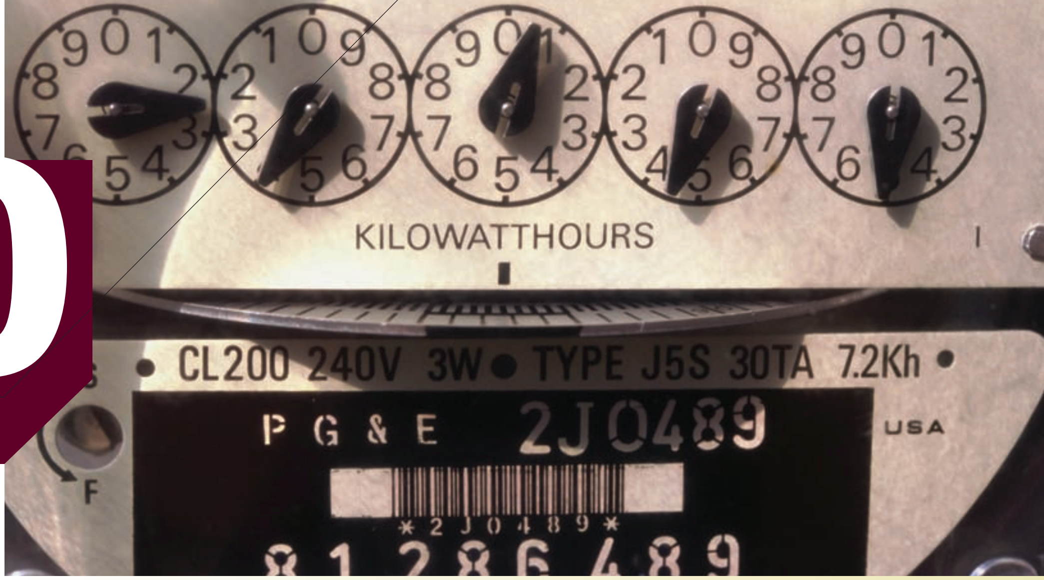
10



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INTRODUCTION TO ESTIMATION

Determining the Sample Size to Estimate the Mean Tree Diameter

A lumber company has just acquired the rights to a large tract of land containing thousands of trees.

A lumber company needs to be able to estimate the amount of lumber it can harvest in a tract of land to determine whether the effort will be profitable. To do so, it must estimate the mean diameter of the trees. It decides to estimate that parameter to within 1 inch with 90% confidence. A forester familiar with the territory guesses that the diameters of the trees are normally distributed with a standard deviation of 6 inches. Using the formula on page 355, he determines that he should sample 98 trees. After sampling those 98 trees, the forester calculates the sample mean to be 25 inches. Suppose that after he has completed his sampling and calculations, he discovers that the actual standard deviation is 12 inches. Will he be satisfied with the result?

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See page 355 for the solution.

Introduction

Having discussed descriptive statistics (Chapter 4), probability distributions (Chapters 7 and 8), and sampling distributions (Chapter 9), we are ready to tackle statistical inference. As we explained in Chapter 1, *statistical inference* is the process by which we acquire information and draw conclusions about populations from samples. There are two general procedures for making inferences about populations: *estimation* and *hypothesis testing*. In this chapter, we introduce the concepts and foundations of estimation and demonstrate them with simple examples. In Chapter 11, we describe the fundamentals of hypothesis testing. Because most of what we do in the remainder of this book applies the concepts of estimation and hypothesis testing, understanding Chapters 10 and 11 is vital to your development as a statistics practitioner.

10.1 Concepts of Estimation

As its name suggests, the objective of estimation is to determine the approximate value of a population parameter on the basis of a sample statistic. For example, the sample mean is employed to estimate the population mean. We refer to the sample mean as the *estimator* of the population mean. Once the sample mean has been computed, its value is called the *estimate*. In this chapter, we will introduce the statistical process whereby we estimate a population mean using sample data. In the rest of the book, we use the concepts and techniques introduced here for other parameters.

Point and Interval Estimators

We can use sample data to estimate a population parameter in two ways. First, we can compute the value of the estimator and consider that value as the estimate of the parameter. Such an estimator is called a *point estimator*:

Point Estimator

A **point estimator** draws inferences about a population by estimating the value of an unknown parameter using a single value or point.

There are three drawbacks to using point estimators. First, it is virtually certain that the estimate will be wrong. (The probability that a continuous random variable will equal a specific value is 0; that is, the probability that $\bar{x}$ will exactly equal μ is 0.) Second, we often need to know how close the estimator is to the parameter. Third, in drawing inferences about a population, it is intuitively reasonable to expect that a large sample will produce more accurate results because it contains more information than a smaller sample does. But point estimators don't have the capacity to reflect the effects of larger sample sizes. As a consequence, we use the second method of estimating a population parameter, the *interval estimator*.

Interval Estimator

An **interval estimator** draws inferences about a population by estimating the value of an unknown parameter using an interval.

As you will see, the interval estimator is affected by the sample size; because it possesses this feature, we will deal mostly with interval estimators in this text.

To illustrate the difference between point and interval estimators, suppose that a statistics professor wants to estimate the mean summer income of his second-year business students. Selecting 25 students at random, he calculates the sample mean weekly income to be \$400. The point estimate is the sample mean. In other words, he estimates the mean weekly summer income of all second-year business students to be \$400. Using the technique described subsequently, he may instead use an interval estimate; he estimates that the mean weekly summer income of second-year business students to lie between \$380 and \$420.

Numerous applications of estimation occur in the real world. For example, television network executives want to know the proportion of television viewers who are tuned in to their networks; an economist wants to know the mean income of university graduates; and a medical researcher wishes to estimate the recovery rate of heart attack victims treated with a new drug. In each of these cases, to accomplish the objective exactly, the statistics practitioner would have to examine each member of the population and then calculate the parameter of interest. For instance, network executives would have to ask each person in the country what he or she is watching to determine the proportion of people who are watching their shows. Because there are millions of television viewers, the task is both impractical and prohibitively expensive. An alternative would be to take a random sample from this population, calculate the sample proportion, and use that as an estimator of the population proportion. The use of the sample proportion to estimate the population proportion seems logical. The selection of the sample statistic to be used as an estimator, however, depends on the characteristics of that statistic. Naturally, we want to use the statistic with the most desirable qualities for our purposes.

One desirable quality of an estimator is *unbiasedness*.

Unbiased Estimator

An **unbiased estimator** of a population parameter is an estimator whose expected value is equal to that parameter.

This means that if you were to take an infinite number of samples and calculate the value of the estimator in each sample, the average value of the estimators would equal the parameter. This amounts to saying that, on average, the sample statistic is equal to the parameter.

We know that the sample mean $\overline{X}$ is an unbiased estimator of the population mean μ. In presenting the sampling distribution of $\overline{X}$ in Section 9.1, we stated that $E(\overline{X}) = \mu$. We also know that the sample proportion is an unbiased estimator of the population proportion because $E(\hat{P}) = p$ and that the difference between two sample means is an unbiased estimator of the difference between two population means because $E(\overline{X}_1 - \overline{X}_2) = \mu_1 - \mu_2$.

Recall that in Chapter 4 we defined the sample variance as

$$s^2 = \sum \frac{(x_i - \bar{x})^2}{n - 1}$$

At the time, it seemed odd that we divided by $n - 1$ rather than by n. The reason for choosing $n - 1$ was to make $E(s^2) = \sigma^2$ so that this definition makes the sample variance an unbiased estimator of the population variance. (The proof of this statement requires about a page of algebraic manipulation, which is more than we would be comfortable presenting here.) Had we defined the sample variance using n in the denominator, the resulting statistic would be a biased estimator of the population variance, one whose expected value is less than the parameter.

Knowing that an estimator is unbiased only assures us that its expected value equals the parameter; it does not tell us how close the estimator is to the parameter. Another desirable quality is that as the sample size grows larger, the sample statistic should come closer to the population parameter. This quality is called *consistency*.

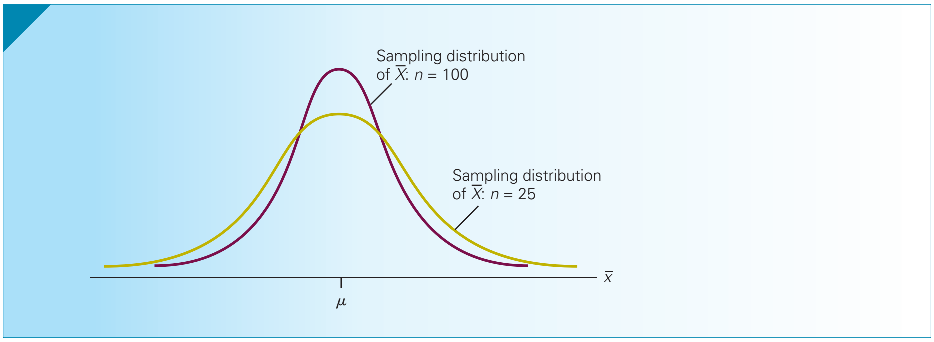
Consistency

An unbiased estimator is said to be **consistent** if the difference between the estimator and the parameter grows smaller as the sample size grows larger.

The measure we use to gauge closeness is the variance (or the standard deviation). Thus, $\overline{X}$ is a consistent estimator of μ because the variance of $\overline{X}$ is σ^2/n. This implies that as n grows larger, the variance of $\overline{X}$ grows smaller. As a consequence, an increasing proportion of sample means falls close to μ.

Figure 10.1 depicts two sampling distributions of $\overline{X}$ when samples are drawn from a population whose mean is 0 and whose standard deviation is 10. One sampling distribution is based on samples of size 25, and the other is based on samples of size 100. The former is more spread out than the latter.

FIGURE **10.1** Sampling Distribution of $\overline{X}$ with $n = 25$ and $n = 100$



Similarly, $\hat{P}$ is a consistent estimator of p because it is unbiased and the variance of $\hat{P}$ is $p(1 - p)/n$, which grows smaller as n grows larger.

A third desirable quality is *relative efficiency*, which compares two unbiased estimators of a parameter.

Relative Efficiency

If there are two unbiased estimators of a parameter, the one whose variance is smaller is said to have **relative efficiency**.

We have already seen that the sample mean is an unbiased estimator of the population mean and that its variance is σ^2/n. In the next section, we will discuss the use of the sample median as an estimator of the population mean. Statisticians have established that the sample median is an unbiased estimator but that its variance is greater than that of the sample mean (when the population is normal). As a consequence, the sample mean is relatively more efficient than the sample median when estimating the population mean.

In the remaining chapters of this book, we will present the statistical inference of a number of different population parameters. In each case, we will select a sample statistic that is unbiased and consistent. When there is more than one such statistic, we will choose the one that is relatively efficient to serve as the estimator.

Developing an Understanding of Statistical Concepts

In this section, we described three desirable characteristics of estimators: unbiasedness, consistency, and relative efficiency. An understanding of statistics requires that you know that there are several potential estimators for each parameter, but that we choose the estimators used in this book because they possess these characteristics.

EXERCISES

10.1 How do point estimators and interval estimators differ?

10.2 Define unbiasedness.

10.3 Draw a sampling distribution of an unbiased estimator.

10.4 Draw a sampling distribution of a biased estimator.

10.5 Define consistency.

10.6 Draw diagrams representing what happens to the sampling distribution of a consistent estimator when the sample size increases.

10.7 Define relative efficiency.

10.8 Draw a diagram that shows the sampling distribution representing two unbiased estimators, one of which is relatively efficient.

10.2 ESTIMATING THE POPULATION MEAN WHEN THE POPULATION STANDARD DEVIATION IS KNOWN

We now describe how an interval estimator is produced from a sampling distribution. We choose to demonstrate estimation with an example that is unrealistic. However, this liability is offset by the example's simplicity. When you understand more about estimation, you will be able to apply the technique to more realistic situations.

Suppose we have a population with mean μ and standard deviation σ. The population mean is assumed to be unknown, and our task is to estimate its value. As we just discussed, the estimation procedure requires the statistics practitioner to draw a random sample of size n and calculate the sample mean $\bar{x}$.

The central limit theorem presented in Section 9.1 stated that $\bar{X}$ is normally distributed if X is normally distributed, or approximately normally distributed if X is nonnormal and n is sufficiently large. This means that the variable

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

is standard normally distributed (or approximately so). In Section 9.1 (page 318) we developed the following probability statement associated with the sampling distribution of the mean:

$$P\left(\mu - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X} < \mu + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

which was derived from

$$P\left(-Z_{\alpha/2} < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < Z_{\alpha/2}\right) = 1 - \alpha$$

Using a similar algebraic manipulation, we can express the probability in a slightly different form:

$$P\left(\overline{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

Notice that in this form the population mean is in the center of the interval created by adding and subtracting $Z_{\alpha/2}$ standard errors to and from the sample mean. It is important for you to understand that this is merely another form of probability statement about the sample mean. This equation says that, with repeated sampling from this population, the proportion of values of $\overline{X}$ for which the interval

$$\overline{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \quad \overline{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

includes the population mean μ is equal to $1 - \alpha$. This form of probability statement is very useful to us because it is the **confidence interval estimator of μ**.

Confidence Interval Estimator of μ*

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \quad \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

The probability $1 - \alpha$ is called the **confidence level**.

$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ is called the **lower confidence limit (LCL)**.

$\bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ is called the **upper confidence limit (UCL)**.

We often represent the confidence interval estimator as

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where the minus sign defines the lower confidence limit and the plus sign defines the upper confidence limit.

To apply this formula, we specify the confidence level $1 - \alpha$, from which we determine α, $\alpha/2$, $z_{\alpha/2}$ (from Table 3 in Appendix B). Because the confidence level is the

*Since Chapter 7, we've been using the convention whereby an uppercase letter (usually X) represents a random variable and a lowercase letter (usually x) represents one of its values. However, in the formulas used in statistical inference, the distinction between the variable and its value becomes blurred. Accordingly, we will discontinue the notational convention and simply use lowercase letters except when we wish to make a probability statement.

probability that the interval includes the actual value of μ, we generally set $1 - \alpha$ close to 1 (usually between .90 and .99).

In Table 10.1, we list four commonly used confidence levels and their associated values of $z_{\alpha/2}$. For example, if the confidence level is $1 - \alpha = .95$, $\alpha = .05$, $\alpha/2 = .025$, and $z_{\alpha/2} = z_{0.25} = 1.96$. The resulting confidence interval estimator is then called the **95% confidence interval estimator of μ.**

TABLE 10.1 Four Commonly Used Confidence Levels and $z_{\alpha/2}$

$1 - \alpha$	α	$\alpha/2$	$z_{\alpha/2}$
.90	.10	.05	$z_{0.05} = 1.645$
.95	.05	.025	$z_{0.25} = 1.96$
.98	.02	.01	$z_{.01} = 2.33$
.99	.01	.005	$z_{.005} = 2.575$

The following example illustrates how statistical techniques are applied. It also illustrates how we intend to solve problems in the rest of this book. The solution process that we advocate and use throughout this book is by and large the same one that statistics practitioners use to apply their skills in the real world. The process is divided into three stages. Simply stated, the stages are (1) the activities we perform before the calculations, (2) the calculations, and (3) the activities we perform after the calculations.

In stage 1, we determine the appropriate statistical technique to employ. Of course, for this example you will have no difficulty identifying the technique because you know only one at this point. (In practice, stage 1 also addresses the problem of *how* to gather the data. The methods used in the examples, exercises, and cases are described in the problem.)

In the second stage we calculate the statistics. We will do this in three ways.* To illustrate how the computations are completed, we will do the arithmetic manually with the assistance of a calculator. Solving problems by hand often provides insights into the statistical inference technique. Additionally, we will use the computer in two ways. First, in Excel we will use the Analysis ToolPak (**Data** menu item **Data Analysis**) or the add-ins we created for this book (**Add-Ins** menu item **Data Analysis Plus**). (Additionally, we will teach how to create do-it-yourself Excel spreadsheets that use built-in statistical functions.) Finally, we will use Minitab, one of the easiest software packages to use.

In the third and last stage of the solution, we intend to interpret the results and deal with the question presented in the problem. To be capable of properly interpreting statistical results, one needs to have an understanding of the fundamental principles underlying statistical inference.

*We anticipate that students in most statistics classes will use only one of the three methods of computing statistics: the choice made by the instructor. If such is the case, readers are directed to ignore the other two.

APPLICATIONS in OPERATIONS MANAGEMENT

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Inventory Management

Operations managers use inventory models to determine the stock level that minimizes total costs. In Section 8.2, we showed how the probabilistic model is used to make the inventory level decision (see page 287). One component of that model is the mean demand during lead time. Recall that *lead time* refers to the interval between the time an order is made and when it is delivered. Demand during lead time is a random variable that is often assumed to be normally distributed. There are several ways to determine mean demand during lead time, but the simplest is to estimate that quantity from a sample.

EXAMPLE 10.1 Doll Computer Company

DATA Xm10-01

The Doll Computer Company makes its own computers and delivers them directly to customers who order them via the Internet. Doll competes primarily on price and speed of delivery. To achieve its objective of speed, Doll makes each of its five most popular computers and transports them to warehouses across the country. The computers are stored in the warehouses from which it generally takes 1 day to deliver a computer to the customer. This strategy requires high levels of inventory that add considerably to the cost. To lower these costs, the operations manager wants to use an inventory model. He notes that both daily demand and lead time are random variables. He concludes that demand during lead time is normally distributed, and he needs to know the mean to compute the optimum inventory level. He observes 25 lead time periods and records the demand during each period. These data are listed here. The manager would like a 95% confidence interval estimate of the mean demand during lead time. From long experience, the manager knows that the standard deviation is 75 computers.

Demand During Lead Time

235	374	309	499	253
421	361	514	462	369
394	439	348	344	330
261	374	302	466	535
386	316	296	332	334

SOLUTION

IDENTIFY

To ultimately determine the optimum inventory level, the manager must know the mean demand during lead time. Thus, the parameter to be estimated is μ. At this point, we have described only one estimator. Thus, the confidence interval estimator that we intend to use is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

The next step is to perform the calculations. As we discussed previously, we will perform the calculations in three ways: manually, using Excel, and using Minitab.

COMPUTE

MANUALLY

We need four values to construct the confidence interval estimate of μ. They are

$$\bar{x},\ z_{\alpha/2},\ \sigma,\ n$$

Using a calculator, we determine the summation $\sum x_i = 9{,}254$. From this, we find

$$\bar{x} = \frac{\sum x_i}{n} = \frac{9{,}254}{25} = 370.16$$

The confidence level is set at 95%; thus, $1 - \alpha = .95$, $\alpha = 1 - .95 = .05$, and $\alpha/2 = .025$.

From Table 3 in Appendix B or from Table 10.1, we find

$$z_{\alpha/2} = z_{.025} = 1.96$$

The population standard deviation is $\sigma = 75$, and the sample size is 25. Substituting $\bar{x}$, $Z_{\alpha/2}$, σ, and n into the confidence interval estimator, we find

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 370.16 \pm z_{.025}\frac{75}{\sqrt{25}} = 370.16 \pm 1.96\frac{75}{\sqrt{25}} = 370.16 \pm 29.40$$

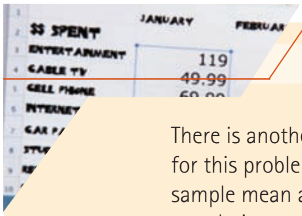
The lower and upper confidence limits are LCL = 340.76 and UCL = 399.56, respectively.

EXCEL

	A	B	C
1	***z*-Estimate: Mean**		
2			
3			*Demand*
4	Mean		370.16
5	Standard Deviation		80.783
6	Observations		25
7	SIGMA		75
8	LCL		340.76
9	UCL		399.56

INSTRUCTIONS

1. Type or import the data into one column. (Open Xm10-01.)
2. Click **Add-Ins, Data Analysis Plus,** and **Z-Estimate: Mean.**
3. Fill in the dialog box: **Input Range** (A1:A26), type the value for the **Standard Deviation** (75), click Labels if the first row contains the name of the variable, and specify the confidence level by typing the value of α (.05).



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DO-IT-YOURSELF **EXCEL**

There is another way to produce the interval estimate for this problem. If you have already calculated the sample mean and know the sample size and population standard deviation, you need not employ the data set and **Data Analysis Plus** described above. Instead you can create a spreadsheet that performs the calculations. Our suggested spreadsheet is shown next.

	A	B	C	D	E
1	z-Estimate of a Mean				
2					
3	Sample mean	370.16	Confidence Interval Estimate		
4	Population standard deviation	75	370.16		29.40
5	Sample size	25	Lower confidence limit		340.76
6	Confidence level	0.95	Upper confidence limit		399.56

Here are the tools (Excel functions) you will need to create this spreadsheet.

SQRT: Syntax: SQRT(*X*) Computes the square root of the quantity in parentheses.

Use the **Insert** and **Ω Symbol** to input the ± sign.

NORMSINV: Syntax: NORMSINV (Probability) This function calculates the value of z such that $P(Z < z)$ = the probability in parentheses. For example, NORMSINV(.95) determines the value of $z_{.05}$, which is 1.645. You will need to figure out how to convert the confidence level specified in Cell B6 into the value for $z_{\alpha/2}$.

We recommend that you save the spreadsheet. It can be used to solve some of the exercises at the end of this section.

In addition to providing another method of using Excel, this spreadsheet allows you to perform a "what-if" analysis; that is, this worksheet provides you the opportunity to learn how changing some of the inputs affects the estimate. For example, type 0.99 in cell B6 to see what happens to the size of the interval when you increase the confidence level. Type 1000 in cell B5 to examine the effect of increasing the sample size. Type 10 in cell B4 to see what happens when the population standard deviation is smaller.

MINITAB

One-Sample *Z*: Demand

The assumed standard deviation = 75

Variable	N	Mean	StDev	SE Mean	95% CI
Demand	25	370.160	80.783	15.000	(340.761, 399.559)

The output includes the sample standard deviation (**StDev** = 80.783), which is not needed for this interval estimate. Also printed is the standard error (**SE Mean** $= \sigma/\sqrt{n} = 15.0$), and last, but not least, the 95% confidence interval estimate of the population mean.

INSTRUCTIONS

1. Type or import that data into one column. (Open Xm10-01.)
2. Click **Stat, Basic Statistics,** and **1-Sample *Z*. . . .**

3. Type or use the **Select** button to specify the name of the variable or the column it is stored in. In the **Samples in columns** box (Demand), type the value of the population standard deviation (75), and click **Options** . . . .
4. Type the value for the confidence level (.95) and in the **Alternative** box select **not equal**.

INTERPRET

The operations manager estimates that the mean demand during lead time lies between 340.76 and 399.56. He can use this estimate as an input in developing an inventory policy. The model discussed in Section 8.2 computes the reorder point, assuming a particular value of the mean demand during lead time. In this example, he could have used the sample mean as a point estimator of the mean demand, from which the inventory policy could be determined. However, the use of the confidence interval estimator allows the manager to use both the lower and upper limits so that he can understand the possible outcomes.

Interpreting the Confidence Interval Estimate

Some people erroneously interpret the confidence interval estimate in Example 10.1 to mean that there is a 95% probability that the population mean lies between 340.76 and 399.56. This interpretation is wrong because it implies that the population mean is a variable about which we can make probability statements. In fact, the population mean is a fixed but unknown quantity. Consequently, we cannot interpret the confidence interval estimate of μ as a probability statement about μ. To translate the confidence interval estimate properly, we must remember that the confidence interval estimator was derived from the sampling distribution of the sample mean. In Section 9.1, we used the sampling distribution to make probability statements about the sample mean. Although the form has changed, the confidence interval estimator is also a probability statement about the sample mean. It states that there is $1 - \alpha$ probability that the sample mean will be equal to a value such that the interval $\bar{x} - z_{\alpha/2}\sigma/\sqrt{n}$ to $\bar{x} + z_{\alpha/2}\sigma/\sqrt{n}$ will include the population mean. Once the sample mean is computed, the interval acts as the lower and upper limits of the interval estimate of the population mean.

As an illustration, suppose we want to estimate the mean value of the distribution resulting from the throw of a fair die. Because we know the distribution, we also know that $\mu = 3.5$ and $\sigma = 1.71$. Pretend now that we know only that $\sigma = 1.71$, that μ is unknown, and that we want to estimate its value. To estimate μ, we draw a sample of size $n = 100$ and calculate $\bar{x}$. The confidence interval estimator of μ is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

The 90% confidence interval estimator is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = \bar{x} \pm 1.645\frac{1.71}{\sqrt{100}} = \bar{x} \pm .281$$

This notation means that, if we repeatedly draw samples of size 100 from this population, 90% of the values of $\bar{x}$ will be such that μ would lie somewhere between $\bar{x} - .281$ and $\bar{x} + .281$, and 10% of the values of $\bar{x}$ will produce intervals that would not include μ. Now, imagine that we draw 40 samples of 100 observations each. The values of $\bar{x}$ and the resulting confidence interval estimates of μ are shown in Table 10.2. Notice that not all the intervals include the true value of the parameter. Samples 5, 16, 22, and 34 produce values of $\bar{x}$ that in turn produce intervals that exclude μ.

Students often react to this situation by asking, What went wrong with samples 5, 16, 22, and 34? The answer is nothing. Statistics does not promise 100% certainty. In fact, in this illustration, we expected 90% of the intervals to include μ and 10% to exclude μ. Since we produced 40 intervals, we expected that 4.0 (10% of 40) intervals would not contain $\mu = 3.5$.* It is important to understand that, even when the statistics practitioner performs experiments properly, a certain proportion (in this example, 10%) of the experiments will produce incorrect estimates by random chance.

TABLE **10.2** 90% Confidence Interval Estimates of μ

SAMPLE	$\bar{x}$	LCL = $\bar{x} - .281$	UCL = $\bar{x} + .281$	DOES INTERVAL INCLUDE $\mu = 3.5$?
1	3.550	3.269	3.831	Yes
2	3.610	3.329	3.891	Yes
3	3.470	3.189	3.751	Yes
4	3.480	3.199	3.761	Yes
5	3.800	3.519	4.081	No
6	3.370	3.089	3.651	Yes
7	3.480	3.199	3.761	Yes
8	3.520	3.239	3.801	Yes
9	3.740	3.459	4.021	Yes
10	3.510	3.229	3.791	Yes
11	3.230	2.949	3.511	Yes
12	3.450	3.169	3.731	Yes
13	3.570	3.289	3.851	Yes
14	3.770	3.489	4.051	Yes
15	3.310	3.029	3.591	Yes
16	3.100	2.819	3.381	No
17	3.500	3.219	3.781	Yes
18	3.550	3.269	3.831	Yes
19	3.650	3.369	3.931	Yes
20	3.280	2.999	3.561	Yes
21	3.400	3.119	3.681	Yes

(Continued)

*In this illustration, exactly 10% of the sample means produced interval estimates that excluded the value of μ, but this will not always be the case. Remember, we expect 10% of the sample means in the long run to result in intervals excluding μ. This group of 40 sample means does not constitute "the long run."

TABLE **10.2** *(Continued)*

SAMPLE	$\bar{x}$	LCL = $\bar{x}$ − .281	UCL = $\bar{x}$ + .281	DOES INTERVAL INCLUDE μ = 3.5?
22	3.880	3.599	4.161	No
23	3.760	3.479	4.041	Yes
24	3.400	3.119	3.681	Yes
25	3.340	3.059	3.621	Yes
26	3.650	3.369	3.931	Yes
27	3.450	3.169	3.731	Yes
28	3.470	3.189	3.751	Yes
29	3.580	3.299	3.861	Yes
30	3.360	3.079	3.641	Yes
31	3.710	3.429	3.991	Yes
32	3.510	3.229	3.791	Yes
33	3.420	3.139	3.701	Yes
34	3.110	2.829	3.391	No
35	3.290	3.009	3.571	Yes
36	3.640	3.359	3.921	Yes
37	3.390	3.109	3.671	Yes
38	3.750	3.469	4.031	Yes
39	3.260	2.979	3.541	Yes
40	3.540	3.259	3.821	Yes

We can improve the confidence associated with the interval estimate. If we let the confidence level $1 - \alpha$ equal .95, the 95% confidence interval estimator is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = \bar{x} \pm 1.96\frac{1.71}{\sqrt{100}} = \bar{x} \pm .335$$

Because this interval is wider, it is more likely to include the value of μ. If you redo Table 10.2, this time using a 95% confidence interval estimator, only samples 16, 22, and 34 will produce intervals that do not include μ. (Notice that we expected 5% of the intervals to exclude μ and that we actually observed $3/40 = 7.5\%$.) The 99% confidence interval estimator is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = \bar{x} \pm 2.575\frac{1.71}{\sqrt{100}} = \bar{x} \pm .440$$

Applying this interval estimate to the sample means listed in Table 10.2 would result in having all 40 interval estimates include the population mean $\mu = 3.5$. (We expected 1% of the intervals to exclude μ; we observed $0/40 = 0\%$.)

SEEING STATISTICS

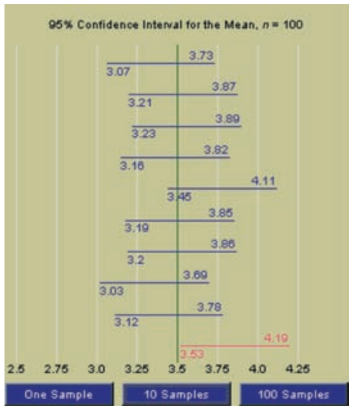
applet 15 Confidence Interval Estimates of a Mean

The simulations used in the applets introduced in Chapter 9 can be used here to demonstrate how confidence interval estimates are interpreted. This applet generates samples of size 100 from the population of the toss of a die. We know that the population mean is $\mu = 3.5$ and that the standard deviation is 1.71. The 95% confidence interval estimator is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = \bar{x} \pm 1.96\frac{1.71}{\sqrt{100}}$$
$$= \bar{x} \pm .335$$

The applet will generate one sample, 10 samples, or 100 samples at a time. The resulting confidence interval is displayed as a horizontal line between the upper and lower ends of the confidence interval. The true mean of 3.5 is the green vertical line. If the confidence interval includes the true population mean of 3.5 (i.e., if the confidence interval line overlaps the green vertical line), it is displayed in blue. If the confidence interval does not include the true mean, it is displayed in red.

After you understand the basics, click the Sample 10 button a few times to see 10 confidence intervals (but not their calculations) at once. Then click on the Sample 100 button to generate 100 samples and confidence intervals.



Applet Exercises

Simulate 100 samples.

15.1 Are all the confidence interval estimates identical?

15.2 Count the number of confidence interval estimates that include the true value of the mean.

15.3 How many intervals did you expect to see that correctly included the mean?

15.4 What do these exercises tell you about the proper interpretation of a confidence interval estimate?

In actual practice, only one sample will be drawn, and thus only one value of $\bar{x}$ will be calculated. The resulting interval estimate will either correctly include the parameter or incorrectly exclude it. Unfortunately, statistics practitioners do not know whether they are correct in each case; they know only that, in the long run, they will incorrectly estimate the parameter some of the time. Statistics practitioners accept that as a fact of life.

We summarize our calculations in Example 10.1 as follows. We estimate that the mean demand during lead time falls between 340.76 and 399.56, and this type of estimator is correct 95% of the time. Thus, the confidence level applies to our estimation procedure and not to any one interval. Incidentally, the media often refer to the 95% figure as "19 times out of 20," which emphasizes the long-run aspect of the confidence level.

Information and the Width of the Interval

Interval estimation, like all other statistical techniques, is designed to convert data into information. However, a wide interval provides little information. For example, suppose that as a result of a statistical study we estimate with 95% confidence that the average starting salary of an accountant lies between $15,000 and $100,000. This interval is so wide that very little information was derived from the data. Suppose, however, that the interval estimate was $52,000 to $55,000. This interval is much narrower, providing accounting students more precise information about the mean starting salary.

The width of the confidence interval estimate is a function of the population standard deviation, the confidence level, and the sample size. Consider Example 10.1, where σ was assumed to be 75. The interval estimate was 370.16 ± 29.40. If σ equaled 150, the 95% confidence interval estimate would become

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 370.16 \pm z_{.025}\frac{150}{\sqrt{25}} = 370.16 \pm 1.96\frac{150}{\sqrt{25}} = 370.16 \pm 58.80$$

Thus, doubling the population standard deviation has the effect of doubling the width of the confidence interval estimate. This result is quite logical. If there is a great deal of variation in the random variable (measured by a large standard deviation), it is more difficult to accurately estimate the population mean. That difficulty is translated into a wider interval.

Although we have no control over the value of σ, we do have the power to select values for the other two elements. In Example 10.1, we chose a 95% confidence level. If we had chosen 90% instead, the interval estimate would have been

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 370.16 \pm z_{.05}\frac{75}{\sqrt{25}} = 370.16 \pm 1.645\frac{75}{\sqrt{25}} = 370.16 \pm 24.68$$

A 99% confidence level results in this interval estimate:

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 370.16 \pm z_{.005}\frac{75}{\sqrt{25}} = 370.16 \pm 2.575\frac{75}{\sqrt{25}} = 370.16 \pm 38.63$$

As you can see, decreasing the confidence level narrows the interval; increasing it widens the interval. However, a large confidence level is generally desirable because that means a larger proportion of confidence interval estimates that will be correct in the long run. There is a direct relationship between the width of the interval and the confidence level. This is because we need to widen the interval to be more confident in the estimate. (The analogy is that to be more likely to capture a butterfly, we need a larger butterfly net.) The trade-off between increased confidence and the resulting wider confidence interval estimates must be resolved by the statistics practitioner. As a general rule, however, 95% confidence is considered "standard."

The third element is the sample size. Had the sample size been 100 instead of 25, the confidence interval estimate would become

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 370.16 \pm z_{.025}\frac{75}{\sqrt{100}} = 370.16 \pm 1.96\frac{75}{\sqrt{100}} = 370.16 \pm 14.70$$

Increasing the sample size fourfold decreases the width of the interval by half. A larger sample size provides more potential information. The increased amount of information is reflected in a narrower interval. However, there is another trade-off: Increasing the sample size increases the sampling cost. We will discuss these issues when we present sample size selection in Section 10.3.

(Optional) Estimating the Population Mean Using the Sample Median

To understand why the sample mean is most often used to estimate a population mean, let's examine the properties of the sampling distribution of the sample median (denoted here as m). The sampling distribution of a sample median is normally distributed provided that the population is normal. Its mean and standard deviation are

$$\mu_m = \mu$$

and

$$\sigma_m = \frac{1.2533\sigma}{\sqrt{n}}$$

Using the same algebraic steps that we used above, we derive the confidence interval estimator of a population mean using the sample median

$$m \pm z_{\alpha/2}\frac{1.2533\sigma}{\sqrt{n}}$$

To illustrate, suppose that we have drawn the following random sample from a normal population whose standard deviation is 2.

1 1 1 3 4 5 6 7 8

The sample mean is $\bar{x} = 4$, and the median is $m = 4$.

The 95% confidence interval estimates using the sample mean and the sample median are

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 4.0 \pm 1.96\frac{2}{\sqrt{9}} = 4 \pm 1.307$$

$$m \pm z_{\alpha/2}\frac{1.2533\sigma}{\sqrt{n}} = 4.0 \pm 1.96\frac{(1.2533)\,(2)}{\sqrt{9}} = 4 \pm 1.638$$

As you can see, the interval based on the sample mean is narrower; as we pointed out previously, narrower intervals provide more precise information. To understand why the sample mean produces better estimators than the sample median, recall how the median is calculated. We simply put the data in order and select the observation that falls in the middle. Thus, as far as the median is concerned the data appear as

1 2 3 4 5 6 7 8 9

By ignoring the actual observations and using their ranks instead, we lose information. With less information, we have less precision in the interval estimators and so ultimately make poorer decisions.

EXERCISES

Developing an Understanding of Statistical Concepts

Exercises 10.9 to 10.16 are "what-if" analyses designed to determine what happens to the interval estimate when the confidence level, sample size, and standard deviation change. These problems can be solved manually, using the spreadsheet you created (that is, if you did create one), or Minitab.

10.9 a. A statistics practitioner took a random sample of 50 observations from a population with a standard deviation of 25 and computed the sample mean to be 100. Estimate the population mean with 90% confidence.
b. Repeat part (a) using a 95% confidence level.
c. Repeat part (a) using a 99% confidence level.
d. Describe the effect on the confidence interval estimate of increasing the confidence level.

10.10 a. The mean of a random sample of 25 observations from a normal population with a standard deviation of 50 is 200. Estimate the population mean with 95% confidence.
b. Repeat part (a) changing the population standard deviation to 25.
c. Repeat part (a) changing the population standard deviation to 10.
d. Describe what happens to the confidence interval estimate when the standard deviation is decreased.

10.11 a. A random sample of 25 was drawn from a normal distribution with a standard deviation of 5. The sample mean is 80. Determine the 95% confidence interval estimate of the population mean.
b. Repeat part (a) with a sample size of 100.

c. Repeat part (a) with a sample size of 400.
d. Describe what happens to the confidence interval estimate when the sample size increases.

10.12 a. Given the following information, determine the 98% confidence interval estimate of the population mean:

$$\bar{x} = 500 \quad \sigma = 12 \quad n = 50$$

b. Repeat part (a) using a 95% confidence level.
c. Repeat part (a) using a 90% confidence level.
d. Review parts (a)–(c) and discuss the effect on the confidence interval estimator of decreasing the confidence level.

10.13 a. The mean of a sample of 25 was calculated as $\bar{x} = 500$. The sample was randomly drawn from a population with a standard deviation of 15. Estimate the population mean with 99% confidence.
b. Repeat part (a) changing the population standard deviation to 30.
c. Repeat part (a) changing the population standard deviation to 60.
d. Describe what happens to the confidence interval estimate when the standard deviation is increased.

10.14 a. A statistics practitioner randomly sampled 100 observations from a population with a standard deviation of 5 and found that $\bar{x}$ is 10. Estimate the population mean with 90% confidence.
b. Repeat part (a) with a sample size of 25.
c. Repeat part (a) with a sample size of 10.
d. Describe what happens to the confidence interval estimate when the sample size decreases.

10.15 a. From the information given here determine the 95% confidence interval estimate of the population mean.

$$\bar{x} = 100 \quad \sigma = 20 \quad n = 25$$

b. Repeat part (a) with $\bar{x} = 200$.
c. Repeat part (a) with $\bar{x} = 500$.
d. Describe what happens to the width of the confidence interval estimate when the sample mean increases.

10.16 a. A random sample of 100 observations was randomly drawn from a population with a standard deviation of 5. The sample mean was calculated as $\bar{x} = 400$. Estimate the population mean with 99% confidence.
b. Repeat part (a) with $\bar{x} = 200$.
c. Repeat part (a) with $\bar{x} = 100$.
d. Describe what happens to the width of the confidence interval estimate when the sample mean decreases.

Exercises 10.17 to 10.20 are based on the optional subsection "Estimating the Population Mean Using the Sample Median." All exercises assume that the population is normal.

10.17 Is the sample median an unbiased estimator of the population mean? Explain.

10.18 Is the sample median a consistent estimator of the population mean? Explain.

10.19 Show that the sample mean is relatively more efficient than the sample median when estimating the population mean.

10.20 a. Given the following information, determine the 90% confidence interval estimate of the population mean using the sample median.

Sample median $= 500$, $\sigma = 12$, and $n = 50$

b. Compare your answer in part (a) to that produced in part (c) of Exercise 10.12. Why is the confidence interval estimate based on the sample median wider than that based on the sample mean?

Applications

The following exercises may be answered manually or with the assistance of a computer. The names of the files containing the data are shown.

10.21 Xr10-21 The following data represent a random sample of 9 marks (out of 10) on a statistics quiz. The marks are normally distributed with a standard deviation of 2. Estimate the population mean with 90% confidence.

7 9 7 5 4 8 3 10 9

10.22 Xr10-22 The following observations are the ages of a random sample of 8 men in a bar. It is known that the ages are normally distributed with a standard deviation of 10. Determine the 95% confidence interval estimate of the population mean. Interpret the interval estimate.

52 68 22 35 30 56 39 48

10.23 Xr10-23 How many rounds of golf do physicians (who play golf) play per year? A survey of 12 physicians revealed the following numbers:

3 41 17 1 33 37 18 15 17 12 29 51

Estimate with 95% confidence the mean number of rounds per year played by physicians, assuming that the number of rounds is normally distributed with a standard deviation of 12.

10.24 Xr10-24 Among the most exciting aspects of a university professor's life are the departmental meetings where such critical issues as the color of the walls will be painted and who gets a new desk are decided.

A sample of 20 professors was asked how many hours per year are devoted to these meetings. The responses are listed here. Assuming that the variable is normally distributed with a standard deviation of 8 hours, estimate the mean number of hours spent at departmental meetings by all professors. Use a confidence level of 90%.

14	17	3	6	17	3	8	4	20	15
7	9	0	5	11	15	18	13	8	4

10.25 Xr10-25 The number of cars sold annually by used car salespeople is normally distributed with a standard deviation of 15. A random sample of 15 salespeople was taken, and the number of cars each sold is listed here. Find the 95% confidence interval estimate of the population mean. Interpret the interval estimate.

79	43	58	66	101	63	79	33	58
71	60	101	74	55	88			

10.26 Xr10-26 It is known that the amount of time needed to change the oil on a car is normally distributed with a standard deviation of 5 minutes. The amount of time to complete a random sample of 10 oil changes was recorded and listed here. Compute the 99% confidence interval estimate of the mean of the population.

11 10 16 15 18 12 25 20 18 24

10.27 Xr10-27 Suppose that the amount of time teenagers spend weekly working at part-time jobs is normally distributed with a standard deviation of 40 minutes. A random sample of 15 teenagers was drawn, and each reported the amount of time spent at part-time jobs (in minutes). These are listed here. Determine the 95% confidence interval estimate of the population mean.

180	130	150	165	90	130	120	60	200
180	80	240	210	150	125			

10.28 Xr10-28 One of the few negative side effects of quitting smoking is weight gain. Suppose that the weight gain in the 12 months following a cessation in smoking is normally distributed with a standard deviation of 6 pounds. To estimate the mean weight gain, a random sample of 13 quitters was drawn; their recorded weights are listed here. Determine the 90% confidence interval estimate of the mean 12-month weight gain for all quitters.

16 23 8 2 14 22 18 11 10 19 5 8 15

10.29 Xr10-29 Because of different sales ability, experience, and devotion, the incomes of real estate agents vary considerably. Suppose that in a large city the annual income is normally distributed with a standard deviation of $15,000. A random sample of 16 real estate agents was asked to report their annual income (in $1,000). The responses are listed here. Determine the 99% confidence interval estimate of the mean annual income of all real estate agents in the city.

65	94	57	111	83	61	50	73	68	80
93	84	113	41	60	77				

The following exercises require the use of a computer and software. The answers may be calculated manually. See Appendix A for the sample statistics.

10.30 Xr10-30 A survey of 400 statistics professors was undertaken. Each professor was asked how much time was devoted to teaching graphical techniques. We believe that the times are normally distributed with a standard deviation of 30 minutes. Estimate the population mean with 95% confidence.

10.31 Xr10-31 In a survey conducted to determine, among other things, the cost of vacations, 64 individuals were randomly sampled. Each person was asked to compute the cost of her or his most recent vacation. Assuming that the standard deviation is $400, estimate with 95% confidence the average cost of all vacations.

10.32 Xr10-32 In an article about *disinflatio*n, various investments were examined. The investments included stocks, bonds, and real estate. Suppose that a random sample of 200 rates of return on real estate investments was computed and recorded. Assuming that the standard deviation of all rates of return on real estate investments is 2.1%, estimate the mean rate of return on all real estate investments with 90% confidence. Interpret the estimate.

10.33 Xr10-33 A statistics professor is in the process of investigating how many classes university students miss each semester. To help answer this question, she took a random sample of 100 university students and asked each to report how many classes he or she had missed in the previous semester. Estimate the mean number of classes missed by all students at the university. Use a 99% confidence level and assume that the population standard deviation is known to be 2.2 classes.

10.34 Xr10-34 As part of a project to develop better lawn fertilizers, a research chemist wanted to determine the mean weekly growth rate of Kentucky bluegrass, a common type of grass. A sample of 250 blades of grass was measured, and the amount of growth in 1 week was recorded. Assuming that weekly growth is normally distributed with a standard deviation of .10 inch, estimate with 99% confidence the mean weekly growth of Kentucky bluegrass. Briefly describe what the interval estimate tells you about the growth of Kentucky bluegrass.

10.35 Xr10-35 A time study of a large production facility was undertaken to determine the mean time required to assemble a cell phone. A random sample of the times to assemble 50 cell phones was recorded. An analysis of the assembly times reveals that they are normally distributed with a standard deviation of 1.3 minutes. Estimate with 95% confidence the mean assembly time for all cell phones. What do your results tell you about the assembly times?

10.36 Xr10-36 The image of the Japanese manager is that of a workaholic with little or no leisure time. In a survey, a random sample of 250 Japanese middle managers was asked how many hours per week they spent in leisure activities (e.g., sports, movies, television). The results of the survey were recorded. Assuming that the population standard deviation is 6 hours, estimate with 90% confidence the mean leisure time per week for all Japanese middle managers. What do these results tell you?

10.37 Xr10-37 One measure of physical fitness is the amount of time it takes for the pulse rate to return to normal after exercise. A random sample of 100 women age 40 to 50 exercised on stationary bicycles for 30 minutes. The amount of time it took for their pulse rates to return to pre-exercise levels was measured and recorded. If the times are normally distributed with a standard deviation of 2.3 minutes, estimate with 99% confidence the true mean pulse-recovery time for all 40- to 50-year-old women. Interpret the results.

10.38 Xr10-38 A survey of 80 randomly selected companies asked them to report the annual income of their presidents. Assuming that incomes are normally distributed with a standard deviation of $30,000, determine the 90% confidence interval estimate of the mean annual income of all company presidents. Interpret the statistical results.

10.39 Xr10-39 To help make a decision about expansion plans, the president of a music company needs to know how many compact discs teenagers buy annually. Accordingly, he commissions a survey of 250 teenagers. Each is asked to report how many CDs he or she purchased in the previous 12 months. Estimate with 90% confidence the mean annual number of CDs purchased by all teenagers. Assume that the population standard deviation is three CDs.

APPLICATIONS in MARKETING

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Advertising

One of the major tools in the promotion mix is advertising. An important decision to be made by the advertising manager is how to allocate the company's total advertising budget among the various competing media types, including television, radio, and newspapers. Ultimately, the manager wants to know, for example, which television programs are most watched by potential customers, and how effective it is to sponsor these programs through advertising. But first the manager must assess the size of the audience, which involves estimating the amount of exposure potential customers have to the various media types, such as television.

10.40 Xr10-40 The sponsors of television shows targeted at the children's market wanted to know the amount of time children spend watching television because the types and number of programs and commercials are greatly influenced by this information. As a result, it was decided to survey 100 North American children and ask them to keep track of the number of hours of television they watch each week. From past experience, it is known that the population standard deviation of the weekly amount of television watched is $\sigma = 8.0$ hours. The television sponsors want an estimate of the amount of television watched by the average North American child. A confidence level of 95% is judged to be appropriate.

10.3 SELECTING THE SAMPLE SIZE

As we discussed in the previous section, if the interval estimate is too wide, it provides little information. In Example 10.1 the interval estimate was 340.76 to 399.56. If the manager is to use this estimate as input for an inventory model, he needs greater precision. Fortunately, statistics practitioners can control the width of the interval by determining the sample size necessary to produce narrow intervals.

To understand how and why we can determine the sample size, we discuss the sampling error.

Error of Estimation

In Chapter 5, we pointed out that sampling error is the difference between the sample and the population that exists only because of the observations that happened to be selected for the sample. Now that we have discussed estimation, we can define the sampling error as the difference between an estimator and a parameter. We can also define this difference as the **error of estimation**. In this chapter, this can be expressed as the difference between $\bar{X}$ and μ. In our derivation of the confidence interval estimator of μ (see page 318), we expressed the following probability,

$$P\left(-Z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < Z_{\alpha/2}\right) = 1 - \alpha$$

which can also be expressed as

$$P\left(-Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X} - \mu < +Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

This tells us that the difference between $\bar{X}$ and μ lies between $-Z_{\alpha/2}\sigma/\sqrt{n}$ and $+Z_{\alpha/2}\sigma/\sqrt{n}$ with probability $1 - \alpha$. Expressed another way, we have with probability $1 - \alpha$,

$$|\bar{X} - \mu| < Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

In other words, the error of estimation is less than $Z_{\alpha/2}\sigma/\sqrt{n}$. We interpret this to mean that $Z_{\alpha/2}\sigma/\sqrt{n}$ is the maximum error of estimation that we are willing to tolerate. We label this value B, which stands for the **bound on the error of estimation**; that is,

$$B = Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Determining the Sample Size

We can solve the equation for n if the population standard deviation σ, the confidence level $1 - \alpha$, and the bound on the error of estimation B are known. Solving for n, we produce the following.

Sample Size to Estimate a Mean

$$n = \left(\frac{z_{\alpha/2}\sigma}{B}\right)^2$$

To illustrate, suppose that in Example 10.1 before gathering the data, the manager had decided that he needed to estimate the mean demand during lead time to within

16 units, which is the bound on the error of estimation. We also have $1 - \alpha = .95$ and $\sigma = 75$. We calculate

$$n = \left(\frac{z_{\alpha/2}\sigma}{B}\right)^2 = \left(\frac{(1.96)\,(75)}{16}\right)^2 = 84.41$$

Because n must be an integer and because we want the bound on the error of estimation to be *no more* than 16, any noninteger value must be rounded up. Thus, the value of n is rounded to 85, which means that to be 95% confident that the error of estimation will be no larger than 16, we need to randomly sample 85 lead time intervals.

Determining the Sample Size to Estimate the Mean Tree Diameter: Solution


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Before the sample was taken, the forester determined the sample size as follows.

The bound on the error of estimation is $B = 1$. The confidence level is 90% ($1 - \alpha = .90$). Thus. $\alpha = .10$ and $\alpha/2 = .05$. It follows that $z_{\alpha/2} = 1.645$. The population standard deviation is assumed to be $\sigma = 6$. Thus,

$$n = \left(\frac{z_{\alpha/2}\sigma}{B}\right)^2 = \left(\frac{1.645 \times 6}{1}\right)^2 = 97.42$$

which is rounded to 98.

However, after the sample is taken the forester discovered that $\sigma = 12$. The 90% confidence interval estimate is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 25 \pm z_{.05}\frac{12}{\sqrt{98}} = 25 \pm 1.645\,\frac{12}{\sqrt{98}} = 25 \pm 2$$

As you can see, the bound on the error of estimation is 2 and not 1. The interval is twice as wide as it was designed to be. The resulting estimate will not be as precise as needed.

In this chapter, we have assumed that we know the value of the population standard deviation. In practice, this is seldom the case. (In Chapter 12, we introduce a more realistic confidence interval estimator of the population mean.) It is frequently necessary to "guesstimate" the value of σ to calculate the sample size; that is, we must use our knowledge of the variable with which we're dealing to assign some value to σ.

Unfortunately, we cannot be very precise in this guess. However, in guesstimating the value of σ, we prefer to err on the high side. For the chapter-opening example, if the forester had determined the sample size using $\sigma = 12$, he would have computed

$$n = \left(\frac{z_{\alpha/2}\sigma}{B}\right)^2 = \left(\frac{(1.645)(12)}{1}\right)^2 = 389.67 \text{ (rounded to 390)}$$

Using $n = 390$ (assuming that the sample mean is again 25), the 90% confidence interval estimate is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 25 \pm 1.645\frac{12}{\sqrt{390}} = 25 \pm 1$$

This interval is as narrow as the forester wanted.

What happens if the standard deviation is *smaller* than assumed? If we discover that the standard deviation is less than we assumed when we determined the sample size, the confidence interval estimator will be narrower and therefore more precise. Suppose that after the sample of 98 trees was taken (assuming again that $\sigma = 6$), the forester discovers that $\sigma = 3$. The confidence interval estimate is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 25 \pm 1.645\frac{3}{\sqrt{98}} = 25 \pm 0.5$$

which is narrower than the forester wanted. Although this means that he would have sampled more trees than needed, the additional cost is relatively low when compared to the value of the information derived.

EXERCISES

Developing an Understanding of Statistical Concepts

10.41 a. Determine the sample size required to estimate a population mean to within 10 units given that the population standard deviation is 50. A confidence level of 90% is judged to be appropriate.
b. Repeat part (a) changing the standard deviation to 100.
c. Redo part (a) using a 95% confidence level.
d. Repeat part (a) wherein we wish to estimate the population mean to within 20 units.

10.42 Review Exercise 10.41. Describe what happens to the sample size when
a. the population standard deviation increases.
b. the confidence level increases.
c. the bound on the error of estimation increases.

10.43 a. A statistics practitioner would like to estimate a population mean to within 50 units with 99% confidence given that the population standard deviation is 250. What sample size should be used?
b. Re-do part (a) changing the standard deviation to 50.
c. Re-do part (a) using a 95% confidence level.
d. Re-do part (a) wherein we wish to estimate the population mean to within 10 units.

10.44 Review the results of Exercise 10.43. Describe what happens to the sample size when
a. the population standard deviation decreases.
b. the confidence level decreases.
c. the bound on the error of estimation decreases.

10.45 a. Determine the sample size necessary to estimate a population mean to within 1 with 90% confidence given that the population standard deviation is 10.
b. Suppose that the sample mean was calculated as 150. Estimate the population mean with 90% confidence.

10.46 a. Repeat part (b) in Exercise 10.45 after discovering that the population standard deviation is actually 5.
b. Repeat part (b) in Exercise 10.45 after discovering that the population standard deviation is actually 20.

10.47 Review Exercises 10.45 and 10.46. Describe what happens to the confidence interval estimate when
a. the standard deviation is equal to the value used to determine the sample size.
b. the standard deviation is smaller than the one used to determine the sample size.
c. the standard deviation is larger than the one used to determine the sample size.

10.48 a. A statistics practitioner would like to estimate a population mean to within 10 units. The confidence level has been set at 95% and $\sigma = 200$. Determine the sample size.
b. Suppose that the sample mean was calculated as 500. Estimate the population mean with 95% confidence.

10.49 a. Repeat part (b) of Exercise 10.48 after discovering that the population standard deviation is actually 100.
b. Repeat part (b) of Exercise 10.48 after discovering that the population standard deviation is actually 400.

10.50 Review Exercises 10.48 and 10.49. Describe what happens to the confidence interval estimate when
a. the standard deviation is equal to the value used to determine the sample size.
b. the standard deviation is smaller than the one used to determine the sample size.
c. the standard deviation is larger than the one used to determine the sample size.

Applications

10.51 A medical statistician wants to estimate the average weight loss of people who are on a new diet plan. In a preliminary study, he guesses that the standard deviation of the population of weight losses is about 10 pounds. How large a sample should he take to estimate the mean weight loss to within 2 pounds, with 90% confidence?

10.52 The operations manager of a large production plant would like to estimate the average amount of time workers take to assemble a new electronic component. After observing a number of workers assembling similar devices, she guesses that the standard deviation is 6 minutes. How large a sample of workers should she take if she wishes to estimate the mean assembly time to within 20 seconds? Assume that the confidence level is to be 99%.

10.53 A statistics professor wants to compare today's students with those 25 years ago. All his current students' marks are stored on a computer so that he can easily determine the population mean. However, the marks 25 years ago reside only in his musty files. He does not want to retrieve all the marks and will be satisfied with a 95% confidence interval estimate of the mean mark 25 years ago. If he assumes that the population standard deviation is 12, how large a sample should he take to estimate the mean to within 2 marks?

10.54 A medical researcher wants to investigate the amount of time it takes for patients' headache pain to be relieved after taking a new prescription painkiller. She plans to use statistical methods to estimate the mean of the population of relief times. She believes that the population is normally distributed with a standard deviation of 20 minutes. How large a sample should she take to estimate the mean time to within 1 minute with 90% confidence?

10.55 The label on 1-gallon cans of paint states that the amount of paint in the can is sufficient to paint 400 square feet. However, this number is quite variable. In fact, the amount of coverage is known to be approximately normally distributed with a standard deviation of 25 square feet. How large a sample should be taken to estimate the true mean coverage of all 1-gallon cans to within 5 square feet with 95% confidence?

10.56 The operations manager of a plant making cellular telephones has proposed rearranging the production process to be more efficient. She wants to estimate the time to assemble the telephone using the new arrangement. She believes that the population standard deviation is 15 seconds. How large a sample of workers should she take to estimate the mean assembly time to within 2 seconds with 95% confidence?

CHAPTER SUMMARY

This chapter introduced the concepts of **estimation** and the **estimator** of a population mean when the population variance is known. It also presented a formula to calculate the sample size necessary to estimate a population mean.

IMPORTANT TERMS

Point estimator 336
Interval estimator 336
Unbiased estimator 337
Consistency 338
Relative efficiency 338
Confidence interval estimator of μ 340
Confidence level 340
Lower confidence limit (LCL) 340
Upper confidence limit (UCL) 340
95% confidence interval estimator of μ 341
Error of estimation 354
Bound on the error of estimation 354

SYMBOLS

Symbol	Pronounced	Represents
$1 - \alpha$	One minus alpha	Confidence level
B		Bound on the error of estimation
$z_{\alpha/2}$	z alpha by 2	Value of Z such that the area to its right is equal to $\alpha/2$

FORMULAS

Confidence interval estimator of μ with σ known

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Sample size to estimate μ

$$n = \left(\frac{z_{\alpha/2}\sigma}{B}\right)^2$$

COMPUTER OUTPUT AND INSTRUCTIONS

Technique	**Excel**	**Minitab**
Confidence interval estimate of μ	343	344

11

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INTRODUCTION TO HYPOTHESIS TESTING

SSA Envelope Plan

DATA
Xm11-00

Federal Express (FedEx) sends invoices to customers requesting payment within 30 days. Each bill lists an address, and customers are expected to use their own envelopes to return their payments. Currently, the mean and standard deviation of the amount of time taken to pay bills are 24 days and 6 days, respectively. The chief financial officer (CFO) believes that including a stamped self-addressed (SSA) envelope would decrease the amount of time. She calculates that the improved cash flow from a 2-day decrease in the payment period would pay for the costs of the envelopes and stamps. Any further decrease in the payment period would generate a profit. To test her belief, she randomly selects 220 customers and includes a stamped self-addressed envelope with their invoices. The numbers of days until payment is received were recorded. Can the CFO conclude that the plan will be profitable?

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After we've introduced the required tools, we'll return to this question and answer it (see page 374).

Introduction

In Chapter 10, we introduced estimation and showed how it is used. Now we're going to present the second general procedure of making inferences about a population—hypothesis testing. The purpose of this type of inference is to determine whether enough statistical evidence exists to enable us to conclude that a belief or hypothesis about a parameter is supported by the data. You will discover that hypothesis testing has a wide variety of applications in business and economics, as well as many other fields. This chapter will lay the foundation upon which the rest of the book is based. As such it represents a critical contribution to your development as a statistics practitioner.

In the next section, we will introduce the concepts of hypothesis testing, and in Section 11.2 we will develop the method employed to test a hypothesis about a population mean when the population standard deviation is known. The rest of the chapter deals with related topics.

11.1 Concepts of Hypothesis Testing

The term **hypothesis testing** is likely new to most readers, but the concepts underlying hypothesis testing are quite familiar. There are a variety of nonstatistical applications of hypothesis testing, the best known of which is a criminal trial.

When a person is accused of a crime, he or she faces a trial. The prosecution presents its case, and a jury must make a decision on the basis of the evidence presented. In fact, the jury conducts a test of hypothesis. There are actually two hypotheses that are tested. The first is called the **null hypothesis** and is represented by H_0 (pronounced *H nought—nought* is a British term for zero). It is

H_0: The defendant is innocent.

The second is called the **alternative hypothesis** (or **research hypothesis**) and is denoted H_1. In a criminal trial it is

H_1: The defendant is guilty.

Of course, the jury does not know which hypothesis is correct. The members must make a decision on the basis of the evidence presented by both the prosecution and the defense. There are only two possible decisions. Convict or acquit the defendant. In statistical parlance, convicting the defendant is equivalent to *rejecting the null hypothesis in favor of the alternative*; that is, the jury is saying that there was enough evidence to conclude that the defendant was guilty. Acquitting a defendant is phrased as *not rejecting the null hypothesis in favor of the alternative*, which means that the jury decided that there was not enough evidence to conclude that the defendant was guilty. Notice that we do not say that we accept the null hypothesis. In a criminal trial, that would be interpreted as finding the defendant *innocent*. Our justice system does not allow this decision.

There are two possible errors. A **Type I error** occurs when we reject a true null hypothesis. A **Type II error** is defined as not rejecting a false null hypothesis. In the criminal trial, a Type I error is made when an innocent person is wrongly convicted. A Type II error occurs when a guilty defendant is acquitted. The probability of a Type I error is denoted by α, which is also called the **significance level**. The probability of a Type II error is denoted by β (Greek letter *beta*). The error probabilities α and β are inversely related, meaning that any attempt to reduce one will increase the other. Table 11.1 summarizes the terminology and the concepts.

TABLE **11.1** Terminology of Hypothesis Testing

DECISION	H_0 IS TRUE (DEFENDANT IS INNOCENT)	H_0 IS FALSE (DEFENDANT IS GUILTY)
REJECT H_0 Convict defendant	Type I Error $P(\text{Type I Error}) = \alpha$	Correct decision
DO NOT REJECT H_0 Acquit defendant	Correct decision	Type II Error $P(\text{Type II Error}) = \beta$

In our justice system, Type I errors are regarded as more serious. As a consequence, the system is set up so that the probability of a Type I error is small. This is arranged by placing the burden of proof on the prosecution (the prosecution must prove guilt—the defense need not prove anything) and by having judges instruct the jury to find the defendant guilty only if there is "evidence beyond a reasonable doubt." In the absence of enough evidence, the jury must acquit even though there may be some evidence of guilt. The consequence of this arrangement is that the probability of acquitting guilty people is relatively large. Oliver Wendell Holmes, a United States Supreme Court justice, once phrased the relationship between the probabilities of Type I and Type II errors in the following way: "Better to acquit 100 guilty men than convict one innocent one." In Justice Holmes's opinion, the probability of a Type I error should be 1/100 of the probability of a Type II error.

The critical concepts in hypothesis testing follow.

1. There are two hypotheses. One is called the null hypothesis, and the other the alternative or research hypothesis.

2. The testing procedure begins with the assumption that the null hypothesis is true.

3. The goal of the process is to determine whether there is enough evidence to infer that the alternative hypothesis is true.

4. There are two possible decisions:

   Conclude that there is enough evidence to support the alternative hypothesis
   Conclude that there is not enough evidence to support the alternative hypothesis

5. Two possible errors can be made in any test. A Type I error occurs when we reject a true null hypothesis, and a Type II error occurs when we don't reject a false null hypothesis. The probabilities of Type I and Type II errors are

$$P(\text{Type I error}) = \alpha$$
$$P(\text{Type II error}) = \beta$$

Let's extend these concepts to statistical hypothesis testing.

In statistics we frequently test hypotheses about parameters. The hypotheses we test are generated by questions that managers need to answer. To illustrate, suppose that in Example 10.1 (page 342) the operations manager did not want to estimate the mean demand during lead time but instead wanted to know whether the mean is different from 350, which may be the point at which the current inventory policy needs to be altered. In other words, the manager wants to determine whether he can infer that μ is not equal to 350. We can rephrase the question so that it now reads, Is there enough evidence to conclude that μ is not equal to 350? This wording is analogous to the criminal trial wherein the jury is asked to determine whether there is enough evidence to conclude that the defendant is guilty. Thus, the alternative (research) hypothesis is

H_1: $\mu \neq 350$

In a criminal trial, the process begins with the assumption that the defendant is innocent. In a similar fashion, we start with the assumption that the parameter equals the value we're testing. Consequently, the operations manager would assume that $\mu = 350$, and the null hypothesis is expressed as

H_0: $\mu = 350$

When we state the hypotheses, we list the null first followed by the alternative hypothesis. To determine whether the mean is different from 350, we test

H_0: $\mu = 350$
H_1: $\mu \neq 350$

Now suppose that in this illustration the current inventory policy is based on an analysis that revealed that the actual mean demand during lead time is 350. After a vigorous advertising campaign, the manager suspects that there has been an increase in demand and thus an increase in mean demand during lead time. To test whether there is evidence of an increase, the manager would specify the alternative hypothesis as

H_1: $\mu > 350$

Because the manager knew that the mean was (and maybe still is) 350, the null hypothesis would state

H_0: $\mu = 350$

Further suppose that the manager does not know the actual mean demand during lead time, but the current inventory policy is based on the assumption that the mean is *less than or equal to* 350. If the advertising campaign increases the mean to a quantity larger than 350, a new inventory plan will have to be instituted. In this scenario, the hypotheses become

H_0: $\mu \leq 350$
H_1: $\mu > 350$

Notice that in both illustrations the alternative hypothesis is designed to determine whether there is enough evidence to conclude that the mean is greater than 350. Although the two null hypotheses are different (one states that the mean is equal to 350, and the other states that the mean is less than or equal to 350), when the test is conducted, the process begins by assuming that the mean is *equal to* 350. In other words, no matter the form of the null hypothesis, we use the equal sign in the null hypothesis. Here is the reason. If there is enough evidence to conclude that the alternative hypothesis (the mean is greater than 350) is true when we assume that the mean is *equal to* 350, we would certainly draw the same conclusion when we assume that the mean is a value that is *less than* 350. As a result, the null hypothesis will always state that the parameter equals the value specified in the alternative hypothesis.

To emphasize this point, suppose the manager now wanted to determine whether there has been a decrease in the mean demand during lead time. We express the null and alternative hypotheses as

H_0: $\mu = 350$

H_1: $\mu < 350$

The hypotheses are often set up to reflect a manager's decision problem wherein the null hypothesis represents the *status quo*. Often this takes the form of some course of action such as maintaining a particular inventory policy. If there is evidence of an increase or decrease in the value of the parameter, a new course of action will be taken. Examples include deciding to produce a new product, switching to a better drug to treat an illness, or sentencing a defendant to prison.

The next element in the procedure is to randomly sample the population and calculate the sample mean. This is called the **test statistic**. The test statistic is the criterion on which we base our decision about the hypotheses. (In the criminal trial analogy, this is equivalent to the evidence presented in the case.) The test statistic is based on the best estimator of the parameter. In Chapter 10, we stated that the best estimator of a population mean is the sample mean.

If the test statistic's value is inconsistent with the null hypothesis, we reject the null hypothesis and infer that the alternative hypothesis is true. For example, if we're trying to decide whether the mean is greater than 350, a large value of $\bar{x}$ (say, 600) would provide enough evidence. If $\bar{x}$ is close to 350 (say, 355), we would say that this does not provide much evidence to infer that the mean is greater than 350. In the absence of sufficient evidence, we do not reject the null hypothesis in favor of the alternative. (In the absence of sufficient evidence of guilt, a jury finds the defendant not guilty.)

In a criminal trial, "sufficient evidence" is defined as "evidence beyond a reasonable doubt." In statistics, we need to use the test statistic's sampling distribution to define "sufficient evidence." We will do so in the next section.

Exercises

Exercises 11.1–11.5 feature nonstatistical applications of hypothesis testing. For each, identify the hypotheses, define Type I and Type II errors, and discuss the consequences of each error. In setting up the hypotheses, you will have to consider where to place the "burden of proof."

11.1 It is the responsibility of the federal government to judge the safety and effectiveness of new drugs. There are two possible decisions: approve the drug or disapprove the drug.

11.2 You are contemplating a Ph.D. in business or economics. If you succeed, a life of fame, fortune, and happiness awaits you. If you fail, you've wasted 5 years of your life. Should you go for it?

11.3 You are the centerfielder of the New York Yankees. It is the bottom of the ninth inning of the seventh game of the World Series. The Yanks lead by 2 with 2 outs and men on second and third. The batter is known to hit for high average and runs very well but only has mediocre power. A single will tie the game, and a hit over your head will likely result in the Yanks losing. Do you play shallow?

11.4 You are faced with two investments. One is very risky, but the potential returns are high. The other is safe, but the potential is quite limited. Pick one.

11.5 You are the pilot of a jumbo jet. You smell smoke in the cockpit. The nearest airport is less than 5 minutes away. Should you land the plane immediately?

11.6 Several years ago in a high-profile case, a defendant was acquitted in a double-murder trial but was subsequently found responsible for the deaths in a civil trial. (Guess the name of the defendant—the answer is in Appendix C.) In a civil trial the plaintiff (the victims' relatives) are required only to show that the preponderance of evidence points to the guilt of the defendant. Aside from the other issues in the cases, discuss why these results are logical.

11.2 TESTING THE POPULATION MEAN WHEN THE POPULATION STANDARD DEVIATION IS KNOWN

To illustrate the process, consider the following example.

EXAMPLE 11.1 Department Store's New Billing System

DATA
Xm11-01

The manager of a department store is thinking about establishing a new billing system for the store's credit customers. After a thorough financial analysis, she determines that the new system will be cost-effective only if the mean monthly account is more than \$170. A random sample of 400 monthly accounts is drawn, for which the sample mean is \$178. The manager knows that the accounts are approximately normally distributed with a standard deviation of \$65. Can the manager conclude from this that the new system will be cost-effective?

SOLUTION

IDENTIFY

This example deals with the population of the credit accounts at the store. To conclude that the system will be cost-effective requires the manager to show that the mean account for all customers is greater than \$170. Consequently, we set up the alternative hypothesis to express this circumstance:

$$H_1: \quad \mu > 170 \quad \text{(Install new system)}$$

If the mean is less than or equal to 170, then the system will not be cost-effective. The null hypothesis can be expressed as

$$H_0: \quad \mu \leq 170 \quad \text{(Do not install new system)}$$

However, as was discussed in Section 11.1, we will actually test $\mu = 170$, which is how we specify the null hypothesis:

$$H_0: \quad \mu = 170$$

As we previously pointed out, the test statistic is the best estimator of the parameter. In Chapter 10, we used the sample mean to estimate the population mean. To conduct this test, we ask and answer the following question: Is a sample mean of 178 sufficiently greater than 170 to allow us to confidently infer that the population mean is greater than 170?

There are two approaches to answering this question. The first is called the *rejection region method*. It can be used in conjunction with the computer, but it is mandatory for those computing statistics manually. The second is the *p-value approach*, which in general can be employed only in conjunction with a computer and statistical software. We recommend, however, that users of statistical software be familiar with both approaches.

Rejection Region

It seems reasonable to reject the null hypothesis in favor of the alternative if the value of the sample mean is large relative to 170. If we had calculated the sample mean to be say, 500, it would be quite apparent that the null hypothesis is false and we would reject it.

On the other hand, values of $\bar{x}$ close to 170, such as 171, do not allow us to reject the null hypothesis because it is entirely possible to observe a sample mean of 171 from a population whose mean is 170. Unfortunately, the decision is not always so obvious. In this example, the sample mean was calculated to be 178, a value apparently neither very far away from nor very close to 170. To make a decision about this sample mean, we set up the *rejection region*.

Rejection Region

The **rejection region** is a range of values such that if the test statistic falls into that range, we decide to reject the null hypothesis in favor of the alternative hypothesis.

Suppose we define the value of the sample mean that is just large enough to reject the null hypothesis as $\bar{x}_L$. The rejection region is

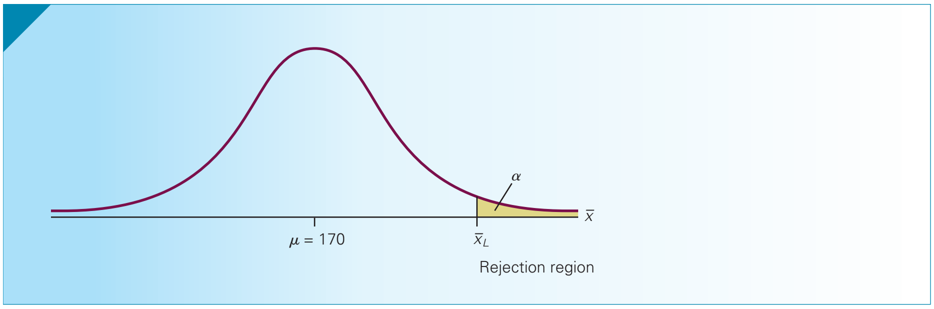
$$\bar{x} > \bar{x}_L$$

Because a Type I error is defined as rejecting a true null hypothesis, and the probability of committing a Type I error is α, it follows that

$$\begin{aligned}\alpha &= P(\text{rejecting } H_0 \text{ given that } H_0 \text{ is true})\\ &= P(\bar{x} > \bar{x}_L \text{ given that } H_0 \text{ is true})\end{aligned}$$

Figure 11.1 depicts the sampling distribution and the rejection region.

FIGURE **11.1** Sampling Distribution for Example 11.1



From Section 9.1, we know that the sampling distribution of $\bar{x}$ is normal or approximately normal, with mean μ and standard deviation $\sigma/\sqrt{n}$. As a result, we can standardize $\bar{x}$ and obtain the following probability:

$$P\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} > \frac{\bar{x}_L-\mu}{\sigma/\sqrt{n}}\right) = P\left(Z > \frac{\bar{x}_L-\mu}{\sigma/\sqrt{n}}\right) = \alpha$$

From Section 8.2, we defined z_α to be the value of a standard normal random variable such that

$$P(Z > z_\alpha) = \alpha$$

Because both probability statements involve the same distribution (standard normal) and the same probability (α), it follows that the limits are identical. Thus,

$$\frac{\bar{x}_L - \mu}{\sigma/\sqrt{n}} = z_\alpha$$

We know that $\sigma = 65$ and $n = 400$. Because the probabilities defined above are conditional on the null hypothesis being true, we have $\mu = 170$. To calculate the rejection region, we need a value of α at the significance level. Suppose that the manager chose α to be 5%. It follows that $z_\alpha = z_{0.5} = 1.645$. We can now calculate the value of $\bar{x}_L$:

$$\frac{\bar{x}_L - \mu}{\sigma/\sqrt{n}} = z_\alpha$$

$$\frac{\bar{x}_L - 170}{65/\sqrt{400}} = 1.645$$

$$\bar{x}_L = 175.34$$

Therefore, the rejection region is

$$\bar{x} > 175.34$$

The sample mean was computed to be 178. Because the test statistic (sample mean) is in the rejection region (it is greater than 175.34), we reject the null hypothesis. Thus, there is sufficient evidence to infer that the mean monthly account is greater than \$170.

Our calculations determined that any value of $\bar{x}$ above 175.34 represents an event that is quite unlikely when sampling (with $n = 400$) from a population whose mean is 170 (and whose standard deviation is 65). This suggests that the assumption that the null hypothesis is true is incorrect, and consequently we reject the null hypothesis in favor of the alternative hypothesis.

Standardized Test Statistic

The preceding test used the test statistic $\bar{x}$; as a result, the rejection region had to be set up in terms of $\bar{x}$. An easier method specifies that the test statistic be the standardized value of $\bar{x}$; that is, we use the **standardized test statistic**.

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

and the rejection region consists of all values of z that are greater than z_α. Algebraically, the rejection region is

$$z > z_\alpha$$

We can redo Example 11.1 using the standardized test statistic.

The rejection region is

$$z > z_\alpha = z_{.05} = 1.645$$

The value of the test statistic is calculated next:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{178 - 170}{65/\sqrt{400}} = 2.46$$

Because 2.46 is greater than 1.645, reject the null hypothesis and conclude that there is enough evidence to infer that the mean monthly account is greater than \$170.

As you can see, the conclusions we draw from using the test statistic $\bar{x}$ and the standardized test statistic z are identical. Figures 11.2 and 11.3 depict the two sampling distributions, highlighting the equivalence of the two tests.

FIGURE **11.2** Sampling Distribution of $\bar{X}$ for Example 11.1

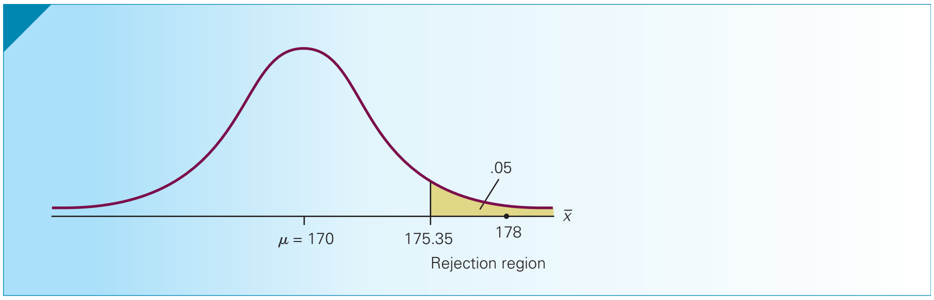
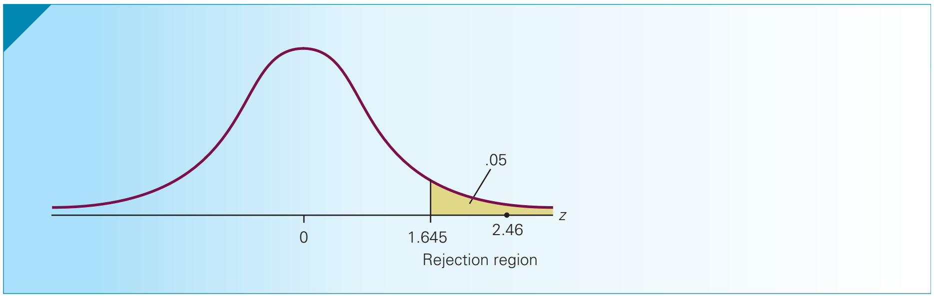


FIGURE **11.3** Sampling Distribution of Z for Example 11.1



Because it is convenient and because statistical software packages employ it, the standardized test statistic will be used throughout this book. For simplicity, we will refer to the *standardized test statistic* simply as the *test statistic*.

Incidentally, when a null hypothesis is rejected, the test is said to be **statistically significant** at whatever significance level the test was conducted. Summarizing Example 11.1, we would say that the test was significant at the 5% significance level.

p-Value

There are several drawbacks to the rejection region method. Foremost among them is the type of information provided by the result of the test. The rejection region method produces a yes or no response to the question, Is there sufficient statistical evidence to infer that the alternative hypothesis is true? The implication is that the result of the test of hypothesis will be converted automatically into one of two possible courses of action: one action as a result of rejecting the null hypothesis in favor of the alternative and another as a result of not rejecting the null hypothesis in favor of the alternative. In Example 11.1, the rejection of the null hypothesis seems to imply that the new billing system will be installed.

In fact, this is not the way in which the result of a statistical analysis is utilized. The statistical procedure is only one of several factors considered by a manager when making a decision. In Example 11.1, the manager discovered that there was enough statistical evidence to conclude that the mean monthly account is greater than \$170. However, before taking any action, the manager would like to consider a number of factors including the cost and feasibility of restructuring the billing system and the possibility of making an error, in this case a Type I error.

What is needed to take full advantage of the information available from the test result and make a better decision is a measure of the amount of statistical evidence supporting the alternative hypothesis so that it can be weighed in relation to the other factors, especially the financial ones. The *p-value of a test* provides this measure.

p-Value

The ***p*-value** of a test is the probability of observing a test statistic at least as extreme as the one computed given that the null hypothesis is true.

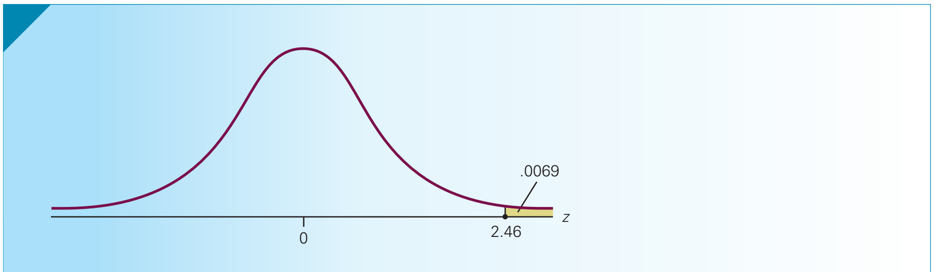
In Example 11.1 the p-value is the probability of observing a sample mean at least as large as 178 when the population mean is 170. Thus,

$$p\text{-value} = P(\overline{X} > 178) = P\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} > \frac{178 - 170}{65/\sqrt{400}}\right) = P(Z > 2.46)$$

$$= 1 - P(Z < 2.46) = 1 - .9931 = .0069$$

Figure 11.4 describes this calculation.

FIGURE **11.4** *p*-Value for Example 11.1



Interpreting the p-Value

To properly interpret the results of an inferential procedure, you must remember that the technique is based on the sampling distribution. The sampling distribution allows us to make probability statements about a sample statistic assuming knowledge of the population parameter. Thus, the probability of observing a sample mean at least as large as 178 from a population whose mean is 170 is .0069, which is very small. In other words, we have just observed an unlikely event, an event so unlikely that we seriously doubt the assumption that began the process—that the null hypothesis is true. Consequently, we have reason to reject the null hypothesis and support the alternative.

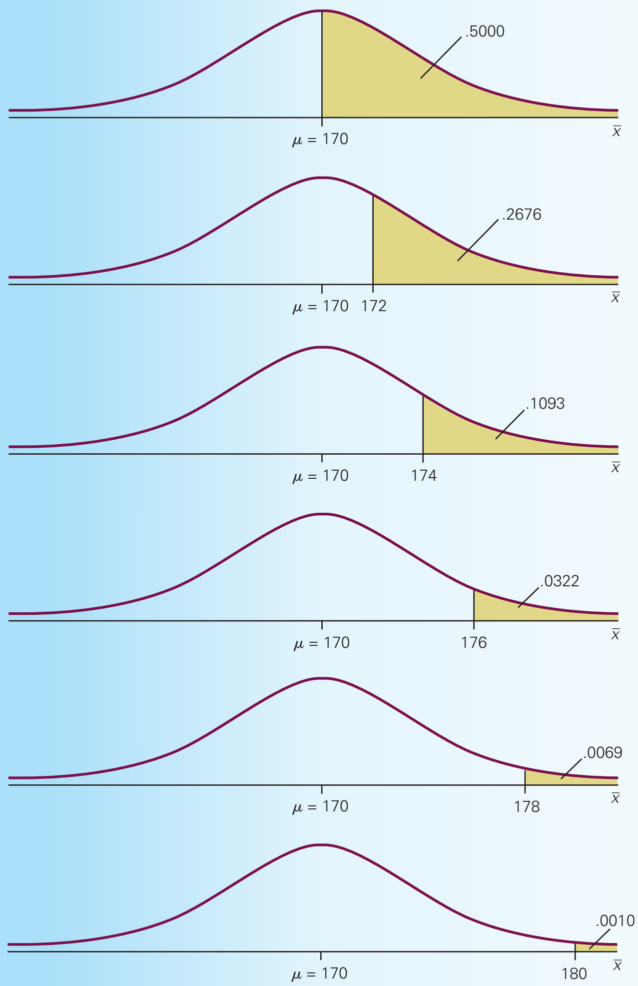
Students may be tempted to simplify the interpretation by stating that the p-value is the probability that the null hypothesis is true. Don't! As was the case with interpreting the confidence interval estimator, you cannot make a probability statement about a parameter. It is not a random variable.

The p-value of a test provides valuable information because it is a measure of the amount of statistical evidence that supports the alternative hypothesis. To understand this interpretation fully, refer to Table 11.2 where we list several values of $\bar{x}$, their z-statistics, and p-values for Example 11.1. Notice that the closer $\bar{x}$ is to the hypothesized mean, 170, the larger the p-value is. The farther $\bar{x}$ is above 170, the smaller the p-value is. Values of $\bar{x}$ far above 170 tend to indicate that the alternative hypothesis is true. Thus, the smaller the p-value, the more the statistical evidence supports the alternative hypothesis. Figure 11.5 graphically depicts the information in Table 11.2.

TABLE **11.2** Test Statistics and p-Values for Example 11.1

SAMPLE MEAN $\bar{x}$	TEST STATISTIC $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{x} - 170}{65/\sqrt{400}}$	p-VALUE
170	0	.5000
172	0.62	.2676
174	1.23	.1093
176	1.85	.0322
178	2.46	.0069
180	3.08	.0010

FIGURE **11.5** p-Values for Example 11.1



This raises the question, How small does the *p*-value have to be to infer that the alternative hypothesis is true? In general, the answer depends on a number of factors, including the costs of making Type I and Type II errors. In Example 11.1, a Type I error would occur if the manager adopts the new billing system when it is not cost-effective. If the cost of this error is high, we attempt to minimize its probability. In the rejection region method, we do so by setting the significance level quite low—say, 1%. Using the *p*-value method, we would insist that the *p*-value be quite small, providing sufficient evidence to infer that the mean monthly account is greater than $170 before proceeding with the new billing system.

Describing the *p*-Value

Statistics practitioners can translate *p*-values using the following descriptive terms:

If the *p*-value is less than .01, we say that there is *overwhelming* evidence to infer that the alternative hypothesis is true. We also say that the test is **highly significant**.

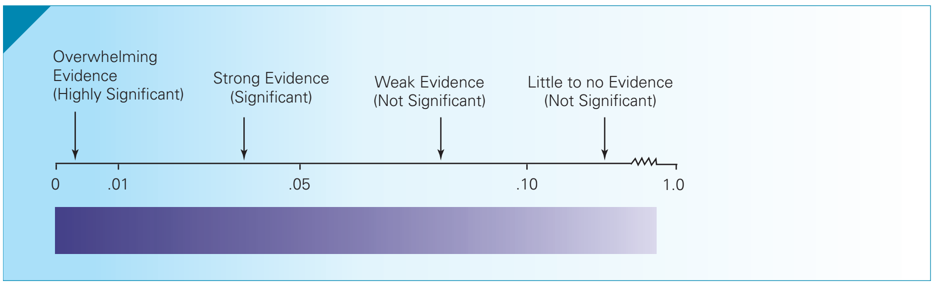
If the *p*-value lies between .01 and .05, there is *strong* evidence to infer that the alternative hypothesis is true. The result is deemed to be **significant**.

If the *p*-value is between .05 and .10, we say that there is *weak* evidence to indicate that the alternative hypothesis is true. When the *p*-value is greater than 5%,we say that the result is **not statistically significant**.

When the *p*-value exceeds .10, we say that there is little to no evidence to infer that the alternative hypothesis is true.

Figure 11.6 summarizes these terms.

FIGURE **11.6** Test Statistics and *p*-Values for Example 11.1



The *p*-Value and Rejection Region Methods

If we so choose, we can use the *p*-value to make the same type of decisions we make in the rejection region method. The rejection region method requires the decision maker to select a significance level from which the rejection region is constructed. We then decide to reject or not reject the null hypothesis. Another way of making that type of decision is to compare the *p*-value with the selected value of the significance level. If the *p*-value is less than α, we judge the *p*-value to be small enough to reject the null hypothesis. If the *p*-value is greater than α, we do not reject the null hypothesis.

Solving Manually, Using Excel, and Using Minitab

As you have already seen, we offer three ways to solve statistical problems. When we perform the calculations manually, we will use the rejection region approach. We will set up the rejection region using the test statistic's sampling distribution and associated table (in Appendix B). The calculations will be performed manually and a reject–do not reject decision will be made. In this chapter, it is possible to compute the *p*-value of the test manually. However, in later chapters we will be using test statistics that are not normally distributed, making it impossible to calculate the *p*-values manually. In these instances, manual calculations require the decision to be made via the rejection region method only.

Most software packages that compute statistics, including Excel and Minitab, print the *p*-value of the test. When we employ the computer, we will not set up the rejection region. Instead we will focus on the interpretation of the *p*-value.

EXCEL

	A	B	C	D
1	**Z-Test: Mean**			
2				
3				*Accounts*
4	Mean			178.00
5	Standard Deviation			68.37
6	Observations			400
7	Hypothesized Mean			170
8	SIGMA			65
9	z Stat			2.46
10	P(Z<=z) one-tail			0.0069
11	z Critical one-tail			1.6449
12	P(Z<=z) two-tail			0.0138
13	z Critical two-tail			1.96

INSTRUCTIONS

1. Type or import the data into one column. (Open Xm11-01.)
2. Click **Add-Ins, Data Analysis Plus,** and **Z-Test: Mean.**
3. Fill in the dialog box: **Input Range** (A1:A401), type the **Hypothesized Mean** (170), type a positive value for the **Standard Deviation** (65), click **Labels** if the first row contains the name of the variable, and type the significance level α (.05).

The first part of the printout reports the statistics and the details of the test. As you can see, the test statistic is $z = 2.46$. The *p*-value* of the test is $P(Z > 2.46) = .0069$. Excel reports this probability as

$$P(Z < = z) \text{ one-tail}$$

Don't take Excel's notation literally. It is not giving us the probability that Z is less than or equal to the value of the z-statistic. Also printed is the critical value of the rejection region shown as

Z Critical one-tail

The printout shown here was produced from the raw data; that is, we input the 400 observations in the data set and the computer calculated the value of the test statistic and the *p*-value. Another way of producing the statistical results is through the use of a spreadsheet that you can create yourself. We describe the required tools for the Do-It-Yourself Excel on page 378.

*Excel provides two probabilities in its printout. The way in which we determine the *p*-value of the test from the printout is somewhat more complicated. Interested students are advised to read Keller's website Appendix Converting Excel's Probabilities to *p*-Values.

MINITAB

One-Sample Z: Accounts

Test of mu = 170 vs > 170
The assumed standard deviation = 65

Variable	N	Mean	StDev	SE Mean	95% Lower Bound	Z	P
Accounts	400	177.997	68.367	3.250	172.651	2.46	0.007

INSTRUCTIONS

1. Type or import the data into one column. (Open Xm11-01.)
2. Click **Stat, Basic Statistics,** and **1-Sample Z . . . .**
3. Type or use the **Select** button to specify the name of the variable or the column in the **Samples in Columns** box (Accounts). Type the value of the **Standard deviation** (65), check the **Perform hypothesis test** box, and type the value of μ under the null hypothesis in the **Hypothesized mean** box (170).
4. Click **Options . . .** and specify the form of the alternative hypothesis in the **Alternative** box (greater than).

Interpreting the Results of a Test

In Example 11.1, we rejected the null hypothesis. Does this prove that the alternative hypothesis is true? The answer is no; because our conclusion is based on sample data (and not on the entire population), we can never *prove* anything by using statistical inference. Consequently, we summarize the test by stating that there is enough statistical evidence to infer that the null hypothesis is false and that the alternative hypothesis is true.

Now suppose that $\bar{x}$ had equaled 174 instead of 178. We would then have calculated $z = 1.23$ (p-value = .1093), which is not in the rejection region. Could we conclude on this basis that there is enough statistical evidence to infer that the null hypothesis is true and hence that $\mu = 170$? Again the answer is "no" because it is absurd to suggest that a sample mean of 174 provides enough evidence to infer that the population mean is 170. (If it proved anything, it would prove that the population mean is 174.) Because we're testing a single value of the parameter under the null hypothesis, we can never have enough statistical evidence to establish that the null hypothesis is true (unless we sample the entire population). (The same argument is valid if you set up the null hypothesis as H_0: $\mu \leq 170$. It would be illogical to conclude that a sample mean of 174 provides enough evidence to conclude that the population mean is *less than or equal to 170*.)

Consequently, if the value of the test statistic does not fall into the rejection region (or the p-value is large), rather than say we accept the null hypothesis (which implies that we're stating that the null hypothesis is true), we state that we do not reject the null hypothesis, and we conclude that not enough evidence exists to show that the alternative hypothesis is true. Although it may appear to be the case, we are not being overly technical. Your ability to set up tests of hypotheses properly and to interpret their results correctly very much depends on your understanding of this point. The point is that the conclusion is based on the alternative hypothesis. In the final analysis, there are only two possible conclusions of a test of hypothesis.

Conclusions of a Test of Hypothesis

If we reject the null hypothesis, we conclude that there is enough statistical evidence to infer that the alternative hypothesis is true.

If we do not reject the null hypothesis, we conclude that there is not enough statistical evidence to infer that the alternative hypothesis is true.

Observe that the alternative hypothesis is the focus of the conclusion. It represents what we are investigating, which is why it is also called the *research hypothesis*. Whatever you're trying to show statistically must be represented by the alternative hypothesis (bearing in mind that you have only three choices for the alternative hypothesis—the parameter is greater than, less than, or not equal to the value specified in the null hypothesis).

When we introduced statistical inference in Chapter 10, we pointed out that the first step in the solution is to identify the technique. When the problem involves hypothesis testing, part of this process is the specification of the hypotheses. Because the alternative hypothesis represents the condition we're researching, we will identify it first. The null hypothesis automatically follows because the null hypothesis must specify equality. However, by tradition, when we list the two hypotheses, the null hypothesis comes first, followed by the alternative hypothesis. All examples in this book will follow that format.

SSA ENVELOPE PLAN: SOLUTION

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IDENTIFY

The objective of the study is to draw a conclusion about the mean payment period. Thus, the parameter to be tested is the population mean μ. We want to know whether there is enough statistical evidence to show that the population mean is less than 22 days. Thus, the alternative hypothesis is

$$H_1: \quad \mu < 22$$

The null hypothesis is

$$H_0: \quad \mu = 22$$

The test statistic is the only one we've presented thus far. It is

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

COMPUTE

MANUALLY

To solve this problem manually, we need to define the rejection region, which requires us to specify a significance level. A 10% significance level is deemed to be appropriate. (We'll discuss our choice later.)

We wish to reject the null hypothesis in favor of the alternative only if the sample mean and hence the value of the test statistic is small enough. As a result, we locate the rejection region in the left tail of the sampling distribution. To understand why, remember that we're trying to decide whether there is enough statistical evidence to infer that the mean is less than 22 (which is the alternative hypothesis). If we observe a large sample mean (and hence a large value of z), do we want to reject the null hypothesis in favor of the alternative? The answer is an emphatic "no." It is illogical to think that if the sample mean is, say, 30, there is enough evidence to conclude that the mean payment period for all customers would be less than 22.

Consequently, we want to reject the null hypothesis only if the sample mean (and hence the value of the test statistic z) is small. How small is small enough? The answer is determined by the significance level and the rejection region. Thus, we set up the rejection region as

$$z < -z_\alpha = -z_{.10} = -1.28$$

Note that the direction of the inequality in the rejection region ($z < -z_\alpha$) matches the direction of the inequality in the alternative hypothesis ($\mu < 22$). Also note that we use the negative sign, because the rejection region is in the left tail (containing values of z less than 0) of the sampling distribution.

From the data, we compute the sum and the sample mean. They are

$$\sum x_i = 4{,}759$$

$$\bar{x} = \frac{\sum x_i}{220} = \frac{4{,}759}{220} = 21.63$$

We will assume that the standard deviation of the payment periods for the SSA plan is unchanged from its current value of $\sigma = 6$. The sample size is $n = 220$, and the value of μ is hypothesized to be 22. We compute the value of the test statistic as

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{21.63 - 22}{6/\sqrt{220}} = -.91$$

Because the value of the test statistic, $z = -.91$, is not less than -1.28, we do not reject the null hypothesis and we do not conclude that the alternative hypothesis is true. There is insufficient evidence to infer that the mean is less than 22 days.

We can determine the p-value of the test as follows:

$$p\text{-value} = P(Z < -.91) = .1814$$

In this type of one-tail (left-tail) test of hypothesis, we calculate the p-value as $P(Z < z)$, where z is the actual value of the test statistic. Figure 11.7 depicts the sampling distribution, rejection region, and p-value.

EXCEL

	A	B	C	D
1	**Z-Test: Mean**			
2				
3				Payment
4	Mean			21.63
5	Standard Deviation			5.84
6	Observations			220
7	Hypothesized Mean			22
8	SIGMA			6
9	z Stat			–0.91
10	P(Z<=z) one-tail			0.1814
11	z Critical one-tail			1.6449
12	P(Z<=z) two-tail			0.3628
13	z Critical two-tail			1.96

MINITAB

One-Sample Z: Payment

Test of mu = 22 vs < 22
The assumed standard deviation = 6

Variable	N	Mean	StDev	SE Mean	95% Upper Bound	Z	P
Payment	220	21.6318	5.8353	0.4045	22.2972	–0.91	0.181

INTERPRET

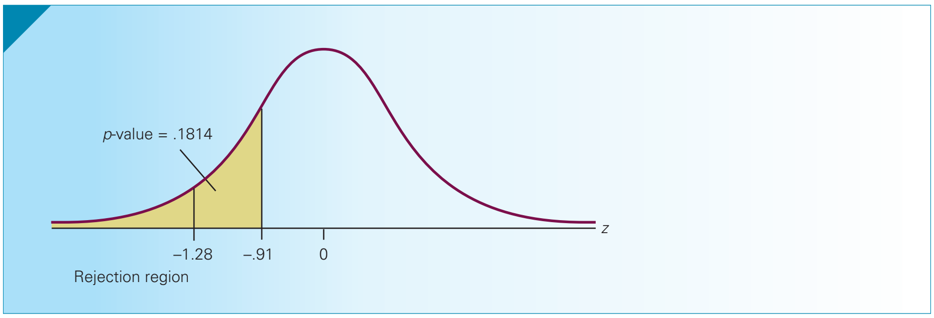
The value of the test statistic is $-.91$, and its *p*-value is .1814, a figure that does not allow us to reject the null hypothesis. Because we were not able to reject the null hypothesis, we say that there is not enough evidence to infer that the mean payment period is less than 22 days. Note that there was some evidence to indicate that the mean of the entire population of payment periods is less than 22 days. We did calculate the sample mean to be 21.63. However, to reject the null hypothesis we need *enough* statistical evidence—and in this case we simply did not have enough reason to reject the null hypothesis in favor of the alternative. In the absence of evidence to show that the mean payment period for all customers sent a stamped self-addressed envelope would be less than 22 days, we cannot infer that the plan would be profitable.

A Type I error occurs when we conclude that the plan works when it actually does not. The cost of this mistake is not high. A Type II error occurs when we don't adopt the SSA envelope plan when it would reduce costs. The cost of this mistake can be high. As a consequence, we would like to minimize the probability of a Type II error. Thus, we chose a large value for the probability of a Type I error; we set

$$\alpha = .10$$

Figure 11.7 exhibits the sampling distribution for this example.

FIGURE **11.7** Sampling Distribution for SSA Envelope Example



One- and Two-Tail Tests

The statistical tests conducted in Example 11.1 and the SSA envelope example are called **one-tail tests** because the rejection region is located in only one tail of the sampling distribution. The *p*-value is also computed by finding the area in one tail of the sampling distribution. The right tail in Example 11.1 is the important one because the alternative hypothesis specifies that the mean is *greater than* 170. In the SSA envelope example, the left tail is emphasized because the alternative hypothesis specifies that the mean is *less than* 22.

We now present an example that requires a **two-tail test**.

EXAMPLE 11.2 Comparison of AT&T and Its Competitor

DATA
Xm11-02

In recent years, several companies have been formed to compete with AT&T in long-distance calls. All advertise that their rates are lower than AT&T's, and as a result their bills will be lower. AT&T has responded by arguing that there will be no difference in billing for the average consumer. Suppose that a statistics practitioner working for AT&T determines that the mean and standard deviation of monthly long-distance bills for all its residential customers are \$17.09 and \$3.87, respectively. He then takes a random sample of 100 customers and recalculates their last month's bill using the rates quoted by a leading competitor. Assuming that the standard deviation of this population is the same as for AT&T, can we conclude at the 5% significance level that there is a difference between the average AT&T bill and that of the leading competitor?

SOLUTION

IDENTIFY

In this problem, we want to know whether the mean monthly long-distance bill is different from \$17.09. Consequently, we set up the alternative hypothesis to express this condition:

$$H_1: \quad \mu \neq 17.09$$

The null hypothesis specifies that the mean is equal to the value specified under the alternative hypothesis. Hence

$$H_0: \quad \mu = 17.09$$

COMPUTE

MANUALLY

To set up the rejection region, we need to realize that we can reject the null hypothesis when the test statistic is large or when it is small. In other words, we must set up a *two-tail rejection region*. Because the total area in the rejection region must be α, we divide this probability by 2. Thus, the rejection region* is

$$z < -z_{\alpha/2} \quad \text{or} \quad z > z_{\alpha/2}$$

For $\alpha = .05$, $\alpha/2 = .025$, and $z_{\alpha/2} = z_{.025} = 1.96$.

$$z < -1.96 \quad \text{or} \quad z > 1.96$$

*Statistics practitioners often represent this rejection region as $|z| > z_{\alpha/2}$, which reads, "the *absolute* value of z is greater than $z_{\alpha/2}$." We prefer our method because it is clear that we are performing a two-tail test.

From the data, we compute

$$\sum x_i = 1{,}754.99$$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{1{,}754.99}{100} = 17.55$$

The value of the test statistic is

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{17.55 - 17.09}{3.87/\sqrt{100}} = 1.19$$

Because 1.19 is neither greater than 1.96 nor less than -1.96, we cannot reject the null hypothesis.

We can also calculate the p-value of the test. Because it is a two-tail test, we determine the p-value by finding the area in both tails; that is,

$$p\text{-value} = P(Z < -1.19) + P(Z > 1.19) = .1170 + .1170 = .2340$$

Or, more simply multiply the probability in one tail by 2.

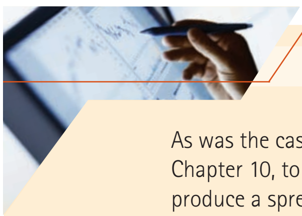
In general, the p-value in a two-tail test is determined by

$$p\text{-value} = 2P(Z > |z|)$$

where z is the actual value of the test statistic and $|z|$ is its absolute value.

EXCEL

	A	B	C	D
1	**Z-Test: Mean**			
2				
3				*Bills*
4	Mean			17.55
5	Standard Deviation			3.94
6	Observations			100
7	Hypothesized Mean			17.09
8	SIGMA			3.87
9	z Stat			1.19
10	P(Z<=z) one-tail			0.1173
11	z Critical one-tail			1.6449
12	P(Z<=z) two-tail			0.2346
13	z Critical two-tail			1.96



DO-IT-YOURSELF **EXCEL**

As was the case with the spreadsheet you created in Chapter 10, to estimate a population mean you can produce a spreadsheet that does the same for testing a population mean.

Tools: **SQRT** and **NORMSINV** are functions described on page 344.

NORMSDIST: Syntax: NORMSDIST(X): This function computes the probability that a standard normal random variable is less than the quantity in parentheses. For example, NORMSDIST(1.19) = $P(z < 1.19)$. However, the quantity we show as **$P(Z< = z)$ one-tail**, which is computed by both Data Analysis and Data Analysis Plus is actually calculated in the following way. Find the probability to the left of the **z Stat** and the probability to its right. **$P(Z< = z)$ one-tail** is the smaller of the two probabilities. **$P(Z< = z)$ two-tail** is twice **$P(Z< = z)$ one-tail**. For more details, we suggest you read the Keller's website Appendix Converting Excel's Probabilities to p-Values.

MINITAB

One-Sample Z: Bills

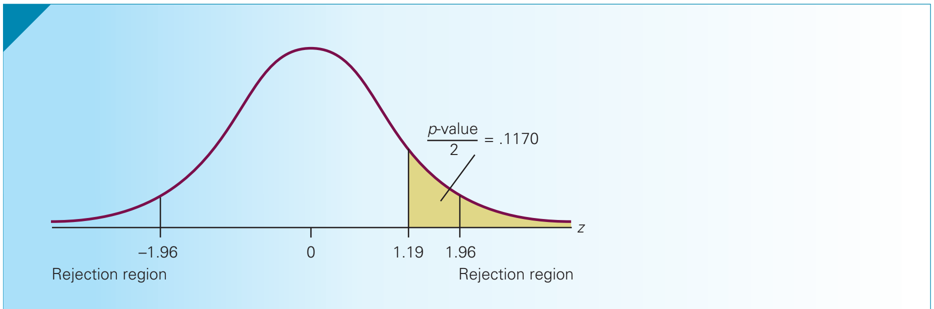
Test of mu = 17.09 vs not = 17.09
The assumed standard deviation = 3.87

Variable	N	Mean	StDev	SE Mean	95% CI	Z	P
Bills	100	17.5499	3.9382	0.3870	(16.7914, 18.3084)	1.19	0.235

INTERPRET

There is not enough evidence to infer that the mean long-distance bill is different from AT&T's mean of $17.09. Figure 11.8 depicts the sampling distribution for this example.

FIGURE **11.8** Sampling Distribution for Example 11.2



When Do We Conduct One- and Two-Tail Tests?

A two-tail test is conducted whenever the alternative hypothesis specifies that the mean is *not equal* to the value stated in the null hypothesis—that is, when the hypotheses assume the following form:

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

There are two one-tail tests. We conduct a one-tail test that focuses on the right tail of the sampling distribution whenever we want to know whether there is enough evidence to infer that the mean is greater than the quantity specified by the null hypothesis—that is, when the hypotheses are

$$H_0: \mu = \mu_0$$
$$H_1: \mu > \mu_0$$

The second one-tail test involves the left tail of the sampling distribution. It is used when the statistics practitioner wants to determine whether there is enough evidence to infer that the mean is less than the value of the mean stated in the null hypothesis. The resulting hypotheses appear in this form:

$$H_0: \mu = \mu_0$$
$$H_1: \mu < \mu_0$$

The techniques introduced in Chapters 12, 13, 16, 17, 18, and 19 require you to decide which of the three forms of the test to employ. Make your decision in the same way as we described the process.

Testing Hypotheses and Confidence Interval Estimators

As you've seen, the test statistic and the confidence interval estimator are both derived from the sampling distribution. It shouldn't be a surprise then that we can use the confidence interval estimator to test hypotheses. To illustrate, consider Example 11.2. The 95% confidence interval estimate of the population mean is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 17.55 \pm 1.96\frac{3.87}{\sqrt{100}} = 17.55 \pm .76$$

$$\text{LCL} = 16.79 \text{ and UCL} = 18.31$$

We estimate that μ lies between \$16.79 and \$18.31. Because this interval includes 17.09, we cannot conclude that there is sufficient evidence to infer that the population mean differs from 17.09.

In Example 11.1, the 95% confidence interval estimate is LCL = 171.63 and UCL = 184.37. The interval estimate excludes 170, allowing us to conclude that the population mean account is not equal to \$170.

As you can see, the confidence interval estimator can be used to conduct tests of hypotheses. This process is equivalent to the rejection region approach. However, instead of finding the critical values of the rejection region and determining whether the test statistic falls into the rejection region, we compute the interval estimate and determine whether the hypothesized value of the mean falls into the interval.

Using the interval estimator to test hypotheses has the advantage of simplicity. Apparently, we don't need the formula for the test statistic; we need only the interval estimator. However, there are two serious drawbacks.

First, when conducting a one-tail test, our conclusion may not answer the original question. In Example 11.1, we wanted to know whether there was enough evidence to infer that the mean is *greater than* 170. The estimate concludes that the mean *differs from* 170. You may be tempted to say that because the entire interval is greater than 170, there is enough statistical evidence to infer that the population mean is greater than 170. However, in attempting to draw this conclusion, we run into the problem of determining the procedure's significance level. Is it 5% or is it 2.5%? We may be able to overcome this problem through the use of **one-sided confidence interval estimators**. However, if the purpose of using confidence interval estimators instead of test statistics is simplicity, one-sided estimators are a contradiction.

Second, the confidence interval estimator does not yield a *p*-value, which we have argued is the better way to draw inferences about a parameter. Using the confidence interval estimator to test hypotheses forces the decision maker into making a reject–don't reject decision rather than providing information about how much statistical evidence exists to be judged with other factors in the decision process. Furthermore, we only postpone the point in time when a test of hypothesis must be used. In later chapters, we will present problems where only a test produces the information we need to make decisions.

Developing an Understanding of Statistical Concepts 1

As is the case with the confidence interval estimator, the test of hypothesis is based on the sampling distribution of the sample statistic. The result of a test of hypothesis is a probability statement about the sample statistic. We assume that the population mean is specified by the null hypothesis. We then compute the test statistic and determine how likely it is to observe this large (or small) a value when the null hypothesis is true. If the probability is small, we conclude that the assumption that the null hypothesis is true is unfounded and we reject it.

Developing an Understanding of Statistical Concepts 2

When we (or the computer) calculate the value of the test statistic

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

we're also measuring the difference between the sample statistic $\bar{x}$ and the hypothesized value of the parameter μ in terms of the standard error $\sigma/\sqrt{n}$. In Example 11.2, we found that the value of the test statistic was $z = 1.19$. This means that the sample mean was 1.19 standard errors above the hypothesized value of μ. The standard normal probability table told us that this value is not considered unlikely. As a result, we did not reject the null hypothesis.

The concept of measuring the difference between the sample statistic and the hypothesized value of the parameter in terms of the standard errors is one that will be used throughout this book.

EXERCISES

Developing an Understanding of Statistical Concepts

In Exercises 11.7–11.12, calculate the value of the test statistic, set up the rejection region, determine the p-value, interpret the result, and draw the sampling distribution.

11.7 H_0: $\mu = 1000$
H_1: $\mu \neq 1000$
$\sigma = 200$, $n = 100$, $\bar{x} = 980$, $\alpha = .01$

11.8 H_0: $\mu = 50$
H_1: $\mu > 50$
$\sigma = 5$, $n = 9$, $\bar{x} = 51$, $\alpha = .03$

11.9 H_0: $\mu = 15$
H_1: $\mu < 15$
$\sigma = 2$, $n = 25$, $\bar{x} = 14.3$, $\alpha = .10$

11.10 H_0: $\mu = 100$
H_1: $\mu \neq 100$
$\sigma = 10$, $n = 100$, $\bar{x} = 100$, $\alpha = .05$

11.11 H_0: $\mu = 70$
H_1: $\mu > 70$
$\sigma = 20$, $n = 100$, $\bar{x} = 80$, $\alpha = .01$

11.12 H_0: $\mu = 50$
H_1: $\mu < 50$
$\sigma = 15$, $n = 100$, $\bar{x} = 48$, $\alpha = .05$

Exercises 11.13 to 11.27 are "what-if analyses" designed to determine what happens to the test statistic and p-value when the sample size, standard deviation, and sample mean change. These problems can be solved manually, using the spreadsheet you created in this section, or by using Minitab.

11.13 a. Compute the p-value in order to test the following hypotheses given that $\bar{x} = 52$, $n = 9$, and $\sigma = 5$.

H_0: $\mu = 50$
H_1: $\mu > 50$

b. Repeat part (a) with $n = 25$.
c. Repeat part (a) with $n = 100$.
d. Describe what happens to the value of the test statistic and its p-value when the sample size increases.

11.14 a. A statistics practitioner formulated the following hypotheses.

H_0: $\mu = 200$
H_1: $\mu < 200$

and learned that $\bar{x} = 190$, $n = 9$, and $\sigma = 50$

Compute the p-value of the test.
b. Repeat part (a) with $\sigma = 30$.
c. Repeat part (a) with $\sigma = 10$
d. Discuss what happens to the value of the test statistic and its p-value when the standard deviation decreases.

11.15 a. Given the following hypotheses, determine the p-value when $\bar{x} = 21$, $n = 25$, and $\sigma = 5$.

$$H_0: \mu = 20$$
$$H_1: \mu \neq 20$$

b. Repeat part (a) with $\bar{x} = 22$.
c. Repeat part (a) with $\bar{x} = 23$.
d. Describe what happens to the value of the test statistic and its p-value when the value of $\bar{x}$ increases.

11.16 a. Test these hypotheses by calculating the p-value given that $\bar{x} = 99$, $n = 100$, and $\sigma = 8$

$$H_0: \mu = 100$$
$$H_1: \mu \neq 100$$

b. Repeat part (a) with $n = 50$.
c. Repeat part (a) with $n = 20$.
d. What is the effect on the value of the test statistic and the p-value of the test when the sample size decreases?

11.17 a. Find the p-value of the following test given that $\bar{x} = 990$, $n = 100$, and $\sigma = 25$.

$$H_0: \mu = 1000$$
$$H_1: \mu < 1000$$

b. Repeat part (a) with $\sigma = 50$.
c. Repeat part (a) with $\sigma = 100$.
d. Describe what happens to the value of the test statistic and its p-value when the standard deviation increases.

11.18 a. Calculate the p-value of the test described here.

$$H_0: \mu = 60$$
$$H_1: \mu > 60$$
$$\bar{x} = 72, \quad n = 25, \quad \sigma = 20$$

b. Repeat part (a) with $\bar{x} = 68$.
c. Repeat part (a) with $\bar{x} = 64$.
d. Describe the effect on the test statistic and the p-value of the test when the value of $\bar{x}$ decreases.

11.19 Redo Example 11.1 with
a. $n = 200$
b. $n = 100$
c. Describe the effect on the test statistic and the p-value when n increases.

11.20 Redo Example 11.1 with
a. $\sigma = 35$
b. $\sigma = 100$
c. Describe the effect on the test statistic and the p-value when σ increases.

11.21 Perform a what-if analysis to calculate the p-values in Table 11.2.

11.22 Redo the SSA example with
a. $n = 100$
b. $n = 500$
c. What is the effect on the test statistic and the p-value when n increases?

11.23 Redo the SSA example with
a. $\sigma = 3$
b. $\sigma = 12$
c. Discuss the effect on the test statistic and the p-value when σ increases.

11.24 For the SSA example, create a table that shows the effect on the test statistic and the p-value of decreasing the value of the sample mean. Use $\bar{x} = 22.0$, 21.8, 21.6, 21.4, 21.2, 21.0, 20.8, 20.6, and 20.4.

11.25 Redo Example 11.2 with
a. $n = 50$
b. $n = 400$
c. Briefly describe the effect on the test statistic and the p-value when n increases.

11.26 Redo Example 11.2 with
a. $\sigma = 2$
b. $\sigma = 10$
c. What happens to the test statistic and the p-value when σ increases?

11.27 Refer to Example 11.2. Create a table that shows the effect on the test statistic and the p-value of changing the value of the sample mean. Use $\bar{x} = 15.0$, 15.5, 16.0, 16.5, 17.0, 17.5, 18.0, 18.5, and 19.0

Applications

The following exercises may be answered manually or with the assistance of a computer. The files containing the data are given.

11.28 Xr11-28 A business student claims that, on average, an MBA student is required to prepare more than five cases per week. To examine the claim, a statistics professor asks a random sample of 10 MBA students to report the number of cases they prepare weekly. The results are exhibited here. Can the professor conclude at the 5% significance level that the claim is true, assuming that the number of cases is normally distributed with a standard deviation of 1.5?

2 7 4 8 9 5 11 3 7 4

11.29 Xr11-29 A random sample of 18 young adult men (20–30 years old) was sampled. Each person was asked how many minutes of sports he watched on television daily. The responses are listed here. It is known that $\sigma = 10$. Test to determine at the 5% significance level whether there is enough statistical evidence to infer that the mean amount of television watched daily by all young adult men is greater than 50 minutes.

50	48	65	74	66	37	45	68	64
65	58	55	52	63	59	57	74	65

11.30 Xr11-30 The club professional at a difficult public course boasts that his course is so tough that the average golfer loses a dozen or more golf balls during a round of golf. A dubious golfer sets out to show that the pro is fibbing. He asks a random sample of 15 golfers who just completed their rounds to report the number of golf balls each lost. Assuming that the number of golf balls lost is normally distributed with a standard deviation of 3, can we infer at the 10% significance level that the average number of golf balls lost is less than 12?

1	14	8	15	17	10	12	6
14	21	15	9	11	4	8	

11.31 Xr11-31 A random sample of 12 second-year university students enrolled in a business statistics course was drawn. At the course's completion, each student was asked how many hours he or she spent doing homework in statistics. The data are listed here. It is known that the population standard deviation is $\sigma = 8.0$. The instructor has recommended that students devote 3 hours per week for the duration of the 12-week semester, for a total of 36 hours. Test to determine whether there is evidence that the average student spent less than the recommended amount of time. Compute the p-value of the test.

31 40 26 30 36 38 29 40 38 30 35 38

11.32 Xr11-32 The owner of a public golf course is concerned about slow play, which clogs the course and results in selling fewer rounds. She believes the problem lies in the amount of time taken to sink putts on the green. To investigate the problem, she randomly samples 10 foursomes and measures the amount of time they spend on the 18th green. The data are listed here. Assuming that the times are normally distributed with a standard deviation of 2 minutes, test to determine whether the owner can infer at the 5% significance level that the mean amount of time spent putting on the 18th green is greater than 6 minutes.

8 11 5 6 7 8 6 4 8 3

11.33 Xr11-33 A machine that produces ball bearings is set so that the average diameter is .50 inch. A sample of 10 ball bearings was measured, with the results shown here. Assuming that the standard deviation is .05 inch, can we conclude at the 5% significance level that the mean diameter is not .50 inch?

.48 .50 .49 .52 .53 .48 .49 .47 .46 .51

11.34 Xr11-34 Spam e-mail has become a serious and costly nuisance. An office manager believes that the average amount of time spent by office workers reading and deleting spam exceeds 25 minutes per day. To test this belief, he takes a random sample of 18 workers and measures the amount of time each spends reading and deleting spam. The results are listed here. If the population of times is normal with a standard deviation of 12 minutes, can the manager infer at the 1% significance level that he is correct?

35	48	29	44	17	21	32	28	34
23	13	9	11	30	42	37	43	48

The following exercises require the use of a computer and software. The answers may be calculated manually. See Appendix A for the sample statistics.

11.35 Xr11-35 A manufacturer of lightbulbs advertises that, on average, its long-life bulb will last more than 5,000 hours. To test the claim, a statistician took a random sample of 100 bulbs and measured the amount of time until each bulb burned out. If we assume that the lifetime of this type of bulb has a standard deviation of 400 hours, can we conclude at the 5% significance level that the claim is true?

11.36 Xr11-36 In the midst of labor–management negotiations, the president of a company argues that the company's blue-collar workers, who are paid an average of \$30,000 per year, are well paid because the mean annual income of all blue-collar workers in the country is less than \$30,000. That figure is disputed by the union, which does not believe that the mean blue-collar income is less than \$30,000. To test the company president's belief, an arbitrator draws a random sample of 350 blue-collar workers from across the country and asks each to report his or her annual income. If the arbitrator assumes that the blue-collar incomes are normally distributed with a standard deviation of \$8,000, can it be inferred at the 5% significance level that the company president is correct?

11.37 Xr11-37 A dean of a business school claims that the Graduate Management Admission Test (GMAT) scores of applicants to the school's MBA program have increased during the past 5 years. Five years ago, the mean and standard deviation of GMAT scores of MBA applicants were 560 and 50, respectively. Twenty applications for this year's program were randomly selected and the GMAT scores recorded. If we assume that the distribution of GMAT scores of this year's applicants is the same as that of 5 years ago, with the possible exception of the mean, can we conclude at the 5% significance level that the dean's claim is true?

11.38 Xr11-38 Past experience indicates that the monthly long-distance telephone bill is normally distributed with a mean of $17.85 and a standard deviation of $3.87. After an advertising campaign aimed at increasing long-distance telephone usage, a random sample of 25 household bills was taken.

a. Do the data allow us to infer at the 10% significance level that the campaign was successful?
b. What assumption must you make to answer part (a)?

11.39 Xr11-39 In an attempt to reduce the number of person-hours lost as a result of industrial accidents, a large production plant installed new safety equipment. In a test of the effectiveness of the equipment, a random sample of 50 departments was chosen. The number of person-hours lost in the month before and the month after the installation of the safety equipment was recorded. The percentage change was calculated and recorded. Assume that the population standard deviation is $\sigma = 6$. Can we infer at the 10% significance level that the new safety equipment is effective?

11.40 Xr11-40 A highway patrol officer believes that the average speed of cars traveling over a certain stretch of highway exceeds the posted limit of 55 mph. The speeds of a random sample of 200 cars were recorded. Do these data provide sufficient evidence at the 1% significance level to support the officer's belief? What is the p-value of the test? (Assume that the standard deviation is known to be 5.)

11.41 Xr11-41 An automotive expert claims that the large number of self-serve gasoline stations has resulted in poor automobile maintenance, and that the average tire pressure is more than 4 pounds per square inch (psi) below its manufacturer's specification. As a quick test, 50 tires are examined, and the number of psi each tire is below specification is recorded. If we assume that tire pressure is normally distributed with $\sigma = 1.5$ psi, can we infer at the 10% significance level that the expert is correct? What is the p-value?

11.42 Xr11-42 For the past few years, the number of customers of a drive-up bank in New York has averaged 20 per hour, with a standard deviation of 3 per hour. This year, another bank 1 mile away opened a drive-up window. The manager of the first bank believes that this will result in a decrease in the number of customers. The number of customers who arrived during 36 randomly selected hours was recorded. Can we conclude at the 5% significance level that the manager is correct?

11.43 Xr11-43 A fast-food franchiser is considering building a restaurant at a certain location. Based on financial analyses, a site is acceptable only if the number of pedestrians passing the location averages more than 100 per hour. The number of pedestrians observed for each of 40 hours was recorded. Assuming that the population standard deviation is known to be 16, can we conclude at the 1% significance level that the site is acceptable?

11.44 Xr11-44 Many Alpine ski centers base their projections of revenues and profits on the assumption that the average Alpine skier skis four times per year. To investigate the validity of this assumption, a random sample of 63 skiers is drawn and each is asked to report the number of times he or she skied the previous year. If we assume that the standard deviation is 2, can we infer at the 10% significance level that the assumption is wrong?

11.45 Xr11-45 The golf professional at a private course claims that members who have taken lessons from him lowered their handicap by more than five strokes. The club manager decides to test the claim by randomly sampling 25 members who have had lessons and asking each to report the reduction in handicap, where a negative number indicates an increase in the handicap. Assuming that the reduction in handicap is approximately normally distributed with a standard deviation of two strokes, test the golf professional's claim using a 10% significance level.

11.46 Xr11-46 The current no-smoking regulations in office buildings require workers who smoke to take breaks and leave the building in order to satisfy their habits. A study indicates that such workers average 32 minutes per day taking smoking breaks. The standard deviation is 8 minutes. To help reduce the average break, rooms with powerful exhausts were installed in the buildings. To see whether these rooms serve their designed purpose, a random sample of 110 smokers was taken. The total amount of time away from their desks was measured for 1 day. Test to determine whether there has been a decrease in the mean time away from their desks. Compute the p-value and interpret it relative to the costs of Type I and Type II errors.

11.47 Xr11-47 A low-handicap golfer who uses Titleist brand golf balls observed that his average drive is 230 yards and the standard deviation is 10 yards. Nike has just introduced a new ball, which has been endorsed by Tiger Woods. Nike claims that the ball will travel farther than Titleist. To test the claim, the golfer hits 100 drives with a Nike ball and measures the distances. Conduct a test to determine whether Nike is correct. Use a 5% significance level.

11.3 CALCULATING THE PROBABILITY OF A TYPE II ERROR

To properly interpret the results of a test of hypothesis, you must be able to specify an appropriate significance level or to judge the *p*-value of a test. However, you also must understand the relationship between Type I and Type II errors. In this section, we describe how the probability of a Type II error is computed and interpreted.

Recall Example 11.1, where we conducted the test using the sample mean as the test statistic and we computed the rejection region (with $\alpha = .05$) as

$$\bar{x} > 175.34$$

A Type II error occurs when a false null hypothesis is not rejected. In Example 11.1, if $\bar{x}$ is less than 175.34, we will not reject the null hypothesis. If we do not reject the null hypothesis, we will not install the new billing system. Thus, the consequence of a Type II error in this example is that we will not install the new system when it would be cost-effective. The probability of this occurring is the probability of a Type II error. It is defined as

$$\beta = P(\bar{X} < 175.34, \text{ given that the null hypothesis is false})$$

The condition that the null hypothesis is false tells us only that the mean is not equal to 170. If we want to compute β, we need to specify a value for μ. Suppose that when the mean account is at least $180, the new billing system's savings become so attractive that the manager would hate to make the mistake of not installing the system. As a result, she would like to determine the probability of not installing the new system when it would produce large cost savings. Because calculating probability from an approximately normal sampling distribution requires a value of μ (as well as σ and n), we will calculate the probability of not installing the new system when μ is *equal* to 180:

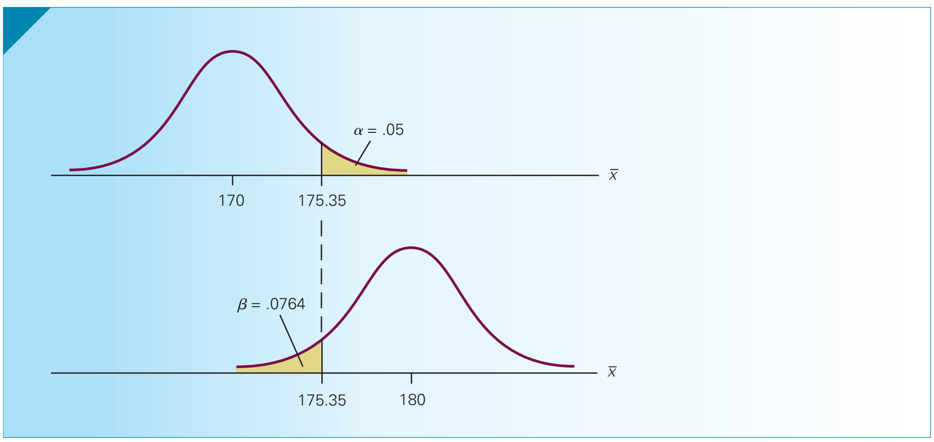
$$\beta = P(\bar{X} < 175.34, \text{ given that } \mu = 180)$$

We know that $\bar{x}$ is approximately normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$. To proceed, we standardize $\bar{x}$ and use the standard normal table (Table 3 in Appendix B):

$$\beta = P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{175.34 - 180}{65/\sqrt{400}}\right) = P(Z < -1.43) = .0764$$

This tells us that when the mean account is actually $180, the probability of incorrectly not rejecting the null hypothesis is .0764. Figure 11.9 graphically depicts

FIGURE **11.9** Calculating β for $\mu = 180$, $\alpha = .05$, and $n = 400$



how the calculation was performed. Notice that to calculate the probability of a Type II error, we had to express the rejection region in terms of the unstandardized test statistic $\bar{x}$, and we had to specify a value for μ other than the one shown in the null hypothesis. In this illustration, the value of μ used was based on a financial analysis indicating that when μ is at least \$180 the cost savings would be very attractive.

Effect on β of Changing α

Suppose that in the previous illustration we had used a significance level of 1% instead of 5%. The rejection region expressed in terms of the standardized test statistic would be

$$z > z_{.01} = 2.33$$

or

$$\frac{\bar{x} - 170}{65/\sqrt{400}} > 2.33$$

Solving for $\bar{x}$, we find the rejection region in terms of the unstandardized test statistic:

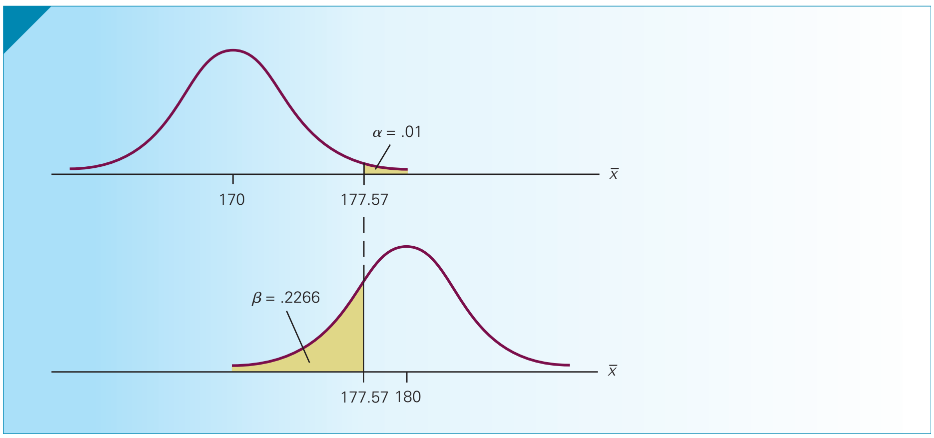
$$\bar{x} > 177.57$$

The probability of a Type II error when $\mu = 180$ is

$$\beta = P\left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} < \frac{177.57 - 180}{65/\sqrt{400}}\right) = P(Z < -.75) = .2266$$

Figure 11.10 depicts this calculation. Compare this figure with Figure 11.9. As you can see, by decreasing the significance level from 5% to 1%, we have shifted the critical value of the rejection region to the right and thus enlarged the area where the null hypothesis is not rejected. The probability of a Type II error increases from .0764 to .2266.

FIGURE **11.10** Calculating β for $\mu = 180$, $\alpha = .01$, and $n = 400$



This calculation illustrates the inverse relationship between the probabilities of Type I and Type II errors alluded to in Section 11.1. It is important to understand this relationship. From a practical point of view, it tells us that if you want to decrease the probability of a Type I error (by specifying a small value of α), you increase the probability of a Type II error. In applications where the cost of a Type I error is considerably larger than the cost of a Type II error, this is appropriate. In fact, a significance level of 1% or less is probably justified. However, when the cost of a Type II error is relatively large, a significance level of 5% or more may be appropriate.

Unfortunately, there is no simple formula to determine what the significance level should be. The manager must consider the costs of both mistakes in deciding what to do. Judgment and knowledge of the factors in the decision are crucial.

Judging the Test

There is another important concept to be derived from this section. A statistical test of hypothesis is effectively defined by the significance level and the sample size, both of which are selected by the statistics practitioner. We can judge how well the test functions by calculating the probability of a Type II error at some value of the parameter. To illustrate, in Example 11.1 the manager chose a sample size of 400 and a 5% significance level on which to base her decision. With those selections, we found β to be .0764 when the actual mean is 180. If we believe that the cost of a Type II error is high and thus that the probability is too large, we have two ways to reduce the probability. We can increase the value of α; however, this would result in an increase in the chance of making a Type I error, which is very costly.

Alternatively, we can increase the sample size. Suppose that the manager chose a sample size of 1,000. We'll now recalculate β with $n = 1000$ (and $\alpha = .05$). The rejection region is

$$z > z_{.05} = 1.645$$

or

$$\frac{\bar{x} - 170}{65/\sqrt{1000}} > 1.645$$

which yields

$$\bar{x} > 173.38$$

The probability of a Type II error is

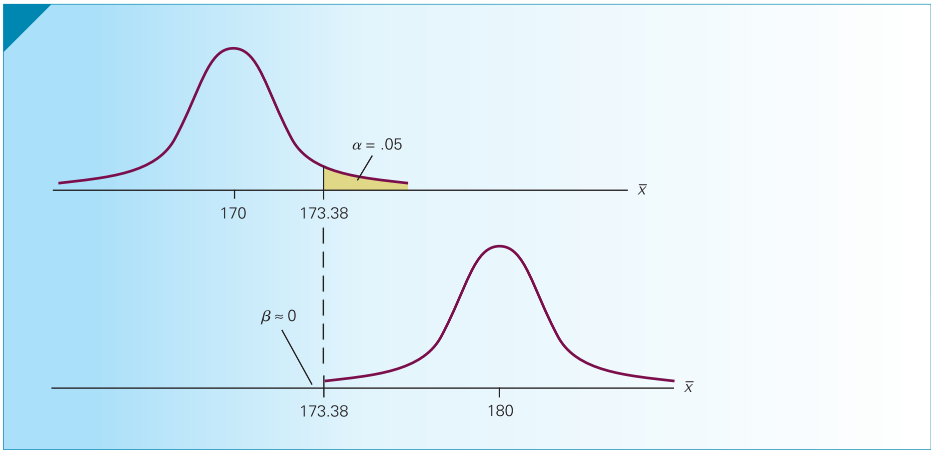
$$\beta = P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{173.38 - 180}{65/\sqrt{1000}}\right) = P(Z < -3.22) = 0 \text{ (approximately)}$$

In this case, we maintained the same value of α (.05), but we reduced the probability of not installing the system when the actual mean account is \$180 to virtually 0.

Developing an Understanding of Statistical Concepts: Larger Sample Size Equals More Information Equals Better Decisions

Figure 11.11 displays the previous calculation. When compared with Figure 11.9, we can see that the sampling distribution of the mean is narrower because the standard error of the mean $\sigma/\sqrt{n}$ becomes smaller as n increases. Narrower distributions represent more information. The increased information is reflected in a smaller probability of a Type II error.

FIGURE **11.11** Calculating β for $\mu = 180$, $\alpha = .05$, and $n = 1{,}000$



The calculation of the probability of a Type II error for $n = 400$ and for $n = 1{,}000$ illustrates a concept whose importance cannot be overstated. By increasing the sample size, we reduce the probability of a Type II error. By reducing the probability of a Type II error, we make this type of error less frequently. Hence, larger sample sizes allow us to make better decisions in the long run. This finding lies at the heart of applied statistical analysis and reinforces the book's first sentence: "Statistics is a way to get information from data."

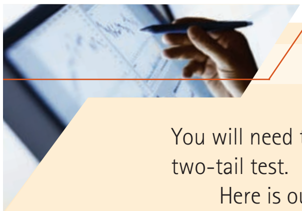
Throughout this book we introduce a variety of applications in accounting, finance, marketing, operations management, human resources management, and economics. In all such applications, the statistics practitioner must make a decision, which involves converting data into information. The more information, the better the decision. Without such information, decisions must be based on guesswork, instinct, and luck. W. Edwards Deming, a famous statistician, said it best: "Without data you're just another person with an opinion."

Power of a Test

Another way of expressing how well a test performs is to report its *power*: the probability of its leading us to reject the null hypothesis when it is false. Thus, the power of a test is $1 - \beta$.

When more than one test can be performed in a given situation, we would naturally prefer to use the test that is correct more frequently. If (given the same alternative hypothesis, sample size, and significance level) one test has a higher power than a second test, the first test is said to be more powerful.

Using the Computer



DO-IT-YOURSELF EXCEL

You will need to create three spreadsheets, one for a left-tail, one for a right-tail, and one for a two-tail test.

Here is our spreadsheet for the right-tail test for Example 11.1.

	A	B	C	D
1	Right-tail Test			
2				
3	H0: MU	170	Critical value	175.35
4	SIGMA	65	Prob(Type II error)	0.0761
5	Sample size	400	Power of the test	0.9239
6	ALPHA	0.05		
7	H1: MU	180		

Tools: **NORMSINV:** Use this function to help compute the critical value in Cell D3.
NORMSDIST: This function is needed to calculate the probability in cell D4.

MINITAB

Minitab computes the power of the test.

```
Power and Sample Size

1-Sample Z Test

Testing mean = null (versus > null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 65


            Sample
Difference    Size     Power
        10     400  0.923938
```

INSTRUCTIONS

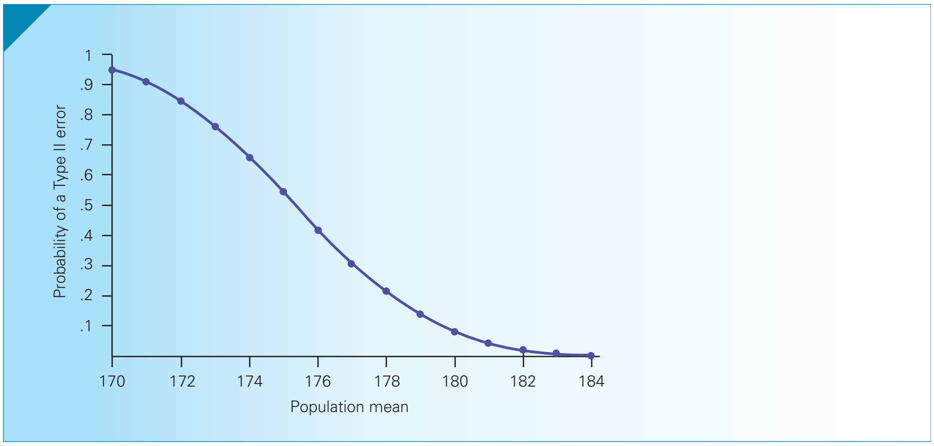
1. Click **Stat, Power and Sample Size**, and **1-Sample Z . . . .**
2. Specify the sample size in the **Sample sizes** box. (You can specify more than one value of n. Minitab will compute the power for each value.) Type the difference between the actual value of μ and the value of μ under the null hypothesis. (You can specify more than one value.) Type the value of the standard deviation in the **Standard deviation** box.
3. Click **Options . . .** and specify the **Alternative Hypothesis** and the **Significance level**.

For Example 11.1, we typed **400** to select the **Sample sizes**, the **Differences** was **10** (=180 − 170), **Standard deviation** was **65**, the **Alternative Hypothesis** was **Greater than**, and the **Significance level** was **0.05**.

Operating Characteristic Curve

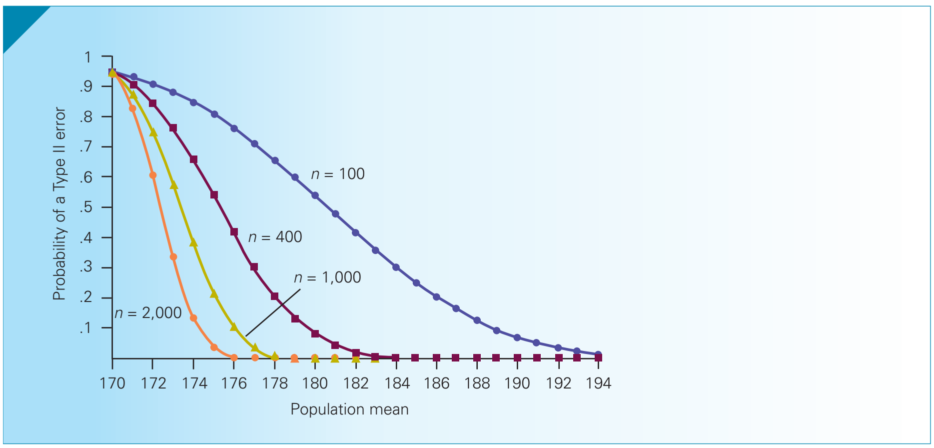
To compute the probability of a Type II error, we must specify the significance level, the sample size, and an alternative value of the population mean. One way to keep track of all these components is to draw the **operating characteristic** (**OC**) **curve**, which plots the values of β versus the values of μ. Because of the time-consuming nature of these calculations, the computer is a virtual necessity. To illustrate, we'll draw the OC curve for Example 11.1. We used Excel (we could have used Minitab instead) to compute the probability of a Type II error in Example 11.1 for $\mu = 170, 171, \ldots, 185$, with $n = 400$. Figure 11.12 depicts this curve. Notice as the alternative value of μ increases the value of β decreases. This tells us that as the alternative value of μ moves farther from the value of μ under the null hypothesis, the probability of a Type II error decreases. In other words, it becomes easier to distinguish between $\mu = 170$ and other values of μ when μ is farther from 170. Note that when $\mu = 170$ (the hypothesized value of μ), $\beta = 1 - \alpha$.

FIGURE **11.12** Operating Characteristic Curve for Example 11.1



The OC curve can also be useful in selecting a sample size. Figure 11.13 shows the OC curve for Example 11.1 with $n = 100$, 400, 1,000, and 2,000. An examination of this chart sheds some light concerning the effect increasing the sample size has on how well the test performs at different values of μ. For example, we can see that smaller sample sizes will work well to distinguish between 170 and values of μ larger than 180. However, to distinguish

FIGURE **11.13** Operating Characteristic Curve for Example 11.1 for n = 100, 400, 1,000, and 2,000



between 170 and smaller values of μ requires larger sample sizes. Although the information is imprecise, it does allow us to select a sample size that is suitable for our purposes.

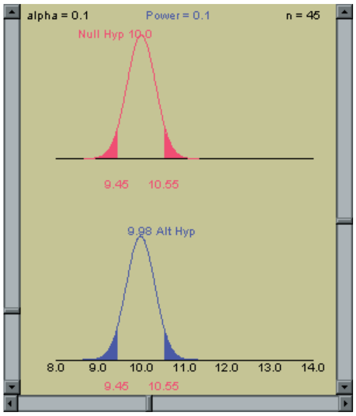
SEEING STATISTICS

applet 16 Power of a z-Test

We are given the following hypotheses to test:

$$H_0: \quad \mu = 10$$
$$H_0: \quad \mu \neq 10$$

The applet allows you to choose the actual value of μ (bottom slider), the value of α (left slider), and the sample size (right slider). The graph shows the effect of changing any of the three values on the two sampling distributions.



Applet Exercises

16.1 Use the left and right sliders to depict the test when $n = 50$ and $\alpha = .10$. Describe what happens to the power of the test (Power $= 1 - \beta$) when the actual value of μ approximately equals the following values:

9.0 9.4 9.8 10.2 10.6 11.0

16.2 Use the bottom and right sliders to depict the test when $\mu = 11$ and $n = 25$. Describe the effect on the test's power when α approximately equals the following:

.01 .03 .05 .10 .20 .30 .40 .50

16.3 Use the bottom and left sliders to depict the test when $\mu = 11$ and $\alpha = .10$. Describe the effect on the test's power when n equals the following:

2 5 10 25 50 75 100

Determining the Alternative Hypothesis to Define Type I and Type II Errors

We've already discussed how the alternative hypothesis is determined. It represents the condition we're investigating. In Example 11.1, we wanted to know whether there was sufficient statistical evidence to infer that the new billing system would be cost-effective—that is, whether the mean monthly account is greater than \$170. In this textbook, you will encounter many problems using similar phraseology. Your job will be to conduct the test that answers the question.

In real life, however, the manager (that's you 5 years from now) will be asking and answering the question. In general, you will find that the question can be posed in two ways. In Example 11.1, we asked whether there was evidence to conclude that the new system would be cost-effective. Another way of investigating the issue is to determine whether there is sufficient evidence to infer that the new system would *not* be cost-effective. We remind you of the criminal trial analogy. In a criminal trial, the burden of proof falls on the prosecution to prove that the defendant is guilty. In other countries with less emphasis on individual rights, the defendant is required to prove his or her innocence. In the United States and Canada (and in other countries), we chose the former because we consider the conviction of an innocent defendant to be the greater error. Thus, the test is set up with the null and alternative hypotheses as described in Section 11.1.

In a statistical test where we are responsible for both asking and answering a question, we must ask the question so that we directly control the error that is more costly. As you have already seen, we control the probability of a Type I error by specifying its value (the significance level). Consider Example 11.1 once again. There are two possible errors: (1) conclude that the billing system is cost-effective when it isn't and

(2) conclude that the system is not cost-effective when it is. If the manager concludes that the billing plan is cost-effective, the company will install the new system. If, in reality, the system is not cost-effective, the company will incur a loss. On the other hand, if the manager concludes that the billing plan is not going to be cost-effective, the company will not install the system. However, if the system is actually cost-effective, the company will lose the potential gain from installing it. Which cost is greater?

Suppose we believe that the cost of installing a system that is not cost-effective is higher than the potential loss of not installing an effective system. The error we wish to avoid is the erroneous conclusion that the system is cost-effective. We define this as a Type I error. As a result, the burden of proof is placed on the system to deliver sufficient statistical evidence that the mean account is greater than $170. The null and alternative hypotheses are as formulated previously:

$$H_0: \mu = 170$$
$$H_1: \mu > 170$$

However, if we believe that the potential loss of not installing the new system when it would be cost-effective is the larger cost, we would place the burden of proof on the manager to infer that the mean monthly account is less than $170. Consequently, the hypotheses would be

$$H_0: \mu = 170$$
$$H_1: \mu < 170$$

This discussion emphasizes the need in practice to examine the costs of making both types of error before setting up the hypotheses. However, it is important for readers to understand that the questions posed in exercises throughout this book have already taken these costs into consideration. Accordingly, your task is to set up the hypotheses to answer the questions.

EXERCISES

Developing an Understanding of Statistical Concepts

11.48 Calculate the probability of a Type II error for the following test of hypothesis, given that $\mu = 203$.

$$H_0: \mu = 200$$
$$H_1: \mu \neq 200$$
$$\alpha = .05, \quad \sigma = 10, \quad n = 100$$

11.49 Find the probability of a Type II error for the following test of hypothesis, given that $\mu = 1{,}050$.

$$H_0: \mu = 1{,}000$$
$$H_1: \mu > 1{,}000$$
$$\alpha = .01, \quad \sigma = 50, \quad n = 25$$

11.50 Determine β for the following test of hypothesis, given that $\mu = 48$.

$$H_0: \mu = 50$$
$$H_1: \mu < 50$$
$$\alpha = .05, \quad \sigma = 10, \quad n = 40$$

11.51 For each of Exercises 11.48–11.50, draw the sampling distributions similar to Figure 11.9.

11.52 A statistics practitioner wants to test the following hypotheses with $\sigma = 20$ and $n = 100$:

$$H_0: \mu = 100$$
$$H_1: \mu > 100$$

a. Using $\alpha = .10$ find the probability of a Type II error when $\mu = 102$.
b. Repeat part (a) with $\alpha = .02$.
c. Describe the effect on β of decreasing α

11.53 a. Calculate the probability of a Type II error for the following hypotheses when $\mu = 37$:

$$H_0: \mu = 40$$
$$H_1: \mu < 40$$

The significance level is 5%, the population standard deviation is 5, and the sample size is 25.

b. Repeat part (a) with $\alpha = 15\%$.
c. Describe the effect on β of increasing α.

11.54 Draw the figures of the sampling distributions for Exercises 11.52 and 11.53.

11.55 a. Find the probability of a Type II error for the following test of hypothesis, given that $\mu = 196$:

$$H_0: \ \mu = 200$$
$$H_1: \ \mu < 200$$

The significance level is 10%, the population standard deviation is 30, and the sample size is 25.
b. Repeat part (a) with $n = 100$.
c. Describe the effect on β of increasing n.

11.56 a. Determine β for the following test of hypothesis, given that $\mu = 310$:

$$H_0: \ \mu = 300$$
$$H_1: \ \mu > 300$$

The statistics practitioner knows that the population standard deviation is 50, the significance level is 5%, and the sample size is 81.
b. Repeat part (a) with $n = 36$.
c. Describe the effect on β of decreasing n.

11.57 For Exercises 11.55 and 11.56, draw the sampling distributions similar to Figure 11.9.

11.58 For the test of hypothesis

$$H_0: \ \mu = 1{,}000$$
$$H_1: \ \mu \neq 1{,}000$$
$$\alpha = .05, \quad \sigma = 200$$

draw the operating characteristic curve for $n = 25$, 100, and 200.

11.59 Draw the operating characteristic curve for $n = 10$, 50, and 100 for the following test:

$$H_0: \ \mu = 400$$
$$H_1: \ \mu > 400$$
$$\alpha = .05, \quad \sigma = 50$$

11.60 Suppose that in Example 11.1 we wanted to determine whether there was sufficient evidence to conclude that the new system would *not* be cost-effective. Set up the null and alternative hypotheses and discuss the consequences of Type I and Type II errors. Conduct the test. Is your conclusion the same as the one reached in Example 11.1? Explain.

Applications

11.61 In Exercise 11.39, we tested to determine whether the installation of safety equipment was effective in reducing person-hours lost to industrial accidents. The null and alternative hypotheses were

$$H_0: \ \mu = 0$$
$$H_1: \ \mu < 0$$

with $\sigma = 6$, $\alpha = .10$, $n = 50$, and μ = the mean percentage change. The test failed to indicate that the new safety equipment is effective. The manager is concerned that the test was not sensitive enough to detect small but important changes. In particular, he worries that if the true reduction in time lost to accidents is actually 2% (i.e., $\mu = -2$), then the firm may miss the opportunity to install very effective equipment. Find the probability that the test with $\sigma = 6$, $\alpha = .10$, and $n = 50$ will fail to conclude that such equipment is effective. Discuss ways to decrease this probability.

11.62 The test of hypothesis in the SSA example concluded that there was not enough evidence to infer that the plan would be profitable. The company would hate to not institute the plan if the actual reduction was as little as 3 days (i.e., $\mu = 21$). Calculate the relevant probability and describe how the company should use this information.

11.63 The fast-food franchiser in Exercise 11.43 was unable to provide enough evidence that the site is acceptable. She is concerned that she may be missing an opportunity to locate the restaurant in a profitable location. She feels that if the actual mean is 104, the restaurant is likely to be very successful. Determine the probability of a Type II error when the mean is 104. Suggest ways to improve this probability.

11.64 Refer to Exercise 11.46. A financial analyst has determined that a 2-minute reduction in the average break would increase productivity. As a result the company would hate to lose this opportunity. Calculate the probability of erroneously concluding that the renovation would not be successful when the average break is 30 minutes. If this probability is high, describe how it can be reduced.

11.65 A school-board administrator believes that the average number of days absent per year among students is less than 10 days. From past experience, he knows that the population standard deviation is 3 days. In testing to determine whether his belief is true, he could use one of the following plans:

i. $n = 100, \quad \alpha = .01$
ii. $n = 75, \quad \alpha = .05$
iii. $n = 50, \quad \alpha = .10$

Which plan has the lowest probability of a Type II error, given that the true population average is 9 days?

11.66 The feasibility of constructing a profitable electricity-producing windmill depends on the mean velocity of the wind. For a certain type of windmill, the mean would have to exceed 20 miles per hour to warrant its construction. The determination of a site's feasibility is a two-stage process. In the first stage, readings of the wind velocity are taken and the mean is calculated. The test is designed to answer the question, "Is the site feasible?" In other words, is there sufficient evidence to conclude that the mean wind velocity exceeds 20 mph? If there is enough evidence, further testing is conducted. If there is not enough evidence, the site is removed from consideration. Discuss the consequences and potential costs of Type I and Type II errors.

11.67 The number of potential sites for the first-stage test in Exercise 11.66 is quite large and the readings can be expensive. Accordingly, the test is conducted with a sample of 25 observations. Because the second-stage cost is high, the significance level is set at 1%. A financial analysis of the potential profits and costs reveals that if the mean wind velocity is as high as 25 mph, the windmill would be extremely profitable. Calculate the probability that the first-stage test will not conclude that the site is feasible when the actual mean wind velocity is 25 mph. (Assume that σ is 8.) Discuss how the process can be improved.

11.4 THE ROAD AHEAD

We had two principal goals to accomplish in Chapters 10 and 11. First, we wanted to present the concepts of estimation and hypothesis testing. Second, we wanted to show how to produce confidence interval estimates and conduct tests of hypotheses. The importance of both goals should not be underestimated. Almost everything that follows this chapter will involve either estimating a parameter or testing a set of hypotheses. Consequently, Sections 10.2 and 11.2 set the pattern for the way in which statistical techniques are applied. It is no exaggeration to state that if you understand how to produce and use confidence interval estimates and how to conduct and interpret hypothesis tests, then you are well on your way to the ultimate goal of being competent at analyzing, interpreting, and presenting data. It is fair for you to ask what more you must accomplish to achieve this goal. The answer, simply put, is much more of the same.

In the chapters that follow, we plan to present about three dozen different statistical techniques that can be (and frequently are) employed by statistics practitioners. To calculate the value of test statistics or confidence interval estimates requires nothing more than the ability to add, subtract, multiply, divide, and compute square roots. If you intend to use the computer, all you need to know are the commands. The key, then, to applying statistics is knowing which formula to calculate or which set of commands to issue. Thus, the real challenge of the subject lies in being able to define the problem and identify which statistical method is the most appropriate one to use.

Most students have some difficulty recognizing the particular kind of statistical problem they are addressing unless, of course, the problem appears among the exercises at the end of a section that just introduced the technique needed. Unfortunately, in practice, statistical problems do not appear already so identified. Consequently, we have adopted an approach to teaching statistics that is designed to help identify the statistical technique.

A number of factors determine which statistical method should be used, but two are especially important: the type of data and the purpose of the statistical inference. In Chapter 2, we pointed out that there are effectively three types of data—interval, ordinal, and nominal. Recall that nominal data represent categories such as marital status, occupation, and gender. Statistics practitioners often record nominal data by assigning numbers to the responses (e.g., 1 = single; 2 = married; 3 = divorced; 4 = widowed). Because these numbers are assigned completely arbitrarily, any calculations performed on them are meaningless. All that we can do with nominal data is count the number of times each category is observed. Ordinal data are obtained from questions whose answers represent a rating or ranking system. For example. if students are asked to rate a university professor, the responses may be excellent, good, fair, or poor. To draw inferences about such data, we convert the responses to numbers. Any numbering system is valid as long as the order of the responses is preserved. Thus "4 = excellent; 3 = good; 2 = fair; 1 = poor" is just as valid as "15 = excellent; 8 = good; 5 = fair; 2 = poor." Because of this feature, the most appropriate statistical procedures for ordinal data are ones based on a ranking process.

Interval data are real numbers, such as those representing income, age, height, weight, and volume. Computation of means and variances is permissible.

The second key factor in determining the statistical technique is the purpose of doing the work. Every statistical method has some specific objective. We address five such objectives in this book.

Problem Objectives

1. **Describe a population.** Our objective here is to describe some property of a population of interest. The decision about which property to describe is generally dictated by the type of data. For example, suppose the population of interest consists of all purchasers of home computers. If we are interested in the purchasers' incomes (for which the data are interval), we may calculate the mean or the variance to describe that aspect of the population. But if we are interested in the brand of computer that has been bought (for which the data are nominal), all we can do is compute the proportion of the population that purchases each brand.

2. **Compare two populations.** In this case, our goal is to compare a property of one population with a corresponding property of a second population. For example, suppose the populations of interest are male and female purchasers of computers. We could compare the means of their incomes, or we could compare the proportion of each population that purchases a certain brand. Once again, the data type generally determines what kinds of properties we compare.

3. **Compare two or more populations.** We might want to compare the average income in each of several locations in order (for example) to decide where to build a new shopping center. Or we might want to compare the proportions of defective items in a number of production lines in order to determine which line is the best. In each case, the problem objective involves comparing two or more populations.

4. **Analyze the relationship between two variables.** There are numerous situations in which we want to know how one variable is related to another. Governments need to know what effect rising interest rates have on the unemployment rate. Companies want to investigate how the sizes of their advertising budgets influence sales volume. In most of the problems in this introductory text, the two variables to be analyzed will be of the same type; we will not attempt to cover the fairly large body of statistical techniques that has been developed to deal with two variables of different types.

5. **Analyze the relationship among two or more variables.** Our objective here is usually to forecast one variable (called the *dependent variable*) on the basis of several other variables (called *independent variables*). We will deal with this problem only in situations in which all variables are interval.

Table 11.3 lists the types of data and the five problem objectives. For each combination, the table specifies the chapter or section where the appropriate statistical technique is presented. For your convenience, a more detailed version of this table is reproduced inside the front cover of this book.

TABLE **11.3** Guide to Statistical Inference Showing Where Each Technique Is Introduced

	DATA TYPE		
PROBLEM OBJECTIVE	NOMINAL	ORDINAL	INTERVAL
Describe a population	Sections 12.3, 15.1	Not covered	Sections 12.1, 12.2
Compare two populations	Sections 13.5, 15.2	Sections 19.1, 19.2	Sections 13.1, 13.3, 13.4, 19.1, 19.2
Compare two or more populations	Section 15.2	Section 19.3	Chapter 14 Section 19.3
Analyze the relationship between two variables	Section 15.2	Section 19.4	Chapter 16
Analyze the relationship among two or more variables	Not covered	Not covered	Chapters 17, 18

Derivations

Because this book is about statistical applications, we assume that our readers have little interest in the mathematical derivations of the techniques described. However, it might be helpful for you to have some understanding about the process that produces the formulas.

As described previously, factors such as the problem objective and the type of data determine the parameter to be estimated and tested. For each parameter, statisticians have determined which statistic to use. That statistic has a sampling distribution that can usually be expressed as a formula. For example, in this chapter, the parameter of interest was the population mean μ, whose best estimator is the sample mean $\bar{x}$. Assuming that the population standard deviation σ is known, the sampling distribution of $\bar{X}$ is normal (or approximately so) with mean μ and standard deviation $\sigma/\sqrt{n}$. The sampling distribution can be described by the formula

$$z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

This formula also describes the test statistic for μ with σ known. With a little algebra, we were able to derive (in Section 10.2) the confidence interval estimator of μ.

In future chapters, we will repeat this process, which in several cases involves the introduction of a new sampling distribution. Although its shape and formula will differ from the sampling distribution used in this chapter, the pattern will be the same. In general, the formula that expresses the sampling distribution will describe the test statistic. Then some algebraic manipulation (which we will not show) produces the interval estimator. Consequently, we will reverse the order of presentation of the two techniques. In other words, we will present the test of hypothesis first, followed by the confidence interval estimator.

CHAPTER SUMMARY

In this chapter, we introduced the concepts of hypothesis testing and applied them to testing hypotheses about a population mean. We showed how to specify the null and alternative hypotheses, set up the rejection region, compute the value of the test statistic, and, finally, to make a decision. Equally as important, we discussed how to interpret the test results. This chapter also demonstrated another way to make decisions; by calculating and using the p-value of the test. To help interpret test results, we showed how to calculate the probability of a Type II error. Finally, we provided a road map of how we plan to present statistical techniques.

IMPORTANT TERMS

Hypothesis testing 361
Null hypothesis 361
Alternative or research hypothesis 361
Type I error 361
Type II error 361
Significance level 361
Test statistic 364
Rejection region 366
Standardized test statistic 367
Statistically significant 368
p-value of a test 369
Highly significant 371
Significant 371
Not statistically significant 371
One-tail test 376
Two-tail test 377
One-sided confidence interval estimator 380
Operating characteristic curve 390

SYMBOLS

Symbol	Pronounced	Represents
H_0	H nought	Null hypothesis
H_1	H one	Alternative (research) hypothesis
α	alpha	Probability of a Type I error
β	beta	Probability of a Type II error
$\bar{x}_L$	X bar sub L or X bar L	Value of $\bar{x}$ large enough to reject H_0
$\|z\|$	Absolute z	Absolute value of z

FORMULA

Test statistic for μ

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

COMPUTER OUTPUT AND INSTRUCTIONS

Technique	Excel	Minitab
Test of μ	372	373
Probability of a Type II error (and Power)	389	389

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12

INFERENCE ABOUT A POPULATION

Nielsen Ratings

DATA
Xm12-00*

Statistical techniques play a vital role in helping advertisers determine how many viewers watch the shows they sponsor. Although several companies sample television viewers to determine what shows they watch, the best known is the A. C. Nielsen firm. The Nielsen ratings are based on a random sample of approximately 5,000 of the 115 million households in the United States with at least one television (in 2010). A meter attached to the televisions in the selected households keeps track of when the televisions are turned on and what channels they are tuned to. The data are sent to the Nielsen's computer every night, from which Nielsen computes the rating and sponsors can determine the number of viewers and the potential value of any commercials.

© Brand X Pictures/Jupiter images

On page 427, we provide a solution to this problem.

The results from Sunday, February 14, 2010 for the time slot 9 to 9:30 P.M. have been recorded using the following codes:

Network	Show	Code
ABC	*Extreme Makeover: Home Edition*	1
CBS	*Undercover Boss*	2
Fox	*Family Guy*	3
NBC	*Vancouver Winter Olympics*	4
Television turned off or watched some other channel		5

Source: tvbythenumbers.com February 15, 2010.

NBC would like to use the data to estimate how many of the households were tuned to its program *Vancouver Winter Olympics*.

Introduction

In the previous two chapters, we introduced the concepts of statistical inference and showed how to estimate and test a population mean. However, the illustration we chose is unrealistic because the techniques require us to use the population standard deviation σ, which, in general, is unknown. The purpose, then, of Chapters 10 and 11 was to set the pattern for the way in which we plan to present other statistical techniques. In other words, we will begin by identifying the parameter to be estimated or tested. We will then specify the parameter's estimator (each parameter has an estimator chosen because of the characteristics we discussed at the beginning of Chapter 10) and its sampling distribution. Using simple mathematics, statisticians have derived the interval estimator and the test statistic. This pattern will be used repeatedly as we introduce new techniques.

In Section 11.4, we described the five problem objectives addressed in this book, and we laid out the order of presentation of the statistical methods. In this chapter, we will present techniques employed when the problem objective is to describe a population. When the data are interval, the parameters of interest are the population mean μ and the population variance σ^2. In Section 12.1, we describe how to make inferences about the population mean under the more realistic assumption that the population standard deviation is unknown. In Section 12.2, we continue to deal with interval data, but our parameter of interest becomes the population variance.

In Chapter 2 and in Section 11.4, we pointed out that when the data are nominal, the only computation that makes sense is determining the proportion of times each value occurs. Section 12.3 discusses inference about the proportion p. In Section 12.4, we present an important application in marketing: market segmentation.

Keller's website Appendix Applications in Accounting: Auditing describes how the statistical techniques introduced in this chapter are used in auditing.

12.1 Inference about a Population Mean When the Standard Deviation Is Unknown

In Sections 10.2 and 11.2, we demonstrated how to estimate and test the population mean when the population standard deviation is known. The confidence interval estimator and the test statistic were derived from the sampling distribution of the sample mean with σ known, expressed as

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

In this section, we take a more realistic approach by acknowledging that if the population mean is unknown, then so is the population standard deviation. Consequently, the previous sampling distribution cannot be used. Instead, we substitute the sample standard deviation s in place of the unknown population standard deviation σ. The result is called a ***t*-statistic** because that is what mathematician William S. Gosset called it. In 1908, Gosset showed that the t-statistic defined as

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

is Student t distributed when the sampled population is normal. (Gosset published his findings under the pseudonym "Student," hence the **Student *t* distribution**.) Recall that we introduced the Student t distribution in Section 8.4.

With exactly the same logic used to develop the test statistic in Section 11.2 and the confidence interval estimator in Section 10.2, we derive the following inferential methods.

Test Statistic for μ When σ Is Unknown

When the population standard deviation is unknown and the population is normal, the test statistic for testing hypotheses about μ is

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

which is Student t distributed with $\nu = n - 1$ degrees of freedom.

Confidence Interval Estimator of μ When σ Is Unknown

$$\bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}} \qquad \nu = n - 1$$

These formulas now make obsolete the test statistic and interval estimator employed in Chapters 10 and 11 to estimate and test a population mean. Although we continue to use the concepts developed in Chapters 10 and 11 (as well as all the other chapters), we will no longer use the z-statistic and the z-estimator of μ. All future inferential problems involving a population mean will be solved using the t-statistic and t-estimator of μ shown in the preceding boxes.

EXAMPLE 12.1 Newspaper Recycling Plant

DATA Xm12-01*

In the near future, nations will likely have to do more to save the environment. Possible actions include reducing energy use and recycling. Currently, most products manufactured from recycled material are considerably more expensive than those manufactured from material found in the earth. For example, it is approximately three times as expensive to produce glass bottles from recycled glass than from silica sand, soda ash, and limestone, all plentiful materials mined in numerous countries. It is more expensive to manufacture aluminum cans from recycled cans than from bauxite. Newspapers are an exception. It can be profitable to recycle newspaper. A major expense is the collection from homes. In recent years, many companies have gone into the business of collecting used newspapers from households and recycling them. A financial analyst for one such company has recently computed that the firm would make a profit if the mean weekly newspaper collection from each household exceeded 2.0 pounds. In a study to determine the feasibility of a recycling plant, a random sample of 148 households was drawn from a large community, and the weekly weight of newspapers discarded for recycling for each household was recorded and listed next. Do these data provide sufficient evidence to allow the analyst to conclude that a recycling plant would be profitable?

Weights of Discarded Newspapers

2.5	0.7	3.4	1.8	1.9	2.0	1.3	1.2	2.2	0.9	2.7	2.9	1.5	1.5	2.2
3.2	0.7	2.3	3.1	1.3	4.2	3.4	1.5	2.1	1.0	2.4	1.8	0.9	1.3	2.6
3.6	0.8	3.0	2.8	3.6	3.1	2.4	3.2	4.4	4.1	1.5	1.9	3.2	1.9	1.6
3.0	3.7	1.7	3.1	2.4	3.0	1.5	3.1	2.4	2.1	2.1	2.3	0.7	0.9	2.7
1.2	2.2	1.3	3.0	3.0	2.2	1.5	2.7	0.9	2.5	3.2	3.7	1.9	2.0	3.7
2.3	0.6	0.0	1.0	1.4	0.9	2.6	2.1	3.4	0.5	4.1	2.2	3.4	3.3	0.0
2.2	4.2	1.1	2.3	3.1	1.7	2.8	2.5	1.8	1.7	0.6	3.6	1.4	2.2	2.2
1.3	1.7	3.0	0.8	1.6	1.8	1.4	3.0	1.9	2.7	0.8	3.3	2.5	1.5	2.2
2.6	3.2	1.0	3.2	1.6	3.4	1.7	2.3	2.6	1.4	3.3	1.3	2.4	2.0	
1.3	1.8	3.3	2.2	1.4	3.2	4.3	0.0	2.0	1.8	0.0	1.7	2.6	3.1	

SOLUTION

IDENTIFY

The problem objective is to describe the population of the amounts of newspaper discarded by each household in the population. The data are interval, indicating that the parameter to be tested is the population mean. Because the financial analyst needs to determine whether the mean is greater than 2.0 pounds, the alternative hypothesis is

$$H_1: \ \mu > 2.0$$

As usual, the null hypothesis states that the mean is equal to the value listed in the alternative hypothesis:

$$H_0: \ \mu = 2.0$$

The test statistic is

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \qquad \nu = n - 1$$

COMPUTE

MANUALLY

The manager believes that the cost of a Type I error (concluding that the mean is greater than 2 when it isn't) is quite high. Consequently, he sets the significance level at 1%. The rejection region is

$$t > t_{\alpha,\nu} = t_{.01,148} \approx t_{.01,150} = 2.351$$

To calculate the value of the test statistic, we need to calculate the sample mean $\bar{x}$ and the sample standard deviation s. From the data, we determine

$$\sum x_i = 322.7 \quad \text{and} \quad \sum x_i^2 = 845.1$$

Thus,

$$\bar{x} = \frac{\sum x_i}{n} = \frac{322.7}{148} = 2.18$$

$$s^2 = \frac{\sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}}{n - 1} = \frac{845.1 - \frac{(322.7)^2}{148}}{148 - 1} = .962$$

and

$$s = \sqrt{s^2} = \sqrt{.962} = .981$$

The value of μ is to be found in the null hypothesis. It is 2.0. The value of the test statistic is

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{2.18 - 2.0}{.981/\sqrt{148}} = 2.23$$

Because 2.23 is not greater than 2.351, we cannot reject the null hypothesis in favor of the alternative. (Students performing the calculations manually can approximate the p-value. Keller's website Appendix Approximating the p-Value from the Student t Table describes how.)

EXCEL

	A	B	C	D
1	t-Test: Mean			
2				
3				Newspaper
4	Mean			2.18
5	Standard Deviation			0.98
6	Hypothesized Mean			2
7	df			147
8	t Stat			2.24
9	P(T<=t) one-tail			0.0134
10	t Critical one-tail			2.3520
11	P(T<=t) two-tail			0.0268
12	t Critical two-tail			2.6097

INSTRUCTIONS

1. Type or import the data into one column*. (Open Xm12-01.)
2. Click **Add-Ins, Data Analysis Plus, and *t*-Test: Mean**.
3. Specify the **Input Range** (A1:A149) the **Hypothesized Mean** (2), and α (.01).

*If the column contains a blank (representing missing data) the row will have to be deleted. See Keller's website Appendix Deleting Blank Rows in Excel.

MINITAB

One-Sample T: Newspaper

Test of mu = 2 vs > 2

Variable	N	Mean	StDev	SE Mean	95% Lower Bound	T	P
Newspaper	148	2.1804	0.9812	0.0807	2.0469	2.24	0.013

INSTRUCTIONS

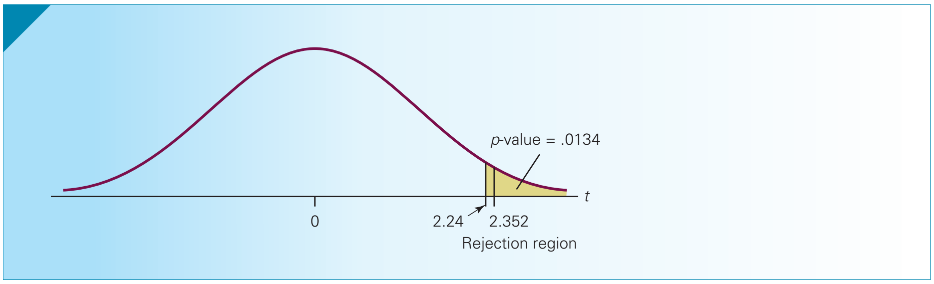
1. Type or import the data into one column. (Open Xm12-01.)
2. Click **Stat, Basic Statistics**, and **1-Sample *t* . . . .**
3. Type or use the **Select** button to specify the name of the variable or the column in the **Samples in columns** box (Newspaper), choose **Perform hypothesis test** and type the value of μ in the **Hypothesized mean** box (2), and click **Options . . . .**
4. Select one of **less than, not equal**, or **greater than** in the **Alternative** box (greater than).

INTERPRET

The value of the test statistic is $t = 2.24$, and its p-value is .0134. There is not enough evidence to infer that the mean weight of discarded newspapers is greater than 2.0. Note that there is some evidence: The p-value is .0134. However, because we wanted the probability of a Type I error to be small, we insisted on a 1% significance level. Thus, we cannot conclude that the recycling plant would be profitable.

Figure 12.1 exhibits the sampling distribution for this example.

FIGURE **12.1** Sampling Distribution for Example 12.1



EXAMPLE 12.2 Tax Collected from Audited Returns

DATA Xm12-02

In 2007 (the latest year reported), 134,543,000 tax returns were filed in the United States (*Source: Statistical Abstract of the United States, 2009*, Table 463). The Internal Revenue Service (IRS) examined 1.03% or 1,385,000 of them to determine if they were correctly done. To determine how well the auditors are performing, a random sample of these returns was drawn and the additional tax was reported, which is listed next. Estimate with 95% confidence the mean additional income tax collected from the 1,385,000 files audited. (Adapted from U.S. Internal Revenue Service, *IRS Data Book*, annual, Publication 55B.)

Additional Income Tax

15731.15	15594.25	8724.17	11374.34	13197.31	10312.43
6364.09	18662.69	8214.82	9316.70	12132.27	15602.60
7116.91	10463.63	12155.59	3977.52	12672.99	10253.46
12890.47	10070.18	4453.51	14034.78	16409.30	20352.98
11853.56	11603.00	10363.78	11830.85	13676.91	9153.78
10665.40	11255.04	8220.39	15968.90	4278.77	16178.15
6635.94	14491.35	13851.38	7313.00	11985.47	17387.08
12254.47	5128.84	9748.55	15078.81	8658.68	13689.50
7619.82	10102.60	15482.87	9904.92	5172.77	7932.38
9524.40	11010.64	10174.46	15923.39	14994.48	10576.01
17041.16	3694.86	10451.61	18292.65	13789.65	16494.25
7648.54	9761.73	16359.09	5318.50	10429.75	1554.77
7678.23	15018.60	14362.03	15467.99	12984.66	14461.00
9198.38	7589.68	13716.94	14588.00	8672.97	12708.45
7951.54	2732.71	12834.86	7977.11	4023.16	16068.56
6660.60	4740.91	11541.49	9952.42	16493.69	15052.86
11493.30	9326.62	11558.31	10007.03	15651.35	12563.35
7792.70	7308.05	7224.16	16132.63	13991.80	4247.18
10147.98	13760.60	9714.45	0.00	18070.00	6996.54
17712.81	7220.72	15002.06	12870.00	13188.00	7863.68
19276.94	22132.00	12613.92	6645.67	12770.00	12971.50
9320.49	14258.93	17276.46	11801.96	4614.75	18461.05
7821.41	2994.88	8126.62	8941.16	9521.19	21480.96
6774.85	11271.67	13054.84	13739.98	10813.72	15999.38
9389.68	2690.57	4978.82	18259.38	14666.33	
10730.14	17390.36	10481.09	15677.43	1974.11	
4798.18	8402.68	6959.23	16069.51	10831.23	
17192.61	0.00	13224.15	11819.80	12071.03	
7730.51	12744.79	12865.30	17389.63	19326.79	
12387.27	16284.14	14898.40	5927.63	11507.15	
17110.60	7100.00	13617.28	15855.37	10443.33	
17415.28	16386.49	11235.86	7666.54	5972.11	

SOLUTION

IDENTIFY

The problem objective is to describe the population of additional income tax. The data are interval, so the parameter is the population mean μ. The question asks us to estimate this parameter. The confidence interval estimator is

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

COMPUTE

MANUALLY

From the data, we determine

$$\sum x_i = 2{,}087{,}080 \quad \text{and} \quad \sum x_i^2 = 27{,}216{,}444{,}599$$

Thus,

$$\bar{x} = \frac{\sum x_i}{n} = \frac{2{,}087{,}080}{184} = 11{,}343$$

and

$$s^2 = \frac{\sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}}{n-1} = \frac{27{,}216{,}444{,}599 - \frac{(2{,}087{,}080)^2}{184}}{184 - 1} = 19{,}360{,}979$$

Thus

$$s = \sqrt{s^2} = \sqrt{19{,}360{,}979} = 4{,}400$$

Because we want a 95% confidence interval estimate, $1 - \alpha = .95$, $\alpha = .05$, $\alpha/2 = .025$, and $t_{\alpha/2,\nu} = t_{.025,208} \approx t_{.25,200} = 1.972$. Thus, the 95% confidence interval estimate of μ is

$$\bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}} = 11{,}343 \pm 1.972\frac{4{,}400}{\sqrt{184}} = 11{,}343 \pm 640$$

or

$$\text{LCL} = \$10{,}703 \qquad \text{UCL} = \$11{,}983$$

EXCEL

	A	B	C
1	**t-Estimate: Mean**		
2			
3			*Taxes*
4	Mean		11,343
5	Standard Deviation		4,400
6	Observations		184
7	Standard Error		324
8	LCL		10,703
9	UCL		11,983

INSTRUCTIONS

1. Type or import the data into one column*. (Open Xm12-02.)
2. Click **Add-Ins, Data Analysis Plus**, and ***t*-Estimate: Mean**.
3. Specify the **Input Range** (A1:A185) and α (.05).

MINITAB

One-Sample T: Taxes

Variable	N	Mean	StDev	SE Mean	95% CI
Taxes	184	11343	4400	324	(10703, 11983)

INSTRUCTIONS

1. Type or import the data into one column. (Open Xm12-02.)
2. Click **Stat, Basic Statistics, and 1-Sample *t* . . . .**
3. Select or type the variable name in the **Samples in columns** box (Taxes) and click **Options . . . .**
4. Specify the **Confidence level** (.95) and **not equal** for the **Alternative**.

*If the column contains a blank (representing missing data) the row will have to be deleted.

INTERPRET

We estimate that the mean additional tax collected lies between $10,703 and $11,983. We can use this estimate to help decide whether the IRS is auditing the individuals who should be audited.

Checking the Required Conditions

When we introduced the Student t distribution, we pointed out that the t-statistic is Student t distributed if the population from which we've sampled is normal. However, statisticians have shown that the mathematical process that derived the Student t distribution is **robust**, which means that if the population is nonnormal, the results of the t-test and confidence interval estimate are still valid provided that the population is not *extremely* nonnormal.* To check this requirement, we draw the histogram and determine whether it is far from bell shaped. Figures 12.2 and 12.3 depict the Excel histograms for Examples 12.1 and 12.2, respectively. (The Minitab histograms are similar.) Both histograms suggest that the variables are not extremely nonnormal.

FIGURE **12.2** Histogram for Example 12.1

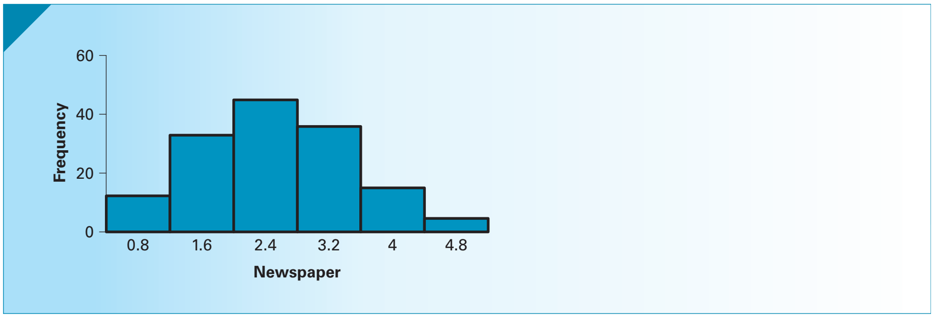
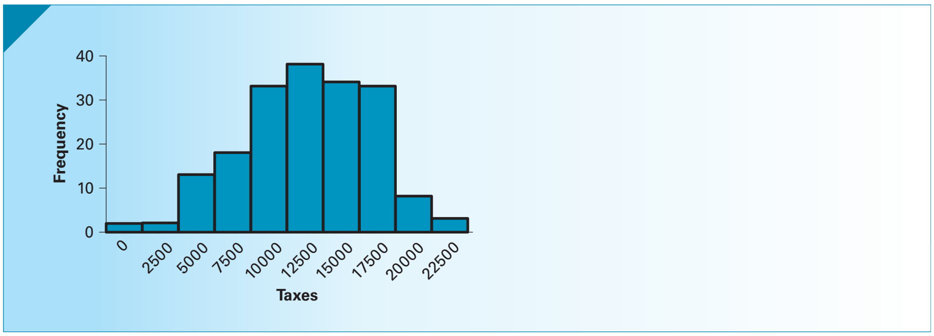


FIGURE **12.3** Histogram for Example 12.2



*Statisticians have shown that when the sample size is large, the results of a t-test and estimator of a mean are valid even when the population is extremely nonnormal. The sample size required depends on the extent of nonnormality.

Estimating the Totals of Finite Populations

The inferential techniques introduced thus far were derived by assuming infinitely large populations. In practice, however, most populations are finite. (Infinite populations are usually the result of some endlessly repeatable process, such as flipping a coin or selecting items with replacement.) When the population is small, we must adjust the test statistic and interval estimator using the finite population correction factor introduced in Chapter 9 (page 313). (In Keller's website Appendix Applications in Accounting: Auditing we feature an application that requires the use of the correction factor.) However, in populations that are large relative to the sample size, we can ignore the correction factor. Large populations are defined as populations that are at least 20 times the sample size.

Finite populations allow us to use the confidence interval estimator of a mean to produce a confidence interval estimator of the population total. To estimate the total, we multiply the lower and upper confidence limits of the estimate of the mean by the population size. Thus, the confidence interval estimator of the total is

$$N\left[\bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}\right]$$

For example, suppose that we wish to estimate the total amount of additional income tax collected from the 1,385,000 returns that were examined. The 95% confidence interval estimate of the total is

$$N\left[\bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}\right] = 1{,}385{,}000(11{,}343 \pm 640)$$

which is

$$\text{LCL} = 14{,}823{,}655{,}000 \quad \text{and} \quad \text{UCL} = 16{,}596{,}455{,}000$$

Developing an Understanding of Statistical Concepts 1

This section introduced the term *degrees of freedom*. We will encounter this term many times in this book, so a brief discussion of its meaning is warranted. The Student t distribution is based on using the sample variance to estimate the unknown population variance. The sample variance is defined as

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n - 1}$$

To compute s^2, we must first determine $\bar{x}$. Recall that sampling distributions are derived by repeated sampling from the same population. To repeatedly take samples to compute s^2, we can choose any numbers for the first $n - 1$ observations in the sample. However, we have no choice on the nth value because the sample mean must be calculated first. To illustrate, suppose that $n = 3$ and we find $\bar{x} = 10$. We can have x_1 and x_2 assume any values without restriction. However, x_3 must be such that $\bar{x} = 10$. For example, if $x_1 = 6$ and $x_2 = 8$, then x_3 must equal 16. Therefore, there are only two degrees of freedom in our selection of the sample. We say that we lose one degree of freedom because we had to calculate $\bar{x}$.

Notice that the denominator in the calculation of s^2 is equal to the number of degrees of freedom. This is not a coincidence and will be repeated throughout this book.

Developing an Understanding of Statistical Concepts 2

The t-statistic like the z-statistic measures the difference between the sample mean $\bar{x}$ and the hypothesized value of μ in terms of the number of standard errors. However, when the population standard deviation σ is unknown we estimate the standard error by $s/\sqrt{n}$.

Developing an Understanding of Statistical Concepts 3

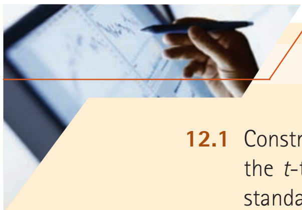
When we introduced the Student t distribution in Section 8.4, we pointed out that it is more widely spread out than the standard normal. This circumstance is logical. The only variable in the z-statistic is the sample mean $\bar{x}$, which will vary from sample to sample. The t-statistic has two variables: the sample mean $\bar{x}$ and the sample standard deviation s, both of which will vary from sample to sample. Because of the greater uncertainty, the t-statistic will display greater variability. Exercises 12.15 – 12.22 address this concept.

We complete this section with a review of how we identify the techniques introduced in this section.

Factors That Identify the *t*-Test and Estimator of μ

1. **Problem objective**: Describe a population.
2. **Data type**: Interval.
3. **Type of descriptive measurement**: Central location.

Exercises



DO-IT-YOURSELF EXCEL

12.1 Construct an Excel spreadsheet that performs the t-test of μ. Inputs: sample mean, sample standard deviation, sample size, hypothesized mean. Outputs: Test statistic, critical values, and one- and two-tail p-values. Tools: **TINV, TDIST.**

12.2 Create a spreadsheet that computes the t-estimate of μ. Inputs: sample mean, sample standard deviation, sample size, and confidence level. Outputs: Upper and lower confidence limits. Tools: **TINV.**

Developing an Understanding of Statistical Concepts

The following exercises are "what-if" analyses designed to determine what happens to the test statistics and interval estimates when elements of the statistical inference change. These problems can be solved manually or using the Do-It-Yourself Excel spreadsheets you created.

12.3 a. A random sample of 25 was drawn from a population. The sample mean and standard deviation are $\bar{x} = 510$ and $s = 125$. Estimate μ with 95% confidence.
b. Repeat part (a) with $n = 50$.
c. Repeat part (a) with $n = 100$.
d. Describe what happens to the confidence interval estimate when the sample size increases.

12.4 a. The mean and standard deviation of a sample of 100 is $\bar{x} = 1500$ and $s = 300$. Estimate the population mean with 95% confidence.
b. Repeat part (a) with $s = 200$.
c. Repeat part (a) with $s = 100$.
d. Discuss the effect on the confidence interval estimate of decreasing the standard deviation s.

12.5 a. A statistics practitioner drew a random sample of 400 observations and found that $\bar{x} = 700$ and $s = 100$. Estimate the population mean with 90% confidence.
b. Repeat part (a) with a 95% confidence level.
c. Repeat part (a) with a 99% confidence level.
d. What is the effect on the confidence interval estimate of increasing the confidence level?

12.6 a. The mean and standard deviation of a sample of 100 are

$$\bar{x} = 10 \quad \text{and} \quad s = 1.$$

Estimate the population mean with 95% confidence.
b. Repeat part (a) with $s = 4$.
c. Repeat part (a) with $s = 10$.
d. Discuss the effect on the confidence interval estimate of increasing the standard deviation s.

12.7 a. A statistics practitioner calculated the mean and standard deviation from a sample of 51. They are $\bar{x} = 120$ and $s = 15$. Estimate the population mean with 95% confidence.
b. Repeat part (a) with a 90% confidence level.
c. Repeat part (a) with an 80% confidence level.
d. What is the effect on the confidence interval estimate of decreasing the confidence level?

12.8 a. The sample mean and standard deviation from a sample of 81 observations are $\bar{x} = 63$ and $s = 8$. Estimate μ with 95% confidence.
b. Repeat part (a) with $n = 64$.
c. Repeat part (a) with $n = 36$.
d. Describe what happens to the confidence interval estimate when the sample size decreases.

12.9 a. The sample mean and standard deviation from a random sample of 10 observations from a normal population were computed as $\bar{x} = 23$ and $s = 9$. Calculate the value of the test statistic (and for Excel users, the p-value) of the test required to determine whether there is enough evidence to infer at the 5% significance level that the population mean is greater than 20.
b. Repeat part (a) with $n = 30$.
c. Repeat part (a) with $n = 50$.
d. Describe the effect on the t-statistic (and for Excel users, the p-value) of increasing the sample size.

12.10 a. A statistics practitioner is in the process of testing to determine whether there is enough evidence to infer that the population mean is different from 180. She calculated the mean and standard deviation of a sample of 200 observations as $\bar{x} = 175$ and $s = 22$. Calculate the value of the test statistic (and for Excel users, the p-value) of the test required to determine whether there is enough evidence at the 5% significance level.
b. Repeat part (a) with $s = 45$.
c. Repeat part (a) with $s = 60$.
d. Discuss what happens to the t statistic (and for Excel users, the p-value) when the standard deviation increases.

12.11 a. Calculate the test statistic (and for Excel users, the p-value) when $\bar{x} = 145$, $s = 50$, and $n = 100$. Use a 5% significance level.

$$H_0: \ \mu = 150$$
$$H_1: \ \mu < 150$$

b. Repeat part (a) with $\bar{x} = 140$.
c. Repeat part (a) with $\bar{x} = 135$.
d. What happens to the t-statistic (and for Excel users, the p-value) when the sample mean decreases?

12.12 a. A random sample of 25 observations was drawn from a normal population. The sample mean and sample standard deviation are $\bar{x} = 52$ and $s = 15$. Calculate the test statistic (and for Excel users, the p-value) of a test to determine if there is enough evidence at the 10% significance level to infer that the population mean is not equal to 50.
b. Repeat part (a) with $n = 15$.
c. Repeat part (a) with $n = 5$.
d. Discuss what happens to the t-statistic (and for Excel users, the p-value) when the sample size decreases.

12.13 a. A statistics practitioner wishes to test the following hypotheses:

$$H_0: \ \mu = 600$$
$$H_1: \ \mu < 600$$

A sample of 50 observations yielded the statistics $\bar{x} = 585$ and $s = 45$. Calculate the test statistic (and for Excel users, the p-value) of a test to determine whether there is enough evidence at the 10% significance level to infer that the alternative hypothesis is true.
b. Repeat part (a) with $\bar{x} = 590$.
c. Repeat part (a) with $\bar{x} = 595$.
d. Describe the effect of decreasing the sample mean.

12.14 a. To test the following hypotheses, a statistics practitioner randomly sampled 100 observations and found $\bar{x} = 106$ and $s = 35$. Calculate the test statistic (and for Excel users, the p-value) of a test to determine whether there is enough evidence at the 1% significance level to infer that the alternative hypothesis is true.

$$H_0: \ \mu = 100$$
$$H_1: \ \mu > 100$$

b. Repeat part (a) with $s = 25$.
c. Repeat part (a) with $s = 15$.
d. Discuss what happens to the t-statistic (and for Excel users, the p-value) when the standard deviation decreases.

12.15 A random sample of 8 observations was drawn from a normal population. The sample mean and sample standard deviation are $\bar{x} = 40$ and $s = 10$.
a. Estimate the population mean with 95% confidence.

b. Repeat part (a) assuming that you know that the population standard deviation is $\sigma = 10$.
c. Explain why the interval estimate produced in part (b) is narrower than that in part (a).

12.16 a. Estimate the population mean with 90% confidence given the following: $\bar{x} = 175$, $s = 30$, and $n = 5$.
b. Repeat part (a) assuming that you know that the population standard deviation is $\sigma = 30$.
c. Explain why the interval estimate produced in part (b) is narrower than that in part (a).

12.17 a. After sampling 1,000 members of a normal population, you find $\bar{x} = 15{,}500$ and $s = 9{,}950$. Estimate the population mean with 90% confidence.
b. Repeat part (a) assuming that you know that the population standard deviation is $\sigma = 9{,}950$.
c. Explain why the interval estimates were virtually identical.

12.18 a. In a random sample of 500 observations drawn from a normal population, the sample mean and sample standard deviation were calculated as $\bar{x} = 350$ and $s = 100$. Estimate the population mean with 99% confidence.
b. Repeat part (a) assuming that you know that the population standard deviation is $\sigma = 100$.
c. Explain why the interval estimates were virtually identical.

12.19 a. A random sample of 11 observations was taken from a normal population. The sample mean and standard deviation are $\bar{x} = 74.5$ and $s = 9$. Can we infer at the 5% significance level that the population mean is greater than 70?
b. Repeat part (a) assuming that you know that the population standard deviation is $\sigma = 90$.
c. Explain why the conclusions produced in parts (a) and (b) differ.

12.20 a. A statistics practitioner randomly sampled 10 observations and found $\bar{x} = 103$ and $s = 17$. Is there sufficient evidence at the 10% significance level to conclude that the population mean is less than 110?
b. Repeat part (a) assuming that you know that the population standard deviation is $\sigma = 17$.
c. Explain why the conclusions produced in parts (a) and (b) differ.

12.21 a. A statistics practitioner randomly sampled 1,500 observations and found $\bar{x} = 14$ and $s = 25$. Test to determine whether there is enough evidence at the 5% significance level to infer that the population mean is less than 15.
b. Repeat part (a) assuming that you know that the population standard deviation is $\sigma = 25$.
c. Explain why the conclusions produced in parts (a) and (b) are virtually identical.

12.22 a. Test the following hypotheses with $\alpha = .05$ given that $\bar{x} = 405$, $s = 100$, and $n = 1{,}000$.

$$H_0: \mu = 400$$
$$H_1: \mu > 400$$

b. Repeat part (a) assuming that you know that the population standard deviation is $\sigma = 100$.
c. Explain why the conclusions produced in parts (a) and (b) are virtually identical.

Applications

The following exercises may be answered manually or with the assistance of a computer. The data are stored in files. Assume that the random variable is normally distributed.

12.23 Xr12-23 A courier service advertises that its average delivery time is less than 6 hours for local deliveries. A random sample of times for 12 deliveries to an address across town was recorded. These data are shown here. Is this sufficient evidence to support the courier's advertisement, at the 5% level of significance?

3.03	6.33	6.50	5.22	3.56	6.76
7.98	4.82	7.96	4.54	5.09	6.46

12.24 Xr12-24 How much money do winners go home with from the television quiz show *Jeopardy*? To determine an answer, a random sample of winners was drawn; the recorded amount of money each won is listed here. Estimate with 95% confidence the mean winnings for all the show's players.

26,650	6,060	52,820	8,490	13,660
25,840	49,840	23,790	51,480	18,960
990	11,450	41,810	21,060	7,860

12.25 Xr12-25 A diet doctor claims that the average North American is more than 20 pounds overweight. To test his claim, a random sample of 20 North Americans was weighed, and the difference between their actual and ideal weights was calculated. The data are listed here. Do these data allow us to infer at the 5% significance level that the doctor's claim is true?

16	23	18	41	22	18	23	19	22	15
18	35	16	15	17	19	23	15	16	26

12.26 Xr12-26 A federal agency responsible for enforcing laws governing weights and measures routinely inspects packages to determine whether the weight of the contents is at least as great as that advertised on the package. A random sample of 18 containers whose packaging states that the contents weigh 8 ounces was drawn. The contents were weighed and the results follow. Can we conclude at the 1% significance level that on average the containers are mislabeled?

7.80	7.91	7.93	7.99	7.94	7.75
7.97	7.95	7.79	8.06	7.82	7.89
7.92	7.87	7.92	7.98	8.05	7.91

12.27 Xr12-27 A parking control officer is conducting an analysis of the amount of time left on parking meters. A quick survey of 15 cars that have just left their metered parking spaces produced the following times (in minutes). Estimate with 95% confidence the mean amount of time left for all the city's meters.

22	15	1	14	0	9	17	31
18	26	23	15	33	28	20	

12.28 Xr12-28 Part of a university professor's job is to publish his or her research. This task often entails reading a variety of journal articles to keep up to date. To help determine faculty standards, a dean of a business school surveyed a random sample of 12 professors across the country and asked them to count the number of journal articles they read in a typical month. These data are listed here. Estimate with 90% confidence the mean number of journal articles read monthly by professors.

9 17 4 23 56 30 41 45 21 10 44 20

12.29 Xr12-29 Most owners of digital cameras store their pictures on the camera. Some will eventually download these to a computer or print them using their own printers or a commercial printer. A film-processing company wanted to know how many pictures were stored on computers. A random sample of 10 digital camera owners produced the data given here. Estimate with 95% confidence the mean number of pictures stored on digital cameras.

25 6 22 26 31 18 13 20 14 2

12.30 Xr12-30 University bookstores order books that instructors adopt for their courses. The number of copies ordered matches the projected demand. However, at the end of the semester, the bookstore has too many copies on hand and must return them to the publisher. A bookstore has a policy that the proportion of books returned should be kept as small as possible. The average is supposed to be less than 10%. To see whether the policy is working, a random sample of book titles was drawn, and the fraction of the total originally ordered that are returned is recorded and listed here. Can we infer at the 10% significance level that the mean proportion of returns is less than 10%?

4 15 11 7 5 9 4 3 5 8

The following exercises require the use of a computer and software. The answers to Exercises 12.31 to 12.45 may be calculated manually. See Appendix A for the sample statistics. Use a 5% significance level unless specified otherwise.

12.31 Xr12-31* A growing concern for educators in the United States is the number of teenagers who have part-time jobs while they attend high school. It is generally believed that the amount of time teenagers spend working is deducted from the amount of time devoted to schoolwork. To investigate this problem, a school guidance counselor took a random sample of 200 15-year-old high school students and asked how many hours per week each worked at a part-time job. Estimate with 95% confidence the mean amount of time all 15-year-old high school students devote per week to part-time jobs.

12.32 Xr12-32 A company that produces universal remote controls wanted to determine the number of remote control devices American homes contain. The company hired a statistician to survey 240 randomly selected homes and determine the number of remote controls. If there are 100 million households, estimate with 99% confidence the total number of remote controls in the United States.

12.33 Xr12-33 A random sample of American adults was asked whether or not they smoked cigarettes. Those who responded affirmatively were asked how many cigarettes they smoked per day. Assuming that there are 50 million American adults who smoke, estimate with 95% confidence the number of cigarettes smoked per day in the United States. (Adapted from the *Statistical Abstract of the United States*, *2009*, Table 196 and Bloomberg News.)

12.34 Xr12-34 Bankers and economists watch for signs that the economy is slowing. One statistic they monitor is consumer debt, particularly credit card debt. The Federal Reserve conducts surveys of consumer finances every 3 years. The last survey determined that 23.8% of American households have no credit cards and another 31.2% of the households paid off their most recent credit card bills. The remainder, approximately 50 million households, did not pay their credit card bills in the previous month. A random sample of these households was drawn. Each household in the sample reported how much credit card debt it currently carries. The Federal Reserve would like an estimate (with 95% confidence) of the total credit card debt in the United States.

12.35 Xr12-35* OfficeMax, a chain that sells a wide variety of office equipment often features sales of products whose prices are reduced because of rebates. Some rebates are so large that the effective price becomes $0. The goal is to lure customers into the store to buy other nonsale items. A secondary objective is to acquire addresses and telephone numbers to sell to telemarketers and other mass marketers. During one week in January, OfficeMax offered a 100-pack of CD-ROMs (regular price $29.99 minus $10 instant rebate, $12 manufacturer's rebate, and $8 OfficeMax mail-in rebate). The number of packages was limited, and no rain checks were issued. In all OfficeMax stores, 2,800 packages were in stock.

All were sold. A random sample of 122 buyers was undertaken. Each was asked to report the total value of the other purchases made that day. Estimate with 95% the total spent on other products purchased by those who bought the CD-ROMs.

12.36 Xr12-36 An increasing number of North Americans regularly take vitamins or herbal remedies daily. To gauge this phenomenon, a random sample of Americans was asked to report the number of vitamin and herbal supplements they take daily. Estimate with 95% confidence the mean number of vitamin and herbal supplements Americans take daily.

12.37 Xr12-37 Generic drug sales make up about half of all prescriptions sold in the United States. The marketing manager for a pharmaceutical company wanted to acquire more information about the sales of generic prescription drugs. To do so, she randomly sampled 900 customers who recently filled prescriptions for generic drugs and recorded the cost of each prescription. Estimate with 95% confidence the mean cost of all generic prescription drugs. (Adapted from the *Statistical Abstract of the United States*, *2009*, Table 151.)

12.38 Xr12-38 Traffic congestion seems to worsen each year. This raises the question, How much does roadway congestion cost the United States annually? The Federal Highway Administration's Highway Performance Monitoring System conducts an analysis to produce an estimate of the total cost. Drivers in the 73 most congested areas in the United States were sampled, and each driver's congestion cost in time and gasoline was recorded. The total number of drivers in these 73 areas was 128,000,000. Estimate with 95% confidence the total cost of congestion in the 73 areas. (Adapted from the *Statistical Abstract of the United States*, *2006*, Table 1082.)

12.39 Xr12-39 To help estimate the size of the disposable razor market, a random sample of men was asked to count the number of shaves they used each razor for. Assume that each razor is used once per day. Estimate with 95% confidence the number of days a pack of 10 razors will last.

12.40 Xr12-40 Because of the enormity of the viewing audience, firms that advertise during the Super Bowl create special commercials that tend to be quite entertaining. Thirty-second commercials cost several million dollars during the Super Bowl game. A random sample of people who watched the game was asked how many commercials they watched in their entirety. Do these data allow us to infer that the mean number of commercials watched is greater than 15?

12.41 Xr12-41 On a per capita basis, the United States spends far more on health care than any other country. To help assess the costs, annual surveys are undertaken. One such survey asks a sample of Americans to report the number of times they visited a health-care professional in the year. The data for 2006 were recorded. In 2006, the United States population was 299,157,000. Estimate with 95% confidence the total number of visits to a health-care professional. (Adapted from the *Statistical Abstract of the United States*, *2009*, Table 158.)

12.42 Xr12-42 Companies that sell groceries over the Internet are called *e-grocers*. Customers enter their orders, pay by credit card, and receive delivery by truck. A potential e-grocer analyzed the market and determined that the average order would have to exceed $85 if the e-grocer were to be profitable. To determine whether an e-grocery would be profitable in one large city, she offered the service and recorded the size of the order for a random sample of customers.

a. Can we infer from these data that an e-grocery will be profitable in this city?
b. Prepare a presentation to the investors who wish to put money into this company. (See Section 3.3 for guidelines in making presentations.)

12.43 Xr12-43 During the last decade, many institutions dedicated to improving the quality of products and services in the United States have been formed. Many of these groups annually give awards to companies that produce high-quality goods and services. An investor believes that publicly traded companies that win awards are likely to outperform companies that do not win such awards. To help determine his return on investment in such companies, he took a random sample of 83 firms that won quality awards the previous year and computed the annual return he would have received had he invested. The investor would like an estimate of the returns he can expect. A 95% confidence level is deemed appropriate.

12.44 Xr12-44 In 2010, most Canadian cities were experiencing a housing boom. As a consequence, home buyers were required to borrow more on their mortgages. To determine the extent of this problem, a survey of Canadian households was undertaken wherein household heads were asked to report their total debt. Assuming that there are 7 million households in Canada, estimate with 95% confidence the total household debt.

12.45 Xr12-45 Refer to Exercise 12.44. In addition to household debt, the survey asked each household to report the debt-to-income ratio. Estimate with 90% confidence the mean debt-to-income ratio.

General Social Survey Exercises

Warning for Excel users: There are blanks representing missing data that must be removed.

12.46 GSS2008* In 2008, respondents were asked to report the number of years of education (EDUC). Do the data provide enough evidence at the 5% significance level to infer that the average American adult completed more than 12 years of education?

12.47 GSS2008* Estimate with 95% confidence the mean numbers of earners (EARNRS) in the household in 2008.

12.48 GSS2008* Can we infer at the 5% significance level that the mean number of hours worked (HRS) among those working full- or part-time is greater than 40?

12.49 GSS2006* Estimate with 95% confidence the mean number of years in current job (YEARSJOB) in 2006.

12.50 GSS2006* Estimate with 90% confidence the mean number of hours American adults spend watching television per day (TVHOURS).

American National Election Survey Exercises

Warning for Excel users: There are blanks representing missing data that must be removed.

12.51 ANES2008* Can we infer with $\alpha = .05$ that the average American has completed more than 12 years of education (EDUC)?

12.52 ANES2008* Estimate with 95% confidence the mean number of days in a typical week (DAYS8) spent by American adults watching the national news on television, not including sports.

12.53 ANES2008* Estimate with 99% confidence the mean amount of time in a typical day spent by American adults watching or reading news on the Internet (TIME1).

12.2 Inference about a Population Variance

In Section 12.1, where we presented the inferential methods about a population mean, we were interested in acquiring information about the central location of the population. As a result, we tested and estimated the population mean. If we are interested instead in drawing inferences about a population's variability, the parameter we need to investigate is the population variance σ^2. Inference about the variance can be used to make decisions in a variety of problems. In an example illustrating the use of the normal distribution in Section 8.2, we showed why variance is a measure of risk. In Section 7.3, we described an important application in finance wherein stock diversification was shown to reduce the variance of a portfolio and, in so doing, reduce the risk associated with that portfolio. In both sections, we assumed that the population variances were known. In this section, we take a more realistic approach and acknowledge that we need to use statistical techniques to draw inferences about a population variance.

Another application of the use of variance comes from operations management. Quality technicians attempt to ensure that their company's products consistently meet specifications. One way of judging the consistency of a production process is to compute the variance of a product's size, weight, or volume; that is, if the variation in size, weight, or volume is large, it is likely that an unsatisfactorily large number of products will lie outside the specifications for that product. We will return to this subject later in this book. In Section 14.6, we discuss how operations managers search for and reduce the variation in production processes.

The task of deriving the test statistic and the interval estimator provides us with another opportunity to show how statistical techniques in general are developed. We begin by identifying the best estimator. That estimator has a sampling distribution, from which we produce the test statistic and the interval estimator.

Statistic and Sampling Distribution

The estimator of σ^2 is the sample variance introduced in Section 4.2. The statistic s^2 has the desirable characteristics presented in Section 10.1; that is, s^2 is an unbiased, consistent estimator of σ^2.

Statisticians have shown that the sum of squared deviations from the mean $\sum(x_i - \bar{x})^2$ [which is equal to $(n-1)s^2$] divided by the population variance is chi-squared distributed with $\nu = n - 1$ degrees of freedom provided that the sampled population is normal. The statistic

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

is called the **chi-squared statistic** (χ^2-statistic). The chi-squared distribution was introduced in Section 8.4.

Testing and Estimating a Population Variance

As we discussed in Section 11.4, the formula that describes the sampling distribution is the formula of the test statistic.

Test Statistic for σ^2

The test statistic used to test hypotheses about σ^2 is

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

which is chi-squared distributed with $\nu = n - 1$ degrees of freedom when the population random variable is normally distributed with variance equal to σ^2.

Using the notation introduced in Section 8.4, we can make the following probability statement:

$$P(\chi^2_{1-\alpha/2} < \chi^2 < \chi^2_{\alpha/2}) = 1 - \alpha$$

Substituting

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

and with some algebraic manipulation, we derive the confidence interval estimator of a population variance.

Confidence Interval Estimator of σ^2

$$\text{Lower confidence limit (LCL)} = \frac{(n-1)s^2}{\chi^2_{\alpha/2}}$$

$$\text{Upper confidence limit (UCL)} = \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

APPLICATIONS in OPERATIONS MANAGEMENT

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Quality

A critical aspect of production is quality. The quality of a final product is a function of the quality of the product's components. If the components don't fit, the product will not function as planned and likely cease functioning before its customers expect it to. For example, if a car door is not made to its specifications, it will not fit. As a result, the door will leak both water and air.

Operations managers attempt to maintain and improve the quality of products by ensuring that all components are made so that there is as little variation as possible. As you have already seen, statisticians measure variation by computing the variance.

Incidentally, an entire chapter (Chapter 21) is devoted to the topic of quality.

EXAMPLE 12.3 Consistency of a Container-Filling Machine, Part 1

DATA
Xm12-03

Container-filling machines are used to package a variety of liquids, including milk, soft drinks, and paint. Ideally, the amount of liquid should vary only slightly because large variations will cause some containers to be underfilled (cheating the customer) and some to be overfilled (resulting in costly waste). The president of a company that developed a new type of machine boasts that this machine can fill 1-liter (1,000 cubic centimeters) containers so consistently that the variance of the fills will be less than 1 cubic centimeter^2. To examine the veracity of the claim, a random sample of 25 l-liter fills was taken and the results (cubic centimeters) recorded. These data are listed here. Do these data allow the president to make this claim at the 5% significance level?

Fills

999.6	1000.7	999.3	1000.1	999.5
1000.5	999.7	999.6	999.1	997.8
1001.3	1000.7	999.4	1000.0	998.3
999.5	1000.1	998.3	999.2	999.2
1000.4	1000.1	1000.1	999.6	999.9

SOLUTION

IDENTIFY

The problem objective is to describe the population of l-liter fills from this machine. The data are interval, and we're interested in the variability of the fills. It follows that the parameter of interest is the population variance. Because we want to determine whether there is enough evidence to support the claim, the alternative hypothesis is

$$H_1: \sigma^2 < 1$$

The null hypothesis is

$$H_0: \sigma^2 = 1$$

and the test statistic we will use is

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

COMPUTE

MANUALLY

Using a calculator, we find

$$\sum x_i = 24{,}992.0 \quad \text{and} \quad \sum x_i^2 = 24{,}984{,}017.76$$

Thus,

$$s^2 = \frac{\sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}}{n-1} = \frac{24{,}984{,}017.76 - \frac{(24{,}992.0)^2}{25}}{25-1} = .6333$$

The value of the test statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{(25-1)(.6333)}{1} = 15.20$$

The rejection region is

$$\chi^2 < \chi^2_{1-\alpha,n-1} = \chi^2_{1-.05,25-1} = \chi^2_{.95,24} = 13.85$$

Because 15.20 is not less than 13.85, we cannot reject the null hypothesis in favor of the alternative.

EXCEL

	A	B	C	D
1	**Chi Squared Test: Variance**			
2				
3				*Fills*
4	Sample Variance			0.6333
5	Hypothesized Variance			1
6	df			24
7	chi-squared Stat			15.20
8	P (CHI<=chi) one-tail			0.0852
9	chi-squared Critical one tail	Left-tail		13.85
10		Right-tail		36.42
11	P (CHI<=chi) two-tail			0.1705
12	chi-squared Critical two tail	Left-tail		12.40
13		Right-tail		39.36

The value of the test statistic is 15.20. P(CHI<=chi) one-tail is the probability $P(\chi^2 < 15.20)$, which is equal to .0852. Because this is a one-tail test, the p-value is .0852.

INSTRUCTIONS

1. Type or import the data into one column*. (Open Xm12-03.)
2. Click **Add-Ins, Data Analysis Plus**, and **Chi-squared Test: Variance**.
3. Specify the **Input Range** (A1:A26), type the **Hypothesized Variance** (1) and the value of α(.05).

*If the column contains a blank (representing missing data) the row will have to be deleted.

MINITAB

Test and CI for One Standard Deviation: Fills

Null hypothesis	sigma = 1
Alternative hypothesis	sigma = < 1

Statistics

Variable	N	StDev	Variance
Fills	25	0.796	0.633

Tests

Variable	Method	Chi-Square	DF	P-Value
Fills	Standard	15.20	24.00	0.085

INSTRUCTIONS

Some of the output has been deleted.

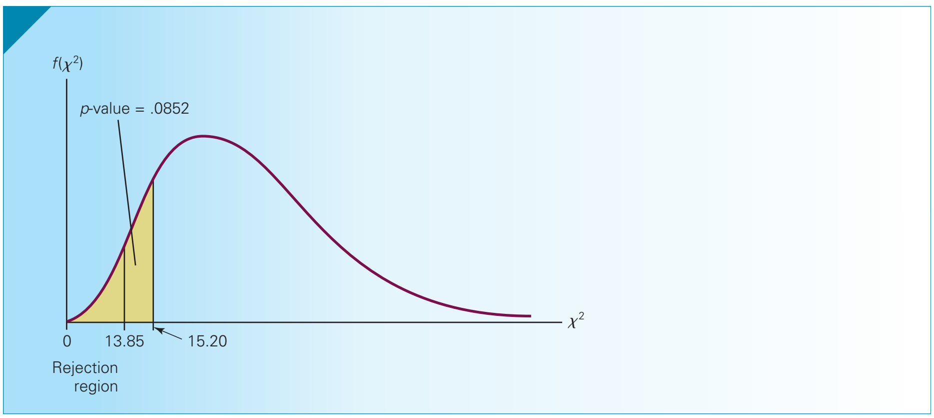
Besides computing the Chi-Squared statistic and *p*-value, and because we're conducting a one-tail test, Minitab calculates a one-sided confidence interval estimate. (See page 379 for a discussion of one-sided confidence interval estimators.)

1. Type or import the data into one column. (Open Xm12-03.)
2. Click **Stat, Basic Statistics**, and **1 Variance . . . .**
3. Type or use the **Select** button to specify the name of the variable or the column in the **Samples in columns** box (Fills), check **Perform hypothesis test**, and type the value of σ in the **Hypothesized standard deviation** box (1).
4. Click **Options . . .** and select one of **less than, not equal**, or **greater than** in the **Alternative** box (less than).

INTERPRET

There is not enough evidence to infer that the claim is true. As we discussed before, the result does not say that the variance is equal to 1; it merely states that we are unable to show that the variance is less than 1. Figure 12.4 depicts the sampling distribution of the test statistic.

FIGURE **12.4** Sampling Distribution for Example 12.3



EXAMPLE 12.4 Consistency of a Container-Filling Machine, Part 2

Estimate with 99% confidence the variance of fills in Example 12.3.

SOLUTION

MANUALLY

In the solution to Example 12.3, we found $(n - 1)s^2$ to be 15.20. From Table 5 in Appendix B, we find

$$\chi^2_{\alpha/2,n-1} = \chi^2_{.005,24} = 45.6$$

$$\chi^2_{1-\alpha/2,n-1} = \chi^2_{.995,24} = 9.89$$

Thus,

$$\text{LCL} = \frac{(n-1)s^2}{\chi^2_{\alpha/2}} = \frac{15.20}{45.6} = .3333$$

$$\text{UCL} = \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}} = \frac{15.20}{9.89} = 1.537$$

We estimate that the variance of fills is a number that lies between .3333 and 1.537.

EXCEL

	A	B
1	**Chi Squared Estimate: Variance**	
2		
3		*Fills*
4	Sample Variance	0.6333
5	df	24
6	LCL	0.3336
7	UCL	1.5375

INSTRUCTIONS

1. Type or import the data into one column*. (Open Xm12-03.)
2. Click **Add-Ins, Data Analysis Plus**, and **Chi-squared Estimate: Variance**.
3. Specify the **Input Range** (A1:A26) and α(.01).

MINITAB

Test and CI for One Standard Deviation: Fills

Statistics

Variable	N	StDev	Variance
Fills	25	0.796	0.633

99% Confidence Intervals

Variable	Method	CI for StDev	CI for Variance
Fills	Standard	(0.578, 1.240)	(0.334, 1.537)

*If the column contains a blank (representing missing data) the row will have to be deleted.

INSTRUCTIONS

Some of the output has been deleted.

1. Type or import the data into one column. (Open Xm12-03.)
2. Click **Stat, Basic Statistics**, and **1 Variance . . . .**
3. Type or use the **Select** button to specify the name of the variable or the column in the **Samples in columns** box (Fills).
4. Click **Options . . .** , type the **Confidence level,** and select **not equal** in the **Alternative** box.

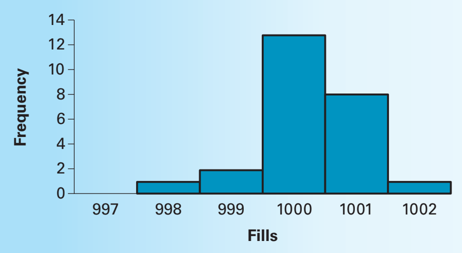
INTERPRET

In Example 12.3, we saw that there was not sufficient evidence to infer that the population variance is less than 1. Here we see that σ^2 is estimated to lie between .3336 and 1.5375. Part of this interval is above 1, which tells us that the variance may be larger than 1, confirming the conclusion we reached in Example 12.3. We may be able to use the estimate to predict the percentage of overfilled and underfilled bottles. This may allow us to choose among competing machines.

Checking the Required Condition

Like the *t*-test and estimator of μ introduced in Section 12.1, the chi-squared test and estimator of σ^2 theoretically require that the sample population be normal. In practice, however, the technique is valid so long as the population is not extremely nonnormal. We can gauge the extent of nonnormality by drawing the histogram. Figure 12.5 depicts Excel's version of this histogram. As you can see, the fills appear to be somewhat asymmetric. However the variable does not appear to be very nonnormal. We conclude that the normality requirement is not seriously violated.

FIGURE **12.5** Histogram for Examples 12.3 and 12.4

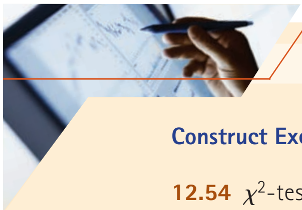


Here is how we recognize when to use the techniques introduced in this section.

Factors That Identify the Chi-Squared Test and Estimator of σ^2

1. **Problem objective**: Describe a population
2. **Data type**: Interval
3. **Type of descriptive measurement**: Variability

EXERCISES



DO-IT-YOURSELF **EXCEL**

Construct Excel spreadsheets that perform the following techniques

12.54 χ^2-test of σ^2. Inputs: sample variance, sample size, and hypothesized variance. Outputs: Test statistic, critical values, and one- and two-tail p-values. Tools: **CHIINV, CHITEST.**

12.55 χ^2-estimate of σ^2. Inputs: sample variance, sample size, and confidence level. Outputs: Upper and lower confidence limits. Tools: **CHIINV.**

Developing an Understanding of Statistical Concepts

The following exercises are "what-if" analyses designed to determine what happens to the test statistics and interval estimates when elements of the statistical inference change. These problems can be solved manually or using the Do-It-Yourself Excel spreadsheets you created.

12.56 a. A random sample of 100 observations was drawn from a normal population. The sample variance was calculated to be $s^2 = 220$. Test with $\alpha = .05$ to determine whether we can infer that the population variance differs from 300.
b. Repeat part (a) changing the sample size to 50.
c. What is the effect of decreasing the sample size?

12.57 a. The sample variance of a random sample of 50 observations from a normal population was found to be $s^2 = 80$. Can we infer at the 1% significance level that σ^2 is less than 100?
b. Repeat part (a) increasing the sample size to 100.
c. What is the effect of increasing the sample size?

12.58 a. Estimate σ^2 with 90% confidence given that $n = 15$ and $s^2 = 12$.
b. Repeat part (a) with $n = 30$.
c. What is the effect of increasing the sample size?

Applications

12.59 Xr12-59 The weights of a random sample of cereal boxes that are supposed to weigh 1 pound are listed here. Estimate the variance of the entire population of cereal box weights with 90% confidence.

1.05 1.03 .98 1.00 .99 .97 1.01 .96

12.60 Xr12-60 After many years of teaching, a statistics professor computed the variance of the marks on her final exam and found it to be $\sigma^2 = 250$. She recently made changes to the way in which the final exam is marked and wondered whether this would result in a reduction in the variance. A random sample of this year's final exam marks are listed here. Can the professor infer at the 10% significance level that the variance has decreased?

57 92 99 73 62 64 75 70 88 60

12.61 Xr12-61 With gasoline prices increasing, drivers are more concerned with their cars' gasoline consumption. For the past 5 years, a driver has tracked the gas mileage of his car and found that the variance from fill-up to fill-up was $\sigma^2 = 23$ mpg^2. Now that his car is 5 years old, he would like to know whether the variability of gas mileage has changed. He recorded the gas mileage from his last eight fill-ups; these are listed here. Conduct a test at a 10% significance level to infer whether the variability has changed.

28 25 29 25 32 36 27 24

12.62 Xr12-62 During annual checkups, physicians routinely send their patients to medical laboratories to have various tests performed. One such test determines the cholesterol level in patients' blood. However, not all tests are conducted in the same way. To acquire more information, a man was sent to 10 laboratories and had his cholesterol level measured in each. The results are listed here. Estimate with 95% confidence the variance of these measurements.

188 193 186 184 190 195 187 190 192 196

The following exercises require the use of a computer and software. The answers may be calculated manually. See Appendix A for the sample statistics.

12.63 Xr12-63 One important factor in inventory control is the variance of the daily demand for the product.

A management scientist has developed the optimal order quantity and reorder point, assuming that the variance is equal to 250. Recently, the company has experienced some inventory problems, which induced the operations manager to doubt the assumption. To examine the problem, the manager took a sample of 25 days and recorded the demand.

a. Do these data provide sufficient evidence at the 5% significance level to infer that the management scientist's assumption about the variance is wrong?
b. What is the required condition for the statistical procedure in part (a)?
c. Does it appear that the required condition is not satisfied?

12.64 Xr12-64 Some traffic experts believe that the major cause of highway collisions is the differing speeds of cars. In other words, when some cars are driven slowly while others are driven at speeds well in excess of the speed limit, cars tend to congregate in bunches, increasing the probability of accidents. Thus, the greater the variation in speeds, the greater will be the number of collisions that occur. Suppose that one expert believes that when the variance exceeds 18 mph^2, the number of accidents will be unacceptably high. A random sample of the speeds of 245 cars on a highway with one of the highest accident rates in the country is taken. Can we conclude at the 10% significance level that the variance in speeds exceeds 18 mph^2?

12.65 Xr12-65 The job placement service at a university observed the not unexpected result of the variance in marks and work experience of the university's graduates: Some graduates received numerous offers, whereas others received far fewer. To learn more about the problem, a survey of 90 recent graduates was conducted wherein each was asked how many job offers he or she received. Estimate with 90% confidence the variance in the number of job offers made to the university's graduates.

12.66 Xr12-66 One problem facing the manager of maintenance departments is when to change the bulbs in streetlamps. If bulbs are changed only when they burn out, it is quite costly to send crews out to change only one bulb at a time. This method also requires someone to report the problem and, in the meantime, the light is off. If each bulb lasts approximately the same amount of time, they can all be replaced periodically, producing significant cost savings in maintenance. Suppose that a financial analysis of the lights at the new Yankee Stadium has concluded that it will pay to replace all of the lightbulbs at the same time if the variance of the lives of the bulbs is less than 200 $hours^2$. The lengths of life of the last 100 bulbs were recorded. What conclusion can be drawn from these data? Use a 5% significance level.

12.67 Xr12-67 Home blood-pressure monitors have been on the market for several years. This device allows people with high blood pressure to measure their own and determine whether additional medication is necessary. Concern has been expressed about inaccurate readings. To judge the severity of the problem, a laboratory technician measured his own blood pressure 25 times using the leading brand of monitors. Estimate the population variance with 95% confidence.

12.3 Inference about a Population Proportion

In this section, we continue to address the problem of describing a population. However, we shift our attention to populations of nominal data, which means that the population consists of nominal or categorical values. For example, in a brand-preference survey in which the statistics practitioner asks consumers of a particular product which brand they purchase, the values of the random variable are the brands. If there are five brands, the values could be represented by their names, by letters (A, B, C, D, and E), or by numbers (1, 2, 3, 4, and 5). When numbers are used, it should be understood that the numbers only represent the name of the brand, are completely arbitrarily assigned, and cannot be treated as real numbers—that is, we cannot calculate means and variances.

Parameter

Recall the discussion of types of data in Chapter 2. When the data are nominal, all that we are permitted to do to describe the population or sample is count the number of occurrences of each value. From the counts, we calculate proportions. Thus, the parameter of interest in describing a population of nominal data is the population proportion p. In Section 7.4, this parameter was used to calculate probabilities based on the binomial experiment. One of the characteristics of the binomial experiment is that there are only two possible outcomes per trial. Most practical applications of inference about p involve more than two outcomes. However, in many cases we're interested in only one outcome, which we label a "success." All other outcomes are labeled as "failures." For example, in brand-preference surveys we are interested in our company's brand. In political surveys, we wish to estimate or test the proportion of voters who will vote for one particular candidate—likely the one who has paid for the survey.

Statistic and Sampling Distribution

The logical statistic used to estimate and test the population proportion is the sample proportion defined as

$$\hat{p} = \frac{x}{n}$$

where x is the number of successes in the sample and n is the sample size. In Section 9.2, we presented the approximate sampling distribution of $\hat{P}$. (The actual distribution is based on the binomial distribution, which does not lend itself to statistical inference.) The sampling distribution of $\hat{P}$ is approximately normal with mean p and standard deviation $\sqrt{p(1 - p)/n}$ [provided that np and $n(1 - p)$ are greater than 5]. We express this sampling distribution as

$$z = \frac{\hat{P} - p}{\sqrt{p(1 - p)/n}}$$

Testing and Estimating a Proportion

As you have already seen, the formula that summarizes the sampling distribution also represents the test statistic.

Test Statistic for *p*

$$z = \frac{\hat{P} - p}{\sqrt{p(1 - p)/n}}$$

which is approximately normal for np and $n(1 - p)$ greater than 5.

Using the same algebra employed in Sections 10.2 and 12.1, we attempt to derive the confidence interval estimator of p from the sampling distribution. The result is

$$\hat{p} \pm z_{\alpha/2}\sqrt{p(1 - p)/n}$$

This formula, although technically correct, is useless. To understand why, examine the standard error of the sampling distribution $\sqrt{p(1 - p)/n}$. To produce the interval estimate, we must compute the standard error, which requires us to know the value of p, the parameter we wish to estimate. This is the first of several statistical techniques where we face the same problem: how to determine the value of the standard error. In this application, the problem is easily and logically solved: Simply estimate the value of p with $\hat{p}$.

Thus, we estimate the standard error with $\sqrt{\hat{p}(1 - \hat{p})/n}$.

Confidence Interval Estimator of p

$$\hat{p} \pm z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}$$

which is valid provided that $n\hat{p}$ and $n(1-\hat{p})$ are greater than 5.

EXAMPLE 12.5

DATA Xm12-05*

Election Day Exit Poll

When an election for political office takes place, the television networks cancel regular programming and instead provide election coverage. When the ballots are counted, the results are reported. However, for important offices such as president or senator in large states, the networks actively compete to see which will be the first to predict a winner. This is done through exit polls,* wherein a random sample of voters who exit the polling booth is asked for whom they voted. From the data, the sample proportion of voters supporting the candidates is computed. A statistical technique is applied to determine whether there is enough evidence to infer that the leading candidate will garner enough votes to win. Suppose that in the exit poll from the state of Florida during the 2000 year elections, the pollsters recorded only the votes of the two candidates who had any chance of winning, Democrat Albert Gore (code = 1) and Republican George W. Bush (code = 2). The polls close at 8:00 P.M. Can the networks conclude from these data that the Republican candidate will win the state? Should the network announce at 8:01 P.M. that the Republican candidate will win?

SOLUTION

IDENTIFY

The problem objective is to describe the population of votes in the state. The data are nominal because the values are "Democrat" (code = 1) and "Republican" (code = 2). Thus the parameter to be tested is the proportion of votes in the entire state that are for the Republican candidate. Because we want to determine whether the network can declare the Republican to be the winner at 8:01 P.M., the alternative hypothesis is

$$H_1:\ p > .5$$

which makes the null hypothesis

$$H_0:\ p = .5$$

The test statistic is

$$z = \frac{\hat{p} - p}{\sqrt{p(1-p)/n}}$$

*Warren Mitofsky is generally credited for creating the election day exit poll in 1967 when he worked for CBS News. Mitofsky claimed to have correctly predicted 2,500 elections and only six wrong. Exit polls are considered so accurate that when the exit poll and the actual election result differ, some newspaper and television reporters claim that the election result is wrong! In the 2004 presidential election, exit polls showed John Kerry leading. However, when the ballots were counted, George Bush won the state of Ohio. Conspiracy theorists now believe that the Ohio election was stolen by the Republicans using the exit poll as their "proof." However, Mitofsky's own analysis found that the exit poll was improperly conducted, resulting in many Republican voters refusing to participate in the poll. Blame was placed on poorly trained interviewers (*Source: Amstat News*, December 2006).

COMPUTE

MANUALLY

It appears that this is a "standard" problem that requires a 5% significance level. Thus, the rejection region is

$$z > z_{\alpha} = z_{.05} = 1.645$$

From the file, we count the number of "successes," which is the number of votes cast for the Republican, and find $x = 407$. The sample size is 765. Hence, the sample proportion is

$$\hat{p} = \frac{x}{n} = \frac{407}{765} = .532$$

The value of the test statistic is

$$z = \frac{\hat{p} - p}{\sqrt{p(1-p)/n}} = \frac{.532 - .5}{\sqrt{.5(1-.5)/765}} = 1.77$$

Because the test statistic is (approximately) normally distributed, we can determine the p-value. It is

$$p\text{-value} = P(z > 1.77) = 1 - p(Z < 1.77) = 1 - .9616 = .0384$$

There is enough evidence at the 5% significance level that the Republican candidate has won.

EXCEL

	A	B	C	D
1	**z-Test: Proportion**			
2				
3				*Votes*
4	Sample Proportion			0.532
5	Observations			765
6	Hypothesized Proportion			0.5
7	z Stat			1.7716
8	P(Z<=z) one-tail			0.0382
9	z Critical one-tail			1.6449
10	P(Z<=z) two-tail			0.0764
11	z Critical two-tail			1.9600

INSTRUCTIONS

1. Type or import the data into one column*. (Open Xm12-05.)
2. Click **Add-Ins, Data Analysis Plus**, and **Z-Test: Proportion**.
3. Specify the **Input Range** (A1:A766), type the **Code for Success** (2), the **Hypothesized Proportion** (.5), and a value of α (.05).

*If the column contains a blank (representing missing data) the row will have to be deleted.

MINITAB

Test and CI for One Proportion: Votes

Test of p = 0.5 vs p > 0.5

Event = 2

Variable	X	N	Sample p	95% Lower Bound	Z-Value	P-Value
Votes	407	765	0.532026	0.502352	1.77	0.038

Using the normal approximation

As was the case with the test of a variance, Minitab calculates a one-sided confidence interval estimate when we're conducting a one-tail test.

INSTRUCTIONS

The data must represent successes and failures. The codes can be numbers or text. There can be only two kinds of entries: one representing success and the other representing failure. If numbers are used, Minitab will interpret the larger one as a success.

1. Type or import the data into one column. (Open Xm12-05.)
2. Click **Stat, Basic Statistics**, and **1 Proportion . . . .**
3. Use the **Select** button or type the name of the variable or its column in the **Samples in columns** box (Votes) and check **Perform hypothesis test** and type the **Hypothesized proportion** (.5).
4. Click **Options . . .** and specify the **Alternative** hypothesis (greater than). To use the normal approximation of the binomial, click **Use test and interval based on normal approximation**.

INTERPRET

The value of the test statistic is $z = 1.77$ and the one-tail p-value $= .0382$. Using a 5% significance level, we reject the null hypothesis and conclude that there is enough evidence to infer that George Bush won the presidential election in the state of Florida.

One of the key issues to consider here is the cost of Type I and Type II errors. A Type I error occurs if we conclude that the Republican will win when in fact he has lost. Such an error would mean that a network would announce at 8:01 P.M. that the Republican has won and then later in the evening would have to admit to a mistake. If a particular network were the only one that made this error, it would cast doubt on their integrity and possibly affect the number of viewers.

This is exactly what happened on the evening of the U.S. presidential elections in November 2000. Shortly after the polls closed at 8:00 P.M., all the networks declared that the Democratic candidate Albert Gore would win the state of Florida. A couple of hours later, the networks admitted that a mistake had been made and that Republican candidate George W. Bush had won. Several hours later, they again admitted a mistake and finally declared the race too close to call. Fortunately for each network, all the networks made the same mistake. However, if one network had not done this, it would have developed a better track record, which could have been used in future advertisements for news shows and would likely draw more viewers.

Missing Data

In real statistical applications, we occasionally find that the data set is incomplete. In some instances, the statistics practitioner may have failed to properly record some observations or some data may have been lost. In other cases, respondents may refuse to answer. For example, in political surveys where the statistics practitioner asks voters for whom they intend to vote in the next election, some people will answer that they haven't decided or that their vote is confidential and refuse to answer. In surveys where respondents are asked to report their income, people often refuse to divulge this information. This is a troublesome issue for statistics practitioners. We can't force people to answer our questions. However, if the number of nonresponses is high, the results of our analysis may be invalid because the sample is no longer truly random. To understand why, suppose that people who are in the top quarter of household incomes regularly refuse to answer questions about their incomes. The resulting estimate of the population household income mean will be lower than the actual value.

The issue can be complicated. There are several ways to compensate for nonresponses. The simplest method is eliminating them. To illustrate, suppose that in a political survey respondents are asked for whom they intend to vote in a two-candidate race. Surveyors record the results as 1 = Candidate A, 2 = Candidate B, 3 = "Don't know," and 4 = "Refuse to say." If we wish to infer something about the proportion of decided voters who will vote for Candidate A, we can simply omit codes 3 and 4. If we're doing the work manually, we will count the number of voters who prefer Candidate A and the number who prefer Candidate B. The sum of these two numbers is the total sample size.

In the language of statistical software, nonresponses that we wish to eliminate are collectively called *missing data*. Software packages deal with missing data in different ways. Keller's website Appendix Excel and Minitab Instructions for Missing Data and Recoding Data describes how to address the problem of missing data in Excel and in Minitab as well as how to recode data.

We have deleted the nonresponses in the General Social Surveys and the American National Election Surveys. In Excel, the nonresponses appear as blanks; in Minitab, they appear as asterisks.

Estimating the Total Number of Successes in a Large Finite Population

As was the case with the inference about a mean, the techniques in this section assume infinitely large populations. When the populations are small, it is necessary to include the finite population correction factor. In our definition a population is small when it is less than 20 times the sample size. When the population is large and finite, we can estimate the total number of successes in the population.

To produce the confidence interval estimator of the total, we multiply the lower and upper confidence limits of the interval estimator of the proportion of successes by the population size. The confidence interval estimator of the total number of successes in a large finite population is

$$N\left(\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

We will use this estimator in the chapter-opening example and several of this section's exercises.

Nielsen Ratings: Solution


© Brand X Pictures/Jupiter images

IDENTIFY

The problem objective is to describe the population of television shows watched by viewers across the country. The data are nominal. The combination of problem objective and data type make the parameter to be estimated the proportion of the entire population that watched *Vancouver Olympic Games* (code = 4). The confidence interval estimator of the proportion is

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

COMPUTE

MANUALLY

To solve manually, we count the number of 4s in the file. We find this value to be 1,319. Thus,

$$\hat{p} = \frac{x}{n} = \frac{1{,}319}{5{,}000} = .2638$$

The confidence level is $1 - \alpha = .95$. It follows that $\alpha = .05$, $\alpha/2 = .025$, and $z_{\alpha/2} = z_{.025} = 1.96$.

The 95% confidence interval estimate of p is

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = .2638 \pm 1.96\sqrt{\frac{(.2638)(1-.2638)}{5{,}000}} = .2638 \pm .0122$$

LCL = .2516 UCL = .2760

EXCEL

	A	B
1	**z-Estimate: Proportion**	
2		*Show*
3	Sample Proportion	0.2638
4	Observations	5000
5	LCL	0.2516
6	UCL	0.2760

INSTRUCTIONS

1. Type or import the data into one column*. (Open Xm12-00.)
2. Click **Add-Ins, Data Analysis Plus,** and **Z-Estimate: Proportion.**
3. Specify the **Input Range** (A1:A5001), the **Code for Success** (4), and the value of α (.05).

MINITAB

Minitab requires that the data set contain only two values, the larger of which would be considered a success. In this example there are five values. If there are more than two codes or if the code for success is smaller than that for failure, we must recode.

Test and CI for One Proportion: Show

Event = 4

Variable	X	N	Sample p	95% CI
Programs	1319	5000	0.263800	(0.251585, 0.276015)

Using the normal approximation.

*If the column contains a blank (representing missing data) the row will have to be deleted.

INSTRUCTIONS

Recode data

1. Click **Data, Code,** and **Numeric to Numeric . . . .**
2. In the **Code data from columns** box, type or **Select** the data you wish to recode.
3. In the **Store coded data in columns** box, type the column where the recoded data are to be stored. (We named the column "Recoded Programs.")
4. Specify the **Original values:** you wish to recode and their **New:** values.

Estimate the proportion

1. Click **Stat, Basic Statistics,** and **1 Proportion . . . .**
2. In the **Samples in columns** box, type or **Select** the data (Show).
3. Click **Options . . . .**
4. Specify the **Confidence level:** (.95), select **Alternative: not equal,** and **Use test and interval based on normal distribution.**

INTERPRET

We estimate that between 25.16% and 27.60% of all households with televisions had tuned to *Vancouver Winter Olympics* on Sunday, February 15, 2010 at 9:00 to 9:30. If we multiply these figures by the total number of television households, 115 million, we produce an interval estimate of the number of televisions tuned to *Vancouver Winter Olympics*. Thus,

$$\text{LCL} = .2516 \times 115 \text{ million} = 28.934 \text{ million}$$

and

$$\text{UCL} = .2760 \times 115 \text{ million} = 31.740 \text{ million}$$

Sponsoring companies can then determine the value of any commercials that appeared on the show.

Selecting the Sample Size to Estimate the Proportion

When we introduced the sample size selection method to estimate a mean in Section 10.3, we pointed out that the sample size depends on the confidence level and the bound on the error of estimation that the statistics practitioner is willing to tolerate. When the parameter to be estimated is a proportion, the bound on the error of estimation is

$$B = z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Solving for n, we produce the required sample size as indicated in the box.

Sample Size to Estimate a Proportion

$$n = \left(\frac{z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})}}{B}\right)^2$$

To illustrate the use of this formula, suppose that in a brand-preference survey we want to estimate the proportion of consumers who prefer our company's brand to within .03

with 95% confidence. This means that the bound on the error of estimation is $B = .03$. Because $1 - \alpha = .95$, $\alpha = .05$, $\alpha/2 = .025$, and $z_{\alpha/2} = z_{.025} = 1.96$,

$$n = \left(\frac{1.96\sqrt{\hat{p}(1-\hat{p})}}{.03}\right)^2$$

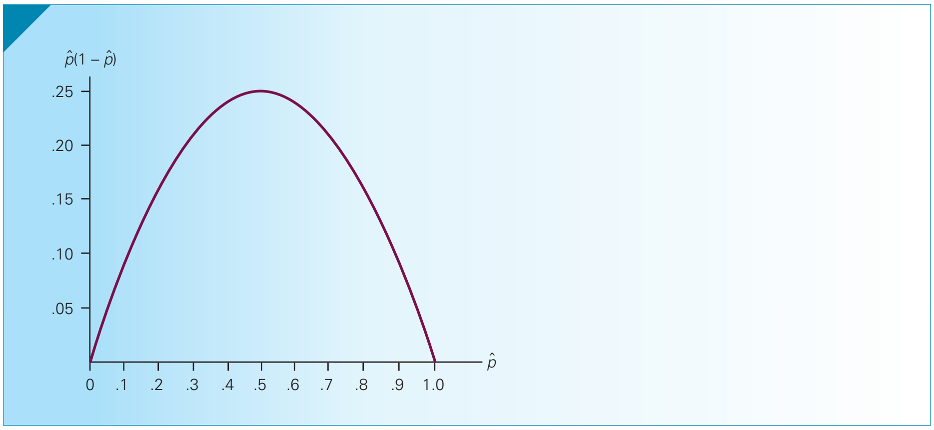
To solve for n, we need to know $\hat{p}$. Unfortunately, this value is unknown, because the sample has not yet been taken. At this point, we can use either of two methods to solve for n.

Method 1 If we have no knowledge of even the approximate value of $\hat{p}$, we let $\hat{p} = .5$. We choose $\hat{p} = .5$ because the product $\hat{p}(1 - \hat{p})$ equals its maximum value at $\hat{p} = .5$. (Figure 12.6 illustrates this point.) This, in turn, results in a conservative value of n; as a result, the confidence interval will be no wider than the interval $\hat{p} \pm .03$. If, when the sample is drawn, $\hat{p}$ does not equal .5, the confidence interval estimate will be better (that is, narrower) than planned. Thus,

$$n = \left(\frac{1.96\sqrt{(.5)(.5)}}{.03}\right)^2 = (32.67)^2 = 1{,}068$$

If it turns out that $\hat{p} = .5$, the interval estimate is $\hat{p} \pm .03$. If not, the interval estimate will be narrower. For instance, if it turns out that $\hat{p} = .2$, then the estimate is $\hat{p} \pm .024$, which is better than we had planned.

FIGURE **12.6** Plot of $\hat{p}$ versus $\hat{p}(1 - \hat{p})$



Method 2 If we have some idea about the value of $\hat{p}$, we can use that quantity to determine n. For example, if we believe that $\hat{p}$ will turn out to be approximately .2, we can solve for n as follows:

$$n = \left(\frac{1.96\sqrt{(.2)(.8)}}{.03}\right)^2 = (26.13)^2 = 683$$

Notice that this produces a smaller value of n (thus reducing sampling costs) than does method 1. If $\hat{p}$ actually lies between .2 and .8, however, the estimate will not be as good as we wanted, because the interval will be wider than desired.

Method 1 is often used to determine the sample size used in public opinion surveys reported by newspapers, magazines, television, and radio. These polls usually estimate proportions to within 3%, with 95% confidence. (The media often state the confidence level as "19 times out of 20.") If you've ever wondered why opinion polls almost always estimate proportions to within 3%, consider the sample size required to estimate a proportion to within 1%:

$$n = \left(\frac{1.96\sqrt{(.5)(.5)}}{.01}\right)^2 = (98)^2 = 9{,}604$$

The sample size 9,604 is 9 times the sample size needed to estimate a proportion to within 3%. Thus, to divide the width of the interval by 3 requires multiplying the sample size by 9. The cost would also increase considerably. For most applications, the increase in accuracy (created by decreasing the width of the confidence interval estimate) does not overcome the increased cost. Confidence interval estimates with 5% or 10% bounds (sample sizes 385 and 97, respectively) are generally considered too wide to be useful. Thus, the 3% bound provides a reasonable compromise between cost and accuracy.

Wilson Estimators (Optional)

When applying the confidence interval estimator of a proportion when success is a relatively rare event, it is possible to find no successes, especially if the sample size is small. To illustrate, suppose that a sample of 100 produced $x = 0$, which means that $\hat{p} = 0$. The 95% confidence interval estimator of the proportion of successes in the population becomes

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0 \pm 1.96\sqrt{\frac{0(1-0)}{100}} = 0 \pm 0$$

This implies that if we find no successes in the sample, then there is no chance of finding a success in the population. Drawing such a conclusion from virtually any sample size is unacceptable. The remedy may be a suggestion made by Edwin Wilson in 1927. The Wilson estimate denoted $\tilde{p}$ (pronounced "p tilde") is computed by adding 2 to the number of successes in the sample and 4 to the sample size. Thus,

$$\tilde{p} = \frac{x+2}{n+4}$$

The standard error of $\tilde{p}$ is

$$\sigma_{\tilde{p}} = \sqrt{\frac{\tilde{p}(1-\tilde{p})}{n+4}}$$

Confidence Interval Estimator of *p* Using the Wilson Estimate

$$\tilde{p} \pm z_{\alpha/2}\sqrt{\frac{\tilde{p}(1-\tilde{p})}{n+4}}$$

Exercises 12.88 – 12.90 require the use of this technique.

We complete this section by reviewing the factors that tell us when to test and estimate a population proportion.

Factors That Identify the z-Test and Interval Estimator of p

1. **Problem objective**: Describe a population
2. **Data type**: Nominal

EXERCISES

DO-IT-YOURSELF **EXCEL**

Construct Excel spreadsheets that perform the following techniques

12.68 z-test of p. Inputs: sample proportion, sample size, and hypothesized proportion. Outputs: Test statistic, critical values, and one- and two-tail p-values. Tools: **NORMINV, NORMSDIST.**

12.69 z-estimate of p. Inputs: sample proportion, sample size, and confidence level. Outputs: Upper and lower confidence limits. Tools: **NORMINV.**

Developing an Understanding of Statistical Concepts

Exercises 12.70 to 12.73 are "what-if" analyses designed to determine what happens to the test statistics and interval estimates when elements of the statistical inference change. These problems can be solved manually or using your Do-It-Yourself Excel spreadsheets.

12.70 a. In a random sample of 500 observations, we found the proportion of successes to be 48%. Estimate with 95% confidence the population proportion of successes.
b. Repeat part (a) with $n = 200$.
c. Repeat part (a) with $n = 1000$.
d. Describe the effect on the confidence interval estimate of increasing the sample size.

12.71 a. The proportion of successes in a random sample of 400 was calculated as 50%. Estimate the population proportion with 95% confidence.
b. Repeat part a with $\hat{p} = 33\%$.
c. Repeat part a with $\hat{p} = 10\%$.
d. Discuss the effect on the width of the confidence interval estimate of reducing the sample proportion.

12.72 a. Calculate the p-value of the test of the following hypotheses given that $\hat{p} = .63$ and $n = 100$:

$$H_0: \ p = .60$$
$$H_1: \ p > .60$$

b. Repeat part (a) with $n = 200$.
c. Repeat part (a) with $n = 400$.
d. Describe the effect on the p-value of increasing the sample size.

12.73 a. A statistics practitioner wants to test the following hypotheses:

$$H_0: \ p = .70$$
$$H_1: \ p > .70$$

A random sample of 100 produced $\hat{p} = .73$. Calculate the p-value of the test.
b. Repeat part (a) with $\hat{p} = .72$.
c. Repeat part (a) with $\hat{p} = .71$.
d. Describe the effect on the z-statistic and its p-value of decreasing the sample proportion.

12.74 Determine the sample size necessary to estimate a population proportion to within .03 with 90% confidence assuming you have no knowledge of the approximate value of the sample proportion.

12.75 Suppose that you used the sample size calculated in Exercise 12.74 and found $\hat{p} = .5$.
a. Estimate the population proportion with 90% confidence.
b. Is this the result you expected? Explain.

12.76 Suppose that you used the sample size calculated in Exercise 12.74 and found $\hat{p} = .75$.
a. Estimate the population proportion with 90% confidence.

b. Is this the result you expected? Explain.
c. If you were hired to conduct this analysis, would the person who hired you be satisfied with the interval estimate you produced? Explain.

12.77 Redo Exercise 12.74 assuming that you know that the sample proportion will be no less than .75.

12.78 Suppose that you used the sample size calculated in Exercise 12.77 and found $\hat{p} = .75$.
a. Estimate the population proportion with 90% confidence.
b. Is this the result you expected? Explain.

12.79 Suppose that you used the sample size calculated in Exercise 12.77 and found $\hat{p} = .92$.
a. Estimate the population proportion with 90% confidence.
b. Is this the result you expected? Explain.
c. If you were hired to conduct this analysis, would the person who hired you be satisfied with the interval estimate you produced? Explain.

12.80 Suppose that you used the sample size calculated in Exercise 12.77 and found $\hat{p} = .5$.
a. Estimate the population proportion with 90% confidence.
b. Is this the result you expected? Explain.
c. If you were hired to conduct this analysis, would the person who hired you be satisfied with the interval estimate you produced? Explain.

Applications

12.81 A statistics practitioner working for major league baseball wants to supply radio and television commentators with interesting statistics. He observed several hundred games and counted the number of times a runner on first base attempted to steal second base. He found 373 such events, 259 of them successful. Estimate with 95% confidence the proportion of all attempted thefts of second base that are successful.

12.82 In some states, the law requires drivers to turn on their headlights when driving in the rain. A highway patrol officer believes that less than one-quarter of all drivers follow this rule. As a test, he randomly samples 200 cars driving in the rain and counts the number whose headlights are turned on. He finds this number to be 41. Does the officer have enough evidence at the 10% significance level to support his belief?

12.83 A dean of a business school wanted to know whether the graduates of her school used a statistical inference technique during their first year of employment after graduation. She surveyed 314 graduates and asked about the use of statistical techniques. After tallying the responses, she found that 204 used statistical inference within one year of graduation. Estimate with 90% confidence the proportion of all business school graduates who use their statistical education within a year of graduation.

12.84 Has the recent drop in airplane passengers resulted in better on-time performance? Before the recent economic downturn, one airline bragged that 92% of its flights were on time. A random sample of 165 flights completed this year reveals that 153 were on time. Can we conclude at the 5% significance level that the airline's on-time performance has improved?

12.85 What type of educational background do CEOs have? In one survey, 344 CEOs of medium and large companies were asked whether they had MBA degrees. There were 97 MBAs. Estimate with 95% confidence the proportion of all CEOs of medium and large companies who have MBAs.

12.86 The GO transportation system of buses and commuter trains operates on the honor system. Train travelers are expected to buy their tickets before boarding the train. Only a small number of people will be checked on the train to see whether they bought a ticket. Suppose that a random sample of 400 train travelers was sampled and 68 of them had failed to buy a ticket. Estimate with 95% confidence the proportion of all train travelers who do not buy a ticket.

12.87 Refer to Exercise 12.86. Assuming that there are 1 million travelers per year and the fare is $3.00, estimate with 95% confidence the amount of revenue lost each year.

The following exercises require the use of the Wilson estimator.

12.88 In Chapter 6, we discussed how an understanding of probability allows one to properly interpret the results of medical screening tests. The use of Bayes's Law requires a set of prior probabilities based on historical records. Suppose that a physician wanted to estimate the probability that a woman under 35 years of age would give birth to a Down syndrome baby. She randomly sampled 200 births and discovered only one such case. Use the Wilson estimator to produce a 95% confidence interval estimate of the proportion of women under 35 who will have a Down syndrome baby.

12.89 Spam is of concern to anyone with an e-mail address. Several companies offer protection by eliminating spam e-mails as soon as they hit an inbox. To examine one such product, a manager randomly sampled his daily e-mails for 50 days after installing spam software. A total of 374 e-mails were received; 3 were spam. Use the Wilson estimator to estimate with 90% confidence the proportion of spam e-mails that get through.

12.90 A management professor was in the process of investigating the relationship between education and managerial level achieved. The source of his data was a survey of 385 CEOs of medium and large companies. He discovered only one CEO who did not have at least one university degree. Estimate (using a Wilson estimator) with 99% confidence the proportion of CEOs of medium and large companies with no university degrees.

Exercises 12.91–12.123 require the use of a computer and software. Use a 5% significance level unless specified otherwise. The answers to Exercises 12.91 to 12.102 may be calculated manually. See Appendix A for the sample statistics.

12.91 Xr12-91* There is a looming crisis in universities and colleges across North America. In most places, enrollments are increasing; this requires more instructors. However, there are not enough PhDs to fill the vacancies now. Moreover, among current professors, a large proportion are nearing retirement age. On top of these problems, some universities allow professors over age 60 to retire early. To help devise a plan to deal with the crisis, a consultant surveyed 521 55- to 64-year-old professors and asked each one whether he or she intended to retire before 65. The responses are 1 = No and 2 = Yes.

a. Estimate with 95% confidence the proportion of professors who plan on early retirement.
b. Write a report for the university president describing your statistical analysis.

12.92 Refer to Exercise 12.91. If the number of professors between the ages of 55 and 64 is 75,000, estimate the total number of such professors who plan to retire early.

12.93 Xr12-93 According to the Internal Revenue Service, in 2009 the top 5% of American income earners earned more than $153,542, and the top 1% earned more than $388,806. The top 1% pay slightly more than 40% of all federal income taxes. To determine whether Americans are aware of these figures, *Investor's Business Daily* randomly sampled American adults and asked, "What share do you think the rich (earning more than $388,806) pay in income taxes?" The categories are (1) 0–10%, (2) 10–20%, (3) 20–30%, (4) 30–40%, and (5) more than 40%. The data are stored using the codes 1 to 5. Estimate with 95% confidence the proportion of Americans who knew that the rich pay more than 40% of all federal income taxes.

12.94 Xr12-94 The results of an annual claimant satisfaction survey of policyholders who have had a claim with State Farm Insurance Company revealed a 90% satisfaction rate for claim service. To check the accuracy of this claim, a random sample of State Farm claimants was asked to rate their satisfaction with the quality of the service (1 = satisfied, 2 = unsatisfied). Can we infer that the satisfaction rate is less than 90%?

12.95 Xr12-95 An increasing number of people are giving gift certificates as Christmas presents. To measure the extent of this practice, a random sample of people was asked (survey conducted December 26–29) whether they had received a gift certificate for Christmas. The responses are recorded as 1 = No and 2 = Yes. Estimate with 95% confidence the proportion of people who received a gift certificate for Christmas.

12.96 Xr12-96* An important decision faces Christmas holiday celebrators: buy a real or artificial tree? A sample of 1,508 male and female respondents age 18 years and older was interviewed. Respondents were asked whether they preferred a real (1) or artificial (2) tree. If 6 million Canadian households buy Christmas trees, estimate with 95% confidence the total number of Canadian households that would prefer artificial Christmas trees. (*Source: Toronto Star*, November 29, 2006.)

12.97 Xr12-97* Because television audiences of newscasts tend to be older (and because older people suffer from a variety of medical ailments), pharmaceutical companies' advertising often appears on national news in the three networks (ABC, CBS, and NBC). The ads concern prescription drugs such as those to treat heartburn. To determine how effective the ads are, a survey was undertaken. Adults 50 and older who regularly watch network newscasts were asked whether they had contacted their physicians to ask about one of the prescription drugs advertised during the newscast. The responses (1 = No and 2 = Yes) were recorded.

a. Estimate with 95% confidence the fraction of adults 50 and older who have contacted their physician to inquire about a prescription drug.
b. Prepare a presentation to the executives of a pharmaceutical company that discusses your analysis.

12.98 Xr12-98 A professor of business statistics recently adopted a new textbook. At the completion of the course, 100 randomly selected students were asked to assess the book. The responses are as follows.

Excellent (1), Good (2), Adequate (3), Poor (4)

The results are stored using the codes in parentheses. Do the data allow us to conclude at the 10% significance level that more than 50% of all business students would rate the book as excellent?

12.99 Refer to Exercise 12.98. Do the data allow us to conclude at the 10% significance level that more than 90% of all business students would rate it as at least adequate?

12.100 Xm12-00* Refer to the chapter-opening example. Estimate with 95% confidence the number of television households that were tuned to the *Extreme Makeover: Home Edition*.

12.101 Xr12-101 According to the American Contract Bridge League (ACBL), bridge hands that contain two four-card suits, one three-card suit and one two-card suit (4-4-3-2) occur with 21.55% probability. Suppose that a bridge-playing statistics professor with much too much time on his hands tracked the number of hands over a one-year period and recorded the following hands with 4-4-3-2 distribution (code 2) and some other distribution (code 1). All hands were shuffled and dealt by the players at a bridge club. Test to determine whether the proportion of 4-4-3-2 hands differ from the theoretical probability. If the answer is yes, propose a reason to explain the result.

12.102 Xr12-102 Chlorofluorocarbons (CFCs) are used in air conditioners. However, CFCs damage the ozone layer, which protects us from the sun's harmful rays. As a result, many jurisdictions have banned the production and use of CFCs. The latest jurisdiction to do so is the province of Ontario, which has banned the use of CFCs in car and truck air conditioners. However, it is not known how many vehicles will be affected by the new legislation. A survey of 650 vehicles was undertaken. Each vehicle was identified as either using CFCs (2) or not (1).

a. If 5 million vehicles are registered in Ontario, estimate with 95% confidence the number of vehicles affected by the new law.
b. Write a report for the premier of the province describing what you have learned about the problem.

General Social Survey Exercises

Warning for Excel users: There are blanks representing missing data that must be removed.

Note: In 2008, there were 230,151,000 American adults (18 years of age and older). (Source: Statistical Abstract of the United States, 2009, Table 7.)

12.103 GSS2008* What is the highest degree you completed (DEGREE)?

0 = Left high school, 1 = High school, 2 = Junior college, 3 = Bachelor's degree, 4 = Graduate degree

Estimate with 95% confidence the number of American adults who did not finish high school.

12.104 GSS2008* "Last week were you working full-time, part-time, going to school, keeping house, or what" (WRKSTAT)? The responses were 1 = Working full-time, 2 = Working part-time, 3 = Temporarily not working, 4 = Unemployed, laid off, 5 = Retired, 6 = School, 7 = Keeping house, 8 = Other.

a. Estimate with 90% confidence the number of American adults who were working full-time.
b. Estimate with 95% confidence the number of American adults who were unemployed or laid off.

12.105 GSS2008* Are you self-employed or do you work for someone else (WRKSLF)? 1 = Self employed, 2 = Someone else. Can we infer that more than 10% of Americans are self-employed?

12.106 GSS2008* Are you employed by the federal, state, or local government or by a private employer (WRKGOVT)? 1 = Government, 2 = Private. Estimate with 90% confidence the number of Americans who work for the government.

Political Questions

PARTYID: Generally speaking, do you think of yourself as a Republican, Democrat, Independent, or what? 0 = Strong Democrat, 1 = Not strong Democrat, 2 = Independent near Democrat, 3 = Independent, 4 = Independent near Republican, 5 = Not strong Republican, 6 = Strong Republican, 7 = Other party.

For the following questions, 0 and 1 represent respondents who identify themselves as Democrats; 2, 3, and 4 represent independents; and 5 and 6 represent Republicans.

12.107 GSS2008* Is there sufficient evidence to infer that in 2008 more Americans saw themselves as Democrats than Republicans?

12.108 GSS2006* Do the data allow us to conclude that in 2006 more Americans identified themselves as Democrats than Republicans?

12.109 GSS2004* Is there enough statistical evidence to conclude that in 2004 there were more Democrats than Republicans?

12.110 GSS2002* Is there sufficient evidence to infer that in 2002 more Americans saw themselves as Democrats rather than as Republicans?

POLVIEW: I'm going to show you a seven-point scale on which the political views that people might hold are arranged from extremely liberal to extremely conservative. Where would you place yourself on this scale? 1 = Extremely liberal, 2 = Liberal, 3 = Slightly Liberal, 4 = Moderate, 5 = Slightly conservative, 6 = Conservative, 7 = Extremely conservative.

For the following questions, responses 1, 2, and 3 represent respondents who identify themselves as liberal, and 5, 6, and 7 represent conservatives.

12.111 GSS2008* Do the data provide enough statistical evidence to conclude that in 2008 more Americans identified themselves as conservatives?

12.112 GSS2006* Do the data provide enough statistical evidence to conclude that in 2006 more Americans identified themselves as conservatives?

12.113 GSS2004* Do the data provide enough statistical evidence to conclude that in 2004 more Americans identified themselves as conservatives?

12.114 GSS2002* Do the data provide enough statistical evidence to conclude that in 2002 more Americans identified themselves as conservatives?

12.115 Write a brief report on what you discovered in Exercises 12.103 to 12.114.

AMERICAN NATIONAL ELECTION SURVEY EXERCISES

Warning for Excel users: There are blanks representing missing data that must be removed.

Note: In 2008, there were 230,151,000 American adults (18 years of age and older). (Source: Statistical Abstract of the United States, 2009, Table 7.)

12.116 ANES2008* PARTY: Do you think of yourself as a Democrat, a Republican, an Independent, or what? 1 = Democrat, 2 = Republican, 3 = Independent, 4 = Other party, 5 = No preference

Is there sufficient evidence to infer that in 2008 more Americans saw themselves as Democrats than as Republicans?

12.117 ANES2004* Repeat Exercise 12.116 for 2004.

Liberal–conservative self-placement (LIBCON: 1 = Extremely liberal; 2 = Liberal; 3 = Slightly liberal; 4 = Moderate, middle of the road; 5 = Slightly conservative; 6 = Conservative; 7 = Extremely conservative. For the following questions, responses 1, 2, and 3 represent respondents who identify themselves as liberal, and 5, 6, and 7 represent conservatives.

12.118 ANES2008* Can we infer that in 2008 more Americans perceived themselves as conservative than as liberal?

12.119 ANES2004* Repeat Exercise 12.118 for 2004.

12.120 ANES2008* Do you currently have any kind of health insurance (HEALTH)? 1 = Yes, 5 = No.

Estimate with 95% confidence the number of American adults who do not have health insurance.

12.121 ANES2008* How often do you vote (OFTEN)? 1 = Always, 2 = Nearly always, 3 = Part of the time, 4 = Seldom. Can we infer from the data that fewer than 50% of American adults always vote?

12.122 ANES2004* In the 2004 presidential election, George W. Bush received 51% of the vote. The American National Election Survey asked some of those surveyed before the election for whom they voted (WHOVOTED). 1 = John Kerry, 3 = George W. Bush, 5 = Ralph Nader, 7 = other. Can we infer that the survey results differ from the actual vote for George W. Bush? If so, suggest possible reasons.

12.123 ANES2008* In the 2008 presidential election, Barack Obama received 53% of the vote. The American National Election Survey asked some of those surveyed before the election for whom they voted (WHOVOTE). 1 = Barack Obama, 3 = John McCain, 7 = other. Can we infer that the survey results differ from the actual vote for Barack Obama? If so, suggest possible reasons.

12.4 (OPTIONAL) APPLICATIONS IN MARKETING: MARKET SEGMENTATION

Mass marketing refers to the mass production and marketing by a company of a single product for the entire market. Mass marketing is especially effective for commodity goods such as gasoline, which are very difficult to differentiate from the competition, except through price and convenience of availability. Generally speaking, however, mass marketing has given way to target marketing, which focuses on satisfying the demands of a particular segment of the entire market. For example, the Coca-Cola Company has moved from the mass marketing of a single beverage to the production of several different beverages. Among the cola products are Coca-Cola Classic, Diet Coke, and Caffeine-Free Diet Coke. Each product is aimed at a different market segment.

Because there is no single way to segment a market, managers must consider several different variables (or characteristics) that could be used to identify segments. Surveys of customers are used to gather data about various aspects of the market, and statistical techniques are applied to define the segments. Market segmentation separates consumers of a product into different groups in such a way that members of each group are similar to each other, and there are differences between groups. Market segmentation grew out of the realization that a single product can seldom satisfy the needs and wants of all consumers. Managers must then formulate a strategy to target these profitable segments, using the four elements of the marketing mix: product, pricing, promotion, and placement.

There are many ways to segment a market. Table 12.1 lists several different segmentation variables and their market segments. For example, car manufacturers can use education levels to segment the market. It is likely that high school graduates would be quite similar to others in this group and that members of this group would differ from university graduates. We would expect those differences to include the types and brands of cars each group would choose to buy. However, it is likely that income level would differentiate more clearly between segments. Statistical techniques can be used to help determine the best way to segment the market. These statistical techniques are more advanced than this textbook. Consequently, we will focus our attention on other statistical applications.

TABLE **12.1** Market Segmentation

SEGMENTATION VARIABLE	SEGMENTS
Geographic	
Countries	Brazil, Canada, China, France, United States
Country regions	Midwest, Northeast, Southwest, Southeast
Demographic	
Age	Under 5, 5–12, 13–19, 20–29, 30–50, older than 50
Education	Some high school, high school graduate, some college, college or university graduate
Income	Under \$20,000, \$20,000–29,999, \$30,000–49,999, more than \$50,000
Marital status	Single, married, divorced, widowed
Social	
Religion	Catholic, Protestant, Jewish, Muslim, Buddhist
Class	Upper class, middle class, working class, lower class
Behavior	
Media usage	TV, Internet, newspaper, magazine
Payment method	Cash, check, Visa, Mastercard

It is important for marketing managers to know the size of the segment because the size (among other parameters) determines its profitability. Not all segments are worth pursuing. In some instances, the size of the segment is too small or the costs of satisfying it may be too high. The size can be determined in several ways. The census provides useful information. For example, we can determine the number of Americans in various age categories or the size of geographic residences. For other segments, we may need to survey members of a general population and use the inferential techniques introduced in the previous section, where we showed how to estimate the total number of successes.

In Section 12.3, we showed how to estimate the total number of successes in a large finite population. The confidence interval estimator is

$$N\left(\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

The following example demonstrates the use of this estimator in market segmentation.

EXAMPLE 12.6

DATA
Xm12-06*

Segmenting the Breakfast Cereal Market

In segmenting the breakfast cereal market, a food manufacturer uses health and diet consciousness as the segmentation variable. Four segments are developed:

1. Concerned about eating healthy foods
2. Concerned primarily about weight
3. Concerned about health because of illness
4. Unconcerned

To distinguish between groups, surveys are conducted. On the basis of a questionnaire, people are categorized as belonging to one of these groups. A recent survey asked a random sample of 1,250 American adults (20 and older) to complete the questionnaire. The categories were recorded using the codes. The most recent census reveals that 194,506,000 Americans are 20 and older. Estimate with 95% confidence the number of American adults who are concerned about eating healthy foods.

SOLUTION

IDENTIFY

The problem objective is to describe the population of American adults. The data are nominal. Consequently, the parameter we wish to estimate is the proportion p of American adults who classify themselves as concerned about eating healthy. The confidence interval estimator we need to employ is

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

from which we will produce the estimate of the size of the market segment.

COMPUTE

MANUALLY

To solve manually, we count the number of 1s in the file. We find this value to be 269. Thus,

$$\hat{p} = \frac{x}{n} = \frac{269}{1{,}250} = .2152$$

The confidence level is $1 - \alpha = .95$. It follows that $\alpha = .05$, $\alpha/2 = .025$, and $z_{\alpha/2} = z_{.025} = 1.96$. The 95% confidence interval estimate of p is

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = .2152 \pm 1.96\sqrt{\frac{(.2152)(1-.2152)}{1{,}250}} = .2152 \pm .0228$$

$$\text{LCL} = .1924 \qquad \text{UCL} = .2380$$

EXCEL

	A	B
1	**z-Estimate: Proportion**	
2		*Group*
3	Sample Proportion	0.2152
4	Observations	1250
5	LCL	0.1924
6	UCL	0.2380

MINITAB

Test and CI for One Proportion:

Sample	X	N	Sample p	95% CI
1	269	1250	0.215200	(0.192418, 0.237982)

Using the normal approximation.

INTERPRET

We estimate that the proportion of American adults who are in group 1 lies between .1924 and .2380. Because there are 194,506,000 adults in the population, we estimate that the number of adults who belong to group 1 falls between

$$\text{LCL} = N\left[\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right] = 194{,}506{,}000(.1924) = 37{,}422{,}954$$

and

$$\text{UCL} = N\left[\hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right] = 194{,}506{,}000(.2380) = 46{,}292{,}428$$

We will return to the subject of market segmentation in other chapters where we demonstrate how statistics can be used to determine whether differences actually exist between segments.

EXERCISES

The following exercises may be solved manually. See Appendix A for the sample statistics.

12.124 Xr12-124 A new credit card company is investigating various market segments to determine whether it is profitable to direct its advertising specifically at each one. One market segment is composed of Hispanic people. The latest census indicates that there are 30,085,000 Hispanic adults (18 and older) in the United States (*Source: Statistical Abstract of the United States, 2009*, Table 8). A survey of 475 Hispanics asked each one how he or she usually pays for product purchases. The responses are

1. Cash
2. Check
3. Visa
4. MasterCard
5. Other credit card

Estimate with 95% confidence the number of Hispanics in the United States who usually pay by credit card.

12.125 Xr12-125* A California university is investigating expanding its evening programs. It wants to target people between 25 and 55 years old who have completed high school but did not complete college or university. To help determine the extent and type of offerings, the university needs to know the size of its target market. A survey of 320 California adults was drawn, and each person was asked to identify his or her highest educational attainment. The responses are

1. Did not complete high school
2. Completed high school only
3. Some college or university
4. College or university graduate

The *Statistical Abstract of the United States, 2009* (Table 16) indicates that 25,179,000 Californians are between ages 25 and 55. Estimate with 95% confidence the number of Californians between 25 and 55 years of age who are in the market segment the university wishes to target.

12.126 Xr12-126* The JC Penney department store chain segments the market for women's apparel by its identification of values. The three segments are

1. Conservative
2. Traditional
3. Contemporary

Questionnaires about personal and family values are used to identify which segment a woman falls into. Suppose that the questionnaire was sent to a random sample of 1,836 women. Each woman was classified using the codes 1, 2, and 3. The latest census reveals that there are 116,878,000 adult women in the United States (*Statistical Abstract of the United States, 2009*, Table 7). Use a 95% confidence level.

a. Estimate the proportion of adult American women who are classified as traditional.
b. Estimate the size of the traditional market segment.

12.127 Xr12-127 Most life-insurance companies are leery about offering policies to people 64 and older. When they do, the premiums must be high enough to overcome the predicted length of life. The president of one life-insurance company was thinking about offering special discounts to Americans 64 and older who held full-time jobs. The plan was based on the belief that full-time workers of this age group are likely to be in good health and would likely live well into their 80s. To help decide what to do, he organized a survey of a random sample of the 38 million American adults age 64 and older (*Statistical Abstract of the United States, 2009*, Table 18). He asked a random sample of 325 of these Americans whether they currently hold a full-time job (1 = No, 2 = Yes).

a. Estimate with 95% confidence the size of this market segment.
b. Write a report to the executives of an insurance company detailing your statistical analysis.

12.128 Xr12-128 An advertising company was awarded the contract to design advertising for Rolls Royce automobiles. An executive in the firm decided to pitch the product not only to the affluent in the United States but also to those who think they are in the top 1% of income earners in the country. A survey was undertaken that, among other questions, asked respondents 25 and older where their annual income ranked. The following responses were given.

1 = Top 1%
2 = Top 5% but not top 1%
3 = Top 10% but not top 5%
4 = Top 25% but not top 10%
5 = Bottom 75%

Estimate with 90% confidence the number of Americans 25 and older who believe they are in the top 1% of income earners. The number of Americans 25 and older is 231 million (*Statistical Abstract of the United States, 2009*, Table 18).

12.129 Xr12-129 Suppose the survey in the previous exercise also asked those who were not in the top 1% whether they believed that within 5 years they would be in the top 1% (1 = will not be in top 1% within 5 years, 2 = will be in top 1% within 5 years). Estimate with 95% confidence the number of Americans who believe that they will be in the top 1% of income earners within 5 years.

Chapter Summary

The inferential methods presented in this chapter address the problem of describing a single population. When the data are interval, the parameters of interest are the population mean μ and the population variance σ^2. The Student t distribution is used to test and estimate the mean when the population standard deviation is unknown. The chi-squared distribution is used to make inferences about a population variance. When the data are nominal, the parameter to be tested and estimated is the population proportion p. The sample proportion follows an approximate normal distribution, which produces the test statistic and the interval estimator. We also discussed how to determine the sample size required to estimate a population proportion. We introduced market segmentation and described how statistical techniques presented in this chapter can be used to estimate the size of a segment.

Important Terms

t-statistic 400
Student t distribution 400
Robust 406
Chi-squared statistic 414

Symbols

Symbol	Pronounced	Represents
ν	nu	Degrees of freedom
χ^2	chi squared	Chi-squared statistic
$\hat{p}$	p hat	Sample proportion
$\tilde{p}$	p tilde	Wilson estimator

Formulas

Test statistic for μ

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

Confidence interval estimator of μ

$$\bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}$$

Test statistic for σ^2

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

Confidence interval estimator of σ^2

$$\text{LCL} = \frac{(n-1)s^2}{\chi^2_{\alpha/2}}$$

$$\text{UCL} = \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

Test statistic for p

$$z = \frac{\hat{p} - p}{\sqrt{p(1-p)/n}}$$

Confidence interval estimator of p

$$\hat{p} \pm z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}$$

Sample size to estimate p

$$n = \left(\frac{z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})}}{B}\right)^2$$

Wilson estimator

$$\tilde{p} = \frac{x+2}{n+4}$$

Confidence interval estimator of p using the Wilson estimator

$$\tilde{p} \pm z_{\alpha/2}\sqrt{\tilde{p}(1-\hat{p})/(n+4)}$$

Confidence interval estimator of the total of a large finite population

$$N\left[\bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}\right]$$

Confidence interval estimator of the total number of successes in a large finite population

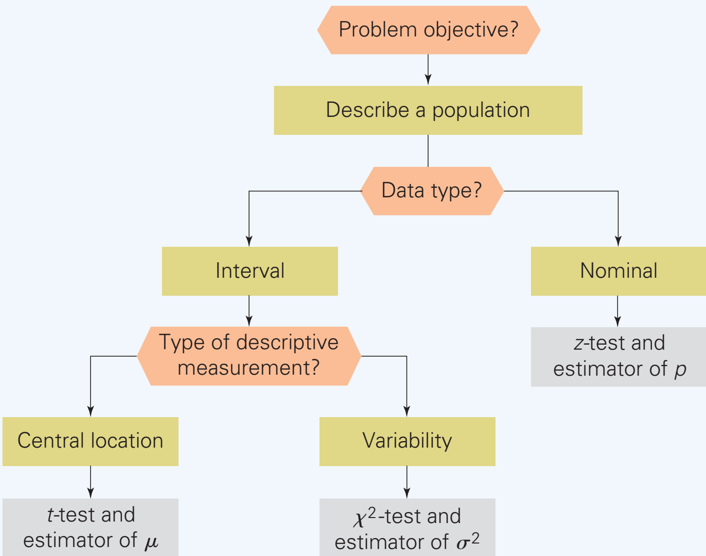
$$N\left[\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p}}{n}}\right]$$

COMPUTER OUTPUT AND INSTRUCTIONS

Technique	Excel	Minitab
t-test of μ	402	403
t-estimator of μ	405	405
Chi-squared test of σ^2	416	417
Chi-squared estimator of σ^2	418	418
z-test of p	424	425
z-estimator of p	438	438

We present the flowchart in Figure 12.7 as part of our ongoing effort to help you identify the appropriate statistical technique. This flowchart shows the techniques introduced in this chapter only. As we add new techniques in the upcoming chapters, we will expand this flowchart until it contains all the statistical inference techniques covered in this book. Use the flowchart to select the correct method in the chapter exercises that follow.

FIGURE **12.7** Flowchart of Techniques: Chapter 12



CHAPTER EXERCISES

The following exercises require the use of a computer and software. Use a 5% significance level unless specified otherwise.

12.130 Xr12-130 One issue that came up in a recent municipal election was the high cost of housing. A candidate seeking to unseat an incumbent claimed that the average family spends more than 30% of its annual income on housing. A housing expert was asked to investigate the claim. A random sample of 125 households was drawn, and each household was asked to report the percentage of household income spent on housing costs.

a. Is there enough evidence to infer that the candidate is correct?

b. Using a confidence level of 95%, estimate the mean percentage of household income spent on housing by all households.

c. What is the required condition for the techniques used in parts (a) and (b)? Use a graphical technique to check whether it is satisfied.

12.131 Xr12-131 The "just-in-time" policy of inventory control (developed by the Japanese) is growing in popularity. For example, General Motors recently spent $2 billion on its Oshawa, Ontario, plant so that it will be less than 1 hour from most suppliers. Suppose that an automobile parts supplier claims to deliver parts to any manufacturer in an average time of less than 1 hour. In an effort to test the claim, a manufacturer recorded the times (in minutes) of 24 deliveries from this supplier. Can we conclude that the supplier's assertion is correct?

12.132 Xr12-132 Robots are being used with increasing frequency on production lines to perform monotonous tasks. To determine whether a robot welder should replace human welders in producing automobiles, an experiment was performed. The time for the robot to complete a series of welds was found to be 38 seconds. A random sample of 20 workers was taken, and the time for each worker to complete the welds was measured. The mean was calculated to be 38 seconds, the same as the robot's time. However, the robot's time did not vary, whereas there was variation among the workers' times. An analysis of the production line revealed that if the variance exceeds 17 seconds^2, there will be problems. Perform an analysis of the data, and determine whether problems using human welders are likely.

12.133 Xr12-133 Opinion Research International surveyed people whose household incomes exceed $50,000 and asked them for their top money-related New Year's resolutions. The responses are

1. Get out of credit card debt
2. Retire before age 65
3. Die broke
4. Make do with current finances
5. Look for higher paying job

Estimate with 90% confidence the proportion of people whose household incomes exceed $50,000 whose top money-related resolution is to get out of credit card debt.

12.134 Xr12-134 Suppose that in a large state university (with numerous campuses) the marks in an introductory statistics course are normally distributed with a mean of 68%. To determine the effect of requiring students to pass a calculus test (which is not currently a prerequisite), a random sample of 50 students who have taken calculus is given a statistics course. The marks out of 100 were recorded.

a. Estimate with 95% confidence the mean statistics mark for all students who have taken calculus.
b. Do these data provide evidence to infer that students with a calculus background would perform better in statistics than students with no calculus?

12.135 Xr12-135 Duplicate bridge is a game in which players compete for master points. When a player receives 300 master points (some of which must be silver, red, and gold), he or she becomes a life master. Because that title comes with a certificate that some people have framed the American Contract Bridge League is interested in knowing the status of nonlife masters. Suppose that a random sample of 80 nonlife masters was asked how many master points they have. The ACBL would like an estimate of the mean number of master points held by all nonlife masters. A confidence level of 90% is considered adequate in this case.

12.136 Xr12-136 A national health-care system was an issue in the 2008 presidential election campaign and is likely to be a subject of debate for many years. The issue arose because of the large number of Americans who have no health insurance. Under the current system, free health care is available to poor people, whereas relatively well-off Americans buy their own health insurance. Those who are considered working poor and who are in the lower-middle-class economic stratum appear to be most unlikely to have adequate medical insurance. To investigate this problem, a statistician surveyed 250 families whose gross incomes last year were between $10,000 and $25,000. Family heads were asked whether they have medical insurance coverage (2 = Has medical insurance, 1 = Doesn't have medical insurance). The statistics practitioner wanted an estimate of the fraction of all families whose incomes are in the range of $10,000 to $25,000 who have medical insurance. Perform the necessary calculations to produce an interval estimate with 90% confidence.

12.137 Xr12-137 The routes of postal deliverers are carefully planned so that each deliverer works between 7 and 7.5 hours per shift. The planned routes assume an average walking speed of 2 miles per hour and no shortcuts across lawns. In an experiment to examine the amount of time deliverers actually spend completing their shifts, a random sample of 75 postal deliverers was secretly timed.

a. Estimate with 99% confidence the mean shift time for all postal deliverers.
b. Check to determine whether the required condition for this statistical inference is satisfied.
c. Is there enough evidence at the 10% significance level to conclude that postal workers are on average spending less than 7 hours per day doing their jobs?

12.138 Xr12-138 As you can easily appreciate, the number of Internet users is rapidly increasing. A recent survey reveals that there are about 50 million Internet users in North America. Suppose that a survey of 200 of these people asked them to report the number of hours they spent on the Internet last week. Estimate with 95% confidence the annual total amount of time spent by all North Americans on the Internet.

12.139 Xr12-139 The manager of a branch of a major bank wants to improve service. She is thinking about giving $1 to any customer who waits in line for a period of time that is considered excessive. (The bank ultimately decided that more than 8 minutes is excessive.) However, to get a better idea about the level of current service, she undertakes a survey of customers. A student is hired to measure the time spent waiting in line by a random sample of 50 customers. Using a stopwatch, the student determined the amount of time between the time the customer joined the line and the time he or she reached the teller. The times were recorded. Construct a 90% confidence interval estimate of the mean waiting time for the bank's customers.

12.140 Xr12-140 In an examination of consumer loyalty in the travel business, 72 first-time visitors to a tourist attraction were asked whether they planned to return. The responses were recorded where 2 = Yes and 1 = No. Estimate with 95% confidence the proportion of all first-time visitors who planned to return to the same destination.

12.141 Xr12-141 Engineers who are in charge of the production of springs used to make car seats are concerned about the variability in the length of the springs. The springs are designed to be 500 mm long. When the springs are too long, they will loosen and fall out. When they are too short, they will not fit into the frames. The springs that are too long and too short must be reworked at considerable additional cost. The engineers have calculated that a standard deviation of 2 mm will result in an acceptable number of springs that must be reworked. A random sample of 100 springs was measured. Can we infer at the 5% significance level that the number of springs requiring reworking is unacceptably large?

12.142 Xr12-142 Refer to Exercise 12.141. Suppose the engineers recoded the data so that springs of the correct length were recorded as 1, springs that were too long were recorded as 2, and springs that were too short were recorded as 3. Can we infer at the 10% significance level that less than 90% of the springs are the correct length?

12.143 Xr12-143 An advertisement for a major home appliance manufacturer claims that its repair personnel are the loneliest in the world because its appliances require the smallest number of service calls. To examine this claim, a researcher drew a random sample of 100 owners of 5-year-old washing machines. The number of service calls made in the 5-year period were recorded. Find the 90% confidence interval estimate of the mean number of service calls for all 5-year-old washing machines.

12.144 Xr12-144 An oil company sends out monthly statements to its customers who purchased gasoline and other items using the company's credit card. Until now, the company has not included a preaddressed envelope for returning payments. The average and the standard deviation of the number of days before payment is received are 9.8 and 3.2, respectively. As an experiment to determine whether enclosing preaddressed envelopes speeds up payment, 150 customers selected at random were sent preaddressed envelopes with their bills. The numbers of days to payment were recorded.

a. Do the data provide sufficient evidence at the 10% level of significance to establish that enclosure of preaddressed envelopes improves the average speed of payments?
b. Can we conclude at the 10% significance level that the variability in payment speeds decreases when a preaddressed envelope is sent?

12.145 A rock promoter is in the process of deciding whether to book a new band for a rock concert. He knows that this band appeals almost exclusively to teenagers. According to the latest census, there are 400,000 teenagers in the area. The promoter decides to do a survey to try to estimate the proportion of teenagers who will attend the concert. How large a sample should be taken in order to estimate the proportion to within .02 with 99% confidence?

12.146 Xr12-146 In Exercise 12.145, suppose that the promoter decided to draw a sample size of 600 (because of financial considerations). Each teenager was asked whether he or she would attend the concert (2 = Yes, I will attend; 1 = No, I will not attend). Estimate with 95% confidence the number of teenagers who will attend the concert.

12.147 Xr12-147 The owner of a downtown parking lot suspects that the person he hired to run the lot is stealing some money. The receipts as provided by the employee indicate that the average number of cars parked in the lot is 125 per day and that, on average, each car is parked for 3.5 hours. To determine whether the employee is stealing, the owner watches the lot for 5 days. On those days, the numbers of cars parked are as follows:

120	130	124	127	128

The time spent on the lot for the 629 cars that the owner observed during the 5 days was recorded. Can the owner conclude at the 1% level of significance that the employee is stealing? (*Hint*: Because there are two ways to steal, two tests should be performed.)

12.148 Xr12-148 Jim Cramer hosts CNBC's *Mad Money* program. Mr. Cramer regularly makes suggestions about which stocks to buy and sell. How well has Mr. Cramer's picks performed over the years 2005 to 2007? To answer the question, a random sample of Mr. Cramer's picks was selected. The name of the stock, the buy price of the stock, the current or sold price, and the percent return were recorded. (*Source: YourMoneyWatch.com.*)

a. Estimate with 95% confidence the mean return for all of Mr. Cramer's selections.
b. Over the two-year period, the Standard and Poor's 500 Index rose by 16%. Is there sufficient evidence to infer that Mr. Cramer's picks have done less well?

12.149 Xr12-149* Unfortunately, it is not uncommon for high school students in the United States to carry weapons (guns, knives, or clubs). To determine how prevalent this practice is, a survey of high school students was undertaken. Students were asked whether they carried a weapon at least once in the previous 30 days (1 = no, 2 = yes anywhere on school property), and genders (1 = male, 2 = female). Estimate with 95% confidence the proportion of all high school students who have carried weapons in the last 30 days. (Adapted from *Statistical Abstract of the United States*, *2009*, Table 239.)

12.150 Xr12-150 In 2006, the average household debt service ratio for homeowners was 14.35. The household debt service ratio is the ratio of debt payments to disposable personal income. Debt payments consist of mortgage payments and payments on consumer debts. To determine whether this economic measure has increased a random sample of Americans was drawn. Can we infer from the data that the debt service ratio has increased since 2006? (Adapted from *Statistical Abstract of the United States*, *2009*, Table 1135.)

12.151 Xr12-151 Refer to Exercise 12.150. Another measure of indebtedness is the financial obligations ratio, which adds automobile lease payments, rental on tenant occupied property, homeowners insurance, and property tax payments to the debt service ratio. In 2005, the ratio for homeowners was 17.62. Can we infer that financial obligations ratio for homeowners has increased between 2005 and 2009? (Adapted from *Statistical Abstract of the United States*, *2009*, Table 1135.)

12.152 Xr12-152 Refer to Exercise 12.151. In 2005, the financial obligations ratio for renters was 25.97. Can we infer that financial obligations ratio for renters has increased between 2005 and 2009? (Adapted from *Statistical Abstract of the United States*, *2009*, Table 1135.)

12.153 Xr12-153 In 2007, there were 116,011,000 households in the United States. There were 78,425,000 family households made up of married-couple, single-male, and single-female households. To determine how many of each type, a survey was undertaken. The results were stored using the codes 1 = married couple, 2 = single male, and 3 = single female. Estimate with 95% confidence the total number of American households with married couples. (Adapted from *Statistical Abstract of the United States*, *2009*, Table 58.)

12.154 Xr12-154 Wages and salaries make up only part of a total compensation. Other parts include paid leave, health insurance, and many others. In 2007, wages and salaries among manufacturers in the United States made up an average of 65.8% of total compensation. To determine if this changed in 2008, a random sample of manufacturing employees was drawn. Can we infer that percentage of total compensation for wages and salaries increased between 2007 and 2008? (Adapted from *Statistical Abstract of the United States*, *2009*, Table 970.)

CASE 12.1 Pepsi's Exclusivity Agreement with a University

© Susan Van Etten

DATA C12-01

In the last few years, colleges and universities have signed exclusivity agreements with a variety of private companies. These agreements bind the university to sell that company's products exclusively on the campus. Many of the agreements involve food and beverage firms.

A large university with a total enrollment of about 50,000 students has offered Pepsi-Cola an exclusivity agreement that would give Pepsi exclusive rights to sell its products at all university facilities for the next year and an option for future years. In return, the university would receive 35% of the on-campus revenues and an additional lump sum of $200,000 per year. Pepsi has been given 2 weeks to respond.

The management at Pepsi quickly reviews what it knows. The market for soft drinks is measured in terms of the equivalent of 12-ounce cans. Pepsi currently sells an average of 22,000 cans or their equivalents per week (over the 40 weeks of the year that the university operates). The cans sell for an average of one dollar each. The costs, including labor, amount to $.30 per can. Pepsi is unsure of its market share but suspects it is considerably less than 50%. A quick analysis reveals that if its current market share were 25%, then with an exclusivity agreement Pepsi would sell 88,000 cans per week. Thus, annual sales would be 3,520,000 cans per year (calculated as 88,000 cans per week × 40 weeks). The gross revenue would be computed as follows*:

Gross revenue = 3,520,000 cans ×
$1.00 revenue/can = $3,520,000

This figure must be multiplied by 65% because the university would rake in 35% of the gross. Thus,

65% × $3,520,000 = $2,288,000

The total cost of 30 cents per can (or $1,056,000) and the annual payment to the university of $200,000 is subtracted to obtain the net profit:

Net profit = $2,288,000 − $1,056,000
− $200,000 = $1,032,000

Its current annual profit is

Current profit = 40 weeks ×
22,000 cans/week × $.70/can
= $616,000

If the current market share is 25%, the potential gain from the agreement is

$1,032,000 − $616,000 = $416,000

The only problem with this analysis is that Pepsi does not know how many soft drinks are sold weekly at the university. In addition, Coke is not likely to supply Pepsi with information about its sales, which together with Pepsi's line of products constitutes virtually the entire market.

A recent graduate of a business program believes that a survey of the university's students can supply the needed information. Accordingly, she organizes a survey that asks 500 students to keep track of the number of soft drinks they purchase on campus over the next 7 days.

Perform a statistical analysis to extract the needed information from the data. Estimate with 95% confidence the parameter that is at the core of the decision problem. Use the estimate to compute estimates of the annual profit. Assume that Coke and Pepsi drinkers would be willing to buy either product in the absence of their first choice.

a. On the basis of maximizing profits from sales of soft drinks at the university, should Pepsi agree to the exclusivity agreement?
b. Write a report to the company's executives describing your analysis.

*We have created an Excel spreadsheet that does the calculations for this case. To access it, click **Excel Workbooks** and **Case 12.1**. The only cell you may alter is cell C3, which contains the average number of soft drinks sold per week per student, assuming a total of 88,000 drinks sold per year.

CASE 12.2 Pepsi's Exclusivity Agreement with a University: The Coke Side of the Equation

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DATA
C12-01

While the executives of Pepsi Cola are trying to decide what to do, the university informs them that a similar offer has gone out to the Coca-Cola Company. Furthermore, if both companies want exclusive rights, a bidding war will take place. The executives at Pepsi would like to know how likely it is that Coke will want exclusive rights under the conditions outlined by the university.

Perform a similar analysis to the one you did in Case 12.1, but this time from Coke's point of view. Is it likely that Coke will want to conclude an exclusivity agreement with the university? Discuss the reasons for your conclusions.

CASE 12.3 Estimating Total Medical Costs


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DATA C12-03

Virtually all countries have universal government-run health-care systems. The United States is one notable exception. This is an issue in every election, with some politicians pushing for the United States to adopt a program similar to Canada's.

In Canada, hospitals are financed and administered by provincial governments. Physicians are paid by the government for each patient service. As a result, Canadians pay nothing for these services. The revenues that support the system are derived through income taxes, corporate taxes, and sales taxes. Despite higher taxes in Canada than those in the United States, the system is chronically underfunded, resulting in long waiting times for sometimes critical procedures. For example, in some provinces, newly diagnosed cancer victims must wait several weeks before treatments can begin. Virtually everyone agrees that more money is needed. No one can agree however, on how much is needed. Unfortunately, the problem is going to worsen. Canada, like the United States, has an aging population because of the large numbers of so-called baby boomers (those born between 1946 and 1966), and because medical costs are generally higher for older people.

One of the first steps in addressing the problem is to forecast medical costs, particularly for the 20-year period starting when the first baby boomers reached age 60 (in 2006). A statistics practitioner has been given the task of making these predictions. Accordingly, random samples of four groups of Canadians were drawn. They are

Group	Ages
1	45–64
2	65–74
3	75–84
4	85 and over

The medical expenses for the previous 12 months were recorded and stored in columns 1 to 4, respectively, in C12-03.

Age Category	2011	2016	2021	2026	2031
45–64	9,718	10,013	10,065	9,996	10,016
65–74	2,644	3,344	3,992	4,511	4,846
75–84	1,600	1,718	2,045	2,627	3,169
85 +	639	738	810	909	1,121

Source: Statistics Canada.

Projections for 2011, 2016, 2021, 2026, and 2031 of the numbers of Canadians (in thousands) in each age category are listed here.

a. Determine the 95% confidence interval estimates of the mean medical costs for each of the four age categories.

b. For each year listed, determine 95% confidence interval estimates of the total medical costs for Canadians 45 years old and older.

CASE 12.4 Estimating the Number of Alzheimer's Cases

DATA
C12-04

As the U.S. population ages, the number of people needing medical care increases. Unless a cure is found in the next decade, one of the most expensive diseases requiring such care is Alzheimer's, a form of dementia. To estimate the total number of Alzheimer's cases in the future, a survey was undertaken. The survey determined the age bracket where 1 = 65–74, 2 = 75–84, 3 = 85 and over and whether the individual had Alzheimer's (1 = no and 2 = yes). (Adapted from the Alzheimer's Association, www.alz.org.)

Here are the projections for the number of Americans (thousands) in each of the three age categories.

Age Category	2015	2020	2025
65–74	26,967	32,312	36,356
75–84	13,578	15,895	20,312
85+	6,292	6,597	7,239

Source: *Statistical Abstract of the United States, 2009*, Table 8.

a. Determine the 95% confidence interval estimates of the proportion of Alzheimer's patients in each of the three age categories.

b. For each year listed, determine 95% confidence interval estimates of the total number of Americans with Alzheimer's disease.

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13

INFERENCE ABOUT COMPARING TWO POPULATIONS

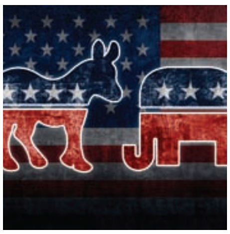
American National Election Survey

Comparing Democrats and Republicans: Who Is More Educated?

DATA ANES2008*

In the business of politics it is important to be able to determine what differences exist between supporters and opponents. In 2008, the American National Election Survey asked people who had indicated that they had completed 13 or more years of education the following question: What is the highest degree earned (DEGREE)?

0. No degree earned
1. Bachelor's degree
2. Master's degree
3. PhD, etc.



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On page 505 we will provide our answer.

4. LLB, JD
5. MD, DDS, etc.
6. JDC, STD, THD
7. Associate's degree

The survey also asked, Do you think of yourself as Democrat, Republican, Independent, or what (PARTY)?

1 = Democrat, 2 = Republican, 3 = Independent, 4 = Other party, 5 = No preference

Do these data allow us to infer that people who identify themselves as Republican Party supporters are more educated than their Democratic counterparts?

INTRODUCTION

We can compare learning how to use statistical techniques to learning how to drive a car. We began by describing what you are going to do in this course (Chapter 1) and then presented the essential background material (Chapters 2–9). Learning the concepts of statistical inference and applying them the way we did in Chapters 10 and 11 is akin to driving a car in an empty parking lot. You're driving, but it's not a realistic experience. Learning Chapter 12 is like driving on a quiet side street with little traffic. The experience represents real driving, but many of the difficulties have been eliminated. In this chapter, you begin to drive for real, with many of the actual problems faced by licensed drivers, and the experience prepares you to tackle the next difficulty.

In this chapter, we present a variety of techniques used to compare two populations. In Sections 13.1 and 13.3, we deal with interval variables; the parameter of interest is the difference between two means. The difference between these two sections introduces yet another factor that determines the correct statistical method—the design of the experiment used to gather the data. In Section 13.1, the samples are independently drawn, whereas in Section 13.3, the samples are taken from a matched pairs experiment. In Section 13.2, we discuss the difference between observational and experimental data, a distinction that is critical to the way in which we interpret statistical results.

Section 13.4 presents the procedures employed to infer whether two population variances differ. The parameter is the ratio σ_1^2/σ_2^2. (When comparing two variances, we use the ratio rather than the difference because of the nature of the sampling distribution.)

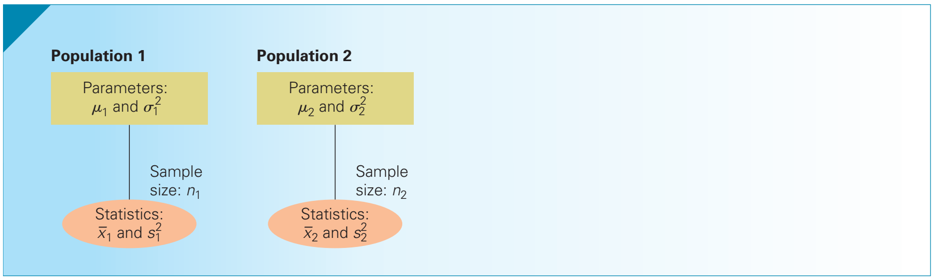
Section 13.5 addresses the problem of comparing two populations of nominal data. The parameter to be tested and estimated is the difference between two proportions.

13.1 INFERENCE ABOUT THE DIFFERENCE BETWEEN TWO MEANS: INDEPENDENT SAMPLES

In order to test and estimate the difference between two population means, the statistics practitioner draws random samples from each of two populations. In this section, we discuss independent samples. In Section 13.3, where we present the matched pairs experiment, the distinction between independent samples and matched pairs will be made clear. For now, we define independent samples as samples completely unrelated to one another.

Figure 13.1 depicts the sampling process. Observe that we draw a sample of size n_1 from population 1 and a sample of size n_2 from population 2. For each sample, we compute the sample means and sample variances.

FIGURE **13.1** Independent Samples from Two Populations



The best estimator of the difference between two population means, $\mu_1 - \mu_2$, is the difference between two sample means, $\bar{x}_1 - \bar{x}_2$. In Section 9.3 we presented the sampling distribution of $\bar{x}_1 - \bar{x}_2$.

Sampling Distribution of $\bar{x}_1 - \bar{x}_2$

1. $\bar{x}_1 - \bar{x}_2$ is normally distributed if the populations are normal and approximately normal if the populations are nonnormal and the sample sizes are large.
2. The expected value of $\bar{x}_1 - \bar{x}_2$ is

$$E(\bar{x}_1 - \bar{x}_2) = \mu_1 - \mu_2$$

3. The variance of $\bar{x}_1 - \bar{x}_2$ is

$$V(\bar{x}_1 - \bar{x}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

The standard error of $\bar{x}_1 - \bar{x}_2$ is

$$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Thus,

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

is a standard normal (or approximately normal) random variable. It follows that the test statistic is

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

The interval estimator is

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

However, these formulas are rarely used because the population variances σ_1^2 and σ_2^2 are virtually always unknown. Consequently, it is necessary to estimate the standard error of the sampling distribution. The way to do this depends on whether the two unknown population variances are equal. When they are equal, the test statistic is defined in the following way.

Test Statistic for $\mu_1 - \mu_2$ when $\sigma_1^2 = \sigma_2^2$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \qquad \nu = n_1 + n_2 - 2$$

where

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

The quantity s_p^2 is called the **pooled variance estimator**. It is the weighted average of the two sample variances with the number of degrees of freedom in each sample used as weights. The requirement that the population variances be equal makes this calculation feasible because we need only one estimate of the common value of σ_1^2 and σ_2^2. It makes sense for us to use the pooled variance estimator because, in combining both samples, we produce a better estimate.

The test statistic is Student t distributed with $n_1 + n_2 - 2$ degrees of freedom, provided that the two populations are normal. The confidence interval estimator is derived by mathematics that by now has become routine.

Confidence Interval Estimator of $\mu_1 - \mu_2$ When $\sigma_1^2 = \sigma_2^2$

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \qquad \nu = n_1 + n_2 - 2$$

We will refer to these formulas as the **equal-variances test statistic** and **confidence interval estimator**, respectively.

When the population variances are unequal, we cannot use the pooled variance estimate. Instead, we estimate each population variance with its sample variance. Unfortunately, the sampling distribution of the resulting statistic

$$\frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

is neither normally nor Student t distributed. However, it can be approximated by a Student t distribution with degrees of freedom equal to

$$\nu = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$$

(It is usually necessary to round this number to the nearest integer.) The test statistic and confidence interval estimator are easily derived from the sampling distribution.

Test Statistic for $\mu_1 - \mu_2$ When $\sigma_1^2 \neq \sigma_2^2$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)}} \qquad \nu = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$$

Confidence Interval Estimator of $\mu_1 - \mu_2$ When $\sigma_1^2 \neq \sigma_2^2$

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)} \qquad \nu = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$$

We will refer to these formulas as the **unequal-variances test statistic** and **confidence interval estimator**, respectively.

The question naturally arises, How do we know when the population variances are equal? The answer is that because σ_1^2 and σ_2^2 are unknown, we can't know for certain whether they're equal. However, we can perform a statistical test to determine whether there is evidence to infer that the population variances differ. We conduct the F-test of the ratio of two variances, which we briefly present here and save the details for Section 13.4.

Testing the Population Variances

The hypotheses to be tested are

$$H_0: \ \sigma_1^2/\sigma_2^2 = 1$$

$$H_1: \ \sigma_1^2/\sigma_2^2 \neq 1$$

The test statistic is the ratio of the sample variances s_1^2/s_2^2, which is F-distributed with degrees of freedom $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 2$. Recall that we introduced the F-distribution in Section 8.4. The required condition is the same as that for the t-test of $\mu_1 - \mu_2$, which is that both populations are normally distributed.

This is a two-tail test so that the rejection region is

$$F > F_{\alpha/2,\nu_1,\nu_2} \qquad \text{or} \qquad F < F_{1-\alpha/2,\nu_1,\nu_2}$$

Put simply, we will reject the null hypothesis that states that the population variances are equal when the ratio of the sample variances is large or if it is small. Table 6 in Appendix B, which lists the critical values of the F-distribution, defines "large" and "small."

Decision Rule: Equal-Variances or Unequal-Variances *t*-Tests and Estimators

Recall that we can never have enough statistical evidence to conclude that the null hypothesis is true. This means that we can only determine whether there is enough evidence to infer that the population variances *differ*. Accordingly, we adopt the following rule: We will use the equal-variances test statistic and confidence interval estimator unless there is evidence (based on the *F*-test of the population variances) to indicate that the population variances are unequal, in which case we will apply the unequal-variances test statistic and confidence interval estimator.

EXAMPLE 13.1*

DATA Xm13-01

Direct and Broker-Purchased Mutual Funds

Millions of investors buy mutual funds (see page 181 for a description of mutual funds), choosing from thousands of possibilities. Some funds can be purchased directly from banks or other financial institutions whereas others must be purchased through brokers, who charge a fee for this service. This raises the question, Can investors do better by buying mutual funds directly than by purchasing mutual funds through brokers? To help answer this question, a group of researchers randomly sampled the annual returns from mutual funds that can be acquired directly and mutual funds that are bought through brokers and recorded the net annual returns, which are the returns on investment after deducting all relevant fees. These are listed next.

Direct					Broker				
9.33	4.68	4.23	14.69	10.29	3.24	3.71	16.4	4.36	9.43
6.94	3.09	10.28	−2.97	4.39	−6.76	13.15	6.39	−11.07	8.31
16.17	7.26	7.1	10.37	−2.06	12.8	11.05	−1.9	9.24	−3.99
16.97	2.05	−3.09	−0.63	7.66	11.1	−3.12	9.49	−2.67	−4.44
5.94	13.07	5.6	−0.15	10.83	2.73	8.94	6.7	8.97	8.63
12.61	0.59	5.27	0.27	14.48	−0.13	2.74	0.19	1.87	7.06
3.33	13.57	8.09	4.59	4.8	18.22	4.07	12.39	−1.53	1.57
16.13	0.35	15.05	6.38	13.12	−0.8	5.6	6.54	5.23	−8.44
11.2	2.69	13.21	−0.24	−6.54	−5.75	−0.85	10.92	6.87	−5.72
1.14	18.45	1.72	10.32	−1.06	2.59	−0.28	−2.15	−1.69	6.95

Can we conclude at the 5% significance level that directly purchased mutual funds outperform mutual funds bought through brokers?

SOLUTION

IDENTIFY

To answer the question, we need to compare the population of returns from direct and the returns from broker-bought mutual funds. The data are obviously interval (we've recorded real numbers). This problem objective–data type combination tells us that the parameter to be tested is the difference between two means, $\mu_1 - \mu_2$. The hypothesis to

**Source:* D. Bergstresser, J. Chalmers, and P. Tufano, "Assessing the Costs and Benefits of Brokers in the Mutual Fund Industry."

be tested is that the mean net annual return from directly purchased mutual funds (μ_1) is larger than the mean of broker-purchased funds (μ_2). Hence, the alternative hypothesis is

$$H_1: \ (\mu_1 - \mu_2) > 0$$

As usual, the null hypothesis automatically follows:

$$H_0: \ (\mu_1 - \mu_2) = 0$$

To decide which of the t-tests of $\mu_1 - \mu_2$ to apply, we conduct the F-test of σ_1^2/σ_2^2.

$$H_0: \ \sigma_1^2/\sigma_2^2 = 1$$

$$H_1: \ \sigma_1^2/\sigma_2^2 \neq 1$$

COMPUTE

MANUALLY

From the data, we calculated the following statistics:

$$s_1^2 = 37.49 \quad \text{and} \quad s_2^2 = 43.34$$

$$\text{Test statistic: } F = s_1^2/s_2^2 = 37.49/43.34 = 0.86$$

$$\text{Rejection region: } F > F_{\alpha/2,\nu_1,\nu_2} = F_{.025,49,49} \approx F_{.025,50,50} = 1.75$$

or

$$F < F_{1-\alpha/2,\nu_1,\nu_2} = F_{.975,49,49} = 1/F_{.025,49,49} \approx 1/F_{.025,50,50} = 1/1.75 = .57$$

Because $F = .86$ is not greater than 1.75 or smaller than .57, we cannot reject the null hypothesis.

EXCEL

	A	B	C
1	F-Test: Two-Sample for Variances		
2			
3		*Direct*	*Broker*
4	Mean	6.63	3.72
5	Variance	37.49	43.34
6	Observations	50	50
7	df	49	49
8	F	0.8650	
9	P(F<=f) one-tail	0.3068	
10	F Critical one-tail	0.6222	

The value of the test statistic is $F = .8650$. Excel outputs the one-tail p-value. Because we're conducting a two-tail test, we double that value. Thus, the p-value of the test we're conducting is $2 \times .3068 = .6136$.

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm13-01.)
2. Click **Data, Data Analysis,** and ***F*-test Two-Sample for Variances.**
3. Specify the **Variable 1 Range** (A1:A51) and the **Variable 2 Range** (B1:B51). Type a value for α (.05).

MINITAB

Test for Equal Variances: Direct, Broker

F-Test (Normal Distribution)
Test statistic = 0.86, p-value = 0.614

INSTRUCTIONS

(*Note:* Some of the printout has been omitted.)

1. Type or import the data into two columns. (Open Xm13-01.)
2. Click **Stat, Basic Statistics,** and **2 Variances . . . .**
3. In the **Samples in different columns** box, select the **First** (Direct) and **Second** (Broker) variables.

INTERPRET

There is not enough evidence to infer that the population variances differ. It follows that we must apply the equal-variances *t*-test of $\mu_1 - \mu_2$.

The hypotheses are

$$H_0: (\mu_1 - \mu_2) = 0$$

$$H_1: (\mu_1 - \mu_2) > 0$$

COMPUTE

MANUALLY

From the data, we calculated the following statistics:

$$\bar{x}_1 = 6.63$$

$$\bar{x}_2 = 3.72$$

$$s_1^2 = 37.49$$

$$s_2^2 = 43.34$$

The pooled variance estimator is

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{(50 - 1)37.49 + (50 - 1)43.34}{50 + 50 - 2}$$

$$= 40.42$$

The number of degrees of freedom of the test statistic is

$$\nu = n_1 + n_2 - 2 = 50 + 50 - 2 = 98$$

The rejection region is

$$t > t_{\alpha,\nu} = t_{.05,98} \approx t_{.05,100} = 1.660$$

We determine that the value of the test statistic is

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{(6.63 - 3.72) - 0}{\sqrt{40.42\left(\frac{1}{50} + \frac{1}{50}\right)}}$$

$$= 2.29$$

EXCEL

	A	B	C
1	t-Test: Two-Sample Assuming Equal Variances		
2			
3		*Direct*	*Broker*
4	Mean	6.63	3.72
5	Variance	37.49	43.34
6	Observations	50	50
7	Pooled Variance	40.41	
8	Hypothesized Mean Difference	0	
9	df	98	
10	t Stat	2.29	
11	P(T<=t) one-tail	0.0122	
12	t Critical one-tail	1.6606	
13	P(T<=t) two-tail	0.0243	
14	t Critical two-tail	1.9845	

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm13-01.)
2. Click **Data, Data Analysis,** and **t-Test: Two-Sample Assuming Equal Variances**.
3. Specify the **Variable 1 Range** (A1:A51) and the **Variable 2 Range** (B1:B51). Type the value of the **Hypothesized Mean Difference*** (0) and type a value for α(.05).

MINITAB

Two-Sample T-Test and CI: Direct, Broker

Two-sample T for Direct vs Broker

	N	Mean	StDev	SE Mean
Direct	50	6.63	6.12	0.87
Broker	50	3.72	6.58	0.93

Difference = mu (Direct) – mu (Broker)
Estimate for difference: 2.91
95% lower bond for difference: 0.80
T-Test of difference = 0 (vs >): T-Value = 2.29 P-Value = 0.012 DF = 98
Both use Pooled StDev = 6.3572

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm13-01.)
2. Click **Stat, Basic Statistics,** and **2-Sample t . . . .**

*This term is technically incorrect. Because we're testing $\mu_1 - \mu_2$, Excel should ask for and output the "Hypothesized Difference between Means."

3. If the data are stacked, use the **Samples in one column** box to specify the names of the variables. If the data are unstacked (as in Example 13.1), specify the **First** and **Second** variables in the **Samples in different columns** box (Direct, Broker). (See the discussion on Data Formats on page 465 for a discussion of stacked and unstacked data.) Click **Assume equal variances**. Click **Options . . . .**
4. In the **Test difference** box, type the value of the parameter under the null hypothesis (0) and select one of **less than, not equal,** or **greater than** for the **Alternative** hypothesis (greater than).

INTERPRET

The value of the test statistic is 2.29. The one-tail p-value is .0122. We observe that the p-value of the test is small (and the test statistic falls into the rejection region). As a result, we conclude that there is sufficient evidence to infer that on average directly purchased mutual funds outperform broker-purchased mutual funds

Estimating $\mu_1 - \mu_2$: Equal-Variances

In addition to testing a value of the difference between two population means, we can also estimate the difference between means. Next we compute the 95% confidence interval estimate of the difference between the mean return for direct and broker mutual funds.

COMPUTE

MANUALLY

The confidence interval estimator of the difference between two means with equal population variances is

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

The 95% confidence interval estimate of the difference between the return for directly purchased mutual funds and the mean return for broker-purchased mutual funds is

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = (6.63 - 3.72) \pm 1.984\sqrt{40.42\left(\frac{1}{50} + \frac{1}{50}\right)}$$

$$= 2.91 \pm 2.52$$

The lower and upper limits are .39 and 5.43.

EXCEL

	A	B	C	D	E	F
1	t-Estimate : Two Means (Equal Variances)					
2						
3				*Direct*	*Broker*	
4	Mean			6.63	3.72	
5	Variance			37.49	43.34	
6	Observations			50	50	
7						
8	Pooled Variance			40.41		
9	Degrees of Freedom			98		
10	Confidence Level			0.95		
11	Confidence Interval Estimate			2.91	±	2.52
12	LCL			0.38		
13	UCL			5.43		

(Continued)

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm13-01.)
2. Click **Add-Ins, Data Analysis Plus**, and **t-Estimate: Two Means**.
3. Specify the **Variable 1 Range** (A1:A51) and the **Variable 2 Range** (B1:B51). Click **Independent Samples with Equal Variances** and the value for α (.05).

MINITAB

Two-Sample T-Test and CI: Direct, Broker

Two-sample T for Direct vs Broker

	N	Mean	StDev	SE Mean
Direct	50	6.63	6.12	0.87
Broker	50	3.72	6.58	0.93

Difference = mu (Direct) – mu (Broker)
Estimate for difference: 2.91
95% CI for difference: (0.38, 5.43)
T-Test of difference = 0 (vs not =): T-Value = 2.29 P-Value = 0.024 DF = 98
Both use Pooled StDev = 6.3572

INSTRUCTIONS

To produce a confidence interval estimate, follow the instructions for the test, but specify **not equal** for the **Alternative**. Minitab will conduct a two-tail test and produce the confidence interval estimate.

INTERPRET

We estimate that the return on directly purchased mutual funds is on average between .38 and 5.43 percentage points larger than broker-purchased mutual funds.

EXAMPLE 13.2†

DATA Xm13-02

Effect of New CEO in Family-Run Businesses

What happens to the family-run business when the boss's son or daughter takes over? Does the business do better after the change if the new boss is the offspring of the owner or does the business do better when an outsider is made chief executive officer (CEO)? In pursuit of an answer, researchers randomly selected 140 firms between 1994 and 2002, 30% of which passed ownership to an offspring and 70% of which appointed an outsider as CEO. For each company, the researchers calculated the operating income as a proportion of assets in the year before and the year after the new CEO took over. The change (operating income after – operating income before) in this variable

†*Source:* M. Bennedsen and K. Nielsen, Copenhagen Business School and D. Wolfenzon, New York University.

was recorded and is listed next. Do these data allow us to infer that the effect of making an offspring CEO is different from the effect of hiring an outsider as CEO?

Offspring			Outsider						
−1.95	0.91	−3.15	0.69	−1.05	1.58	−2.46	3.33	−1.32	−0.51
0	−2.16	3.27	−0.95	−4.23	−1.98	1.59	3.2	5.93	8.68
0.56	1.22	−0.67	−2.2	−0.16	4.41	−2.03	0.55	−0.45	1.43
1.44	0.67	2.61	2.65	2.77	4.62	−1.69	−1.4	−3.2	−0.37
1.5	−0.39	1.55	5.39	−0.96	4.5	0.55	2.79	5.08	−0.49
1.41	−1.43	−2.67	4.15	1.01	2.37	0.95	5.62	0.23	−0.08
−0.32	−0.48	−1.91	4.28	0.09	2.44	3.06	−2.69	−2.69	−1.16
−1.7	0.24	1.01	2.97	6.79	1.07	4.83	−2.59	3.76	1.04
−1.66	0.79	−1.62	4.11	1.72	−1.11	5.67	2.45	1.05	1.28
−1.87	−1.19	−5.25	2.66	6.64	0.44	−0.8	3.39	0.53	1.74
−1.38	1.89	0.14	6.31	4.75	1.36	1.37	5.89	3.2	−0.14
0.57	−3.7	2.12	−3.04	2.84	0.88	0.72	−0.71	−3.07	−0.82
3.05	−0.31	2.75	−0.42	−2.1	0.33	4.14	4.22	−4.34	0
2.98	−1.37	0.3	−0.89	2.07	−5.96	3.04	0.46	−1.16	2.68

SOLUTION

IDENTIFY

The objective is to compare two populations, and the data are interval. It follows that the parameter of interest is the difference between two population means $\mu_1 - \mu_2$, where μ_1 is the mean difference for companies where the owner's son or daughter became CEO and μ_2 is the mean difference for companies who appointed an outsider as CEO.

To determine whether to apply the equal or unequal variances t-test, we use the F-test of two variances.

$$H_0: \ \sigma_1^2/\sigma_2^2 = 1$$
$$H_1: \ \sigma_1^2/\sigma_2^2 \neq 1$$

COMPUTE

MANUALLY

From the data, we calculated the following statistics:

$s_1^2 = 3.79$ and $s_2^2 = 8.03$

Test statistic: $F = s_1^2/s_2^2 = 3.79/8.03 = 0.47$

The degrees of freedom are $\nu_1 = n_1 - 1 = 42 - 1 = 41$ and $\nu_2 = n_2 - 1 = 98 - 1 = 97$

Rejection region: $F > F_{\alpha/2,\nu_1,\nu_2} = F_{.025,41,97} \approx F_{.025,40,100} = 1.64$

or

$$F < F_{1-\alpha/2,\nu_1,\nu_2} = F_{.975,41,97} = 1/F_{.025,97,41} \approx 1/F_{.025,100,40} = 1/1.74 = .57$$

Because $F = .47$ is less than $.57$, we reject the null hypothesis.

EXCEL

	A	B	C
1	F-Test: Two-Sample for Variances		
2			
3		*Offspring*	*Outsider*
4	Mean	–0.10	1.24
5	Variance	3.79	8.03
6	Observations	42	98
7	df	41	97
8	F	0.47	
9	P(F<=f) one-tail	0.0040	
10	F Critical one-tail	0.6314	

The value of the test statistic is $F = .47$, and the p-value $= 2 \times .0040 = .0080$.

MINITAB

Test for Equal Variances: Offspring, Outsider

F-Test (Normal Distribution)
Test statistic = 0.47, p-value = 0.008

INTERPRET

There is enough evidence to infer that the population variances differ. The appropriate technique is the unequal-variances t-test of $\mu_1 - \mu_2$.

Because we want to determine whether there is a *difference* between means, the alternative hypothesis is

$$H_1: \quad (\mu_1 - \mu_2) \neq 0$$

and the null hypothesis is

$$H_0: \quad (\mu_1 - \mu_2) = 0$$

COMPUTE

MANUALLY

From the data, we calculated the following statistics:

$$\bar{x}_1 = -.10$$
$$\bar{x}_2 = 1.24$$
$$s_1^2 = 3.79$$
$$s_2^2 = 8.03$$

The number of degrees of freedom of the test statistic is

$$\nu = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}}$$

$$= \frac{(3.79/42 + 8.03/98)^2}{\dfrac{(3.79/42)^2}{42 - 1} + \dfrac{(8.03/98)^2}{98 - 1}}$$

$$= 110.69 \text{ rounded to } 111$$

The rejection region is

$$t < -t_{\alpha/2,\nu} = -t_{.025,111} \approx -t_{.025,110} = -1.982 \quad \text{or} \quad t > t_{\alpha/2,\nu} = t_{.025,111} \approx 1.982$$

The value of the test statistic is computed next:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)}}$$

$$= \frac{(-.10 - 1.24) - (0)}{\sqrt{\left(\dfrac{3.79}{42} + \dfrac{8.03}{98}\right)}} = -3.22$$

EXCEL

	A	B	C
1	t-Test: Two-Sample Assuming Unequal Variances		
2			
3		*Offspring*	*Outsider*
4	Mean	–0.10	1.24
5	Variance	3.79	8.03
6	Observations	42	98
7	Hypothesized Mean Difference	0	
8	df	111	
9	t Stat	–3.22	
10	P(T<=t) one-tail	0.0008	
11	t Critical one-tail	1.6587	
12	P(T<=t) two-tail	0.0017	
13	t Critical two-tail	1.9816	

INSTRUCTIONS

Follow the instructions for Example 13.1, except at step 2 click **Data, Data Analysis,** and **t-Test: Two-Sample Assuming Unequal Variances**.

MINITAB

Two-Sample T-Test and CI: Offspring, Outsider

Two-sample T for Offspring vs Outsider

	N	Mean	StDev	SE Mean
Offspring	42	–0.10	1.95	0.30
Outsider	98	1.24	2.83	0.29

Difference = mu (Offspring) – mu (Outsider)
Estimate for difference: –1.336
95% CI for difference: (–2.158, –0.514)
T-Test of difference = 0 (vs not =): T-Value = –3.22 P-Value = 0.002 DF = 110

INSTRUCTIONS

Follow the instructions for Example 13.1 except at step 3 do not click **Assume equal variances**.

INTERPRET

The *t*-statistic is −3.22, and its *p*-value is .0017. Accordingly, we conclude there is sufficient evidence to infer that the mean changes in operating income differ.

Estimating $\mu_1 - \mu_2$: Unequal-Variances

We can also draw inferences about the difference between the two population means by calculating the confidence interval estimator. We use the unequal-variances confidence interval estimator of $\mu_1 - \mu_2$ and a 95% confidence level.

COMPUTE

MANUALLY

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)}$$

$$= (-.10 - 1.24) \pm 1.982\sqrt{\left(\frac{3.79}{42} + \frac{8.03}{98}\right)}$$

$$= -1.34 \pm .82$$

$$\text{LCL} = -2.16 \quad \text{and} \quad \text{UCL} = -.52$$

EXCEL

	A	B	C	D
1	t-Estimate : Two Means (Unequal Variances)			
2				
3		*Offspring*	*Outsider*	
4	Mean	–0.10	1.24	
5	Variance	3.79	8.03	
6	Observations	42	98	
7				
8	Degrees of Freedom	110.75		
9	Confidence Level	0.95		
10	Confidence Interval Estimate	–1.34	±	0.82
11	LCL	–2.16		
12	UCL	–0.51		

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm13-01.)
2. Click **Add-Ins, Data Analysis Plus**, and **t-Estimate: Two Means**.
3. Specify the **Variable 1 Range** (A1:A43) and the **Variable 2 Range** (B1:B99). Click **Independent Samples with Unequal Variances** and the value for α (.05).

MINITAB

Minitab prints the confidence interval estimate as part of the output of the test statistic. However, you must specify the **Alternative** hypothesis as **not equal** to produce a two-sided interval.

INTERPRET

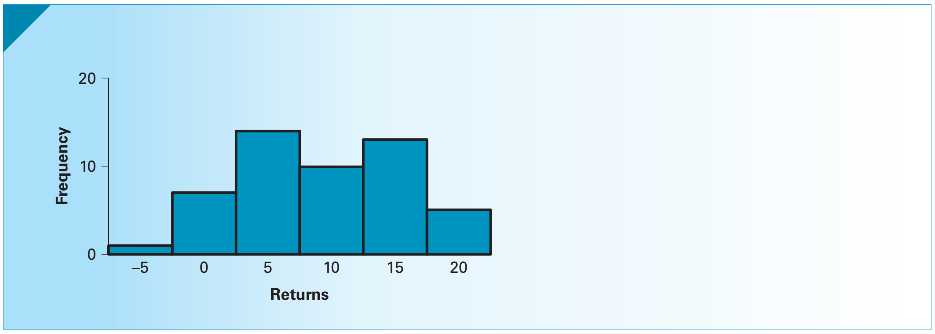
We estimate that the mean change in operating incomes for outsiders exceeds the mean change in the operating income for offspring lies between .51 and 2.16 percentage points.

Checking the Required Condition

Both the equal-variances and unequal-variances techniques require that the populations be normally distributed.* As before, we can check to see whether the requirement is satisfied by drawing the histograms of the data.

To illustrate, we used Excel (Minitab histograms are almost identical) to create the histograms for Example 13.1 (Figures 13.2 and 13.3) and Example 13.2 (Figures 13.4

FIGURE **13.2** Histogram of Rates of Return for Directly Purchased Mutual Funds in Example 13.1



*As we pointed out in Chapter 12 large sample sizes can overcome the effects of extreme nonnormality.

and 13.5). Although the histograms are not perfectly bell shaped, it appears that in both examples the data are at least approximately normal. Because this technique is robust, we can be confident in the validity of the results.

FIGURE **13.3** Histogram of Rates of Return for Broker-Purchased Mutual Funds in Example 13.1

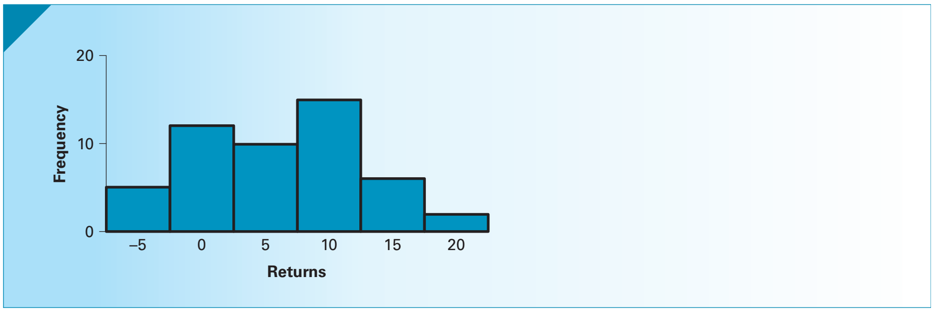
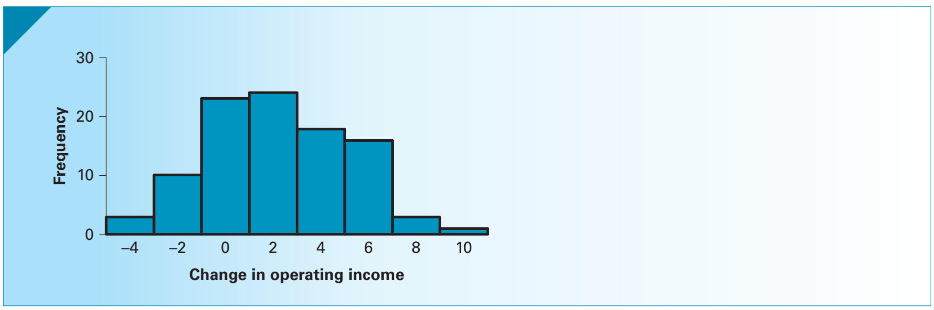


FIGURE **13.4** Histogram of Change in Operating Income for Offspring-Run Businesses in Example 13.2

FIGURE **13.5** Histogram of Change in Operating Income for Outsider-Run Businesses in Example 13.2



Violation of the Required Condition

When the normality requirement is unsatisfied, we can use a nonparametric technique: the Wilcoxon rank sum test (Chapter 19*) to replace the equal-variances test of $\mu_1 - \mu_2$. We have no alternative to the unequal-variances test of $\mu_1 - \mu_2$ when the populations are very nonnormal.

Data Formats

There are two formats for storing the data when drawing inferences about the difference between two means. The first, which you have seen demonstrated in both Examples 13.1 and 13.2, is called *unstacked*, wherein the observations from sample 1 are stored in one column and the observations from sample 2 are stored in a second column. We may also store the data in stacked format. In this format, all the observations are stored in one column. A second column contains the codes, usually 1 and 2, that indicate from which sample the corresponding observation was drawn. Here is an example of unstacked data.

Column 1 (Sample 1)	Column 2 (Sample 2)
12	18
19	23
13	25

Here are the same data in stacked form.

Column 1	Column 2
12	1
19	1
13	1
18	2
23	2
25	2

It should be understood that the data need not be in order. Hence, they could have been stored in this way:

Column 1	Column 2
18	2
25	2
13	1
12	1
23	2
19	1

If there are two populations to compare and only one variable, then it is probably better to record the data in unstacked form. However, it is frequently the case that we want to observe several variables and compare them. For example, suppose that we survey male and female MBAs and ask each to report his or her age, income, and number of years of experience. These data are usually stored in stacked form using the following format.

*Instructors who wish to teach the use of nonparametric techniques for testing the difference between two means when the normality requirement is not satisfied should use Keller's website Appendix Introduction to Nonparametric Techniques and Keller's website Appendix Wilcoxon Rank Sum Test and Wilcoxon Signed Rank Sum Test.

Column 1: Code identifying female (1) and male (2)

Column 2: Age

Column 3: Income

Column 4: Years of experience

To compare ages, we would use columns 1 and 2. Columns 1 and 3 are used to compare incomes, and columns 1 and 4 are used to compare experience levels.

Most statistical software requires one format or the other. Some but not all of Excel's techniques require unstacked data. Some of Minitab's procedures allow either format, whereas others specify only one. Fortunately, both of our software packages allow the statistics practitioner to alter the format. (See Keller's website Appendix Excel and Minitab Instructions for Stacking and Unstacking Data.) We say "fortunately" because this allowed us to store the data in either form on our website. In fact, we've used both forms to allow you to practice your ability to manipulate the data as necessary. You will need this ability to perform statistical techniques in this and other chapters in this book.

Developing an Understanding of Statistical Concepts 1

The formulas in this section are relatively complicated. However, conceptually both test statistics are based on the techniques we introduced in Chapter 11 and repeated in Chapter 12: The value of the test statistic is the difference between the statistic $\bar{x}_1 - \bar{x}_2$ and the hypothesized value of the parameter $\mu_1 - \mu_2$ measured in terms of the standard error.

Developing an Understanding of Statistical Concepts 2

The standard error must be estimated from the data for all inferential procedures introduced here. The method we use to compute the standard error of $\bar{x}_1 - \bar{x}_2$ depends on whether the population variances are equal. When they are equal we calculate and use the pooled variance estimator s_p^2. We are applying an important principle here, and we will do so again in Section 13.5 and in later chapters. The principle can be loosely stated as follows: Where possible, it is advantageous to pool sample data to estimate the standard error. In Example 13.1, we are able to pool because we assume that the two samples were drawn from populations with a common variance. Combining both samples increases the accuracy of the estimate. Thus, s_p^2 is a better estimator of the common variance than either s_1^2 or s_2^2 separately. When the two population variances are unequal, we cannot pool the data and produce a common estimator. We must compute s_1^2 and s_2^2 and use them to estimate σ_1^2 and σ_2^2, respectively.

Here is a summary of how we recognize the techniques presented in this section.

Factors That Identify the Equal-Variances *t*-Test and Estimator of $\mu_1 - \mu_2$

1. **Problem objective**: Compare two populations
2. **Data type:** Interval
3. **Descriptive measurement**: Central location
4. **Experimental design**: Independent samples
5. **Population variances**: Equal

Factors That Identify the Unequal-Variances *t*-Test and Estimator of $\mu_1 - \mu_2$

1. **Problem objective**: Compare two populations
2. **Data type**: Interval
3. **Descriptive measurement**: Central location
4. **Experimental design**: Independent samples
5. **Population variances**: Unequal

EXERCISES

DO-IT-YOURSELF EXCEL

Construct Excel spreadsheets for each of the following:

13.1 Equal-variance *t*-test of $\mu_1 - \mu_2$. Inputs: Sample means, sample standard deviations, sample sizes, hypothesized difference between means. Outputs: Test statistic, critical values, and one- and two-tail *p*-values. Tools: **TINV, TDIST**

13.2 Equal-variance *t*-estimator of $\mu_1 - \mu_2$. Inputs: Sample means, sample standard deviations, sample sizes, and confidence level. Outputs: Upper and lower confidence limits. Tools: **TINV**

13.3 Unequal-variance *t*-test of $\mu_1 - \mu_2$. Inputs: Sample means, sample standard deviations, sample sizes, hypothesized difference between means. Outputs: Test statistic, critical values, and one- and two-tail *p*-values. Tools: **TINV, TDIST**

13.4 Unequal-variance *t*-estimator of $\mu_1 - \mu_2$. Inputs: Sample means, sample standard deviations, sample sizes, and confidence level. Outputs: Upper and lower confidence limits. Tools: **TINV**

Developing an Understanding of Statistical Concepts

Exercises 13.5 to 13.10 are "what-if" analyses designed to determine what happens to the test statistics and interval estimates when elements of the statistical inference change. These problems can be solved manually, using the Excel spreadsheets you created or Minitab.

13.5 In random samples of 25 from each of two normal populations, we found the following statistics:

$\bar{x}_1 = 524$ $\quad s_1 = 129$

$\bar{x}_2 = 469$ $\quad s_2 = 141$

a. Estimate the difference between the two population means with 95% confidence.
b. Repeat part (a) increasing the standard deviations to $s_1 = 255$ and $s_2 = 260$.
c. Describe what happens when the sample standard deviations get larger.
d. Repeat part (a) with samples of size 100.
e. Discuss the effects of increasing the sample size.

13.6 In random samples of 12 from each of two normal populations, we found the following statistics:

$\bar{x}_1 = 74$ $\quad s_1 = 18$

$\bar{x}_2 = 71$ $\quad s_2 = 16$

a. Test with $\alpha = .05$ to determine whether we can infer that the population means differ.
b. Repeat part (a) increasing the standard deviations to $s_1 = 210$ and $s_2 = 198$.
c. Describe what happens when the sample standard deviations get larger.
d. Repeat part (a) with samples of size 150.
e. Discuss the effects of increasing the sample size.
f. Repeat part (a) changing the mean of sample 1 to $\bar{x}_1 = 76$.
g. Discuss the effect of increasing $\bar{x}_1$.

13.7 Random sampling from two normal populations produced the following results:

$\bar{x}_1 = 63$ $\quad s_1 = 18$ $\quad n_1 = 50$

$\bar{x}_2 = 60$ $\quad s_2 = 7$ $\quad n_2 = 45$

a. Estimate with 90% confidence the difference between the two population means.
b. Repeat part (a) changing the sample standard deviations to 41 and 15, respectively.
c. What happens when the sample standard deviations increase?
d. Repeat part (a) doubling the sample sizes.
e. Describe the effects of increasing the sample sizes.

13.8 Random sampling from two normal populations produced the following results:

$\bar{x}_1 = 412$ $\quad s_1 = 128$ $\quad n_1 = 150$

$\bar{x}_2 = 405$ $\quad s_2 = 54$ $\quad n_2 = 150$

a. Can we infer at the 5% significance level that μ_1 is greater than μ_2?
b. Repeat part (a) decreasing the standard deviations to $s_1 = 31$ and $s_2 = 16$.
c. Describe what happens when the sample standard deviations get smaller.
d. Repeat part (a) with samples of size 20.
e. Discuss the effects of decreasing the sample size.
f. Repeat part (a) changing the mean of sample 1 to $\bar{x}_1 = 409$
g. Discuss the effect of decreasing $\bar{x}_1$.

13.9 For each of the following, determine the number of degrees of freedom assuming equal population variances and unequal population variances.
a. $n_1 = 15, n_2 = 15, s_1^2 = 25, s_2^2 = 15$
b. $n_1 = 10, n_2 = 16, s_1^2 = 100, s_2^2 = 15$
c. $n_1 = 50, n_2 = 50, s_1^2 = 8, s_2^2 = 14$
d. $n_1 = 60, n_2 = 45, s_1^2 = 75, s_2^2 = 10$

13.10 Refer to Exercise 13.9.
a. Confirm that in each case the number of degrees of freedom for the equal-variances test statistic and confidence interval estimator is larger than that for the unequal-variances test statistic and confidence interval estimator.
b. Try various combinations of sample sizes and sample variances to illustrate that the number of degrees of freedom for the equal-variances test statistic and confidence interval estimator is larger than that for the unequal-variances test statistic and confidence interval estimator.

Applications

13.11 Xr13-11 Every month a clothing store conducts an inventory and calculates losses from theft. The store would like to reduce these losses and is considering two methods. The first is to hire a security guard, and the second is to install cameras. To help decide which method to choose, the manager hired a security guard for 6 months. During the next 6-month period, the store installed cameras. The monthly losses were recorded and are listed here. The manager decided that because the cameras were cheaper than the guard, he would install the cameras unless there was enough evidence to infer that the guard was better. What should the manager do?

Security guard	355	284	401	398	477	254
Cameras	486	303	270	386	411	435

13.12 Xr13-12 A men's softball league is experimenting with a yellow baseball that is easier to see during night games. One way to judge the effectiveness is to count the number of errors. In a preliminary experiment, the yellow baseball was used in 10 games and the traditional white baseball was used in another 10 games. The number of errors in each game was recorded and is listed here. Can we infer that there are fewer errors on average when the yellow ball is used?

Yellow	5	2	6	7	2	5	3	8	4	9
White	7	6	8	5	9	11	8	3	6	10

13.13 Xr13-13 A number of restaurants feature a device that allows credit card users to swipe their cards at the table. It allows the user to specify a percentage or a dollar amount to leave as a tip. In an experiment to see how it works, a random sample of credit card users was drawn. Some paid the usual way, and some used the new device. The percent left as a tip was recorded and listed below. Can we infer that users of the device leave larger tips?

Usual	10.3	15.2	13.0	9.9	12.1	13.4	12.2	14.9	13.2	12.0	
Device	13.6	15.7	12.9	13.2	12.9	13.4	12.1	13.9	15.7	15.4	17.4

13.14 Xr13-14 Who spends more on their vacations, golfers or skiers? To help answer this question, a travel agency surveyed 15 customers who regularly take their spouses on either a skiing or a golfing vacation. The amounts spent on vacations last year are shown here. Can we infer that golfers and skiers differ in their vacation expenses?

Golfer	2,450	3,860	4,528	1,944	3,166	3,275
	4,490	3,685	2,950			
Skier	3,805	3,725	2,990	4,357	5,550	4,130

13.15 Xr13-15 A growing concern among fans and owners is the amount of time to complete a major league baseball game. To assess the extent of the problem, a statistician recorded the amount of time (in minutes) to complete a random sample of games 5 years ago and this year. Can we conclude that games take longer to complete this year than 5 years ago?

5 Years Ago											
169	160	174	161	187	172	177	187	153	169	161	194
This Year											
153	182	162	190	163	189	171	197	159	180	197	178

13.16 Xr13-16 How do drivers react to sudden large increases in the price of gasoline? To help answer the question, a statistician recorded the speeds of cars as they passed a large service station. He recorded the speeds (mph) in the same location after the service station sign showed that the price of gasoline had risen by 15 cents. Can we conclude that the speeds differ?

Speeds Before Price Increase

43	36	31	30	28	36	27	36	35	30	32	36

Speeds After Price Increase

32	33	36	31	32	29	28	39	26	30	32	30

Exercises 13.17–13.44 require the use of a computer and software. Use a 5% significance level unless specified otherwise. The answers to Exercises 13.17–13.37 may be calculated manually using the sample statistics listed in Appendix A.

13.17 Xr13-17 The president of Tastee Inc., a baby-food producer, claims that her company's product is superior to that of her leading competitor because babies gain weight faster with her product. (This is a good thing for babies.) To test this claim, a survey was undertaken. Mothers of newborn babies were asked which baby food they intended to feed their babies. Those who responded Tastee or the leading competitor were asked to keep track of their babies' weight gains over the next 2 months. There were 15 mothers who indicated that they would feed their babies Tastee and 25 who responded that they would feed their babies the product of the leading competitor. Each baby's weight gain (in ounces) was recorded.

a. Can we conclude, using weight gain as our criterion, that Tastee baby food is indeed superior?
b. Estimate with 95% confidence the difference between the mean weight gains of the two products.
c. Check to ensure that the required condition(s) is satisfied.

13.18 Xr13-18 Is eating oat bran an effective way to reduce cholesterol? Early studies indicated that eating oat bran daily reduces cholesterol levels by 5% to 10%. Reports of this study resulted in the introduction of many new breakfast cereals with various percentages of oat bran as an ingredient. However, an experiment performed by medical researchers in Boston cast doubt on the effectiveness of oat bran. In that study, 120 volunteers ate oat bran for breakfast, and another 120 volunteers ate another grain cereal for breakfast. At the end of 6 weeks, the percentage of cholesterol reduction was computed for both groups. Can we infer that oat bran is different from other cereals in terms of cholesterol reduction?

13.19 Xr13-19* In assessing the value of radio advertisements, sponsors consider not only the total number of listeners but also their ages. The 18-to-34 age group is considered to spend the most money. To examine the issue, the manager of an FM station commissioned a survey. One objective was to measure the difference in listening habits between the 18-to-34 and 35-to-50 age groups. The survey asked 250 people in each age category how much time they spent listening to FM radio per day. The results (in minutes) were recorded and stored in stacked format (column 1 = Age group and column 2 = Listening times).

a. Can we conclude that a difference exists between the two groups?
b. Estimate with 95% confidence the difference in mean time listening to FM radio between the two age groups.
c. Are the required conditions satisfied for the techniques you used in parts (a) and (b)?

13.20 Xr13-20 The cruise ship business is rapidly increasing. Although cruises have long been associated with seniors, it now appears that younger people are choosing a cruise as their vacations. To determine whether this is true, an executive for a cruise line sampled passengers 2 years ago and this year and determined their ages.

a. Do these data allow the executive to infer that cruise ships are attracting younger customers?
b. Estimate with 99% confidence the difference in ages between this year and 2 years ago.

13.21 Xr13-21* Automobile insurance companies take many factors into consideration when setting rates. These factors include age, marital status, and miles driven per year. To determine the effect of gender, a random sample of young (under 25, with at least 2 years of driving experience) male and female drivers was surveyed. Each was asked how many miles he or she had driven in the past year. The distances (in thousands of miles) are stored in stacked format (column 1 = driving distances and column 2 identifies the gender where 1 = male and code 2 = female).

a. Can we conclude that male and female drivers differ in the numbers of miles driven per year?
b. Estimate with 95% confidence the difference in mean distance driven by male and female drivers.
c. Check to ensure that the required condition(s) of the techniques used in parts (a) and (b) is satisfied.

13.22 Xr13-22 The president of a company that manufactures automobile air conditioners is considering switching his supplier of condensers. Supplier A, the current producer of condensers for the manufacturer, prices its product 5% higher than supplier B. Because the president wants to maintain his company's reputation for quality, he wants to be sure that supplier B's condensers last at least as long as supplier A's. After a careful analysis, the president decided to retain supplier A if there is sufficient statistical evidence that supplier A's condensers last longer on average than supplier B's. In an experiment, 30 midsize cars were equipped with air conditioners using type A condensers while another 30 midsize cars were equipped with type B condensers. The number of miles (in thousands) driven by each car before the condenser broke down was recorded. Should the president retain supplier A?

13.23 Xr13-23 An important function of a firm's human resources manager is to track worker turnover. As a general rule, companies prefer to retain workers. New workers frequently need to be trained, and it often takes time for new workers to learn how to perform their jobs. To investigate nationwide results, a human resources manager organized a survey wherein a random sample of men and women was asked how long they had worked for their current employers. Can we infer that men and women have different job tenures? (Adapted from the *Statistical Abstract of the United States, 2000*, Table 664).

13.24 Xr13-24 A statistics professor is about to select a statistical software package for her course. One of the most important features, according to the professor, is the ease with which students learn to use the software. She has narrowed the selection to two possibilities: software A, a menu-driven statistical package with some high-powered techniques, and software B, a spreadsheet that has the capability of performing most techniques. To help make her decision, she asks 40 statistics students selected at random to choose one of the two packages. She gives each student a statistics problem to solve by computer and the appropriate manual. The amount of time (in minutes) each student needed to complete the assignment was recorded.

a. Can the professor conclude from these data that the two software packages differ in the amount of time needed to learn how to use them? (Use a 1% significance level.)
b. Estimate with 95% confidence the difference in the mean amount of time needed to learn to use the two packages.
c. What are the required conditions for the techniques used in parts (a) and (b)?
d. Check to see whether the required conditions are satisfied.

13.25 Xr13-25 One factor in low productivity is the amount of time wasted by workers. Wasted time includes time spent cleaning up mistakes, waiting for more material and equipment, and performing any other activity not related to production. In a project designed to examine the problem, an operations-management consultant took a survey of 200 workers in companies that were classified as successful (on the basis of their latest annual profits) and another 200 workers from unsuccessful companies. The amount of time (in hours) wasted during a standard 40-hour workweek was recorded for each worker.

a. Do these data provide enough evidence at the 1% significance level to infer that the amount of time wasted in unsuccessful firms exceeds that of successful ones?
b. Estimate with 95% confidence how much more time is wasted in unsuccessful firms than in successful ones.

13.26 Xr13-26 Recent studies seem to indicate that using a cell phone while driving is dangerous. One reason for this is that a driver's reaction time may slow while he or she is talking on the phone. Researchers at Miami (Ohio) University measured the reaction times of a sample of drivers who owned a cell phone. Half the sample was tested while on the phone and the other half was tested while not on the phone. Can we conclude that reaction times are slower for drivers using cell phones?

13.27 Xr13-27 Refer to Exercise 13.26. To determine whether the type of phone usage affects reaction times, another study was launched. A group of drivers was asked to participate in a discussion. Half the group engaged in simple chitchat, and the other half participated in a political discussion. Once again, reaction times were measured. Can we infer that the type of telephone discussion affects reaction times?

13.28 Xr13-28 Most consumers who require someone to perform various professional services undertake research before making their selection. A random sample of people who recently selected a financial planner and a random sample of individuals who chose a stockbroker were asked to report the amount of time they spent researching before deciding. Can we infer that people spend more time researching for a financial planner than they do for a stockbroker? (*Source*: Yankelovich Partners.)

13.29 Xr13-29 Xr13-23 A recent study by researchers at North Carolina State University found thousands of errors in 12 of the most widely used high school science texts. For example, the Statue of Liberty is left-handed; volume is equal to length multiplied by depth (*Time Magazine*, February 12, 2001). The books are so bad that Philip Sadler, director of science education at the Harvard-Smithsonian Center for Astrophysics, decided to conduct a study of their effects. He recorded the physics marks of college students who had used a textbook in high school and the marks of students who did not have a high school textbook. Do these data allow us to infer that students without high school textbooks in science outperform students who used textbooks?

13.30 Xr13-30 Between Wendy's and McDonald's, which fast-food drive-through window is faster? To answer the question, a random sample of service times for each restaurant was measured. Can we infer from these data that there are differences in service times between the two chains? (*Source*: 2000 QSR Drive-Thru Time Study.)

13.31 Xr13-31 Lack of sleep is a serious medical problem. It has been linked to heart attacks and automobile collisions. A Statistics Canada study asked a random sample of Canadian adults to report the amount of sleep they normally get. Can we conclude from the data that men and women differ in the amount of sleep?

13.32 Xr13-32 It is often useful for companies to know who their customers are and how they became customers. In a study of credit card use, random samples were drawn of cardholders who applied for the credit card and credit cardholders who were contacted by telemarketers or by mail. The total purchases made by each last month were recorded. Can we conclude from these data that differences exist on average between the two types of customers?

13.33 Xr13-33 Tire manufacturers are constantly researching ways to produce tires that last longer. New innovations are tested by professional drivers on racetracks. However, any promising inventions are also test-driven by ordinary drivers. The latter tests are closer to what the tire company's customers will actually experience. Suppose that to determine whether a new steel-belted radial tire lasts longer than the company's current model, two new-design tires were installed on the rear wheels of 20 randomly selected cars and two existing-design tires were installed on the rear wheels of another 20 cars. All drivers were told to drive in their usual way until the tires wore out. The number of miles driven by each driver was recorded. Can the company infer that the new tire will last longer on average than the existing tire?

13.34 Xr13-34 It is generally believed that salespeople who are paid on a commission basis outperform salespeople who are paid a fixed salary. Some management consultants argue, however, that in certain industries the fixed-salary salesperson may sell more because the consumer will feel less sales pressure and respond to the salesperson less as an antagonist. In an experiment to study this, a random sample of 180 salespeople from a retail clothing chain was selected. Of these, 90 salespeople were paid a fixed salary, and the remaining 90 were paid a commission on each sale. The total dollar amount of 1 month's sales for each was recorded. Can we conclude that the commission salesperson outperforms the fixed-salary salesperson?

13.35 Xr13-35 Credit scorecards were designed to be used to help financial institutions make decisions about loan applications (see page 63). However, some insurance companies have suggested that credit scores could also be used to determine insurance premiums, particularly car insurance. The Massachusetts Public Interest Research Group has come out against this proposal. To acquire more information, an executive for a car-insurance company gathered data about a random sample of the company's customers. She recorded whether the individual was involved in an accident in the last 3 years and determined the credit score. Can the executive infer that there is a difference in scores between those who did and those who did not have accidents in a 3-year period?

13.36 Xr13-36* Traditionally, wine has been sold in glass bottles with cork stoppers. The stoppers are supposed to keep air out of the bottle because oxygen is the enemy of wine, particularly red wine. Recent research appears to indicate that metal screw caps are more effective in keeping air out of the bottle. However, metal caps are perceived to be inferior and usually associated with cheaper brands of wine. To determine if this perception is wrong, a random sample of 130 people who drink at least one bottle per week on average was asked to participate in an experiment. All were given the same wine in two types of bottles. One group was given a corked bottle, and the other was given a bottle with a metal cap and asked to taste the wine and indicate what they think the retail price of the wine should be. Determine whether there is enough evidence to conclude that bottles of wine with metal caps are perceived to be cheaper.

13.37 Xr13-37 Studies have shown that tired children have trouble learning because neurons become incapable of forming new synaptic connections that are necessary to encode memory. The problem is that the school day starts too early. Awakened at dawn, teenage brains are still releasing melatonin, which makes them sleepy. Several years ago, Edina, Minnesota, changed its high school start from 7:25 A.M. to 8:30 A.M. The SAT scores for a random sample of students taken before the change and a random sample of SAT scores after the change were recorded. Can we infer from the data that SAT scores increased after the change in the school start time?

General Social Survey Exercises

13.38 GSS2008* Study after study indicates that men earn higher incomes than women. To determine the extent of the differential in 2008, estimate with 95% confidence the difference between male and female (SEX: 1 = Male, 2 = Female) annual incomes (INCOME).

13.39 GSS2006* Repeat Exercise 13.38 using data from the 2006 General Social Survey.

13.40 [Ch03:\\CPI-Annual] Use the CPI annual to allow a comparison of the results of Exercises 13.38 and 13.39. Is the income differential decreasing?

13.41 GSS2008* One of the major economic issues in 2010 was the growing size of federal, state, and municipal payrolls. One issue is that people who work for the government earn more than those who work in the private sector. Conduct a test using the 2008 General Social Survey to determine whether we can infer that government employees (WRKGOVT: 1 = Government, 2 = Private) earn more income (INCOME) than other workers?

American National Election Survey Exercises

13.42 ANES2008* The chapter-opening example compares Republicans and Democrats in terms of whether they had graduated from high school. Another way of judging is to measure the number of years of education (EDUC). Conduct a test to determine whether Republicans have more years of education than do Democrats (PARTY: 1 = Democrat and 2 = Republicans)?

13.43 GSS2008* Do the data from the American National Election Survey in 2008 allow us to infer than males have higher incomes than females (INCOME).

13.44 ANES04* Repeat Exercise 13.43 using the ANES data from 2004.

13.2 Observational and Experimental Data

As we've pointed out several times, the ability to properly interpret the results of a statistical technique is a crucial skill for students to develop. This ability is dependent on your understanding of Type I and Type II errors and the fundamental concepts that are part of statistical inference. However, there is another component that must be understood: the difference between **observational data** and **experimental data**. The difference results from the way the data are generated. The following example will demonstrate the difference between the two types.

EXAMPLE 13.3

DATA Xm13-03

Dietary Effects of High-Fiber Breakfast Cereals

Despite some controversy, scientists generally agree that high-fiber cereals reduce the likelihood of various forms of cancer. However, one scientist claims that people who eat high-fiber cereal for breakfast will consume, on average, fewer calories for lunch than people who don't eat high-fiber cereal for breakfast. If this is true, high-fiber cereal manufacturers will be able to claim another advantage of eating their product—potential weight reduction for dieters. As a preliminary test of the claim, 150 people were randomly selected and asked what they regularly eat for breakfast and lunch. Each person was identified as either a consumer or a nonconsumer of high-fiber cereal, and the number of calories consumed at lunch was measured and recorded. These data are listed here. Can the scientist conclude at the 5% significance level that his belief is correct?

Calories Consumed at Lunch by Consumers of High-Fiber Cereal

568	646	607	555	530	714	593	647	650
498	636	529	565	566	639	551	580	629
589	739	637	568	687	693	683	532	651
681	539	617	584	694	556	667	467	
540	596	633	607	566	473	649	622	

Calories Consumed at Lunch by Nonconsumers of High-Fiber Cereal

705	754	740	569	593	637	563	421	514	536
819	741	688	547	723	553	733	812	580	833
706	628	539	710	730	620	664	547	624	644
509	537	725	679	701	679	625	643	566	594
613	748	711	674	672	599	655	693	709	596
582	663	607	505	685	566	466	624	518	750
601	526	816	527	800	484	462	549	554	582
608	541	426	679	663	739	603	726	623	788
787	462	773	830	369	717	646	645	747	
573	719	480	602	596	642	588	794	583	
428	754	632	765	758	663	476	490	573	

SOLUTION

The appropriate technique is the unequal-variances t-test of $\mu_1 - \mu_2$, where μ_1 is the mean of the number of calories for lunch by consumers of high-fiber cereal for breakfast and μ_2 is the mean of the number of calories for lunch by nonconsumers of high-fiber cereal for breakfast. [The F-test of the ratio of two variances (not shown here) yielded $F = .3845$ and p-value $= .0008$.]

The hypotheses are

$$H_0: \quad (\mu_1 - \mu_2) = 0$$
$$H_1: \quad (\mu_1 - \mu_2) < 0$$

The Excel printout is shown next. The manually calculated and Minitab-produced results are identical.

	A	B	C
1	t-Test: Two-Sample Assuming Unequal Variances		
2			
3		*Consumers*	*Nonconsumers*
4	Mean	604.02	633.23
5	Variance	4103	10670
6	Observations	43	107
7	Hypothesized Mean Difference	0	
8	df	123	
9	t Stat	–2.09	
10	P(T<=t) one-tail	0.0193	
11	t Critical one-tail	1.6573	
12	P(T<=t) two-tail	0.0386	
13	t Critical two-tail	1.9794	

INTERPRET

The value of the test statistic is −2.09. The one-tail *p*-value is .0193. We observe that the *p*-value of the test is small (and the test statistic falls into the rejection region). As a result, we conclude that there is sufficient evidence to infer that consumers of high-fiber cereal do eat fewer calories at lunch than do nonconsumers. From this result, we're inclined to believe that eating a high-fiber cereal at breakfast may be a way to reduce weight. However, other interpretations are plausible. For example, people who eat fewer calories are probably more health conscious, and such people are more likely to eat high-fiber cereal as part of a healthy breakfast. In this interpretation, high-fiber cereals do not necessarily lead to fewer calories at lunch. Instead, another factor, general health consciousness, leads to both fewer calories at lunch and high-fiber cereal for breakfast. Notice that the conclusion of the statistical procedure is unchanged. On average, people who eat high-fiber cereal consume fewer calories at lunch. However, because of the way the data were gathered, we have more difficulty interpreting this result.

Suppose that we redo Example 13.3 using the experimental approach. We randomly select 150 people to participate in the experiment. We randomly assign 75 to eat high-fiber cereal for breakfast and the other 75 to eat something else. We then record the number of calories each person consumes at lunch. Ideally, in this experiment both groups will be similar in all other dimensions, including health consciousness. (Larger sample sizes increase the likelihood that the two groups will be similar.) If the statistical result is about the same as in Example 13.3, we may have some valid reason to believe that high-fiber cereal at breakfast leads to a decrease in caloric intake at lunch.

Experimental data are usually more expensive to obtain because of the planning required to set up the experiment; observational data usually require less work to gather. Furthermore, in many situations it is impossible to conduct a controlled experiment. For example, suppose that we want to determine whether an undergraduate degree in engineering better prepares students for an MBA than does an arts degree. In a controlled experiment, we would randomly assign some students to achieve a degree in engineering and other students to obtain an arts degree. We would then make them sign up for an MBA program where we would record their grades. Unfortunately for statistical despots (and fortunately for the rest of us), we live in a democratic society, which makes the coercion necessary to perform this controlled experiment impossible.

To answer our question about the relative performance of engineering and arts students, we have no choice but to obtain our data by observational methods. We would take a random sample of engineering students and arts students who have already entered MBA programs and record their grades. If we find that engineering students do better, we may tend to conclude that an engineering background better prepares students for an MBA program. However, it may be true that better students tend to choose engineering as their undergraduate major and that better students achieve higher grades in all programs, including the MBA program.

Although we've discussed observational and experimental data in the context of the test of the difference between two means, you should be aware that the issue of how the data are obtained is relevant to the interpretation of all the techniques that follow.

EXERCISES

13.45 Refer to Exercise 13.17. If the data are observational, describe another conclusion besides the one that infers that Tastee is better for babies.

13.46 Are the data in Exercise 13.18 observational or experimental? Explain. If the data are observational, describe a method of producing experimental data.

13.47 Refer to Exercise 13.24.
a. Are the data observational or experimental?
b. If the data are observational, describe a method of answering the question with experimental data?
c. If the data are observational produce another explanation for the statistical outcome.

13.48 Suppose that you wish to test to determine whether one method of teaching statistics is better than another.
a. Describe a data-gathering process that produces observational data.
b. Describe a data-gathering process that produces experimental data.

13.49 Put yourself in place of the director of research and development for a pharmaceutical company. When a new drug is developed, it undergoes a number of tests. One test is designed to determine whether the drug is safe and effective. Your company has just developed a drug that is designed to alleviate the symptoms of degenerative diseases such as multiple sclerosis. Design an experiment that tests the new drug.

13.50 You wish to determine whether MBA graduates who majored in finance attract higher starting salaries than MBA graduates who majored in marketing.
a. Describe a data-gathering process that produces observational data.
b. Describe a data-gathering process that produces experimental data.
c. If observational data indicate that finance majors attract higher salaries than do marketing majors, provide two explanations for this result.

13.51 Suppose that you are analyzing one of the hundreds of statistical studies that link smoking with lung cancer. The study analyzed thousands of randomly selected people, some of whom had lung cancer. The statistics indicate that those who have lung cancer smoked on average significantly more than those who did not have lung cancer.
a. Explain how you know that the data are observational.
b. Is there another interpretation of the statistics besides the obvious one that smoking causes lung cancer? If so, what is it? (Students who produce the best answers will be eligible for a job in the public relations department of a tobacco company.)
c. Is it possible to conduct a controlled experiment to produce data that address the question of the relationship between smoking and lung cancer? If so, describe the experiment.

13.3 INFERENCE ABOUT THE DIFFERENCE BETWEEN TWO MEANS: MATCHED PAIRS EXPERIMENT

We continue our presentation of statistical techniques that address the problem of comparing two populations of interval data. In Section 13.1, the parameter of interest was the difference between two population means, where the data were generated from independent samples. In this section, the data are gathered from a matched pairs experiment. To illustrate why matched pairs experiments are needed and how we deal with data produced in this way, consider the following example.

EXAMPLE 13.4 Comparing Salary Offers for Finance and Marketing MBA Majors, Part 1

DATA Xm13-04

In the last few years, a number of web-based companies that offer job placement services have been created. The manager of one such company wanted to investigate the job offers recent MBAs were obtaining. In particular, she wanted to know whether finance majors were being offered higher salaries than marketing majors. In a preliminary study, she randomly sampled 50 recently graduated MBAs, half of whom majored in finance and half in marketing. From each she obtained the highest salary offer (including benefits). These data are listed here. Can we infer that finance majors obtain higher salary offers than do marketing majors among MBAs?

Highest salary offer made to finance majors

61,228	51,836	20,620	73,356	84,186	79,782	29,523	80,645	76,125
62,531	77,073	86,705	70,286	63,196	64,358	47,915	86,792	75,155
65,948	29,392	96,382	80,644	51,389	61,955	63,573		

Highest salary offer made to marketing majors

73,361	36,956	63,627	71,069	40,203	97,097	49,442	75,188	59,854
79,816	51,943	35,272	60,631	63,567	69,423	68,421	56,276	47,510
58,925	78,704	62,553	81,931	30,867	49,091	48,843		

SOLUTION

IDENTIFY

The objective is to compare two populations of interval data. The parameter is the difference between two means $\mu_1 - \mu_2$ (where μ_1 = mean highest salary offer to finance majors and μ_2 = mean highest salary offer to marketing majors). Because we want to determine whether finance majors are offered higher salaries, the alternative hypothesis will specify that μ_1 is greater than μ_2. The F-test for variances was conducted, and the results indicate that there is not enough evidence to infer that the population variances differ. Hence we use the equal-variances test statistic:

$$H_0: \quad (\mu_1 - \mu_2) = 0$$
$$H_1: \quad (\mu_1 - \mu_2) > 0$$

$$\text{Test statistic: } t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

COMPUTE

MANUALLY

From the data, we calculated the following statistics:

$$\bar{x}_1 = 65{,}624$$
$$\bar{x}_2 = 60{,}423$$
$$s_1^2 = 360{,}433{,}294$$
$$s_2^2 = 262{,}228{,}559$$
$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$
$$= \frac{(25 - 1)(360{,}433{,}294) + (25 - 1)(262{,}228{,}559)}{25 + 25 - 2}$$
$$= 311{,}330{,}926$$

The value of the test statistic is computed next:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{(65{,}624 - 60{,}423) - (0)}{\sqrt{311{,}330{,}926\left(\frac{1}{25} + \frac{1}{25}\right)}}$$

$$= 1.04$$

The number of degrees of freedom of the test statistic is

$$\nu = n_1 + n_2 - 2 = 25 + 25 - 2 = 48$$

The rejection region is

$$t > t_{\alpha,\nu} = t_{.05,48} \approx 1.676$$

EXCEL

	A	B	C
1	t-Test: Two-Sample Assuming Equal Variances		
2			
3		*Finance*	*Marketing*
4	Mean	65,624	60,423
5	Variance	360,433,294	262,228,559
6	Observations	25	25
7	Pooled Variance	311,330,926	
8	Hypothesized Mean Difference	0	
9	df	48	
10	t Stat	1.04	
11	P(T<=t) one-tail	0.1513	
12	t Critical one-tail	1.6772	
13	P(T<=t) two-tail	0.3026	
14	t Critical two-tail	2.0106	

MINITAB

Two-Sample T-Test and CI: Finance, Marketing

Two-sample T for Finance vs Marketing

	N	Mean	StDev	SE Mean
Finance	25	65624	18985	3797
Marketing	25	60423	16193	3239

Difference = mu (Finance) – mu (Marketing)
Estimate for difference: 5201.00
95% lower bound for difference: –3169.42
T-Test of difference = 0 (vs >): T-Value = 1.04 P-Value = 0.151 DF = 48
Both use Pooled StDev = 17644.5722

INTERPRET

The value of the test statistic ($t = 1.04$) and its p-value (.1513) indicate that there is very little evidence to support the hypothesis that finance majors receive higher salary offers than marketing majors.

Notice that we have some evidence to support the alternative hypothesis. The difference in sample means is

$$(\bar{x}_1 - \bar{x}_2) = (65{,}624 - 60{,}423) = 5{,}201$$

However, we judge the difference between sample means in relation to the standard error of $\bar{x}_1 - \bar{x}_2$. As we've already calculated,

$$s_p^2 = 311{,}330{,}926$$

and

$$\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = 4{,}991$$

Consequently, the value of the test statistic is $t = 5{,}201/4{,}991 = 1.04$, a value that does not allow us to infer that finance majors attract higher salary offers. We can see that although the difference between the sample means was quite large, the variability of the data as measured by s_p^2 was also large, resulting in a small test statistic value.

EXAMPLE 13.5 Comparing Salary Offers for Finance and Marketing MBA Majors, Part 2

DATA
Xm13-05

Suppose now that we redo the experiment in the following way. We examine the transcripts of finance and marketing MBA majors. We randomly select a finance and a marketing major whose grade point average (GPA) falls between 3.92 and 4 (based on a maximum of 4). We then randomly select a finance and a marketing major whose GPA is between 3.84 and 3.92. We continue this process until the 25th pair of finance and marketing majors is selected whose GPA fell between 2.0 and 2.08. (The minimum GPA required for graduation is 2.0.) As we did in Example 13.4, we recorded the highest salary offer. These data, together with the GPA group, are listed here. Can we conclude from these data that finance majors draw larger salary offers than do marketing majors?

Group	Finance	Marketing
1	95,171	89,329
2	88,009	92,705
3	98,089	99,205
4	106,322	99,003
5	74,566	74,825
6	87,089	77,038
7	88,664	78,272
8	71,200	59,462
9	69,367	51,555
10	82,618	81,591
11	69,131	68,110
12	58,187	54,970
13	64,718	68,675
14	67,716	54,110
15	49,296	46,467
16	56,625	53,559
17	63,728	46,793
18	55,425	39,984
19	37,898	30,137
20	56,244	61,965
21	51,071	47,438

Group	Finance	Marketing
22	31,235	29,662
23	32,477	33,710
24	35,274	31,989
25	45,835	38,788

SOLUTION

The experiment described in Example 13.4 is one in which the samples are independent. In other words, there is no relationship between the observations in one sample and the observations in the second sample. However, in this example the experiment was designed in such a way that each observation in one sample is matched with an observation in the other sample. The matching is conducted by selecting finance and marketing majors with similar GPAs. Thus, it is logical to compare the salary offers for finance and marketing majors in each group. This type of experiment is called **matched pairs**. We now describe how we conduct the test.

For each GPA group, we calculate the matched pair difference between the salary offers for finance and marketing majors.

Group	Finance	Marketing	Difference
1	95,171	89,329	5,842
2	88,009	92,705	−4,696
3	98,089	99,205	−1,116
4	106,322	99,003	7,319
5	74,566	74,825	−259
6	87,089	77,038	10,051
7	88,664	78,272	10,392
8	71,200	59,462	11,738
9	69,367	51,555	17,812
10	82,618	81,591	1,027
11	69,131	68,110	1,021
12	58,187	54,970	3,217
13	64,718	68,675	−3,957
14	67,716	54,110	13,606
15	49,296	46,467	2,829
16	56,625	53,559	3,066
17	63,728	46,793	16,935
18	55,425	39,984	15,441
19	37,898	30,137	7,761
20	56,244	61,965	−5,721
21	51,071	47,438	3,633
22	31,235	29,662	1,573
23	32,477	33,710	−1,233
24	35,274	31,989	3,285
25	45,835	38,788	7,047

In this experimental design, the parameter of interest is the **mean of the population of differences**, which we label μ_D. Note that μ_D does in fact equal $\mu_1 - \mu_2$, but we test μ_D because of the way the experiment was designed. Hence, the hypotheses to be tested are

$$H_0\colon \ \mu_D = 0$$
$$H_1\colon \ \mu_D > 0$$

We have already presented inferential techniques about a population mean. Recall that in Chapter 12 we introduced the t-test of μ. Thus, to test hypotheses about μ_D, we use the following test statistic.

Test Statistic for μ_D

$$t = \frac{\bar{x}_D - \mu_D}{s_D/\sqrt{n_D}}$$

which is Student t distributed with $\nu = n_D - 1$ degrees of freedom, provided that the differences are normally distributed.

Aside from the subscript D, this test statistic is identical to the one presented in Chapter 12. We conduct the test in the usual way.

COMPUTE

MANUALLY

Using the differences computed above, we find the following statistics:

$$\bar{x}_D = 5{,}065$$

$$s_D = 6{,}647$$

from which we calculate the value of the test statistic:

$$t = \frac{\bar{x}_D - \mu_D}{s_D/\sqrt{n_D}} = \frac{5{,}065 - 0}{6{,}647/\sqrt{25}} = 3.81$$

The rejection region is

$$t > t_{\alpha,\nu} = t_{.05,24} = 1.711$$

EXCEL

	A	B	C
1	t-Test: Paired Two Sample for Means		
2			
3		*Finance*	*Marketing*
4	Mean	65,438	60,374
5	Variance	444,981,810	469,441,785
6	Observations	25	25
7	Pearson Correlation	0.9520	
8	Hypothesized Mean Difference	0	
9	df	24	
10	t Stat	3.81	
11	P(T<=t) one-tail	0.0004	
12	t Critical one-tail	1.7109	
13	P(T<=t) two-tail	0.0009	
14	t Critical two-tail	2.0639	

Excel prints the sample means, variances, and sample sizes for each sample (as well as the coefficient of correlation), which implies that the procedure uses these statistics. It doesn't. The technique is based on computing the paired differences from which the mean, variance, and sample size are determined. Excel should have printed these statistics.

INSTRUCTIONS

1. Type or import the data into two columns*. (Open Xm13-05.)
2. Click **Data, Data Analysis,** and **t-Test: Paired Two-Sample for Means**.

*If one or both columns contain a blank (representing missing data) the row must be deleted.

3. Specify the **Variable 1 Range** (B1:B26) and the **Variable 2 Range** (C1:C26)**.** Type the value of **the Hypothesized Mean Difference** (0) and specify a value for α (.05).

Warning: If there are blank spaces (representing missing data) in any of the rows in either **Variable 1** or **Variable 2 Range**, Excel will produce the wrong answer. You must delete all rows that contain one or two blanks**.** See Keller's website appendix Deleting blank rows in Excel.

MINITAB

Paired T-Test and CI: Finance, Marketing

Paired T for Finance - Marketing

	N	Mean	StDev	SE Mean
Finance	25	65438.2	21094.6	4218.9
Marketing	25	60373.7	21666.6	4333.3
Difference	25	5064.52	6646.90	1329.38

95% lower bound for mean difference: 2790.11
T-Test of mean difference = 0 (vs > 0): T-Value = 3.81 P-Value = 0.000

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm13-05.)
2. **Click Stat, Basic Statistics,** and **Paired t . . . .**
3. Select the variable names of the **First sample** (Finance) and **Second sample** (Marketing). Click **Options . . . .**
4. In the **Test Mean** box, type the hypothesized mean of the paired difference (0), and specify the **Alternative** (greater than)**.**

INTERPRET

The value of the test statistic is $t = 3.81$ with a p-value of .0004. There is now overwhelming evidence to infer that finance majors obtain higher salary offers than marketing majors. By redoing the experiment as matched pairs, we were able to extract this information from the data.

Estimating the Mean Difference

We derive the confidence interval estimator of μ_D using the usual form for the confidence interval.

Confidence Interval Estimator of μ_D

$$\bar{x}_D \pm t_{\alpha/2}\frac{s_D}{\sqrt{n_D}}$$

EXAMPLE 13.6 Comparing Salary Offers for Finance and Marketing MBA Majors, Part 3

DATA
Xm13-05

Compute the 95% confidence interval estimate of the mean difference in salary offers between finance and marketing majors in Example 13.5.

SOLUTION

COMPUTE

MANUALLY

The 95% confidence interval estimate of the mean difference is

$$\bar{x}_D \pm t_{\alpha/2}\frac{s_D}{\sqrt{n_D}} = 5{,}065 \pm 2.064\frac{6{,}647}{\sqrt{25}} = 5{,}065 \pm 2{,}744$$

$$\text{LCL} = 2{,}321 \quad \text{and} \quad \text{UCL} = 7{,}809$$

EXCEL

	A	B	C	D
1	t-Estimate : Two Means (Matched Pairs)			
2				
3		*Difference*		
4	Mean	5065		
5	Variance	44181217		
6	Observations	25		
7	Degrees of Freedom	24		
8	Confidence Level	0.95		
9	Confidence Interval Estimate	5065	±	2744
10	LCL	2321		
11	UCL	7808		

INSTRUCTIONS

1. Type or import the data into two columns*. (Open Xm13-05.)
2. Click **Add-Ins, Data Analysis Plus**, and **t-Estimate: Two Means**.
3. Specify the **Variable 1 Range** (B1:B51) and the **Variable 2 Range** (C1:C51). Click **Matched Pairs** and the value for α (.05).

MINITAB

Paired T-Test and CI: Finance, Marketing

Paired T for Finance - Marketing

	N	Mean	StDev	SE Mean
Finance	25	65438.2	21094.6	4218.9
Marketing	25	60373.7	21666.6	4333.3
Difference	25	5064.52	6646.90	1329.38

95% CI for mean difference: (2320.82, 7808.22)
T-Test of mean difference = 0 (vs not = 0): T-Value = 3.81 P-Value = 0.001

*If one or both columns contain a blank (representing missing data) the row must be deleted.

INSTRUCTIONS

Follow the instructions to test the paired difference. However, you must specify **not equal** for the **Alternative** hypothesis to produce the two-sided confidence interval estimate of the mean difference.

INTERPRET

We estimate that the mean salary offer to finance majors exceeds the mean salary offer to marketing majors by an amount that lies between \$2,321 and \$7,808 (using the computer output).

Independent Samples or Matched Pairs: Which Experimental Design Is Better?

Examples 13.4 and 13.5 demonstrated that the experimental design is an important factor in statistical inference. However, these two examples raise several questions about experimental designs.

1. Why does the matched pairs experiment result in concluding that finance majors receive higher salary offers than do marketing majors, whereas the independent samples experiment could not?
2. Should we always use the matched pairs experiment? In particular, are there disadvantages to its use?
3. How do we recognize when a matched pairs experiment has been performed?

Here are our answers.

1. The matched pairs experiment worked in Example 13.5 by reducing the variation in the data. To understand this point, examine the statistics from both examples. In Example 13.4, we found $\bar{x}_1 - \bar{x}_2 = 5{,}201$. In Example 13.5, we computed $\bar{x}_D = 5{,}065$. Thus, the numerators of the two test statistics were quite similar. However, the test statistic in Example 13.5 was much larger than the test statistic in Example 13.4 because of the standard errors. In Example 13.4, we calculated

$$s_p^2 = 311{,}330{,}926 \quad \text{and} \quad \sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = 4{,}991$$

Example 13.5 produced

$$s_D = 6{,}647 \quad \text{and} \quad \frac{s_D}{\sqrt{n_D}} = 1{,}329$$

As you can see, the difference in the test statistics was caused not by the numerator, but by the denominator. This raises another question: Why was the variation in the data of Example 13.4 so much greater than the variation in the data of Example 13.5? If you examine the data and statistics from Example 13.4, you will find that there was a great deal of variation *between* the salary offers in each sample. In other words, some MBA graduates received high salary offers and others relatively low ones. This high level of variation, as expressed by s_p^2, made the difference between the sample means appear to be small. As a result, we could not conclude that finance majors attract higher salary offers.

Looking at the data from Example 13.5, we see that there is very little variation between the observations of the paired differences. The variation caused by different GPAs has been decreased markedly. The smaller variation causes the value of the test statistic to be larger. Consequently, we conclude that finance majors obtain higher salary offers.

2. Will the matched pairs experiment always produce a larger test statistic than the independent samples experiment? The answer is, not necessarily. Suppose that in our example we found that companies did not consider grade point averages when making decisions about how much to offer the MBA graduates. In such circumstances, the matched pairs experiment would result in no significant decrease in variation when compared to independent samples. It is possible that the matched pairs experiment may be less likely to reject the null hypothesis than the independent samples experiment. The reason can be seen by calculating the degrees of freedom. In Example 13.4, the number of degrees of freedom was 48, whereas in Example 13.5, it was 24. Even though we had the same number of observations (25 in each sample), the matched pairs experiment had half the number of degrees of freedom as the equivalent independent samples experiment. For exactly the same value of the test statistic, a smaller number of degrees of freedom in a Student t distributed test statistic yields a larger p-value. What this means is that if there is little reduction in variation to be achieved by the matched pairs experiment, the statistics practitioner should choose instead to conduct the experiment with independent samples.

3. As you've seen, in this book we deal with questions arising from experiments that have already been conducted. Consequently, one of your tasks is to determine the appropriate test statistic. In the case of comparing two populations of interval data, you must decide whether the samples are independent (in which case the parameter is $\mu_1 - \mu_2$) or matched pairs (in which case the parameter is μ_D) to select the correct test statistic. To help you do so, we suggest you ask and answer the following question: Does some natural relationship exist between each pair of observations that provides a logical reason to compare the first observation of sample 1 with the first observation of sample 2, the second observation of sample 1 with the second observation of sample 2, and so on? If so, the experiment was conducted by matched pairs. If not, it was conducted using independent samples.

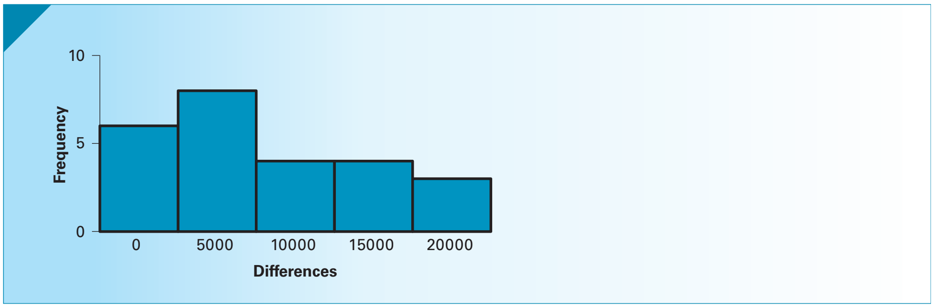
Observational and Experimental Data

The points we made in Section 13.2 are also valid in this section: We can design a matched pairs experiment where the data are gathered using a controlled experiment or by observation. The data in Examples 13.4 and 13.5 are observational. As a consequence, when the statistical result provided evidence that finance majors attracted higher salary offers, it did not necessarily mean that students educated in finance are more attractive to prospective employers. It may be, for example, that better students major in finance and better students achieve higher starting salaries.

Checking the Required Condition

The validity of the results of the t-test and estimator of μ_D depends on the normality of the differences (or large enough sample sizes). The histogram of the differences (Figure 13.6) is positively skewed but not enough so that the normality requirement is violated.

FIGURE **13.6** Histogram of Differences in Example 13.5



Violation of Required Condition

If the differences are very nonnormal, we cannot use the *t*-test of μ_D. We can, however, employ a nonparametric technique—the Wilcoxon signed rank sum test for matched pairs, which we present in Chapter 19.*

Developing an Understanding of Statistical Concepts 1

Two of the most important principles in statistics were applied in this section. The first is the concept of analyzing sources of variation. In Examples 13.4 and 13.5, we showed that by reducing the variation between salary offers in each sample we were able to detect a real difference between the two majors. This was an application of the more general procedure of analyzing data and attributing some fraction of the variation to several sources. In Example 13.5, the two sources of variation were the GPA and the MBA major. However, we were not interested in the variation between graduates with differing GPAs. Instead, we only wanted to eliminate that source of variation, making it easier to determine whether finance majors draw larger salary offers.

In Chapter 14, we will introduce a technique called the *analysis of variance* that does what its name suggests: It analyzes sources of variation in an attempt to detect real differences. In most applications of this procedure, we will be interested in each source of variation and not simply in reducing one source. We refer to the process as *explaining the variation*. The concept of explained variation is also applied in Chapters 16–18, where we introduce regression analysis.

Developing an Understanding of Statistical Concepts 2

The second principle demonstrated in this section is that statistics practitioners can design data-gathering procedures in such a way that they can analyze sources of variation. Before conducting the experiment in Example 13.5, the statistics practitioner suspected that there were large differences between graduates with different GPAs. Consequently, the experiment was organized so that the effects of those differences were mostly eliminated. It is also possible to design experiments that allow for easy detection

*Instructors who wish to teach the use of nonparametric techniques for testing the mean difference when the normality requirement is not satisfied should use Keller's website Appendix Introduction to Nonparametric Techniques and Keller's website Appendix Wilcoxon Rank Sum Test and Wilcoxon Signed Rank Sum Test.

of real differences and minimize the costs of data gathering. Unfortunately, we will not present this topic. However, you should understand that the entire subject of the design of experiments is an important one, because statistics practitioners often need to be able to analyze data to detect differences, and the cost is almost always a factor.

Here is a summary of how we determine when to use these techniques.

Factors That Identify the t-Test and Estimator of μ_D

1. **Problem objective**: Compare two populations
2. **Data type**: Interval
3. **Descriptive measurement**: Central location
4. **Experimental design**: Matched pairs

Exercises

Applications

Conduct all tests of hypotheses at the 5% significance level.

13.52 Xr13-52 Many people use scanners to read documents and store them in a Word (or some other software) file. To help determine which brand of scanner to buy, a student conducts an experiment in which eight documents are scanned by each of the two scanners he is interested in. He records the number of errors made by each. These data are listed here. Can he infer that brand A (the more expensive scanner) is better than brand B?

Document	1	2	3	4	5	6	7	8
Brand A	17	29	18	14	21	25	22	29
Brand B	21	38	15	19	22	30	31	37

13.53 Xr13-53 How effective is an antilock braking system (ABS), which pumps very rapidly rather than lock and thus avoid skids? As a test, a car buyer organized an experiment. He hit the brakes and, using a stopwatch, recorded the number of seconds it took to stop an ABS-equipped car and another identical car without ABS. The speeds when the brakes were applied and the number of seconds each took to stop on dry pavement are listed here. Can we infer that ABS is better?

Speeds	20	25	30	35	40	45	50	55
ABS	3.6	4.1	4.8	5.3	5.9	6.3	6.7	7.0
Non-ABS	3.4	4.0	5.1	5.5	6.4	6.5	6.9	7.3

13.54 Xr13-54 In a preliminary study to determine whether the installation of a camera designed to catch cars that go through red lights affects the number of violators, the number of red-light runners was recorded for each day of the week before and after the camera was installed. These data are listed here. Can we infer that the camera reduces the number of red-light runners?

Day	Sunday	Monday	Tuesday	Wednesday
Before	7	21	27	18
After	8	18	24	19

Day	Thursday	Friday	Saturday
Before	20	24	16
After	16	19	16

13.55 Xr13-55 In an effort to determine whether a new type of fertilizer is more effective than the type currently in use, researchers took 12 two-acre plots of land scattered throughout the county. Each plot was divided into two equal-sized subplots, one of which was treated with the current fertilizer and the other with the new fertilizer. Wheat was planted, and the crop yields were measured.

Plot	1	2	3	4	5	6	7	8	9	10	11	12
Current fertilizer	56	45	68	72	61	69	57	55	60	72	75	66
New fertilizer	60	49	66	73	59	67	61	60	58	75	72	68

a. Can we conclude at the 5% significance level that the new fertilizer is more effective than the current one?
b. Estimate with 95% confidence the difference in mean crop yields between the two fertilizers.
c. What is the required condition(s) for the validity of the results obtained in parts (a) and (b)?

d. Is the required condition(s) satisfied?
e. Are these data experimental or observational? Explain.
f. How should the experiment be conducted if the researchers believed that the land throughout the county was essentially the same?

13.56 Xr13-56 The president of a large company is in the process of deciding whether to adopt a lunchtime exercise program. The purpose of such programs is to improve the health of workers and thus reduce medical expenses. To get more information, he instituted an exercise program for the employees in one office. The president knows that during the winter months medical expenses are relatively high because of the incidence of colds and flu. Consequently, he decides to use a matched pairs design by recording medical expenses for the 12 months before the program and for 12 months after the program. The "before" and "after" expenses (in thousands of dollars) are compared on a month-to-month basis and shown here.
a. Do the data indicate that exercise programs reduce medical expenses? (Test with $\alpha = .05$.)
b. Estimate with 95% confidence the mean savings produced by exercise programs.
c. Was it appropriate to conduct a matched pairs experiment? Explain.

Month	Jan	Feb	Mar	Apr	May	Jun
Before program	68	44	30	58	35	33
After program	59	42	20	62	25	30

Month	Jul	Aug	Sep	Oct	Nov	Dec
Before program	52	69	23	69	48	30
After program	56	62	25	75	40	26

Exercises 13.57–13.72 require the use of a computer and software. Use a 5% significance level unless specified otherwise. The answers to Exercises 13.57 to 13.69 may be calculated manually. See Appendix A for the sample statistics.

13.57 Xr13-57 One measure of the state of the economy is the amount of money homeowners pay on their mortgage each month. To determine the extent of change between this year and 5 years ago, a random sample of 150 homeowners was drawn. The monthly mortgage payments for each homeowner for this year and for 5 years ago were recorded. (The amounts have been adjusted so that we're comparing constant dollars.) Can we infer that mortgage payments have risen over the past 5 years?

13.58 Xr13-58 Do waiters or waitresses earn larger tips? To answer this question, a restaurant consultant undertook a preliminary study. The study involved measuring the percentage of the total bill left as a tip for one randomly selected waiter and one randomly selected waitress in each of 50 restaurants during a 1-week period. What conclusions can be drawn from these data?

13.59 Xr13-59 To determine the effect of advertising in the Yellow Pages, Bell Telephone took a sample of 40 retail stores that did not advertise in the Yellow Pages last year but did so this year. The annual sales (in thousands of dollars) for each store in both years were recorded.
a. Estimate with 90% confidence the improvement in sales between the 2 years.
b. Can we infer that advertising in the Yellow Pages improves sales?
c. Check to ensure that the required condition(s) of the techniques used in parts (a) and (b) is satisfied.
d. Would it be advantageous to perform this experiment with independent samples? Explain why or why not.

13.60 Xr13-60 Because of the high cost of energy, homeowners in northern climates need to find ways to cut their heating costs. A building contractor wanted to investigate the effect on heating costs of increasing the insulation. As an experiment, he located a large subdevelopment built around 1970 with minimal insulation. His plan was to insulate some of the houses and compare the heating costs in the insulated homes with those that remained uninsulated. However, it was clear to him that the size of the house was a critical factor in determining heating costs. Consequently, he found 16 pairs of identical-sized houses ranging from about 1,200 to 2,800 square feet. He insulated one house in each pair (levels of R20 in the walls and R32 in the attic) and left the other house unchanged. The heating cost for the following winter season was recorded for each house.
a. Do these data allow the contractor to infer at the 10% significance level that the heating cost for insulated houses is less than that for the uninsulated houses?
b. Estimate with 95% confidence the mean savings due to insulating the house.
c. What is the required condition for the use of the techniques in parts (a) and (b)?

13.61 Xr13-61 The cost of health care is rising faster than most other items. To learn more about the problem, a survey was undertaken to determine whether differences in health-care expenditures exist between men and women. The survey randomly sampled men and women aged 21, 22, . . . , 65 and determined the total amount spent on health care. Do these data allow us to infer that men and women spend different amounts on health care? (*Source*: Bureau of Labor Statistics, Consumer Expenditure Survey.)

13.62 Xr13-62 The fluctuations in the stock market induce some investors to sell and move their money into more stable investments. To determine the degree to which recent fluctuations affected ownership, a random sample of 170 people who confirmed that they owned some stock was surveyed. The values of the holdings were recorded at the end of last year and at the end of the year before. Can we infer that the value of the stock holdings has decreased?

13.63 Xr13-63 Are Americans more deeply in debt this year compared to last year? To help answer this question, a statistics practitioner randomly sampled Americans this year and last year. The sampling was conducted so that the samples were matched by the age of the head of the household. For each, the ratio of debt payments to household income was recorded. Can we infer that the ratios are higher this year than last?

13.64 Xr13-64 Every April Americans and Canadians fill out their tax return forms. Many turn to tax preparation companies to do this tedious job. The question arises, Are there differences between companies? In an experiment, two of the largest companies were asked to prepare the tax returns of a sample of 55 taxpayers. The amounts of tax payable were recorded. Can we conclude that company 1's service results in higher tax payable?

13.65 Xr13-65 Refer to Exercise 13.33. Suppose now we redo the experiment in the following way. On 20 randomly selected cars, one of each type of tire is installed on the rear wheels and, as before, the cars are driven until the tires wear out. The number of miles until wear-out occurred was recorded. Can we conclude from these data that the new tire is superior?

13.66 Refer to Exercises 13.33 and 13.65. Explain why the matched pairs experiment produced significant results whereas the independent samples *t*-test did not.

13.67 Xr13-67 Refer to Examples 13.4 and 13.5. Suppose that another experiment is conducted. Finance and marketing MBA majors were matched according to their undergraduate GPA. As in the previous examples, the highest starting salary offers were recorded. Can we infer from these data that finance majors attract higher salary offers than marketing majors?

13.68 Discuss why the experiment in Example 13.5 produced a significant test result whereas the one in Exercise 13.67 did not.

13.69 Xr13-69 Refer to Example 13.2.The actual after and before operating incomes were recorded.

a. Test to determine whether there is enough evidence to infer that for companies where an offspring takes the helm there is a decrease in operating income.

b. Is there sufficient evidence to conclude that when an outsider becomes CEO the operating income increases?

General Social Survey Exercises

Warning: Some rows contain blanks representing missing data.

13.70 GSS2008* The general trend over the last century is that each generation is more educated that its predecessor. Has this trend continued? To answer this question, determine whether there is sufficient evidence that Americans are more educated than their fathers (EDUC and PAEDUC).

13.71 GSS2008* Is there sufficient evidence to infer that Americans are more educated than their mothers (EDUC and MAEDUC)?

13.72 GSS2008* If it is true that this generation is more educated than its parents, does it follow that its members have more prestigious occupations? To help answer this question, conduct a statistical procedure to determine whether adults today have more prestigious jobs than their fathers (PRESTG80 and PAPRES80).

American National Election Survey Exercise

Warning: Some rows contain blanks representing missing data.

13.73 ANES2008* Estimate with 95% confidence the average difference between the amount of time spent watching news on television (not including sports) and the amount of time spent reading news in a printed newspaper during a typical day (TIME2 and TIME3).

13.4 INFERENCE ABOUT THE RATIO OF TWO VARIANCES

In Sections 13.1 and 13.3, we dealt with statistical inference concerning the difference between two population means. The problem objective in each case was to compare two populations of interval data, and our interest was in comparing measures of central location. This section discusses the statistical technique to use when the problem objective and the data type are the same as in Sections 13.1 and 13.3, but our interest is in comparing variability. Here we will study the ratio of two population variances. We make inferences about the ratio because the sampling distribution is based on ratios rather than differences.

We have already encountered this technique when we used the F-test of two variances to determine which t-test and estimator of the difference between two means to use. In this section, we apply the technique to other problems where our interest is in comparing the variability in two populations.

In the previous chapter, we presented the procedures used to draw inferences about a single population variance. We pointed out that variance can be used to address problems where we need to judge the consistency of a production process. We also use variance to measure the risk associated with a portfolio of investments. In this section, we compare two variances, enabling us to compare the consistency of two production processes. We can also compare the relative risks of two sets of investments.

We will proceed in a manner that is probably becoming quite familiar.

Parameter

As you will see shortly, we compare two population variances by determining the ratio. Consequently, the parameter is σ_1^2/σ_2^2.

Statistic and Sampling Distribution

We have previously noted that the sample variance (defined in Chapter 4) is an unbiased and consistent estimator of the population variance. Not surprisingly, the estimator of the parameter σ_1^2/σ_2^2 is the ratio of the two sample variances drawn from their respective populations s_1^2/s_2^2.

The sampling distribution of s_1^2/s_2^2 is said to be F-distributed provided that we have independently sampled from two normal populations. (The F-distribution was introduced in Section 8.4.)

Statisticians have shown that the ratio of two independent chi-squared variables divided by their degrees of freedom is F-distributed. The degrees of freedom of the F-distribution are identical to the degrees of freedom for the two chi-squared distributions. In Section 12.2, we pointed out that $(n - 1)\, s^2 / \sigma^2$ is chi-squared distributed, provided that the sampled population is normal. If we have independent samples drawn from two normal populations, then both $(n_1 - 1)s_1^2/\sigma_1^2$ and $(n_2 - 1)s_2^2/\sigma_2^2$ are chi-squared distributed. If we divide each by their respective number of degrees of freedom and take the ratio, we produce

$$\frac{\dfrac{(n_1 - 1)s_1^2/\sigma_1^2}{(n_1 - 1)}}{\dfrac{(n_2 - 1)s_2^2/\sigma_2^2}{(n_2 - 1)}}$$

which simplifies to

$$\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2}$$

This statistic is F-distributed with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom. Recall that ν_1 is called the **numerator degrees of freedom** and ν_2 is called the **denominator degrees of freedom**.

Testing and Estimating a Ratio of Two Variances

In this book, our null hypothesis will always specify that the two variances are equal. As a result, the ratio will equal 1. Thus, the null hypothesis will always be expressed as

$$H_0: \quad \sigma_1^2/\sigma_2^2 = 1$$

The alternative hypothesis can state that the ratio σ_1^2/σ_2^2 is either not equal to 1, greater than 1, or less than 1. Technically, the test statistic is

$$F = \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2}$$

However, under the null hypothesis, which states that $\sigma_1^2/\sigma_2^2 = 1$, the test statistic becomes as follows.

Test Statistic for σ_1^2/σ_2^2

The test statistic employed to test that σ_1^2/σ_2^2 is equal to 1 is

$$F = \frac{s_1^2}{s_2^2}$$

which is F-distributed with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom provided that the populations are normal.

With the usual algebraic manipulation, we can derive the confidence interval estimator of the ratio of two population variances.

Confidence Interval Estimator of σ_1^2/σ_2^2

$$\text{LCL} = \left(\frac{s_1^2}{s_2^2}\right)\frac{1}{F_{\alpha/2,\nu_1,\nu_2}}$$

$$\text{UCL} = \left(\frac{s_1^2}{s_2^2}\right)F_{\alpha/2,\nu_2,\nu_1}$$

where $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$

EXAMPLE 13.7 Testing the Quality of Two-Bottle Filling Machines

DATA Xm13-07

In Example 12.3, we applied the chi-squared test of a variance to determine whether there was sufficient evidence to conclude that the population variance was less than 1.0. Suppose that the statistics practitioner also collected data from another container-filling machine and recorded the fills of a randomly selected sample. Can we infer at the 5% significance level that the second machine is superior in its consistency?

SOLUTION

IDENTIFY

The problem objective is to compare two populations where the data are interval. Because we want information about the consistency of the two machines, the parameter we wish to test is σ_1^2/σ_2^2, where σ_1^2 is the variance of machine 1 and σ_2^2 is the variance for machine 2. We need to conduct the F-test of σ_1^2/σ_2^2 to determine whether the variance of population 2 is less than that of population 1. Expressed differently, we wish to determine whether there is enough evidence to infer that σ_1^2 is larger than σ_2^2. Hence, the hypotheses we test are

$$H_0: \quad \sigma_1^2/\sigma_2^2 = 1$$

$$H_1: \quad \sigma_1^2/\sigma_2^2 > 1$$

COMPUTE

MANUALLY

The sample variances are $s_1^2 = .6333$ and $s_2^2 = .4528$.

The value of the test statistic is

$$F = \frac{s_1^2}{s_2^2} = \frac{.6333}{.4528} = 1.40$$

The rejection region is

$$F > F_{\alpha,\nu_1,\nu_2} = F_{.05,24,24} = 1.98$$

Because the value of the test statistic is not greater than 1.98, we cannot reject the null hypothesis.

EXCEL

	A	B	C
1	F-Test Two-Sample for Variances		
2			
3		*Machine 1*	*Machine 2*
4	Mean	999.7	999.8
5	Variance	0.6333	0.4528
6	Observations	25	25
7	df	24	24
8	F	1.3988	
9	P(F<=f) one-tail	0.2085	
10	F Critical one-tail	1.9838	

The value of the test statistic is $F = 1.3988$. Excel outputs the one-tail p-value, which is .2085.

(Continued)

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm13-07.)
2. Click **Data, Data Analysis**, and ***F*-test Two-Sample for Variances**.
3. Specify the **Variable 1 Range** (A1:A26) and the **Variable 2 Range** (B1:B26). Type a value for α (.05).

MINITAB

Test for Equal Variances: Machine 1, Machine 2

F-Test (Normal Distribution)
Test statistic = 1.40, p-value = 0.417

Note that Minitab conducts a two-tail test only. Thus, the p-value $= .417/2 = .2085$.

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm13-07.)
2. Click **Stat, Basic Statistics,** and **2 Variances . . . .**
3. In the **Samples in different columns** box, select the **First** (Machine 1) and **Second** (Machine 2) variables.

INTERPRET

There is not enough evidence to infer that the variance of machine 2 is less than the variance of machine 1.

The histograms (not shown) appear to be sufficiently bell shaped to satisfy the normality requirement.

EXAMPLE 13.8 Estimating the Ratio of the Variances in Example 13.7

DATA
Xm13-07

Determine the 95% confidence interval estimate of the ratio of the two population variances in Example 13.7.

SOLUTION

COMPUTE

MANUALLY

We find

$$F_{\alpha/2,\nu_1,\nu_2} = F_{.025,24,24} = 2.27$$

Thus,

$$\text{LCL} = \left(\frac{s_1^2}{s_2^2}\right)\frac{1}{F_{\alpha/2,\nu_1,\nu_2}} = \left(\frac{.6333}{.4528}\right)\frac{1}{2.27} = .616$$

$$\text{UCL} = \left(\frac{s_1^2}{s_2^2}\right)F_{\alpha/2,\nu_2,\nu_1} = \left(\frac{.6333}{.4528}\right)2.27 = 3.17$$

We estimate that σ_1^2/σ_2^2 lies between .616 and 3.17.

EXCEL

	A	B	C
1	F-Estimate : Two Variances		
2			
3		*Machine 1*	*Machine 2*
4	Mean	999.7	999.8
5	Variance	0.6333	0.4528
6	Observations	25	25
7	df	24	24
8	LCL	0.6164	
9	UCL	3.1743	

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm13-07.)
2. Click **Add-ins, Data Analysis Plus,** and **F Estimate 2 Variances**.
3. Specify the **Variable 1 Range** (A1:A26) and the **Variable 2 Range** (B1:B26). Type a value for α (.05).

MINITAB

Minitab does not compute the estimate of the ratio of two variances.

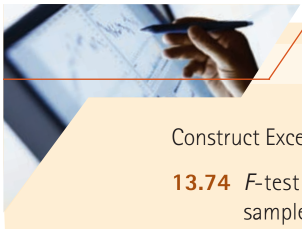
INTERPRET

As we pointed out in Chapter 11, we can often use a confidence interval estimator to test hypotheses. In this example, the interval estimate excludes the value of 1. Consequently, we can draw the same conclusion as we did in Example 13.7.

Factors That Identify the *F*-Test and Estimator of σ_1^2/σ_2^2

1. **Problem objective**: Compare two populations
2. **Data type**: Interval
3. **Descriptive measurement**: Variability

EXERCISES



DO-IT-YOURSELF EXCEL

Construct Excel spreadsheets for each of the following:

13.74 F-test of σ_1^2/σ_2^2. Inputs: Sample variances, sample sizes, and hypothesized ratio of population variances. Outputs: Test statistic, critical values, and one- and two-tail p-values. Tools: **FINV, FDIST**

13.74 F-estimator of σ_1^2/σ_2^2. Inputs: Sample variances, sample sizes, and confidence level. Outputs: Upper and lower confidence limits, Tools: **FINV**

Developing an Understanding of Statistical Concepts

Exercises 13.76 and 13.77 are "what-if" analyses designed to determine what happens to the test statistics and interval estimates when elements of the statistical inference change. These problems can be solved manually, using Do-It-Yourself Excel spreadsheets you just created, or Minitab.

13.76 Random samples from two normal populations produced the following statistics:

$$s_1^2 = 350 \quad n_1 = 30 \quad s_2^2 = 700 \quad n_2 = 30$$

a. Can we infer at the 10% significance level that the two population variances differ?
b. Repeat part (a) changing the sample sizes to $n_1 = 15$ and $n_2 = 15$.
c. Describe what happens to the test statistic and the conclusion when the sample sizes decrease.

13.77 Random samples from two normal populations produced the following statistics:

$$s_1^2 = 28 \quad n_1 = 10 \quad s_2^2 = 19 \quad n_2 = 10$$

a. Estimate with 95% confidence the ratio of the two population variances.
b. Repeat part (a) changing the sample sizes to $n_1 = 25$ and $n_2 = 25$.
c. Describe what happens to the width of the confidence interval estimate when the sample sizes increase.

Applications

Use a 5% significance level in all tests unless specified otherwise.

13.78 Xr13-78 The manager of a dairy is in the process of deciding which of two new carton-filling machines to use. The most important attribute is the consistency of the fills. In a preliminary study, she measured the fills in the 1-liter carton and listed them here. Can the manager infer that the two machines differ in their consistency of fills?

Machine 1	.998	.997	1.003	1.000	.999	
	1.000	.998	1.003	1.004	1.000	
Machine 2	1.003	1.004	.997	.996	.999	1.003
	1.000	1.005	1.002	1.004	.996	

13.79 Xr13-79 An operations manager who supervises an assembly line has been experiencing problems with the sequencing of jobs. The problem is that bottlenecks are occurring because of the inconsistency of sequential operations. He decides to conduct an experiment wherein two different methods are used to complete the same task. He measures the times (in seconds). The data are listed here. Can he infer that the second method is more consistent than the first method?

Method 1	8.8	9.6	8.4	9.0	8.3	9.2	9.0	8.7	8.5	9.4
Method 2	9.2	9.4	8.9	9.6	9.7	8.4	8.8	8.9	9.0	9.7

13.80 Xr13-80 A statistics professor hypothesized that not only would the means vary but also so would the variances if the business statistics course was taught in two different ways but had the same final exam. He organized an experiment wherein one section of the course was taught using detailed PowerPoint slides whereas the other required students to read the book and answer questions in class discussions. A sample of the marks was recorded and listed next. Can we infer that the variances of the marks differ between the two sections?

Class 1	64	85	80	64	48	62	75	77	50	81	90
Class 2	73	78	66	69	79	81	74	59	83	79	84

The following exercises require the use of a computer and software. The answers may be calculated manually. See Appendix A for the sample statistics.

13.81 Xr13-81 A new highway has just been completed, and the government must decide on speed limits. There are several possible choices. However, on advice from police who monitor traffic, the objective was to reduce the variation in speeds, which it is thought to contribute to the number of collisions. It has been acknowledged that speed contributes to the severity of collisions. It is decided to conduct an experiment to acquire more information. Signs are posted for 1 week indicating that the speed limit is 70 mph. A random sample of cars' speeds is measured. During the second week, signs are posted indicating that the maximum speed is 70 mph and that the minimum speed is 60 mph. Once again a random sample of speeds is measured. Can we infer that limiting the minimum and maximum speeds reduces the variation in speeds?

13.82 Xr13-82 In Exercise 12.66, we described the problem of whether to change all the lightbulbs at Yankee Stadium or change them one by one as they burn out. There are two brands of bulbs that can be used. Because both the mean and the variance of the lengths of life are important, it was decided to test the two brands. A random sample of both brands was drawn and left on until they burned out. The times were recorded. Can the Yankee Stadium management conclude that the variances differ?

13.83 Xr13-83 In deciding where to invest her retirement fund, an investor recorded the weekly returns of two portfolios for 1 year. Can we conclude that portfolio 2 is riskier than portfolio 1?

13.84 Xr13-84 An important statistical measurement in service facilities (such as restaurants and banks) is the variability in service times. As an experiment, two bank tellers were observed, and the service times for each of 100 customers were recorded. Do these data allow us to infer at the 10% significance level that the variance in service times differs between the two tellers?

13.5 Inference about the Difference between Two Population Proportions

In this section, we present the procedures for drawing inferences about the difference between populations whose data are nominal. The number of applications of these techniques is almost limitless. For example, pharmaceutical companies test new drugs by comparing the new and old or the new versus a placebo. Marketing managers compare market shares before and after advertising campaigns. Operations managers compare defective rates between two machines. Political pollsters measure the difference in popularity before and after an election.

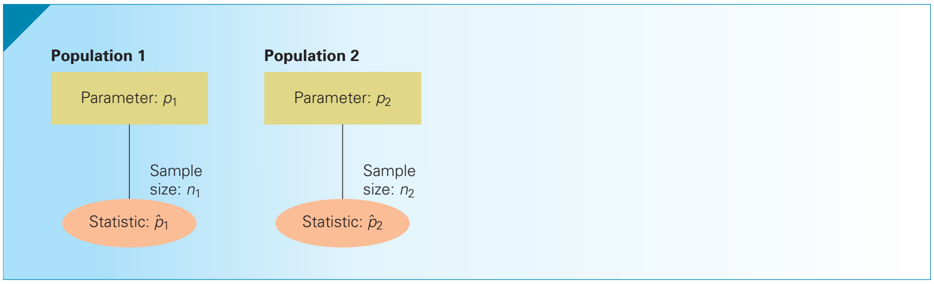
Parameter

When data are nominal, the only meaningful computation is to count the number of occurrences of each type of outcome and calculate proportions. Consequently, the parameter to be tested and estimated in this section is the difference between two population proportions $p_1 - p_2$.

Statistic and Sampling Distribution

To draw inferences about $p_1 - p_2$, we take a sample of size n_1 from population 1 and a sample of size n_2 from population 2 (Figure 13.7 depicts the sampling process).

FIGURE **13.7** Sampling From Two Populations of Nominal Data



For each sample, we count the number of successes (recall that we call anything we're looking for a success), which we label x_1 and x_2, respectively. The sample proportions are then computed:

$$\hat{p}_1 = \frac{x_1}{n_1} \quad \text{and} \quad \hat{p}_2 = \frac{x_2}{n_2}$$

Statisticians have proven that the statistic $\hat{p}_1 - \hat{p}_2$ is an unbiased consistent estimator of the parameter $p_1 - p_2$. Using the same mathematics as we did in Chapter 9 to derive the sampling distribution of the sample proportion $\hat{p}$, we determine the sampling distribution of the difference between two sample proportions.

Sampling Distribution of $\hat{p}_1 - \hat{p}_2$

1. The statistic $\hat{p}_1 - \hat{p}_2$ is approximately normally distributed provided that the sample sizes are large enough so that n_1p_1, $n_1(1 - p_1)$, n_2p_2, and $n_2(1 - p_2)$ are all greater than or equal to 5. [Because p_1 and p_2 are unknown, we express the sample size requirement as $n_1\hat{p}_1$, $n_1(1 - \hat{p}_1)$, $n_2\hat{p}_2$, and $n_2(1 - \hat{p}_2)$ are greater than or equal to 5.]
2. The mean of $\hat{p}_1 - \hat{p}_2$ is

$$E(\hat{p}_1 - \hat{p}_2) = p_1 - p_2$$

3. The variance of $\hat{p}_1 - \hat{p}_2$ is

$$V(\hat{p}_1 - \hat{p}_2) = \frac{p_1(1 - p_1)}{n_1} + \frac{p_2(1 - p_2)}{n_2}$$

The standard error is

$$\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{p_1(1 - p_1)}{n_1} + \frac{p_2(1 - p_2)}{n_2}}$$

Thus, the variable

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\dfrac{p_1(1 - p_1)}{n_1} + \dfrac{p_2(1 - p_2)}{n_2}}}$$

is approximately standard normally distributed.

Testing and Estimating the Difference between Two Proportions

We would like to use the z-statistic just described as our test statistic; however, the standard error of $\hat{p}_1 - \hat{p}_2$, which is

$$\sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

is unknown because both p_1 and p_2 are unknown. As a result, the standard error of $\hat{p}_1 - \hat{p}_2$ must be estimated from the sample data. There are two different estimators of this quantity, and the determination of which one to use depends on the null hypothesis. If the null hypothesis states that $p_1 - p_2 = 0$, the hypothesized equality of the two population proportions allows us to pool the data from the two samples to produce an estimate of the common value of the two proportions p_1 and p_2. The **pooled proportion estimate** is defined as

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

Thus, the estimated standard error of $\hat{p}_1 - \hat{p}_2$ is

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n_1} + \frac{\hat{p}(1-\hat{p})}{n_2}} = \sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

The principle used in estimating the standard error of $\hat{p}_1 - \hat{p}_2$ is analogous to that applied in Section 13.1 to produce the pooled variance estimate s_p^2, which is used to test $\mu_1 - \mu_2$ with σ_1^2 and σ_2^2 unknown but equal. The principle roughly states that, where possible, pooling data from two samples produces a better estimate of the standard error. Here, pooling is made possible by hypothesizing (under the null hypothesis) that $p_1 = p_2$. (In Section 13.1, we used the pooled variance estimate because we assumed that $\sigma_1^2 = \sigma_2^2$.) We will call this application Case 1.

Test Statistic for $p_1 - p_2$: Case 1

If the null hypothesis specifies

$$H_0: \ (p_1 - p_2) = 0$$

the test statistic is

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

Because we hypothesize that $p_1 - p_2 = 0$, we simplify the test statistic to

$$z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

The second case applies when, under the null hypothesis, we state that $p_1 - p_2 = D$, where D is some value other than 0. Under such circumstances, we cannot pool the sample data to estimate the standard error of $\hat{p}_1 - \hat{p}_2$. The appropriate test statistic is described next as Case 2.

Test Statistic for $p_1 - p_2$: Case 2

If the null hypothesis specifies

$$H_0: \ (p_1 - p_2) = D \quad (D \neq 0)$$

the test statistic is

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\dfrac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \dfrac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}}$$

which can also be expressed as

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - D}{\sqrt{\dfrac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \dfrac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}}$$

Notice that this test statistic is determined by simply substituting the sample statistics $\hat{p}_1$ and $\hat{p}_2$ in the standard error of $\hat{p}_1 - \hat{p}_2$.

You will find that, in most practical applications (including the exercises in this book), Case 1 applies—in most problems, we want to know whether the two population proportions differ: that is,

$$H_1: \ (p_1 - p_2) \neq 0$$

or if one proportion exceeds the other; that is,

$$H_1: \ (p_1 - p_2) > 0 \qquad \text{or} \qquad H_1: \ (p_1 - p_2) < 0$$

In some other problems, however, the objective is to determine whether one proportion exceeds the other by a specific nonzero quantity. In such situations, Case 2 applies.

We derive the interval estimator of $p_1 - p_2$ in the same manner we have been using since Chapter 10.

Confidence Interval Estimator of $p_1 - p_2$

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}$$

This formula is valid when $n_1\hat{p}_1$, $n_1(1 - \hat{p}_1)$, $n_2\hat{p}_2$, and $n_2(1 - \hat{p}_2)$ are greater than or equal to 5.

Notice that the standard error is estimated using the individual sample proportions rather than the pooled proportion. In this procedure we cannot assume that the population proportions are equal as we did in the Case 1 test statistic.

APPLICATIONS in MARKETING

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Test Marketing

Marketing managers frequently make use of test marketing to assess consumer reaction to a change in a characteristic (such as price or packaging) of an existing product, or to assess consumers' preferences regarding a proposed new product. *Test marketing* involves experimenting with changes to the marketing mix in a small, limited test market and assessing consumers' reaction in the test market before undertaking costly changes in production and distribution for the entire market.

EXAMPLE 13.9 Test Marketing of Package Designs, Part 1

DATA Xm13-09

The General Products Company produces and sells a variety of household products. Because of stiff competition, one of its products, a bath soap, is not selling well. Hoping to improve sales, General Products decided to introduce more attractive packaging. The company's advertising agency developed two new designs. The first design features several bright colors to distinguish it from other brands. The second design is light green in color with just the company's logo on it. As a test to determine which design is better, the marketing manager selected two supermarkets. In one supermarket, the soap was packaged in a box using the first design; in the second supermarket, the second design was used. The product scanner at each supermarket tracked every buyer of soap over a 1-week period. The supermarkets recorded the last four digits of the scanner code for each of the five brands of soap the supermarket sold. The code for the General Products brand of soap is 9077 (the other codes are 4255, 3745, 7118, and 8855). After the trial period, the scanner data were transferred to a computer file. Because the first design is more expensive, management has decided to use this design only if there is sufficient evidence to allow it to conclude that design is better. Should management switch to the brightly colored design or the simple green one?

SOLUTION

IDENTIFY

The problem objective is to compare two populations. The first is the population of soap sales in supermarket 1, and the second is the population of soap sales in supermarket 2. The data are nominal because the values are "buy General Products soap" and "buy other companies' soap." These two factors tell us that the parameter to be tested is the difference between two population proportions $p_1 - p_2$ (where p_1 and p_2 are the proportions of soap sales that are a General Products brand in supermarkets 1 and 2, respectively). Because we want to know whether there is enough evidence to adopt the brightly colored design, the alternative hypothesis is

$$H_1: \ (p_1 - p_2) > 0$$

The null hypothesis must be

$$H_0: \ (p_1 - p_2) = 0$$

which tells us that this is an application of Case 1. Thus, the test statistic is

$$z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

COMPUTE

MANUALLY

To compute the test statistic manually requires the statistics practitioner to tally the number of successes in each sample, where success is represented by the code 9077. Reviewing all the sales reveals that

$$x_1 = 180 \qquad n_1 = 904 \qquad x_2 = 155 \qquad n_2 = 1{,}038$$

The sample proportions are

$$\hat{p}_1 = \frac{180}{904} = .1991$$

and

$$\hat{p}_2 = \frac{155}{1{,}038} = .1493$$

The pooled proportion is

$$\hat{p} = \frac{180 + 155}{904 + 1{,}038} = \frac{335}{1{,}942} = .1725$$

The value of the test statistic is

$$z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{(.1991 - .1493)}{\sqrt{(.1725)(1 - .1725)\left(\frac{1}{904} + \frac{1}{1{,}038}\right)}} = 2.90$$

A 5% significance level seems to be appropriate. Thus, the rejection region is

$$z > z_\alpha = z_{.05} = 1.645$$

EXCEL

	A	B	C	D
1	**z-Test: Two Proportions**			
2				
3			*Supermarket 1*	*Supermarket 2*
4	Sample Proportions		0.1991	0.1493
5	Observations		904	1038
6	Hypothesized Difference		0	
7	z Stat		2.90	
8	P(Z<=z) one tail		0.0019	
9	z Critical one-tail		1.6449	
10	P(Z<=z) two-tail		0.0038	
11	z Critical two-tail		1.96	

INSTRUCTIONS

1. Type or import the data into two adjacent columns*. (Open Xm13-09.)
2. Click **Add-Ins, Data Analysis Plus,** and **Z-Test: 2 Proportions**.
3. Specify the **Variable 1 Range** (A1:A905) and the **Variable 2 Range** (B1:B1039). Type the **Code for Success** (9077), the **Hypothesized Difference** (0), and a value for α (.05).

*If one or both columns contain a blank (representing missing data) the row will have to be deleted.

MINITAB

Test and CI for Two Proportions: Supermarket 1, Supermarket 2

Event = 9077

Variable	X	N	Sample p
Supermarket 1	180	904	0.199115
Supermarket 2	155	1038	0.149326

Difference = p (Supermarket 1) – p (Supermarket 2)
Estimate for difference: 0.0497894
95% lower bound for difference: 0.0213577
Test for difference = 0 (vs > 0): Z = 2.90 P-Value = 0.002

INSTRUCTIONS

1. Type or import the data into two adjacent columns. (Open Xm13-09.) Recode the data if necessary. (Minitab requires that there be only two codes and the higher value is deemed to be a success. See Keller's website Appendix Excel and Minitab Instructions for Missing Data and Recoding data.)
2. Click **Stat, Basic Statistics,** and **2 Proportions . . . .**
3. In the **Samples in different columns** specify the **First** (Supermarket 1) and **Second** (Supermarket 2) samples. Click **Options . . . .**
4. Type the value of the **Test difference** (0), specify the **Alternative** hypothesis (greater than), and click **Use pooled estimate of *p* for test**.

Warning: If there are asterisks representing missing data, Minitab will be unable to conduct either the test or the estimate of the difference between two proportions. Click **Data** and **Sort**, which will eliminate the asterisks.

INTERPRET

The value of the test statistic is $z = 2.90$; its p-value is .0019. There is enough evidence to infer that the brightly colored design is more popular than the simple design. As a result, it is recommended that management switch to the first design.

EXAMPLE 13.10 Test Marketing of Package Designs, Part 2

DATA
Xm13-09

Suppose that in Example 13.9 the additional cost of the brightly colored design requires that it outsell the simple design by more than 3%. Should management switch to the brightly colored design?

SOLUTION

IDENTIFY

The alternative hypothesis is

$$H_1: \quad (p_1 - p_2) > .03$$

and the null hypothesis follows as

$$H_0: \ (p_1 - p_2) = .03$$

Because the null hypothesis specifies a nonzero difference, we would apply the Case 2 test statistic.

COMPUTE

MANUALLY

The value of the test statistic is

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\dfrac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \dfrac{\hat{p}_2(1-\hat{p}_2)}{n_2}}} = \frac{(.1991 - .1493) - (.03)}{\sqrt{\dfrac{.1991(1 - .1991)}{904} + \dfrac{.1493(1 - .1493)}{1{,}038}}} = 1.15$$

EXCEL

	A	B	C	D
1	**z-Test: Two Proportions**			
2				
3			*Supermarket 1*	*Supermarket 2*
4	Sample Proportions		0.1991	0.1493
5	Observations		904	1038
6	Hypothesized Difference		0.03	
7	z Stat		1.14	
8	P(Z<=z) one tail		0.1261	
9	z Critical one-tail		1.6449	
10	P(Z<=z) two-tail		0.2522	
11	z Critical two-tail		1.96	

INSTRUCTIONS

Use the same commands we used previously, except specify that the **Hypothesized Difference** is .03. Excel will apply the Case 2 test statistic when a nonzero value is typed.

MINITAB

Test and CI for Two Proportions: Supermarket 1, Supermarket 2

Event = 9077

Variable	X	N	Sample p
Supermarket 1	180	904	0.199115
Supermarket 2	155	1038	0.149326

Difference = p (Supermarket 1) – p (Supermarket 2)
Estimate for difference: 0.0497894
95% lower bound for difference: 0.0213577
Test for difference = 0.03 (vs > 0.03): Z = 1.14 P-Value = 0.126

INSTRUCTIONS

Use the same commands detailed previously except at step 4, specify that the **Test difference** is .03 and do not click **Use pooled estimate of p for test**.

INTERPRET

There is not enough evidence to infer that the proportion of soap customers who buy the product with the brightly colored design is more than 3% higher than the proportion of soap customers who buy the product with the simple design. In the absence of sufficient evidence, the analysis suggests that the product should be packaged using the simple design.

EXAMPLE 13.11 Test Marketing of Package Designs, Part 3

DATA
Xm13-09

To help estimate the difference in profitability, the marketing manager in Examples 13.9 and 13.10 would like to estimate the difference between the two proportions. A confidence level of 95% is suggested.

SOLUTION

IDENTIFY

The parameter is $p_1 - p_2$, which is estimated by the following confidence interval estimator:

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

COMPUTE

MANUALLY

The sample proportions have already been computed. They are

$$\hat{p}_1 = \frac{180}{904} = .1991$$

and

$$\hat{p}_2 = \frac{155}{1038} = .1493$$

The 95% confidence interval estimate of $p_1 - p_2$ is

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$= (.1991 - .1493) \pm 1.96\sqrt{\frac{.1991(1-.1991)}{904} + \frac{.1493(1-.1493)}{1{,}038}}$$

$$= .0498 \pm .0339$$

$$\text{LCL} = .0159 \quad \text{and} \quad \text{UCL} = .0837$$

EXCEL

	A	B	C	D
1	z-Estimate: Two Proportions			
2				
3			*Supermarket 1*	*Supermarket 2*
4	Sample Proportions		0.1991	0.1493
5	Observations		904	1038
6				
7	LCL		0.0159	
8	UCL		0.0837	

INSTRUCTIONS

1. Type or import the data into two adjacent columns*. (Open Xm13-09.)
2. Click **Add-Ins, Data Analysis Plus,** and **Z-Estimate: 2 Proportions**.
3. Specify the **Variable 1 Range** (A1:A905) and the **Variable 2 Range** (B1:B1039). Specify the **Code for Success** (9077) and a value for α (.05).

MINITAB

Test and CI for Two Proportions: Supermarket 1, Supermarket 2

Event = 9077

Variable	X	N	Sample p
Supermarket 1	180	904	0.199115
Supermarket 2	155	1038	0.149326

Difference = p (Supermarket 1) – p (Supermarket 2)
Estimate for difference: 0.0497894
95% CI for difference: (0.0159109, 0.0836679)
Test for difference = 0 (vs not = 0): Z = 2.88 P-Value = 0.004

INSTRUCTIONS

Follow the commands to test hypotheses about two proportions. Specify the alternative hypothesis as **not equal** and do not click **Use pooled estimate of *p* for test**.

INTERPRET

We estimate that the market share for the brightly colored design is between 1.59% and 8.37% larger than the market share for the simple design.

*If one or both columns contain a blank (representing missing data) the row must be deleted.

American National Election Survey

Comparing Democrats and Republicans: Who Is More Educated?

The problem objective is to compare two populations (Democrats and Republicans). The data are nominal. We've recoded the data so that all categories greater than 0 are represented by 2, which will be our definition of success. The parameter is $p_1 - p_2$, where p_1 = proportion of Democrats with at least a bachelor's degree and p_2 = proportion of Republicans with at least a bachelor's degree. The hypotheses are

$$H_0: \quad (p_1 - p_2) = 0$$
$$H_1: \quad (p_1 - p_2) < 0$$

The null hypothesis tells us that this is an application of Case 1. Thus, the test statistic is

$$z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

EXCEL

	A	B	C	D
1	**z-Test: Two Proportions**			
2				
3			*Democrats*	*Republicans*
4	Sample Proportions		0.6246	0.7085
5	Observations		341	271
6	Hypothesized Difference		0	
7	z Stat		−2.18	
8	P(Z<=z) one tail		0.0147	
9	z Critical one-tail		1.6449	
10	P(Z<=z) two-tail		0.0294	
11	z Critical two-tail		1.96	

We copied the variables DEGREE and PARTY into a new spreadsheet and sorted the two columns by party. We collected the data for code 1 (Democrats) and code 2 (Republicans), recoded the data, and conducted the z-test of $p_1 - p_2$, using 2 as a success.

MINITAB

Test and CI for Two Proportions: Dem, Rep

Event = 2

Variable	X	N	Sample p
Dem	213	341	0.624633
Rep	192	271	0.708487

Difference = p (Dem) – p (Rep)
Estimate for difference: – 0.0838537
95% lower bound for difference: – 0.0212260
Test for difference = 0 (vs < 0): Z = –2.18 P-Value = 0.015

We copied the variables DEGREE and PARTY into new columns and sorted the two columns by party. We collected the data for code 1 (Democrats) and code 2 (Republicans). We then sorted each column to remove the asterisks. Finally, we coded the data so that codes 1 to 7 became 2 and 0 remained 0. We then conducted the z-test of $p_1 - p_2$.

INTERPRET

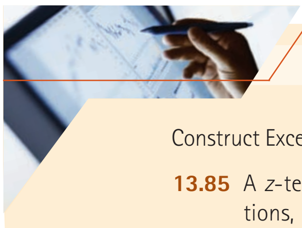
There is sufficient evidence to infer that the proportion of Republicans with at least a bachelor's degree is greater than the proportion of Democrats with at least a bachelor's degree. The popular perception (judging from the media, some politicians, and some comedians) that Democrats are more educated than Republicans is not supported by these data. At the end of this section, you will have the opportunity to test this perception again.

The factors that identify the inference about the difference between two proportions are listed below.

Factors That Identify the z-Test and Estimator of $p_1 - p_2$

1. **Problem objective**: Compare two populations
2. **Data type**: Nominal

EXERCISES



DO-IT-YOURSELF EXCEL

Construct Excel spreadsheets for each of the following:

13.85 A z-test of $p_1 - p_2$. Inputs: Sample proportions, sample sizes, and hypothesized difference between two populations. Outputs: Test statistic, critical values, and one- and two-tail p-values. Tools: **NORMSINV, NORMSDIST**

13.85 A z-estimate of $p_1 - p_2$. Inputs: Sample proportions, sample sizes, and confidence level. Outputs: Test statistic, one- and two-tail p-values. Tools: **NORMSINV**

Developing an Understanding of Statistical Concepts

Exercises 13.87 to 13.89 are "what-if" analyses designed to determine what happens to the test statistics and interval estimates when elements of the statistical inference change. These problems can be solved manually, using Do-It-Yourself Excel spreadsheets you created, or using Minitab.

13.87 Random samples from two binomial populations yielded the following statistics:

$$\hat{p}_1 = .45 \quad n_1 = 100 \quad \hat{p}_2 = .40 \quad n_2 = 100$$

a. Calculate the p-value of a test to determine whether we can infer that the population proportions differ.
b. Repeat part (a) increasing the sample sizes to 400.
c. Describe what happens to the p-value when the sample sizes increase.

13.88 These statistics were calculated from two random samples:

$$\hat{p}_1 = .60 \quad n_1 = 225 \quad \hat{p}_2 = .55 \quad n_2 = 225$$

a. Calculate the p-value of a test to determine whether there is evidence to infer that the population proportions differ.
b. Repeat part (a) with $\hat{p}_1 = .95$ and $\hat{p}_2 = .90$.
c. Describe the effect on the p-value of increasing the sample proportions.

d. Repeat part (a) with $\hat{p}_1 = .10$ and $\hat{p}_2 = .05$.
e. Describe the effect on the p-value of decreasing the sample proportions.

13.89 After sampling from two binomial populations we found the following.

$\hat{p}_1 = .18 \quad n_1 = 100 \quad \hat{p}_2 = .22 \quad n_2 = 100$

a. Estimate with 90% confidence the difference in population proportions.
b. Repeat part (a) increasing the sample proportions to .48 and .52, respectively.
c. Describe the effects of increasing the sample proportions.

Applications

13.90 Many stores sell extended warranties for products they sell. These are very lucrative for store owners. To learn more about who buys these warranties, a random sample was drawn of a store's customers who recently purchased a product for which an extended warranty was available. Among other variables, each respondent reported whether he or she paid the regular price or a sale price and whether he or she purchased an extended warranty.

	Regular Price	Sale Price
Sample size	229	178
Number who bought extended warranty	47	25

Can we conclude at the 10% significance level that those who paid the regular price are more likely to buy an extended warranty?

13.91 A firm has classified its customers in two ways: (1) according to whether the account is overdue and (2) whether the account is new (less than 12 months) or old. To acquire information about which customers are paying on time and which are overdue, a random sample of 292 customer accounts was drawn. Each was categorized as either a new account or an old account, and whether the customer has paid or is overdue. The results are summarized next.

	New Account	Old Account
Sample size	83	209
Overdue account	12	49

Is there enough evidence at the 5% significance level to infer that new and old accounts are different with respect to overdue accounts?

13.92 Credit scorecards are used by financial institutions to help decide to whom loans should be granted (see the Applications in Banking: Credit Scorecards summary on page 63). An analysis of the records of a random sample of loans at one bank produced the following results:

	Score Below 600	Score 600 or More
Sample size	562	804
Number defaulted	11	7

Do these results allow us to conclude that those who score below 600 are more likely to default than those who score 600 or more? Use a 10% significance level.

13.93 Surveys have been widely used by politicians around the world as a way of monitoring the opinions of the electorate. Six months ago, a survey was undertaken to determine the degree of support for a national party leader. Of a sample of 1,100, 56% indicated that they would vote for this politician. This month, another survey of 800 voters revealed that 46% now support the leader.
a. At the 5% significance level, can we infer that the national leader's popularity has decreased?
b. At the 5% significance level, can we infer that the national leader's popularity has decreased by more than 5%?
c. Estimate with 95% confidence the decrease in percentage support between now and 6 months ago.

13.94 The process that is used to produce a complex component used in medical instruments typically results in defective rates in the 40% range. Recently, two innovative processes have been developed to replace the existing process. Process 1 appears to be more promising, but it is considerably more expensive to purchase and operate than process 2. After a thorough analysis of the costs, management decides that it will adopt process 1 only if the proportion of defective components it produces is more than 8% smaller than that produced by process 2. In a test to guide the decision, both processes were used to produce 300 components. Of the 300 components produced by process 1, 33 were found to be defective, whereas 84 out of the 300 produced by process 2 were defective. Conduct a test using a significance level of 1% to help management make a decision.

APPLICATIONS in OPERATIONS MANAGEMENT

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Pharmaceutical and Medical Experiments

When new products are developed, they are tested in several ways. First, does the new product work? Second, is it better than the existing product? Third, will customers buy it at a price that is profitable? Performing a customer survey or some other experiment that yields the information needed often tests the last question. This experiment is usually the domain of the marketing manager.

The other two questions are dealt with by the developers of the new product, which usually means the research department or the operations manager. When the product is a new drug, there are particular ways in which the data are gathered. The sample is divided into two groups. One group is assigned the new drug and the other is assigned a placebo, a pill that contains no medication. The experiment is often called "double-blind" because neither the subjects who take the drug nor the physician or scientist who provides the drug knows whether any individual is taking the drug or the placebo. At the end of the experiment, the data that are compiled allow statistics practitioners to do their work. Exercises 13.95–13.99 are examples of this type of statistical application. Exercise 13.100 describes a health-related problem where the use of a placebo is not possible.

13.95 Cold and allergy medicines have been available for a number of years. One serious side effect of these medications is that they cause drowsiness, which makes them dangerous for industrial workers. In recent years, a nondrowsy cold and allergy medicine has been developed. One such product, Hismanal, is claimed by its manufacturer to be the first once-a-day nondrowsy allergy medicine. The nondrowsy part of the claim is based on a clinical experiment in which 1,604 patients were given Hismanal and 1,109 patients were given a placebo. Of the first group, 7.1% reported drowsiness; of the second group, 6.4% reported drowsiness. Do these results allow us to infer at the 5% significance level that Hismanal's claim is false?

13.96 Plavix is a drug that is given to angioplasty patients to help prevent blood clots. A researcher at McMaster University organized a study that involved 12,562 patients in 482 hospitals in 28 countries. All the patients had acute coronary syndrome, which produces mild heart attacks or unstable angina, chest pain that may precede a heart attack. The patients were divided into two equal groups. Group 1 received daily Plavix pills; group 2 received a placebo. After 1 year, 9.3% of patients on Plavix suffered a stroke or new heart attack or had died of cardiovascular disease, compared with 11.5% of those who took the placebo.

a. Can we infer that Plavix is effective?

b. Describe your statistical analysis in a report to the marketing manager of the pharmaceutical company.

13.97 In a study that was highly publicized, doctors discovered that aspirin seems to help prevent heart attacks. The research project, which was scheduled to last for 5 years, involved 22,000 American physicians (all male). Half took an aspirin tablet three times per week, and the other half took a placebo on the same schedule. The researchers tracked all of the volunteers and updated the records regularly. Among the physicians who took aspirin, 104 suffered heart attacks; 189 physicians who took the placebo had heart attacks.

a. Determine whether these results indicate that aspirin is effective in reducing the incidence of heart attacks.

b. Write a report that describes the results of this experiment.

13.98 Exercise 13.97 described the experiment that determined that taking aspirin daily reduces one's probability of suffering a heart attack. The study was conducted in 1982; at that time, the mean age of the physicians was 50. In the years following the experiment, the physicians were monitored for other medical conditions. One of these was the incidence of cataracts. There were 1,084 cataracts in the aspirin group and 997 in the placebo group. Do these statistics allow researchers to conclude that aspirin leads to more cataracts?

13.99 According to the Canadian Cancer Society, more than 21,000 women will be diagnosed with breast cancer every year and more than 5,000 will die. (U.S. figures are more than 10 times those in Canada.) Surgery is generally considered the first method of treatment. However, many women suffer recurrences of cancer. For this reason, many women are treated with tamoxifen. But after 5 years, tumors develop a resistance to tamoxifen. A new drug called *letrozole* was developed by Novartis Pharmaceuticals to replace tamoxifen. To determine its effectiveness, a study involving 5,187 breast cancer survivors from Canada, the United States, and Europe was undertaken. Half the sample received letrozole and the other half a placebo. The study was to run for 5 years. However, after only 2.5 years, it was determined that 132 women receiving the placebo and 75 taking the drug had recurrences of their cancers. (The study was published in the *New England Journal of Medicine*.)

a. Do these results provide sufficient evidence to infer that letrozole works?
b. Prepare a presentation to the board of directors of Novartis describing your analysis.

13.100 A study described in the *British Medical Journal* (January 2004) sought to determine whether exercise would help extend the lives of patients with heart failure. A sample of 801 patients with heart failure was recruited; 395 received exercise training and 406 did not. There were 88 deaths among the exercise group and 105 among those who did not exercise. Can researchers infer that exercise training reduces mortality?

Exercises 13.101–13.125 require the use of a computer and software. Use a 5% significance level unless specified otherwise. The answers to Exercises 13.101 to 13.112 may be calculated manually. See Appendix A for the sample statistics.

13.101 Xr13-101 Automobile magazines often compare models and rate them in various ways. One question that is often asked of car owners, Would you buy the same model again? Suppose that a researcher for one magazine asked a random sample of Lexus owners and a random sample of Acura owners whether they plan to buy another Lexus or Acura the next time they shop for a new car. The responses (1 = no and 2 = yes) were recorded. Do these data allow the researcher to infer that the two populations of car owners differ in their satisfaction levels?

13.102 Xr13-102 An insurance company is thinking about offering discounts on its life-insurance policies to nonsmokers. As part of its analysis, the company randomly selects 200 men who are 60 years old and asks them whether they smoke at least one pack of cigarettes per day and if they have ever suffered from heart disease (2 = suffer from heart disease, and 1 = do not suffer from heart disease).

a. Can the company conclude at the 10% significance level that smokers have a higher incidence of heart disease than nonsmokers?
b. Estimate with 90% confidence the difference in the proportions of men suffering from heart disease between smokers and nonsmokers.

13.103 Xr13-103 Has the illicit use of drugs decreased over the past 10 years? Government agencies have undertaken surveys of Americans 12 years of age and older. Each was asked whether he or she used drugs at least once in the previous month. The results of this year's survey and the results of the survey completed 10 years ago were recorded as 1 = no and 2 = yes. Can we infer that the use of illicit drugs in the United States has increased in the past decade? (Adapted from the U.S. Substance Abuse and Mental Health Services Administration, National Household Survey on Drug Abuse.)

13.104 Xr13-104 It has been estimated that the oil sands in Alberta, Canada, contain 2 trillion barrels of oil. However, recovering the oil damages the environment. A survey of Canadians and Americans was asked, What is more important to you with regards to the oil sands: (1) environmental concerns or (2) the potential of a secure nonforeign supply of oil to North America? Do these data allow you to conclude that Canadians and Americans differ in their responses to this question? (Source: Flieshman-Hillard Oilsands Survey.)

13.105 Xr13-105 An operations manager of a computer chip maker is in the process of selecting a new machine to replace several older ones. Although technological innovations have improved the production process, it is quite common for the machines to produce defective chips. The operations manager must choose between two machines. The cost of machine A is several thousand dollars greater than the cost of machine B. After an analysis of the costs, it was determined that machine A is warranted, provided that its defective rate is more than 2% less than that of machine B. To help decide, both machines are used to produce 200 chips each. Each chip was examined, and whether it was defective (code = 2) or not (code = 1) was recorded. Should the operations manager select machine A?

13.106 Xr13-106 Parents often urge their children to get more education, not only for the increased income but also to perhaps work less hard. A survey asked a random sample of Canadians whether they work 11 or more hours a day (1 = no, 2 = yes) and whether they completed high school only or completed post-secondary education. Can we infer that those with more education are less likely to work 11 hours or more per day? (Source: Harris/Decima survey.)

13.107 Xr13-107 Are Americans becoming more unhappy at work? A survey of Americans in 2008 and again this year asked whether they were satisfied with their jobs (1 = no, 2 = yes). Can we infer that more Americans are unhappy compared to 2008?

Public Opinion about Global Warming and Climate Change

In Chapters 3 and 4, we described the issue of global warming and pointed out that Earth has not warmed since 1998, explaining why the media now refer to the problem as *climate change*, and that a weak linear relationship exists between temperature anomalies and CO_2 levels. In the last few years, news stories have appeared that seem to cast doubt on the entire theory. To measure the effect on public opinion, several surveys have been conducted. Below we describe two surveys in the United States, Canada, and Britain. For each exercise and each country, determine whether there is sufficient evidence that the belief that global warming is real has fallen.

13.108 Xr13-108 The following question was asked in the three countries in November 2009 and December 2009.

Which of the following statements comes closest to your view of global warming (or climate change)?

1. Global warming is a fact and is mostly caused by emissions from vehicles and industrial facilities.
2. Global warming is a fact and is mostly caused by natural changes.
3. Global warming is a theory and has not yet been proven.
4. Not sure

13.109 Xr13-109 Do you agree (1 = Yes, 2 = No) that climate change and how we respond to it are among the biggest issues that you worry about today? The question was asked in the three countries in November 2008 and November 2009.

APPLICATIONS in MARKETING


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Market Segmentation

In Section 12.4, we introduced market segmentation and described how the size of market segments can be estimated. Once the segments have been defined we can use statistical techniques to determine whether members of the segments differ in their purchases of a firm's products.

13.110 Xr13-110* The market for breakfast cereals has been divided into several segments related to health. One company identified a segment as health-conscious adults. The marketing manager would like to know whether this segment is more likely to purchase its Special X cereal, which is pitched toward the health-conscious segment. A survey of adults was undertaken. On the basis of several probing questions, each was classified as either a member of the health-conscious group (code = 1) or not (code = 2). Each respondent was also asked whether he or she buys Special X (1 = no, 2 = yes). The data were recorded in stacked format. Can we infer from these data that health-conscious adults are more likely to buy Special X?

13.111 Xr13-111* Quik Lube is a company that offers an oil-change service while the customer waits. Its market has been broken down into the following segments:

1: Working men and women too busy to wait at a dealer or service center
2: Spouses who work in the home
3: Retired persons
4: Other

A random sample of car owners was drawn. All owners classified their market segment and also reported whether they usually use such services as Quik Lube (1 = yes, and 2 = no). These data are stored in stacked format.

a. Determine whether members of segment 1 are more likely than members of segment 4 to respond that they usually use the service?
b. Can we infer that retired persons and spouses who work in the home differ in their use of services such as Quik Lube?

13.112 Xr13-112 Telemarketers obtain names and telephone numbers from several sources. To determine whether one particular source is better than a second, a random sample of names and numbers from the two different sources was obtained. For each potential customer, a statistics practitioner recorded whether that individual made a purchase (code = 2) or not (code = 1). Can we infer that differences exist between the two sources?

GENERAL SOCIAL SURVEY EXERCISES

13.113 GSS2008* A generation ago, men were more likely to attend a university and acquire a graduate degree than women. However, women now appear to be attending universities in greater numbers than men. To gauge the extent of the difference, test to determine whether men and women (SEX: 1 = Male and 2 = Female) differ in completing a graduate degree (DEGREE: 4 = Graduate).

13.114 GSS2008* The deep recession of 2008–2010 may have changed patterns of employment. Because of the large number of layoffs an increasing number of individuals have chosen to work for themselves. The question arises, Do men and women (SEX: 1 = Male and 2 = Female) differ in their decision to work for themselves? (WRKSLF: 1 = self-employed, 2 = someone else.) Conduct a test to answer the question.

For each of the following four exercises, determine whether men and women are likely to differ in answering each question correctly.

13.115 GSS2008* A doctor tells a couple that there is one chance in four that their child will have an inherited disease. Does this mean that if the first child has the illness, the next three will not (ODDS1)? 1 = Yes, 2 = No. Correct answer: No.

13.116 GSS2008* A doctor tells a couple that there is one chance in four that their child will have an inherited disease. Does this mean that each of the couple's children will have the same risk of suffering the illness (ODDS2)? 1 = Yes, 2 = No. Correct answer: Yes.

13.117 GSS2008* True or false—Earth's center is very hot. 1 = True, 2 = False. Correct answer: True.

13.118 GSS2008* Does Earth go around the Sun or does the Sun go around Earth? 1 = Earth around Sun, 2 = Sun around Earth. Correct answer: Earth around Sun.

For each of the following variables, conduct a test to determine whether there is a difference between 2008 and 2006.

13.119 GSS2008* GSS2006* WRKGOVT: Are (were) you employed by the federal, state, or local government or by a private employer (including not-for-profit organizations)? 1 = Government, 2 = Private.

13.120 GSS2008* GSS2006* CAPPUN: Do you favor capital punishment for murder? 1 = Favor, 2 = Oppose.

13.121 GSS2008* GSS2006* GUNLAW: Do you favor requiring a police permit to buy a gun? 1 = Favor, 2 = Oppose.

13.122 GSS2002* GSS2004* GSS2006* GSS2008* Test to determine whether Democrats and Republicans (PARTYID: 0 and 1 = Democrat and 5 and 6 = Republicans) differ in each of the years 2002, 2004, 2006, and 2008 in completing a graduate degree (DEGREE: 4 = Graduate).

American National Election Survey Exercises

For each of the following variables, conduct a test to determine whether Democrats and Republicans (PARTY: 1 = Democrat and 2 = Republicans) differ.

13.123 ANES2008* Likely to be employed (EMPLOY: 1 = Working now, 2–8 = Other categories).

13.124 ANES2008* Have health insurance (HEALTH 1 = Yes, 5 = No).

13.125 ANES2008* Always vote (OFTEN: 1 = Always, 2, 3, 4 = Other categories).

Chapter Summary

In this chapter, we presented a variety of techniques that allow statistics practitioners to compare two populations. When the data are interval and we are interested in measures of central location, we encountered two more factors that must be considered when choosing the appropriate technique. When the samples are independent, we can use either the **equal-variances** or **unequal-variances formulas**. When the samples are **matched pairs**, we have only one set of formulas. We introduced the ***F*-statistic**, which is used to make inferences about two population variances. When the data are nominal, the parameter of interest is the difference between two proportions. For this parameter, we had two test statistics and one interval estimator. Finally, we discussed **observational** and **experimental data**, important concepts in attempting to interpret statistical findings.

Important Terms

Pooled variance estimator 451
Equal-variances test statistic 451
Equal-variances confidence interval estimator 451
Unequal-variances test statistic 452
Unequal-variances confidence interval estimator 452
Observational data 472
Experimental data 472
Matched pairs experiment 479
Mean of the population of differences 479
Numerator degrees of freedom 490
Denominator degrees of freedom 490
Pooled proportion estimate 497

SYMBOLS

Symbol	Pronounced	Represents
s_p^2	s sub p squared	Pooled variance estimator
μ_D	mu sub D or mu D	Mean of the paired differences
$\bar{x}_D$	x bar sub D or x bar D	Sample mean of the paired differences
s_D	s sub D or s D	Sample standard deviation of the paired differences
n_D	n sub D or n D	Sample size of the paired differences
$\hat{p}$	p hat	Pooled proportion

FORMULAS

Equal-variances t-test of $\mu_1 - \mu_2$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \qquad \nu = n_1 + n_2 - 2$$

Equal-variances interval estimator of $\mu_1 - \mu_2$

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \qquad \nu = n_1 + n_2 - 2$$

Unequal-variances t-test of $\mu_1 - \mu_2$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)}} \qquad \nu = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$$

Unequal-variances interval estimator of $\mu_1 - \mu_2$

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \qquad \nu = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$$

t-test of μ_D

$$t = \frac{\bar{x}_D - \mu_D}{s_D/\sqrt{n_D}} \qquad \nu = n_D - 1$$

t-estimator of μ_D

$$\bar{x}_D \pm t_{\alpha/2}\frac{s_D}{\sqrt{n_D}} \qquad \nu = n_D - 1$$

F-test of σ_1^2/σ_2^2

$$F = \frac{s_1^2}{s_2^2} \qquad \nu_1 = n_1 - 1 \text{ and } \nu_2 = n_2 - 1$$

F-estimator of σ_1^2/σ_2^2

$$\text{LCL} = \left(\frac{s_1^2}{s_2^2}\right)\frac{1}{F_{\alpha/2,\nu_1,\nu_2}}$$

$$\text{UCL} = \left(\frac{s_1^2}{s_2^2}\right)F_{\alpha/2,\nu_2,\nu_1}$$

z-test and estimator of $p_1 - p_2$

$$\text{Case 1:} \quad z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$\text{Case 2:} \quad z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}}$$

z-estimator of $p_1 - p_2$

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}$$

COMPUTER OUTPUT AND INSTRUCTIONS

Technique	Excel	Minitab
Unequal-variances t-test of $\mu_1 - \mu_2$	461	462
Unequal-variances estimator of $\mu_1 - \mu_2$	463	463
Equal-variances t-test of $\mu_1 - \mu_2$	456	456
Equal-variances estimator of $\mu_1 - \mu_2$	457	458
t-test of μ_D	480	481
t-estimator of μ_D	482	482
F-test of σ_1^2/σ_2^2	491	492
F-estimator of σ_1^2/σ_2^2	493	493
z-test of $p_1 - p_2$ (Case 1)	500	501
z-test of $p_1 - p_2$ (Case 2)	502	502
z-estimator of $p_1 - p_2$	504	504

CHAPTER EXERCISES

The following exercises require the use of a computer and software. Use a 5% significance level unless specified otherwise.

13.126 Xr13-126 Obesity among children has quickly become an epidemic across North America. Television and video games are part of the problem. To gauge to what extent nonparticipation in organized sports contributes to the crisis, surveys of children 5 to 14 years old were conducted this year and 10 years ago. The gender of the child and whether he or she participated in organized sports (1 = No, 2 = Yes) were recorded.

a. Can we conclude that there has been a decrease in participation among boys over the past 10 years?
b. Repeat part (a) for girls.
c. Can we infer that girls are less likely to participate than boys this year?

13.127 Xr13-127 A restaurant located in an office building decides to adopt a new strategy for attracting customers to the restaurant. Every week it advertises in the city newspaper. To assess how well the advertising is working, the restaurant owner recorded the weekly gross sales for the 15 weeks after the campaign began and the weekly gross sales for the 24 weeks immediately prior to the campaign. Can the restaurateur conclude that the advertising campaign is successful?

13.128 Refer to Exercise 13.127. Assume that the profit is 20% of the gross. If the ads cost $50 per week, can the restaurateur conclude that the ads are profitable?

13.129 Xr13-129 How important to your health are regular vacations? In a study, a random sample of men and women were asked how frequently they take vacations. The men and women were divided into two groups each. The members of group 1 had suffered a heart attack; the members of group 2 had not. The number of days of vacation last year was recorded for each person. Can we infer that men and women who suffer heart attacks vacation less than those who did not suffer a heart attack?

13.130 Xr13-130 Research scientists at a pharmaceutical company have recently developed a new nonprescription sleeping pill. They decide to test its effectiveness by measuring the time it takes for people to fall asleep after taking the pill. Preliminary analysis indicates that the time to fall asleep varies considerably from one person to another. Consequently, the researchers organize the experiment in the following way. A random sample of 100 volunteers who regularly suffer from insomnia is chosen. Each person is given one pill containing the newly developed drug and one placebo. (They do not know whether the pill they are taking is the placebo or the real thing, and the order of use is random.) Each participant is fitted with a device that measures the time until sleep occurs. Can we conclude that the new drug is effective?

13.131 Xr13-131 The city of Toronto boasts four daily newspapers. Not surprisingly, competition is keen. To help learn more about newspaper readers, an advertiser selected a random sample of people who bought their newspapers from a street vendor and people who had the newspaper delivered to their homes. All were asked how many minutes they spent reading their newspapers. Can we infer that the amount of time reading differs between the two groups?

13.132 Xr13-132 In recent years, a number of state governments have passed mandatory seat-belt laws. Although the use of seat belts is known to save lives and reduce serious injuries, compliance with seat-belt laws is not universal. In an effort to increase the use of seat belts, a government agency sponsored a 2-year study. Among its objectives was to determine whether there was enough evidence to infer that seat-belt usage increased between last year and this year. To test this belief, random samples of drivers last year and this year were asked whether they always use their seat belts (2 = wear seat belt, 1 = do not wear seat belt). Can we infer that seat-belt usage has increased over the last year?

13.133 Xr13-133 An important component of the cost of living is the amount of money spent on housing. Housing costs include rent (for tenants), mortgage payments and property tax (for home owners), heating, electricity, and water. An economist undertook a 5-year study to determine how housing costs have changed. Five years ago, he took a random sample of 200 households and recorded the percentage of total income spent on housing. This year, he took another sample of 200 households.

a. Conduct a test (with $\alpha = .10$) to determine whether the economist can infer that housing cost as a percentage of total income has increased over the last 5 years.
b. Use whatever statistical method you deem appropriate to check the required condition(s) of the test used in part (a).

13.134 Xr13-134 In designing advertising campaigns to sell magazines, it is important to know how much time each of several demographic groups spends reading magazines. In a preliminary study, 40 people were randomly selected. Each was asked how much time per week he or she spends reading magazines; in addition, each was categorized by both gender and income level (high or low). The data are stored in the following way: column 1 = time spent reading magazines per week in minutes for all respondents; column 2 = gender (1 = male, 2 = female); column 3 = income level (1 = low, 2 = high).

a. Is there sufficient evidence at the 10% significance level to conclude that men and women differ in the amount of time spent reading magazines?
b. Is there sufficient evidence at the 10% significance level to conclude that high-income individuals devote more time to reading magazines than low-income people?

13.135 Xr13-135 In a study to determine whether gender affects salary offers for graduating MBA students, 25 pairs of students were selected. Each pair consisted of a female and a male student who were matched according to their grade point averages, courses taken, ages, and previous work experience. The highest salary offered (in thousands of dollars) to each graduate was recorded.

a. Is there enough evidence at the 10% significance level to infer that gender is a factor in salary offers?
b. Discuss why the experiment was organized in the way it was.
c. Is the required condition for the test in part (a) satisfied?

13.136 Xr13-136 Have North Americans grown to distrust television and newspaper journalists? A study was conducted this year to compare what Americans currently think of the news media versus what they said 3 years ago. The survey asked respondents whether they agreed that the news media tends to favor one side when reporting on political and social issues. A random sample of people was asked to participate in this year's survey. The results of a survey of another random sample taken 3 years ago are also available. The responses are 2 = agree and 1 = disagree. Can we conclude at the 10% significance level that Americans have become more distrustful of television and newspaper reporting this year than they were 3 years ago?

13.137 Xr13-137 Before deciding which of two types of stamping machines should be purchased, the plant manager of an automotive parts manufacturer wants to determine the number of units that each produces. The two machines differ in cost, reliability, and productivity. The firm's accountant has calculated that machine A must produce 25 more nondefective units per hour than machine B to warrant buying machine A. To help decide, both machines were operated for 24 hours. The total number of units and the number of defective units produced by each machine per hour were recorded. These data are stored in the following way: column 1 = total number of units produced by machine A, column 2 = number of defectives produced by machine A, column 3 = total number of units produced by machine B, and column 4 = number of defectives produced by machine B. Determine which machine should be purchased.

13.138 Refer to Exercise 13.137. Can we conclude that the defective rate differs between the two machines?

13.139 Xr13-139 The growing use of bicycles to commute to work has caused many cities to create exclusive bicycle lanes. These lanes are usually created by disallowing parking on streets that formerly allowed curbside parking. Merchants on such streets complain that the removal of parking will cause their businesses to suffer. To examine this problem, the mayor of a large city decided to launch an experiment on one busy street that had 1-hour parking meters. The meters were removed, and a bicycle lane was created. The mayor asked the three businesses (a dry cleaner, a doughnut shop, and a convenience store) in one block to record daily sales for two complete weeks (Sunday to Saturday) before the change and two complete weeks after the change. The data are stored as follows: column 1 = day of the week, column 2 = sales before change for dry cleaner, column 3 = sales after change for dry cleaner, column 4 = sales before change for doughnut shop, column 5 = sales after change for doughnut shop, column 6 = sales before change for convenience store, and column 7 = sales after change for convenience store. What conclusions can you draw from these data?

13.140 Xr13-140 Researchers at the University of Ohio surveyed 219 students and found that 148 had Facebook accounts. All students were asked for their current grade point average. Do the data allow us to infer that Facebook users have lower GPAs?

13.141 Xr13-141 Clinical depression is linked to several other diseases. Scientists at Johns Hopkins University undertook a study to determine whether heart disease is one of these. A group of 1,190 male medical students was tracked over a 40-year period. Of these, 132 had suffered clinically diagnosed depression. For each student, the scientists recorded whether the student died of a heart attack (code = 2) or did not (code = 1).

a. Can we infer at the 1% significance level that men who are clinically depressed are more likely to die from heart disease?
b. If the answer to part (a) is "yes," can you interpret this to mean that depression causes heart disease? Explain.

13.142 Xr13-142 High blood pressure (hypertension) is a leading cause of strokes. Medical researchers are constantly seeking ways to treat patients suffering from this condition. A specialist in hypertension claims that regular aerobic exercise can reduce high blood pressure just as successfully as drugs, with none of the adverse side effects. To test the claim, 50 patients who suffer from high blood pressure were chosen to participate in an experiment. For 60 days, half the sample exercised three times per week for 1 hour and did not take medication; the other half took the standard medication. The percentage reduction in blood pressure was recorded for each individual.

a. Can we conclude at the 1% significance level that exercise is more effective than medication in reducing hypertension?
b. Estimate with 95% confidence the difference in mean percentage reduction in blood pressure between drugs and exercise programs.
c. Check to ensure that the required condition(s) of the techniques used in parts (a) and (b) is satisfied.

13.143 Xr13-143 Most people exercise in order to lose weight. To determine better ways to lose weight, a random sample of male and female exercisers was divided into groups. The first group exercised vigorously twice a week. The second group exercised moderately four times per week. The weight loss for each individual was recorded. Can we infer that people who exercise moderately more frequently lose more weight than people who exercise vigorously?

13.144 Xr13-144 After observing the results of the test in Exercise 13.143, a statistics practitioner organized another experiment. People were matched according to gender, height, and weight. One member of each matched pair then exercised vigorously twice a week, and the other member exercised moderately four times per week. The weight losses were recorded. Can we infer that people who exercise moderately lose more weight?

13.145 Xr13-145 "Pass the Lotion," a long-running television commercial for Special K cereal, features a flabby sunbather who asks his wife to smear sun lotion on his back. A random sample of Special K customers and a random sample of people who do not buy Special K were asked to indicate whether they liked (code = 1) or disliked (code = 2) the ad. Can we infer that Special K buyers like the ad more than nonbuyers?

13.146 Xr13-146 Refer to Exercise 13.145. The respondents were also asked whether they thought the ad would be effective in selling the product. The responses (1 = Yes and 2 = No) were recorded. Can we infer that Special K buyers are more likely to respond yes than nonbuyers?

13.147 Xr13-147 Most English professors complain that students don't write very well. In particular, they point out that students often confuse quality and quantity. A study at the University of Texas examined this claim. In the study, undergraduate students were asked to compare the cost benefits of Japanese and American cars. All wrote their analyses on computers. Unbeknownst to the students, the computers were rigged so that some students would have to type twice as many words to fill a single page. The number of words used by each student was recorded. Can we conclude that students write in such a way as to fill the allotted space?

13.148 Xr13-148 Approximately 20 million Americans work for themselves. Most run single-person businesses out of their homes. One-quarter of these individuals use personal computers in their businesses. A market research firm, Computer Intelligence InfoCorp, wanted to know whether single-person businesses that use personal computers are more successful than those with no computer. They surveyed 150 single-person firms and recorded their annual incomes. Can we infer at the 10% significance level that single-person businesses that use a personal computer earn more than those that do not?

13.149 Xr13-149 Many small retailers advertise in their neighborhoods by sending out flyers. People deliver these to homes and are paid according to the number of flyers delivered. Each deliverer is given several streets whose homes become their responsibility. One of the ways retailers use to check the performance of deliverers is to randomly sample some of the homes and ask the home owner whether he or she received the flyer. Recently, university students started a new delivery service. They have promised better service at a competitive price. A retailer wanted to know whether the new company's delivery rate is better than that of the existing firm. She had both companies deliver her flyers. Random samples of homes were drawn, and each was asked whether he or she received the flyer (2 = yes and 1 = no). Can the retailer conclude that the new company is better? (Test with $\alpha = .10$.)

13.150 Xr13-150 Medical experts advocate the use of vitamin and mineral supplements to help fight infections. A study undertaken by researchers at Memorial University (reported in the British journal *Lancet*, November 1992) recruited 96 men and women age 65 and older. One-half of them received daily supplements of vitamins and minerals, whereas the other half received placebos. The supplements contained the daily recommended amounts of 18 vitamins and minerals, including vitamins B-6, B-12, C, and D, as well as thiamine, riboflavin, niacin, calcium, copper, iodine, iron, selenium, magnesium, and zinc. The doses of vitamins A and E were slightly less than the daily requirements. The supplements included four times the amount of beta-carotene than the average person ingests daily. The number of days of illness from infections (ranging from colds to pneumonia) was recorded for each person. Can we infer that taking vitamin and mineral supplements daily increases the body's immune system?

13.151 Xr13-151 An inspector for the Atlantic City Gaming Commission suspects that a particular blackjack dealer may be cheating (in favor of the casino) when he deals at expensive tables. To test her belief, she observed 500 hands each at the $100-limit table and the $3,000-limit table. For each hand, she recorded whether the dealer won (code = 2) or lost (code = 1). When a tie occurs, there is no winner or loser. Can the inspector conclude at the 10% significance level that the dealer is cheating at the more expensive table?

13.152 Xr13-152 In 2005 Larry Summers, then president of Harvard University, received an avalanche of criticism for his attempt to explain why there are more male professors than female professors in mathematics. He suggested that there were innate differences that might permanently thwart the search for a more perfect gender balance. In an attempt to refute Dr. Summers's hypothesis, several researchers conducted large-scale mathematics tests of male and female students. Suppose the results were recorded. Conduct whatever tests you deem necessary to draw conclusions from these data. (*Note:* The data are simulated but represent actual results.)

Exercises 13.153 and 13.154 require access to the data files introduced in previous exercises.

13.153 Xr12-31* Exercise 12.31 dealt with the amount of time high school students spend per week at part-time jobs. In addition to the hours of part-time work, the school guidance counselor recorded the gender of the student surveyed (1 = female and 2 = male). Can we conclude that female and male high school students differ in the amount of time spent at part-time jobs?

13.154 Xm12-01* The company that organized the survey to determine the amount of discarded newspaper (Example 12.1) kept track of the type of neighborhood (1 = city and 2 = suburbs). Do these data allow the company management to infer that city households discard more newspaper than do suburban households?

APPLICATIONS IN **MARKETING**

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Market Segmentation

In Section 12.4, we introduced market segmentation. The following exercises address the problem of determining whether two market segments differ in their pattern of purchases of a particular product or service.

13.155 Xr13-155 Movie studios segment their markets by age. Two segments that are particularly important to this industry are teenagers and 20-to-30-year-olds. To assess markets and guide the making of movies, a random sample of teenagers and 20-to-30-year-olds was drawn. All were asked to report the number of movies they saw in theaters last year. Do these data allow us to infer that teenagers see more movies than 20-to-30-year-olds?

The following exercises employ data files associated with examples and exercises seen previously in this book.

13.156 Xr12.125* In addition to asking about educational attainment, the survey conducted in Exercise 12.125 also asked whether the respondent had plans in the next 2 years to take a course (1 = no and 2 = yes). Can we conclude that Californians who did not complete high school are less likely to take a course in the university's evening program?

13.157 Xm12-06* The objective in the survey conducted in Example 12.6 was to estimate the size of the market segment of adults who are concerned about eating healthy foods. As part of the survey, each respondent was asked how much they

(Continued)

spend on breakfast cereal in an average month. The marketing manager of a company that produces several breakfast cereals would like to know whether on average the market segment concerned about eating health foods outspends the other market segments. Write a brief report detailing your findings.

13.158 Xr12-35* In Exercise 12.35, we described how the office equipment chain OfficeMax offers rebates on some products. The goal in that exercise was to estimate the total amount spent by customers who bought the package of 100 CD-ROMs. In addition to tracking these amounts, an executive also determined the amounts spent in the store by another sample of customers who purchased a fax machine/copier (regular price $89.99 minus $40 manufacturer's rebate and $10 OfficeMax mail-in rebate). Can OfficeMax conclude that those who buy the fax/copier outspend those who buy the package of CD-ROMs? Write a brief memo to the executives of OfficeMax describing your findings and any possible recommendations.

13.159 Xr12-91* In addition to recording whether faculty members who are between 55 and 64 plan to retire before they reach 65 in Exercise 12.91, the consultant asked each to report his or her annual salary. Can the president infer that professors aged 55 to 64 who plan to retire early have higher salaries than those who don't plan to retire early?

13.160 Xr12-96* In Exercise 12.96, the statistics practitioner also recorded the gender of the respondents where 1 = female and 2 = male. Can we infer that men and women differ in their choices of Christmas trees?

APPENDIX 13 / REVIEW OF CHAPTERS 12 AND 13

As you may have already discovered, the ability to identify the correct statistical technique is critical; any calculation performed without it is useless. When you solved problems at the end of each section in the preceding chapters (you *have* been solving problems at the end of each section covered, haven't you?), you probably had no great difficulty identifying the correct technique to use. You used the statistical technique introduced in that section. Although those exercises provided practice in setting up hypotheses, producing computer output of tests of hypothesis and confidence interval estimators, and interpreting the results, you did not address a fundamental question faced by statistics practitioners: Which technique should I use? If you still do not appreciate the dimension of this problem, examine Table A13.1, which lists all the inferential methods covered thus far.

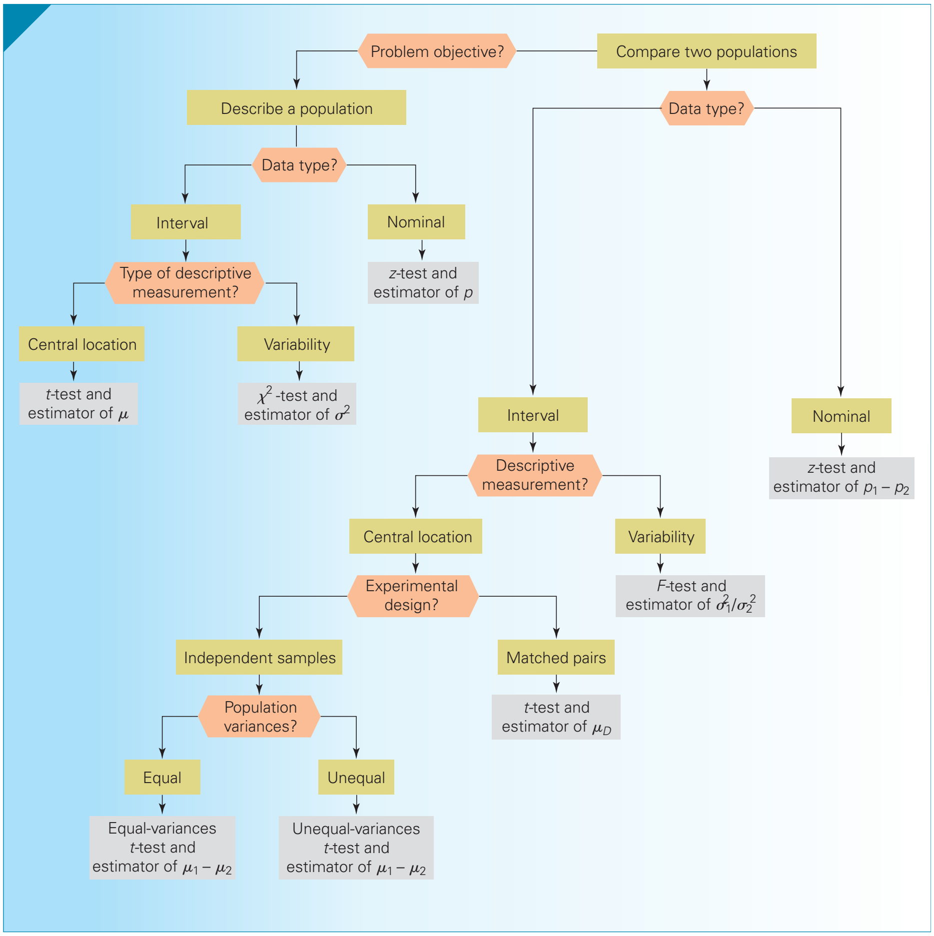
TABLE A13.1 Summary of Statistical Techniques in Chapters 12 and 13

t-test of μ
Estimator of μ (including estimator of $N\mu$)
z-test of p
Estimator of p (including estimator of Np)
χ^2-test of σ^2
Estimator of σ^2
Equal-variances t-test of $\mu_1 - \mu_2$
Equal-variances estimator of $\mu_1 - \mu_2$
Unequal-variances t-test of $\mu_1 - \mu_2$
Unequal-variances estimator of $\mu_1 - \mu_2$
t-test of μ_D
Estimator of μ_D
F-test of σ_1^2/σ_2^2
Estimator of σ_1^2/σ_2^2
z-test of $p_1 - p_2$ (Case 1)
z-test of $p_1 - p_2$ (Case 2)
Estimator of $p_1 - p_2$

Counting tests and confidence interval estimators of a parameter as two different techniques, a total of 17 statistical procedures have been presented thus far, and there is much left to be done. Faced with statistical problems that require the use of some of these techniques (such as in real-world applications or on a quiz or midterm test), most students need some assistance in identifying the appropriate method. In this appendix and the appendixes of five more chapters, you will have the opportunity to practice your decision skills; we've provided exercises and cases that require all the inferential techniques introduced in Chapters 12 and 13. Solving these problems will require you to do what statistics practitioners must do: analyze the problem, identify the technique or techniques, employ statistical software and a computer to yield the required statistics, and interpret the results.

The flowchart in Figure A13.1 represents the logical process that leads to the identification of the appropriate method. Of course, it only shows the techniques covered to this point. Chapters 14, 15, 16, 17, and 19 will include appendixes that review all the techniques introduced up to that chapter. The list and the flowchart will be expanded in each appendix, and all appendixes will contain review exercises. (Some will contain cases.)

FIGURE **A13.1** Flowchart of Techniques in Chapters 12 and 13



As we pointed out in Chapter 11, the two most important factors in determining the correct statistical technique are the problem objective and the data type. In some situations, once these have been recognized, the technique automatically follows. In other cases, however, several additional factors must be identified before you can proceed. For example, when the problem objective is to compare two populations and the data are interval, three other significant issues must be addressed: the descriptive measurement (central location or variability), whether the samples are independently drawn, and, if so, whether the unknown population variances are equal.

EXERCISES

The purpose of the exercises that follow is twofold. First, the exercises provide you with practice in the critical skill of identifying the correct technique. Second, they allow you to improve your ability to determine the statistics needed to answer the question and interpret the results. We believe that the first skill is underdeveloped because up to now you have had little practice. The exercises you've worked on have appeared at the end of sections and chapters where the correct techniques have just been presented. Determining the correct technique should not have been difficult. Because the exercises that follow were selected from the types that you have already encountered in Chapters 12 and 13, they will help you develop your technique-identification skills.

You will note that in the exercises that require a test of hypothesis, we do not specify a significance level. We have left this decision to you. After analyzing the issues raised in the exercise, use your own judgment to determine whether the p-value is small enough to reject the null hypothesis.

A13.1 XrA13-01 Shopping malls are more than places where we buy things. We go to malls to watch movies; buy breakfast, lunch, and dinner; exercise; meet friends; and, in general, to socialize. To study the trends, a sociologist took a random sample of 100 mall shoppers and asked a variety of questions. This survey was first conducted 3 years ago with another sample of 100 shoppers. In both surveys, respondents were asked to report the number of hours they spend in malls during an average week. Can we conclude that the amount of time spent at malls has decreased over the past 3 years?

A13.2 XrA13-02 It is often useful for retailers to determine why their potential customers choose to visit their store. Possible reasons include advertising, advice from a friend, or previous experience. To determine the effect of full-page advertisements in the local newspaper, the owner of an electronic-equipment store asked 200 randomly selected people who visited the store whether they had seen the ad. He also determined whether the customers had bought anything, and, if so, how much they spent. There were 113 respondents who saw the ad. Of these, 49 made a purchase. Of the 87 respondents who did not see the ad, 21 made a purchase. The amounts spent were recorded.

a. Can the owner conclude that customers who see the ad are more likely to make a purchase than those who do not see the ad?
b. Can the owner conclude that customers who see the ad spend more than those who do not see the ad (among those who make a purchase)?
c. Estimate with 95% confidence the proportion of all customers who see the ad who then make a purchase.
d. Estimate with 95% confidence the mean amount spent by customers who see the ad and make a purchase.

A13.3 XrA13-03 In an attempt to reduce the number of person-hours lost as a result of industrial accidents, a large multiplant corporation installed new safety equipment in all departments and all plants. To test the effectiveness of the equipment, a random sample of 25 plants was drawn. The number of person-hours lost in the month before installation of the safety equipment and in the month after installation was recorded. Can we conclude that the equipment is effective?

A13.4 XrA13-04 Is the antilock braking system (ABS) now available as a standard feature on many cars really effective? The ABS works by automatically pumping brakes extremely quickly on slippery surfaces so the brakes do not lock and thus avoiding an uncontrollable skid. If ABS is effective, we would expect that cars equipped with ABS would have fewer accidents, and the costs of repairs for the accidents that do occur would be smaller. To investigate the effectiveness of ABS, the Highway Loss Data Institute gathered data on a random sample of 500 General Motors cars that did not have ABS and 500 GM cars that were equipped with ABS. For each year, the institute recorded whether the car was involved in an accident and, if so, the cost of making repairs. Forty-two cars without ABS and 38 ABS-equipped cars were involved in accidents. The costs of repairs were recorded. Using frequency of accidents and cost of repairs as measures of effectiveness, can we conclude that ABS is effective? If so, estimate how much better are cars equipped with ABS compared to cars without ABS.

A13.5 XrA13-05 The electric company is considering an incentive plan to encourage its customers to pay their bills promptly. The plan is to discount the bills 1% if the customer pays within 5 days as opposed to the usual 25 days. As an experiment, 50 customers are offered the discount on their September bill. The amount of time each takes to pay his or her bill is recorded. The amount of time a random sample of 50 customers not offered the discount take to pay their bills is also recorded. Do these data allow us to infer that the discount plan works?

A13.6 XrA13-06 Traffic experts are always looking for ways to control automobile speeds. Some communities have experimented with "traffic-calming" techniques. These include speed bumps and various obstructions that force cars to slow down to drive around them. Critics point out that the techniques are counterproductive because they cause drivers to speed on other parts of these roads. In an analysis of the effectiveness of speed bumps, a statistics practitioner organized a study over a 1-mile stretch of city road that had 10 stop signs. He then took a random sample of 100 cars and recorded their average speed (the speed limit was 30 mph) and the number of proper stops at the stop signs. He repeated the observations for another sample of 100 cars after speed bumps were placed on the road. Do these data allow the statistics practitioner to conclude that the speed bumps are effective?

A13.7 XrA13-07 The proliferation of self-serve pumps at gas stations has generally resulted in poorer automobile maintenance. One feature of poor maintenance is low tire pressure, which results in shorter tire life and higher gasoline consumption. To examine this problem, an automotive expert took a random sample of cars across the country and measured the tire pressure. The difference between the recommended tire pressure and the observed tire pressure was recorded. [A recording of 8 means that the pressure of the tire is 8 pounds per square inch (psi) less than the amount recommended by the tire manufacturer.] Suppose that for each psi below recommendation, tire life decreases by 100 miles and gasoline consumption increases by 0.1 gallon per mile. Estimate with 95% confidence the effect on tire life and gasoline consumption.

A13.8 XrA13-08 Many North American cities encourage the use of bicycles as a way to reduce pollution and traffic congestion. So many people now regularly use bicycles to get to work and for exercise that some jurisdictions have enacted bicycle helmet laws that specify that all bicycle riders must wear helmets to protect against head injuries. Critics of these laws complain that it is a violation of individual freedom and that helmet laws tend to discourage bicycle usage. To examine this issue, a researcher randomly sampled 50 bicycle users and asked each to record the number of miles he or she rode weekly. Several weeks later, the helmet law was enacted. The number of miles each of the 50 bicycle riders rode weekly was recorded for the week after the law was passed. Can we infer from these data that the law discourages bicycle usage?

A13.9 XrA13-09 Cardizem CD is a prescription drug that is used to treat high blood pressure and angina. One common side effect of such drugs is the occurrence of headaches and dizziness. To determine whether its drug has the same side effects, the drug's manufacturer, Marion Merrell Dow, Inc., undertook a study. A random sample of 908 high-blood-pressure sufferers was recruited; 607 took Cardizem CD and 301 took a placebo. Each reported whether they suffered from headaches or dizziness (2 = yes, 1 = no). Can the pharmaceutical company scientist infer that Cardizem CD users are more likely to suffer headache and dizziness side effects than nonusers?

A13.10 XrA13-10 A fast-food franchiser is considering building a restaurant at a downtown location. Based on a financial analysis, a site is acceptable only if the number of pedestrians passing the location during the work day averages more than 200 per hour. To help decide whether to build on the site, a statistics practitioner observes the number of pedestrians who pass the site each hour over a 40-hour workweek. Should the franchiser build on this site?

A13.11 XrA13-11 Most people who quit smoking cigarettes do so for health reasons. However, some quitters find that they gain weight after quitting, and scientists estimate that the health risks of smoking two packs of cigarettes per day or carrying 65 extra pounds of weight are about equivalent. In an attempt to learn more about the effects of quitting smoking, the U.S. Centers for Disease Control conducted a study (reported in *Time*, March 25, 1991). A sample of 1,885 smokers was taken. During the course of the experiment, some of the smokers quit their habit. The amount of weight gained by all the subjects was recorded. Do these data allow us to conclude that quitting smoking results in weight gains?

A13.12 XrA13-12 Golf-equipment manufacturers compete against one another by offering a bewildering array of new products and innovations. Oversized clubs, square grooves, and graphite shafts are examples of such innovations. The effect of these new products on the average golfer is, however, much in doubt. One product, a perimeter-weighted iron, was designed to increase the consistency of distance and accuracy. The most important aspect of irons is consistency, which means that ideally there should be no variation in distance from shot to shot. To examine the relative merits of two brands of perimeter-weighted irons, an average golfer used the 7-iron, hitting 100 shots using each of two brands. The distance in yards was recorded. Can the golfer conclude that brand B is superior to brand A?

A13.13 XrA13-13 Managers are frequently called on to negotiate in a variety of settings. This calls for an ability to think logically, which requires an ability to concentrate and ignore distractions. In a study of the effect of distractions, a random sample of 208 students was drawn by psychologists at McMaster University (reported in the *National Post*, December 11, 2003). The male students were shown pictures of women of varying attractiveness. The female students were shown pictures of men of varying attractiveness. All students were then offered a choice of an immediate reward of $15 or a wait of 8 months for a reward of $75. The choices of the male and of the female students (1 = immediate reward, 2 = larger reward 8 months later) were recorded. The results are stored in the following way:

Column 1: Choices of males shown most attractive women

Column 2: Choices of males shown less attractive women

Column 3: Choices of females shown most attractive men

Column 4: Choices of females shown less attractive men

a. Can we infer that men's choices are affected by the attractiveness of women's pictures?
b. Can we infer that women's choices are affected by the attractiveness of men's pictures?

A13.14 XrA13-14 Throughout the day, many exercise shows appear on television. These usually feature attractive and fit men and women performing various exercises and urging viewers to duplicate the activity at home. Some viewers are exercisers. However, some people like to watch the shows without exercising (which explains why attractive people are used as demonstrators). Various companies sponsor the shows, and there are commercial breaks. One sponsor wanted to determine whether there are differences between exercisers and nonexercisers in terms of how well they remember the sponsor's name. A random sample of viewers was selected and called after the exercise show was over. Each was asked to report whether he or she exercised or only watched. They were also asked to name the sponsor's brand name (2 = yes, they could; 1 = no, they couldn't). Can the sponsor conclude that exercisers are more likely to remember the sponsor's brand name than those who only watch?

A13.15 XrA13-15 According to the latest census, the number of households in a large metropolitan area is 425,000. The home-delivery department of the local newspaper reports that 104,320 households receive daily home delivery. To increase home-delivery sales, the marketing department launches an expensive advertising campaign. A financial analyst tells the publisher that for the campaign to be successful, home-delivery sales must increase to more than 110,000 households. Anxious to see whether the campaign is working, the publisher authorizes a telephone survey of 400 households within 1 week of the beginning of the campaign and asks each household head whether he or she has the newspaper delivered. The responses were recorded where 2 = yes and 1 = no.

a. Do these data indicate that the campaign will increase home-delivery sales?
b. Do these data allow the publisher to conclude that the campaign will be successful?

A13.16 XrA13-16 The Scholastic Aptitude Test (SAT), which is organized by the Educational Testing Service (ETS), is important to high school students seeking admission to colleges and universities throughout the United States. A number of companies offer courses to prepare students for the SAT. The Stanley H. Kaplan Educational Center claims that its students gain, on average, more than 110 points by taking its course. ETS, however, insists that preparatory courses can improve a score by no more than 40 points. (The minimum and maximum scores of the SAT are 400 and 1,600, respectively.) Suppose a random sample of 40 students wrote the exam, then took the Kaplan preparatory course, and then took the exam again.

a. Do these data provide sufficient evidence to refute the ETS claim?
b. Do these data provide sufficient evidence to refute Kaplan's claim?

A13.17 XrA13-17 A potato chip manufacturer has contracted for the delivery of 15,000,000 kilograms of potatoes. The supplier agrees to deliver the potatoes in 15,000 equal truckloads. The manufacturer suspects that the supplier will attempt to cheat him. He has the weight of the first 50 truckloads recorded.

a. Can the manufacturer conclude from these data that the supplier is cheating him?
b. Estimate with 95% confidence the total weight of potatoes for all 15,000 truckloads.

General Social Survey Exercises

A13.18 GSS2008* Is there sufficient evidence to conclude that people who work for the government (WRKGOVT: 1 = Government, 2 = Private) work fewer hours (HRS)?

For each of the following variables, conduct a test to determine whether Democrats and Republicans (PARTY 1 = Democrat, 3 = Republican) differ in their correct answers to the following questions.

A13.19 GSS2008* Correct answers to ODDS1: A doctor tells a couple that there is one chance in four that their child will have an inherited disease. Does this mean that if the first child has the illness, the next three will not? 1 = Yes, 2 = No. Correct answer: No.

A13.20 GSS2008* Correct answers to ODDS2: A doctor tells a couple that there is one chance in four that their child will have an inherited disease. Does this mean that each of the couple's children will have the same risk of suffering the illness? 1 = Yes, 2 = No. Correct answer: Yes.

A13.21 GSS2008* Correct answers to HOTCORE: The center of the earth is very hot. 1 = True, 2 = False. Correct answer: True.

A13.22 GSS2008* Correct answers to EARTHSUN: Does Earth go around the Sun or does the Sun go around Earth? 1 = Earth around Sun, 2 = Sun around Earth. Correct answer: Earth around Sun.

A13.23 GSS2008* Estimate with 95% confidence Americans mean position on the following question: Should government reduce income differences between rich and poor (EQWLTH: 1 = government should reduce differences, 2, 3, 4, 5, 6, 7 = No government action)?.

A13.24 Estimate with 95% confidence the mean number of years with current employer (CUREMPYR).

A13.25 Estimate with 90% confidence the proportion of Americans whose income is at least $75,000 (INCOME06).

A13.26 GSS2006* GSS2008* Can we infer from the data that the proportion of Americans earning at least $75,000 is greater in 2008 than in 2006 (INCOME06)?

AMERICAN NATIONAL ELECTION SURVEY EXERCISES

A13.27 ANES2008* Conduct a test to determine whether Democrats and Republicans (PARTY: 1 = Democrat and 2 = Republican) differ in their intention to vote (DEFINITE: 1 = Definitely will not vote, 2, 3, 4, 5, 6, 7, 8 9, 10 = Definitely will vote).

A13.28 ANES2008* Estimate with 99% confidence the mean amount of time in a typical day spent by American adults watching news on television, not including sports (TIME2).

A13.29 ANES2008* Conduct a test to determine whether Democrats and Republicans (PARTY: 1 = Democrat and 2 = Republican) differ in how much they thought about the upcoming election for president (THOUGHT: 1 = Quite a lot, 5 = Only a little).

A13.30 ANES2008* Estimate with 95% confidence the proportion of Americans earning at least $100,000.

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14

ANALYSIS OF VARIANCE

General Social Survey: Liberal–Conservative Spectrum and Income

DATA
GSS2008*

Are Americans' political views affected by their incomes, or perhaps vice versa? If so, we would expect that incomes would differ between groups who define themselves somewhere on the following scale (POLVIEW).

1 = Extremely liberal
2 = Liberal
3 = Slightly liberal
4 = Moderate
5 = Slightly conservative
6 = Conservative
7 = Extremely conservative

© AP Photo/Chris Carlson

The question to be answered (on page 537) is, Are there differences in income between the seven groups of political views?

INTRODUCTION

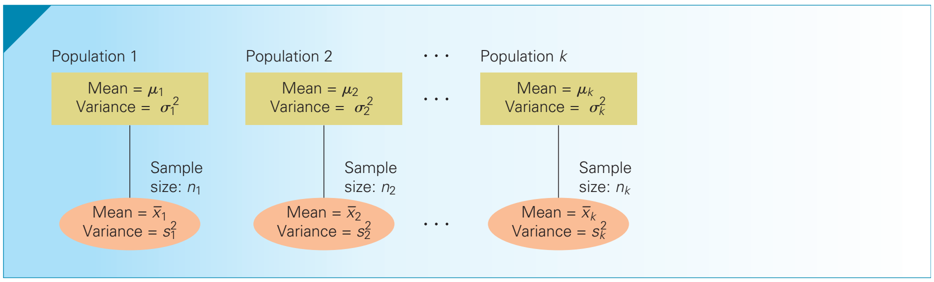
The technique presented in this chapter allows statistics practitioners to compare two or more populations of interval data. The technique is called the **analysis of variance**, and it is an extremely powerful and commonly used procedure. The analysis of variance technique determines whether differences exist between population means. Ironically, the procedure works by analyzing the sample variance, hence the name. We will examine several different forms of the technique.

One of the first applications of the analysis of variance was conducted in the 1920s to determine whether different treatments of fertilizer produced different crop yields. The terminology of that original experiment is still used. No matter what the experiment, the procedure is designed to determine whether there are significant differences between the **treatment means**.

14.1 ONE-WAY ANALYSIS OF VARIANCE

The analysis of variance is a procedure that tests to determine whether differences exist between two or more population means. The name of the technique derives from the way in which the calculations are performed; that is, the technique analyzes the variance of the data to determine whether we can infer that the population means differ. As in Chapter 13, the experimental design is a determinant in identifying the proper method to use. In this section, we describe the procedure to apply when the samples are independently drawn. The technique is called the **one-way analysis of variance**. Figure 14.1 depicts the sampling process for drawing independent samples. The mean and variance of population j ($j = 1, 2, \ldots, k$) are labeled μ_j and σ_j^2, respectively. Both parameters are unknown. For each population, we draw independent random samples. For each sample, we can compute the mean $\bar{x}_j$ and the variance s_j^2.

FIGURE **14.1** Sampling Scheme for Independent Samples



EXAMPLE 14.1* Proportion of Total Assets Invested in Stocks

DATA Xm14-01

In the last decade, stockbrokers have drastically changed the way they do business. Internet trading has become quite common, and online trades can cost as little as $7. It is now easier and cheaper to invest in the stock market than ever before. What are the effects of these changes? To help answer this question, a financial analyst randomly sampled 366 American households and asked each to report the age category of the head of

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*Adapted from U.S. Census Bureau, "Asset Ownership of Households, May 2003," *Statistical Abstract of the United States*, *2006*, Table 700.

the household and the proportion of its financial assets that are invested in the stock market. The age categories are

Young (less than 35)

Early middle age (35 to 49)

Late middle age (50 to 65)

Senior (older than 65)

The analyst was particularly interested in determining whether the ownership of stocks varied by age. Some of the data are listed next. Do these data allow the analyst to determine that there are differences in stock ownership between the four age groups?

Young	Early Middle Age	Late Middle Age	Senior
24.8	28.9	81.5	66.8
35.5	7.3	0.0	77.4
68.7	61.8	61.3	32.9
42.2	53.6	0.0	74.0
⋮	⋮	⋮	⋮

SOLUTION

You should confirm that the data are interval (percentage of total assets invested in the stock market) and that the problem objective is to compare four populations (age categories). The parameters are the four population means: μ_1, μ_2, μ_3, and μ_4. The null hypothesis will state that there are no differences between the population means. Hence,

$$H_0: \quad \mu_1 = \mu_2 = \mu_3 = \mu_4$$

The analysis of variance determines whether there is enough statistical evidence to show that the null hypothesis is false. Consequently, the alternative hypothesis will always specify the following:

$$H_1: \quad \text{At least two means differ}$$

The next step is to determine the test statistic, which is somewhat more involved than the test statistics we have introduced thus far. The process of performing the analysis of variance is facilitated by the notation in Table 14.1.

TABLE 14.1 Notation for the One-Way Analysis of Variance

	TREATMENT				
	1	2		j		k
	x_{11}	x_{12}	$\cdots$	x_{1j}	$\cdots$	x_{1k}
	x_{21}	x_{22}	$\cdots$	x_{2j}	$\cdots$	x_{2k}
	⋮	⋮		⋮		⋮
	$x_{n_1 1}$	$x_{n_2 2}$		$x_{n_j j}$		$x_{n_k k}$
Sample size	n_1	n_2		n_j		n_k
Sample mean	$\bar{x}_1$	$\bar{x}_2$		$\bar{x}_j$		$\bar{x}_k$

x_{ij} = ith observation of the jth sample

n_j = number of observations in the sample taken from the jth population

$\bar{x}_j$ = mean of the jth sample = $\dfrac{\sum_{i=1}^{n_j} x_{ij}}{n_j}$

$\bar{\bar{x}}$ = grand mean of all the observations = $\dfrac{\sum_{j=1}^{k}\sum_{i=1}^{n_j} x_{ij}}{n}$ where $n = n_1 + n_2 + \cdots + n_k$, and k is the number of populations

The variable X is called the **response variable**, and its values are called **responses**. The unit that we measure is called an **experimental unit**. In this example, the response variable is the percentage of assets invested in stocks, and the experimental units are the heads of households sampled. The criterion by which we classify the populations is called a **factor**. Each population is called a factor **level**. The factor in Example 14.1 is the age category of the head of the household and there are four levels. Later in this chapter, we'll discuss an experiment where the populations are classified using two factors. In this section, we deal with single-factor experiments only.

Test Statistic

The test statistic is computed in accordance with the following rationale. If the null hypothesis is true, the population means would all be equal. We would then expect that the sample means would be close to one another. If the alternative hypothesis is true, however, there would be large differences between some of the sample means. The statistic that measures the proximity of the sample means to each other is called the **between-treatments variation**; it is denoted **SST**, which stands for **sum of squares for treatments**.

Sum of Squares for Treatments

$$\text{SST} = \sum_{j=1}^{k} n_j(\bar{x}_j - \bar{\bar{x}})^2$$

As you can deduce from this formula, if the sample means are close to each other, all of the sample means would be close to the grand mean; as a result, SST would be small. In fact, SST achieves its smallest value (zero) when all the sample means are equal. In other words, if

$$\bar{x}_1 = \bar{x}_2 = \cdots = \bar{x}_k$$

then

$$\text{SST} = 0$$

It follows that a small value of SST supports the null hypothesis. In this example, we compute the sample means and the grand mean as

$$\bar{x}_1 = 44.40$$
$$\bar{x}_2 = 52.47$$
$$\bar{x}_3 = 51.14$$
$$\bar{x}_4 = 51.84$$
$$\bar{\bar{x}} = 50.18$$

The sample sizes are

$$n_1 = 84$$
$$n_2 = 131$$
$$n_3 = 93$$
$$n_4 = 58$$
$$n = n_1 + n_2 + n_3 + n_4 = 84 + 131 + 93 + 58 = 366$$

Then

$$\begin{aligned}\text{SST} &= \sum_{j=1}^{k} n_j(\bar{x}_j - \bar{\bar{x}})^2 \\ &= 84(44.40 - 50.18)^2 + 131(52.47 - 50.18)^2 \\ &\quad + 93(51.14 - 50.18)^2 + 58(51.84 - 50.18)^2 \\ &= 3{,}738.8\end{aligned}$$

If large differences exist between the sample means, at least some sample means differ considerably from the grand mean, producing a large value of SST. It is then reasonable to reject the null hypothesis in favor of the alternative hypothesis. The key question to be answered in this test (as in all other statistical tests) is, How large does the statistic have to be for us to justify rejecting the null hypothesis? In our example, SST = 3,738.8. Is this value large enough to indicate that the population means differ? To answer this question, we need to know how much variation exists in the percentage of assets, which is measured by the **within-treatments variation**, which is denoted by **SSE** (**sum of squares for error**). The within-treatments variation provides a measure of the amount of variation in the response variable that is not caused by the treatments. In this example, we are trying to determine whether the percentages of total assets invested in stocks vary by the age of the head of the household. However, other variables also affect the responses variable. We would expect that variables such as household income, occupation, and the size of the family would play a role in determining how much money families invest in stocks. All of these (as well as others we may not even be able to identify) are sources of variation, which we would group together and call the error. This source of variation is measured by the sum of squares for error.

Sum of Squares for Error

$$\text{SSE} = \sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij} - \bar{x}_j)^2$$

When SSE is partially expanded, we get

$$\text{SSE} = \sum_{i=1}^{n_1}(x_{i1} - \bar{x}_1)^2 + \sum_{i=1}^{n_2}(x_{i2} - \bar{x}_2)^2 + \cdots + \sum_{i=1}^{n_k}(x_{ik} - \bar{x}_k)^2$$

If you examine each of the k components of SSE, you'll see that each is a measure of the variability of that sample. If we divide each component by $n_j - 1$, we obtain the sample variances. We can express this by rewriting SSE as

$$\text{SSE} = (n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + \cdots + (n_k - 1)s_k^2$$

where s_j^2 is the sample variance of sample j. SSE is thus the combined or pooled variation of the k samples. This is an extension of a calculation we made in Section 13.1, where we tested and estimated the difference between two means using the pooled estimate of the common population variance (denoted s_p^2). One of the required conditions for that statistical technique is that the population variances are equal. That same condition is now necessary for us to use SSE; that is, we require that

$$\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2$$

Returning to our example, we calculate the sample variances as follows:

$$s_1^2 = 386.55$$
$$s_2^2 = 469.44$$
$$s_3^2 = 471.82$$
$$s_4^2 = 444.79$$

Thus,

$$\begin{aligned} \text{SSE} &= (n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + (n_3 - 1)s_3^2 + (n_4 - 1)s_4^2 \\ &= (84 - 1)(386.55) + (131 - 1)(469.44) \\ &\quad + (93 - 1)(471.82) + (58 - 1)(444.79) \\ &= 161{,}871.3 \end{aligned}$$

The next step is to compute quantities called the **mean squares**. The **mean square for treatments** is computed by dividing SST by the number of treatments minus 1.

Mean Square for Treatments

$$\text{MST} = \frac{\text{SST}}{k - 1}$$

The **mean square for error** is determined by dividing SSE by the total sample size (labeled n) minus the number of treatments.

Mean Square for Error

$$\text{MSE} = \frac{\text{SSE}}{n - k}$$

Finally, the test statistic is defined as the ratio of the two mean squares.

Test Statistic

$$F = \frac{\text{MST}}{\text{MSE}}$$

Sampling Distribution of the Test Statistic

The test statistic is F-distributed with $k - 1$ and $n - k$ degrees of freedom, provided that the response variable is normally distributed. In Section 8.4, we introduced the F-distribution, and in Section 13.4 we used it to test and estimate the ratio of two population variances. The test statistic in that application was the ratio of two sample variances s_1^2 and s_2^2. If you examine the definitions of SST and SSE, you will see that both measure variation similar to the numerator in the formula used to calculate the sample variance s^2 used throughout this book. When we divide SST by $k - 1$ and SSE by $n - k$ to calculate MST and MSE, respectively, we're actually computing unbiased estimators of the common population variance, assuming (as we do) that the null hypothesis is true. Thus, the ratio $F = \text{MST/MSE}$ is the ratio of two sample variances. The degrees of freedom for this application are the denominators in the mean squares; that is, $\nu_1 = k - 1$ and $\nu_2 = n - k$. For Example 14.1, the degrees of freedom are

$$\nu_1 = k - 1 = 4 - 1 = 3$$

$$\nu_2 = n - k = 366 - 4 = 362$$

In our example, we found

$$\text{MST} = \frac{\text{SST}}{k-1} = \frac{3{,}738.8}{3} = 1{,}246.27$$

$$\text{MSE} = \frac{\text{SSE}}{n-k} = \frac{161{,}871.3}{362} = 447.16$$

$$F = \frac{\text{MST}}{\text{MSE}} = \frac{1{,}246.27}{447.16} = 2.79$$

Rejection Region and *p*-Value

The purpose of calculating the ***F*-statistic** is to determine whether the value of SST is large enough to reject the null hypothesis. As you can see, if SST is large, F will be large. Hence, we reject the null hypothesis only if

$$F > F_{\alpha,k-1,n-k}$$

If we let $\alpha = .05$, the rejection region for Example 14.1 is

$$F > F_{\alpha,k-1,n-k} = F_{.05,3,362} \approx F_{.05,3,\infty} = 2.61$$

We found the value of the test statistic to be $F = 2.79$. Thus, there is enough evidence to infer that the mean percentage of total assets invested in the stock market differs between the four age groups.

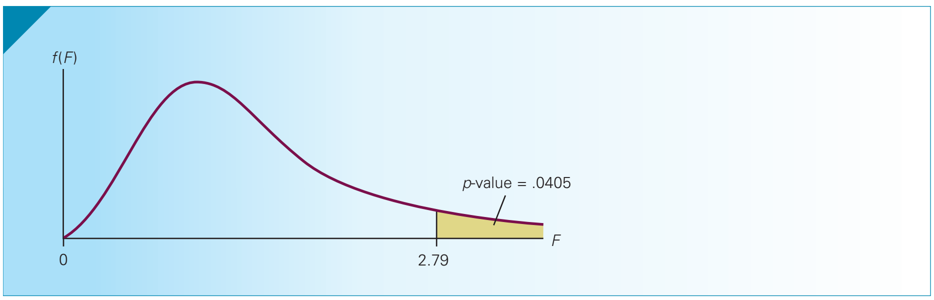
The p-value of this test is

$$P(F > 2.79)$$

A computer is required to calculate this value, which is .0405.

Figure 14.2 depicts the sampling distribution for Example 14.1.

FIGURE **14.2** Sampling Distribution for Example 14.1



The results of the analysis of variance are usually reported in an **analysis of variance (ANOVA) table**. Table 14.2 shows the general organization of the ANOVA table, and Table 14.3 shows the ANOVA table for Example 14.1.

TABLE **14.2** ANOVA Table for the One-Way Analysis of Variance

SOURCE OF VARIATION	DEGREES OF FREEDOM	SUMS OF SQUARES	MEAN SQUARES	F-STATISTIC
Treatments	$k - 1$	SST	MST = SST/$(k - 1)$	F = MST/MSE
Error	$n - k$	SSE	MSE = SSE/$(n - k)$	
Total	$n - 1$	SS(Total)		

TABLE **14.3** ANOVA Table for Example 14.1

SOURCE OF VARIATION	DEGREES OF FREEDOM	SUMS OF SQUARES	MEAN SQUARES	F-STATISTIC
Treatments	3	3,738.8	1,246.27	2.79
Error	362	161,871.3	447.16	
Total	365	165,610.1		

The terminology used in the ANOVA table (and for that matter, in the test itself) is based on the partitioning of the sum of squares. Such partitioning is derived from the following equation (whose validity can be demonstrated by using the rules of summation):

$$\sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij} - \bar{\bar{x}})^2 = \sum_{j=1}^{k} n_j(\bar{x}_j - \bar{\bar{x}})^2 + \sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij} - \bar{x}_j)^2$$

The term on the left represents the **total variation** of all the data. This expression is denoted **SS(Total)**. If we divide SS(Total) by the total sample size minus 1 (that is, by $n - 1$), we would obtain the sample variance (assuming that the null hypothesis is true). The first term on the right of the equal sign is SST, and the second term is SSE. As you can see, the total variation SS(Total) is partitioned into two sources of variation. The sum of squares for treatments (SST) is the variation attributed to the differences between the treatment means, whereas the sum of squares for error (SSE) measures the variation within the samples. The preceding equation can be restated as

$$\text{SS(Total)} = \text{SST} + \text{SSE}$$

The test is then based on the comparison of the mean squares of SST and SSE.

Recall that in discussing the advantages and disadvantages of the matched pairs experiment in Section 13.3, we pointed out that statistics practitioners frequently seek ways to reduce or explain the variation in a random variable. In the analysis of variance introduced in this section, the sum of squares for treatments explains the variation attributed to the treatments (age categories). The sum of squares for error measures the amount of variation that is unexplained by the different treatments. If SST explains a significant portion of the total variation, we conclude that the population means differ. In Sections 14.4 and 14.5, we will introduce other experimental designs of the analysis of variance—designs that attempt to reduce or explain even more of the variation.

If you've felt some appreciation of the computer and statistical software sparing you the need to manually perform the statistical techniques in earlier chapters, your appreciation should now grow, because the computer will allow you to avoid the incredibly time-consuming and boring task of performing the analysis of variance by hand. As usual, we've solved Example 14.1 using Excel and Minitab, whose outputs are shown here.

COMPUTE

EXCEL

	A	B	C	D	E	F	G
1	Anova: Single Factor						
2							
3	SUMMARY						
4	*Groups*	*Count*	*Sum*	*Average*	*Variance*		
5	Young	84	3729.5	44.40	386.55		
6	Early Middle Age	131	6873.9	52.47	469.44		
7	Late Middle Age	93	4755.9	51.14	471.82		
8	Senior	58	3006.6	51.84	444.79		
9							
10							
11	ANOVA						
12	*Source of Variation*	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
13	Between Groups	3741.4	3	1247.12	2.79	0.0405	2.6296
14	Within Groups	161871.0	362	447.16			
15							
16	Total	165612.3	365				

INSTRUCTIONS

1. Type or import the data into adjacent columns. (Open Xm14-01.)
2. Click **Data, Data Analysis,** and **Anova: Single Factor.**
3. Specify the **Input Range** (A1:D132) and a value for α (.05).

MINITAB

One-way ANOVA: Young, Early Middle Age, Late Middle Age, Senior

```
Source   DF      SS    MS     F      P
Factor    3    3741  1247  2.79  0.041
Error   362  161871   447
Total   365  165612

S = 21.15   R-Sq = 2.26%   R-Sq(adj) = 1.45%

Level              N   Mean  StDev
Young             84  44.40  19.66
Early Middle Age 131  52.47  21.67
Late Middle Age   93  51.14  21.72
Senior            58  51.84  21.09

                       Individual 95% CIs For Mean Based on Pooled StDev
Level                  +-------------+-------------+-------------+-------------
Young                  (------------*------------)
Early Middle Age                                (---------*---------)
Late Middle Age                             (-----------*-----------)
Senior                                     (--------------*--------------)
                       +-------------+-------------+-------------+-------------
                     40.0          45.0          50.0          55.0

Pooled StDev = 21.15
```

(Continued)

INSTRUCTIONS

If the data are unstacked:

1. Type or import the data. (Open Xm14-01.)
2. Click **Stat**, **ANOVA**, and **Oneway (Unstacked) . . . .**
3. In the **Responses (in separate columns)** box, type or select the variable names of the treatments (Young, Early Middle Age, Late Middle Age, Senior).

If the data are stacked:

1. Type or import the data in two columns.
2. Click **Stat**, **ANOVA**, and **Oneway . . . .**
3. Type the variable name of the response variable and the name of the factor variable.

INTERPRET

The value of the test statistic is $F = 2.79$, and its p-value is .0405, which means there is evidence to infer that the percentage of total assets invested in stocks are different in at least two of the age categories.

Note that in this example the data are observational. We cannot conduct a controlled experiment. To do so would require the financial analyst to randomly assign households to each of the four age groups, which is impossible.

Incidentally, when the data are obtained through a controlled experiment in the one-way analysis of variance, we call the experimental design the **completely randomized design** of the analysis of variance.

Checking the Required Conditions

The F-test of the analysis of variance requires that the random variable be normally distributed with equal variances. The normality requirement is easily checked graphically by producing the histograms for each sample. From the Excel histograms in Figure 14.3, we can see that there is no reason to believe that the requirement is not satisfied.

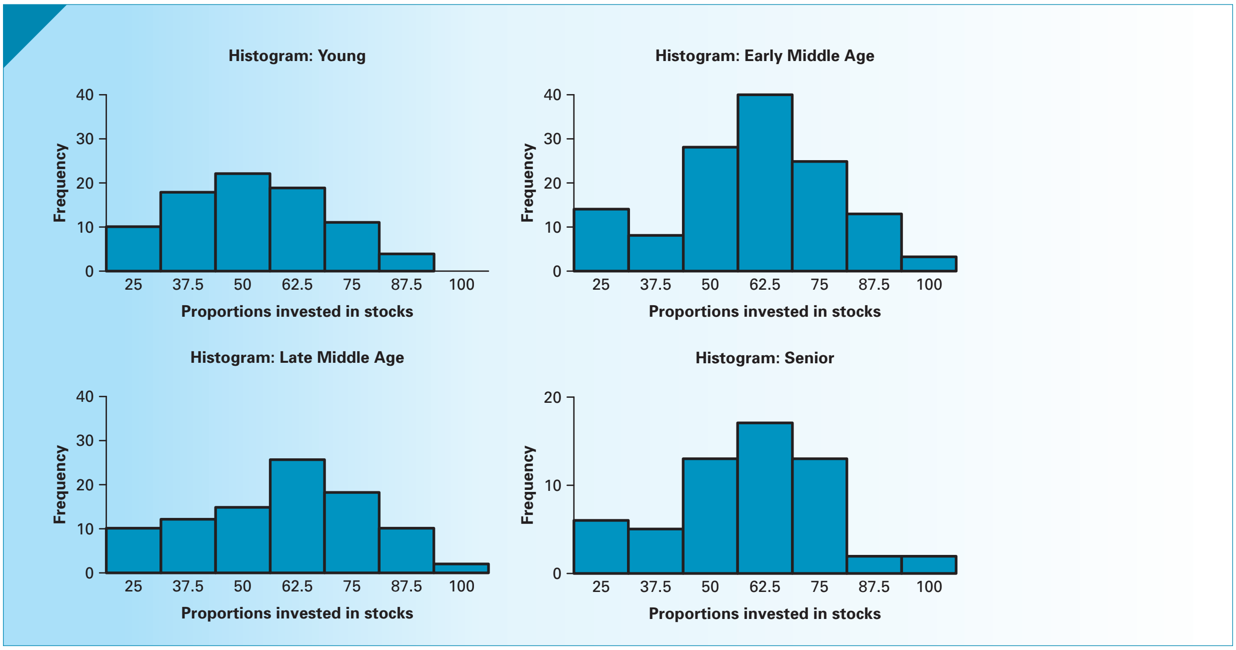
The equality of variances is examined by printing the sample standard deviations or variances. Excel output includes the variances, and Minitab calculates the standard deviations. The similarity of sample variances allows us to assume that the population variances are equal. In Keller's website Appendix Bartlett's Test, we present a statistical procedure designed to test for the equality of variances.

Violation of the Required Conditions

If the data are not normally distributed, we can replace the one-way analysis of variance with its nonparametric counterpart, which is the Kruskal–Wallis Test. (See Section 19.3.†) If the population variances are unequal, we can use several methods to correct the problem. However, these corrective measures are beyond the level of this book.

†Instructors who wish to teach the use of nonparametric techniques for testing the difference between two or more means when the normality requirement is not satisfied should use Keller's website Appendix Kruskal–Wallis Test and Friedman Test.

FIGURE **14.3** Histograms for Example 14.1



Can We Use the *t*-Test of the Difference between Two Means Instead of the Analysis of Variance?

The analysis of variance tests to determine whether there is evidence of differences between two or more population means. The *t*-test of $\mu_1 - \mu_2$ determines whether there is evidence of a difference between two population means. The question arises, Can we use *t*-tests instead of the analysis of variance? In other words, instead of testing all the means in one test as in the analysis of variance, why not test each pair of means? In Example 14.1, we would test $(\mu_1 - \mu_2)$, $(\mu_1 - \mu_3)$, $(\mu_1 - \mu_4)$, $(\mu_2 - \mu_3)$, $(\mu_2 - \mu_4)$, and $(\mu_3 - \mu_4)$. If we find no evidence of a difference in each test, we would conclude that none of the means differ. If there was evidence of a difference in at least one test, we would conclude that some of the means differ.

There are two reasons why we don't use multiple *t*-tests instead of one *F*-test. First, we would have to perform many more calculations. Even with a computer, this extra work is tedious. Second, and more important, conducting multiple tests increases the probability of making Type I errors. To understand why, consider a problem where we want to compare six populations, all of which are identical. If we conduct an analysis of variance where we set the significance level at 5%, there is a 5% chance that we would reject the true null hypothesis; that is, there is a 5% chance that we would conclude that differences exist when, in fact, they don't.

To replace the *F*-test, we would perform 15 *t*-tests. [This number is derived from the number of combinations of pairs of means to test, which is $C_2^6 = (6 \times 5)/2 = 15$.] Each test would have a 5% probability of erroneously rejecting the null hypothesis. The probability of committing one or more Type I errors is about 54%.‡

‡The probability of committing at least one Type I error is computed from a binomial distribution with $n = 15$ and $p = .05$. Thus, $P(X \geq 1) = 1 - P(X = 0) = 1 - .463 = .537$.

One remedy for this problem is to decrease the significance level. In this illustration, we would perform the t-tests with $\alpha = .05/15$, which is equal to .0033. (We will use this procedure in Section 14.2 when we discuss multiple comparisons.) Unfortunately, this would increase the probability of a Type II error. Regardless of the significance level, performing multiple t-tests increases the likelihood of making mistakes. Consequently, when we want to compare more than two populations of interval data, we use the analysis of variance.

Now that we've argued that the t-tests cannot replace the analysis of variance, we need to argue that the analysis of variance cannot replace the t-test.

Can We Use the Analysis of Variance Instead of the t-Test of $\mu_1 - \mu_2$?

The analysis of variance is the first of several techniques that allow us to compare two or more populations. Most of the examples and exercises deal with more than two populations. However, it should be noted that, like all other techniques whose objective is to compare two or more populations, the analysis of variance can be used to compare only two populations. If that's the case, then why do we need techniques to compare exactly two populations? Specifically, why do we need the t-test of $\mu_1 - \mu_2$ when the analysis of variance can be used to test two population means?

To understand why, we still need the t-test to make inferences about $\mu_1 - \mu_2$. Suppose that we plan to use the analysis of variance to test two population means. The null and alternative hypotheses are

H_0: $\mu_1 = \mu_2$

H_1: At least two means differ

Of course, the alternative hypothesis specifies that $\mu_1 \neq \mu_2$. However, if we want to determine whether μ_1 is greater than μ_2 (or vice versa), we cannot use the analysis of variance because this technique allows us to test for a difference only. Thus, if we want to test to determine whether one population mean exceeds the other, we must use the t-test of $\mu_1 - \mu_2$ (with $\sigma_1^2 = \sigma_2^2$). Moreover, the analysis of variance requires that the population variances are equal. If they are not, we must use the unequal variances test statistic.

Relationship between the F-Statistic and the t-Statistic

It is probably useful for you to understand the relationship between the t-statistic and the F-statistic. The test statistic for testing hypotheses about $\mu_1 - \mu_2$ with equal variances is

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

If we square this quantity, the result is the F-statistic: $F = t^2$. To illustrate this point, we'll redo the calculation of the test statistic in Example 13.1 using the analysis of variance. Recall that because we were able to assume that the population variances were equal, the test statistic was as follows:

$$t = \frac{(6.63 - 3.72) - 0}{\sqrt{40.42\left(\frac{1}{50} + \frac{1}{50}\right)}} = 2.29$$

Using the analysis of variance (the Excel output is shown here; Minitab's is similar), we find that the value of the test statistic is $F = 5.23$, which is $(2.29)^2$. Notice though that the analysis of variance p-value is .0243, which is twice the t-test p-value, which is .0122. The reason: The analysis of variance is conducting a test to determine whether the population means *differ*. If Example 13.1 had asked to determine whether the means differ, we would have conducted a two-tail test and the p-value would be .0243, the same as the analysis of variance p-value.

Excel Analysis of Variance Output for Example 13.1

	A	B	C	D	E	F	G
1	Anova: Single Factor						
2							
3	SUMMARY						
4	*Groups*	*Count*	*Sum*	*Average*	*Variance*		
5	Direct	50	331.6	6.63	37.49		
6	Broker	50	186.2	3.72	43.34		
7							
8							
9	ANOVA						
10	*Source of Variation*	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
11	Between Groups	211.4	1	211.41	5.23	0.0243	3.9381
12	Within Groups	3960.5	98	40.41			
13							
14	Total	4172.0	99				

Developing an Understanding of Statistical Concepts

Conceptually and mathematically, the F-test of the independent samples' single-factor analysis of variance is an extension of the t-test of $\mu_1 - \mu_2$. Moreover, if we simply want to determine whether a difference between two means exists, we can use the analysis of variance. The advantage of using the analysis of variance is that we can partition the total sum of squares, which enables us to measure how much variation is attributable to differences between populations and how much variation is attributable to differences within populations. As we pointed out in Section 13.3, explaining the variation is an extremely important topic, one that we will see again in other experimental designs of the analysis of variance and in regression analysis (Chapters 16, 17, and 18).

General Social Survey: Liberal–Conservative Spectrum And Income

© AP Photo/Chris Carlson

IDENTIFY

The variable is income (INCOME) of American adults, which is interval. The problem objective is to compare seven populations (the political views) and the experimental design is independent samples. Thus, we apply the one-way analysis of variance.

COMPUTE

EXCEL

	A	B	C	D	E	F	G
1	Anova: Single Factor						
2							
3	SUMMARY						
4	*Groups*	*Count*	*Sum*	*Average*	*Variance*		
5	E Liberal	42	1,857,750	44,232	1,420,296,929		
6	Liberal	154	6,681,000	43,383	1,644,698,667		
7	S Liberal	152	5,613,500	36,931	1,003,760,097		
8	Moderate	442	16,946,750	38,341	1,290,463,769		
9	S Conservative	156	7,920,750	50,774	1,943,227,241		
10	Conservative	183	8,179,750	44,698	1,726,326,961		
11	E Conservative	32	1,947,750	60,867	2,886,806,389		
12							
13							
14	ANOVA						
15	*Source of Variation*	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
16	Between Groups	34,931,882,213	6	5,821,980,369	3.87	0.0008	2.1064
17	Within Groups	1,735,416,094,316	1154	1,503,826,772			
18							
19	Total	1,770,347,976,529	1160				

MINITAB

One-way ANOVA: Income versus POLVIEWS

```
Source      DF           SS          MS     F      P
Polviews     6  34931882213  5821980369  3.87  0.001
Error     1154  1.73542E+12  1503826772
Total     1160  1.77035E+12

S = 38779  R-Sq = 1.97%    R-Sq(adj) = 1.46%


                            Individual 95% CIs For Mean Based on
                            Pooled StDev
Level     N   Mean  StDev   ---------+---------+---------+---------+-----
1        42  44232  37687                (-------*------)
2       154  43383  40555            (-------*------)
3       152  36931  31682           (-------*------)
4       442  38341  35923       (-------*------)
5       156  50774  44082              (-------*------)
6       183  44698  41549              (-------*------)
7        32  60867  53729        (-------*------)
                            ---------+---------+---------+---------+-----
                               36000     48000     60000     72000

Pooled StDev = 38779
```

INTERPRET

The p-value is .0008. There is sufficient evidence to infer that the incomes differ between the seven political views. It appears that conservatives have higher incomes than liberals.

Let's review how we recognize the need to use the techniques introduced in this section.

Factors That Identify the One-Way Analysis of Variance

1. **Problem objective**: Compare two or more populations
2. **Data type**: Interval
3. **Experimental design**: Independent samples

EXERCISES

Developing an Understanding of Statistical Concepts

Exercises 14.1–14.3 are "what-if" analyses designed to determine what happens to the test statistic when the means, variances, and sample sizes change. These problems can be solved manually or by creating an Excel worksheet.

14.1 A statistics practitioner calculated the following statistics:

	Treatment		
Statistic	1	2	3
n	5	5	5
$\bar{x}$	10	15	20
s^2	50	50	50

a. Complete the ANOVA table.
b. Repeat part (a) changing the sample sizes to 10 each.
c. Describe what happens to the F-statistic when the sample sizes increase.

14.2 You are given the following statistics:

	Treatment		
Statistic	1	2	3
n	4	4	4
$\bar{x}$	20	22	25
s^2	10	10	10

a. Complete the ANOVA table.
b. Repeat part (a) changing the variances to 25 each.
c. Describe the effect on the F-statistic of increasing the sample variances.

14.3 The following statistics were calculated:

	Treatment			
Statistic	1	2	3	4
n	10	14	11	18
$\bar{x}$	30	35	33	40
s^2	10	10	10	10

a. Complete the ANOVA table.
b. Repeat part (a) changing the sample means to 130, 135, 133, and 140.
c. Describe the effect on the F-statistic of increasing the sample means by 100.

Applications

14.4 Xr14-04 How does an MBA major affect the number of job offers received? An MBA student randomly sampled four recent graduates, one each in finance, marketing, and management, and asked them to report the number of job offers. Can we conclude at the 5% significance level that there are differences in the number of job offers between the three MBA majors?

Finance	Marketing	Management
3	1	8
1	5	5
4	3	4
1	4	6

14.5 Xr14-05 A consumer organization was concerned about the differences between the advertised sizes of containers and the actual amount of product. In a preliminary study, six packages of three different brands of margarine that are supposed to contain 500 ml were measured. The differences from 500 ml are listed here. Do these data provide sufficient evidence to conclude that differences exist between the three brands? Use $\alpha = .01$.

Brand 1	Brand 2	Brand 3
1	2	1
3	2	2
3	4	4
0	3	2
1	0	3
0	4	4

14.6 Xr14-06 Many college and university students obtain summer jobs. A statistics professor wanted to determine whether students in different degree programs earn different amounts. A random sample of 5 students in the BA, BSc, and BBA programs were asked to report what they earned the previous summer. The results (in $1,000s) are listed here. Can the professor infer at the 5% significance level that students in different degree programs differ in their summer earnings?

B.A.	B.Sc.	B.B.A.
3.3	3.9	4.0
2.5	5.1	6.2
4.6	3.9	6.3
5.4	6.2	5.9
3.9	4.8	6.4

14.7 Xr14-07 Spam is the price we pay for being able to easily communicate by e-mail. Does spam affect everyone equally? In a preliminary study, university professors, administrators, and students were randomly sampled. Each person was asked to count the number of spam messages received that day. The results follow. Can we infer at the 2.5% significance level that the differing university communities differ in the amount of spam they receive in their e-mails?

Professors	Administrators	Students
7	5	12
4	9	4
0	12	5
3	16	18
18	10	15

14.8 Xr14-08 A management scientist believes that one way of judging whether a computer came equipped with enough memory is to determine the age of the computer. In a preliminary study, random samples of computer users were asked to identify the brand of computer and its age (in months). The categorized responses are shown here. Do these data provide sufficient evidence to conclude that there are differences in age between the computer brands? (Use $\alpha = .05$.)

IBM	Dell	Hewlett-Packard	Other
17	8	6	24
10	4	15	12
13	21	8	15

Exercises 14.9–14.32 require the use of a computer and software. Use a 5% significance level unless specified otherwise. The answers to Exercises 14.9–14.20 may be calculated manually. See Appendix A for the sample statistics.

14.9 Xr14-09 Because there are no national or regional standards, it is difficult for university admission committees to compare graduates of different high schools. University administrators have noted that an 80% average at a high school with low standards may be equivalent to a 70% average at another school with higher standards of grading. In an effort to more equitably compare applications, a pilot study was initiated. Random samples of students who were admitted the previous year from four local high schools were drawn. All the students entered the business program with averages between 70% and 80%. Their average grades in the first year at the university were computed.

a. Can the university admissions officer conclude that there are differences in grading standards between the four high schools?
b. What are the required conditions for the test conducted in part (a)?
c. Does it appear that the required conditions of the test in part (a) are satisfied?

14.10 Xr14-10 The friendly folks at the Internal Revenue Service (IRS) in the United States and Canada Revenue Agency (CRA) are always looking for ways to improve the wording and format of its tax return forms. Three new forms have been developed recently. To determine which, if any, are superior to the current form, 120 individuals were asked to participate in an experiment. Each of the three new forms and the currently used form were filled out by 30 different people. The amount of time (in minutes) taken by each person to complete the task was recorded.

a. What conclusions can be drawn from these data?
b. What are the required conditions for the test conducted in part (a)?
c. Does it appear that the required conditions of the test in part (a) are satisfied?

14.11 Xr14-11 Are proficiency test scores affected by the education of the child's parents? (Proficiency tests are administered to a sample of students in private and public schools. Test scores can range from 0 to 500.) To answer this question, a random sample of 9-year-old children was drawn. Each child's test score and the educational level of the parent with the higher level were recorded. The education categories are less than high school, high school graduate, some college, and college graduate. Can we infer that there are differences in test scores between children whose parents have different educational levels? (Adapted from the *Statistical Abstract of the United States*, *2000*, Table 286.)

14.12 Xr14-12 A manufacturer of outdoor brass lamps and mailboxes has received numerous complaints about premature corrosion. The manufacturer has identified the cause of the problem as the low-quality lacquer used to coat the brass. He decides to replace his current lacquer supplier with one of five possible alternatives. To judge which is best, he uses each of the five lacquers to coat 25 brass mailboxes and puts all 125 mailboxes outside. He records, for each, the number of days until the first sign of corrosion is observed.

a. Is there sufficient evidence at the 1% significance level to allow the manufacturer to conclude that differences exist between the five lacquers?
b. What are the required conditions for the test conducted in part (a)?
c. Does it appear that the required conditions of the test in part (a) are satisfied?

14.13 Xr14-13 In early 2001, the economy was slowing down and companies were laying off workers. A Gallup poll asked a random sample of workers how long it would be before they had significant financial hardships if they lost their jobs and couldn't find new ones. They also classified their income. The classifications are

More than $50,000
$30,000 to $50,000
$20,000 to $30,000
Less than $20,000

Can we infer that differences exist between the four groups?

14.14 Xr14-14 In the introduction to this chapter, we mentioned that the first use of the analysis of variance was in the 1920s. It was employed to determine whether different amounts of fertilizer yielded different amounts of crop. Suppose that a scientist at an agricultural college wanted to redo the original experiment using three different types of fertilizer. Accordingly, she applied fertilizer A to 20 1-acre plots of land, fertilizer B to another 20 plots, and fertilizer C to yet another 20 plots of land. At the end of the growing season, the crop yields were recorded. Can the scientist infer that differences exist between the crop yields?

14.15 Xr14-15 A study performed by a Columbia University professor (described in *Report on Business*, August 1991) counted the number of times per minute professors from three different departments said "uh" or "ah" during lectures to fill gaps between words. The data derived from observing 100 minutes from each of the three departments were recorded. If we assume that the more frequent use of "uh" and "ah" results in more boring lectures, can we conclude that some departments' professors are more boring than others?

14.16 Xr14-16 Does the level of success of publicly traded companies affect the way their board members are paid? Publicly traded companies were divided into four quarters using the rate of return in their stocks to differentiate among the companies. The annual payment (in $1,000s) to their board members was recorded. Can we infer that the amount of payment differs between the four groups of companies?

14.17 Xr14-17 In 1994, the chief executive officers of the major tobacco companies testified before a U.S. Senate subcommittee. One of the accusations made was that tobacco firms added nicotine to their cigarettes, which made them even more addictive to smokers. Company scientists argued that the amount of nicotine in cigarettes depended completely on the size of the tobacco leaf: During poor growing seasons, the tobacco leaves would be smaller than in normal or good growing seasons. However, because the amount of nicotine in a leaf is a fixed quantity, smaller leaves would result in cigarettes having more nicotine (because a greater fraction of the leaf would be used to make a cigarette). To examine the issue, a university chemist took random samples of tobacco leaves that were grown in greenhouses where the amount of water was allowed to vary. Three different groups of tobacco leaves were grown. Group 1 leaves were grown with about an average season's rainfall. Group 2 leaves were given about 67% of group 1's water, and group 3 leaves were given 33% of group 1's water. The size of the leaf (in grams) and the amount of nicotine in each leaf were measured.
a. Test to determine whether the leaf sizes differ between the three groups.
b. Test to determine whether the amounts of nicotine differ in the three groups.

14.18 Xr14-18 There is a bewildering number of breakfast cereals on the market. Each company produces several different products in the belief that there are distinct markets. For example, there is a market composed primarily of children, another for diet-conscious adults, and another for health-conscious adults. Each cereal the companies produce has at least one market as its target. However, consumers make their own decisions, which may or may not match the target predicted by the cereal maker. In an attempt to distinguish between consumers, a survey of adults between the ages of 25 and 65 was undertaken. Each was asked several questions, including age, income, and years of education, as well as which brand of cereal they consumed most frequently. The cereal choices are

1. Sugar Smacks, a children's cereal
2. Special K, a cereal aimed at dieters
3. Fiber One, a cereal that is designed and advertised as healthy
4. Cheerios, a combination of healthy and tasty

The results of the survey were recorded using the following format:

Column 1: Cereal choice
Column 2: Age of respondent
Column 3: Annual household income
Column 4: Years of education

a. Determine whether there are differences between the ages of the consumers of the four cereals.
b. Determine whether there are differences between the incomes of the consumers of the four cereals.
c. Determine whether there are differences between the educational levels of the consumers of the four cereals.
d. Summarize your findings in parts (a) through (c) and prepare a report describing the differences between the four groups of cereal consumers.

APPLICATIONS in MARKETING

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Test Marketing

In Chapter 13, we introduced test marketing, which allows us to determine whether changing some of the elements of the marketing mix yields different sales. In the next exercise, we apply the technique to discover the effect of different prices.

14.19 Xr14-19 A manufacturer of novelty items is undecided about the price to charge for a new product. The marketing manager knows that it should sell for about $10 but is unsure of whether sales will vary significantly if it is priced at either $9 or $11. To conduct a pricing experiment, she distributes the new product to a sample of 60 stores belonging to a certain chain of variety stores. These 60 stores are all located in similar neighborhoods. The manager randomly selects 20 stores in which to sell the item at $9, 20 stores to sell it at $10, and the remaining 20 stores to sell it at $11. Sales at the end of the trial period were recorded. What should the manager conclude?

APPLICATIONS in MARKETING

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Marketing Segmentation

Section 12.4 introduced market segmentation. In Chapter 13 we demonstrated how to use statistical analyses to determine whether two segments differ in their buying behavior. The next exercise requires you to apply the analysis of variance to determine whether several segments differ.

14.20 Xr14-20 After determining in Exercise 13.155 that teenagers watch more movies than do 20 to 30 year olds, teenagers were further segmented into three age groups: 12 to 14, 15 to 16, and 17 to 19. Random samples were drawn from each segment, and the number of movies each teenager saw last year was recorded. Do these data allow a marketing manager of a movie studio to conclude that differences exist between the three segments?

GENERAL SOCIAL SURVEY EXERCISES

14.21 GSS2002* GSS2004* GSS2006* GSS2008* Have educational levels kept uniform over the years 2002, 2004, 2006, and 2008? Conduct a test to determine whether the number of years of education (EDUC) differ in the four years.

14.22 GSS2008* Television networks and their advertisers are constantly measuring viewers to determine their likes and dislikes and how much time adults spend watching television per day. Do the data from the General Social Survey in 2008 allow us to infer that the amount of television (TVHOURS) differs by race (RACE)?

14.23 GSS2008* How are income and degree related? The General Social Survey asked respondents to identify the highest degree completed (DEGREE: 0 = Left high school, 1 = High school, 2 = Junior college, 3 = Bachelor's degree, 4 = Graduate degree). Is there enough statistical evidence to conclude that there are differences in income (INCOME) between people with different completed degrees?

14.24 GSS2002* GSS2004* GSS2006* GSS2008* Has the amount of time Americans devote to work weekly (HRS) changed over the years 2002, 2004, 2006, and 2008? Perform a statistical analysis to answer the question.

14.25 GSS2008* Who earns more money: married people, single people, widows and widowers, or divorcees? Conduct an appropriate statistical technique to determine whether there is enough evidence to conclude that incomes (INCOME) vary by marital status (MARITAL).

14.26 GSS2002* GSS2004* GSS2006* GSS2008* Has the amount of television American adults watch been constant over the years 2002, 2004, 2006, and 2008, or has the amount varied? Test to determine whether the number of hours of television per day (TVHOURS) changed over the 8-year span.

American National Election Survey Exercises

14.27 ANES2008* Repeat the chapter-opening example using the data from the American National Election Survey of 2008. Test to determine whether annual incomes (INCOME) differ between the seven political views (LIBCON).

14.28 ANES2008* Who has the most and least education among Democrats, Independents, and Republicans? Conduct a statistical test to determine if there is evidence of a difference in education (EDUC) between the three political affiliations (PARTY3).

14.29 ANES2008* Who reads newspapers more: Democrats, Independents, or Republicans? Test to determine whether differences exist in the number of days reading a newspaper (DAYS9) between the three political affiliations (PARTY3).

14.30 ANES2008* How are income and degree related? The American National Election Survey asked respondents who reported at least 13 years of education to identify the highest degree completed (DEGREE: 0 = No degree earned; 1 = Bachelor's degree; 2 = Master's degree; 3 = PhD, etc.; 4 = LLB, JD; 5 = MD, DDS, etc.; 6 = JDC, STD, THD; 7 = Associate's degree). Is there enough statistical evidence to conclude that there are differences in income (INCOME) between people with different completed degrees?

14.31 ANES2008* Are marital status and education related? If so, we would expect that the amount of education in at least two of the marital status categories to be different. Conduct a statistical procedure to determine whether there is enough evidence to infer that the amount of education (EDUC) differs between marital status (MARITAL) categories.

14.32 ANES2008* How definite is one's intention to vote in the presidential election, and is that intention related to party affiliation? To answer this question, conduct a test to determine whether the intention to vote (DEFINITE) varies between Democrats, Independents, and Republicans (PARTY3).

14.2 Multiple Comparisons

When we conclude from the one-way analysis of variance that at least two treatment means differ, we often need to know which treatment means are responsible for these differences. For example, if an experiment is undertaken to determine whether different locations within a store produce different mean sales, the manager would be keenly interested in determining which locations result in significantly higher sales and which locations result in lower sales. Similarly, a stockbroker would like to know which one of several mutual funds outperforms the others, and a television executive would like to know which television commercials hold the viewers' attention and which are ignored.

Although it may appear that all we need to do is examine the sample means and identify the largest or the smallest to determine which population means are largest or smallest, this is not the case. To illustrate, suppose that in a five-treatment analysis of variance, we discover that differences exist and that the sample means are as follows:

$$\bar{x}_1 = 20 \quad \bar{x}_2 = 19 \quad \bar{x}_3 = 25 \quad \bar{x}_4 = 22 \quad \bar{x}_5 = 17$$

The statistics practitioner wants to know which of the following conclusions are valid:

1. μ_3 is larger than the other means.
2. μ_3 and μ_4 are larger than the other means.
3. μ_5 is smaller than the other means.
4. μ_5 and μ_2 are smaller than the other means.
5. μ_3 is larger than the other means, and μ_5 is smaller than the other means.

From the information we have, it is impossible to determine which, if any, of the statements are true. We need a statistical method to make this determination. The technique is called **multiple comparisons**.

EXAMPLE 14.2 Comparing the Costs of Repairing Car Bumpers

DATA Xm14-02

Because of foreign competition, North American automobile manufacturers have become more concerned with quality. One aspect of quality is the cost of repairing damage caused by accidents. A manufacturer is considering several new types of bumpers. To test how well they react to low-speed collisions, 10 bumpers of each of four different types were installed on mid-size cars, which were then driven into a wall at 5 miles per hour. The cost of repairing the damage in each case was assessed. The data are shown below.

a. Is there sufficient evidence at the 5% significance level to infer that the bumpers differ in their reactions to low-speed collisions?

b. If differences exist, which bumpers differ?

Bumper 1	Bumper 2	Bumper 3	Bumper 4
610	404	599	272
354	663	426	405
234	521	429	197
399	518	621	363
278	499	426	297
358	374	414	538
379	562	332	181
548	505	460	318
196	375	494	412
444	438	637	499

SOLUTION

IDENTIFY

The problem objective is to compare four populations. The data are interval, and the samples are independent. The correct statistical method is the one-way analysis of variance, which we perform using Excel and Minitab.

COMPUTE

EXCEL

	A	B	C	D	E	F	G
1	Anova: Single Factor						
2							
3	SUMMARY						
4	*Groups*	*Count*	*Sum*	*Average*	*Variance*		
5	Bumper 1	10	3800	380.0	16,924		
6	Bumper 2	10	4859	485.9	8,197		
7	Bumper 3	10	4838	483.8	10,426		
8	Bumper 4	10	3482	348.2	14,049		
9							
10							
11	ANOVA						
12	*Source of Variation*	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
13	Between Groups	150,884	3	50,295	4.06	0.0139	2.8663
14	Within Groups	446,368	36	12,399			
15							
16	Total	597,252	39				

MINITAB

One-way ANOVA: Bumper 1, Bumper 2, Bumper 3, Bumper 4

```
Source  DF      SS     MS     F      P
Factor   3  150884  50295  4.06  0.014
Error   36  446368  12399
Total   39  597252

S = 111.4   R-Sq = 25.26%   R-Sq(adj) = 19.03%

                                Individual 95% CIs For Mean Based on Pooled StDev
Level      N   Mean  StDev   ---------+-----------------+-----------------+-----------------+--------
Bumper 1  10  380.0  130.1          (---------------*--------------)
Bumper 2  10  485.9   90.5                                (----------------*----------------)
Bumper 3  10  483.8  102.1                                (--------------*----------------)
Bumper 4  10  348.2  118.5   (-----------------*--------------)
                             ---------+-----------------+-----------------+-----------------+--------
                                     320               400               480               560
```

INTERPRET

The test statistic is $F = 4.06$ and the p-value $= .0139$. There is enough statistical evidence to infer that there are differences between some of the bumpers. The question is now, Which bumpers differ?

There are several statistical inference procedures that deal with this problem. We will present three methods that allow us to determine which population means differ. All three methods apply to the one-way experiment only.

Fisher's Least Significant Difference (LSD) Method

The **least significant difference (LSD)** method was briefly introduced in Section 14.1 (page 535). To determine which population means differ, we could perform a series of t-tests of the difference between two means on all pairs of population means to determine which are significantly different. In Chapter 13, we introduced the equal-variances t-test of the difference between two means. The test statistic and confidence interval estimator are, respectively,

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

with degrees of freedom $\nu = n_1 + n_2 - 2$.

Recall that s_p^2 is the pooled variance estimate, which is an unbiased estimator of the variance of the two populations. (Recall that the use of these techniques requires that the population variances be equal.) In this section, we modify the test statistic and interval estimator.

Earlier in this chapter, we pointed out that MSE is an unbiased estimator of the common variance of the populations we're testing. Because MSE is based on all the observations in the k samples, it will be a better estimator than s_p^2 (which is based on only two samples). Thus, we could draw inferences about every pair of means by substituting MSE for s_p^2 in the formulas for test statistic and confidence interval estimator shown previously. The number of degrees of freedom would also change to $\nu = n - k$ (where n is the total sample size). The test statistic to determine whether μ_i and μ_j differ is

$$t = \frac{(\bar{x}_i - \bar{x}_j) - (\mu_i - \mu_j)}{\sqrt{\text{MSE}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}}$$

The confidence interval estimator is

$$(\bar{x}_i - \bar{x}_j) \pm t_{\alpha/2}\sqrt{\text{MSE}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

with degrees of freedom $\nu = n - k$.

We define the least significant difference LSD as

$$\text{LSD} = t_{\alpha/2}\sqrt{\text{MSE}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

A simple way of determining whether differences exist between each pair of population means is to compare the absolute value of the difference between their two sample means and LSD. In other words, we will conclude that μ_i and μ_j differ if

$$|\bar{x}_i - \bar{x}_j| > \text{LSD}$$

LSD will be the same for all pairs of means if all k sample sizes are equal. If some sample sizes differ, LSD must be calculated for each combination.

In Section 14.1 we argued that this method is flawed because it will increase the probability of committing a Type I error. That is, it is more likely than the analysis of variance to conclude that a difference exists in some of the population means when in fact none differ. On page 535, we calculated that if $k = 6$ and all population means are equal, the probability of erroneously inferring at the 5% significance level that at least two means differ is about 54%. The 5% figure is now referred to as the *comparisonwise Type I error rate*. The true probability of making at least one Type I error is called the *experimentwise Type I error rate*, denoted α_E. The experimentwise Type I error rate can be calculated as

$$\alpha_E = 1 - (1 - \alpha)^C$$

Here C is the number of pairwise comparisons, which can be calculated by $C = k(k - 1)/2$. Mathematicians have proven that

$$\alpha_E \le C\alpha$$

which means that if we want the probability of making at least one Type I error to be no more than α_E, we simply specify $\alpha = \alpha_E/C$. The resulting procedure is called the **Bonferroni adjustment**.

Bonferroni Adjustment to LSD Method

The adjustment is made by dividing the specified experimentwise Type I error rate by the number of combinations of pairs of population means. For example, if $k = 6$, then

$$C = \frac{k(k-1)}{2} = \frac{6(5)}{2} = 15$$

If we want the true probability of a Type I error to be no more than 5%, we divide this probability by C. Thus, for each test we would use a value of α equal to

$$\alpha = \frac{\alpha_E}{C} = \frac{.05}{15} = .0033$$

We use Example 14.2 to illustrate Fisher's LSD method and the Bonferroni adjustment. The four sample means are

$$\bar{x}_1 = 380.0$$
$$\bar{x}_2 = 485.9$$
$$\bar{x}_3 = 483.8$$
$$\bar{x}_4 = 348.2$$

The pairwise absolute differences are

$$|\bar{x}_1 - \bar{x}_2| = |380.0 - 485.9| = |-105.9| = 105.9$$
$$|\bar{x}_1 - \bar{x}_3| = |380.0 - 483.8| = |-103.8| = 103.8$$
$$|\bar{x}_1 - \bar{x}_4| = |380.0 - 348.2| = |31.8| = 31.8$$
$$|\bar{x}_2 - \bar{x}_3| = |485.9 - 483.8| = |2.1| = 2.1$$
$$|\bar{x}_2 - \bar{x}_4| = |485.9 - 348.2| = |137.7| = 137.7$$
$$|\bar{x}_3 - \bar{x}_4| = |483.8 - 348.2| = |135.6| = 135.6$$

From the computer output, we learn that MSE = 12,399 and $\nu = n - k = 40 - 4 = 36$. If we conduct the LSD procedure with $\alpha = .05$ we find $t_{\alpha/2,n-k} = t_{.025,36} \approx t_{.025,35} = 2.030$. Thus,

$$t_{\alpha/2}\sqrt{\text{MSE}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)} = 2.030\sqrt{12{,}399\left(\frac{1}{10} + \frac{1}{10}\right)} = 101.09$$

We can see that four pairs of sample means differ by more than 101.09. In other words, $|\bar{x}_1 - \bar{x}_2| = 105.9$, $|\bar{x}_1 - \bar{x}_3| = 103.8$, $|\bar{x}_2 - \bar{x}_4| = 137.7$, and $|\bar{x}_3 - \bar{x}_4| = 135.6$. Hence, μ_1 and μ_2, μ_1 and μ_3, μ_2 and μ_4, and μ_3 and μ_4 differ. The other two pairs—μ_1 and μ_4, and μ_2 and μ_3—do not differ.

If we perform the LSD procedure with the Bonferroni adjustment, the number of pairwise comparisons is 6 (calculated as $C = k(k-1)/2 = 4(3)/2$). We set $\alpha = .05/6 = .0083$. Thus $t_{\alpha/2,36} = t_{.0042,36} = 2.794$ (available from Excel and difficult to approximate manually) and

$$\text{LSD} = t_{\alpha/2}\sqrt{\text{MSE}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)} = 2.794\sqrt{12{,}399\left(\frac{1}{10} + \frac{1}{10}\right)} = 139.13$$

Now no pair of means differ because all the absolute values of the differences between sample means are less than 139.19.

The drawback to the LSD procedure is that we increase the probability of at least one Type I error. The Bonferroni adjustment corrects this problem. However, recall that the probabilities of Type I and Type II errors are inversely related. The Bonferroni adjustment uses a smaller value of α, which results in an increased probability of a Type II error. A Type II error occurs when a difference between population means exists, yet we cannot detect it. This may be the case in this example. The next multiple comparison method addresses this problem.

Tukey's Multiple Comparison Method

A more powerful test is **Tukey's multiple comparison method**. This technique determines a critical number similar to LSD for Fisher's test, denoted by ω (Greek letter *omega*) such that, if any pair of sample means has a difference greater than ω, we conclude that the pair's two corresponding population means are different.

The test is based on the Studentized range, which is defined as the variable

$$q = \frac{\bar{x}_{\max} - \bar{x}_{\min}}{s/\sqrt{n}}$$

where $\bar{x}_{\max}$ and $\bar{x}_{\min}$ are the largest and smallest sample means, respectively, assuming that there are no differences between the population means. We define ω as follows.

Critical Number ω

$$\omega = q_\alpha(k,\nu)\sqrt{\frac{\text{MSE}}{n_g}}$$

where

k = Number of treatments
n = Number of observations ($n = n_1 + n_2 + \cdots + n_k$)
ν = Number of degrees of freedom associated with MSE ($\nu = n - k$)
n_g = Number of observations in each of k samples
α = Significance level
$q_\alpha(k, \nu)$ = Critical value of the Studentized range

Theoretically, this procedure requires that all sample sizes be equal. However, if the sample sizes are different, we can still use this technique provided that the sample sizes are at least similar. The value of n_g used previously is the *harmonic mean* of the sample sizes; that is,

$$n_g = \frac{k}{\dfrac{1}{n_1} + \dfrac{1}{n_2} + \cdots + \dfrac{1}{n_k}}$$

Table 7 in Appendix B provides values of $q_\alpha\,(k,\nu)$ for a variety of values of k and ν, and for $\alpha = .01$ and .05. Applying Tukey's method to Example 14.2, we find

$$\begin{aligned} k &= 4 \\ n_1 = n_2 = n_3 = n_4 = n_g &= 10 \\ \nu = n - k = 40 - 4 &= 36 \\ \text{MSE} &= 12{,}399 \\ q_{.05}(4, 37) \approx q_{.05}(4, 40) &= 3.79 \end{aligned}$$

Thus,

$$\omega = q_\alpha(k,\nu)\sqrt{\frac{\text{MSE}}{n_g}} = (3.79)\sqrt{\frac{12{,}399}{10}} = 133.45$$

There are two absolute values larger than 133.45. Hence, we conclude that μ_2 and μ_4, and μ_3 and μ_4 differ. The other four pairs do not differ.

EXCEL

	A	B	C	D	E
1	**Multiple Comparisons**				
2					
3				LSD	Omega
4	Treatment	Treatment	Difference	Alpha = 0.05	Alpha = 0.05
5	*Bumper 1*	*Bumper 2*	–105.9	100.99	133.45
6		*Bumper 3*	–103.8	100.99	133.45
7		*Bumper 4*	31.8	100.99	133.45
8	*Bumper 2*	*Bumper 3*	2.1	100.99	133.45
9		*Bumper 4*	137.7	100.99	133.45
10	*Bumper 3*	*Bumper 4*	135.6	100.99	133.45

Tukey and Fisher's LSD with the Bonferroni Adjustment (α = .05/6 = 0083)

	A	B	C	D	E
1	**Multiple Comparisons**				
2					
3				LSD	Omega
4	Treatment	Treatment	Difference	Alpha = 0.0083	Alpha = 0.05
5	*Bumper 1*	*Bumper 2*	–105.9	139.11	133.45
6		*Bumper 3*	–103.8	139.11	133.45
7		*Bumper 4*	31.8	139.11	133.45
8	*Bumper 2*	*Bumper 3*	2.1	139.11	133.45
9		*Bumper 4*	137.7	139.11	133.45
10	*Bumper 3*	*Bumper 4*	135.6	139.11	133.45

The printout includes ω (Tukey's method), the differences between sample means for each combination of populations, and Fisher's LSD. (The Bonferroni adjustment is made by specifying another value for α.)

INSTRUCTIONS

1. Type or import the data into adjacent columns. (Open Xm14-02.)
2. Click **Add-Ins, Data Analysis Plus,** and **Multiple Comparisons**.
3. Specify the **Input Range** (A1:D11). Type the value of α. To use the Bonferroni adjustment divide α by $C = k(k - 1)/2$. For Tukey, Excel computes ω only for $\alpha = .05$.

MINITAB

```
Tukey 95% Simultaneous Confidence Intervals
All Pairwise Comparisons

Individual confidence level = 98.38%

Bumper 1 subtracted from:

           Lower  Center   Upper  ---------+---------+---------+---------+-
Bumper 2   -28.3   105.9   240.1                (---------*---------)
Bumper 3   -30.4   103.8   238.0                (---------*---------)
Bumper 4  -166.0   -31.8   102.4       (---------*---------)
                                  ---------+---------+---------+---------+-
                                        -150         0       150       300

Bumper 2 subtracted from:

           Lower  Center   Upper  ---------+---------+---------+---------+-
Bumper 3  -136.3    -2.1   132.1          (---------*---------)
Bumper 4  -271.9  -137.7    -3.5  (---------*---------)
                                  ---------+---------+---------+---------+--
                                        -150         0       150       300

Bumper 3 subtracted from:

           Lower  Center   Upper  ---------+---------+---------+---------+-
Bumper 4  -269.8  -135.6    -1.4  (---------*---------)
                                  ---------+---------+---------+---------+-
                                        -150         0       150       300

Fisher 99.17% Individual Confidence Intervals
All Pairwise Comparisons

Simultaneous confidence level = 96.04%

Bumper 1 subtracted from:

           Lower  Center   Upper  ---------+---------+---------+---------+-
Bumper 2   -33.2   105.9   245.0                (---------*---------)
Bumper 3   -35.3   103.8   242.9                (---------*---------)
Bumper 4  -170.9   -31.8   107.3       (---------*---------)
                                  ---------+---------+---------+---------+-
                                        -150         0       150       300

Bumper 2 subtracted from:

           Lower  Center   Upper  ---------+---------+---------+---------+-
Bumper 3  -141.2    -2.1   137.0        (---------*---------)
Bumper 4  -276.8  -137.7     1.4  (---------*---------)
                                  ---------+---------+---------+---------+-
                                        -150         0       150       300

Bumper 3 subtracted from:

           Lower  Center   Upper  ---------+---------+---------+---------+-
Bumper 4  -274.4  -135.6     3.5  (---------*---------)
                                  ---------+---------+---------+---------+-
                                        -150         0       150       300
```

Minitab reports the results of Tukey's multiple comparisons by printing interval estimates of the differences between each pair of means. The estimates are computed by calculating the pairwise difference between sample means minus ω for the lower limit and plus ω for the upper limit. The calculations are described in the following table.

Tukey's Method

Pair of Population Means Compared	Difference	Lower Limit	Upper Limit
Bumper 2–Bumper 1	105.9	−28.3	240.1
Bumper 3–Bumper 1	103.8	−30.4	238.0
Bumper 4–Bumper 1	−31.8	−166.0	102.4
Bumper 3–Bumper 2	−2.1	−136.3	132.1
Bumper 4–Bumper 2	−137.7	−271.9	−3.5
Bumper 4–Bumper 3	−135.6	−269.8	−1.4

A similar calculation is performed for Fisher's method replacing ω by LSD.

Fisher's Method

Pair of Population Means Compared	Difference	Lower Limit	Upper Limit
Bumper 2–Bumper 1	105.9	−33.2	245.0
Bumper 3–Bumper 1	103.8	−35.3	242.9
Bumper 4–Bumper 1	−31.8	−170.9	107.3
Bumper 3–Bumper 2	−2.1	−141.2	137.0
Bumper 4–Bumper 2	−137.7	−276.8	1.4
Bumper 4–Bumper 3	−135.6	274.7	3.5

We interpret the test results in the following way. If the interval includes 0, there is not enough evidence to infer that the pair of means differ. If the entire interval is above or the entire interval is below 0, we conclude that the pair of means differ.

INSTRUCTIONS

1. Type or import the data either in stacked or unstacked format. (Open Xm14-02.)
2. Click **Stat, ANOVA**, and **Oneway (Unstacked) . . . .**
3. Type or **Select** the variables in the **Responses (in separate columns)** box (Bumper 1, Bumper 2, Bumper 3, Bumper 4).
4. Click **Comparisons . . .** Select Tukey's method and specify α. Select Fisher's method and specify α. For the Bonferroni adjustment divide α by $C = k(k - 1)/2$.

INTERPRET

Using the Bonferroni adjustment of Fisher's LSD method, we discover that none of the bumpers differ. Tukey's method tells us that bumper 4 differs from both bumpers 2 and 3. Based on this sample, bumper 4 appears to have the lowest cost of repair. Because there was not enough evidence to conclude that bumpers 1 and 4 differ, we would consider using bumper 1 if it has other advantages over bumper 4.

Which Multiple Comparison Method to Use

Unfortunately, no one procedure works best in all types of problems. Most statisticians agree with the following guidelines:

If you have identified two or three pairwise comparisons that you wish to make before conducting the analysis of variance, use the Bonferroni method. This means that if there are 10 populations in a problem but you're particularly interested in comparing, say, populations 3 and 7 and populations 5 and 9, use Bonferroni with $C = 2$.

If you plan to compare all possible combinations, use Tukey.

When do we use Fisher's LSD? If the purpose of the analysis is to point to areas that should be investigated further, Fisher's LSD method is indicated.

Incidentally, to employ Fisher's LSD or the Bonferroni adjustment, you must perform the analysis of variance first. Tukey's method can be employed instead of the analysis of variance.

EXERCISES

Developing an Understanding of Statistical Concepts

14.33 a. Use Fisher's LSD method with $\alpha = .05$ to determine which population means differ in the following problem.

$k = 3 \quad n_1 = 10 \quad n_2 = 10 \quad n_3 = 10$

$\text{MSE} = 700 \quad \bar{x}_1 = 128.7 \quad \bar{x}_2 = 101.4 \quad \bar{x}_3 = 133.7$

b. Repeat part (a) using the Bonferroni adjustment.
c. Repeat part (a) using Tukey's multiple comparison method.

14.34 a. Use Fisher's LSD procedure with $\alpha = .05$ to determine which population means differ given the following statistics:

$k = 5 \quad n_1 = 5 \quad n_2 = 5 \quad n_3 = 5$

$\text{MSE} = 125 \quad \bar{x}_1 = 227 \quad \bar{x}_2 = 205 \quad \bar{x}_3 = 219$

$n_4 = 5 \quad n_5 = 5$

$\bar{x}_4 = 248 \quad \bar{x}_5 = 202$

b. Repeat part (a) using the Bonferroni adjustment.
c. Repeat part (a) using Tukey's multiple comparison method

Applications

14.35 Apply Tukey's method to determine which brands differ in Exercise 14.5.

14.36 Refer to Exercise 14.6.
a. Employ Fisher's LSD method to determine which degrees differ (use $\alpha = .10$).
b. Repeat part (a) using the Bonferroni adjustment.

Exercises 14.37–14.50 require the use of a computer and software. Use a 5% significance level unless specified otherwise. The answers to Exercises 14.37–14.42 may be calculated manually. See Appendix A for the sample statistics.

14.37 Xr14-09 a. Apply Fisher's LSD method with the Bonferroni adjustment to determine which schools differ in Exercise 14.9.
b. Repeat part (a) applying Tukey's method instead.

14.38 Xr14-10 a. Apply Tukey's multiple comparison method to determine which forms differ in Exercise 14.10.
b. Repeat part (a) applying the Bonferroni adjustment.

14.39 Xr14-39 Police cars, ambulances, and other emergency vehicles are required to carry road flares. One of the most important features of flares is their burning times. To help decide which of four brands on the market to use, a police laboratory technician measured the burning time for a random sample of 10 flares of each brand. The results were recorded to the nearest minute.
a. Can we conclude that differences exist between the burning times of the four brands of flares?
b. Apply Fisher's LSD method with the Bonferroni adjustment to determine which flares are better.
c. Repeat part (b) using Tukey's method.

14.40 Xr14-12 Refer to Exercise 14.12.
a. Apply Fisher's LSD method with the Bonferroni adjustment to determine which lacquers differ.
b. Repeat part (a) applying Tukey's method instead.

14.41 Xr14-41 An engineering student who is about to graduate decided to survey various firms in Silicon Valley to see which offered the best chance for early promotion and career advancement. He surveyed 30 small firms (size level is based on gross revenues), 30 medium-size firms, and 30 large firms and determined how much time must elapse before an average engineer can receive a promotion.
a. Can the engineering student conclude that speed of promotion varies between the three sizes of engineering firms?
b. If differences exist, which of the following is true? Use Tukey's method.
   i. Small firms differ from the other two.
   ii. Medium-size firms differ from the other two.
   iii. Large firms differ from the other two.
   iv. All three firms differ from one another.
   v. Small firms differ from large firms.

14.42 Xr14-14 a. Apply Tukey's multiple comparison method to determine which fertilizers differ in Exercise 14.14.
b. Repeat part (a) applying the Bonferroni adjustment.

GENERAL SOCIAL SURVEY EXERCISES

14.43 GSS2002* GSS2004* GSS2006* GSS2008* Refer to Exercise 14.21. Use an appropriate statistical technique to determine which years differ with respect to the amount of education (EDUC).

14.44 GSS2008* Refer to Exercise 14.22. Use Tukey's multiple comparison method to determine which races differ in the amount of television watched (TVHOURS).

14.45 GSS2008* Refer to Exercise 14.23. Use a suitable multiple comparison method to determine which degrees differ in annual incomes (INCOME).

14.46 GSS2008* Refer to Exercise 14.25. Apply a suitable multiple comparison method to determine which categories of marital status differ.

American National Election Survey Exercises

14.47 ANES2008* Refer to Exercise 14.27. Apply Tukey's multiple comparison method to determine which positions on the liberal–conservative spectrum (LIBCON) differ with respect to income (INCOME).

14.48 ANES2008* Refer to Exercise 14.28. Use a multiple comparison method to determine which of the three parties (PARTY3) differ with respect to education (EDUC).

14.49 ANES2008* Refer to Exercise 14.31. Use Tukey's multiple comparison method to find which categories of marital status (MARITAL) differ with respect to education (EDUC).

14.50 ANES2008* Refer to Exercise 14.32. Use an appropriate multiple comparison method to determine which of the three parties (PARTY3) differ with respect to intention to vote (DEFINITE).

14.3 Analysis of Variance Experimental Designs

Since we introduced the matched pairs experiment in Section 13.3, the experimental design has been one of the factors that determines which technique we use. Statistics practitioners often design experiments to help extract the information they need to assist them in making decisions. The one-way analysis of variance introduced in Section 14.1 is only one of many different experimental designs of the analysis of variance. For each type of experiment, we can describe the behavior of the response variable using a mathematical expression or model. Although we will not exhibit the mathematical expressions in this chapter (we introduce models in Chapter 16), we think it is useful for you to be aware of the elements that distinguish one experimental design or model from another. In this section, we present some of these elements; in so doing, we introduce two of the experimental designs that will be presented later in this chapter.

Single-Factor and Multifactor Experimental Designs

As we pointed out in Section 14.1, the criterion by which we identify populations is called a *factor*. The experiment described in Section 14.1 is a single-factor analysis of variance because it addresses the problem of comparing two or more populations defined on the basis of only one factor. A **multifactor experiment** is one in which two or more factors define the treatments. The experiment described in Example 14.1 is a single-factor design because we had one treatment: age of the head of the household. In other words, the factor is the age, and the four age categories were the levels of this factor.

Suppose that we can also look at the gender of the household head in another study. We would then develop a two-factor analysis of variance in which the first factor, age, has four levels, and the second factor, gender, has two levels. We will discuss two-factor experiments in Section 14.5.

Independent Samples and Blocks

In Section 13.3, we introduced statistical techniques where the data were gathered from a matched pairs experiment. This type of experimental design reduces the variation within the samples, making it easier to detect differences between the two populations.

When the problem objective is to compare more than two populations, the experimental design that is the counterpart of the matched pairs experiment is called the **randomized block design**. The term *block* refers to a matched group of observations from each population. Suppose that in Examples 13.4 and 13.5 we had wanted to compare the salary offers for finance, marketing, accounting, and operations management majors. To redo Example 13.5 we would conduct a randomized block experiment where the blocks are the 25 GPA groups and the treatments are the four MBA majors.

Once again, the experimental design should reduce the variation in each treatment to make it easier to detect differences.

We can also perform a blocked experiment by using the same subject (person, plant, and store) for each treatment. For example, we can determine whether sleeping pills are effective by giving three brands of pills to the same group of people to measure the effects. Such experiments are called **repeated measures** designs. Technically, this is a different design than the randomized block. However, the data are analyzed in the same way for both designs. Hence, we will treat repeated measures designs as randomized block designs.

The randomized block experiment is also called the **two-way analysis of variance**. In Section 14.4, we introduce the technique used to calculate the test statistic for this type of experiment.

Fixed and Random Effects

If our analysis includes all possible levels of a factor, the technique is called a **fixed-effects analysis of variance**. If the levels included in the study represent a random sample of all the levels that exist, the technique is called a **random-effects analysis of variance**. In Example 14.2, there were only four possible bumpers. Consequently, the study is a fixed-effects experiment. However, if there were other bumpers besides the four described in the example, and we wanted to know whether there were differences in repair costs between all bumpers, the application would be a random-effects experiment. Here's another example.

To determine whether there is a difference in the number of units produced by the machines in a large factory, 4 machines out of 50 in the plant are randomly selected for study. The number of units each produces per day for 10 days will be recorded. This experiment is a random-effects experiment because we selected a random sample of four machines and the statistical results thus allow us to determine whether there are differences between the 50 machines.

In some experimental designs, there are no differences in calculations of the test statistic between fixed and random effects. However, in others, including the two-factor experiment presented in Section 14.5, the calculations are different.

14.4 Randomized Block (Two-Way) Analysis of Variance

The purpose of designing a randomized block experiment is to reduce the within-treatments variation to more easily detect differences between the treatment means. In the one-way analysis of variance, we partitioned the total variation into the between-treatments and the within-treatments variation; that is,

$$\text{SS(Total)} = \text{SST} + \text{SSE}$$

In the randomized block design of the analysis of variance, we partition the total variation into three sources of variation,

$$\text{SS(Total)} = \text{SST} + \text{SSB} + \text{SSE}$$

where **SSB**, the **sum of squares for blocks**, measures the variation between the blocks. When the variation associated with the blocks is removed, SSE is reduced, making it easier to determine whether differences exist between the treatment means.

At this point in our presentation of statistical inference, we will deviate from our usual procedure of solving examples in three ways: manually, using Excel, and using Minitab. The calculations for this experimental design and for the experiment presented in the next section are so time consuming that solving them by hand adds little to your understanding of the technique. Consequently, although we will continue to present the concepts by discussing how the statistics are calculated, we will solve the problems only by computer.

To help you understand the formulas, we will use the following notation:

$\bar{x}[T]_j$ = Mean of the observations in the jth treatment ($j = 1, 2, \ldots, k$)

$\bar{x}[B]_i$ = Mean of the observations in the ith block ($i = 1, 2, \ldots, b$)

b = Number of blocks

Table 14.4 summarizes the notation we use in this experimental design.

TABLE 14.4 Notation for the Randomized Block Analysis of Variance

	TREATMENTS				
BLOCK	1	2		k	BLOCK MEAN
1	x_{11}	x_{12}	$\cdots$	x_{1k}	$\bar{x}[B]_1$
2	x_{21}	x_{22}	$\cdots$	x_{2k}	$\bar{x}[B]_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
b	x_{b1}	x_{b2}	$\cdots$	x_{bk}	$\bar{x}[B]_b$
Treatment mean	$\bar{x}[T]_1$	$\bar{x}[T]_2$	$\cdots$	$\bar{x}[T]_k$	

The definitions of SS(Total) and SST in the randomized block design are identical to those in the independent samples design. SSE in the independent samples design is equal to the sum of SSB and SSE in the randomized block design.

Sums of Squares in the Randomized Block Experiment

$$\text{SS(Total)} = \sum_{j=1}^{k}\sum_{i=1}^{b}(x_{ij} - \bar{\bar{x}})^2$$

$$\text{SST} = \sum_{j=1}^{k} b(\bar{x}[T]_j - \bar{\bar{x}})^2$$

$$\text{SSB} = \sum_{i=1}^{b} k(\bar{x}[B]_i - \bar{\bar{x}})^2$$

$$\text{SSE} = \sum_{j=1}^{k}\sum_{i=1}^{b}(x_{ij} - \bar{x}[T]_j - \bar{x}[B]_i + \bar{\bar{x}})^2$$

The test is conducted by determining the mean squares, which are computed by dividing the sums of squares by their respective degrees of freedom.

Mean Squares for the Randomized Block Experiment

$$\text{MST} = \frac{\text{SST}}{k - 1}$$

$$\text{MSB} = \frac{\text{SSB}}{b - 1}$$

$$\text{MSE} = \frac{\text{SSE}}{n - k - b + 1}$$

Finally, the test statistic is the ratio of mean squares, as described in the box.

Test Statistic for the Randomized Block Experiment

$$F = \frac{\text{MST}}{\text{MSE}}$$

which is F-distributed with $\nu_1 = k - 1$ and $\nu_2 = n - k - b + 1$ degrees of freedom.

An interesting, and sometimes useful, by-product of the test of the treatment means is that we can also test to determine whether the block means differ. This will allow us to determine whether the experiment should have been conducted as a randomized block design. (If there are no differences between the blocks, the randomized block design is less likely to detect real differences between the treatment means.) Such a discovery could be useful in future similar experiments. The test of the block means is almost identical to that of the treatment means except the test statistic is

$$F = \frac{\text{MSB}}{\text{MSE}}$$

which is F-distributed with $\nu_1 = b - 1$ and $\nu_2 = n - k - b + 1$ degrees of freedom.

As with the one-way experiment, the statistics generated in the randomized block experiment are summarized in an ANOVA table, whose general form is exhibited in Table 14.5.

TABLE **14.5** ANOVA Table for the Randomized Block Analysis of Variance

SOURCE OF VARIATION	DEGREES OF FREEDOM	SUMS OF SQUARES	MEAN SQUARES	F-STATISTIC
Treatments	$k - 1$	SST	MST = SST/$(k - 1)$	F = MST/MSE
Blocks	$b - 1$	SSB	MSB = SSB/$(b - 1)$	F = MSB/MSE
Error	$n - k - b + 1$	SSE	MSE = SSE/$(n - k - b + 1)$	
Total	$n - 1$	SS(Total)		

EXAMPLE 14.3

DATA Xm14-03

Comparing Cholesterol-Lowering Drugs

Many North Americans suffer from high levels of cholesterol, which can lead to heart attacks. For those with very high levels (above 280), doctors prescribe drugs to reduce cholesterol levels. A pharmaceutical company has recently developed four such drugs. To determine whether any differences exist in their benefits, an experiment was organized. The company selected 25 groups of four men, each of whom had cholesterol levels in excess of 280. In each group, the men were matched according to age and weight. The drugs were administered over a 2-month period, and the reduction in cholesterol was recorded. Do these results allow the company to conclude that differences exist between the four new drugs?

Group	Drug 1	Drug 2	Drug 3	Drug 4
1	6.6	12.6	2.7	8.7
2	7.1	3.5	2.4	9.3
3	7.5	4.4	6.5	10
4	9.9	7.5	16.2	12.6
5	13.8	6.4	8.3	10.6
6	13.9	13.5	5.4	15.4
7	15.9	16.9	15.4	16.3
8	14.3	11.4	17.1	18.9
9	16	16.9	7.7	13.7
10	16.3	14.8	16.1	19.4
11	14.6	18.6	9	18.5
12	18.7	21.2	24.3	21.1
13	17.3	10	9.3	19.3
14	19.6	17	19.2	21.9
15	20.7	21	18.7	22.1
16	18.4	27.2	18.9	19.4
17	21.5	26.8	7.9	25.4
18	20.4	28	23.8	26.5
19	21.9	31.7	8.8	22.2
20	22.5	11.9	26.7	23.5
21	21.5	28.7	25.2	19.6
22	25.2	29.5	27.3	30.1
23	23	22.2	17.6	26.6
24	23.7	19.5	25.6	24.5
25	28.4	31.2	26.1	27.4

SOLUTION

IDENTIFY

The problem objective is to compare four populations, and the data are interval. Because the researchers recorded the cholesterol reduction for each drug for each member of the similar groups of men, we identify the experimental design as randomized block. The response variable is the cholesterol reduction, the treatments are the drugs, and the blocks are the 25 similar groups of men. The hypotheses to be tested are as follows.

H_0: $\mu_1 = \mu_2 = \mu_3 = \mu_4$

H_1: At least two means differ

COMPUTE

EXCEL

	A	B	C	D	E	F	G
36	ANOVA						
37	*Source of Variation*	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
38	Rows	3848.66	24	160.36	10.11	9.70E-15	1.6695
39	Columns	195.95	3	65.32	4.12	0.0094	2.7318
40	Error	1142.56	72	15.87			
41							
42	Total	5187.17	99				

Note the use of scientific notation for one of the *p*-values. The number 9.70E-15 (E stands for *exponent*) is 9.70 multiplied by 10 raised to the power − 15, that is, 9.70×10^{-15}. You can increase or decrease the number of decimal places, and you can convert the number into a regular number, but you would need many decimal places, which is why Excel uses scientific notation when the number is very small. (Excel also uses scientific notation for very large numbers.)

The output includes block and treatment statistics (sums, averages, and variances, which are not shown here), and the ANOVA table. The *F*-statistic to determine whether differences exist between the four drugs (**Columns**) is 4.12. Its *p*-value is .0094. The other *F*-statistic, 10.11 (p-value $= 9.70 \times 10^{-15}$ = virtually 0), indicates that there are differences between the groups of men (**Rows**).

INSTRUCTIONS

1. Type or import the data into adjacent columns*. (Open Xm14-03.)
2. Click **Data, Data Analysis . . .**, and **Anova: Two-Factor Without Replication**.
3. Specify the **Input Range** (A1:E26). Click **Labels** if applicable. If you do, both the treatments and blocks must be labeled (as in Xm14-03). Specify the value of α (.05).

MINITAB

Two-way ANOVA: Reduction versus Group, Drug

Analysis of Variance for Reduction

Source	DF	SS	MS	F	P
Group	24	3848.7	160.4	10.11	0.000
Drug	3	196.0	65.3	4.12	0.009
Error	72	1142.6	15.9		
Total	99	5187.2			

The *F*-statistic for **Drug** is 4.12 with a *p*-value of .009. The *F*-statistic for the blocks (**Group**) is 10.11, with a *p*-value of 0.

INSTRUCTIONS

The data must be in stacked format in three columns. One column contains the responses, another contains codes for the levels of the blocks, and a third column contains codes for the levels of the treatments.

1. Click **Stat, ANOVA,** and **Twoway . . . .**
2. Specify the **Responses, Row factor,** and **Column factor**

*If one or more columns contain a blank (representing missing data) the entire row must be deleted.

INTERPRET

A Type I error occurs when you conclude that differences exist when, in fact, they do not. A Type II error is committed when the test reveals no difference when at least two means differ. It would appear that both errors are equally costly. Accordingly, we judge the p-value against a standard of 5%. Because the p-value = .0094, we conclude that there is sufficient evidence to infer that at least two of the drugs differ. An examination reveals that cholesterol reduction is greatest using drugs 2 and 4. Further testing is recommended to determine which is better.

Checking the Required Conditions

The F-test of the randomized block design of the analysis of variance has the same requirements as the independent samples design. That is, the random variable must be normally distributed and the population variances must be equal. The histograms (not shown) appear to support the validity of our results; the reductions appear to be normal. The equality of variances requirement also appears to be met.

Violation of the Required Conditions

When the response is not normally distributed, we can replace the randomized block analysis of variance with the Friedman test, which is introduced in Section 19.4.

Criteria for Blocking

In Section 13.3, we listed the advantages and disadvantages of performing a matched pairs experiment. The same comments are valid when we discuss performing a blocked experiment. The purpose of blocking is to reduce the variation caused by differences between the experimental units. By grouping the experimental units into homogeneous blocks with respect to the response variable, the statistics practitioner increases the chances of detecting actual differences between the treatment means. Hence, we need to find criteria for blocking that significantly affect the response variable. For example, suppose that a statistics professor wants to determine which of four methods of teaching statistics is best. In a one-way experiment, he might take four samples of 10 students, teach each sample by a different method, grade the students at the end of the course, and perform an F-test to determine whether differences exist. However, it is likely that there are very large differences between the students within each class that may hide differences between classes. To reduce this variation, the statistics professor must identify variables that are linked to a student's grade in statistics. For example, overall ability of the student, completion of mathematics courses, and exposure to other statistics courses are all related to performance in a statistics course.

The experiment could be performed in the following way. The statistics professor selects four students at random whose average grade before statistics is 95–100. He then randomly assigns the students to one of the four classes. He repeats the process with students whose average is 90–95, 85–90, . . . , and 50–55. The final grades would be used to test for differences between the classes.

Any characteristics that are related to the experimental units are potential blocking criteria. For example, if the experimental units are people, we may block according to age, gender, income, work experience, intelligence, residence (country, county, or city), weight, or height. If the experimental unit is a factory and we're measuring number of units produced hourly, blocking criteria include workforce experience, age of the plant, and quality of suppliers.

Developing an Understanding of Statistical Concepts

As we explained previously, the randomized block experiment is an extension of the matched pairs experiment discussed in Section 13.3. In the matched pairs experiment, we simply remove the effect of the variation caused by differences between the experimental units. The effect of this removal is seen in the decrease in the value of the standard error (compared to the standard error in the test statistic produced from independent samples) and the increase in the value of the t-statistic. In the randomized block experiment of the analysis of variance, we actually measure the variation between the blocks by computing SSB. The sum of squares for error is reduced by SSB, making it easier to detect differences between the treatments. In addition, we can test to determine whether the blocks differ—a procedure we were unable to perform in the matched pairs experiment.

To illustrate, let's return to Examples 13.4 and 13.5, which were experiments to determine whether there was a difference in starting salaries offered to finance and marketing MBA majors. (In fact, we tested to determine whether finance majors draw higher salary offers than do marketing majors. However, the analysis of variance can test only for differences.) In Example 13.4 (independent samples), there was insufficient evidence to infer a difference between the two types of majors. In Example 13.5 (matched pairs experiment), there was enough evidence to infer a difference. As we pointed out in Section 13.3, matching by grade point average allowed the statistics practitioner to more easily discern a difference between the two types of majors. If we repeat Examples 13.4 and 13.5 using the analysis of variance, we come to the same conclusion. The Excel outputs are shown here. (Minitab's printouts are similar.)

Excel Analysis of Variance Output for Example 13.4

	A	B	C	D	E	F	G
9	ANOVA						
10	*Source of Variation*	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
11	Between Groups	338,130,013	1	338,130,013	1.09	0.3026	4.0427
12	Within Groups	14,943,884,470	48	311,330,926			
13							
14	Total	15,282,014,483	49				

Excel Analysis of Variance Output for Example 13.5

	A	B	C	D	E	F	G
34	ANOVA						
35	*Source of Variation*	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
36	Rows	21,415,991,654	24	892,332,986	40.39	4.17E-14	1.9838
37	Columns	320,617,035	1	320,617,035	14.51	0.0009	4.2597
38	Error	530,174,605	24	22,090,609			
39							
40	Total	22,266,783,295	49				

In Example 13.4, we partition the total sum of squares [SS(Total) = 15,282,014,483] into two sources of variation: SST = 338,130,013 and SSE = 14,943,884,470. In Example 13.5, the total sum of squares is SS(Total) = 22,266,783,295, SST (sum of squares for majors) = 320,617,035, SSB (sum of squares for GPA) = 21,415,991,654, and SSE = 530,174,605. As you can see, the sums of squares for treatments are approximately equal (338,130,013 and 320,617,035). However, the two calculations differ in the sums of squares for error. SSE in Example 13.5 is much smaller than SSE in Example 13.4 because the randomized block experiment allows us to measure and remove the effect of the variation between MBA students with the same majors. The sum of squares for blocks (sum of squares for GPA groups) is 21,415,991,654, a statistic that measures how much variation exists between the salary offers within majors. As a result of removing this variation, SSE is small. Thus, we conclude in Example 13.5 that the salary offers differ between majors whereas there was not enough evidence in Example 13.4 to draw the same conclusion.

Notice that in both examples the t-statistic squared equals the F-statistic. in Example 13.4, $t = 1.04$, which when squared equals 1.09, which is the F-statistic (rounded). In Example 13.5, $t = 3.81$, which when squared equals 14.51, the F-statistic for the test of the treatment means. Moreover, the p-values are also the same.

We now complete this section by listing the factors that we need to recognize to use this experiment of the analysis of variance.

Factors That Identify the Randomized Block of the Analysis of Variance

1. **Problem objective:** Compare two or more populations
2. **Data type:** Interval
3. **Experimental design:** Blocked samples

EXERCISES

Developing an Understanding of Statistical Concepts

14.51 The following statistics were generated from a randomized block experiment with $k = 3$ and $b = 7$:

SST = 100 SSB = 50 SSE = 25

a. Test to determine whether the treatment means differ. (Use $\alpha = .05$.)
b. Test to determine whether the block means differ. (Use $\alpha = .05$.)

14.52 A randomized block experiment produced the following statistics:

$k = 5$ $b = 12$ SST = 1,500 SSB = 1,000 SS(Total) = 3,500

a. Test to determine whether the treatment means differ. (Use $\alpha = .01$.)
b. Test to determine whether the block means differ. (Use $\alpha = .01$.)

14.53 Suppose the following statistics were calculated from data gathered from a randomized block experiment with $k = 4$ and $b = 10$:

SS(Total) = 1,210 SST = 275 SSB = 625

a. Can we conclude from these statistics that the treatment means differ? (Use $\alpha = .01$.)
b. Can we conclude from these statistics that the block means differ? (Use $\alpha = .01$.)

14.54 A randomized block experiment produced the following statistics.

$k = 3$ $b = 8$ SST = 1,500 SS(Total) = 3,500

a. Test at the 5% significance level to determine whether the treatment means differ given that SSB = 500.
b. Repeat part (a) with SSB = 1,000.
c. Repeat part (a) with SSB = 1,500.
d. Describe what happens to the test statistic as SSB increases.

14.55 Xr14-55 a. Assuming that the data shown here were generated from a randomized block experiment, calculate SS(Total), SST, SSB, and SSE.
b. Assuming that the data below were generated from a one-way (independent samples) experiment, calculate SS(Total), SST, and SSE.

c. Why does SS(Total) remain the same for both experimental designs?
d. Why does SST remain the same for both experimental designs?
e. Why does SSB + SSE in part (a) equal SSE in part (b)?

Treatment		
1	2	3
7	12	8
10	8	9
12	16	13
9	13	6
12	10	11

14.56 Xr14-56 a. Calculate SS(Total), SST, SSB, and SSE, assuming that the accompanying data were generated from a randomized block experiment.
b. Calculate SS(Total), SST, and SSE, assuming that the data below were generated from a one-way (independent samples) experiment.
c. Explain why SS(Total) remains the same for both experimental designs.
d. Explain why SST remains the same for both experimental designs.
e. Explain why SSB + SSE in part (a) equals SSE in part (b).

Treatment			
1	2	3	4
6	5	4	4
8	5	5	6
7	6	5	6

Applications

14.57 Xr14-57 As an experiment to understand measurement error, a statistics professor asks four students to measure the height of the professor, a male student, and a female student. The differences (in centimeters) between the correct dimension and the ones produced by the students are listed here. Can we infer that there are differences in the errors between the subjects being measured? (Use $\alpha = .05$.)

	Errors in Measuring Heights of		
Student	Professor	Male Student	Female Student
1	1.4	1.5	1.3
2	3.1	2.6	2.4
3	2.8	2.1	1.5
4	3.4	3.6	2.9

14.58 Xr14-58 How well do diets work? In a preliminary study, 20 people who were more than 50 pounds overweight were recruited to compare four diets. The people were matched by age. The oldest four became block 1, the next oldest four became block 2, and so on. The number of pounds that each person lost are listed in the following table. Can we infer at the 1% significance level that there are differences between the four diets?

	Diet			
Block	1	2	3	4
1	5	2	6	8
2	4	7	8	10
3	6	12	9	2
4	7	11	16	7
5	9	8	15	14

Exercises 14.59–14.67 require the use of a computer and software. Use a 5% significance level unless specified otherwise. The answers to Exercises 14.59–14.65 may be calculated manually. See Appendix A for the sample statistics.

14.59 Xr14-59 In recent years, lack of confidence in the U.S. Postal Service has led many companies to send all of their correspondence by private courier. A large company is in the process of selecting one of 3 possible couriers to act as its sole delivery method. To help in making the decision, an experiment was performed in which letters were sent using each of the 3 couriers at 12 different times of the day to a delivery point across town. The number of minutes required for delivery was recorded.
a. Can we conclude that there are differences in delivery times between the three couriers?
b. Did the statistics practitioner choose the correct design? Explain.

14.60 Xr14-60 Refer to Exercise 14.14. Despite failing to show that differences in the three types of fertilizer exist, the scientist continued to believe that there were differences, and that the differences were masked by the variation between the plots of land. Accordingly, she conducted another experiment. In the second experiment, she found 20 three-acre plots of land scattered across the county. She divided each into three plots and applied the three types of fertilizer on each of the 1-acre plots. The crop yields were recorded.
a. Can the scientist infer that there are differences between the three types of fertilizer?
b. What do these test results reveal about the variation between the plots?

14.61 Xr14-61 A recruiter for a computer company would like to determine whether there are differences in sales ability between business, arts, and science graduates. She takes a random sample of 20 business graduates who have been working for the company for the past 2 years. Each is then matched with an arts graduate and a science graduate with similar educational and working experience. The commission earned by each (in $1,000s) in the last year was recorded.

a. Is there sufficient evidence to allow the recruiter to conclude that there are differences in sales ability between the holders of the three types of degrees?
b. Conduct a test to determine whether an independent samples design would have been a better choice.
c. What are the required conditions for the test in part (a)?
d. Are the required conditions satisfied?

14.62 Xr14-62 Exercise 14.10 described an experiment that involved comparing the completion times associated with four different income tax forms. Suppose the experiment is redone in the following way. Thirty people are asked to fill out all four forms. The completion times (in minutes) are recorded.

a. Is there sufficient evidence at the 1% significance level to infer that differences in the completion times exist between the four forms?
b. Comment on the suitability of this experimental design in this problem.

14.63 Xr14-63 The advertising revenues commanded by a radio station depend on the number of listeners it has. The manager of a station that plays mostly hard rock music wants to learn more about its listeners—mostly teenagers and young adults. In particular, he wants to know whether the amount of time they spend listening to radio music varies by the day of the week. If the manager discovers that the mean time per day is about the same, he will schedule the most popular music evenly throughout the week. Otherwise, the top hits will be played mostly on the days that attract the greatest audience. An opinion survey company is hired, and it randomly selects 200 teenagers and asks them to record the amount of time spent listening to music on the radio for each day of the previous week. What can the manager conclude from these data?

14.64 Xr14-64 Do medical specialists differ in the amount of time they devote to patient care? To answer this question, a statistics practitioner organized a study. The numbers of hours of patient care per week were recorded for five specialists. The experimental design was randomized blocks. The physicians were blocked by age. (Adapted from the *Statistical Abstract of the United States, 2000*, Table 190.)

a. Can we infer that there are differences in the amount of patient care between medical specialties?
b. Can we infer that blocking by age was appropriate?

14.65 Xr14-65 Refer to Exercise 14.9. Another study was conducted in the following way. Students from each of the high schools who were admitted to the business program were matched according to their high school averages. The average grades in the first year were recorded. Can the university admissions officer conclude that there are differences in grading standards between the four high schools?

American National Election Survey Exercises

14.66 ANES2008* Is there sufficient evidence to infer that there are differences between the number of days Americans watch national news on television (DAYS1), watch local television news in the afternoon or early evening (DAYS2), watch local television news in the late evening (DAYS3), and read a daily newspaper (DAYS4)?

Warning: There are blanks representing missing data that must be removed.

14.67 ANES2004* Repeat Exercise 14.66 for 2004.

14.5 Two-Factor Analysis of Variance

In Section 14.1, we addressed problems where the data were generated from single-factor experiments. In Example 14.1, the treatments were the four age categories. Thus, there were four levels of a single factor. In this section, we address the problem where the experiment features two factors. The general term for such data-gathering procedures is **factorial experiment**. In factorial experiments, we can examine the effect on the response variable of two or more factors, although in this book we address the problem of only two factors. We can use the analysis of variance to determine whether the levels of each factor are different from one another.

We will present the technique for fixed effects only. That means we will address problems where all the levels of the factors are included in the experiment. As was the case with the randomized block design, calculating the test statistic in this type of experiment is quite time consuming. As a result, we will use Excel and Minitab to produce our statistics.

EXAMPLE 14.4*

Comparing the Lifetime Number of Jobs by Educational Level

DATA
Xm14-04

One measure of the health of a nation's economy is how quickly it creates jobs. One aspect of this issue is the number of jobs individuals hold. As part of a study on job tenure, a survey was conducted in which Americans aged between 37 and 45 were asked how many jobs they have held in their lifetimes. Also recorded were gender and educational attainment. The categories are

Less than high school (E1)

High school (E2)

Some college/university but no degree (E3)

At least one university degree (E4)

The data are shown for each of the eight categories of gender and education. Can we infer that differences exist between genders and educational levels?

Male E1	Male E2	Male E3	Male E4	Female E1	Female E2	Female E3	Female E4
10	12	15	8	7	7	5	7
9	11	8	9	13	12	13	9
12	9	7	5	14	6	12	3
16	14	7	11	6	15	3	7
14	12	7	13	11	10	13	9
17	16	9	8	14	13	11	6
13	10	14	7	13	9	15	10
9	10	15	11	11	15	5	15
11	5	11	10	14	12	9	4
15	11	13	8	12	13	8	11

SOLUTION

IDENTIFY

We begin by treating this example as a one-way analysis of variance. Notice that there are eight treatments. However, the treatments are defined by two different factors. One factor is gender, which has two levels. The second factor is educational attainment, which has four levels.

We can proceed to solve this problem in the same way we did in Section 14.1: We test the following hypotheses.

H_0: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_8$
H_1: At least two means differ

*Adapted from the *Statistical Abstract of the United States, 2006*, Table 598.

COMPUTE

EXCEL

	A	B	C	D	E	F	G
1	Anova: Single Factor						
2							
3	SUMMARY						
4	*Groups*	*Count*	*Sum*	*Average*	*Variance*		
5	Male E1	10	126	12.60	8.27		
6	Male E2	10	110	11.00	8.67		
7	Male E3	10	106	10.60	11.60		
8	Male E4	10	90	9.00	5.33		
9	Female E1	10	115	11.50	8.28		
10	Female E2	10	112	11.20	9.73		
11	Female E3	10	94	9.40	16.49		
12	Female E4	10	81	8.10	12.32		
13							
14							
15	ANOVA						
16	*Source of Variation*	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
17	Between Groups	153.35	7	21.91	2.17	0.0467	2.1397
18	Within Groups	726.20	72	10.09			
19							
20	Total	879.55	79				

MINITAB

One-way ANOVA: Male E1, Male E2, Male E3, Male E4, Female E1, Female E2, ...

```
Source  DF     SS    MS     F      P
Factor   7  153.4  21.9  2.17  0.047
Error   72  726.2  10.1
Total   79  879.5

S = 3.176   R-Sq = 17.44%   R-Sq(adj) = 9.41%


                                    Individual 95% CIs For Mean Based on
                                    Pooled StDev
Level       N    Mean   StDev  -------+---------+---------+---------+---
Male E1    10  12.600   2.875                    (-------*-------)
Male E2    10  11.000   2.944              (-------*-------)
Male E3    10  10.600   3.406            (-------*-------)
Male E4    10   9.000   2.309      (-------*-------)
Female E1  10  11.500   2.877                (-------*-------)
Female E2  10  11.200   3.120               (-------*-------)
Female E3  10   9.400   4.061        (-------*-------)
Female E4  10   8.100   3.510  (-------*-------)
                               -------+---------+---------+---------+---
                                    7.5      10.0      12.5      15.0

Pooled StDev = 3.176
```

INTERPRET

The value of the test statistic is $F = 2.17$ with a p-value of .0467. We conclude that there are differences in the number of jobs between the eight treatments.

This statistical result raises more questions—namely, can we conclude that the differences in the mean number of jobs are caused by differences between males and females? Or are they caused by differences between educational levels? Or, perhaps, are there combinations, called **interactions**, of gender and education that result in especially high or low numbers? To show how we test for each type of difference, we need to develop some terminology.

A **complete factorial experiment** is an experiment in which the data for all possible combinations of the levels of the factors are gathered. That means that in Example 14.4 we measured the number of jobs for all eight combinations. This experiment is called a complete 2 × 4 factorial experiment.

In general, we will refer to one of the factors as factor A (arbitrarily chosen). The number of levels of this factor will be denoted by *a*. The other factor is called factor B, and its number of levels is denoted by *b*. This terminology becomes clearer when we present the data from Example 14.4 in another format. Table 14.6 depicts

TABLE **14.6** Two-Way Classification for Example 14.4

	MALE	FEMALE
Less than high school	10	7
	9	13
	12	14
	16	6
	14	11
	17	14
	13	13
	9	11
	11	14
	15	12
High School	12	7
	11	12
	9	6
	14	15
	12	10
	16	13
	10	9
	10	15
	5	12
	11	13
Less than bachelor's degree	15	5
	8	13
	7	12
	7	3
	7	13
	9	11
	14	15
	15	5
	11	9
	13	8
At least one bachelor's degree	8	7
	9	9
	5	3
	11	7
	13	9
	8	6
	7	10
	11	15
	10	4
	8	11

the layout for a *two-way classification*, which is another name for the complete factorial experiment. The number of observations for each combination is called a **replicate**. The number of replicates is denoted by r. In this book, we address only problems in which the number of replicates is the same for each treatment. Such a design is called **balanced**.

Thus, we use a complete factorial experiment where the number of treatments is ab with r replicates per treatment. In Example 14.4, $a = 2$, $b = 4$, and $r = 10$. As a result, we have 10 observations for each of the eight treatments.

If you examine the ANOVA table, you can see that the total variation is SS(Total) = 879.55, the sum of squares for treatments is SST = 153.35, and the sum of squares for error is SSE = 726.20. The variation caused by the treatments is measured by SST. To determine whether the differences result from factor A, factor B, or some interaction between the two factors, we need to partition SST into three sources. These are SS(A), SS(B), and SS(AB).

For those whose mathematical confidence is high, we have provided an explanation of the notation as well as the definitions of the sums of squares. Learning how the sums of squares are calculated is useful but hardly essential to your ability to conduct the tests. Uninterested readers should jump to the box on page 569 where we describe the individual F-tests.

How the Sums of Squares for Factors A and B and Interaction are Computed

To help you understand the formulas, we will use the following notation:

x_{ijk} = kth observation in the ijth treatment

$\bar{x}[\text{AB}]_{ij}$ = Mean of the response variable in the ijth treatment (mean of the treatment when the factor A level is i and the factor B level is j)

$\bar{x}[\text{A}]_i$ = Mean of the observations when the factor A level is i

$\bar{x}[\text{B}]_j$ = Mean of the observations when the factor B level is j

$\bar{\bar{x}}$ = Mean of all the observations

a = Number of factor A levels

b = Number of factor B levels

r = Number of replicates

In this notation, $\bar{x}[\text{AB}]_{11}$ is the mean of the responses for factor A level 1 and factor B level 1. The mean of the responses for factor A level 1 is $\bar{x}[\text{A}]_1$. The mean of the responses for factor B level 1 is $\bar{x}[\text{B}]_1$.

Table 14.7 describes the notation for the two-factor analysis of variance.

TABLE **14.7** Notation for Two-Factor Analysis of Variance

	Factor A				
Factor B	**1**	**2**	**. . .**	***a***	
1	x_{111} x_{112} . . . x_{11r} $\bar{x}[\mathrm{AB}]_{11}$	x_{211} x_{212} . . . x_{21r} $\bar{x}[\mathrm{AB}]_{21}$		x_{a11} x_{a12} . . . x_{a1r} $\bar{x}[\mathrm{AB}]_{a1}$	$\bar{x}[\mathrm{B}]_1$
2	x_{121} x_{122} . . . x_{12r} $\bar{x}[\mathrm{AB}]_{12}$	x_{221} x_{222} . . . x_{22r} $\bar{x}[\mathrm{AB}]_{22}$		x_{a21} x_{a22} . . . x_{a2r} $\bar{x}[\mathrm{AB}]_{a2}$	$\bar{x}[\mathrm{B}]_2$
. . .					
b	x_{1b1} x_{1b2} . . . x_{1br} $\bar{x}[\mathrm{AB}]_{1b}$	x_{2b1} x_{2b2} . . . x_{2br} $\bar{x}[\mathrm{AB}]_{2b}$		x_{ab1} x_{ab2} . . . x_{abr} $\bar{x}[\mathrm{AB}]_{ab}$	$\bar{x}[\mathrm{B}]_b$
	$\bar{x}[\mathrm{A}]_1$	$\bar{x}[\mathrm{A}]_2$		$\bar{x}[\mathrm{A}]_a$	$\bar{\bar{x}}$

The sums of squares are defined as follows.

Sums of Squares in the Two-Factor Analysis of Variance

$$\mathrm{SS(Total)} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}(x_{ijk} - \bar{\bar{x}})^2$$

$$\mathrm{SS(A)} = rb\sum_{i=1}^{a}(\bar{x}[\mathrm{A}]_i - \bar{\bar{x}})^2$$

$$\mathrm{SS(B)} = ra\sum_{j=1}^{b}(\bar{x}[\mathrm{B}]_j - \bar{\bar{x}})^2$$

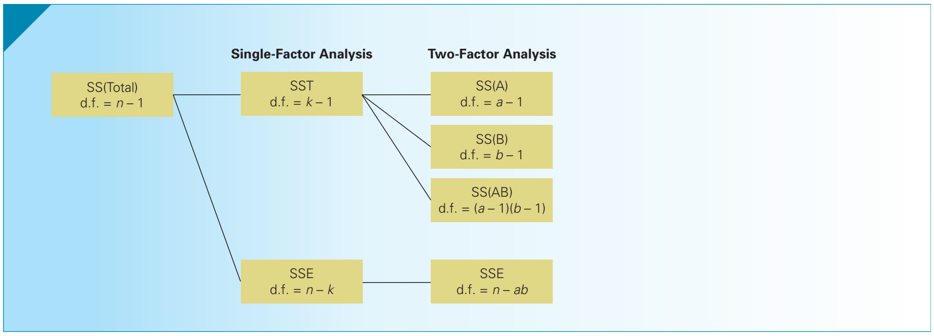
$$\mathrm{SS(AB)} = r\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{x}[\mathrm{AB}]_{ij} - \bar{x}[\mathrm{A}]_i - \bar{x}[\mathrm{B}]_j + \bar{\bar{x}})^2$$

$$\mathrm{SSE} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}(x_{ijk} - \bar{x}[\mathrm{AB}]_{ij})^2$$

To compute SS(A), we calculate the sum of the squared differences between the factor A level means, which are denoted $\bar{x}[\mathrm{A}]_i$, and the grand mean, $\bar{\bar{x}}$. The sum of squares for factor B, SS(B), is defined similarly. The interaction sum of squares, SS(AB), is calculated by taking each treatment mean (a treatment consists of a combination of a level of factor A and a level of factor B), subtracting the factor A level mean, subtracting the factor B level mean, adding the grand mean, squaring this quantity, and adding. The sum of squares for error, SSE, is calculated by subtracting the treatment means from the observations, squaring, and adding.

To test for each possibility, we conduct several F-tests similar to the one performed in Section 14.1. Figure 14.4 illustrates the partitioning of the total sum of squares that leads to the F-tests. We've included in this figure the partitioning used in the one-way study. When the one-way analysis of variance allows us to infer that differences between the treatment means exist, we continue our analysis by partitioning the treatment sum of squares into three sources of variation. The first is sum of squares for factor A, which we label SS(A), which measures the variation between the levels of factor A. Its degrees of freedom are $a - 1$. The second is the sum of squares for factor B, whose degrees of freedom are $b - 1$. SS(B) is the variation between the levels of factor B. The interaction sum of squares is labeled SS(AB), which is a measure of the amount of variation between the combinations of factors A and B; its degrees of freedom are $(a - 1) \times (b - 1)$. The sum of squares for error is SSE, and its degrees of freedom are $n - ab$. (Recall that n is the total sample size, which in this experiment is $n = abr$.) Notice that SSE and its number of degrees of freedom are identical in both partitions. As in the previous experiment, SSE is the variation within the treatments.

FIGURE **14.4** Partitioning SS(Total) in Single-Factor and Two-Factor Analysis of Variance



F-Tests Conducted in Two-Factor Analysis of Variance

Test for Differences between the Levels of Factor A

H_0: The means of the a levels of factor A are equal

H_1: At least two means differ

$$\text{Test statistic: } F = \frac{\text{MS(A)}}{\text{MSE}}$$

Test for Differences between the Levels of Factor B

H_0: The means of the b levels of factor B are equal

H_1: At least two means differ

$$\text{Test statistic: } F = \frac{\text{MS(B)}}{\text{MSE}}$$

Test for Interaction between Factors A and B

H_0: Factors A and B do not interact to affect the mean responses

H_1: Factors A and B do interact to affect the mean responses

$$\text{Test statistic: } F = \frac{\text{MS(AB)}}{\text{MSE}}$$

Required Conditions

1. The distribution of the response is normally distributed.
2. The variance for each treatment is identical.
3. The samples are independent.

As in the two previous experimental designs of the analysis of variance, we summarize the results in an ANOVA table. Table 14.8 depicts the general form of the table for the complete factorial experiment.

TABLE **14.8** ANOVA Table for the Two-Factor Experiment

SOURCE OF VARIATION	DEGREES OF FREEDOM	SUMS OF SQUARES	MEAN SQUARES	*F*-STATISTIC
Factor A	$a - 1$	SS(A)	MS(A) = SS(A)/$(a - 1)$	F = MS(A)/MSE
Factor B	$b - 1$	SS(B)	MS(B) = SS(B)/$(b - 1)$	F = MS(B)/MSE
Interaction	$(a - 1)(b - 1)$	SS(AB)	MS(AB) = SS(AB)/$[(a - 1)(b - 1)]$	F = MS(AB)/MSE
Error	$n - ab$	SSE	MSE = SSE/$(n - ab)$	
Total	$n - 1$	SS(Total)		

We'll illustrate the techniques using the data in Example 14.4. All calculations will be performed by Excel and Minitab.

EXCEL

	A	B	C	D	E	F	G
1	Anova: Two-Factor with Replication						
2							
3	SUMMARY	Male	Female	Total			
4	*Less than HS*						
5	Count	10	10	20			
6	Sum	126	115	241			
7	Average	12.6	11.5	12.1			
8	Variance	8.27	8.28	8.16			
9							
10	*High School*						
11	Count	10	10	20			
12	Sum	110	112	222			
13	Average	11.0	11.2	11.1			
14	Variance	8.67	9.73	8.73			
15							
16	*Less than Bachelor's*						
17	Count	10	10	20			
18	Sum	106	94	200			
19	Average	10.6	9.4	10.0			
20	Variance	11.6	16.49	13.68			
21							
22	*Bachelor's or more*						
23	Count	10	10	20			
24	Sum	90	81	171			
25	Average	9.0	8.1	8.6			
26	Variance	5.33	12.32	8.58			
27							
28	*Total*						
29	Count	40	40				
30	Sum	432	402				
31	Average	10.8	10.1				
32	Variance	9.50	12.77				
33							
34	ANOVA						
35	*Source of Variation*	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
36	Sample	135.85	3	45.28	4.49	0.0060	2.7318
37	Columns	11.25	1	11.25	1.12	0.2944	3.9739
38	Interaction	6.25	3	2.08	0.21	0.8915	2.7318
39	Within	726.20	72	10.09			
40							
41	Total	879.55	79				

In the ANOVA table, **Sample** refers to factor B (educational level) and **Columns** refers to factor A (gender). Thus, MS(B) = 45.28, MS(A) = 11.25, MS(AB) = 2.08, and MSE = 10.09. The *F*-statistics are 4.49 (educational level), 1.12 (gender), and .21 (interaction).

INSTRUCTIONS

1. Type or import the data using the same format as Xm14-04a. (*Note:* You must label the rows and columns as we did.)
2. Click **Data, Data Analysis,** and **Anova:Two-Factor with Replication**.
3. Specify the **Input Range** (A1:C41). Type the number of replications in the **Rows per sample** box (10).
4. Specify a value for α (.05).

MINITAB

```
Two-way ANOVA: Jobs versus Gender, Education

Source       DF      SS       MS     F      P
Gender        1   11.25  11.2500  1.12  0.294
Education     3  135.85  45.2833  4.49  0.006
Interaction   3    6.25   2.0833  0.21  0.892
Error        72  726.20  10.0861
Total        79  879.55

S = 3.176   R-Sq = 17.44%   R-Sq(adj) = 9.41%

                 Individual 95% CIs For Mean Based on Pooled StDev
Gender   Mean    -+-------------+-------------+-------------+------------
1       10.80                   (---------------------*---------------------)
2       10.05     (---------------------*--------------------)
                 -+-------------+-------------+-------------+------------
                 9.10          9.80         10.50         11.20

                    Individual 95% CIs For Mean Based on Pooled StDev
Education   Mean    --------+-------------+-------------+-------------+------
1          12.05                                (-------------*-------------)
2          11.10                           (------------*------------)
3          10.00                   (-------------*------------)
4           8.55    (-----------*------------)
                    --------+-------------+-------------+-------------+------
                           8.0           9.6          11.2          12.8
```

INSTRUCTIONS

1. Type or import the data in stacked format in three columns. One column contains the responses, another contains codes for the levels of factor A, and a third column contains codes for the levels of factor B. (Open Xm14-04b.)
2. Click **Stat, ANOVA,** and **Twoway . . . .**
3. Specify the **Responses** (Jobs), **Row factor** (Gender), and **Column factor** (Education).
4. To produce the graphics check **Display means**.

Test for Differences in Number of Jobs between Men and Women

H_0: The means of the two levels of factor A are equal
H_1: At least two means differ

$$\text{Test statistic: } F = \frac{\text{MS(A)}}{\text{MSE}}$$

Value of the test statistic: From the computer output, we have

MS(A) = 11.25, MSE = 10.09, and $F = 11.25/10.09 = 1.12$ (p-value = .2944)

There is not evidence at the 5% significance level to infer that differences in the number of jobs exist between men and women.

Test for Differences in Number of Jobs between Education Levels

H_0: The means of the four levels of factor B are equal
H_1: At least two means differ

Test statistic: $F = \dfrac{\text{MS(B)}}{\text{MSE}}$

Value of the test statistic: From the computer output, we find

MS(B) = 45.28 and MSE = 10.09. Thus, $F = 45.28/10.09 = 4.49$ (p-value = .0060).

There is sufficient evidence at the 5% significance level to infer that differences in the number of jobs exist between educational levels.

Test for Interaction between Factors A and B

H_0: Factors A and B do not interact to affect the mean number of jobs
H_1: Factors A and B do interact to affect the mean number of jobs

Test statistic: $F = \dfrac{\text{MS(AB)}}{\text{MSE}}$

Value of the test statistic: From the printouts,

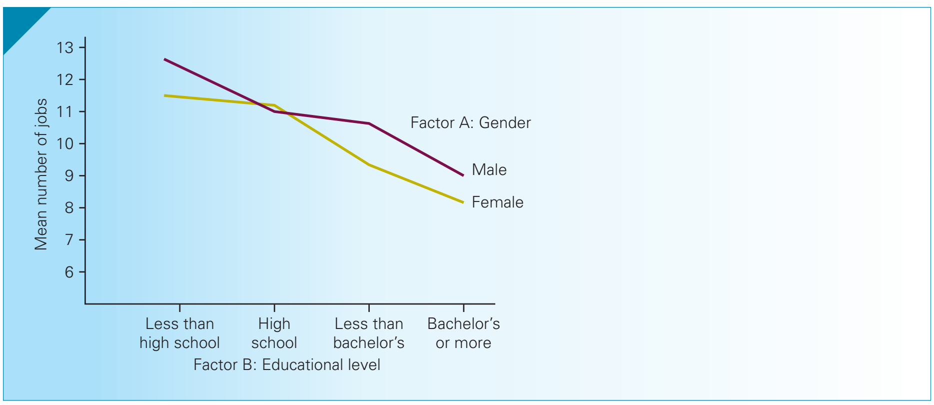
MS(AB) = 2.08, MSE = 10.09, and $F = 2.08/10.09 = .21$ (p-value = .8915).

There is not enough evidence to conclude that there is an interaction between gender and education.

INTERPRET

Figure 14.5 is a graph of the mean responses for each of the eight treatments. As you can see, there are small (not significant) differences between males and females. There are significant differences between men and women with different educational backgrounds. Finally, there is no interaction.

FIGURE **14.5** Mean Responses for Example 14.4



What Is Interaction?

To more fully understand interaction we have changed the sample associated with men who have not finished high school (Treatment 1). We subtracted 6 from the original numbers so that the sample in treatment 1 is

4, 3, 6, 10, 8, 11, 7, 3, 5, 9

The new data are stored in Xm14-04c (Excel format) and Xm14-04d (Minitab format).The mean is 6.6. Here are the Excel and Minitab ANOVA tables.

EXCEL

	A	B	C	D	E	F	G
35	ANOVA						
36	*Source of Variation*	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
37	Sample	75.85	3	25.28	2.51	0.0657	2.7318
38	Columns	11.25	1	11.25	1.12	0.2944	3.9739
39	Interaction	120.25	3	40.08	3.97	0.0112	2.7318
40	Within	726.20	72	10.09			
41							
42	Total	933.55	79				

MINITAB

Two-way ANOVA: Jobs versus Gender, Education

Source	DF	SS	MS	F	P
Gender	1	11.25	11.2500	1.12	0.294
Education	3	75.85	25.2833	2.51	0.066
Interaction	3	120.25	40.0833	3.97	0.011
Error	72	726.20	10.0861		
Total	79	933.55			

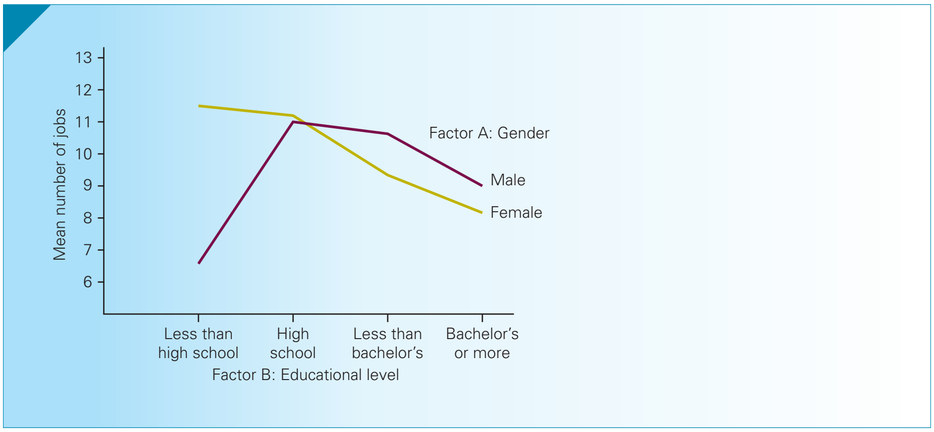
INTERPRET

In this example there is not enough evidence (at the 5% significance level) to infer that there are differences between men and women and between the educational levels. However, there is sufficient evidence to conclude that there is interaction between gender and education.

	Male	Female
Less than high school	6.6	11.5
High school	11.0	11.2
Less than bachelor's	10.6	9.4
Bachelor's or more	9.0	8.1

Compare Figures 14.5 and 14.6. In Figure 14.5, the lines joining the response means for males and females are quite similar. In particular we see that the lines are almost parallel. However, in Figure 14.6 the lines are no longer almost parallel. It is apparent that

FIGURE **14.6** Mean Responses for Example 14.4a



the mean of treatment 1 is smaller; the pattern is different. For whatever reason, in this case men with less than high school have a smaller number of jobs.

Conducting the Analysis of Variance for the Complete Factorial Experiment

In addressing the problem outlined in Example 14.4, we began by conducting a one-way analysis of variance to determine whether differences existed between the eight treatment means. This was done primarily for pedagogical reasons to enable you to see that when the treatment means differ, we need to analyze the reasons for the differences. However, in practice, we generally do not conduct this test in the complete factorial experiment (although it should be noted that some statistics practitioners prefer this "two-stage" strategy). We recommend that you proceed directly to the two-factor analysis of variance.

In the two versions of Example 14.4, we conducted the tests of each factor and then the test for interaction.

However, if there is evidence of interaction, the tests of the factors are irrelevant. There may or may not be differences between the levels of factor A and of the levels of factor B. Accordingly, we change the order of conducting the *F*-tests.

Order of Testing in the Two-Factor Analysis of Variance

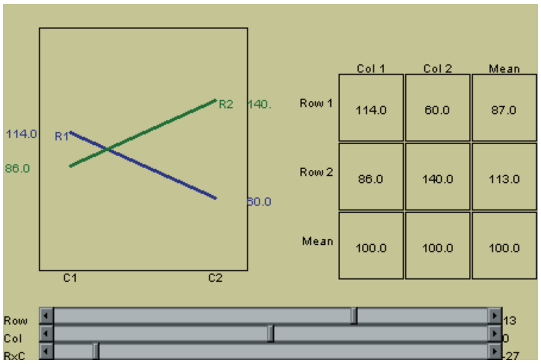
Test for interaction first. If there is enough evidence to infer that there is interaction, do not conduct the other tests.

If there is not enough evidence to conclude that there is interaction, proceed to conduct the *F*-tests for factors A and B.

SEEING STATISTICS

applet 17 Plots of Two-Way ANOVA Effects

This applet provides a graph similar to those in Figures 14.5 and 14.6. There are three sliders: one for rows, one for columns, and one for interaction. Moving the top slider changes the difference between the row means. The second slider changes the difference between the column means. The third slider allows us to see the effects of interaction.



Applet Exercises

Label the columns factor A and the rows factor B. Move the sliders to arrange for each of the following differences. Describe what the resulting figure tells you about differences between levels of factor A, levels of factor B, and interaction.

	ROW	COL	R × C
17.1	−30	0	0
17.2	0	25	0
17.3	0	0	−20
17.4	25	−30	0
17.5	30	0	30
17.6	30	0	−30
17.7	0	20	20
17.8	0	20	−20
17.9	30	30	30
17.10	30	30	−30

Developing an Understanding of Statistical Concepts

You may have noticed that there are similarities between the two-factor experiment and the randomized block experiment. In fact, when the number of replicates is one, the calculations are identical. (Minitab uses the same command.) This raises the question, What is the difference between a factor in a multifactor study and a block in a randomized block experiment? In general, the difference between the two experimental designs is that in the randomized block experiment, blocking is performed specifically to reduce variation, whereas in the two-factor model the effect of the factors on the response variable is of interest to the statistics practitioner. The criteria that define the blocks are always characteristics of the experimental units. Consequently, factors that are characteristics of the experimental units will be treated not as factors in a multifactor study, but as blocks in a randomized block experiment.

Let's review how we recognize the need to use the procedure described in this section.

Factors That Identify the Independent Samples Two-Factor Analysis of Variance

1. **Problem objective**: Compare two or more populations (populations are defined as the combinations of the levels of two factors)
2. **Data type**: Interval
3. **Experimental design**: Independent samples

EXERCISES

14.68 A two-factor analysis of variance experiment was performed with $a = 3$, $b = 4$, and $r = 20$. The following sums of squares were computed:

SS(Total) = 42,450 SS(A) = 1,560
SS(B) = 2,880 SS(AB) = 7,605

a. Determine the one-way ANOVA table.
b. Test at the 1% significance level to determine whether differences exist between the 12 treatments.
c. Conduct whatever test you deem necessary at the 1% significance level to determine whether there are differences between the levels of factor A, the levels of factor B, or interaction between factors A and B.

14.69 A statistics practitioner conducted a two-factor analysis of variance experiment with a = 4, $b = 3$, and $r = 8$. The sums of squares are listed here:

SS(Total) = 9,420 SS(A) = 203 SS(B) = 859
SS(AB) = 513

a. Test at the 5% significance level to determine whether factors A and B interact.
b. Test at the 5% significance level to determine whether differences exist between the levels of factor A.
c. Test at the 5% significance level to determine whether differences exist between the levels of factor B.

14.70 Xr14-70 The following data were generated from a 2×2 factorial experiment with three replicates:

	Factor B	
Factor A	**1**	**2**
1	6	12
	9	10
	7	11
2	9	15
	10	14
	5	10

a. Test at the 5% significance level to determine whether factors A and B interact.
b. Test at the 5% significance level to determine whether differences exist between the levels of factor A.
c. Test at the 5% significance level to determine whether differences exist between the levels of factor B.

14.71 Xr14-71 The data shown here were taken from a 2×3 factorial experiment with four replicates:

	Factor B	
Factor A	**1**	**2**
1	23	20
	18	17
	17	16
	20	19
2	27	29
	23	23
	21	27
	28	25
3	23	27
	21	19
	24	20
	16	22

a. Test at the 5% significance level to determine whether factors A and B interact.
b. Test at the 5% significance level to determine whether differences exist between the levels of factor A.
c. Test at the 5% significance level to determine whether differences exist between the levels of factor B.

14.72 Xr14-72 Refer to Example 14.4. We've revised the data by adding 2 to each of the numbers of the men. What do these data tell you?

14.73 Xr14-73 Refer to Example 14.4. We've altered the data by subtracting 4 from the numbers of treatment 8. What do these data tell you?

Applications

The following exercises require the use of a computer and software.

14.74 Xr14-74 Refer to Exercise 14.10. Suppose that the experiment is redone in the following way. Thirty taxpayers fill out each of the four forms. However, 10 taxpayers in each group are in the lowest income bracket, 10 are in the next income bracket, and the remaining 10 are in the highest bracket. The amount of time needed to complete the returns is recorded.

Column 1: Group number
Column 2: Times to complete form 1 (first 10 rows = low income, next 10 rows = next income bracket, and last 10 rows = highest bracket)
Column 3: Times to complete form 2 (same format as column 2)
Column 4: Times to complete form 3 (same format as column 2)
Column 5: Times to complete form 4 (same format as column 2)

a. How many treatments are there in this experiment?
b. How many factors are there? What are they?
c. What are the levels of each factor?
d. Is there evidence at the 5% significance level of interaction between the two factors?
e. Can we conclude at the 5% significance level that differences exist between the four forms?
f. Can we conclude at the 5% significance level that taxpayers in different brackets require different amounts of time to complete their tax forms?

14.75 Xr14-75 Detergent manufacturers frequently make claims about the effectiveness of their products. A consumer protection service decided to test the five best-selling brands of detergent, where each manufacturer claims that its product produces the "whitest whites" in all water temperatures. The experiment was conducted in the following way. One hundred fifty white sheets were equally soiled. Thirty sheets were washed in each brand—l0 with cold water, 10 with warm water, and 10 with hot water. After washing, the "whiteness" scores for each sheet were measured with laser equipment.

Column 1: Water temperature code
Column 2: Scores for detergent 1 (first 10 rows = cold water, middle 10 rows = warm, and last 10 rows = hot)
Column 2: Scores for detergent 2 (same format as column 2)
Column 3: Scores for detergent 3 (same format as column 2)
Column 4: Scores for detergent 4 (same format as column 2)
Column 5: Scores for detergent 5 (same format as column 2)

a. What are the factors in this experiment?
b. What is the response variable?
c. Identify the levels of each factor.
d. Perform a statistic analysis using a 5% significance level to determine whether there is sufficient statistical evidence to infer that there are differences in whiteness scores between the five detergents, differences in whiteness scores between the three water temperatures, or interaction between detergents and temperatures.

14.76 Xr14-76 Headaches are one of the most common, but least understood, ailments. Most people get headaches several times per month; over-the-counter medication is usually sufficient to eliminate their pain. However, for a significant proportion of people, headaches are debilitating and make their lives almost unbearable. Many such people have investigated a wide spectrum of possible treatments, including narcotic drugs, hypnosis, biofeedback, and acupuncture, with little or no success. In the last few years, a promising new treatment has been developed. Simply described, the treatment involves a series of injections of a local anesthetic to the occipital nerve (located in the back of the neck). The current treatment procedure is to schedule the injections once a week for 4 weeks. However, it has been suggested that another procedure may be better—one that features one injection every other day for a total of four injections. In addition, some physicians recommend other combinations of drugs that may increase the effectiveness of the injections. To analyze the problem, an experiment was organized. It was decided to test for a difference between the two schedules of injection and to determine whether there are differences between four drug mixtures. Because of the possibility of an interaction between the schedule and the drug, a complete factorial experiment was chosen. Five headache patients were randomly selected for each combination of schedule and drug. Forty patients were treated, and each was asked to report the frequency, duration, and severity of his or her headache prior to treatment and for the 30 days following the last injection. An index ranging from 0 to 100 was constructed for each patient, with 0 indicating no headache pain and 100 specifying the worst headache pain. The improvement in the headache index for each patient was recorded and reproduced in the accompanying table. (A negative value indicates a worsening condition.) (The author is grateful to Dr. Lorne Greenspan for his help in writing this example.)

a. What are the factors in this experiment?
b. What is the response variable?
c. Identify the levels of each factor.
d. Analyze the data and conduct whichever tests you deem necessary at the 5% significance level to determine whether there is sufficient statistical evidence to infer that there are differences in the improvement in the headache index between the two schedules, differences in the improvement in the headache index between the four drug mixtures, or interaction between schedules and drug mixtures.

Improvement in Headache Index

	Drug Mixture			
Schedule	**1**	**2**	**3**	**4**
One Injection	17	24	14	10
Every Week	6	15	9	−1
(Four Weeks)	10	10	12	0
	12	16	0	3
	14	14	6	−1
One Injection	18	−2	20	−2
Every Two Days	9	0	16	7
(Four Days)	17	17	12	10
	21	2	17	6
	15	6	18	7

14.77 Xr14-77 Most college instructors prefer to have their students participate actively in class. Ideally, students will ask their professor questions and answer their professor's questions, making the classroom experience more interesting and useful. Many professors seek ways to encourage their students to participate in class. A statistics professor at a community college in upper New York state believes that several external factors affect student participation. He believes that the time of day and the configuration of seats are two such factors. Consequently, he organized the following experiment. Six classes of about 60 students each were scheduled for one semester. Two classes were scheduled at 9 A.M., two at 1 P.M., and two at 4 P.M. At each of the three times, one of the classes was assigned to a room where the seats were arranged in rows of 10 seats. The other class was a U-shaped, tiered room, where students not only face the instructor but also face their fellow students. In each of the six classrooms, over 5 days, student participation was measured by counting the number of times students asked and answered questions. These data are displayed in the accompanying table.

a. How many factors are there in this experiment? What are they?
b. What is the response variable?
c. Identify the levels of each factor.
d. What conclusions can the professor draw from these data?

	Time		
Class Configuration	**9 A.M.**	**1 P.M.**	**4 P.M.**
Rows	10	9	7
	7	12	12
	9	12	9
	6	14	20
	8	8	7
U-Shape	15	4	7
	18	4	4
	11	7	9
	13	4	8
	13	6	7

14.6 (Optional) Applications in Operations Management: Finding and Reducing Variation

In the introduction to Example 12.3, we pointed out that variation in the size, weight, or volume of a product's components causes the product to fail or not function properly. Unfortunately, it is impossible to eliminate all variation. Designers of products and the processes that make the products understand this phenomenon. Consequently, when they specify the length, weight, or some other measurable characteristic of the product, they allow for some variation, which is called the *tolerance*. For example, the diameters of the piston rings of a car are supposed to be .826 millimeter (mm) with a tolerance of .006 mm; that is, the product will function provided that the diameter is between $.826 - .006 = .820$ and $.826 + .006 = .832$ mm. These quantities are called the *lower* and *upper specification limits* (LSL and USL), respectively.

Suppose that the diameter of the piston rings is actually a random variable that is normally distributed with a mean of .826 and a standard deviation of .003 mm. We can compute the probability that a piston ring's diameter is between the specification limits. Thus,

$$\begin{aligned} P(.820 < X < .832) &= P\left(\frac{.820 - .826}{.003} < \frac{X - \mu}{\sigma} < \frac{.832 - .826}{.003}\right) \\ &= P(-2.0 < Z < 2.0) \\ &= .9772 - .0228 \\ &= .9544 \end{aligned}$$

The probability that the diameter does not meet specifications is $1 - .9544 = .0456$. This probability is a measure of the process capability.

If we can decrease the standard deviation, a greater proportion of piston rings will have diameters that meet specification. Suppose that the operations manager has decreased the diameter's standard deviation to .002. The proportion of piston rings that do not meet specifications is .0026. When the probabilities are quite low, we express the probabilities as the number of defective units per million or per billion. Thus, if the standard deviation is .002, the number of defective piston rings is expected to be 2,600 per million. The goal of many firms is to reduce the standard deviation so that the lower specification and upper specification limits are at least 6 standard deviations away from the mean. If the standard deviation is .001, the proportion of nonconforming piston rings is $1 - P(-6 < Z < 6)$, which is 2 per billion. (Incidentally, this figure is often erroneously quoted as 3.4 per million.) The goal is called *six sigma*. Figure 14.7 depicts the proportion of conforming and nonconforming piston rings for $\sigma = .003, .002$, and $.001$.

Another way to measure how well the process works is the process capability index, denoted by C_p, *which is defined as*

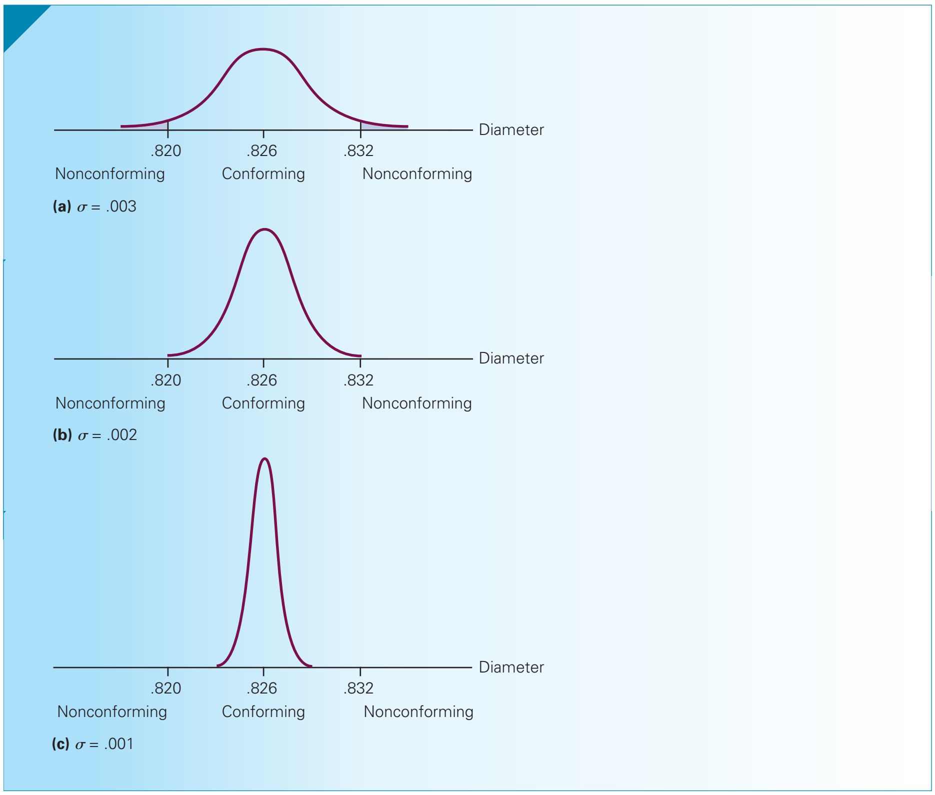
$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma}$$

Thus, in the illustration USL = .832 and LSL = .820. If the standard deviation is .002 then

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma} = \frac{.832 - .820}{6(.002)} = 1.0$$

The larger the process capability index, the more capable is the process in meeting specifications. A value of 1.0 describes a production process where the specification limits are equal to 3 standard deviations above and below the mean. A process capability index of 2.0 means that the upper and lower limits are 6 standard deviations above and below the mean. This is the goal for many firms.

FIGURE **14.7** **Proportion of Conforming and Nonconforming Piston Rings**



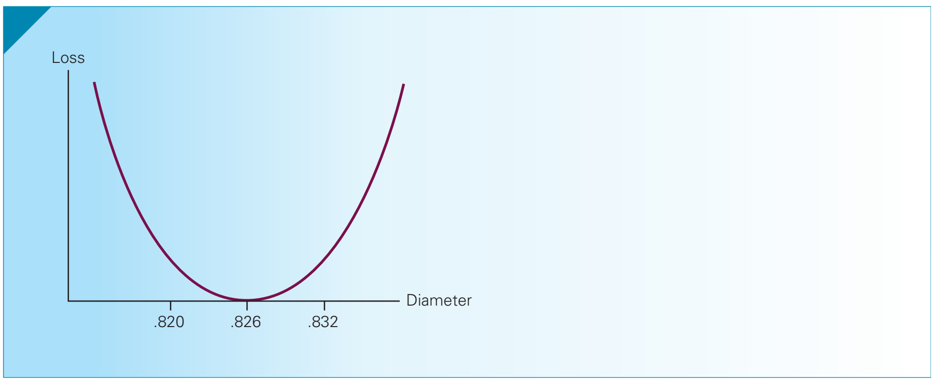
In practice, the standard deviation must be estimated from the data. We will address this issue again in Chapter 21.

Taguchi Loss Function

Historically, operations managers applied the "goalpost" philosophy, a name derived from the game of football. If the ball is kicked *anywhere* between the goalposts, the kick is equally as successful as one that is in the center of the goalposts. Under this philosophy, a piston ring that has a diameter of .821 works as well as one that is exactly .826. In other words, the company sustains a loss only when the product falls outside the goalposts. Products that lie between the goalposts suffer no financial loss. For many firms, this philosophy has now been replaced by the Taguchi loss function (named for Genichi Taguchi, a Japanese statistician whose ideas and techniques permeate any discussion of statistical applications in quality management).

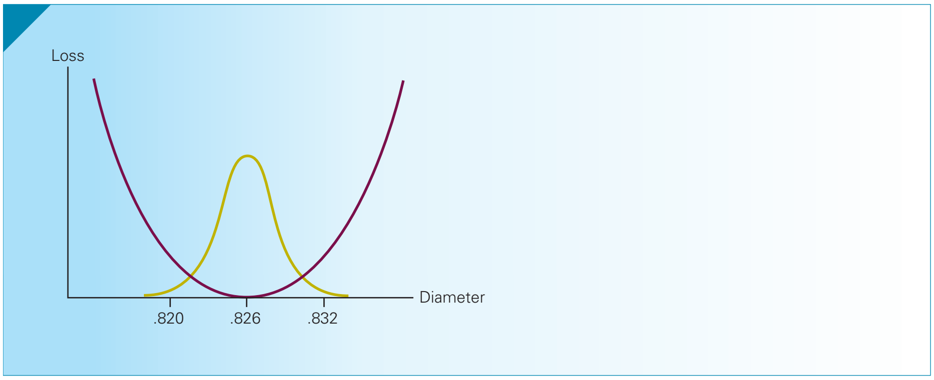
Products whose length or weight fall within the tolerances of their specifications do not all function in exactly the same way. There is a difference between a product that barely falls between the goalposts and one that is in the exact center. The Taguchi loss function recognizes that any deviation from the target value results in a financial loss. In addition, the farther the product's variable is from the target value, the greater the loss. The piston ring described previously is specified to have a diameter of exactly .826 mm, an amount specified by the manufacturer to work at the optimum level. Any deviation will cause that part and perhaps other parts to wear out prematurely. Although customers will not know the reason for the problem, they will know that the unit had to be replaced. The greater the deviation, the more quickly the part will wear and need replacing. If the part is under warranty, the company will incur a loss in replacing it. If the warranty has expired, customers will have to pay to replace the unit, causing some degree of displeasure that may result in them buying another company's product in the future. In either case, the company loses money. Figure 14.8 depicts the loss function. As you can see, any deviation from the target value results in some loss, with large deviations resulting in larger losses.

FIGURE **14.8** Taguchi Loss Function



Management scientists have shown that the loss function can be expressed as a function of the production process mean and variance. In Figure 14.9 we describe a normal distribution of the diameter of the machined part with a target value of .826 mm. When

FIGURE **14.9** Taguchi Loss Function and the Distribution of Piston Rings



the mean of the distribution is .826, any loss is caused by the variance. The statistical techniques introduced in Chapter 21 are usually employed to center the distribution on the target value. However, reducing the variance is considerably more difficult. To reduce variation, it is necessary to first find the sources of variation. We do so by conducting experiments. The principles are quite straightforward, drawing on the concepts developed in the previous section.

An important function of operations management is production design in which decisions are made about how a product is manufactured. The objective is to produce the highest quality product at a reasonable cost. This objective is achieved by choosing the machines, materials, methods, and "manpower" (personnel), the so-called 4 M's. By altering some or all of these elements, the operations manager can alter the size, weight, or volume and, ultimately, the quality of the product.

EXAMPLE 14.5

Causes of Variation

DATA
Xm14-05

A critical component in an aircraft engine is a steel rod that must be 41.387 cm long. The operations manager has noted that there has been some variation in the lengths. In some cases, the steel rods had to be discarded or reworked because they were either too short or too long. The operations manager believes that some of the variation is caused by the way the production process has been designed. Specifically, he believes that the rods vary from machine to machine and from operator to operator. To help unravel the truth, he organizes an experiment. Each of the three operators produces five rods on each of the four machines. The lengths are measured and recorded. Determine whether the machines or the operators (or both) are indeed sources of variation.

SOLUTION

IDENTIFY

The response variable is the length of the rods. The two factors are the operators and the machines. There are three levels of operators and four levels of machines. The model we employ is the two-factor model with interaction. The computer output is shown here.

COMPUTE

EXCEL

	A	B	C	D	E	F	G
1	Anova: Two-Factor With Replication						
2							
3	ANOVA						
4	*Source of Variation*	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
5	Sample	0.0151	2	0.0076	6.98	0.0022	3.1907
6	Columns	0.0034	3	0.0011	1.04	0.3856	2.7981
7	Interaction	0.0046	6	0.0008	0.71	0.6394	2.2946
8	Within	0.0520	48	0.0011			
9							
10	Total	0.0751	59				

MINITAB

Xm14-00a stores the data in Minitab format.

Two-way Analysis of Variance

Analysis of Variance for Rods

Source	DF	SS	MS	F	P
Machines	3	3363	1121	1.04	0.386
Operator	2	15133	7566	6.98	0.002
Interaction	6	4646	774	0.71	0.639
Error	48	51995	1083		
Total	59	75137			

INTERPRET

The test for interaction yields $F = .71$ and a p-value of .6394. There is not enough evidence to infer that the two factors interact. The F-statistic for the operator factor (Sample) is 6.98 (p-value = .0022). The F-statistic for the machine factor (Columns) is 1.04 (p-value = .3856). We conclude that there are differences only between the levels of the operators. Thus, the only source of variation here is the different operators. The operations manager can now focus on reducing or eliminating this variation. For example, the manager may use only one operator in the future or investigate why the operators differ.

The causes of variation example that opened this chapter illustrate this strategy. Because we have limited our discussion to the two-factor model, the example features this experimental design. It should be understood, however, that more complicated models are needed to fully investigate sources of variation.

Design of Experiments and Taguchi Methods

In the example just discussed, the experiment used only two factors. In practice, there are frequently many more factors. The problem is that the total number of treatments or combinations can be quite high, making any experimentation both time consuming and expensive. For example, if there are 10 factors each with 2 levels, the number of treatments is $2^{10} = 1{,}024$. If we measure each treatment with 10 replicates, the number of observations, 10,240, makes this experiment prohibitive. Fortunately, it is possible to reduce this number considerably. Through the use of *orthogonal arrays*, we can conduct *fractional factorial experiments* that can produce useful results at a small fraction of the cost. The experimental designs and statistical analyses are beyond the level of this book. Interested readers can find a variety of books at different levels of mathematical and statistical sophistication to learn more about this application.

EXERCISES

Applications

The following exercises require the use of a computer and software. Use a 5% significance level.

14.78 Xr14-78 The headrests on a car's front seats are designed to protect the driver and front-seat passenger from whiplash when the car is hit from behind. The frame of the headrest is made from metal rods. A machine is used to bend the rod into a U shape exactly 440 millimeters wide. The width is critical; too wide or too narrow, and the rod won't fit into the holes drilled into the car seat frame. The company has experimented with several different metal alloys in the hope of finding a material that will result in more headrest frames that fit. Another possible source of variation is the machines used. To learn more about the process, the operations manager conducts an experiment. Both of the machines are used to produce 10 headrests from each of the five metal alloys now being used. Each frame is measured, and the data (in millimeters) are recorded using the format shown here. Analyze the data to determine whether the alloys, machines, or both are sources of variation.

Column 1: Machine 1, rows 1 to 10 alloy A, rows 11 to 20 alloy B
Column 2: Machine 2, rows 1 to 10 alloy A, rows 11 to 20 alloy B

14.79 Xr14-79 A paint manufacturer is attempting to improve the process that fills the 1-gallon containers. The foreperson has suggested that the nozzle can be made from several different alloys. Furthermore, the way that the process "knows" when to stop the flow of paint can be accomplished in two ways: by setting a predetermined amount or by measuring the amount of paint already in the can. To determine what factors lead to variation, an experiment is conducted. For each of the four alloys that could be used to make the nozzles and the two measuring devices, five cans are filled. The amount of paint in each container is precisely measured. The data in liters were recorded in the following way:

Column 1: Device 1, rows 1 to 5 alloy A, rows 6 to 10 alloy B, etc.
Column 2: Device 2, rows 1 to 5 alloy A, rows 6 to 10 alloy B, etc.

Can we infer that the alloys, the measuring devices, or both are sources of variation?

14.80 Xr14-80 The marketing department of a firm that manufactures office furniture has ascertained that there is a growing market for a specialized desk that houses the various parts of a computer system. The operations manager is summoned to put together a plan that will produce high-quality desks at low cost. The characteristics of the desk have been dictated by the marketing department, which has specified the material that the desk will be made from and the machines used to produce the parts. However, three methods can be utilized. Moreover, because of the complexity of the operation, the manager realizes that it is possible that different skill levels of the workers can yield different results. Accordingly, he organized an experiment. Workers from each of three skill levels were chosen. These groups were further divided into two subgroups. Each subgroup assembled the desks using methods A and B. The amount of time taken to assemble each of eight desks was recorded as follows. Columns 1 and 2 contain the times for methods A and B; rows 1 to 8, 9 to16, and 17 to 24 store the times for the three skill levels. What can we infer from these data?

CHAPTER SUMMARY

The analysis of variance allows us to test for differences between populations when the data are interval. The analyses of the results of three different experimental designs were presented in this chapter. The one-way analysis of variance defines the populations on the basis of one factor. The second experimental design also defines the treatments on the basis of one factor. However, the randomized block design uses data gathered by observing the results of a matched or blocked experiment (two-way analysis of variance). The third design is the two-factor experiment wherein the treatments are defined as the combinations of the levels of two factors. All the analyses of variance are based on partitioning the total sum of squares into sources of variation from which the mean squares and F-statistics are computed.

In addition, we introduced three multiple comparison methods that allow us to determine which means differ in the one-way analysis of variance.

Finally, we described an important application in operations management that employs the analysis of variance.

IMPORTANT TERMS

SYMBOLS

Symbol	Pronounced	Represents
$\bar{\bar{x}}$	*x* double bar	Overall or grand mean
q		Studentized range
ω	Omega	Critical value of Tukey's multiple comparison method
$q_\alpha(k,\nu)$	*q* sub alpha *k* *ν*	Critical value of the Studentized range
n_g		Number of observations in each of *k* samples
$\bar{x}[T]_j$	*x* bar *T* sub *j*	Mean of the *j*th treatment
$\bar{x}[B]_i$	*x* bar *B* sub *i*	Mean of the *i*th block
$\bar{x}[AB]_{ij}$	*x* bar *A B* sub *i j*	Mean of the *ij*th treatment
$\bar{x}[A]_i$	*x* bar *A* sub *i*	Mean of the observations when the factor A level is *i*
$\bar{x}[B]_j$	*x* bar *B* sub *j*	Mean of the observations when the factor B level is *j*

FORMULAS

One-way analysis of variance

$$SST = \sum_{j=1}^{k} n_j(\bar{x}_j - \bar{\bar{x}})^2$$

$$SSE = \sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij} - \bar{x}_j)^2$$

$$MST = \frac{SST}{k-1}$$

$$MST = \frac{SSE}{n-k}$$

$$F = \frac{MST}{MSE}$$

Least significant difference comparison method

$$LSD = t_{\alpha/2}\sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

Tukey's multiple comparison method

$$\omega = q_\alpha(k,\nu)\sqrt{\frac{MSE}{n_g}}$$

Two-way analysis of variance (randomized block design of experiment)

$$SS(Total) = \sum_{j=1}^{k}\sum_{i=1}^{b}(x_{ij} - \bar{\bar{x}})^2$$

$$SST = \sum_{j=1}^{k} b(\bar{x}[T]_j - \bar{\bar{x}})^2$$

$$SSB = \sum_{i=1}^{b} k(\bar{x}[B]_i - \bar{\bar{x}})^2$$

$$SSE = \sum_{j=1}^{k}\sum_{i=1}^{b}(x_{ij} - \bar{x}[T]_j - \bar{x}[B]_i + \bar{\bar{x}})^2$$

$$MST = \frac{SST}{k - 1}$$

$$MSB = \frac{SSB}{b - 1}$$

$$MSE = \frac{SSE}{n - k - b + 1}$$

$$F = \frac{MST}{MSE}$$

$$F = \frac{MSB}{MSE}$$

Two-factor analysis of variance

$$SS(\text{Total}) = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}(x_{ijk} - \bar{\bar{x}})^2$$

$$SS(A) = rb\sum_{i=1}^{a}(\bar{x}[A]_i - \bar{\bar{x}})^2$$

$$SS(B) = ra\sum_{j=1}^{b}(\bar{x}[B]_j - \bar{\bar{x}})^2$$

$$SS(AB) = r\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{x}[AB]_{ij} - \bar{x}[A]_i - \bar{x}[B]_j + \bar{\bar{x}})^2$$

$$SSE = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}(x_{ijk} - \bar{x}[AB]_{ij})^2$$

$$MS(A) = \frac{SS(A)}{a - 1}$$

$$MS(B) = \frac{SS(B)}{b - 1}$$

$$MS(AB) = \frac{SS(AB)}{(a - 1)(b - 1)}$$

$$F = \frac{MS(A)}{MSE}$$

$$F = \frac{MS(B)}{MSE}$$

$$F = \frac{MS(AB)}{MSE}$$

COMPUTER OUTPUT AND INSTRUCTIONS

Technique	Excel	Minitab
One-way ANOVA	533	533
Multiple comparisons (LSD, Bonferroni adjustment, and Tukey)	549	550
Two-way (randomized block) ANOVA	558	558
Two-factor ANOVA	570	571

CHAPTER EXERCISES

The following exercises require the use of a computer and software. Use a 5% significance level.

14.81 Xr14-81 Each year billions of dollars are lost because of worker injuries on the job. Costs can be decreased if injured workers can be rehabilitated quickly. As part of an analysis of the amount of time taken for workers to return to work, a sample was taken of male blue-collar workers aged 35 to 45 who suffered a common wrist fracture. The researchers believed that the mental and physical condition of the individual affects recovery time. Each man was asked to complete a questionnaire that measured whether he tended to be optimistic or pessimistic. The men's physical condition was also evaluated and categorized as very physically fit, average, or in poor condition. The number of days until the wrist returned to full function was measured for each individual. These data were recorded in the following way:

Column 1: Time to recover for optimists (rows 1–10) = very fit, rows 11–20 = in average condition, rows 21–30 = poor condition

Column 2: Time to recover for pessimists (same format as column 1)

a. What are the factors in this experiment? What are the levels of each factor?
b. Can we conclude that pessimists and optimists differ in their recovery times?
c. Can we conclude that physical condition affects recovery times?

14.82 Xr14-82 In the past decade, American companies have spent nearly $1 trillion on computer systems. However, productivity gains have been quite small. During the 1980s, productivity in U.S. service industries (where most computers are used) grew by only .7% annually. In the 1990s, this figure rose to 1.5%. (Source: *New York Times Service*, February 22, 1995.) The problem of small productivity increases may be caused by the trouble employees experience in learning how to use the computer. Suppose that in an experiment to examine the problem, 100 firms were studied. Each company had bought a new computer system 5 years earlier. The companies reported their increase in productivity over the 5-year period and were also classified as offering extensive employee training, some employee training, little employee training, or no formal employee training in the use of computers. (There were 25 firms in each group.)

a. Can we conclude that differences in productivity gain exist between the four groups of companies?
b. If there are differences, what are they?

14.83 Xr14-83 The possible imposition of a residential property tax has been a sensitive political issue in a large city that consists of five boroughs. Currently, property tax is based on an assessment system that dates back to 1950. This system has produced numerous inequities whereby newer homes tend to be assessed at higher values than older homes. A new system based on the market value of the house has been proposed. Opponents of the plan argue that residents of some boroughs would have to pay considerably more on the average, while residents of other boroughs would pay less. As part of a study examining this issue, several homes in each borough were assessed under both plans. The percentage increase (a decrease is represented by a negative increase) in each case was recorded.

a. Can we conclude that there are differences in the effect the new assessment system would have on the five boroughs?
b. If differences exist, which boroughs differ? Use Tukey's multiple comparison method.
c. What are the required conditions for your conclusions to be valid?
d. Are the required conditions satisfied?

14.84 Xr14-84 The editor of the student newspaper was in the process of making some major changes in the newspaper's layout. He was also contemplating changing the typeface of the print used. To help himself make a decision, he set up an experiment in which 20 individuals were asked to read four newspaper pages, with each page printed in a different typeface. If the reading speed differed, then the typeface that was read fastest would be used. However, if there was not enough evidence to allow the editor to conclude that such differences existed, the current typeface would be continued. The times (in seconds) to completely read one page were recorded. What should the editor do?

14.85 Xr14-85 In marketing children's products, it is extremely important to produce television commercials that hold the attention of the children who view them. A psychologist hired by a marketing research firm wants to determine whether differences in attention span exist between children watching advertisements for different types of products. One hundred fifty children less than 10 were recruited for an experiment. One-third watched a 60-second commercial for a new computer game, one-third watched a commercial for a breakfast cereal, and one-third watched a commercial for children's clothes. Their attention spans (in seconds) were measured and recorded. Do these data provide enough evidence to conclude that there are differences in attention span between the three products advertised?

14.86 Xr14-86 On reconsidering the experiment in Exercise 14.85, the psychologist decides that the age of the child may influence the attention span. Consequently, the experiment is redone in the following way. Three children of each age (10 year olds, 9 year olds, 8 year olds, 7 year olds, 6 year olds, 5 year olds, and 4 year olds) are randomly assigned to watch one of the commercials, and their attention spans are measured. Do the results indicate that there are differences in the abilities of the products advertised to hold children's attention?

14.87 Xr14-87 It is important for salespeople to be knowledgeable about how people shop for certain products. Suppose that a new car salesperson believes that the age and gender of a car shopper affect the way he or she makes an offer on a car. He records the initial offers made by a group of men and women shoppers on a $30,000 Honda Accord. Besides the gender of the shopper, the salesman also notes the age category.

The amount of money below the asking price that each person offered initially for the car was recorded using the following format: Column 1 contains the data for the less than 30 group, the first 25 rows store the results for female shoppers, and the last 25 rows are the male shoppers. Columns 2 and 3 store the data for the 30–45 and older than 45 categories, respectively. What can we conclude from these data?

14.88 Xr14-88 Many of you reading this page probably learned how to read using the whole-language method. This strategy maintains that the natural and effective way is to be exposed to whole words in context. Students learn how to read by recognizing words they have seen before. In the past generation, this has been the dominant teaching strategy throughout North America. It replaced phonics, wherein children were taught to sound out the letters to form words. The whole language method was instituted with little or no research and has been severely criticized in the past. A recent study may have resolved the question of which method should be employed. Barbara Foorman, an educational psychologist at the University of Houston described the experiment at the annual meeting of the American Association for the Advancement of Science. The subjects were 375 low-achieving, poor, first-grade students in Houston schools. The students were divided into three groups. One was educated according to the whole language philosophy, a second group was taught using a pure phonics strategy, and the third was taught employing a mixed or embedded phonics technique. At the end of the term, students were asked to read words on a list of 50 words. The number of words each child could read was recorded.

a. Can we infer that differences exist between the effects of the three teaching strategies?
b. If differences exist, identify which method appears to be best.

14.89 Xr14-89 Are babies who are exposed to music before their birth smarter than those who are not? And, if so, what kind of music is best? Researchers at the University of Wisconsin conducted an experiment with rats. The researchers selected a random sample of pregnant rats and divided the sample into three groups. Mozart works were played to one group, a second group was exposed to white noise (a steady hum with no musical elements), and the third group listened to Philip Glass music (very simple compositions). The researchers then trained the young rats to run a maze in search of food. The amount of time for the rats to complete the maze was measured for all three groups.

a. Can we infer from these data that there are differences between the three groups?
b. If there are differences, determine which group is best.

14.90 Xr14-90 Increasing tuition has resulted in some students being saddled with large debts on graduation. To examine this issue, a random sample of recent graduates was asked to report whether they had student loans; if so, how much was the debt at graduation? Those who reported they owed money were also asked whether their degrees were BAs, BScs, BBAs, or other. Can we conclude that debt levels differ between the four types of degree?

14.91 Xr14-91 Studies indicate that single male investors tend to take the most risk, whereas married female investors tend to be conservative. This raises the question, Which does best? The risk-adjusted returns for single and married men, and for single and married women were recorded. Can we infer that differences exist between the four groups of investors?

14.92 Xr14-92 Like all other fine restaurants Ye Olde Steak House in Windsor, Ontario, attempts to have three "seatings" on weekend nights. Three seatings means that each table gets three different sets of customers. Obviously, any group that lingers over dessert and coffee may result in the loss of one seating and profit for the restaurant. In an effort to determine which types of groups tend to linger, a random sample of 150 groups was drawn. For each group, the number of members and the length of time that the group stayed were recorded in the following way.

Column A: Length of time for 2 people
Column B: Length of time for 3 people
Column C: Length of time for 4 people
Column D: Length of time for more than 4 people

Do these data allow us to infer that the length of time in the restaurant depends on the size of the party?

14.93 Xr14-93 When the stock market has a large 1-day decline, does it bounce back the next day or does the bad news endure? To answer this question, an economist examined a random sample of daily changes to the Toronto Stock Index (TSE). He recorded the percent change. He classified declines as

down by less than 0.5%
down by 0.5% to 1.5%
down by 1.5% to 2.5%
down by more than 2.5%

For each of these days, he recorded the percent loss the following day. Do these data allow us to infer that there are differences in changes to the TSE depending on the loss the previous day? (This exercise is based on a study undertaken by Tim Whitehead, an economist for Left Bank Economics, a consulting firm near Paris, Ontario.)

14.94 Xr14-94 Stock market investors are always seeking the "Holy Grail," a sign that tells them the market has bottomed out or achieved its highest level. There are several indicators. One is the buy signal developed by Gerald Appel, who believed that a bottom has been reached when the difference between the weekly close of the New York Stock Exchange index and the 10-week moving average (see Chapter 20) is −4.0 points or more. Another bottom indicator is based on identifying a certain pattern in the line chart of the stock market index. As an experiment, a financial analyst randomly selected 100 weeks. For each week, he determined whether there was an Appel buy, a chart buy, or no indication. For each type of week, he recorded the percentage change over the next 4 weeks. Can we infer that the two buy indicators are not useful?

14.95 Xr14-95 Millions of North Americans spend up to several hours a day commuting to and from work. Aside from the wasted time, are there other negative effects associated with fighting traffic? A study by Statistics Canada may shed light on the issue. A random sample of adults was surveyed. Among other questions, each was asked how much time he or she slept and how much time was spent commuting. The categories for commuting time are 1 to 30 minutes, 31 to 60 minutes, and more than 60 minutes. Is there sufficient evidence to conclude that the amount of sleep differs between commuting categories?

The following exercises use data files associated with three exercises seen previously in this book.

14.96 Xr12-126* In Exercise 12.126, marketing managers for the JC Penney department store chain segmented the market for women's apparel on the basis of personal and family values. The segments are labeled Conservative, Traditional, and Contemporary. Recall that the classification was done on the basis of questionnaires. In addition to identifying the segment via the questionnaire, each woman was also asked to report family income (in $1,000s). Do these data allow us to infer that family incomes differ between the three market segments?

14.97 Xr13-21* Exercise 13.21 addressed the problem of determining whether the distances young (less than 25) males and females drive annually differ. Included in the data is also the number of accidents that each person was involved in the past 2 years. Responses are 0, 1, or 2 or more. Do the data allow us to infer that the distances driven differ between the drivers who have had 0, 1, or 2 or more accidents?

14.98 Xr13-111* The objective in Exercise 13.111 was to determine whether various market segments were more likely to use the Quik Lube service. Included with the data is also the age (in months) of the car. Do the data allow us to conclude that there are differences in the age between the four market segments?

CASE 14.1 Comparing Three Methods of Treating Childhood Ear Infections*

©AP Photo/Chris Carlson

DATA C14-01

Acute otitis media, an infection of the middle ear, is a common childhood illness. There are various ways to treat the problem. To help determine the best way, researchers conducted an experiment. One hundred and eighty children between 10 months and 2 years with recurrent acute otitis media were divided into three equal groups. Group 1 was treated by surgically removing the adenoids (adenoidectomy), the second was treated with the drug Sulfafurazole, and the third with a placebo. Each child was tracked for 2 years, during which time all symptoms and episodes of acute otitis media were recorded. The data were recorded in the following way:

Column 1: ID number
Column 2: Group number
Column 3: Number of episodes of the illness
Column 4: Number of visits to a physician because of any infection
Column 5: Number of prescriptions
Column 6: Number of days with symptoms of respiratory infection

a. Are there differences between the three groups with respect to the number of episodes, number of physician visits, number of prescriptions, and number of days with symptoms of respiratory infection?

b. Assume that you are working for the company that makes the drug Sulfafurazole. Write a report to the company's executives discussing your results.

*This case is adapted from the *British Medical Journal*, February 2004.

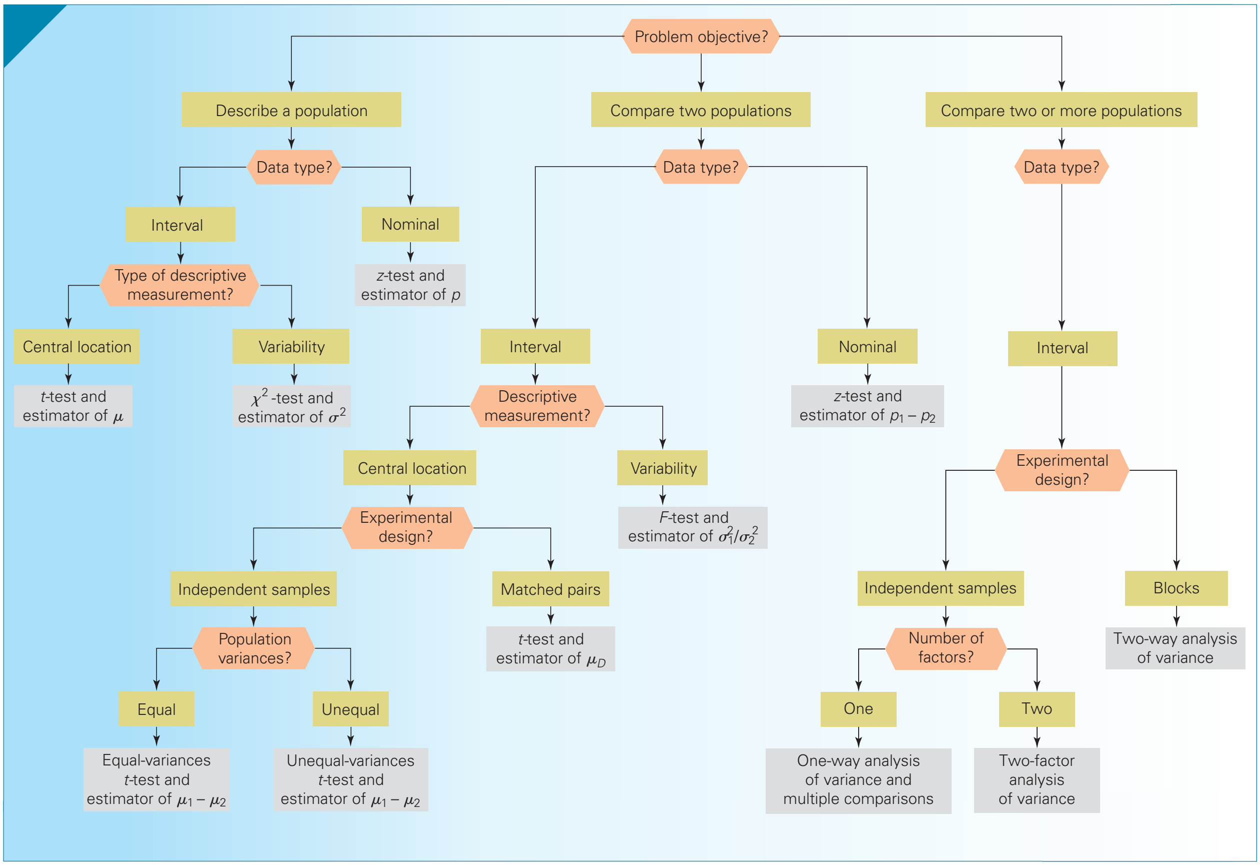
APPENDIX 14 / REVIEW OF CHAPTERS 12 TO 14

The number of techniques introduced in Chapters 12 to 14 is up to 20. As we did in Appendix 13, we provide a table of the techniques with formulas and required conditions, a flowchart to help you identify the correct technique, and 25 exercises to give you practice in how to choose the appropriate method. The table and the flowchart have been amended to include the three analysis of variance techniques introduced in this chapter and the three multiple comparison methods.

TABLE **A14.1** Summary of Statistical Techniques in Chapters 12 to 14

t-test of μ
Estimator of μ (including estimator of $N\mu$)
χ^2 test of σ^2
Estimator of σ^2
z-test of p
Estimator of p (including estimator of Np)
Equal-variances t-test of $\mu_1 - \mu_2$
Equal-variances estimator of $\mu_1 - \mu_2$
Unequal-variances t-test of $\mu_1 - \mu_2$
Unequal-variances estimator of $\mu_1 - \mu_2$
t-test of μ_D
Estimator of μ_D
F-test of σ_1^2/σ_2^2
Estimator of σ_1^2/σ_2^2
z-test of $p_1 - p_2$ (Case 1)
z-test of $p_1 - p_2$ (Case 2)
Estimator of $p_1 - p_2$
One-way analysis of variance (including multiple comparisons)
Two-way (randomized blocks) analysis of variance
Two-factor analysis of variance

FIGURE **A14.1** Summary of Statistical Techniques in Chapters 12 to 14



EXERCISES

Note that as we did in Appendix 13, we do not specify a significance level in exercises requiring a test of hypothesis. We leave this decision to you. After analyzing the issues raised in the exercise, use your own judgment to determine whether the p-value is small enough to reject the null hypothesis.

A14.1 XrA14-01 Sales of a product may depend on its placement in a store. Candy manufacturers frequently offer discounts to retailers who display their products more prominently than competing brands. To examine this phenomenon more carefully, a candy manufacturer (with the assistance of a national chain of restaurants) planned the following experiment. In 20 restaurants, the manufacturer's brand was displayed behind the cashier's counter with all the other brands (this was called position 1). In another 20 restaurants, the brand was placed separately but close to the other brands (position 2). In a third group of 20 restaurants, the candy was placed in a special display next to the cash register (position 3). The number of packages sold during 1 week at each restaurant was recorded. Is there sufficient evidence to infer that sales of candy differ according to placement?

A14.2 XrA14-02 Advertising is critical in the residential real estate industry. Agents are always seeking ways to increase sales through improved advertising methods. A particular agent believes that he can increase the number of inquiries (and thus the probability of making a sale) by describing the house for sale without indicating its asking price. To support his belief, he conducted an experiment in which 100 houses for sale were advertised in two ways—with and without the asking price. The number of inquiries for each house was recorded as well as whether the customer saw the ad with or without the asking price shown. Do these data allow the real estate agent to infer that ads with no price shown are more effective in generating interest in a house?

A14.3 XrA14-03 A professor of statistics hands back his graded midterms in class by calling out the name of each student and personally handing the exam over to its owner. At the end of the process, he notes that there are several exams left over, the result of students missing that class. He forms the theory that the absence is caused by a poor performance by those students on the test. If the theory is correct, the leftover papers will have lower marks than those papers handed back. He recorded the marks (out of 100) for the leftover papers and the marks of the returned papers. Do the data support the professor's theory?

A14.4 XrA14-04 A study was undertaken to determine whether a drug commonly used to treat epilepsy could help alcoholics to overcome their addiction. The researchers took a sample of 103 hardcore alcoholics. Fifty-five drinkers were given topiramate and the remaining 48 were given a placebo. The following variables were recorded after 6 months:

Column 1: Identification number
Column 2: 1 = Topiramate and 2 = placebo
Column 3: Abstain from alcohol for one month (1 = no, 2 = yes)
Column 4: Did not binge in final month (1 = no, 2 = yes)

Do these data provide sufficient evidence to infer that topiramate is effective in

a. causing abstinence for the first month?
b. causing alcoholics to refrain from binge drinking in the final month?

A14.5 XrA14-05 Health-care costs in the United States and Canada are concerns for citizens and politicians. The question is, How can we devise a system wherein people's medical bills are covered but individuals attempt to reduce costs? An American company has come up with a possible solution. Golden Rule is an insurance company in Indiana with 1,300 employees. The company offered its employees a choice of programs. One choice was a medical savings account (MSA) plan. Here's how it works. To ensure that a major illness or accident does not financially destroy an employee, Golden Rule offers catastrophic insurance—a policy that covers all expenses above $2,000 per year. At the beginning of the year, the company deposits $1,000 (for a single employee) and $2,000 (for an employee with a family) into the MSA. For minor expenses, the employee pays from his or her MSA. As an incentive for the employee to spend wisely, any money left in the MSA at the end of the year can be withdrawn by the employee. To determine how well it works, a random sample of employees who opted for the medical savings account plan was compared to employees who chose the regular plan. At the end of the year, the medical expenses for each employee were recorded. Critics of MSA say that the plan leads to poorer health care, and as a result employees are less likely to be in excellent health. To address this issue, each employee was examined. The results of the examination were recorded where 1 = excellent health and 2 = not in excellent health

a. Can we infer from these data that MSA is effective in reducing costs?
b. Can we infer that the critics of MSA are correct?

A14.6 XrA14-06 Discrimination in hiring has been illegal for many years. It is illegal to discriminate against any person on the basis of race, gender, or religion. It is also illegal to discriminate because of a person's handicap if it in no way prevents that person from performing that job. In recent years, the definition of "handicap" has widened. Several applicants have successfully sued companies because they were denied employment for no other reason than that they were overweight. A study was conducted to examine attitudes toward overweight people. The experiment involved showing a number of subjects videotape of an applicant being interviewed for a job. Before the interview, the subject was given a description of the job. Following the interview, the subject was asked to score the applicant in terms of how well the applicant was suited for the job. The score was out of 100, where higher scores described greater suitability. (The scores are interval data.) The same procedure was repeated for each subject. However, the gender and weight (average and overweight) of the applicant varied. The results were recorded using the following format:

Column 1: Score for average weight males
Column 2: Score for overweight males
Column 3: Score for average weight females
Column 4: Score for overweight females

a. Can we infer that the scores of the four groups of applicants differ?
b. Are the differences detected in part (a) because of weight, gender, or some interaction?

A14.7 XrA14-07 Most automobile repair shops now charge according to a schedule that is claimed to be based on average times. This means that instead of determining the actual time to make a repair and multiplying this value by their hourly rate, repair shops determine the cost from a schedule that is calculated from average times. A critic of this policy is examining how closely this schedule adheres to the actual time to complete a job. He randomly selects five jobs. According to the schedule, these jobs should take 45 minutes, 60 minutes, 80 minutes, 100 minutes, and 125 minutes, respectively. The critic then takes a random sample of repair shops and records the actual times for each of 20 cars for each job. For each job, can we infer that the time specified by the schedule is greater than the actual time?

A14.8 XrA14-08 Automobile insurance appraisers examine cars that have been involved in accidental collisions and estimate the cost of repairs. An insurance executive claims that there are significant differences in the estimates from different appraisers. To support his claim, he takes a random sample of 25 cars that have recently been damaged in accidents. Three appraisers then estimated the repair costs of each car. The estimates were recorded for each appraiser. From the data, can we conclude that the executive's claim is true?

A14.9 XrA14-09 The widespread use of salt on roads in Canada and the northern United States during the winter and acid precipitation throughout the year combine to cause rust on cars. Car manufacturers and other companies offer rustproofing services to help purchasers preserve the value of their cars. A consumer protection agency decides to determine whether there are any differences between the rust protection provided by automobile manufacturers and that provided by two competing types of rustproofing services. As an experiment, 60 identical new cars are selected. Of these, 20 are rustproofed by the manufacturer. Another 20 are rustproofed using a method that applies a liquid to critical areas of the car. The liquid hardens, forming a (supposedly) lifetime bond with the metal. The last 20 are treated with oil and are retreated every 12 months. The cars are then driven under similar conditions in a Minnesota city. The number of months until the first rust appears was recorded. Is there sufficient evidence to conclude that at least one rustproofing method is different from the others?

A14.10 XrA14-10 One of the ways in which advertisers measure the value of television commercials is by telephone surveys conducted shortly after commercials are aired. Respondents who watched a certain television station at a given time period, during which the commercial appeared, are asked whether they can recall the name of the product in the commercial. Suppose an advertiser wants to compare the recall proportions of two commercials. The first commercial is relatively inexpensive. A second commercial shown a week later is quite expensive to produce. The advertiser decides that the second commercial is viable only if its recall proportion is more than 15% higher than the recall proportion of the first commercial. Two surveys of 500 television viewers each were conducted after each commercial was aired. Each person was asked whether he or she remembered the product name. The results are stored in columns 1 (commercial 1) and 2 (commercial 2) (2 = remembered the product name, 1 = did not remember the product name). Can we infer that the second commercial is viable?

A14.11 XrA14-11 In the door-to-door selling of vacuum cleaners, various factors influence sales. The Birk Vacuum Cleaner Company considers its sales pitch and overall package to be extremely important. As a result, it often thinks of new ways to sell its product. Because the company's management develops so many new sales pitches each year, there is a two-stage testing process. In stage 1, a new plan is tested with a relatively small sample. If there is sufficient evidence that the plan increases sales, a second, considerably larger, test is undertaken. In a stage 1 test to determine whether the inclusion of a "free" 10-year service contract increases sales, 100 sales representatives were selected at random from the company's list of several thousand. The monthly sales of these representatives were recorded for 1 month before the use of the new sales pitch and for 1 month after its introduction. Should the company proceed to stage 2?

A14.12 XrA14-12 The cost of workplace injuries is high for the individual worker, for the company, and for society. It is in everyone's interest to rehabilitate the injured worker as quickly as possible. A statistician working for an insurance company has investigated the problem. He believes that physical condition is a major determinant in how quickly a worker returns to his or her job after sustaining an injury. To help determine whether he is on the right track, he organized an experiment. He took a random sample of male and female workers who were injured during the preceding year. He recorded their gender, their physical condition, and the number of working days until they returned to their job. These data were recorded in the following way. Columns 1 and 2 store the number of working days until return to work for men and women, respectively. In each column, the first 25 observations relate to those who are physically fit, the next 25 rows relate to individuals who are moderately fit, and the last 25 observations are for those who are in poor physical shape. Can we infer that the six groups differ? If differences exist, determine whether the differences result from gender, physical fitness, or some combination of gender and physical fitness.

A14.13 XrA14-13 Does driving an ABS-equipped car change the behavior of drivers? To help answer this question, the following experiment was undertaken. A random sample of 200 drivers who currently operate cars without ABS was selected. Each person was given an identical car to drive for 1 year. Half the sample were given cars that had ABS, and the other half were given cars with standard-equipment brakes. Computers on the cars recorded the average speed (in miles per hour) during the year. Can we infer that operating an ABS-equipped car changes the behavior of the driver?

A14.14 XrA14-14 We expect the demand for a product depends on its price: The higher the price, the lower the demand. However, this may not be entirely true. In an experiment conducted by professors at Northwestern University and MIT, a mail-order dress was available at the prices $34, $39, and $44. The number of dresses sold weekly over a 20-week period was recorded. The prices were randomized over 60 weeks. Conduct a test to determine whether demand differed and, if so, which price elicited the highest sales.

A14.15 XrA14-15 Researchers at the University of Washington conducted an experiment to determine whether the herbal remedy Echinacea is effective in treating children's colds and other respiratory infection (*National Post*, December 3, 2003). A sample of 524 children was recruited. Half the sample treated their colds with Echinacea, and the other half was given a placebo. For each infection, the duration of the colds (in days) were measured and recorded. Can we conclude that Echinacea is effective?

A14.16 XrA14-16 The marketing manager of a large ski resort wants to advertise that his ski resort has the shortest lift lines of any resort in the area. To avoid the possibility of a false advertising liability suit, he collects data on the times skiers wait in line at his resort and at each of two competing resorts on each of 14 days.

a. Can he conclude that there are differences in waiting times between the three resorts?
b. What are the required conditions for these techniques?
c. How would you check to determine that the required conditions are satisfied?

A14.17 XrA14-17 A popularly held belief about university professors is that they don't work very hard and that the higher their rank, the less work they do. A statistics student decided to determine whether the belief is true. She took a random sample of 20 university instructors in the faculties of business, engineering, arts, and sciences. In each sample of 20, 5 were instructors, 5 were assistant professors, 5 were associate professors, and 5 were full professors. Each professor was surveyed and asked to report confidentially the number of weekly hours of work. These data were recorded in the following way:

Column 1: hours of work for business professors (first 5 rows = instructors, next 5 rows = assistant professors, next 5 rows = associate professors, and last 5 rows = full professors)
Column 2: hours of work for engineering professors (same format as column 1)
Column 3: hours of work for arts professors (same format as column 1)
Column 4: hours of work for science professors (same format as column 1)

a. If we conduct the test under the single-factor analysis of variance, how many levels are there? What are they?
b. Test to determine whether differences exist using a single-factor analysis of variance.
c. If we conduct tests using the two-factor analysis of variance, what are the factors? What are their levels?
d. Is there evidence of interaction?
e. Are there differences between the four ranks of instructor?
f. Are there differences between the four faculties?

A14.18 XrA14-18 Billions of dollars are spent annually by Americans for the care and feeding of pets. A survey conducted by the American Veterinary Medical Association drew a random sample of 1,328 American households and asked whether they owned a pet and, if so, the type of animal. In addition, each was asked to report the veterinary expenditures for the previous 12 months. Column 1 contains the expenditures for dogs, and column 2 stores the expenditures for cats. The results are that 474 households reported that they owned at least one dog and 419 owned at least one cat. The latest census indicates that there are 112 million households in the United States. (*Source: Statistical Abstract of the United States, 2006*, Table 1232.)

a. Estimate with 95% confidence the total number of households owning at least one dog.
b. Repeat part (a) for cats.
c. Assume that there are 40 million households with at least one dog and estimate with 95% confidence the total amount spent on veterinary expenditures for dogs.
d. Assume that there are 35 million households with at least one cat and estimate with 95% confidence the total amount spent on veterinary expenditures for cats.

General Social Survey Exercises

A14.19 GSS2008* Can we infer from the data that the majority of Americans support capital punishment for murderers? (CAPPUN: 1 = Favor, 2 = Oppose)

A14.20 GSS2008* Test to determine whether Democrats and Republicans differ in their answers to the question, Have you ever taken any drugs by injection (heroine, cocaine, etc.)? (EVIDU: 1 = Yes, 2 = No)

A14.21 GSS2008* Is there enough evidence to infer that differences in the amount of television watched (TVHOURS) differs between classes (CLASS)?

A14.22 GSS2008* Do the data provide enough statistical evidence to conclude that differences in number of hours worked (HRS) exist between the three races (RACE)?

A14.23 GSS2006* GSS2008* Is there sufficient evidence to infer that on average Americans have aged (AGE) between 2006 and 2008?

American National Election Survey Exercises

A14.24 ANES2008* Can we conclude that differences in having access to the Internet (ACCESS: 1 = Yes, 5 = No) differs between Republicans and Democrats (PARTY: 1 Democrat, 2 = Republican)?

A14.25 ANES2008* Can we conclude that differences in age (AGE) exist between liberals, moderates, and conservatives (LIBCON3)?

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15 CHI-SQUARED TESTS

General Social Surveys

Has Support for Capital Punishment for Murderers Changed since 2002?

DATA
GSS2002*
GSS2004*
GSS2006*
GSS2008*

The issue of capital punishment for murderers in the United States has been argued for many years. A few states have abolished it, and others have kept their laws on the books but rarely use them. Where does the public stand on the issue, and has public support been constant or has it changed from year to year? One of the questions asked in the General Social Survey was

> Do you favor capital punishment for murder (CAPPUN)? The responses are
>
> 1 = Favor, 2 = Oppose

Conduct a test to determine whether public support varies from year to year.

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On page 611 we solve this problem.

INTRODUCTION

We have seen a variety of statistical techniques that are used when the data are nominal. In Chapter 2, we introduced bar and pie charts, both graphical techniques to describe a set of nominal data. Later in Chapter 2, we showed how to describe the relationship between two sets of nominal data by producing a frequency table and a bar chart. However, these techniques simply describe the data, which may represent a sample or a population. In this chapter, we deal with similar problems, but the goal is to use statistical techniques to make inferences about populations from sample data.

This chapter develops two statistical techniques that involve nominal data. The first is a *goodness-of-fit test* applied to data produced by a *multinomial experiment*, a generalization of a binomial experiment. The second uses data arranged in a table (called a *contingency table*) to determine whether two classifications of a population of nominal data are statistically independent; this test can also be interpreted as a comparison of two or more populations. The sampling distribution of the test statistics in both tests is the chi-squared distribution introduced in Chapter 8.

15.1 CHI-SQUARED GOODNESS-OF-FIT TEST

This section presents another test designed to describe a population of nominal data. The first such test was introduced in Section 12.3, where we discussed the statistical procedure employed to test hypotheses about a population proportion. In that case, the nominal variable could assume one of only two possible values: success or failure. Our tests dealt with hypotheses about the proportion of successes in the entire population. Recall that the experiment that produces the data is called a *binomial experiment*. In this section, we introduce the **multinomial experiment**, which is an extension of the binomial experiment, wherein there are two or more possible outcomes per trial.

Multinomial Experiment

A multinomial experiment is one that possesses the following properties.

1. The experiment consists of a fixed number n of trials.
2. The outcome of each trial can be classified into one of k categories, called *cells*.
3. The probability p_i that the outcome will fall into cell i remains constant for each trial. Moreover, $p_1 + p_2 + \cdots + p_k = 1$
4. Each trial of the experiment is independent of the other trials.

When $k = 2$, the multinomial experiment is identical to the binomial experiment. Just as we count the number of successes (recall that we label the number of successes x) and failures in a binomial experiment, we count the number of outcomes falling into each of the k cells in a multinomial experiment. In this way, we obtain a set of observed frequencies $f_1, f_2, \ldots, f_k$ where f_i is the observed frequency of outcomes falling into cell i, for $i = 1, 2, \ldots, k$. Because the experiment consists of n trials and an outcome must fall into some cell,

$$f_1 + f_2 + \cdots + f_k = n$$

Just as we used the number of successes x (by calculating the sample proportion $\hat{p}$, which is equal to x/n) to draw inferences about p, so we use the observed frequencies to draw inferences about the cell probabilities. We'll proceed in what by now has become a standard procedure. We will set up the hypotheses and develop the test statistic and its sampling distribution. We'll demonstrate the process with the following example.

EXAMPLE 15.1 Testing Market Shares

Company A has recently conducted aggressive advertising campaigns to maintain and possibly increase its share of the market (currently 45%) for fabric softener. Its main competitor, company B, has 40% of the market, and a number of other competitors account for the remaining 15%. To determine whether the market shares changed after the advertising campaign, the marketing manager for company A solicited the preferences of a random sample of 200 customers of fabric softener. Of the 200 customers, 102 indicated a preference for company A's product, 82 preferred company B's fabric softener, and the remaining 16 preferred the products of one of the competitors. Can the analyst infer at the 5% significance level that customer preferences have changed from their levels before the advertising campaigns were launched?

SOLUTION

The population in question is composed of the brand preferences of the fabric softener customers. The data are nominal because each respondent will choose one of three possible answers: product A, product B, or other. If there were only two categories, or if we were interested only in the proportion of one company's customers (which we would label as successes and label the others as failures), we would identify the technique as the z-test of p. However, in this problem we're interested in the proportions of all three categories. We recognize this experiment as a multinomial experiment, and we identify the technique as the **chi-squared goodness-of-fit test**.

Because we want to know whether the market shares have changed, we specify those precampaign market shares in the null hypothesis.

$$H_0: \quad p_1 = .45, p_2 = .40, p_3 = .15$$

The alternative hypothesis attempts to answer our question, Have the proportions changed? Thus,

$$H_1: \quad \text{At least one } p_i \text{ is not equal to its specified value}$$

Test Statistic

If the null hypothesis is true, we would expect the number of customers selecting brand A, brand B, and other to be 200 times the proportions specified under the null hypothesis; that is,

$$e_1 = 200(.45) = 90$$

$$e_2 = 200(.40) = 80$$

$$e_3 = 200(.15) = 30$$

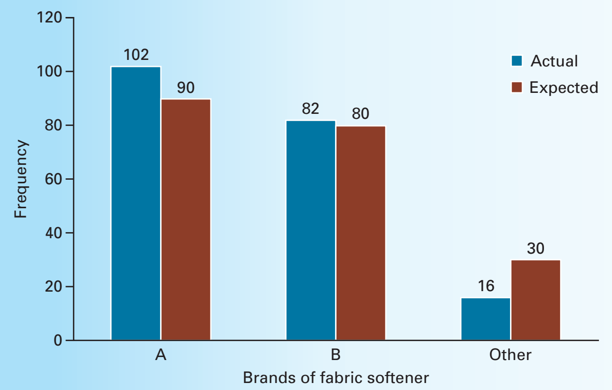
In general, the **expected frequency** for each cell is given by

$$e_i = np_i$$

This expression is derived from the formula for the expected value of a binomial random variable, introduced in Section 7.4.

Figure 15.1 is a bar chart (created by Excel) showing the comparison of actual and expected frequencies.

FIGURE **15.1** Bar Chart for Example 15.1



If the expected frequencies e_i and the **observed frequencies** f_i are quite different, we would conclude that the null hypothesis is false, and we would reject it. However, if the expected and observed frequencies are similar, we would not reject the null hypothesis. The test statistic defined in the box measures the similarity of the expected and observed frequencies.

Chi-Squared Goodness-of-Fit Test Statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(f_i - e_i)^2}{e_i}$$

The sampling distribution of the test statistic is approximately chi-squared distributed with $\nu = k - 1$ degrees of freedom, provided that the sample size is large. We will discuss this required condition later. (The chi-squared distribution was introduced in Section 8.4.)

The following table demonstrates the calculation of the test statistic. Thus, the value $\chi^2 = 8.18$. As usual, we judge the size of this test statistic by specifying the rejection region or by determining the p-value.

Company	Observed Frequency f_i	Expected Frequency e_i	$(f_i - e_i)$	$\frac{(f_i - e_i)^2}{e_i}$
A	102	90	12	1.60
B	82	80	2	0.05
Other	16	30	−14	6.53
Total	200	200		$\chi^2 = 8.18$

When the null hypothesis is true, the observed and expected frequencies should be similar, in which case the test statistic will be small. Thus, a small test statistic supports the null hypothesis. If the null hypothesis is untrue, some of the observed and expected frequencies will differ and the test statistic will be large. Consequently, we want to reject the null hypothesis when χ^2 is greater than $\chi^2_{\alpha,k-1}$. In other words, the rejection region is

$$\chi^2 > \chi^2_{\alpha,k-1}$$

In Example 15.1, $k = 3$; the rejection region is

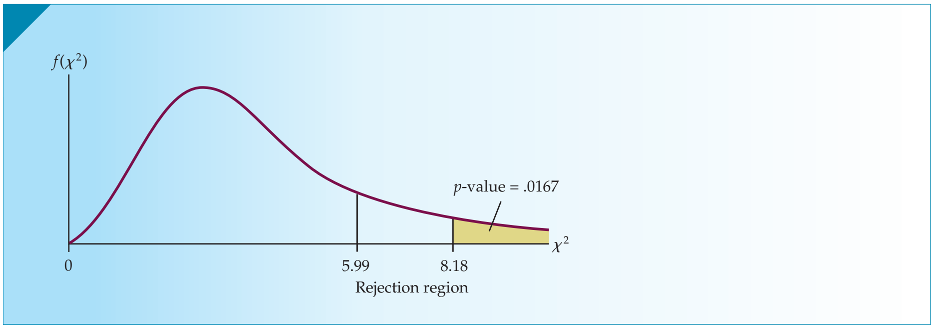
$$\chi^2 > \chi^2_{\alpha,k-1} = \chi^2_{.05,2} = 5.99$$

Because the test statistic is $\chi^2 = 8.18$, we reject the null hypothesis. The p-value of the test is

$$p\text{-value} = P(\chi^2 > 8.18)$$

Unfortunately, Table 5 in Appendix B does not allow us to perform this calculation (except for approximation by interpolation). The p-value must be produced by computer. Figure 15.2 depicts the sampling distribution, rejection region, and p-value.

FIGURE **15.2** Sampling Distribution for Example 15.1



EXCEL

The output from the commands listed here is the p-value of the test. It is .0167.

INSTRUCTIONS

1. Type the observed values into one column and the expected values into another column. (If you wish, you can type the cell probabilities specified in the null hypothesis and let Excel convert these into expected values by multiplying by the sample size.)
2. Activate an empty cell and type

   = **CHITEST**(**[Actual_range], [Expected_range]**)

   where the ranges are the cells containing the actual observations and the expected values.

You can also perform what-if analyses to determine for yourself the effect of changing some of the observed values and the sample size.

If we have the raw data representing the nominal responses we must first determine the frequency of each category (the observed values) using the **COUNTIF** function described on page 20.

MINITAB

Chi-Square Goodness-of-Fit Test for Observed Counts in Variable: C1

Category	Observed	Test Proportion	Expected	Contribution to Chi-Sq
1	102	0.45	90	1.60000
2	82	0.40	80	0.05000
3	16	0.15	30	6.53333

N	DF	Chi-Sq	P-Value
200	2	8.18333	0.017

INSTRUCTIONS

1. Click **Stat, Tables**, and **Chi-square Goodness-of-Fit Test (One Variable) . . . .**
2. Type the observed values into the **Observed counts:** box (102 82 16). If you have a column of data click **Categorical data:** and specify the column or variable name.
3. Click **Proportions specified by historical counts** and **Input constants**. Type the values of the proportions under the null hypothesis (.45 .40 .15).

INTERPRET

There is sufficient evidence at the 5% significance level to infer that the proportions have changed since the advertising campaigns were implemented. If the sampling was conducted properly, we can be quite confident in our conclusion. This technique has only one required condition, which is satisfied. (See the next subsection.) It is probably a worthwhile exercise to determine the nature and causes of the changes. The results of this analysis will determine the design and timing of other advertising campaigns.

Required Condition

The actual sampling distribution of the test statistic defined previously is discrete, but it can be approximated by the chi-squared distribution provided that the sample size is large. This requirement is similar to the one we imposed when we used the normal approximation to the binomial in the sampling distribution of a proportion. In that approximation we needed np and $n(1 - p)$ to be 5 or more. A similar rule is imposed for the chi-squared test statistic. It is called the *rule of five*, which states that the sample size must be large enough so that the expected value for each cell must be 5 or more. Where necessary, cells should be combined to satisfy this condition. We discuss this required condition and provide more details on its application in Keller's website Appendix Rule of Five.

Factors That Identify the Chi-Squared Goodness-of-Fit Test

1. **Problem objective**: Describe a single population
2. **Data type**: Nominal
3. **Number of categories**: 2 or more

Exercises

Developing an Understanding of Statistical Concepts

Exercises 15.1–15.6 are "what-if" analyses designed to determine what happens to the test statistic of the goodness-of-fit test when elements of the statistical inference change. These problems can be solved manually or using Excel's **CHITEST**.

15.1 Consider a multinomial experiment involving $n = 300$ trials and $k = 5$ cells. The observed frequencies resulting from the experiment are shown in the accompanying table, and the null hypothesis to be tested is as follows:

$$H_0\text{:}\quad p_1 = .1, p_2 = .2, p_3 = .3, p_4 = .2, p_5 = .2$$

Test the hypothesis at the 1% significance level.

Cell	1	2	3	4	5
Frequency	24	64	84	72	56

15.2 Repeat Exercise 15.1 with the following frequencies:

Cell	1	2	3	4	5
Frequency	12	32	42	36	28

15.3 Repeat Exercise 15.1 with the following frequencies:

Cell	1	2	3	4	5
Frequency	6	16	21	18	14

15.4 Review the results of Exercises 15.1–15.3. What is the effect of decreasing the sample size?

15.5 Consider a multinomial experiment involving $n = 150$ trials and $k = 4$ cells. The observed frequencies resulting from the experiment are shown in the accompanying table, and the null hypothesis to be tested is as follows:

$$H_0\text{:}\quad p_1 = .3, p_2 = .3, p_3 = .2, p_4 = .2$$

Cell	1	2	3	4
Frequency	38	50	38	24

Test the hypotheses, using $\alpha = .05$.

15.6 For Exercise 15.5, retest the hypotheses, assuming that the experiment involved twice as many trials ($n = 300$) and that the observed frequencies were twice as high as before, as shown here.

Cell	1	2	3	4
Frequency	76	100	76	48

Exercises 15.7–15.21 require the use of a computer and software. Use a 5% significance level unless specified otherwise. The answers to Exercises 15.7–15.16 may be calculated manually. See Appendix A for the sample statistics.

15.7 Xr15-07 The results of a multinomial experiment with $k = 5$ were recorded. Each outcome is identified by the numbers 1 to 5. Test to determine whether there is enough evidence to infer that the proportions of outcomes differ.

15.8 Xr15-08 A multinomial experiment was conducted with $k = 4$. Each outcome is stored as an integer from 1 to 4 and the results of a survey were recorded. Test the following hypotheses.

$$H_0\text{:}\quad p_1 = .15, p_2 = .40, p_3 = .35, p_4 = .10$$

H_1: At least one p_i is not equal to its specified value

15.9 Xr15-09 To determine whether a single die is balanced, or fair, the die was rolled 600 times. Is there sufficient evidence to allow you to conclude that the die is not fair?

Applications

15.10 Xr15-10 Grades assigned by an economics instructor have historically followed a symmetrical distribution: 5% A's, 25% B's, 40% C's, 25% D's, and 5% F's. This year, a sample of 150 grades was drawn and the grades (1 = A, 2 = B, 3 = C, 4 = D, and 5 = F) were recorded. Can you conclude, at the 10% level of significance, that this year's grades are distributed differently from grades in the past?

15.11 Xr15-11 Pat Statsdud is about to write a multiple-choice exam but as usual knows absolutely nothing. Pat plans to guess one of the five choices. Pat has been given one of the professor's previous exams with the correct answers marked. The correct choices were recorded where 1 = (a), 2 = (b), 3 = (c), 4 = (d), and 5 = (e). Help Pat determine whether this professor does not randomly distribute the correct answer over the five choices? If this is true, how does it affect Pat's strategy?

15.12 Xr15-12 Financial managers are interested in the speed with which customers who make purchases on credit pay their bills. In addition to calculating the average number of days that unpaid bills (called *accounts receivable*) remain outstanding, they often prepare an aging schedule. An aging schedule classifies outstanding accounts receivable according to the time that has elapsed since billing and records the proportion of accounts receivable belonging to each classification. A large firm has determined its aging schedule for the past 5 years. These results are shown in the accompanying table. During the past few months, however, the economy has taken a downturn. The company would like to know whether the recession has affected the aging schedule. A random sample of 250 accounts receivable was drawn and each account was classified as follows:

1 = 0–14 days outstanding

2 = 15–29 days outstanding

3 = 30–59 days outstanding

4 = 60 or more days outstanding

Number of Days Outstanding	Proportion of Accounts Receivable Past 5 Years
0–14	.72
15–29	.15
30–59	.10
60 and more	.03

Determine whether the aging schedule has changed.

15.13 Xr15-13 License records in a county reveal that 15% of cars are subcompacts (1), 25% are compacts (2), 40% are midsize (3), and the rest are an assortment of other styles and models (4). A random sample of accidents involving cars licensed in the county was drawn. The type of car was recorded using the codes in parentheses. Can we infer that certain sizes of cars are involved in a higher than expected percentage of accidents?

15.14 Xr15-14 In an election held last year that was contested by three parties. Party A captured 31% of the vote, party B garnered 51%, and party C received the remaining votes. A survey of 1,200 voters asked each to identify the party that they would vote for in the next election. These results were recorded where 1 = party A, 2 = party B, and 3 = party C. Can we infer at the 10% significance level that voter support has changed since the election?

15.15 Xr15-15 In a number of pharmaceutical studies volunteers who take placebos (but are told they have taken a cold remedy) report the following side effects:

Headache (1)	5%
Drowsiness (2)	7%
Stomach upset (3)	4%
No side effect (4)	84%

A random sample of 250 people who were given a placebo (but who thought they had taken an anti-inflammatory) reported whether they had experienced each of the side effects. These responses were recorded using the codes in parentheses. Do these data provide enough evidence to infer that the reported side effects of the placebo for an anti-inflammatory differ from that of a cold remedy?

APPLICATIONS in MARKETING

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Market Segmentation

Market segmentation was introduced in Section 12.4, where a statistical technique was used to estimate the size of a segment. In Chapters 13 and 14, statistical procedures were applied to determine whether market segments differ in their purchases of products and services. Exercise 15.16 requires you to apply the chi-squared goodness-of-fit test to determine whether the relative sizes of segments have changed.

15.16 Xr12-125* Refer to Exercise 12.125 where the statistics practitioner estimated the size of market segments based on education among California adults. Suppose that census figures from 10 years ago showed the education levels and the proportions of California adults, as follows:

Level	Proportion
1. Did not complete high school	.23
2. Completed high school only	.40
3. Some college or university	.15
4. College or university graduate	.22

Determine whether there has been a change in these proportions.

American National Election Survey Exercise

15.17 ANES2008* According to the *Statistical Abstract of the United States, 2009*, Table 55, the proportions for each category of marital status in 2007 was

Never married (including partnered, not married) 25%
Married (including separated, but not divorced) 58%
Widowed 6%
Divorced 11%

Can we infer that the American National Election Survey in 2008 overrepresented at least one category of marital status (MARITAL)?

General Social Survey Exercises

According to the *Statistical Abstract of the United States, 2009*, Table 7, the racial mix in the United States in 2007 was

White 79%
Black 13%
Other 8%

15.18 GSS2008* Test to determine whether there is sufficient evidence that the General Social Survey in 2008 overrepresented at least one race (RACE).

15.19 GSS2006* Is there sufficient evidence to conclude that the General Social Survey in 2006 overrepresented at least one race (RACE)?

According to the *Statistical Abstract of the United States, 2009*, Table 55, the proportions for each category of marital status in 2007 was

Never married 25%
Married (including separated, but not divorced) 58%
Widowed 6%
Divorced 11%

15.20 GSS2008* Can we infer that the General Social Survey in 2008 overrepresented at least one category of marital status (MARITAL)?

15.21 GSS2006* Is there sufficient evidence to conclude that the General Social Survey in 2006 overrepresented at least one category of marital status (MARITAL)?

15.2 Chi-Squared Test of a Contingency Table

In Chapter 2, we developed the **cross-classification table** as a first step in graphing the relationship between two nominal variables (see page 32). Our goal was to determine whether the two variables were related. In this section we extend the technique to statistical inference. We introduce another chi-squared test, this one designed to satisfy two different problem objectives. The **chi-squared test of a contingency table** is used to determine whether there is enough evidence to infer that two nominal variables are related and to infer that differences exist between two or more populations of nominal variables. Completing both objectives entails classifying items according to two different criteria. To see how this is done, consider the following example.

EXAMPLE 15.2

Relationship between Undergraduate Degree and MBA Major

DATA
Xm15-02

The MBA program was experiencing problems scheduling its courses. The demand for the program's optional courses and majors was quite variable from one year to the next. In one year, students seem to want marketing courses; in other years, accounting or finance are the rage. In desperation, the dean of the business school turned to a statistics professor for assistance. The statistics professor believed that the problem may be the variability in the academic background of the students and that the undergraduate degree affects the choice of major. As a start, he took a random sample of last year's MBA students and recorded the undergraduate degree and the major selected in the graduate program. The undergraduate degrees were BA, BEng, BBA, and several others. There are three possible majors for the MBA students: accounting, finance, and marketing. The results were summarized in a cross-classification table, which is shown here. Can the statistician conclude that the undergraduate degree affects the choice of major?

	MBA Major			
Undergraduate Degree	**Accounting**	**Finance**	**Marketing**	**Total**
B.A.	31	13	16	60
B.Eng.	8	16	7	31
B.B.A.	12	10	17	39
Other	10	5	7	22
Total	61	44	47	152

SOLUTION

One way to solve the problem is to consider that there are two variables: undergraduate degree and MBA major. Both are nominal. The values of the undergraduate degree are BA, BEng, BBA, and other. The values of MBA major are accounting, finance, and marketing. The problem objective is to analyze the relationship between the two variables. Specifically, we want to know whether one variable is related to the other.

Another way of addressing the problem is to determine whether differences exist between BA's, BEng's, BBA's, and others. In other words, we treat the holders of each undergraduate degree as a separate population. Each population has three possible values represented by the MBA major. The problem objective is to compare four populations. (We can also answer the question by treating the MBA majors as populations and the undergraduate degrees as the values of the random variable.)

As you will shortly discover, both objectives lead to the same test. Consequently, we address both objectives at the same time.

The null hypothesis will specify that there is no relationship between the two variables. We state this in the following way:

H_0: The two variables are independent

The alternative hypothesis specifies one variable affects the other, expressed as

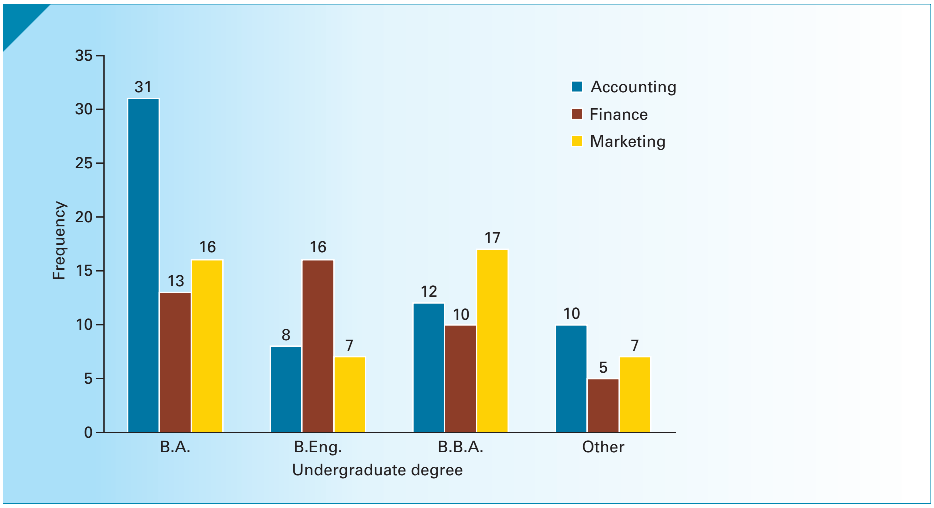
H_1: The two variables are dependent

Graphical Technique

Figure 15.3 depicts the graphical technique introduced in Chapter 2 to show the relationship (if any) between the two nominal variables.

The bar chart displays the data from the sample. It does appear that there is a relationship between the two nominal variables in the sample. However, to draw inferences about the population of MBA students we need to apply an inferential technique.

FIGURE **15.3** Bar Chart for Example 15.2



Test Statistic

The test statistic is the same as the one used to test proportions in the goodness-of-fit test; that is, the test statistic is

$$\chi^2 = \sum_{i=1}^{k} \frac{(f_i - e_i)^2}{e_i}$$

where k is the number of cells in the cross-classification table. If you examine the null hypothesis described in the goodness-of-fit test and the one described above, you will discover a major difference. In the goodness-of-fit test, the null hypothesis lists values for the probabilities p_i. The null hypothesis for the chi-squared test of a contingency table only states that the two variables are independent. However, we need the probabilities to compute the expected values e_i, which in turn are needed to calculate the value of the test statistic. (The entries in the table are the observed values f_i.) The question immediately arises, From where do we get the probabilities? The answer is that they must come from the data after we assume that the null hypothesis is true.

In Chapter 6 we introduced independent events and showed that if two events A and B are independent, the joint probability P(A and B) is equal to the product of P(A) and P(B). That is,

$$P(\text{A and B}) = P(\text{A}) \times P(\text{B})$$

The events in this example are the values each of the two nominal variables can assume. Unfortunately, we do not have the probabilities of A and B. However, these probabilities can be estimated from the data. Using relative frequencies, we calculate the estimated probabilities for the MBA major.

$$P(\text{Accounting}) = \frac{61}{152} = .401$$

$$P(\text{Finance}) = \frac{44}{152} = .289$$

$$P(\text{Marketing}) = \frac{47}{152} = .309$$

We calculate the estimated probabilities for the undergraduate degree.

$$P(\text{BA}) = \frac{60}{152} = .395$$

$$P(\text{BEng}) = \frac{31}{152} = .204$$

$$P(\text{BBA}) = \frac{39}{152} = .257$$

$$P(\text{Other}) = \frac{22}{152} = .145$$

Assuming that the null hypothesis is true, we can compute the estimated joint probabilities. To produce the expected values, we multiply the estimated joint probabilities by the sample size, $n = 152$. The results are listed in a **contingency table**, the word *contingency* derived by calculating the expected values contingent on the assumption that the null hypothesis is true (the two variables are independent).

Undergraduate	MBA Major			
Degree	**Accounting**	**Finance**	**Marketing**	**Total**
B.A.	$152 \times \frac{60}{152} \times \frac{61}{152} = 24.08$	$152 \times \frac{60}{152} \times \frac{44}{152} = 17.37$	$152 \times \frac{60}{152} \times \frac{47}{152} = 18.55$	60
B.Eng.	$152 \times \frac{31}{152} \times \frac{61}{152} = 12.44$	$152 \times \frac{31}{152} \times \frac{44}{152} = 8.97$	$152 \times \frac{31}{152} \times \frac{47}{152} = 9.59$	31
B.B.A.	$152 \times \frac{39}{152} \times \frac{61}{152} = 15.65$	$152 \times \frac{39}{152} \times \frac{44}{152} = 11.29$	$152 \times \frac{39}{152} \times \frac{47}{152} = 12.06$	39
Other	$152 \times \frac{22}{152} \times \frac{61}{152} = 8.83$	$152 \times \frac{22}{152} \times \frac{44}{152} = 6.37$	$152 \times \frac{22}{152} \times \frac{47}{152} = 6.80$	22
Total	61	44	47	152

As you can see, the expected value for each cell is computed by multiplying the row total by the column total and dividing by the sample size. For example, the BA and Accounting cell expected value is

$$152 \times \frac{60}{152} \times \frac{61}{152} = \frac{60 \times 61}{152} = 24.08$$

All the other expected values would be determined similarly.

Expected Frequencies for a Contingency Table

The expected frequency of the cell in row i and column j is

$$e_{ij} = \frac{\text{Row } i \text{ total} \times \text{Column } j \text{ total}}{\text{Sample size}}$$

The expected cell frequencies are shown in parentheses in the following table. As in the case of the goodness-of-fit test, the expected cell frequencies should satisfy the rule of five.

	MBA Major		
Undergraduate Degree	**Accounting**	**Finance**	**Marketing**
B.A.	31 (24.08)	13 (17.37)	16 (18.55)
B.Eng.	8 (12.44)	16 (8.97)	7 (9.59)
B.B.A.	12 (15.65)	10 (11.29)	17 12.06)
Other	10 (8.83)	5 (6.37)	7 (6.80)

We can now calculate the value of the test statistic:

$$\chi^2 = \sum_{i=1}^{k} \frac{(f_i - e_i)^2}{e_i} = \frac{(31 - 24.08)^2}{24.08} + \frac{(13 - 17.37)^2}{17.37} + \frac{(16 - 18.55)^2}{18.55}$$

$$+ \frac{(8 - 12.44)^2}{12.44} + \frac{(16 - 8.97)^2}{8.97} + \frac{(7 - 9.59)^2}{9.59} + \frac{(12 - 15.65)^2}{15.65}$$

$$+ \frac{(10 - 11.29)^2}{11.29} + \frac{(17 - 12.06)^2}{12.06} + \frac{(10 - 8.83)^2}{8.83}$$

$$+ \frac{(5 - 6.37)^2}{6.37} + \frac{(7 - 6.80)^2}{6.80}$$

$$= 14.70$$

Notice that we continue to use a single subscript in the formula of the test statistic when we should use two subscripts, one for the rows and one for the columns. We believe that it is clear, that for each cell we must calculate the squared difference between the observed and expected frequencies divided by the expected frequency. We don't believe that the satisfaction of using the mathematically correct notation overcomes the unnecessary complication.

Rejection Region and *p*-Value

To determine the rejection region we must know the number of degrees of freedom associated with the chi-squared statistic. The number of degrees of freedom for a contingency table with r rows and c columns is $\nu = (r - 1)(c - 1)$. For this example, the number of degrees of freedom is $\nu = (r - 1)(c - 1) = (4 - 1)(3 - 1) = 6$.

If we employ a 5% significance level, the rejection region is

$$\chi^2 > \chi^2_{\alpha,\nu} = \chi^2_{.05,6} = 12.6$$

Because $\chi^2 = 14.70$, we reject the null hypothesis and conclude that there is evidence of a relationship between undergraduate degree and MBA major.

The p-value of the test statistic is

$$P(\chi^2 > 14.70)$$

Unfortunately, we cannot determine the p-value manually.

Using the Computer

Excel and Minitab can produce the chi-squared statistic either from a cross-classification table whose frequencies have already been calculated or from raw data. The respective printouts are almost identical.

File Xm15-02 contains the raw data using the following codes:

Column1 (Undergraduate Degree)	Column 2 (MBA Major)
1 = B.A.	1 = Accounting
2 = B.Eng.	2 = Finance
3 = B.B.A.	3 = Marketing
4 = Other	

EXCEL

	A	B	C	D	E	F
1	Contingency Table					
2						
3		*Degree*				
4	*MBA Major*		1	2	3	TOTAL
5		1	31	13	16	60
6		2	8	16	7	31
7		3	12	10	17	39
8		4	10	5	7	22
9		TOTAL	61	44	47	152
10						
11						
12		chi-squared Stat			14.70	
13		df			6	
14		p-value			0.0227	
15		chi-squared Critical			12.5916	

INSTRUCTIONS (RAW DATA)

1. Type or import the data into two adjacent columns*. (Open Xm15-02.) The codes must be positive integers greater than 0.
2. Click **Add-Ins**, **Data Analysis Plus**, and **Contingency Table (Raw Data)**.
3. Specify the **Input Range** (A1:B153) and specify the value of α(.05).

INSTRUCTIONS (COMPLETED TABLE)

1. Type the frequencies into adjacent columns.
2. Click **Add-Ins**, **Data Analysis Plus**, and **Contingency Table**.
3. Specify the **Input Range**. Click **Labels** if the first row and first column of the input range contain the names of the categories. Specify the value for α.

MINITAB

Tabulated statistics: Degree, MBA Major

Rows: Degree   Columns: MBA Major

	1	2	3	All
1	31	13	16	60
2	8	16	7	31
3	12	10	17	39
4	10	5	7	22
All	61	44	47	152

Cell Contents: Count

Pearson Chi-Square = 14.702, DF = 6, P-Value = 0.023
Likelihood Ratio Chi-Square = 13.781, DF = 6, P-Value = 0.032

INSTRUCTIONS (RAW DATA)

1. Type or import the data into two columns. (Open Xm15-02.)
2. Click **Stat**, **Tables**, and **Cross Tabulation and Chi-Square . . . .**

(Continued)

*If one or both columns contain a blank (representing missing data) the row must be deleted.

3. In the **Categorical variables** box, select or type the variables **For rows** (Degree) and **For columns** (MBA Major). Click **Chi-Square . . . .**
4. Under **Display** click **Chi-Square analysis.** Specify **Chi-Square analysis**.

INSTRUCTIONS (COMPLETED TABLE)

1. Type the observed frequencies into adjacent columns.
2. Click **Stat**, **Tables**, and **Chi-Square Test (Table in Worksheet) . . . .**
3. Select or type the names of the variables representing the columns.

INTERPRET

There is strong evidence to infer that the undergraduate degree and MBA major are related. This suggests that the dean can predict the number of optional courses by counting the number of MBA students with each type of undergraduate degree. We can see that BA's favor accounting courses, BEng's prefer finance, BBA's are partial to marketing, and others show no particular preference.

If the null hypothesis is true, undergraduate degree and MBA major are independent of one another. This means that whether an MBA student earned a BA, BEng, BBA, or other degree does not affect his or her choice of major program in the MBA. Consequently, there is no difference in major choice among the graduates of the undergraduate programs. If the alternative hypothesis is true, undergraduate degree does affect the choice of MBA major. Thus, there are differences between the four undergraduate degree categories.

Rule of Five

In the previous section, we pointed out that the expected values should be at least 5 to ensure that the chi-squared distribution provides an adequate approximation of the sampling distribution. In a contingency table where one or more cells have expected values of less than 5, we need to combine rows or columns to satisfy the rule of five. This subject is discussed in Keller's website Appendix Rule of Five.

Data Formats

In Example 15.2, the data were stored in two columns, one column containing the values of one nominal variable and the second column storing the values of the second nominal variable. The data can be stored in another way. In Example 15.2, we could have recorded the data in three columns, one column for each MBA major. The columns would contain the codes representing the undergraduate degree. Alternatively, we could have stored the data in four columns, one column for each undergraduate degree. The columns would contain the codes for the MBA majors. In either case, we have to count the number of each value and construct the cross-tabulation table using the counts. Both Excel and Minitab can calculate the chi-squared statistic and its *p*-value from the cross-tabulation table. We will illustrate this approach with the solution to the chapter-opening example.

General Social Surveys

Has Support for Capital Punishment for Murderers Changed since 2002?

© Bob Daemmerich/PhotoEdit

IDENTIFY

The problem objective is to compare public opinion in four different years. The variable is nominal since its values are Favor and Oppose represented by 1 and 2, respectively. The appropriate technique is the chi-squared test of a contingency table. The hypotheses are

H_0 : The two variables are independent
H_1 : The two variables are dependent

In this application, the two variables are year (2002, 2004, 2006, and 2008) and the answer to the question posed by the General Social Survey (Favor and Oppose).

Unlike Example 15.2, the data are not stored in two columns. To produce the statistical result we will need to count the number of Americans in favor and the number opposed in each of the four years. The following table was determined by counting the numbers of 1's and 2's for each year.

	Year			
	2002	2004	2006	2008
Favor	899	855	1,885	1,263
Oppose	409	402	930	639

EXCEL

	A	B	C	D	E	F	G
1	Contingency Table						
2							
3							
4			Year 2002	Year 2004	Year 2006	Year 2008	TOTAL
5		Favor	899	855	1885	1263	4902
6		Oppose	409	402	930	639	2380
7		TOTAL	1308	1257	2815	1902	7282
8							
9							
10		Chi-Squared Statistic		2.3517			
11		Degrees Of Freedom		3			
12		P-Value		0.5027			
13		Chi-Squared Critical		7.8147			

INTERPRET

The *p*-value is .5027. There is not enough evidence to infer that the two variables are independent. Thus, there is not enough evidence to conclude that support for capital punishment for murder varies from year to year.

Here is a summary of the factors that tell us when to apply the chi-squared test of a contingency table. Note that there are two problem objectives satisfied by this statistical procedure.

Factors That Identify the Chi-Squared Test of a Contingency Table

1. **Problem objectives**: Analyze the relationship between two variables and compare two or more populations
2. **Data type**: Nominal

EXERCISES

Developing an Understanding of Statistical Concepts

15.22 Conduct a test to determine whether the two classifications L and M are independent, using the data in the accompanying cross-classification table. (Use $\alpha = .05$.)

	M_1	M_2
L_1	28	68
L_2	56	36

15.23 Repeat Exercise 15.22 using the following table:

	M_1	M_2
L_1	14	34
L_2	28	18

15.24 Repeat Exercise 15.22 using the following table:

	M_1	M_2
L_1	7	17
L_2	14	9

15.25 Review the results of Exercises 15.22–15.24. What is the effect of decreasing the sample size?

15.26 Conduct a test to determine whether the two classifications R and C are independent, using the data in the accompanying cross-classification table. (Use $\alpha = .10$.)

	C_1	C_1	C_3
R_1	40	32	48
R_2	30	48	52

Applications

Use a 5% significance level unless specified otherwise.

15.27 The trustee of a company's pension plan has solicited the opinions of a sample of the company's employees about a proposed revision of the plan. A breakdown of the responses is shown in the accompanying table. Is there enough evidence to infer that the responses differ between the three groups of employees?

Responses	Blue-Collar Workers	White-Collar Workers	Managers
For	67	32	11
Against	63	18	9

15.28 The operations manager of a company that manufactures shirts wants to determine whether there are differences in the quality of workmanship among the three daily shifts. She randomly selects 600 recently made shirts and carefully inspects them. Each shirt is classified as either perfect or flawed, and the shift that produced it is also recorded. The accompanying table summarizes the number of shirts that fell into each cell. Do these data provide sufficient evidence to infer that there are differences in quality between the three shifts?

	Shift		
Shirt Condition	1	2	3
Perfect	240	191	139
Flawed	10	9	11

15.29 One of the issues that came up in a recent national election (and is likely to arise in many future elections) is how to deal with a sluggish economy. Specifically, should governments cut spending, raise taxes, inflate the economy (by printing more money) or do none of the above and let the deficit rise? And as with most other issues, politicians need to know which parts of the electorate support these options. Suppose that a random sample of 1,000 people was asked which option they support and their political affiliations. The possible responses to the question about political affiliation were Democrat, Republican, and Independent (which included a variety of political persuasions). The responses are summarized in the accompanying table. Do these results allow us to conclude at the 1% significance level that political affiliation affects support for the economic options?

Economic Options	Political Affiliation		
	Democrat	Republican	Independent
Cut spending	101	282	61
Raise taxes	38	67	25
Inflate the economy	131	88	31
Let deficit increase	61	90	25

15.30 Econetics Research Corporation, a well-known Montreal-based consulting firm, wants to test how it can influence the proportion of questionnaires returned from surveys. In the belief that the inclusion of an inducement to respond may be important, the firm sends out 1,000 questionnaires: Two hundred promise to send respondents a summary of the survey results, 300 indicate that 20 respondents (selected by lottery) will be awarded gifts, and 500 are accompanied by no inducements. Of these, 80 questionnaires promising a summary, 100 questionnaires offering gifts, and 120 questionnaires offering no inducements are returned. What can you conclude from these results?

Exercises 15.31–15.46 require the use of a computer and software. Use a 5% significance level unless specified otherwise. The answers to Exercises 15.31–15.38 may be calculated manually. See Appendix A for the sample statistics.

15.31 Xm02-04 (Example 2.4 revisited) A major North American city has four competing newspapers: the *Globe and Mail* (G&M), *Post*, *Sun*, and *Star*. To help design advertising campaigns, the advertising managers of the newspapers need to know which segments of the newspaper market are reading their papers. A survey was conducted to analyze the relationship between newspapers read and occupation. A sample of newspaper readers was asked to report which newspaper they read: *Globe and Mail* (1) *Post* (2), *Star* (3), *Sun* (4), and to indicate whether they were blue-collar workers (1), white-collar workers (2), or professionals (3). Can we infer that occupation and newspaper are related?

15.32 Xr15-32 An investor who can correctly forecast the direction and size of changes in foreign currency exchange rates is able to reap huge profits in the international currency markets. A knowledgeable reader of the *Wall Street Journal* (in particular, of the currency futures market quotations) can determine the direction of change in various exchange rates that is predicted by all investors, viewed collectively. Predictions from 216 investors, together with the subsequent actual directions of change, were recorded in the following way: Column 1: predicted change where 1 = positive and 2 = negative; column 2: actual change where 1 = positive and 2 = negative.

a. Can we infer at the 10% significance level that a relationship exists between the predicted and actual directions of change?
b. To what extent would you make use of these predictions in formulating your forecasts of future exchange rate changes?

15.33 Xr02-43 (Exercise 2.43 revisited) Is there brand loyalty among car owners in their purchases of gasoline? To help answer the question, a random sample of car owners was asked to record the brand of gasoline in their last two purchases: 1 = Exxon, 2 = Amoco, 3 = Texaco, 4 = Other. Can we conclude that there is brand loyalty in gasoline purchases?

15.34 Xr15-34 During the past decade, many cigarette smokers have attempted to quit. Unfortunately, nicotine is highly addictive. Smokers use a large number of different methods to help them quit. These include nicotine patches, hypnosis, and various forms of therapy. A researcher for the Addiction Research Council wanted to determine why some people quit while others attempted to quit but failed. He surveyed 1,000 people who planned to quit smoking. He determined their educational level and whether they continued to smoke 1 year later. Educational level was recorded in the following way:

1 = Did not finish high school
2 = High school graduate
3 = University or college graduate
4 = Completed a postgraduate degree

A continuing smoker was recorded as 1; a quitter was recorded as 2. Can we infer that the amount of education is a factor in determining whether a smoker will quit?

15.35 Xr15-35 Because television audiences of newscasts tend to be older (and because older people suffer from a variety of medical ailments), pharmaceutical companies' advertising often appears on national news on the three networks (ABC, CBS, and NBC). To determine how effective the ads are a survey was undertaken. Adults over 50 were asked about their primary sources of news. The responses are

1. ABC News 2. CBS News 3. NBC News
4. Newspapers 5. Radio 6. None of the above

Each person was also asked whether they suffer from heartburn, and if so, what remedy they take. The answers were recorded as follows:

1. Do not suffer from heartburn
2. Suffer from heartburn but take no remedy

3. Suffer from heartburn and take an over-the-counter remedy (e.g., Tums, Gavoscol)
4. Suffer from heartburn and take a prescription pill (e.g., Nexium)

Is there a relationship between an adult's source of news and his or her heartburn condition?

15.36 Xr02-42 (Exercise 2.42 revisited) The associate dean of a business school was looking for ways to improve the quality of the applicants to its MBA program. In particular, she wanted to know whether the undergraduate degree of applicants differed among her school and the three nearby universities with MBA programs. She sampled 100 applicants of her program and an equal number from each of the other universities. She recorded their undergraduate degrees (1 = BA, 2 = BEng, 3 = BBA, 4 = other) as well as universities (codes 1, 2, 3, and 4). Do these data provide sufficient evidence to infer that undergraduate degree and the university each person applied are related?

15.37 Xr15-37 The relationship between drug companies and medical researchers is under scrutiny because of possible conflict of interest. The issue that started the controversy was a 1995 case control study that suggested that the use of calcium-channel blockers to treat hypertension led to an increase risk of heart disease. This led to an intense debate both in technical journals and in the press. Researchers writing in the *New England Journal of Medicine* ("Conflict of Interest in the Debate over Calcium Channel Antagonists," January 8, 1998, p. 101) looked at the 70 reports that appeared during 1996–1997, classifying them as favorable, neutral, or critical toward the drugs. The researchers then contacted the authors of the reports and questioned them about financial ties to drug companies. The results were recorded in the following way:

Column 1: Results of the scientific study; 1 = favorable, 2 = neutral, 3 = critical
Column 2: 1 = financial ties to drug companies, 2 = no ties to drug companies

Do these data allow us to infer that the research findings for calcium-channel blockers are affected by whether the research is funded by drug companies?

15.38 Xr15-38 After a thorough analysis of the market, a publisher of business and economics statistics books has divided the market into three general approaches to teach applied statistics. These are (1) use of a computer and statistical software with no manual calculations, (2) traditional teaching of concepts and solution of problems by hand, and (3) mathematical approach with emphasis on derivations and proofs. The publisher wanted to know whether this market could be segmented on the basis of the educational background of the instructor. As a result, the statistics editor organized a survey that asked 195 professors of business and economics statistics to report their approach to teaching and which one of the following categories represents their highest degree:

1. Business (MBA or Ph.D. in business)
2. Economics
3. Mathematics or engineering
4. Other

a. Can the editor infer that there are differences in type of degree among the three teaching approaches? If so, how can the editor use this information?
b. Suppose that you work in the marketing department of a textbook publisher. Prepare a report for the editor that describes this analysis.

GENERAL SOCIAL SURVEY EXERCISES

15.39 GSS2002* GSS2004* GSS2006* GSS2008* The issue of gun control in the United States is often debated, particularly during elections. The question arises, What does the public think about the issue and does support vary from year to year? Test to determine whether there is enough evidence to conclude that support for gun laws (GUNLAW) varied from year to year.

15.40 GSS2002* GSS2004* GSS2006* GSS2008* Can we conclude that Americans' marital status (MARITAL) distribution has changed from year to year?

15.41 GSS2008* In the last two decades, an increasing proportion of women have entered the workforce. Determine whether there is enough evidence to conclude that men and women (SEX) differ in their work status (WRKSTA).

15.42 GSS2008* Is there sufficient evidence to infer that support for capital punishment (CAPPUN) is related to political affiliation (PARTYID3: 1 = Democrat, 2 = Republican, 3 = Independent)?

American National Election Survey Exercises

For each of the following variables, conduct a test to determine whether there are differences between the three political party affiliations (PARTY: 1 = Democrat, 2 = Republican, 3 = Independent).

15.43 ANES2008* Know where to vote (KNOW)

15.44 ANES2008* Read about campaign in newspaper (READ)

15.45 ANES2008* Have health insurance (HEALTH)

15.46 ANES2008* Have access to the Internet (ACCESS)

15.3 Summary of Tests on Nominal Data

At this point in the textbook, we've described four tests that are used when the data are nominal:

z-test of p (Section 12.3)
z-test of $p_1 - p_2$ (Section 13.5)
Chi-squared goodness-of-fit test (Section 15.1)
Chi-squared test of a contingency table (Section 15.2)

In the process of presenting these techniques, it was necessary to concentrate on one technique at a time and focus on the kinds of problems each addresses. However, this approach tends to conflict somewhat with our promised goal of emphasizing the "when" of statistical inference. In this section, we summarize the statistical tests on nominal data to ensure that you are capable of selecting the correct method.

There are two critical factors in identifying the technique used when the data are nominal. The first, of course, is the problem objective. The second is the number of categories that the nominal variable can assume. Table 15.1 provides a guide to help select the correct technique.

TABLE **15.1** Statistical Techniques for Nominal Data

PROBLEM OBJECTIVE	NUMBER OF CATEGORIES	STATISTICAL TECHNIQUE
Describe a population	2	z-test of p or the chi-squared goodness-of-fit test
Describe a population	More than 2	Chi-squared goodness-of-fit test
Compare two populations	2	z-test of $p_1 - p_2$ or chi-squared test of a contingency table
Compare two populations	More than 2	Chi-squared test of a contingency table
Compare two or more populations	2 or more	Chi-squared test of a contingency table
Analyze the relationship between two variables	2 or more	Chi-squared test of a contingency table

Notice that when we describe a population of nominal data with exactly two categories, we can use either of two techniques. We can employ the z-test of p or the chi-squared goodness-of-fit test. These two tests are equivalent because if there are only two categories, the multinomial experiment is actually a binomial experiment (one of the categorical outcomes is labeled *success*, and the other is labeled *failure*). Mathematical statisticians have established that if we square the value of z, the test statistic for the test of p, we produce the χ^2-statistic; that is, $z^2 = \chi^2$. Thus, if we want to conduct a two-tail test of a population proportion, we can employ either technique. However, the chi-squared goodness-of-fit test can test only to determine whether the hypothesized values of p_1 (which we can label p) and p_2 (which we call $1 - p$) are not equal to their specified values. Consequently, to perform a one-tail test of a population proportion, we must use the z-test of p. (This issue was discussed in Chapter 14 when we pointed out that we can use either the t-test of $\mu_1 - \mu_2$ or the analysis of variance to conduct a test to determine whether two population means differ.)

When we test for differences between two populations of nominal data with two categories, we can also use either of two techniques: the z-test of $p_1 - p_2$ (Case 1) or the chi-squared test of a contingency table. Once again, we can use either technique to perform a two-tail test about $p_1 - p_2$. (Squaring the value of the z-statistic yields the value of the χ^2-statistic.) However, one-tail tests must be conducted by the z-test of $p_1 - p_2$. The rest of the table is quite straightforward. Notice that when we want to compare two populations when there are more than two categories, we use the chi-squared test of a contingency table.

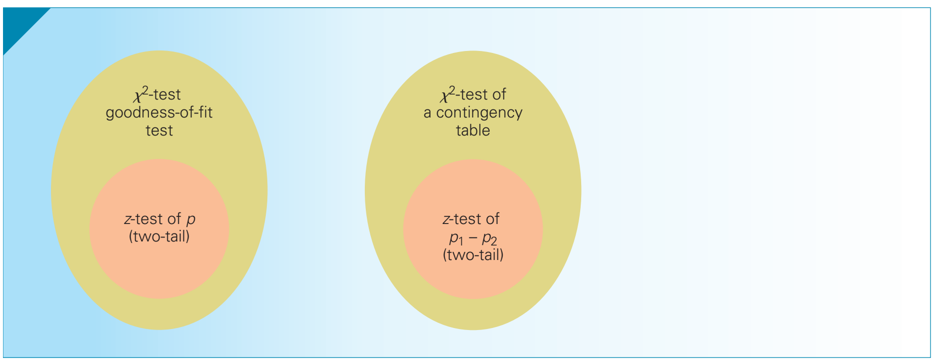
Figure 15.4 offers another summary of the tests that deal with nominal data introduced in this book. There are two groups of tests: those that test hypotheses about single populations and those that test either for differences or for independence. In the first set, we have the z-test of p, which can be replaced by the chi-squared test of a multinomial experiment. The latter test is employed when there are more than two categories.

To test for differences between two proportions, we apply the z-test of $p_1 - p_2$. Instead we can use the chi-squared test of a contingency table, which can be applied to a variety of other problems.

Developing an Understanding of Statistical Concepts

Table 15.1 and Figure 15.4 summarize how we deal with nominal data. We determine the frequency of each category and use these frequencies to compute test statistics. We can then compute proportions to calculate z-statistics or use the frequencies to calculate

FIGURE **15.4** Tests on Nominal Data



χ^2-statistics. Because squaring a standard normal random variable produces a chi-squared variable, we can employ either statistic to test for differences. As a consequence, when you encounter nominal data in the problems described in this book (and other introductory applied statistics books), the most logical starting point in selecting the appropriate technique will be either a z-statistic or a χ^2-statistic. However, you should know that there are other statistical procedures that can be applied to nominal data, techniques that are not included in this book.

15.4 (Optional) Chi-Squared Test for Normality

We can use the goodness-of-fit test presented in Section 15.1 in another way. We can test to determine whether data were drawn from any distribution. The most common application of this procedure is a test of normality.

In the examples and exercises shown in Section 15.1, the probabilities specified in the null hypothesis were derived from the question. In Example 15.1, the probabilities p_1, p_2, and p_3 were the market shares before the advertising campaign. To test for normality (or any other distribution), the probabilities must first be calculated using the hypothesized distribution. To illustrate, consider Example 12.1, where we tested the mean amount of discarded newspaper using the Student t distribution. The required condition for this procedure is that the data must be normally distributed. To determine whether the 148 observations in our sample were indeed taken from a normal distribution, we must calculate the theoretical probabilities assuming a normal distribution. To do so, we must first calculate the sample mean and standard deviation: $\bar{x} = 2.18$ and $s = .981$. Next, we find the probabilities of an arbitrary number of intervals. For example, we can find the probabilities of the following intervals:

Interval 1: $X \leq .709$
Interval 2: $.709 < X \leq 1.69$
Interval 3: $1.69 < X \leq 2.67$
Interval 4: $2.67 < X \leq 3.65$
Interval 5: $X > 3.65$

We will discuss the reasons for our choices of intervals later.

The probabilities are computed using the normal distribution and the values of $\bar{x}$ and s as estimators of μ and σ. We calculated the sample mean and standard deviation as $\bar{x} = 2.18$ and $s = .981$. Thus,

$$P(X \leq .709) = P\left(\frac{X-\mu}{\sigma} \leq \frac{.709 - 2.18}{.981}\right) = P(Z \leq -1.5) = .0668$$

$$P(.709 < X \leq 1.69) = P\left(\frac{.709 - 2.18}{.981} < \frac{X-\mu}{\sigma} \leq \frac{1.69 - 2.18}{.981}\right)$$

$$= P(-1.5 < Z \leq -.5) = .2417$$

$$P(1.69 < X \leq 2.67) = P\left(\frac{1.69 - 2.18}{.981} < \frac{X-\mu}{\sigma} \leq \frac{2.67 - 2.18}{.981}\right)$$

$$= P(-.5 < Z \leq .5) = .3829$$

$$P(2.67 < X \leq 3.65) = P\left(\frac{2.67 - 2.18}{.981} < \frac{X - \mu}{\sigma} \leq \frac{3.65 - 2.18}{.981}\right)$$

$$= P(.5 < Z \leq 1.5) = .2417$$

$$P(X > 3.65) = P\left(\frac{X - \mu}{\sigma} > \frac{3.65 - 2.18}{.981}\right) = P(Z > 1.5) = .0668$$

To test for normality is to test the following hypotheses:

H_0: $p_1 = .0668, p_2 = .2417, p_3 = .3829, p_4 = .2417, p_5 = .0668$

H_1: At least two proportions differ from their specified values

We complete the test as we did in Section 15.1, except that the number of degrees of freedom associated with the chi-squared statistic is the number of intervals minus 1 minus the number of parameters estimated, which in this illustration is two. (We estimated the population mean μ and the population standard deviation σ.) Thus, in this case, the number of degrees of freedom is $k - 1 - 2 = 5 - 1 - 2 = 2$.

The expected values are

$$e_1 = np_1 = 148(.0668) = 9.89$$
$$e_2 = np_2 = 148(.2417) = 35.78$$
$$e_3 = np_3 = 148(.3829) = 56.67$$
$$e_4 = np_4 = 148(.2417) = 35.78$$
$$e_5 = np_5 = 148(.0668) = 9.89$$

The observed values are determined manually by counting the number of values in each interval. Thus,

$$f_1 = 10$$
$$f_2 = 36$$
$$f_3 = 54$$
$$f_4 = 39$$
$$f_5 = 9$$

The chi-squared statistic is

$$\chi^2 = \sum_{i=1}^{k} \frac{(f_i - e_i)^2}{e_i} = \frac{(10 - 9.89)^2}{9.89} + \frac{(36 - 35.78)^2}{35.78} + \frac{(54 - 56.67)^2}{56.67}$$
$$+ \frac{(39 - 35.78)^2}{35.78} + \frac{(9 - 9.89)^2}{9.89}$$
$$= .50$$

The rejection region is

$$\chi^2 > \chi^2_{\alpha,k-3} = \chi^2_{.05,2} = 5.99$$

There is not enough evidence to conclude that these data are not normally distributed.

Class Intervals

In practice you can use any intervals you like. We chose the intervals we did to facilitate the calculation of the normal probabilities. The number of intervals was chosen to comply with the rule of five, which requires that all expected values be at least equal to 5. Because the number of degrees of freedom is $k - 3$, the minimum number of intervals is $k = 4$.

Using the Computer

EXCEL

	A	B	C	D
1	Chi-Squared Test of Normality			
2				
3		*Newspaper*		
4	Mean	2.18		
5	Standard deviation	0.981		
6	Observations	148		
7				
8	Intervals	Probability	Expected	Observed
9	(z <= −1.5)	0.0668	9.89	10
10	(−1.5 < z <= −0.5)	0.2417	35.78	36
11	(−0.5 < z <= 1.5)	0.3829	56.67	54
12	(0.5 < z <=1.5)	0.2417	35.78	39
13	(z > 1.5)	0.0668	9.89	9
14				
15				
16	chi-squared Stat	0.50		
17	df	2		
18	p-value	0.7792		
19	chi-squared Critical	5.9915		

We programmed Excel to calculate the value of the test statistic so that the expected values are at least 5 (where possible) and the minimum number of intervals is 4. Hence, if the number of observations is more than 220, the intervals and probabilities are

Interval	Probability
$Z \le -2$	.0228
$-2 < Z \le -1$	.1359
$-1 < Z \le 0$	.3413
$0 < Z \le 1$	.3413
$1 < Z \le 2$	.1359
$Z > 2$	.0228

If the sample size is less than or equal to 220 and greater than 80, the intervals are

Interval	Probability
$Z \le -1.5$	.0668
$-1.5 < Z \le -0.5$	.2417
$-0.5 < Z \le 0.5$	.3829
$0.5 < Z \le 1.5$	.2417
$Z > 1.5$	.0668

If the sample size is less than or equal to 80, we employ the minimum number of intervals, 4. When the sample size is less than 32, at least one expected value will be less than 5. The intervals are

(Continued)

Interval	Probability
$Z \leq -1$	.1587
$-1 < Z \leq 0$	.3413
$0 < Z \leq 1$	.3413
$Z > 1$	.1587

INSTRUCTIONS

1. Type or import the data into one column. (Open Xm12-01.)
2. Click **Add-Ins**, **Data Analysis Plus**, and **Chi-Squared Test of Normality**.
3. Specify the **Input Range** (A1:A149) and the value of α(.05).

MINITAB

Minitab does not conduct this procedure.

Interpreting the Results of a Chi-Squared Test for Normality

In the example above, we found that there was little evidence to conclude that the weight of discarded newspaper is not normally distributed. However, had we found evidence of nonnormality, this would not necessarily invalidate the *t*-test we conducted in Example 12.1. As we pointed out in Chapter 12, the *t*-test of a mean is a robust procedure, which means that only if the variable is extremely nonnormal and the sample size is small can we conclude that the technique is suspect. The problem here is that if the sample size is large and the variable is only slightly nonnormal, the chi-squared test for normality will, in many cases, conclude that the variable is not normally distributed. However, if the variable is even quite nonnormal and the sample size is large, the *t*-test will still be valid. Although there are situations in which we need to know whether a variable is nonnormal, we continue to advocate that the way to decide if the normality requirement for almost all statistical techniques applied to interval data is satisfied is to draw histograms and look for shapes that are far from bell shaped (e.g., highly skewed or bimodal). We will use this approach in Chapter 19 when we introduce nonparametric techniques that are used when interval data are nonnormal.

EXERCISES

15.47 Suppose that a random sample of 100 observations was drawn from a population. After calculating the mean and standard deviation, each observation was standardized and the number of observations in each of the following intervals was counted. Can we infer at the 5% significance level that the data were not drawn from a normal population?

Interval	Frequency
$Z \leq 1.5$	10
$-1.5 < Z \leq -0.5$	18
$-0.5 < Z \leq 0.5$	48
$0.5 < Z \leq 1.5$	16
$Z > 1.5$	8

15.48 A random sample of 50 observations yielded the following frequencies for the standardized intervals:

Interval	Frequency
$Z \leq -1$	6
$-1 < Z \leq 0$	27
$0 < Z \leq 1$	14
$Z > 1$	3

Can we infer that the data are not normal? (Use $\alpha = .10$.)

The following exercises require the use of a computer and software.

15.49 Xr12-31 Refer to Exercise 12.31. Test at the 10% significance level to determine whether the amount of time spent working at part-time jobs is normally distributed. If there is evidence of nonnormality, is the t-test invalid?

15.50 Xr12-37 The t-test in Exercise 12.37 requires that the costs of prescriptions is normally distributed. Conduct a test with $\alpha = .05$ to determine whether the required condition is unsatisfied. If there is enough evidence to conclude that the requirement is not satisfied, does this indicate that the t-test is invalid?

15.51 Xr13-25 Exercise 13.25 required you to conduct a t-test of the difference between two means. Each sample's productivity data are required to be normally distributed. Is that required condition violated? Test with $\alpha = .05$.

15.52 Xr13-26 Exercise 13.26 asked you to conduct a t-test of the difference between two means (reaction times). Test to determine whether there is enough evidence to infer that the reaction times are not normally distributed. A 5% significance level is judged to be suitable.

15.53 Xr13-59 In Exercise 13.59, you performed a test of the mean matched pairs difference. The test result depends on the requirement that the differences are normally distributed. Test with a 10% significance level to determine whether the requirement is violated.

CHAPTER SUMMARY

This chapter introduced three statistical techniques. The first is the chi-squared goodness-of-fit test, which is applied when the problem objective is to describe a single population of nominal data with two or more categories. The second is the chi-squared test of a contingency table. This test has two objectives: to analyze the relationship between two nominal variables and to compare two or more populations of nominal data. The last procedure is designed to test for normality.

IMPORTANT TERMS

Multinomial experiment 597
Chi-squared goodness-of-fit test 598
Expected frequency 598
Observed frequencies 599
Cross-classification table 604
Chi-squared test of a contingency table 604
Contingency table 607

SYMBOLS

Symbol	Pronounced	Represents
f_i	f sub i	Frequency of the ith category
e_i	e sub i	Expected value of the ith category
χ^2	Chi squared	Test statistic

FORMULA

Test statistic for all procedures

$$\chi^2 = \sum_{i=1}^{k} \frac{(f_i - e_i)^2}{e_i}$$

COMPUTER OUTPUT AND INSTRUCTIONS

Technique	Excel	Minitab
Chi-squared goodness-of-fit test	600	601
Chi-squared test of a contingency table (raw data)	609	609
Chi-squared test of a contingency table	609	609
Chi-squared test of normality	619	620

CHAPTER EXERCISES

Use a 5% significance level unless specified otherwise.

15.54 An organization dedicated to ensuring fairness in television game shows is investigating *Wheel of Fortune*. In this show, three contestants are required to solve puzzles by selecting letters. Each contestant gets to select the first letter and continues selecting until he or she chooses a letter that is not in the hidden word, phrase, or name. The order of contestants is random. However, contestant 1 gets to start game 1, contestant 2 starts game 2, and so on. The contestant who wins the most money is declared the winner, and he or she is given an opportunity to win a grand prize. Usually, more than three games are played per show, and as a result it appears that contestant 1 has an advantage: Contestant 1 will start two games, whereas contestant 3 will usually start only one game. To see whether this is the case, a random sample of 30 shows was taken, and the starting position of the winning contestant for each show was recorded. These are shown in the following table:

Starting position	1	2	3
Number of wins	14	10	6

Do the tabulated results allow us to conclude that the game is unfair?

15.55 It has been estimated that employee absenteeism costs North American companies more than $100 billion per year. As a first step in addressing the rising cost of absenteeism, the personnel department of a large corporation recorded the weekdays during which individuals in a sample of 362 absentees were away over the past several months. Do these data suggest that absenteeism is higher on some days of the week than on others?

Day of the week	Monday	Tuesday	Wednesday	Thursday	Friday
Number absent	87	62	71	68	74

15.56 Suppose that the personnel department in Exercise 15.55 continued its investigation by categorizing absentees according to the shift on which they worked, as shown in the accompanying table. Is there sufficient evidence at the 10% significance level of a relationship between the days on which employees are absent and the shift on which the employees work?

Shift	Monday	Tuesday	Wednesday	Thursday	Friday
Day	52	28	37	31	33
Evening	35	34	34	37	41

15.57 A management behavior analyst has been studying the relationship between male–female supervisory structures in the workplace and the level of employees' job satisfaction. The results of a recent survey are shown in the accompanying table. Is there sufficient evidence to infer that the level of job satisfaction depends on the boss–employee gender relationship?

	Boss/Employee			
Level of Satisfaction	Female/Male	Female/Female	Male/Male	Male/Female
Satisfied	21	25	54	71
Neutral	39	49	50	38
Dissatisfied	31	48	10	11

The following exercises require the use of a computer and software. The answers may be calculated manually. See Appendix A for the sample statistics. Use a 5% significance level unless specified otherwise.

15.58 Xr15-58 Stress is a serious medical problem that costs businesses and government billions of dollars annually. As a result, it is important to determine the causes and possible cures. It would be helpful to know whether the causes are universal or if they vary from country to country. In a survey, American and Canadian adults were asked to report their primary source of stress in their lives. The responses are

1 = Job, 2 = Finances, 3 = Health,
4 = Family life, 5 = Other

The data were recorded using the codes above plus 1 = American and 2 = Canadian. Do these data provide sufficient evidence to conclude that Americans and Canadians differ in their sources of stress?

15.59 Xr15-59 According to NBC News (March 11, 1994) more than 3,000 Americans quit smoking each day. (Unfortunately, more than 3,000 Americans start smoking each day.) Because nicotine is one of the most addictive drugs, quitting smoking is a difficult and frustrating task. It usually takes several tries before success is achieved. There are various methods, including cold turkey, nicotine patches, hypnosis, and group therapy sessions. In an experiment to determine how these methods differ, a random sample of smokers who have decided to quit is selected. Each smoker has chosen one of the methods listed above. After one year, the respondents report whether they have quit (1 = yes and 2 = no) and which method they used (1 = cold turkey, 2 = nicotine patch, 3 = hypnosis, 4 = group therapy sessions). Is there sufficient evidence to conclude that the four methods differ in their success?

15.60 Xr15-60 A newspaper publisher, trying to pinpoint his market's characteristics, wondered whether the way people read a newspaper is related to the reader's educational level. A survey asked adult readers which section of the paper they read first and asked them to report their highest educational level. These data were recorded (Column 1 = first section read where 1 = front page, 2 = sports, 3 = editorial, and 4 = other; and column 2 = educational level where 1 = did not complete high school, 2 = high school graduate, 3 = university or college graduate, and 4 = postgraduate degree). What do these data tell the publisher about how educational level affects the way adults read the newspaper?

15.61 Xr15-61 Every week, the Florida Lottery draws six numbers between 1 and 49. Lottery ticket buyers are naturally interested in whether certain numbers are drawn more frequently than others. To assist players, the *Sun-Sentinel* publishes the number of times each of the 49 numbers has been drawn in the past 52 weeks. The numbers and the frequency with which each occurred were recorded.

a. If the numbers are drawn from a uniform distribution, what is the expected frequency for each number?
b. Can we infer that the data were not generated from a uniform distribution?

In Section 15.4, we showed how to test for normality. However, we can use the same process to test for any other distribution.

15.62 Xr15-62 A scientist believes that the gender of a child is a binomial random variable with probability = .5 for a boy and .5 for a girl. To help test her belief, she randomly samples 100 families with five children. She records the number of boys. Can the scientist infer that the number of boys in families with five children is not a binomial random variable with $p = .5$? (*Hint*: Find the probability of $X = 0, 1, 2, 3, 4,$ and 5 from a binomial distribution with $n = 5$ and $p = .5$).

15.63 Xr15-63 Given the high cost of medical care, research that points the way to avoid illness is welcome. Previously performed research tells us that stress affects the immune system. Two scientists at Carnegie Mellon Hospital in Pittsburgh asked 114 healthy adults about their social circles; they were asked to list every group they had contact with at least once every 2 weeks—family, co-workers, neighbors, friends, and religious and community groups. Participants also reported negative life events over the past year, including the death of a friend or relative, divorce, or job-related problems. The participants were divided into four groups:

Group 1: Highly social and highly stressed
Group 2: Not highly social and highly stressed
Group 3: Highly social and not highly stressed
Group 4: Not highly social and not highly stressed

Each individual was classified in this way. In addition, whether each person contracted a cold over the next 12 weeks was recorded (1 = cold, 2 = no cold). Can we infer that there are differences between the four groups in terms of contracting a cold?

The following exercises employ data files associated with examples and exercises seen earlier in this book.

15.64 Xr12-91* Exercise 12.91 described the problem of a looming shortage of professors, possibly made worse by professors desiring to retire before the age of 65. A survey asked a random sample of professors whether they intended to retire before 65. The responses are "no" (1) "yes" (2). In addition, the survey asked to which faculty each professor belonged (1 = Arts, 2 = Science, 3 = Business, 4 = Engineering, 5 = other). Do these provide sufficient evidence to infer that whether a professor wishes to retire is related to the faculty?

15.65 Xr12-95* Refer to Exercise 12.95. Determine whether there is enough evidence to infer that there are differences in the choice of Christmas tree between the three age categories.

15.66 Xr12-97* Exercise 12.97 described a study to determine whether viewers (older than 50) of the network news had contacted their physician to ask about one of the prescription drugs advertised during the newscast. The responses (1 = no and 2 = yes) were recorded. Also recorded were which of the three networks they normally watch (1 = ABC, 2 = CBS, 3 = NBC). Can we conclude that there are differences in responses between the three network news shows?

15.67 Xr13-110* Exercise 13.110 described a survey of adults wherein, on the basis of several probing questions, each was classified as being or not being a member of the health-conscious group (belonging = 1, not belonging = 2) and whether he or she buys Special X (1 = no, 2 = yes). In addition, his or her educational attainment was recorded (1 = did not finish high school, 2 = finished high school, 3 = finished college or university, 4 = postgraduate degree).

a. Do the data allow the surveyor to conclude that there are differences in educational attainment between those who do and those who do not belong to the health-conscious group?
b. Can we infer that there is a relationship between the four educational groups and whether or not a person buys Special X?

15.68 Xm12-05* Example 12.5 described exit polls wherein people are asked whether they voted for the Democrat or Republican candidate for president. The surveyors also record gender (1 = female, 2 = male), educational attainment (1 = did not finish high school, 2 = completed high school, 3 = completed college or university, 4 = postgraduate degree), and income level (1 = less than $25,000, 2 = $25,000 to $49,999, 3 = $50,000 to $75,000), 4 = more than $75,000).

a. Is there sufficient evidence to infer that voting and gender are related?
b. Do the data allow the conclusion that voting and educational level are related?
c. Can we infer that voting and income are related?

APPLICATIONS in MARKETING

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Market Segmentation

In Section 12.4 and Chapters 13 and 14, we described how marketing managers use statistical analyses to estimate the size of market segments and determine whether there are differences between segments.

The following exercises require the application of the chi-squared test of a contingency table to determine whether market segments differ with respect to some nominal variable.

15.69 Xr12-126* Exercise 12.126 described the market segments defined by JC Penney. Another question included in the questionnaire that classified the women surveyed asked whether each worked outside the home. The responses were

1. No
2. Part-time job
3. Full-time job

These data plus the classifications (1 = conservative, 2 = traditional, and 3 = contemporary) were recorded. Can we infer from these data that there are differences in employment status between the three market segments?

15.70 Xr12-126* Refer to Exercise 12.126. The women in the survey were also asked to define value by identifying what they considered to be the most important attribute of value. The responses are

1. Price
2. Quality
3. Fashion

The responses and the classifications of segments (1 = conservative, 2 = traditional, and 3 = contemporary) were recorded. Do these data allow us to infer that there are differences in the definition of value between the three market segments?

15.71 Xm12-06* Refer to Example 12.6. In segmenting the breakfast cereal market, a food manufacturer uses health and diet consciousness as the segmentation variable. Four segments are developed:

1. Concerned about eating healthy foods
2. Concerned primarily about weight
3. Concerned about health because of illness
4. Unconcerned

A survey was undertaken, and each person was asked how often they ate a healthy breakfast (defined as cereal with or without fruit). The responses are

1. Never
2. Seldom
3. Often
4. Always

The responses and the market segments of each respondent were recorded. Can we infer that there are differences in frequency of healthy breakfasts between the market segments?

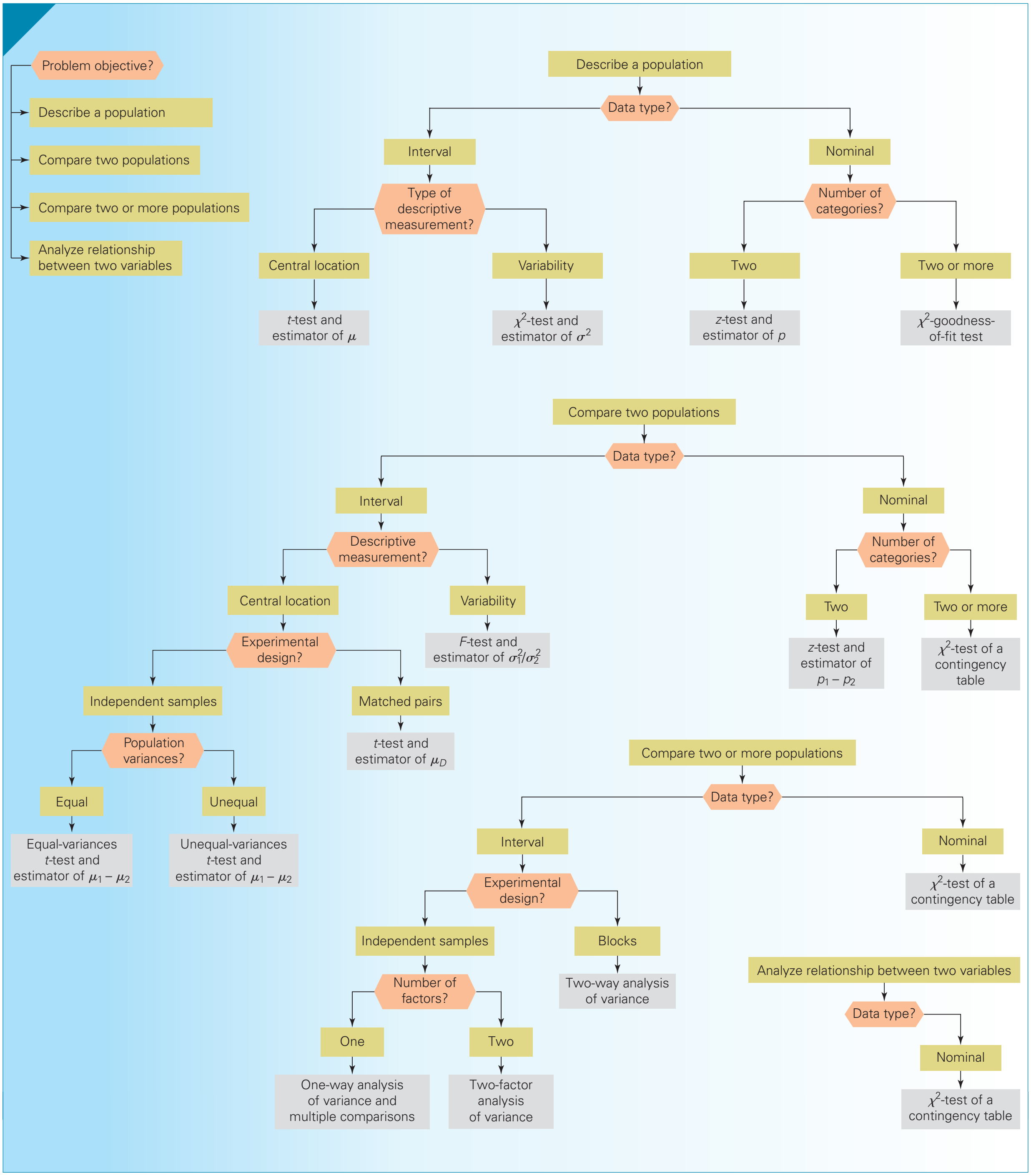
APPENDIX 15 / Review of Chapters 12 to 15

Here are the updated list of statistical techniques (Table A15.1) and the flowchart (Figure A15.1) for Chapters 12 to 15. Counting the two techniques of chi-squared tests introduced here (we do not include the chi-squared test for normality), we have covered 22 statistical methods.

TABLE **A15.1** Summary of Statistical Techniques in Chapters 12 to 15

t-test of μ
Estimator of μ (including estimator of $N\mu$)
χ^2-test of σ^2
Estimator of σ^2
z-test of p
Estimator of p (including estimator of Np)
Equal-variances t-test of $\mu_1 - \mu_2$
Equal-variances estimator of $\mu_1 - \mu_2$
Unequal-variances t-test of $\mu_1 - \mu_2$
Unequal-variances estimator of $\mu_1 - \mu_2$
t-test of μ_D
Estimator of μ_D
F-test of σ_1^2/σ_2^2
Estimator of σ_1^2/σ_2^2
z-test of $p_1 - p_2$ (Case 1)
z-test of $p_1 - p_2$ (Case 2)
Estimator of $p_1 - p_2$
One-way analysis of variance (including multiple comparisons)
Two-way (randomized blocks) analysis of variance
Two-factor analysis of variance
χ^2-goodness-of-fit test
χ^2-test of a contingency table

FIGURE A15.1 Summary of Statistical Techniques in Chapters 12 to 15



EXERCISES

We remind you that we do not specify significance levels in the exercise that follow. Choose your own.

A15.1 XrA15-01 An analysis of the applicants of all MBA programs in North America reveals that the proportions of each type of undergraduate degree are as follows:

Undergraduate Degree	Proportion (%)
B.A. (1)	50
B.B.A. (2)	20
B.Sc. (3)	15
B.Eng. (4)	10
Other (5)	5

The director of Wilfrid Laurier University's (WLU's) MBA program recorded the undergraduate degree of the applicants for this year using the codes in parentheses. Do these data indicate that applicants to WLU's MBA program are different in terms of their undergraduate degrees from the population of MBA applicants?

A15.2 XrA15-02 The experiment to determine the effect of taking a preparatory course to improve SAT scores in Exercise A13.16 was criticized by other statisticians. They argued that the first test would provide a valuable learning experience that would produce a higher test score from the second exam even without the preparatory course. Consequently, another experiment was performed. Forty students wrote the SAT without taking any preparatory course. At the next scheduled exam (3 months later), these same students took the exam again (again with no preparatory course). The scores for both exams were recorded in columns 1 (first test scores) and 2 (second test scores). Can we infer that repeating the SAT produces higher exam scores even without the preparatory course?

A15.3 XrA15-03 How does dieting affect the brain? This question was addressed by researchers in Australia. The experiment used 40 middle-age women in Adelaide, Australia; half were on a diet and half were not (*National Post*, December 1, 2003). The mental arithmetic part of the experiment required the participants to add two three-digit numbers. The amount of time taken to solve the 48 problems was recorded. The participants were given another test that required them to repeat a string of five letters they had been told 10 seconds earlier. They were asked to repeat the test with five words told to them 10 seconds earlier. The data were recorded in the following way:

Column 1: Identification number
Column 2: 1 = dieting, 2 = not dieting
Column 3: Time to solve 48 problems (seconds)
Column 4: Repeat string of 5 letters (1 = no, 2 = yes)
Column 5: Repeat string of 5 words (1 = no, 2 = yes)

Is there sufficient evidence to infer that dieting adversely affects the brain?

A15.4 XrA15-04 A small but important part of a university library's budget is the amount collected in fines on overdue books. Last year, a library collected \$75,652.75 in fine payments; however, the head librarian suspects that some employees are not bothering to collect the fines on overdue books. In an effort to learn more about the situation, she asked a sample of 400 students (out of a total student population of 50,000) how many books they had returned late to the library in the previous 12 months. They were also asked how many days overdue the books had been. The results indicated that the total number of days overdue ranged from 0 to 55 days. The number of days overdue was recorded.

a. Estimate with 95% confidence the average number of days overdue for all 50,000 students at the university.
b. If the fine is 25 cents per day, estimate the amount that should be collected annually. Should the librarian conclude that not all the fines were collected?

A15.5 XrA15-05 An apple juice manufacturer has developed a new product—a liquid concentrate that produces 1 liter of apple juice when mixed with water. The product has several attractive features. First, it is more convenient than bottled apple juice, which is the way apple juice is currently sold. Second, because the apple juice that is sold in cans is actually made from concentrate, the quality of the new product is at least as high as that of bottled apple juice. Third, the cost of the new product is slightly lower than that of bottled apple juice. The marketing manager has to decide how to market the new product. She can create advertising that emphasizes convenience, quality, or price. To facilitate a decision, she conducts an experiment in three different small cities. In one city, she launches the product with advertising stressing the convenience of the liquid concentrate (e.g., easy to carry from store to home and takes up less room in the freezer). In the second city, the advertisements emphasize the quality of the product ("average" shoppers are depicted discussing how good the apple juice tastes). Advertising that highlights the relatively low cost of the liquid concentrate is used in the third city. The number of packages sold weekly is recorded for the 20 weeks following the beginning of the campaign. The marketing manager wants to know whether differences in sales exist between the three advertising strategies. (We will assume that except for the type of advertising, the three cities are identical.)

A15.6 XrA15-06 Mutual funds are a popular way of investing in the stock market. A financial analyst wanted to determine the effect income had on ownership of mutual funds and whether the relationship had changed from four years earlier. She took a random sample of adults 25 years of age and older and asked each person whether he or she owned mutual funds (No = 1 and Yes = 2) and to report the annual household income. The categories are

1. Less than $25,000
2. $25,000 to $34,999
3. $35,000 to $49,999
4. $50,000 to $74,999
5. $75,000 to $100,000
6. More than $100,000

Can we infer from the data that household income and ownership of mutual funds are related? (Adapted from the *Statistical Abstract of the United States, 2006*, Table 1200.)

A15.7 XrA15-07 Refer to Exercise A15.5. Suppose that in addition to varying the marketing strategy, the manufacturer also decided to advertise in one of the two media that are available: television and newspapers. As a consequence, the experiment was repeated in the following way. Six different small cities were selected. In city 1, the marketing emphasized convenience, and all the advertising was conducted on television. In city 2, marketing also emphasized convenience, but all the advertising was conducted in the daily newspaper. Quality was emphasized in cities 3 and 4. City 3 learned about the product from television commercials, and city 4 saw newspaper advertising. Price was the marketing emphasis in cities 5 and 6. City 5 saw television commercials, and city 6 saw newspaper advertisements. In each city, the weekly sales for each of 10 weeks were recorded. What conclusions can be drawn from these data?

A15.8 XrA15-08 After a recent study, researchers reported on the effects of folic acid on the occurrence of spina bifida—a birth defect in which there is incomplete formation of the spine. A sample of 2,000 women who gave birth to children with spina bifida and who were planning another pregnancy was recruited. Before attempting to get pregnant again, half the sample was given regular doses of folic acid, and the other half was given a placebo. After 18 months, researchers recorded the result for each woman: 1 = birth to normal baby, 2 = birth to baby with spina bifida, 3 = not pregnant or no baby yet delivered. Can we infer that folic acid reduces the incidence of spina bifida in newborn babies?

A15.9 XrA15-09 Slow play of golfers is a serious problem for golf clubs. Slow play results in fewer rounds of golf and less profits for public course owners. To examine this problem, a random sample of British and American golf courses was selected. The amount of time taken (in minutes) was recorded for a random sample of British and American golfers. Can we conclude that British golfers play golf in less time than do American golfers? (Source: *Golf Magazine*, July 2001.)

A15.10 XrA15-10 The United States and Canada (among others) are countries in which a significant proportion of citizens are immigrants. Many arrive in North America with few assets but quickly adapt to a changed economic environment. The question often arises, How quickly do immigrants increase their standard of living? A study initiated by Statistics Canada surveyed three different types of families:

1. Immigrants who arrived before 1976
2. Immigrants who came to Canada after 1986
3. Canadian-born families

The survey measured family wealth, which includes houses, cars, income, and savings and recorded the results (in $1,000s). Can we infer that differences exist between the three groups? If so, what are those differences?

A15.11 XrA15-11 During the decade of the 1980s, professional baseball thrived in North America. However, in the 1990s attendance dropped, and the number of television viewers also decreased. To examine the popularity of baseball relative to other sports, surveys were performed. In 1985 and again in 1992, a Harris Poll asked a random sample of 500 people to name their favorite sport. The results, which were published in the *Wall Street Journal* (July 6, 1993), were recorded in the following way: favorite sport (1 = professional football, 2 = baseball, 3 = professional basketball, 4 = college basketball, 5 = college football, 6 = golf, 7 = auto racing, 8 = tennis, and 9 = other); year (1 = 1985, 2 = 1992). Do these results indicate that North Americans changed their favorite sport between 1985 and 1992?

A15.12 XrA15-12 In an attempt to learn more about traffic congestion in a large North American city, the number of cars passing through intersections was determined (*National Post*, October 18, 2006). The number of cars was counted in 5-minute samples throughout several days. The counts for one busy intersection were recorded. Estimate with 95% confidence the mean number of cars in 5 minutes. Use the result to estimate the counts for a 24-hour day.

A15.13 XrA15-13 Organizations that sponsor various leisure activities need to know the number of people who wish to participate. Bureaucrats need to know the number because many organizations apply for government grants to pay the costs. The U.S. National Endowment for the Arts conducts surveys of American adults to acquire this type of information. One part of the survey asked a random sample of adults whether they participated in exercise programs. The responses (1 = yes and 2 = no) were recorded. A recent census reveals that there are 205.9 million adults in the United States. Estimate with 95% confidence the number of American adults who participate in exercise programs. (Adapted from the *Statistical Abstract of the United States, 2006*, Table 1227.)

A15.14 XrA15-14 Low back pain is a common medical problem that sometimes results in disability and absence from work. Any method of treatment that decreases absence would be welcome by individuals and insurance companies. A randomized control study (published in *Annals of Internal Medicine*, January 2004) was undertaken to determine whether an alternate form of treatment is effective. The study examined 134 workers who were absent from work because of low back pain. Half the sample was assigned to graded activity, a physical exercise program designed to stimulate rapid return to work. The other half was assigned to the usual care, which involves mostly rest. For each worker, the number of days absent from work because of low back pain in the following 6 months was recorded. Do these data provide sufficient evidence to infer that the graded activity is effective?

A15.15 XrA15-15 Clinical depression is a serious and sometimes debilitating disease. It is often treated by antidepressants such as Prozac and Zoloft. Recent studies may indicate another possible remedy. Researchers took a random sample of people who are clinically depressed and divided them into three groups. The first group was treated with antidepressants and light therapy, the second was treated with a placebo and light therapy, and the third group treated with a placebo. Whether the patient showed improvement (code = 1) or not (code = 2) and the group number were recorded. Can we infer that there are differences between the three groups?

A15.16 How well do airlines keep to their schedules? To help answer this question, an economist conducted a survey of 780 takeoffs in the United States and determined that 77.4% of them departed on time (defined as a departure that is within 15 minutes of its scheduled time). There were 7,140,596 flight departures in the United States in 2005. Estimate with 95% confidence the total number of on-time departures.

General Social Survey Exercises

A15.17 GSS2006 GSS2008* During 2008, the United States was in the throes of a deep recession. The unemployment rate rose sharply. How did this affect job tenure (the amount of time a worker has been with his or her current employer)? Is there sufficient evidence to conclude that job tenure changed between 2006 (YEARSJOB) and 2008 (CUREMPYR)?

A15.18 GSS2008* Capital punishment for murderers exists in most U.S. states. However, a few states ban this form of punishment. Politicians often need to know which members of the public support and which oppose. Can we conclude from the data that there is a difference between Democrats, Republicans, and Independents (PARTYID: 0, 1 = Democrat, 2, 3, 4 = Independent, 5, 6 = Republican) in terms of support for capital punishment (CAPPUN)?

A15.19 GSS2008* Are married couples postponing bearing children? One way to measure this is to determine how old people are when their first child is born. Estimate with 95% confidence the average age of Americans when their first child is born (AGEKDBRN).

A15.20 GSS2008* In Chapter 2, we used a graphical technique and data from the American National Election

Survey to attempt to determine whether men and women differ in their political affiliation. Use a suitable statistical inference technique to determine whether there is sufficient evidence to infer that men and women (SEX) differ in their political affiliations (PARTYID).

A15.21 Do teenage and adult children living with their parents contribute to household income by holding down full- or part-time jobs? And is it more likely that they do so for affluent than for less affluent families? To answer the question, test to determine whether the data allow us to conclude that there are differences in the number of family members earning money (EARNRS) between the four classes (CLASS).

A15.22 GSS2002* GSS2004* GSS2006* A generally accepted method of finding whether Americans have improved financially over a multiyear period is to calculate inflation-adjusted incomes. Using the General Social Survey data, can we infer that American inflation-adjusted incomes (CONRINC) varied from year to year in 2002, 2004, and 2006 ?

A15.23 GSS2008* Does the race of an individual affect whether he or she is likely to be self-employed? Can we conclude that differences in whether an individual works for him- or herself (WRKSLF: 1 = Self-employed, 2 = Someone else) exists between the races (RACE)?

A15.24 GSS2008* Does being unemployed for any period of time affect an individual's political persuasion? Using the GSS 2008 data, determine whether there is enough evidence to infer that Americans who have been unemployed in the last 10 years (UNEMP) have different party affiliations (PARTYID: 0, 1 = Democrat, 2, 3, 4 = Independent, 5, 6 = Republican) than those who have not been unemployed.

American National Election Survey Exercises

A15.25 ANES2008* In recent years, the proportion of eligible voters in the United States who actually vote for president has hovered around 50%. Turning out the vote is considered a critical function for most political voters. Are there differences between Liberals and Conservatives in their intention to vote? Conduct a test to determine whether there is sufficient evidence to infer that liberals and conservatives (LIBCON: 1, 2, 3 = liberal, 5, 6, 7 = conservative) differ in their intention to vote (DEFINITE).

A15.26 ANES2004* ANES2008* The economy in 2004 was strong, with growth in the economy and unemployment low. By 2008, the U.S. economy was in recession. Can we conclude that employment status (EMPLOY) has changed between 2004 and 2008?

A15.27 ANES2008* Do the data provide sufficient evidence to conclude that Americans who consider themselves strong Democrat or Republicans (STRENGTH: 1 = Strong, 5 = Not very strong) have more education (EDUC) than those who do not?

CASE A15.1 Which Diets Work?

DATA CA15-01

Every year, millions of people start new diets. There is a bewildering array of diets to choose from. The question for many people is, which ones work? Researchers at Tufts University in Boston made an attempt to point dieters in the right direction. Four diets were used:

1. Atkins low-carbohydrate diet
2. Zone high-protein, moderate-carbohydrate diet
3. Weight Watchers diet
4. Dr. Ornish's low-fat diet

The study recruited 160 overweight people and randomly assigned 40 to each diet. The average weight before dieting was 220 pounds, and all needed to lose between 30 and 80 pounds. All volunteers agreed to follow their diets for 2 months. No exercise or regular meetings were required. The following variables were recorded for each dieter using the format shown here:

Column 1: Identification number
Column 2: Diet
Column 3: Percent weight loss
Column 4: Percent low-density lipoprotein (LDL)—"bad" cholesterol—decrease
Column 5: Percent high-density lipoprotein (HDL)—"good" cholesterol—increase
Column 6: Quit after 2 months? 1 = yes, 2 = no
Column 7: Quit after 1 year? 1 = yes, 2 = no

Is there enough evidence to conclude that there are differences between the diets with respect to

a. percent weight loss?
b. percent LDL decrease?
c. percent HDL increase?
d. proportion quitting within 2 months?
e. proportion quitting after 1 year?

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SIMPLE LINEAR REGRESSION AND CORRELATION

Education and Income: How Are They Related?

DATA
GSS2008*

If you're taking this course, you're probably a student in an undergraduate or graduate business or economics program. Your plan is to graduate, get a good job, and draw a high salary. You have probably assumed that more education equals better job equals higher income. Is this true? Fortunately, the General Social Survey recorded two variables that will help determine whether education and income are related and, if so, what the value of an additional year of education might be.

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On page 663, we will provide our answer.

INTRODUCTION

Regression analysis is used to predict the value of one variable on the basis of other variables. This technique may be the most commonly used statistical procedure because, as you can easily appreciate, almost all companies and government institutions forecast variables such as product demand, interest rates, inflation rates, prices of raw materials, and labor costs.

The technique involves developing a mathematical equation or model that describes the relationship between the variable to be forecast, which is called the **dependent variable**, and variables that the statistics practitioner believes are related to the dependent variable. The dependent variable is denoted Y, whereas the related variables are called **independent variables** and are denoted $X_1, X_2, \ldots, X_k$ (where k is the number of independent variables).

If we are interested only in determining whether a relationship exists, we employ correlation analysis, a technique that we have already introduced. In Chapter 3, we presented the graphical method to describe the association between two interval variables—the scatter diagram. We introduced the coefficient of correlation and covariance in Chapter 4.

Because regression analysis involves many new techniques and concepts, we divided the presentation into three chapters. In this chapter, we present techniques that allow us to determine the relationship between only two variables. In Chapter 17, we expand our discussion to more than two variables; in Chapter 18, we discuss how to build regression models.

Here are three illustrations of the use of regression analysis.

Illustration 1 The product manager in charge of a particular brand of children's breakfast cereal would like to predict the demand for the cereal during the next year. To use regression analysis, she and her staff list the following variables as likely to affect sales:

Price of the product

Number of children 5 to 12 years of age (the target market)

Price of competitors' products

Effectiveness of advertising (as measured by advertising exposure)

Annual sales this year

Annual sales in previous years

Illustration 2 A gold speculator is considering a major purchase of gold bullion. He would like to forecast the price of gold 2 years from now (his planning horizon), using regression analysis. In preparation, he produces the following list of independent variables:

Interest rates

Inflation rate

Price of oil

Demand for gold jewelry

Demand for industrial and commercial gold

Dow Jones Industrial Average

Illustration 3 A real estate agent wants to predict the selling price of houses more accurately. She believes that the following variables affect the price of a house:

Size of the house (number of square feet)

Number of bedrooms

Frontage of the lot

Condition

Location

In each of these illustrations, the primary motive for using regression analysis is forecasting. Nonetheless, analyzing the relationship among variables can also be quite useful in managerial decision making. For instance, in the first application, the product manager may want to know how price is related to product demand so that a decision about a prospective change in pricing can be made.

Regardless of why regression analysis is performed, the next step in the technique is to develop a mathematical equation or model that accurately describes the nature of the relationship that exists between the dependent variable and the independent variables. This stage—which is only a small part of the total process—is described in the next section. In the ensuing sections of this chapter (and in Chapter 17), we will spend considerable time assessing and testing how well the model fits the actual data. Only when we're satisfied with the model do we use it to estimate and forecast.

16.1 Model

The job of developing a mathematical equation can be quite complex, because we need to have some idea about the nature of the relationship between each of the independent variables and the dependent variable. The number of different mathematical models that could be proposed is virtually infinite. Here is an example from Chapter 4.

$$\text{Profit} = (\text{Price per unit} - \text{variable cost per unit}) \times \text{Number of units sold} - \text{Fixed costs}$$

You may encounter the next example in a finance course:

$$F = P(1 + i)^n$$

where

F = Future value of an investment

P = principle or present value

i = interest rate per period

n = number of periods

These are all examples of **deterministic models**, so named because such equations allow us to determine the value of the dependent variable (on the left side of the equation) from the values of the independent variables. In many practical applications of interest to us, deterministic models are unrealistic. For example, is it reasonable to believe that we can determine the selling price of a house solely on the basis of its size? Unquestionably, the size of a house affects its price, but many other variables (some of which may not be measurable) also influence price. What must be included in most practical models is a method to represent the randomness that is part of a real-life process. Such a model is called a **probabilistic model**.

To create a probabilistic model, we start with a deterministic model that approximates the relationship we want to model. We then add a term that measures the random error of the deterministic component.

Suppose that in illustration 3, the real estate agent knows that the cost of building a new house is about \$100 per square foot and that most lots sell for about \$100,000. The approximate selling price would be

$$y = 100{,}000 + 100x$$

where y = selling price and x = size of the house in square feet. A house of 2,000 square feet would therefore be estimated to sell for

$$y = 100{,}000 + 100(2{,}000) = 300{,}000$$

We know, however, that the selling price is not likely to be exactly \$300,000. Prices may actually range from \$200,000 to \$400,000. In other words, the deterministic model is not really suitable. To represent this situation properly, we should use the probabilistic model

$$y = 100{,}000 + 100x + \varepsilon$$

where ε (the Greek letter epsilon) represents the **error variable**—the difference between the actual selling price and the estimated price based on the size of the house. The error thus accounts for all the variables, measurable and immeasurable, that are not part of the model. The value of ε will vary from one sale to the next, even if x remains constant. In other words, houses of exactly the same size will sell for different prices because of differences in location and number of bedrooms and bathrooms, as well as other variables.

In the three chapters devoted to regression analysis, we will present only probabilistic models. In this chapter, we describe only the straight-line model with one independent variable. This model is called the **first-order linear model**—sometimes called the **simple linear regression model**.*

First-Order Linear Model

$$y = \beta_0 + \beta_1 x + \varepsilon$$

where

y = dependent variable

x = independent variable

β_0 = y-intercept

β_1 = slope of the line (defined as rise/run)

ε = error variable

The problem objective addressed by the model is to analyze the relationship between two variables, x and y, both of which must be interval. To define the relationship between x and y, we need to know the value of the coefficients β_0 and β_1. However, these coefficients are population parameters, which are almost always unknown. In the next section, we discuss how these parameters are estimated.

*We use the term *linear* in two ways. The "linear" in linear regression refers to the form of the model wherein the terms form a linear combination of the coefficients β_0 and β_1. Thus, for example, the model $y = \beta_0 + \beta_1 x^2 + \varepsilon$ is a linear combination whereas $y = \beta_0 + \beta_1^2 x + \varepsilon$ is not. The simple linear regression model $y = \beta_0 + \beta_1 x + \varepsilon$ describes a straight-line or linear relationship between the dependent variable and one independent variable. In this book, we use the linear regression technique only. Hence, when we use the word *linear* we will be referring to the straight-line relationship between the variables.

16.2 ESTIMATING THE COEFFICIENTS

We estimate the parameters β_0 and β_1 in a way similar to the methods used to estimate all the other parameters discussed in this book. We draw a random sample from the population of interest and calculate the sample statistics we need. However, because β_0 and β_1 represent the coefficients of a straight line, their estimators are based on drawing a straight line through the sample data. The straight line that we wish to use to estimate β_0 and β_1 is the "best" straight line—best in the sense that it comes closest to the sample data points. This best straight line, called the *least squares line*, is derived from calculus and is represented by the following equation:

$$\hat{y} = b_0 + b_1 x$$

Here b_0 is the y-intercept, b_1 is the slope, and $\hat{y}$ is the predicted or fitted value of y. In Chapter 4, we introduced the **least squares method**, which produces a straight line that minimizes the sum of the squared differences between the points and the line. The coefficients b_0 and b_1 are calculated so that the sum of squared deviations

$$\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

is minimized. In other words, the values of $\hat{y}$ on average come closest to the observed values of y. The calculus derivation is available in Keller's website appendix, Deriving the Normal Equations, which shows how the following formulas, first shown in Chapter 4, were produced.

Least Squares Line Coefficients

$$b_1 = \frac{s_{xy}}{s_x^2}$$

$$b_0 = \bar{y} - b_1 \bar{x}$$

where

$$s_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n - 1}$$

$$s_x^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n - 1}$$

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$$

In Chapter 4, we provided shortcut formulas for the sample variance (page 110) and the sample covariance (page 127). Combining them provides a shortcut method to manually calculate the slope coefficient.

Shortcut Formula for b_1

$$b_1 = \frac{s_{xy}}{s_x^2}$$

$$s_{xy} = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i y_i - \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n}\right]$$

$$s_x^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}\right]$$

Statisticians have shown that b_0 and b_1 are unbiased estimators of β_0 and β_1, respectively.

Although the calculations are straightforward, we would rarely compute the regression line manually because the work is time consuming. However, we illustrate the manual calculations for a very small sample.

EXAMPLE 16.1 Annual Bonus and Years of Experience

DATA Xm16-01

The annual bonuses ($1,000s) of six employees with different years of experience were recorded as follows. We wish to determine the straight-line relationship between annual bonus and years of experience.

Years of experience *x*	1	2	3	4	5	6
Annual bonus *y*	6	1	9	5	17	12

SOLUTION

To apply the shortcut formula, we need to compute four summations. Using a calculator, we find

$$\sum_{i=1}^{n} x_i = 21$$

$$\sum_{i=1}^{n} y_i = 50$$

$$\sum_{i=1}^{n} x_i y_i = 212$$

$$\sum_{i=1}^{n} x_i^2 = 91$$

The covariance and the variance of x can now be computed:

$$s_{xy} = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i y_i - \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n}\right] = \frac{1}{6-1}\left[212 - \frac{(21)(50)}{6}\right] = 7.4$$

$$s_x^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}\right] = \frac{1}{6-1}\left[91 - \frac{(21)^2}{6}\right] = 3.5$$

The sample slope coefficient is calculated next:

$$b_1 = \frac{s_{xy}}{s_x^2} = \frac{7.4}{3.5} = 2.114$$

The y-intercept is computed as follows:

$$\bar{x} = \frac{\sum x_i}{n} = \frac{21}{6} = 3.5$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{50}{6} = 8.333$$

$$b_0 = \bar{y} - b_1\bar{x} = 8.333 - (2.114)(3.5) = .934$$

Thus, the least squares line is

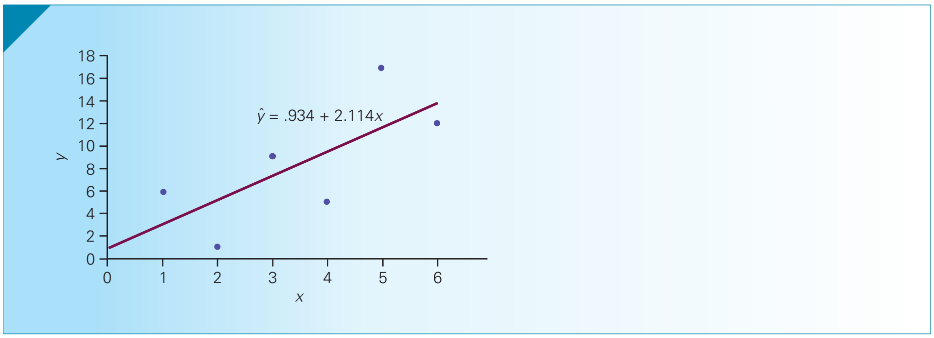
$$\hat{y} = .934 + 2.114x$$

Figure 16.1 depicts the least squares (or regression) line. As you can see, the line fits the data reasonably well. We can measure how well by computing the value of the minimized sum of squared deviations. The deviations between the actual data points and the line are called **residuals**, denoted e_i; that is,

$$e_i = y_i - \hat{y}_i$$

The residuals are observations of the error variable. Consequently, the minimized sum of squared deviations is called the **sum of squares for error**, denoted SSE.

FIGURE **16.1** Scatter Diagram with Regression Line for Example 16.1



The calculation of the residuals in this example is shown in Figure 16.2. Notice that we compute $\hat{y}_i$ by substituting x_i into the formula of the regression line. The residuals are the differences between the observed values of y_i and the fitted or predicted values of $\hat{y}_i$. Table 16.1 describes these calculations.

Thus, SSE = 81.104. No other straight line will produce a sum of squared deviations as small as 81.104. In that sense, the regression line fits the data best. The sum of squares for error is an important statistic because it is the basis for other statistics that assess how well the linear model fits the data. We will introduce these statistics in Section 16.4.

FIGURE **16.2** Calculation of Residuals in Example 16.1

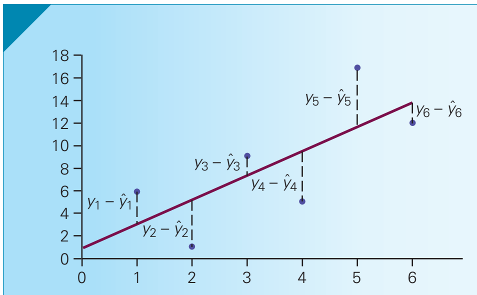


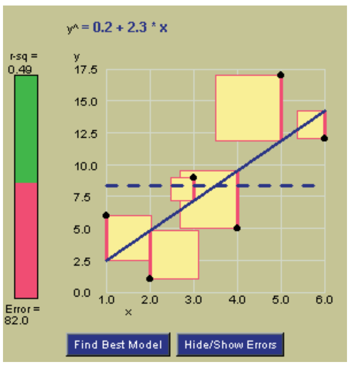
TABLE **16.1** Calculation of Residuals in Example 16.1

x_i	y_i	$\hat{y}_i = .934 + 2.114x_i$	$y_i - \hat{y}_i$	$(y_i - \hat{y}_i)^2$
1	6	3.048	2.952	8.714
2	1	5.162	−4.162	17.322
3	9	7.276	1.724	2.972
4	5	9.390	−4.390	19.272
5	17	11.504	5.496	30.206
6	12	13.618	−1.618	2.618
				$\sum(y_i - \hat{y}_i)^2 = 81.104$

SEEING STATISTICS

applet 18 Fitting the Regression Line

This applet allows you to experiment with the data in Example 16.1. Click or drag the mouse in the graph to change the slope of the line. The errors are measured by the red lines. The squares represent the squared errors. (You can hide or show them by clicking on the **Hide/Show Errors button**.) The error meter on the left keeps track of your progress. The amount of the error that turns green is the proportion of the squared error you eliminate by finding a better regression line. The sum of squared errors is shown at the bottom. The coefficient of correlation squared (which is the coefficient of determination, explained



in Section 16.4) is shown at the top. Change the slope until the sum of squares for error as indicated in the error meter is minimized. If you need help, click the **Find Best Model** button.

Applet Exercises

Change the slope (if necessary) so that the line is horizontal.

17.1 What is the slope of this line?

17.2 What is the y-intercept?

17.3 The y-intercept is equal to $\bar{y}$. What does this tell you about predicting the value of y?

17.4 Drag the mouse to change the slope to 1. What is the sum of squared errors?

17.5 Drag the mouse to change the slope to .5. What is the sum of squared errors?

17.6 Experiment with different lines. What point is common to all the lines?

EXAMPLE 16.2 Odometer Reading and Prices of Used Toyota Camrys, Part 1

DATA
Xm16-02*

Car dealers across North America use the so-called Blue Book to help them determine the value of used cars that their customers trade in when purchasing new cars. The book, which is published monthly, lists the trade-in values for all basic models of cars. It provides alternative values for each car model according to its condition and optional features. The values are determined on the basis of the average paid at recent used-car auctions, the source of supply for many used-car dealers. However, the Blue Book does not indicate the value determined by the odometer reading, despite the fact that a critical factor for used-car buyers is how far the car has been driven. To examine this issue, a used-car dealer randomly selected 100 3-year old Toyota Camrys that were sold at auction during the past month. Each car was in top condition and equipped with all the features that come standard with this car. The dealer recorded the price ($1,000) and the number of miles (thousands) on the odometer. Some of these data are listed here. The dealer wants to find the regression line.

Car	Price ($1,000)	Odometer (1,000 mi)
1	14.6	37.4
2	14.1	44.8
3	14.0	45.8
⋮	⋮	⋮
98	14.5	33.2
99	14.7	39.2
100	14.3	36.4

SOLUTION

IDENTIFY

Notice that the problem objective is to analyze the relationship between two interval variables. Because we believe that the odometer reading affects the selling price, we identify the former as the independent variable, which we label x, and the latter as the dependent variable, which we label y.

COMPUTE

MANUALLY

From the data set, we find

$$\sum_{i=1}^{n} x_i = 3{,}601.1$$

$$\sum_{i=1}^{n} y_i = 1{,}484.1$$

$$\sum_{i=1}^{n} x_i y_i = 53{,}155.93$$

$$\sum_{i=1}^{n} x_i^2 = 133{,}986.59$$

Next we calculate the covariance and the variance of the independent variable x:

$$s_{xy} = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i y_i - \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n}\right]$$

$$= \frac{1}{100-1}\left[53{,}155.93 - \frac{(3{,}601.1)(1{,}484.1)}{100}\right] = -2.909$$

$$s_x^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}\right]$$

$$= \frac{1}{100-1}\left[133{,}986.59 - \frac{(3{,}601.1)^2}{100}\right] = 43.509$$

The sample slope coefficient is calculated next:

$$b_1 = \frac{s_{xy}}{s_x^2} = \frac{-2.909}{43.509} = -.0669$$

The y-intercept is computed as follows:

$$\bar{x} = \frac{\sum x_i}{n} = \frac{3{,}601.1}{100} = 36.011$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{1{,}484.1}{100} = 14.841$$

$$b_0 = \bar{y} - b_1\bar{x} = 14.841 - (-.0669)(36.011) = 17.250$$

The sample regression line is

$$\hat{y} = 17.250 - 0.0669x$$

EXCEL

	A	B	C	D	E	F
1	SUMMARY OUTPUT					
2						
3	*Regression Statistics*					
4	Multiple R	0.8052				
5	R Square	0.6483				
6	Adjusted R Square	0.6447				
7	Standard Error	0.3265				
8	Observations	100				
9						
10	ANOVA					
11		*df*	*SS*	*MS*	*F*	*Significance F*
12	Regression	1	19.26	19.26	180.64	5.75E-24
13	Residual	98	10.45	0.11		
14	Total	99	29.70			
15						
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	
17	Intercept	17.25	0.182	94.73	3.57E-98	
18	Odometer	−0.0669	0.0050	−13.44	5.75E-24	

INSTRUCTIONS

1. Type or import data into two columns*, one storing the dependent variable and the other the independent variable. (Open Xm16-02.)
2. Click **Data**, **Data Analysis,** and **Regression**.
3. Specify the **Input *Y* Range** (A1:A101) and the **Input *X* Range** (B1:B101).

To draw the scatter diagram follow the instructions provided in Chapter 3 on page 76.

MINITAB

Regression Analysis: Price versus Odometer

The regression equation is
Price = 17.2 - 0.0669 Odometer

Predictor	Coef	SE Coef	T	P
Constant	17.2487	0.1821	94.73	0.000
Odometer	–0.066861	0.004975	–13.44	0.000

S = 0.326489 R-Sq = 64.8% R-Sq(adj) = 64.5%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	19.256	19.256	180.64	0.000
Residual Error	98	10.446	0.107		
Total	99	29.702			

INSTRUCTIONS

1. Type or import the data into two columns. (Open Xm16-02.)
2. Click **Stat**, **Regression**, and **Regression . . . .**
3. Type the name of the dependent variable in the **Response** box (Price) and the name of the independent variable in the **Predictors** box (Odometer).

To draw the scatter diagram click **Stat**, **Regression,** and **Fitted Line Plot**. Alternatively, follow the instructions provide in Chapter 3.

The printouts include more statistics than we need right now. However, we will be discussing the rest of the printouts later.

INTERPRET

The slope coefficient b_1 is -0.0669, which means that for each additional 1,000 miles on the odometer, the price decreases by an average of \$.0669 thousand. Expressed more simply, the slope tells us that for each additional mile on the odometer, the price decreases on average by \$.0669 or 6.69 cents.

The intercept is $b_0 = 17.250$. Technically, the intercept is the point at which the regression line and the y-axis intersect. This means that when $x = 0$ (i.e., the car was not driven at all) the selling price is \$17.250 thousand or \$17,250. We might be tempted to

*If one or both columns contain a blank (representing missing data) the row must be deleted.

interpret this number as the price of cars that have not been driven. However, in this case, the intercept is probably meaningless. Because our sample did not include any cars with zero miles on the odometer, we have no basis for interpreting b_0. As a general rule, we cannot determine the value of $\hat{y}$ for a value of x that is far outside the range of the sample values of x. In this example, the smallest and largest values of x are 19.1 and 49.2, respectively. Because $x = 0$ is not in this interval, we cannot safely interpret the value of $\hat{y}$ when $x = 0$.

It is important to bear in mind that the interpretation of the coefficients pertains only to the sample, which consists of 100 observations. To infer information about the population, we need statistical inference techniques, which are described subsequently.

In the sections that follow, we will return to this problem and the computer output to introduce other statistics associated with regression analysis.

EXERCISES

16.1 The term *regression* was originally used in 1885 by Sir Francis Galton in his analysis of the relationship between the heights of children and parents. He formulated the "law of universal regression," which specifies that "each peculiarity in a man is shared by his kinsmen, but on average in a less degree." (Evidently, people spoke this way in 1885.) In 1903, two statisticians, K. Pearson and A. Lee, took a random sample of 1,078 father–son pairs to examine Galton's law ("On the Laws of Inheritance in Man, I. Inheritance of Physical Characteristics," *Biometrika* 2:457–462). Their sample regression line was

$$\text{Son's height} = 33.73 + .516 \times \text{Father's height}$$

a. Interpret the coefficients.
b. What does the regression line tell you about the heights of sons of tall fathers?
c. What does the regression line tell you about the heights of sons of short fathers?

16.2 Xr16-02 Attempting to analyze the relationship between advertising and sales, the owner of a furniture store recorded the monthly advertising budget ($ thousands) and the sales ($ millions) for a sample of 12 months. The data are listed here.

Advertising	23	46	60	54	28	33
Sales	9.6	11.3	12.8	9.8	8.9	12.5

Advertising	25	31	36	88	90	99
Sales	12.0	11.4	12.6	13.7	14.4	15.9

a. Draw a scatter diagram. Does it appear that advertising and sales are linearly related?
b. Calculate the least squares line and interpret the coefficients.

16.3 Xr16-03 To determine how the number of housing starts is affected by mortgage rates an economist recorded the average mortgage rate and the number of housing starts in a large county for the past 10 years. These data are listed here.

Rate	8.5	7.8	7.6	7.5	8.0
Starts	115	111	185	201	206

Rate	8.4	8.8	8.9	8.5	8.0
Starts	167	155	117	133	150

a. Determine the regression line.
b. What do the coefficients of the regression line tell you about the relationship between mortgage rates and housing starts?

16.4 Xr16-04 Critics of television often refer to the detrimental effects that all the violence shown on television has on children. However, there may be another problem. It may be that watching television also reduces the amount of physical exercise, causing weight gains. A sample of 15 10-year-old children was taken. The number of pounds each child was overweight was recorded (a negative number indicates the child is underweight). In addition, the number of hours of television viewing per week was also recorded. These data are listed here.

Television	42	34	25	35	37	38	31	33
Overweight	18	6	0	−1	13	14	7	7

Television	19	29	38	28	29	36	18
Overweight	−9	8	8	5	3	14	−7

a. Draw the scatter diagram.
b. Calculate the sample regression line and describe what the coefficients tell you about the relationship between the two variables.

16.5 Xr16-05 To help determine how many beers to stock the concession manager at Yankee Stadium wanted to know how the temperature affected beer sales. Accordingly, she took a sample of 10 games and recorded the number of beers sold and the temperature in the middle of the game.

Temperature	80	68	78	79	87
Number of beers	20,533	1,439	13,829	21,286	30,985

Temperature	74	86	92	77	84
Number of beers	17,187	30,240	37,596	9,610	28,742

a. Compute the coefficients of the regression line.
b. Interpret the coefficients.

The exercises that follow were created to allow you to see how regression analysis is used to solve realistic problems. As a result, most feature a large number of observations. We anticipate that most students will solve these problems using a computer and statistical software. However, for students without these resources, we have computed the means, variances, and covariances that will permit them to complete the calculations manually. (See Appendix A.)

16.6 Xr16-06* In television's early years, most commercials were 60 seconds long. Now, however, commercials can be any length. The objective of commercials remains the same—to have as many viewers as possible remember the product in a favorable way and eventually buy it. In an experiment to determine how the length of a commercial is related to people's memory of it, 60 randomly selected people were asked to watch a 1-hour television program. In the middle of the show, a commercial advertising a brand of toothpaste appeared. Some viewers watched a commercial that lasted for 20 seconds, others watched one that lasted for 24 seconds, 28 seconds, . . . , 60 seconds. The essential content of the commercials was the same. After the show, each person was given a test to measure how much he or she remembered about the product. The commercial times and test scores (on a 30-point test) were recorded.

a. Draw a scatter diagram of the data to determine whether a linear model appears to be appropriate.
b. Determine the least squares line.
c. Interpret the coefficients.

16.7 Xr16-07 Florida condominiums are popular winter retreats for many North Americans. In recent years, the prices have steadily increased. A real estate agent wanted to know why prices of similar-sized apartments in the same building vary. A possible answer lies in the floor. It may be that the higher the floor, the greater the sale price of the apartment. He recorded the price (in $1,000s) of 1,200 sq. ft. condominiums in several buildings in the same location that have sold recently and the floor number of the condominium.

a. Determine the regression line.
b. What do the coefficients tell you about the relationship between the two variables?

16.8 Xr16-08 In 2010, the United States conducted a census of the entire country. The census is completed by mail. To help ensure that the questions are understood, a random sample of Americans take the questionnaire before it is sent out. As part of their analysis, they record the amount of time and ages of the sample. Use the least squares method to analyze the relationship between the amount of time taken to complete the questionnaire and the age of the individual answering the questions. What do the coefficients tell you about the relationship between the two variables?

APPLICATIONS in HUMAN RESOURCES MANAGEMENT

© Brand X Pictures/Getty Images

Retaining Workers

Human resource managers are responsible for a variety of tasks within organizations. As we pointed out in the introduction in Chapter 1, personnel or human resource managers are involved with recruiting new workers, determining which applicants are most suitable to hire, and helping with various aspects of monitoring the workforce, including absenteeism and worker turnover. For many firms, worker turnover is a costly problem. First, there is the cost of recruiting and attracting qualified workers. The firm must advertise vacant positions and make certain that applicants are judged properly. Second, the cost

(Continued)

of training hirees can be high, particularly in technical areas. Third, new employees are often not as productive and efficient as experienced employees. Consequently, it is in the interests of the firm to attract and keep the best workers. Any information that the personnel manager can obtain is likely to be useful.

16.9 Xr16-09 The human resource manager of a telemarketing firm is concerned about the rapid turnover of the firm's telemarketers. It appears that many telemarketers do not work very long before quitting. There may be a number of reasons, including relatively low pay, personal unsuitability for the work, and the low probability of advancement. Because of the high cost of hiring and training new workers, the manager decided to examine the factors that influence workers to quit. He reviewed the work history of a random sample of workers who have quit in the last year and recorded the number of weeks on the job before quitting and the age of each worker when originally hired.

a. Use regression analysis to describe how the work period and age are related.
b. Briefly discuss what the coefficients tell you.

16.10 Xr16-10 Besides their known long-term effects, do cigarettes also cause short-term illnesses such as colds? To help answer this question, a sample of smokers was drawn. Each person was asked to report the average number of cigarettes smoked per day and the number of days absent from work due to colds last year.

a. Determine the regression line.
b. What do the coefficients tell you about the relationship between smoking cigarettes and sick days because of colds?

16.11 Xr16-11 Fire damage in the United States amounts to billions of dollars, much of it insured. The time taken to arrive at the fire is critical. This raises the question, Should insurance companies lower premiums if the home to be insured is close to a fire station? To help make a decision, a study was undertaken wherein a number of fires were investigated. The distance to the nearest fire station (in miles) and the percentage of fire damage were recorded. Determine the least squares line and interpret the coefficients.

16.12 Xr16-12* A real estate agent specializing in commercial real estate wanted a more precise method of judging the likely selling price (in $1,000s) of apartment buildings. As a first effort, she recorded the price of a number of apartment buildings sold recently and the number of square feet (in 1,000s) in the building.

a. Calculate the regression line.
b. What do the coefficients tell you about the relationship between price and square footage?

16.13 Xr16-13 Millions of boats are registered in the United States. As is the case with automobiles, there is an active used-boat market. Many of the boats purchased require bank financing, and, as a result, it is important for financial institutions to be capable of accurately estimating the price of boats. One variable that affects the price is the number of hours the engine has been run. To determine the effect of the hours on the price, a financial analyst recorded the price (in $1,000s) of a sample of 2007 24-foot Sea Ray cruisers (one of the most popular boats) and the number of hours they had been run. Determine the least squares line and explain what the coefficients tell you.

16.14 Xr03-54 (Exercise 3.54 revisited) In an attempt to determine the factors that affect the amount of energy used, 200 households were analyzed. In each, the number of occupants and the amount of electricity used were measured. Determine the regression line and interpret the results.

16.15 Xr16-15 An economist for the federal government is attempting to produce a better measure of poverty than is currently in use. To help acquire information, she recorded the annual household income (in $1,000s) and the amount of money spent on food during one week for a random sample of households. Determine the regression line and interpret the coefficients.

16.16 Xr16-16* An economist wanted to investigate the relationship between office rents (the dependent variable) and vacancy rates. Accordingly, he took a random sample of monthly office rents and the percentage of vacant office space in 30 different cities.

a. Determine the regression line.
b. Interpret the coefficients.

16.17 Xr03-56 (Exercise 3.56 revisited) One general belief held by observers of the business world is that taller men earn more money than shorter men. In a University of Pittsburgh study, 250 MBA graduates, all about 30 years old, were polled and asked to report their height (in inches) and their annual income (to the nearest $1,000).

a. Determine the regression line.
b. What do the coefficients tell you?

APPLICATIONS in HUMAN RESOURCES MANAGEMENT

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Testing Job Applicants

The recruitment process at many firms involves tests to determine the suitability of candidates. The tests may be written to determine whether the applicant has sufficient knowledge in his or her area of expertise to perform well on the job. There may be oral tests to determine whether the applicant's personality matches the needs of the job. Manual or technical skills can be tested through a variety of physical tests. The test results contribute to the decision to hire. In some cases, the test result is the only criterion to hire. Consequently, it is vital to ensure that the test is a reliable predictor of job performance. If the tests are poor predictors, they should be discontinued. Statistical analyses allow personnel managers to examine the link between the test results and job performance.

16.18 Xr16-18 Although a large number of tasks in the computer industry are robotic, many operations require human workers. Some jobs require a great deal of dexterity to properly position components into place. A large North American computer maker routinely tests applicants for these jobs by giving a dexterity test that involves a number of intricate finger and hand movements. The tests are scored on a 100-point scale. Only those who have scored above 70 are hired. To determine whether the tests are valid predictors of job performance, the personnel manager drew a random sample of 45 workers who were hired 2 months ago. He recorded their test scores and the percentage of nondefective computers they produced in the last week. Determine the regression line and interpret the coefficients.

16.3 ERROR VARIABLE: REQUIRED CONDITIONS

In the previous section, we used the least squares method to estimate the coefficients of the linear regression model. A critical part of this model is the error variable ε. In the next section, we will present an inferential method that determines whether there is a relationship between the dependent and independent variables. Later we will show how we use the regression equation to estimate and predict. For these methods to be valid, however, four requirements involving the probability distribution of the error variable must be satisfied.

Required Conditions for the Error Variable

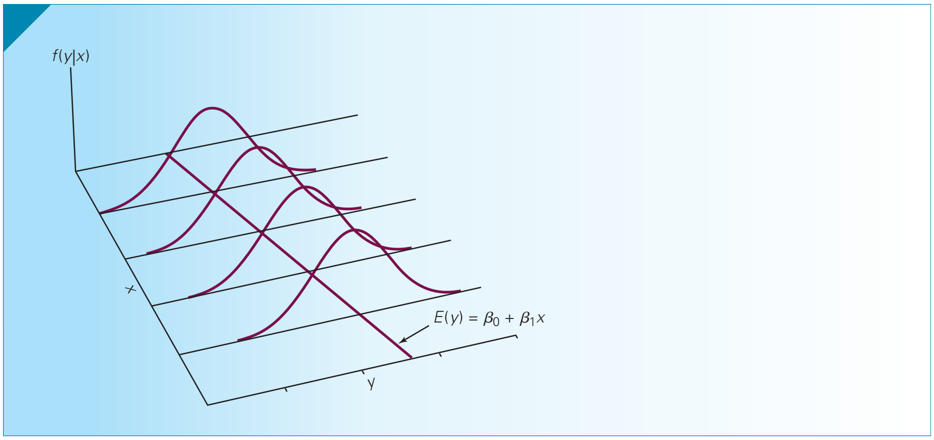
1. The probability distribution of ε is normal.
2. The mean of the distribution is 0; that is, $E(\varepsilon) = 0$.
3. The standard deviation of ε is σ_ε, which is a constant regardless of the value of x.
4. The value of ε associated with any particular value of y is independent of ε associated with any other value of y.

Requirements 1, 2, and 3 can be interpreted in another way: For each value of x, y is a normally distributed random variable whose mean is

$$E(y) = \beta_0 + \beta_1 x$$

and whose standard deviation is σ_ε. Notice that the mean depends on x. The standard deviation, however, is not influenced by x because it is a constant over all values of x. Figure 16.3 depicts this interpretation. Notice that for each value of x, $E(y)$ changes, but the shape of the distribution of y remains the same. In other words, for each x, y is normally distributed with the same standard deviation.

FIGURE **16.3** Distribution of y Given x



In Section 16.6, we will discuss how departures from these required conditions affect the regression analysis and how they are identified.

Observational and Experimental Data

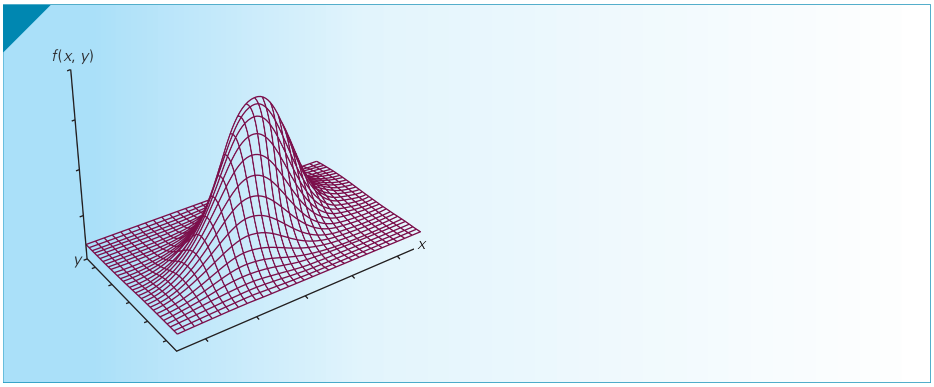
In Chapter 5 and again in Chapter 13, we described the difference between observational and experimental data. We pointed out that statistics practitioners often design controlled experiments to enable them to interpret the results of their analyses more clearly than would be the case after conducting an observational study. Example 16.2 is an illustration of observational data. In that example, we merely observed the odometer reading and auction selling price of 100 randomly selected cars.

If you examine Exercise 16.6, you will see experimental data gathered through a controlled experiment. To determine the effect of the length of a television commercial on its viewers' memories of the product advertised, the statistics practitioner arranged for 60 television viewers to watch a commercial of differing lengths and then tested their memories of that commercial. Each viewer was randomly assigned a commercial length. The values of x ranged from 20 to 60 and were set by the statistics practitioner as part of the experiment. For each value of x, the distribution of the memory test scores is assumed to be normally distributed with a constant variance.

We can summarize the difference between the experiment described in Example 16.2 and the one described in Exercise 16.6. In Example 16.2, both the odometer reading and the auction selling price are random variables. We hypothesize that for each possible odometer reading, there is a theoretical population of auction selling prices that are normally distributed with a mean that is a linear function of the odometer reading and a variance that is constant. In Exercise 16.6, the length of the commercial is not a random variable but a series of values selected by the statistics practitioner. For each commercial length, the memory test scores are required to be normally distributed with a constant variance.

Regression analysis can be applied to data generated from either observational or controlled experiments. In both cases, our objective is to determine how the independent variable is related to the dependent variable. However, observational data can be analyzed in another way. When the data are observational, both variables are random variables. We need not specify that one variable is independent and the other is dependent. We can simply determine *whether* the two variables are related. The equivalent of the required conditions described in the previous box is that the two variables are bivariate normally distributed. (Recall that in Section 7.2 we introduced the bivariate distribution, which describes the joint probability of two variables.) A bivariate normal distribution is described in Figure 16.4. As you can see, it is a three-dimensional bell-shaped curve. The dimensions are the variables x, y, and the joint density function $f(x,y)$.

FIGURE **16.4** **Bivariate Normal Distribution**



In Section 16.4, we will discuss the statistical technique that is used when both x and y are random variables and they are bivariate normally distributed. In Chapter 19, we will introduce a procedure applied when the normality requirement is not satisfied.

EXERCISES

16.19 Describe what the required conditions mean in Exercise 16.6. If the conditions are satisfied, what can you say about the distribution of memory test scores?

16.20 What are the required conditions for Exercise 16.8? Do these seem reasonable?

16.21 Assuming that the required conditions are satisfied in Exercise 16.13, what does this tell you about the distribution of used boat prices?

16.4 Assessing the Model

The least squares method produces the best straight line. However, there may, in fact, be no relationship or perhaps a nonlinear relationship between the two variables. If so, a straight-line model is likely to be impractical. Consequently, it is important for us to assess how well the linear model fits the data. If the fit is poor, we should discard the linear model and seek another one.

Several methods are used to evaluate the model. In this section, we present two statistics and one test procedure to determine whether a linear model should be employed. They are the **standard error of estimate**, the t-test of the slope, and the coefficient of determination. All these methods are based on the sum of squares for error.

Sum of Squares for Error

The least squares method determines the coefficients that minimize the sum of squared deviations between the points and the line defined by the coefficients. Recall from Section 16.2 that the minimized sum of squared deviations is called the *sum of squares for error*, denoted SSE. In that section, we demonstrated the direct method of calculating SSE. For each value of x, we compute the value of $\hat{y}$. In other words, for $i = 1$ to n, we compute

$$\hat{y}_i = b_0 + b_1 x_i$$

For each point, we then compute the difference between the actual value of y and the value calculated at the line, which is the residual. We square each residual and sum the squared values. Table 16.1 on page 640 shows these calculations for Example 16.1. To calculate SSE manually requires a great deal of arithmetic. Fortunately, there is a shortcut method available that uses the sample variances and the covariance.

Shortcut Calculation of SSE

$$\text{SSE} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = (n-1)\left(s_y^2 - \frac{s_{xy}^2}{s_x^2}\right)$$

where s_y^2 is the sample variance of the dependent variable.

Standard Error of Estimate

In Section 16.3, we pointed out that the error variable ε is normally distributed with mean 0 and standard deviation σ_ε. If σ_ε is large, some of the errors will be large, which implies that the model's fit is poor. If σ_ε is small, the errors tend to be close to the mean (which is 0); as a result, the model fits well. Hence, we could use σ_ε to measure the suitability of using a linear model. Unfortunately, σ_ε is a population parameter and, like most other parameters, is unknown. We can, however, estimate σ_ε from the data. The estimate is based on SSE. The unbiased estimator of the variance of the error variable σ_ε^2 is

$$s_\varepsilon^2 = \frac{\text{SSE}}{n-2}$$

The square root of s_ε^2 is called the *standard error of estimate*.

Standard Error of Estimate

$$s_\varepsilon = \sqrt{\frac{\text{SSE}}{n-2}}$$

EXAMPLE 16.3

Odometer Reading and Prices of Used Toyota Camrys—Part 2

Find the standard error of estimate for Example 16.2 and describe what it tells you about the model's fit.

SOLUTION

COMPUTE

MANUALLY

To compute the standard error of estimate, we must compute SSE, which is calculated from the sample variances and the covariance. We have already determined the covariance and the variance of x: -2.909 and 43.509, respectively. The sample variance of y (applying the shortcut method) is

$$s_y^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} y_i^2 - \frac{\left(\sum_{i=1}^{n} y_i\right)^2}{n}\right]$$

$$= \frac{1}{100-1}\left[22{,}055.23 - \frac{(1{,}484.1)^2}{100}\right]$$

$$= .300$$

$$\text{SSE} = (n-1)\left(s_y^2 - \frac{s_{xy}^2}{s_x^2}\right)$$

$$= (100-1)\left(.300 - \frac{[-2.909]^2}{43.509}\right)$$

$$= 10.445$$

The standard error of estimate follows:

$$s_\varepsilon = \sqrt{\frac{\text{SSE}}{n-2}} = \sqrt{\frac{10.445}{98}} = .3265$$

EXCEL

	A	B
7	Standard Error	0.3265

This part of the Excel printout was copied from the complete printout on page 642.

MINITAB

S = 0.326489

This part of the Minitab printout was copied from the complete printout on page 643.

INTERPRET

The smallest value that s_ε can assume is 0, which occurs when SSE = 0, that is, when all the points fall on the regression line. Thus, when s_ε is small, the fit is excellent, and the linear model is likely to be an effective analytical and forecasting tool. If s_ε is large, the model is a poor one, and the statistics practitioner should improve it or discard it.

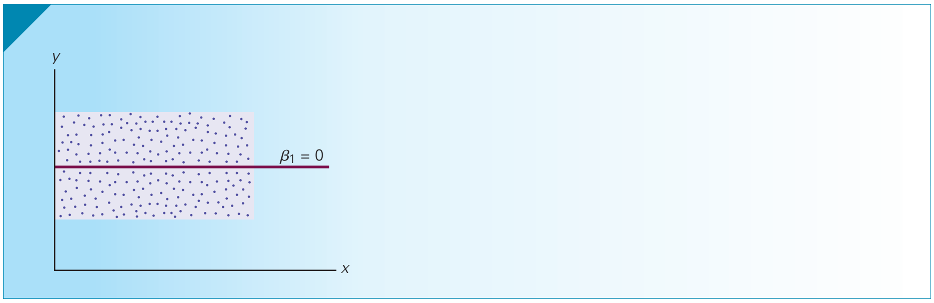
We judge the value of s_ε by comparing it to the values of the dependent variable y or more specifically to the sample mean $\bar{y}$. In this example, because $s_\varepsilon = .3265$ and $\bar{y} = 14.841$, it does appear that the standard error of estimate is small. However, because there is no predefined upper limit on s_ε, it is often difficult to assess the model in this way. In general, the standard error of estimate cannot be used as an absolute measure of the model's utility.

Nonetheless, s_ε is useful in comparing models. If the statistics practitioner has several models from which to choose, the one with the smallest value of s_ε should generally be the one used. As you'll see, s_ε is also an important statistic in other procedures associated with regression analysis.

Testing the Slope

To understand this method of assessing the linear model, consider the consequences of applying the regression technique to two variables that are not at all linearly related. If we could observe the entire population and draw the regression line, we would observe the scatter diagram shown in Figure 16.5. The line is horizontal, which means that no matter what value of x is used, we would estimate the same value for $\hat{y}$; thus, y is not linearly related to x. Recall that a horizontal straight line has a slope of 0, that is, $\beta_1 = 0$.

FIGURE **16.5** Scatter Diagram of Entire Population with $\beta_1 = 0$



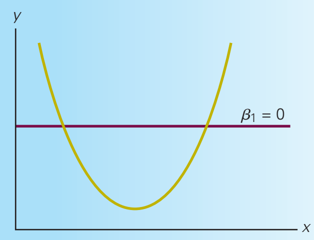
Because we rarely examine complete populations, the parameters are unknown. However, we can draw inferences about the population slope β_1 from the sample slope b_1.

The process of testing hypotheses about β_1 is identical to the process of testing any other parameter. We begin with the hypotheses. The null hypothesis specifies that there is no linear relationship, which means that the slope is 0. Thus, we specify

$$H_0: \quad \beta_1 = 0$$

It must be noted that if the null hypothesis is true, it does not necessarily mean that no relationship exists. For example, a quadratic relationship described in Figure 16.6 may exist where $\beta_1 = 0$.

FIGURE **16.6** Quadratic Relationship



We can conduct one- or two-tail tests of β_1. Most often, we perform a two-tail test to determine whether there is sufficient evidence to infer that a linear relationship exists.* We test the alternative hypothesis

$$H_1: \quad \beta_1 \neq 0$$

Estimator and Sampling Distribution

In Section 16.2, we pointed out that b_1 is an unbiased estimator of β_1; that is,

$$E(b_1) = \beta_1$$

The estimated standard error of b_1 is

$$s_{b_1} = \frac{s_\varepsilon}{\sqrt{(n-1)s_x^2}}$$

where s_ε is the standard error of estimate and s_x^2 is the sample variance of the independent variable. If the required conditions outlined in Section 16.3 are satisfied, the sampling distribution of the t-statistic

$$t = \frac{b_1 - \beta_1}{s_{b_1}}$$

is Student t with degrees of freedom $\nu = n - 2$. Notice that the standard error of b_1 decreases when the sample size increases (which makes b_1 a consistent estimator of β_1) or the variance of the independent variable increases.

Thus, the test statistic and confidence interval estimator are as follows.

Test Statistic for β_1

$$t = \frac{b_1 - \beta_1}{s_{b_1}} \qquad \nu = n - 2$$

*If the alternative hypothesis is true it may be that a linear relationship exists or that a nonlinear relationship exists, but that the relationship can be approximated by a straight line.

Confidence Interval Estimator of β_1

$$b_1 \pm t_{\alpha/2} s_{b_1} \quad \nu = n - 2$$

EXAMPLE 16.4 Are Odometer Reading and Price of Used Toyota Camrys Related?

Test to determine whether there is enough evidence in Example 16.2 to infer that there is a linear relationship between the auction price and the odometer reading for all 3-year-old Toyota Camrys. Use a 5% significance level.

SOLUTION

We test the hypotheses

$$H_0: \quad \beta_1 = 0$$

$$H_1: \quad \beta_1 \neq 0$$

If the null hypothesis is true, no linear relationship exists. If the alternative hypothesis is true, some linear relationship exists.

COMPUTE

MANUALLY

To compute the value of the test statistic, we need b_1 and s_{b_1}. In Example 16.2, we found

$$b_1 = -.0669$$

and

$$s_x^2 = 43.509$$

Thus,

$$s_{b_1} = \frac{s_\varepsilon}{\sqrt{(n-1)s_x^2}} = \frac{.3265}{\sqrt{(99)(43.509)}} = .00497$$

The value of the test statistic is

$$t = \frac{b_1 - \beta_1}{s_{b_1}} = \frac{-.0669 - 0}{.00497} = -13.46$$

The rejection region is

$$t < -t_{\alpha/2,\nu} = -t_{.025,98} \approx -1.984 \quad \text{or} \quad t > t_{\alpha/2,\nu} = t_{.025,98} \approx 1.984$$

EXCEL

	A	B	C	D	E
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*
17	Intercept	17.25	0.182	94.73	3.57E-98
18	Odometer	–0.0669	0.0050	–13.44	5.75E-24

MINITAB

Predictor	Coef	SE Coef	T	P
Constant	17.2487	0.1821	94.73	0.000
Odometer	-0.066861	0.004975	-13.44	0.000

INTERPRET

The value of the test statistic is $t = -13.44$, with a p-value of 0. (Excel uses scientific notation, which in this case is 5.75×10^{-24}, which is approximately 0.) There is overwhelming evidence to infer that a linear relationship exists. What this means is that the odometer reading may affect the auction selling price of the cars. (See the subsection on cause-and-effect relationship on page 659.)

As was the case when we interpreted the y-intercept, the conclusion we draw here is valid only over the range of the values of the independent variable. We can infer that there is a relationship between odometer reading and auction price for the 3-year-old Toyota Camrys whose odometer readings lie between 19.1 (thousand) and 49.2 (thousand) miles (the minimum and maximum values of x in the sample). Because we have no observations outside this range, we do not know how, or even whether, the two variables are related.

Notice that the printout includes a test for β_0. However, as we pointed out before, interpreting the value of the y-intercept can lead to erroneous, if not ridiculous, conclusions. Consequently, we generally ignore the test of β_0.

We can also acquire information about the relationship by estimating the slope coefficient. In this example, the 95% confidence interval estimate (approximating $t_{.025}$ with 98 degrees of freedom with $t_{.025}$ with 100 degrees of freedom) is

$$b_1 \pm t_{\alpha/2} s_{b_1} = -.0669 \pm 1.984(.00497) = -.0669 \pm .0099$$

We estimate that the slope coefficient lies between $-.0768$ and $-.0570$.

One-Tail Tests

If we wish to test for positive or negative linear relationships, we conduct one-tail tests. To illustrate, suppose that in Example 16.2 we wanted to know whether there is evidence of a negative linear relationship between odometer reading and auction selling price. We would specify the hypotheses as

$$H_0: \quad \beta_1 = 0$$

$$H_1: \quad \beta_1 < 0$$

The value of the test statistic would be exactly as computed previously (Example 16.4). However, in this case the p-value would be the two-tail p-value divided by 2; using Excel's p-value, this would be $(5.75 \times 10^{-24})/2 = 2.875 \times 10^{-24}$, which is still approximately 0.

Coefficient of Determination

The test of β_1 addresses only the question of whether there is enough evidence to infer that a linear relationship exists. In many cases, however, it is also useful to measure the strength of that linear relationship, particularly when we want to compare several different models. The statistic that performs this function is the **coefficient of determination**, which is denoted R^2. Statistics practitioners often refer to this statistic as the "R-square." Recall that we introduced the coefficient of determination in Chapter 4, where we pointed out that this statistic is a measure of the amount of variation in the dependent variable that is explained by the variation in the independent variable. However, we did not describe why we interpret the R-square in this way.

Coefficient of Determination

$$R^2 = \frac{s_{xy}^2}{s_x^2 s_y^2}$$

With a little algebra, statisticians can show that

$$R^2 = 1 - \frac{\text{SSE}}{\sum (y_i - \bar{y})^2}$$

We'll return to Example 16.1 to learn more about how to interpret the coefficient of determination. In Chapter 14, we partitioned the total sum of squares into two sources of variation. We do so here as well. We begin by adding and subtracting $\hat{y}_i$ from the deviation between y_i from the mean $\bar{y}$; that is,

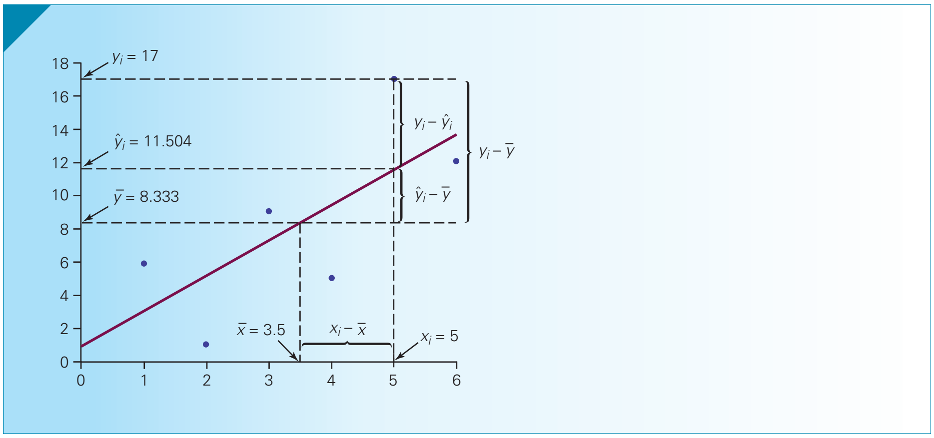
$$(y_i - \bar{y}) = (y_i - \bar{y}) + \hat{y}_i - \hat{y}_i$$

We observe that by rearranging the terms, the deviation between y_i and $\bar{y}$ can be decomposed into two parts; that is,

$$(y_i - \bar{y}) = (y_i - \hat{y}_i) + (\hat{y}_i - \bar{y})$$

This equation is represented graphically (for $i = 5$) in Figure 16.7.

FIGURE **16.7** Partitioning the Deviation for $i = 5$



Now we ask why the values of y are different from one another. From Figure 16.7, we see that part of the difference between y_i and $\bar{y}$ is the difference between $\hat{y}_i$ and $\bar{y}$, which is accounted for by the difference between x_i and $\bar{x}$. In other words, some of the variation in y is explained by the changes to x. The other part of the difference between y_i and $\bar{y}$, however, is accounted for by the difference between y_i and $\hat{y}_i$. This difference is the residual, which represents variables not otherwise represented by the model. As a result, we say that this part of the difference is *unexplained* by the variation in x.

If we now square both sides of the equation, sum over all sample points, and perform some algebra, we produce

$$\sum (y_i - \bar{y})^2 = \sum (y_i - \hat{y}_i)^2 + \sum (\hat{y}_i - \bar{y})^2$$

The quantity on the left side of this equation is a measure of the variation in the dependent variable y. The first quantity on the right side of the equation is SSE, and the second term is denoted SSR, for sum of squares for regression. We can rewrite the equation as

$$\text{Variation in } y = \text{SSE} + \text{SSR}$$

As we did in the analysis of variance, we partition the variation of y into two parts: SSE, which measures the amount of variation in y that remains unexplained; and SSR, which measures the amount of variation in y that is explained by the variation in the independent variable x. We can incorporate this analysis into the definition of R^2.

Coefficient of Determination

$$R^2 = 1 - \frac{\text{SSE}}{\sum (y_i - \bar{y})^2} = \frac{\sum (y_i - \bar{y})^2 - \text{SSE}}{\sum (y_i - \bar{y})^2} = \frac{\text{Explained variation}}{\text{Variation in } y}$$

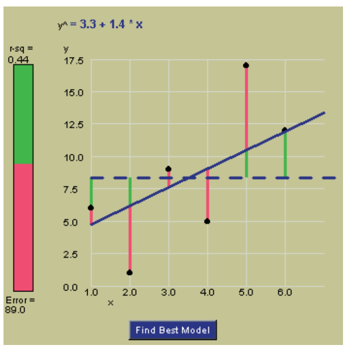
It follows that R^2 measures the proportion of the variation in y that can be explained by the variation in x.

SEEING STATISTICS

applet 19 Analysis of Regression Deviations

This applet provides another way to understand the coefficient of determination.

Move the regression line to reduce the sum of squared errors. The vertical line from each point to the horizontal line depicts the deviation from the mean. In regression this is divided into two parts—the green part, which is the deviation that is eliminated by using the regression line, and the red part, which is the deviation remaining. Note that for some points the deviations become larger.



Applet Exercises

Change the slope (if necessary) so that the line is horizontal.

19.1 How much of the variation in y is explained by the variation in x? Why is this so?

Move the line so that it goes through the sixth point ($x = 6$).

19.2 What is the value of R^2?

19.3 How much of the variation between y_6 and $\bar{y}$ is explained by the variation between x_6 and $\bar{x}$? Why is this so?

Produce the least squares line. (Click the **Find Best Model** button.)

19.4 How much of the variation in y is explained by the variation in x?

EXAMPLE 16.5

Measuring the Strength of the Linear Relationship between Odometer Reading and Price of Used Toyota Camrys

Find the coefficient of determination for Example 16.2 and describe what this statistic tells you about the regression model.

SOLUTION

COMPUTE

MANUALLY

We have already calculated all the necessary components of this statistic. In Example 16.2 we found

$$s_{xy} = -2.909$$
$$s_x^2 = 43.509$$

and from Example 16.3

$$s_y^2 = .300$$

Thus,

$$R^2 = \frac{s_{xy}^2}{s_x^2 s_y^2} = \frac{(-2.909)^2}{(43.509)(.300)} = .6483$$

EXCEL

	A	B
5	R Square	0.6483

MINITAB

R-Sq = 64.8%

Both Minitab and Excel print a second R^2 statistic called the *coefficient of determination adjusted for degrees of freedom*. We will define and describe this statistic in Chapter 17.

INTERPRET

We found that R^2 is equal to .6483. This statistic tells us that 64.83% of the variation in the auction selling prices is explained by the variation in the odometer readings. The remaining 35.17% is unexplained. Unlike the value of a test statistic, the coefficient of determination does not have a critical value that enables us to draw conclusions. In general, the higher the value of R^2, the better the model fits the data. From the t-test of β_1 we already know that there is evidence of a linear relationship. The coefficient of determination merely supplies us with a measure of the strength of that relationship. As you will discover in the next chapter, when we improve the model, the value of R^2 increases.

Other Parts of the Computer Printout

The last part of the printout shown on pages 642 and 643 relates to our discussion of the interpretation of the value of R^2, when its meaning is derived from the partitioning of the variation in y. The values of SSR and SSE are shown in an analysis of variance table similar to the tables introduced in Chapter 14. The general form of the table is shown in Table 16.2. The F-test performed in the ANOVA table will be explained in Chapter 17.

TABLE **16.2** **General Form of the ANOVA Table in the Simple Linear Regression Model**

SOURCE	d.f.	SUMS OF SQUARES	MEAN SQUARES	F-STATISTIC
Regression	1	SSR	MSR = SSR/1	F = MSR/MSE
Error	$n - 2$	SSE	MSE = SSE/($n - 2$)	
Total	$n - 1$	Variation in y		

Note: Excel uses the word "Residual" to refer to the second source of variation, which we called "Error."

Developing an Understanding of Statistical Concepts

Once again, we encounter the concept of explained variation. We first discussed the concept in Chapter 13 when we introduced the matched pairs experiment, where the experiment was designed to reduce the variation among experimental units. This concept was extended in the analysis of variance, where we partitioned the total variation into two or more sources (depending on the experimental design). And now in regression analysis, we use the concept to measure how the dependent variable is related to the independent variable. We partition the variation of the dependent variable into the sources: the variation explained by the variation in the independent variable and the unexplained variation. The greater the explained variation, the better the model is. We often refer to the coefficient of determination as a measure of the explanatory power of the model.

Cause-and-Effect Relationship

A common mistake is made by many students when they attempt to interpret the results of a regression analysis when there is evidence of a linear relationship. They imply that changes in the independent variable cause changes in the dependent variable. It must be emphasized that we cannot infer a causal relationship from statistics alone. Any inference about the cause of the changes in the dependent variable must be justified by a reasonable theoretical relationship. For example, statistical tests established that the more one smoked, the greater the probability of developing lung cancer. However, this analysis did not prove that smoking causes lung cancer. It only demonstrated that smoking and lung cancer were somehow related. Only when medical investigations established the connection were scientists able to confidently declare that smoking causes lung cancer.

As another illustration, consider Example 16.2 where we showed that the odometer reading is linearly related to the auction price. Although it seems reasonable to conclude that decreasing the odometer reading would cause the auction price to rise, the conclusion may not be entirely true. It is theoretically possible that the price is determined by the overall condition of the car and that the condition generally worsens when the car is driven longer. Another analysis would be needed to establish the veracity of this conclusion.

Be cautious about the use of the terms *explained variation* and *explanatory power of the model*. Do not interpret the word *explained* to mean *caused*. We say that the coefficient of determination measures the amount of variation in y that is explained (not caused) by the variation in x. Thus, regression analysis can only show that a statistical relationship exists. We cannot infer that one variable causes another.

Recall that we first pointed this out in Chapter 3 using the following sentence:

Correlation is not causation.

Testing the Coefficient of Correlation

When we introduced the coefficient of correlation (also called the **Pearson coefficient of correlation**) in Chapter 4, we observed that it is used to measure the strength of association between two variables. However, the coefficient of correlation can be useful in another way. We can use it to test for a linear relationship between two variables.

When we are interested in determining *how* the independent variable is related to the dependent variable, we estimate and test the linear regression model. The t-test of the slope presented previously allows us to determine whether a linear relationship actually exists. As we pointed out in Section 16.3, the statistical test requires that for each value of x, there exists a population of values of y that are normally distributed with a constant variance. This condition is required whether the data are experimental or observational.

In many circumstances, we're interested in determining only *whether* a linear relationship exists and not the form of the relationship. When the data are observational and the two variables are bivariate normally distributed (See Section 16.3.) we can calculate the coefficient of correlation and use it to test for linear association.

As we noted in Chapter 4, the population coefficient of correlation is denoted ρ (the Greek letter *rho*). Because ρ is a population parameter (which is almost always unknown), we must estimate its value from the sample data. Recall that the sample coefficient of correlation is defined as follows.

Sample Coefficient of Correlation

$$r = \frac{s_{xy}}{s_x s_y}$$

When there is no linear relationship between the two variables, $\rho = 0$. To determine whether we can infer that ρ is 0, we test the hypotheses

H_0: $\rho = 0$

H_1: $\rho \neq 0$

The test statistic is defined in the following way.

Test Statistic for Testing $\rho = 0$

$$t = r\sqrt{\frac{n-2}{1-r^2}}$$

which is Student t distributed with $\nu = n - 2$ degrees of freedom provided that the variables are bivariate normally distributed.

EXAMPLE 16.6

Are Odometer Reading and Price of Used Toyota Camrys Linearly Related? Testing the Coefficient of Correlation

Conduct the t-test of the coefficient of correlation to determine whether odometer reading and auction selling price are linearly related in Example 16.2. Assume that the two variables are bivariate normally distributed.

SOLUTION

COMPUTE

MANUALLY

The hypotheses to be tested are

$$H_0: \quad \rho = 0$$

$$H_1: \quad \rho \neq 0$$

In Example 16.2, we found $s_{xy} = -2.909$ and $s_x^2 = 43.509$. In Example 16.5, we determined that $s_y^2 = .300$. Thus,

$$s_x = \sqrt{43.509} = 6.596$$

$$s_y = \sqrt{.300} = .5477$$

The coefficient of correlation is

$$r = \frac{s_{xy}}{s_x s_y} = \frac{-2.909}{(6.596)(.5477)} = -.8052$$

The value of the test statistic is

$$t = r\sqrt{\frac{n-2}{1-r^2}} = -.8052\sqrt{\frac{100-2}{1-(-.8052)^2}} = -13.44$$

Notice that this is the same value we produced in the t-test of the slope in Example 16.4. Because both sampling distributions are Student t with 98 degrees of freedom, the p-value and conclusion are also identical.

EXCEL

	A	B
1	**Correlation**	
2		
3	*Price and Odometer*	
4	Pearson Coefficient of Correlation	–0.8052
5	t Stat	–13.44
6	df	98
7	P(T<=t) one tail	0
8	t Critical one tail	1.6606
9	P(T<=t) two tail	0
10	t Critical two tail	1.9845

INSTRUCTIONS

1. Type or import the data into two adjacent columns*. (Open Xm16-02.)
2. Click **Add-ins, Data Analysis Plus**, and **Correlation (Pearson).**
3. Specify the **Variable 1 Input Range** (A1:A101), **Variable 2 Input Range** (B1:B101), and α (.05).

MINITAB

Correlations: Odometer, Price

Pearson correlation of Price and Odometer = –0.805
P-Value = 0.000

INSTRUCTIONS

1. Type or import the data into two adjacent columns. (Open Xm16-02.)
2. Click **Stat, Basic Statistics**, and **Correlation**.
3. Type the names of the variables in the **Variables** box (Odometer Price).

Notice that the t-test of ρ and the t-test of β_1 in Example 16.4 produced identical results. This should not be surprising because both tests are conducted to determine whether there is evidence of a linear relationship. The decision about which test to use is based on the type of experiment and the information we seek from the statistical analysis. If we're interested in discovering the relationship between two variables, or if we've conducted an experiment where we controlled the values of the independent variable (as in Exercise 16.6), the t-test of β_1 should be applied. If we're interested only in determining *whether* two random variables that are bivariate normally distributed are linearly related, the t-test of ρ should be applied.

As is the case with the t-test of the slope, we can also conduct one-tail tests. We can test for a positive or a negative linear relationship.

*If one or both columns contain a blank (representing missing data) the row must be deleted.

Education and Income: How Are They Related?

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IDENTIFY

The problem objective is to analyze the relationship between two interval variables. Because we want to know how education affects income the independent variable is education (EDUC) and the dependent variable is income (INCOME).

COMPUTE

EXCEL

	A	B	C	D	E	F
1	SUMMARY OUTPUT					
2						
3	*Regression Statistics*					
4	Multiple R	0.3790				
5	R Square	0.1436				
6	Adjusted R Square	0.1429				
7	Standard Error	35,972				
8	Observations	1189				
9						
10	ANOVA					
11		*df*	*SS*	*MS*	*F*	*Significance F*
12	Regression	1	257,561,051,309	257,561,051,309	199.04	6.702E-42
13	Residual	1187	1,535,986,496,000	1,294,007,158		
14	Total	1188	1,793,547,547,309			
15						
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	
17	Intercept	–28926	5117	–5.65	1.971E-08	
18	EDUC	5111	362	14.11	6.702E-42	

MINITAB

Regression Analysis: INCOME versus EDUC

The regression equation is
Income = –28926 + 5111 EDUC
1189 cases used, 834 cases contain missing values

Predictor	Coef	SE Coef	T	P
Constant	–28926	5117	–5.65	0.000
EDUC	5110.7	362.2	14.11	0.000

S = 35972.3 R-Sq = 14.4% R-Sq(adj) = 14.3%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	2.57561E+11	2.57561E+11	199.04	0.000
Residual Error	1187	1.53599E+12	1294007158		
Total	1188	1.79355E+12			

INTERPRET

The regression equation is $\hat{y} = -28926 + 5111x$. The slope coefficient tells us that on average for each additional year of education income increases by $5,111. We test to determine whether there is evidence of a linear relationship.

$$H_0: \ \beta_1 = 0$$
$$H_1: \ \beta_1 \neq 0$$

The test statistic is $t = 14.11$ and the p-value is $6.702 \text{ X } 10^{-42}$, which is virtually 0. The coefficient of determination is $R^2 = .1436$, which means that 14.36% of the variation in income is explained by the variation in education and the remaining 85.64% is not explained.

Violation of the Required Condition

When the normality requirement is unsatisfied, we can use a nonparametric technique—the Spearman rank correlation coefficient (Chapter 19*)—to replace the t-test of ρ.

*Instructors who wish to teach the use of the Spearman rank correlation coefficient here can use Keller's website Appendix Spearman Rank Correlation Coefficient and Test.

EXERCISES

Use a 5% significance level for all tests of hypotheses.

16.22 You have been given the following data:

x	1	3	4	6	9	8	10
y	1	8	15	33	75	70	95

a. Draw the scatter diagram. Does it appear that x and y are related? If so, how?
b. Test to determine whether there is evidence of a linear relationship.

16.23 Suppose that you have the following data:

x	3	5	2	6	1	4
y	25	110	9	250	3	71

a. Draw the scatter diagram. Does it appear that x and y are related? If so, how?
b. Test to determine whether there is evidence of a linear relationship.

16.24 Refer to Exercise 16.2.
a. Determine the standard error of estimate.
b. Is there evidence of a linear relationship between advertising and sales?
c. Estimate β_1 with 95% confidence.
d. Compute the coefficient of determination and interpret this value.
e. Briefly summarize what you have learned in parts (a) through (d).

16.25 Calculate the coefficient of determination and conduct a test to determine whether a linear relationship exists between housing starts and mortgage interest in Exercise 16.3.

16.26 Is there evidence of a linear relationship between the number of hours of television viewing and how overweight the child is in Exercise 16.4?

16.27 Determine whether there is evidence of a negative linear relationship between temperature and the number of beers sold at Yankee Stadium in Exercise 16.5.

Exercises 16.28–16.53 require the use of a computer and software. The answers to Exercises 16.28 to 16.44 may be calculated manually. See Appendix A for the sample statistics.

16.28 Refer to Exercise 16.6.
a. What is the standard error of estimate? Interpret its value.
b. Describe how well the memory test scores and length of television commercial are linearly related.
c. Are the memory test scores and length of commercial linearly related? Test using a 5% significance level.
d. Estimate the slope coefficient with 90% confidence.

16.29 Refer to Exercise 16.7. Apply the three methods of assessing the model to determine how well the linear model fits.

16.30 Is there enough evidence to infer that age and the amount of time needed to complete the questionnaire are linearly related in Exercise 16.8?

16.31 Refer to Exercise 16.9. Use two statistics to measure the strength of the linear association. What do these statistics tell you?

16.32 Is there evidence of a linear relationship between number of cigarettes smoked and number of sick days in Exercise 16.10?

16.33 Refer to Exercise 16.11.
a. Test to determine whether there is evidence of a linear relationship between distance to the nearest fire station and percentage of damage.
b. Estimate the slope coefficient with 95% confidence.
c. Determine the coefficient of determination. What does this statistic tell you about the relationship?

16.34 Refer to Exercise 16.12.
a. Determine the standard error of estimate, and describe what this statistic tells you about the regression line.
b. Can we conclude that the size and price of the apartment building are linearly related?
c. Determine the coefficient of determination and discuss what its value tells you about the two variables.

16.35 Is there enough evidence to infer that as the number of hours of engine use increases, the price decreases in Exercise 16.13?

16.36 Assess fit of the regression line in Exercise 16.14.

16.37 Refer to Exercise 16.15.
a. Determine the coefficient of determination and describe what it tells you.
b. Conduct a test to determine whether there is evidence of a linear relationship between household income and food budget.

16.38 Can we infer that office rents and vacancy rates are linearly related in Exercise 16.16?

16.39 Are height and income in Exercise 16.17 positively linearly related?

16.40 Refer to Exercise 16.18.
a. Compute the coefficient of determination and describe what it tells you.
b. Can we infer that aptitude test scores and percentages of nondefectives are linearly related?

16.41 Repeat Exercise 16.13 using the *t*-test of the coefficient of correlation to determine whether there is a negative linear relationship between the number of hours of engine use and the selling price of the used boats.

16.42 Repeat Exercise 16.6 using the *t*-test of the coefficient of correlation. Is this result identical to the one you produced in Exercise 16.6?

16.43 Are food budget and household income in Exercise 16.15 linearly related? Employ the *t*-test of the coefficient of correlation to answer the question.

16.44 Refer to Exercise 16.10. Use the *t*-test of the coefficient of correlation to determine whether there is evidence of a positive linear relationship between number of cigarettes smoked and the number of sick days.

AMERICAN NATIONAL ELECTION SURVEY EXERCISES

16.45 ANES2008* Do more educated people spend more time watching or reading news on the Internet? Conduct a regression analysis to determine whether there is enough statistical evidence to conclude that the more education (EDUC) one has the more one watches or reads news on the Internet (TIME1)?

16.46 ANES2008* In the Chapter 16 opening example, we analyzed the relationship between income and education using the 2008 General Social Survey of 2008. Conduct a similar analysis using the 2008 American National Election Survey.

16.47 ANES2008* National news on television features commercials describing pharmaceutical drugs that treat ailments that plague older people. Apparently, the major networks believe that older people tend to watch national newscasts. Is there sufficient evidence to conclude age (AGE) and number of days watching national news on television (DAYS1) are positively related?

16.48 ANES2008* In most presidential elections in the United States, the voter turnout is quite low, often in the neighborhood of 50%. Political workers would like to be able to predict who is likely to vote. Thus, it is important to know which variables are related to intention to vote. One candidate is age. Is there sufficient evidence to infer that age (AGE) and intention to vote (DEFINITE) are linearly related?

16.49 ANES2008* Do more affluent people get their news from radio? Answer the question by conducting an analysis of the relationship between income (INCOME) and time listening to news on the radio (TIME4).

General Social Survey Exercises

16.50 GSS2008* Does income affect people's positions on the question, Should the government reduce income differences between rich and poor (EQWLTH)? Answer the question by testing the relationship between income (INCOME) and EQWLTH.

16.51 GSS2008* Conduct an analysis of the relationship between income (INCOME) and age (AGE). Estimate with 95% confidence the average increase in income for each additional year of age.

16.52 GSS2008* Is there sufficient evidence to conclude that more educated people (EDUC) watch less television (TVHOURS)?

16.53 GSS2006* Use the 2006 survey data to determine whether more education (EDUC) leads to higher income (INCOME).

16.5 Using the Regression Equation

Using the techniques in Section 16.4, we can assess how well the linear model fits the data. If the model fits satisfactorily, we can use it to forecast and estimate values of the dependent variable. To illustrate, suppose that in Example 16.2, the used-car dealer wanted to predict the selling price of a 3-year-old Toyota Camry with 40 (thousand) miles on the odometer. Using the regression equation, with $x = 40$, we get

$$\hat{y} = 17.250 - .0669x = 17.250 - 0.0669(40) = 14.574$$

We call this value the **point prediction**, and $\hat{y}$ is the point estimate or predicted value for y when $x = 40$. Thus, the dealer would predict that the car would sell for $14,574.

By itself, however, the point prediction does not provide any information about how closely the value will match the true selling price. To discover that information, we must use an interval. In fact, we can use one of two intervals: the prediction interval of a particular value of y or the confidence interval estimator of the expected value of y.

Predicting the Particular Value of y for a Given x

The first confidence interval we present is used whenever we want to predict a one-time occurrence for a particular value of the dependent variable when the independent variable is a given value x_g. This interval, often called the **prediction interval**, is calculated in the usual way (point estimator $\pm$ bound on the error of estimation). Here the point estimate for y is $\hat{y}$, and the bound on the error of estimation is shown below.

Prediction Interval

$$\hat{y} \pm t_{\alpha/2,n-2}s_\varepsilon\sqrt{1 + \frac{1}{n} + \frac{(x_g - \bar{x})^2}{(n-1)s_x^2}}$$

where x_g is the given value of x and $\hat{y} = b_0 + b_1x_g$

Estimating the Expected Value of *y* for a Given *x*

The conditions described in Section 16.3 imply that, for a given value of x, there is a population of values of y whose mean is

$$E(y) = \beta_0 + \beta_1 x$$

To estimate the mean of y or long-run average value of y we would use the following interval referred to simply as the confidence interval. Again, the point estimator is $\hat{y}$, but the bound on the error of estimation is different from the prediction interval shown below.

Confidence Interval Estimator of the Expected Value of *y*

$$\hat{y} \pm t_{\alpha/2,n-2} s_\varepsilon \sqrt{\frac{1}{n} + \frac{(x_g - \bar{x})^2}{(n-1)s_x^2}}$$

Unlike the formula for the prediction interval, this formula does not include the 1 under the square-root sign. As a result, the **confidence interval estimate of the expected value of *y*** will be narrower than the prediction interval for the same given value of x and confidence level. This is because there is less error in estimating a mean value as opposed to predicting an individual value.

EXAMPLE 16.7

Predicting the Price and Estimating the Mean Price of Used Toyota Camrys

a. A used-car dealer is about to bid on a 3-year-old Toyota Camry equipped with all the standard features and with 40,000 ($x_g = 40$) miles on the odometer. To help him decide how much to bid, he needs to predict the selling price.

b. The used-car dealer mentioned in part (a) has an opportunity to bid on a lot of cars offered by a rental company. The rental company has 250 Toyota Camrys all equipped with standard features. All the cars in this lot have about 40,000 ($x_g = 40$) miles on their odometers. The dealer would like an estimate of the selling price of all the cars in the lot.

SOLUTION

IDENTIFY

a. The dealer would like to predict the selling price of a single car. Thus, he must employ the prediction interval

$$\hat{y} \pm t_{\alpha/2,n-2} s_\varepsilon \sqrt{1 + \frac{1}{n} + \frac{(x_g - \bar{x})^2}{(n-1)s_x^2}}$$

b. The dealer wants to determine the mean price of a large lot of cars, so he needs to calculate the confidence interval estimator of the expected value:

$$\hat{y} \pm t_{\alpha/2,n-2}s_\varepsilon\sqrt{\frac{1}{n} + \frac{(x_g - \bar{x})^2}{(n-1)s_x^2}}$$

Technically, this formula is used for infinitely large populations. However, we can interpret our problem as attempting to determine the average selling price of all Toyota Camrys equipped as described above, all with 40,000 miles on the odometer. The crucial factor in part (b) is the need to estimate the mean price of a number of cars. We arbitrarily select a 95% confidence level.

COMPUTE

MANUALLY

From previous calculations, we have the following:

$$\hat{y} = 17.250 - .0669(40) = 14.574$$

$$s_\varepsilon = .3265$$

$$s_x^2 = 43.509$$

$$\bar{x} = 36.011$$

From Table 4 in Appendix B, we find

$$t_{\alpha/2} = t_{.025,98} \approx t_{.025,100} = 1.984$$

a. The 95% prediction interval is

$$\hat{y} \pm t_{\alpha/2,n-2}s_\varepsilon\sqrt{1 + \frac{1}{n} + \frac{(x_g - \bar{x})^2}{(n-1)s_x^2}}$$

$$= 14.574 \pm 1.984 \times .3265\sqrt{1 + \frac{1}{100} + \frac{(40 - 36.011)^2}{(100-1)(43.509)}}$$

$$= 14.574 \pm .652$$

The lower and upper limits of the prediction interval are \$13,922 and \$15,226, respectively.

b. The 95% confidence interval estimator of the mean price is

$$\hat{y} \pm t_{\alpha/2,n-2}s_\varepsilon\sqrt{\frac{1}{n} + \frac{(x_g - \bar{x})^2}{(n-1)s_x^2}}$$

$$= 14.574 \pm 1.984 \times .3265\sqrt{\frac{1}{100} + \frac{(40 - 36.011)^2}{(100-1)(43.509)}}$$

$$= 14.574 \pm .076$$

The lower and upper limits of the confidence interval estimate of the expected value are \$14,498 and 14,650, respectively.

EXCEL

	A	B	C
1	**Prediction Interval**		
2			
3			Price
4			
5	Predicted value		14.574
6			
7	Prediction Interval		
8	Lower limit		13.922
9	Upper limit		15.227
10			
11	Interval Estimate of Expected Value		
12	Lower limit		14.498
13	Upper limit		14.650

INSTRUCTIONS

1. Type or import the data into two columns*. (Open Xm16-02.)
2. Type the given value of x into any cell. We suggest the next available row in the column containing the independent variable.
3. Click **Add-Ins, Data Analysis Plus,** and **Prediction Interval**.
4. Specify the **Input *Y* Range** (A1:A101), the **Input X Range** (B1:B101), the **Given X Range** (B102), and the **Confidence Level** (.95).

MINITAB

```
Predicted Values for New Observations

New
Obs    Fit     SE Fit        95% CI                95% PI
  1  14.5743   0.0382  (14.4985, 14.6501)  (13.9220, 15.2266)


Values of Predictors for New Observations

New
Obs   Odometer
  1      40.0
```

The output includes the predicted value $\hat{y}$ (**Fit**), the standard deviation of $\hat{y}$ (**SE Fit**), the 95% confidence interval estimate of the expected value of y (**CI**), and the 95% prediction interval (**PI**).

INSTRUCTIONS

1. Proceed through the three steps of regression analysis described on page 642. Do not click **OK**. Click **Options . . . .**
2. Specify the given value of x in the **Prediction intervals for new observations** box (40).
3. Specify the confidence level (.95).

INTERPRET

We predict that one car will sell for between $13,925 and $15,226. The average selling price of the population of 3-year-old Toyota Camrys is estimated to lie between $14,498 and $14,650. Because predicting the selling price of one car is more difficult than estimating the mean selling price of all similar cars, the prediction interval is wider than the interval estimate of the expected value.

*If one or both columns contain a blank (representing missing data) the row must be deleted.

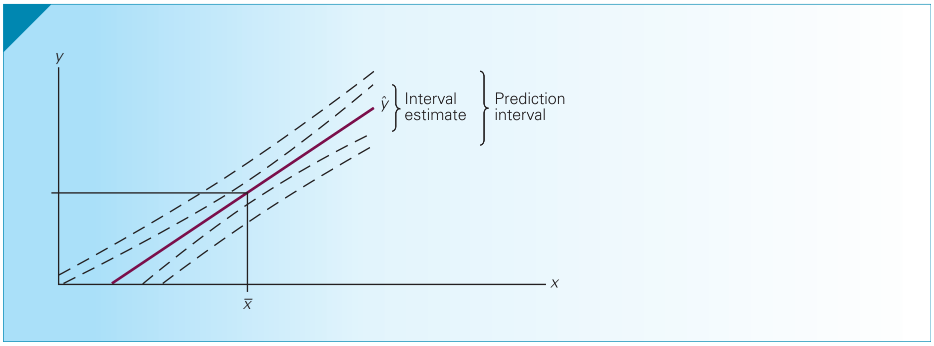
Effect of the Given Value of *x* on the Intervals

Calculating the two intervals for various values of x results in the graph in Figure 16.8. Notice that both intervals are represented by curved lines. This is because the farther the given value of x is from $\bar{x}$, the greater the estimated error becomes. This part of the estimated error is measured by

$$\frac{(x_g - \bar{x})^2}{(n-1)s_x^2}$$

which appears in both the prediction interval and the interval estimate of the expected value.

FIGURE **16.8** Interval Estimates and Prediction Intervals



EXERCISES

16.54 Briefly describe the difference between predicting a value of y and estimating the expected value of y.

16.55 Use the regression equation in Exercise 16.2 to predict with 90% confidence the sales when the advertising budget is $90,000.

16.56 Estimate with 90% confidence the mean monthly number of housing starts when the mortgage interest rate is 8% in Exercise 16.3.

16.57 Refer to Exercise 16.4.
a. Predict with 90% confidence the number of pounds overweight for a child who watches 30 hours of television per week.
b. Estimate with 90% confidence the mean number of pounds overweight for children who watch 30 hours of television per week.

16.58 Refer to Exercise 16.5. Predict with 90% confidence the number of beers to be sold when the temperature is 80 degrees.

Exercises 16.59–16.80 require the use of a computer and software. The answers to Exercises 16.59–16.72 may be calculated manually. See Appendix A for the sample statistics.

16.59 Refer to Exercise 16.6.
a. Predict with 95% confidence the memory test score of a viewer who watches a 36-second commercial.
b. Estimate with 95% confidence the mean memory test score of people who watch 36-second commercials.

16.60 Refer to Exercise 16.7.
a. Predict with 95% confidence the selling price of a 1,200 sq. ft. condominium on the 25th floor.
b. Estimate with 99% confidence the average selling price of a 1,200 sq. ft. condominium on the 12th floor.

16.61 Refer to Exercise 16.8. Estimate with 90% confidence the mean amount of time for 50-year old Americans to complete the census.

16.62 Refer to Exercise 16.9. The company has just hired a 25-year-old telemarketer. Predict with 95% confidence how long he will stay with the company.

16.63 Refer to Exercise 16.10. Predict with 95% confidence the number of sick days for individuals who smoke on average 30 cigarettes per day.

16.64 Refer to Exercise 16.11.

a. Predict with 95% confidence the percentage loss resulting from fire for a house that is 5 miles away from the nearest fire station.

b. Estimate with 95% confidence the average percentage loss resulting from fire for houses that are 2 miles away from the nearest fire station.

16.65 Refer to Exercise 16.12. Estimate with 95% confidence the mean price of 50,000 sq. ft. apartment buildings.

16.66 Refer to Exercise 16.13. Predict with 99% confidence the price of a 1999 24-foot Sea Ray cruiser with 500 hours of engine use.

16.67 Refer to Exercise 16.14. Estimate with 90% confidence the mean electricity consumption for households with 5 occupants.

16.68 Refer to Exercise 16.15. Predict the food budget of a family whose household income is $50,000. Use a 90% confidence level.

16.69 Refer to Exercise 16.16. Predict with 95% confidence the monthly office rent in a city when the vacancy rate is 10%.

16.70 Refer to Exercise 16.17

a. Estimate with 95% confidence the mean annual income of 6-foot-tall men.

b. Suppose that an individual is 5 feet 6 inches tall. Predict with 95% confidence his annual income.

16.71 Refer to Exercise 16.18. Estimate with 95% confidence the mean percentage of defectives for workers who score 75 on the dexterity test.

16.72 Refer to Exercise 16.18. Predict with 90% confidence the percentage of defectives for a worker who scored 80 on the dexterity test.

American National Election Survey Exercise

16.73 ANES2008* Refer to Exercise 16.45. Predict with 90% confidence the amount of time spent watching or reading news on the Internet by a person with 15 years of education

16.74 ANES2008* Refer to Exercise 16.46. Estimate with 95% confidence the income of average of people who have 10 years of education.

16.75 ANES2008* Refer to Exercise 16.47. Estimate with 99% confidence the mean number of days watching national news on television by 50-year-old people.

16.76 ANES2008* Refer to Exercise 16.49. Predict with 95% confidence the amount of time listening to news on the radio by individuals who earn $50,000 annually.

General Social Survey Exercises

16.77 GSS2008* Refer to Exercise 16.51. Predict the annual income of someone who is 45 years old.

16.78 GSS2008* Refer to Exercise 16.52. Estimate with 90% confidence the average number of hours of television watching per day for people with 12 years of education.

16.79 GSS2006* Refer to Exercise 16.53. Use the General Social Survey of 2006 to predict with 99% confidence the annual income of someone with 17 years of education.

16.80 GSS2008* Refer to Exercise 16.52. Predict with 90% confidence the number of hours of television watching per day for someone with 8 years of education.

16.6 Regression Diagnostics—I

In Section 16.3, we described the required conditions for the validity of regression analysis. Simply put, the error variable must be normally distributed with a constant variance, and the errors must be independent of each other. In this section, we show how to diagnose violations. In addition, we discuss how to deal with observations that are unusually large or small. Such observations must be investigated to determine whether an error was made in recording them.

Residual Analysis

Most departures from required conditions can be diagnosed by examining the residuals, which we discussed in Section 16.4. Most computer packages allow you to output the values of the residuals and apply various graphical and statistical techniques to this variable.

We can also compute the standardized residuals. We standardize residuals in the same way we standardize all variables, by subtracting the mean and dividing by the standard deviation. The mean of the residuals is 0, and because the standard deviation σ_ε is unknown, we must estimate its value. The simplest estimate is the standard error of estimate s_ε. Thus,

$$\text{Standardized residuals for point } i = \frac{e_i}{s_\varepsilon}$$

EXCEL

Excel calculates the standardized residuals by dividing the residuals by the standard deviation of the residuals. (The difference between the standard error of estimate and the standard deviation of the residuals is that in the formula of the former the denominator is $n - 2$, whereas in the formula for the latter, the denominator is $n - 1$.)

Part of the printout (we show only the first five and last five values) for Example 16.2 follows.

	A	B	C	D
1	RESIDUAL OUTPUT			
2				
3	*Observation*	*Predicted Price*	*Residuals*	*Standard Residuals*
4	1	14.748	−0.148	−0.456
5	2	14.253	−0.153	−0.472
6	3	14.186	−0.186	−0.574
7	4	15.183	0.417	1.285
8	5	15.129	0.471	1.449
9				
10				
11				
12	95	15.149	−0.049	−0.152
13	96	14.828	−0.028	−0.087
14	97	14.962	−0.362	−1.115
15	98	15.029	−0.529	−1.628
16	99	14.628	0.072	0.222
17	100	14.815	−0.515	−1.585

INSTRUCTIONS

Proceed with the three steps of regression analysis described on page 642. Before clicking **OK**, select **Residuals** and **Standardized Residuals**. The predicted values, residuals, and standardized residuals will be printed.

We can also standardize by computing the standard deviation of each residual. Statisticians have determined that the standard deviation of the residual for observation i is defined as follows.

Standard Deviation of the *i*th Residual

$$s_{e_i} = s_\varepsilon\sqrt{1 - h_i}$$

where

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{(n - 1)s_x^2}$$

The quantity h_i should look familiar; it was used in the formula for the prediction interval and confidence interval estimate of the expected value of y in Section 16.6. Minitab computes this version of the standardized residuals. Part of the printout (we show only the first five and last five values) for Example 16.2 is shown below.

MINITAB

Obs	Odometer	Price	Fit	SE Fit	Residual	St Resid
1	37.4	14.6000	14.7481	0.0334	−0.1481	−0.46
2	44.8	14.1000	14.2534	0.0546	−0.1534	−0.48
3	45.8	14.0000	14.1865	0.0586	−0.1865	−0.58
4	30.9	15.6000	15.1827	0.0414	0.4173	1.29
5	31.7	15.6000	15.1292	0.0391	0.4708	1.45
96	36.2	14.8000	14.8284	0.0327	−0.0284	−0.09
97	34.2	14.6000	14.9621	0.0339	−0.3621	−1.12
98	33.2	14.5000	15.0289	0.0355	−0.5289	−1.63
99	39.2	14.7000	14.6278	0.0363	0.0722	0.22
100	36.4	14.3000	14.8150	0.0327	−0.5150	−1.59

INSTRUCTIONS

Proceed with the three steps of regression analysis as described on page 642. After specifying the **Response** and **Predictors**, click **Results. . .** , and **In addition, the full table of fits and residuals**.

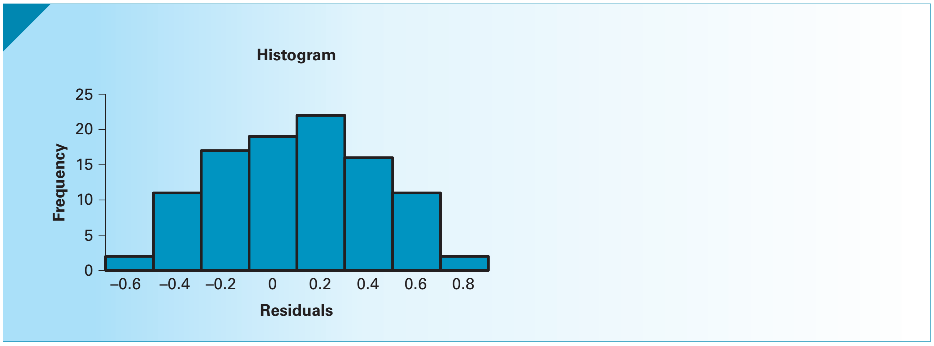
The predicted values, residuals, and standardized residuals will be printed.

An analysis of the residuals will allow us to determine whether the error variable is nonnormal, whether the error variance is constant, and whether the errors are independent. We begin with nonnormality.

Nonnormality

As we've done throughout this book, we check for normality by drawing the histogram of the residuals. Figure 16.9 is Excel's version (Minitab's is similar). As you can see, the histogram is bell shaped, leading us to believe that the error is normally distributed.

FIGURE **16.9** Histogram of Residuals for Example 16.2



Heteroscedasticity

The variance of the error variable σ_ε^2 is required to be constant. When this requirement is violated, the condition is called **heteroscedasticity**. (You can impress friends and relatives by using this term. If you can't pronounce it, try **homoscedasticity**, which refers to the condition where the requirement is satisfied.) One method of diagnosing heteroscedasticity is to plot the residuals against the predicted values of y. We then look for a change in the spread of the plotted points.* Figure 16.10 describes such a situation. Notice that in this illustration, σ_ε^2 appears to be small when $\hat{y}$ is small and large when $\hat{y}$ is large. Of course, many other patterns could be used to depict this problem.

FIGURE **16.10** Plot of Residuals Depicting Heteroscedasticity

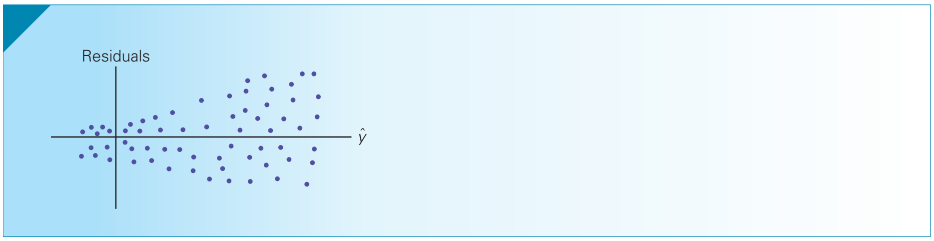
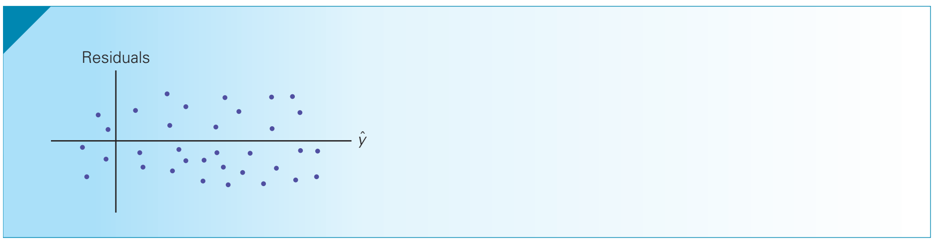


Figure 16.11 illustrates a case in which σ_ε^2 is constant. As a result, there is no apparent change in the variation of the residuals.

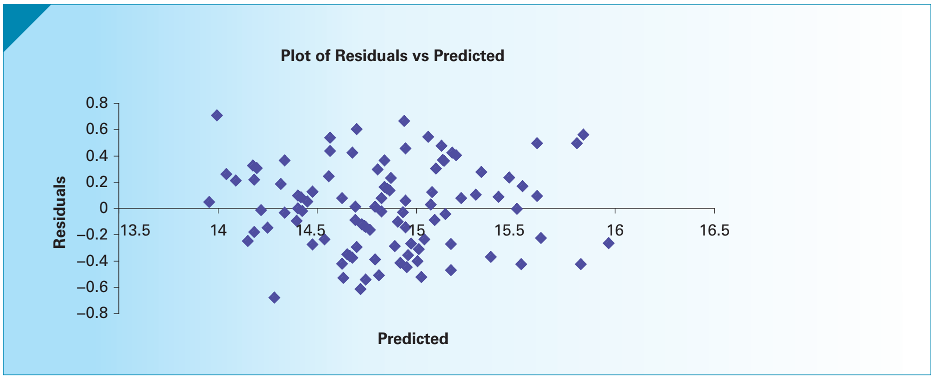
FIGURE **16.11** Plot of Residuals Depicting Homoscedasticity



Excel's plot of the residuals versus the predicted values of y for Example 16.2 is shown in Figure 16.12. There is no sign of heteroscedasticity.

*Keller's website Appendix Szroeter's Test describes a test for heteroscedasticity.

FIGURE **16.12** Plot of Predicted Values versus Residuals for Example 16.2



Nonindependence of the Error Variable

In Chapter 3, we briefly described the difference between cross-sectional and time-series data. Cross-sectional data are observations made at approximately the same time, whereas a time series is a set of observations taken at successive points of time. The data in Example 16.2 are cross-sectional because all of the prices and odometer readings were taken at about the same time. If we were to observe the auction price of cars every week for, say, a year, that would constitute a time series.

Condition 4 states that the values of the error variable are independent. When the data are time series, the errors often are correlated. Error terms that are correlated over time are said to be **autocorrelated** or **serially correlated**. For example, suppose that, in an analysis of the relationship between annual gross profits and some independent variable, we observe the gross profits for the years 1991 to 2010. The observed values of y are denoted $y_1, y_2, \ldots y_{20}$, where y_1 is the gross profit for 1991, y_2 is the gross profit for 1992, and so on. If we label the residuals $e_1, e_2, \ldots, e_{20}$, then—if the independence requirement is satisfied—there should be no relationship among the residuals. However, if the residuals are related it is likely that autocorrelation exists.

We can often detect autocorrelation by graphing the residuals against the time periods. If a pattern emerges, it is likely that the independence requirement is violated. Figures 16.13 (alternating positive and negative residuals) and 16.14 (increasing residuals) exhibit patterns indicating autocorrelation. (Notice that we joined the points to make it easier to see the patterns.) Figure 16.15 shows no pattern (the residuals appear to be randomly distributed over the time periods) and thus likely represent the occurrence of independent errors.

In Chapter 17, we introduce the Durbin-Watson test, which is another statistical test to determine whether one form of autocorrelation is present.

FIGURE **16.13** Plot of Residuals versus Time Indicating Autocorrelation (Alternating)

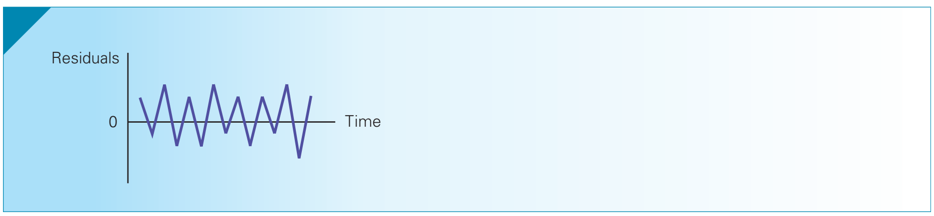


FIGURE 16.14 Plot of Residuals versus Time Indicating Autocorrelation (Increasing)

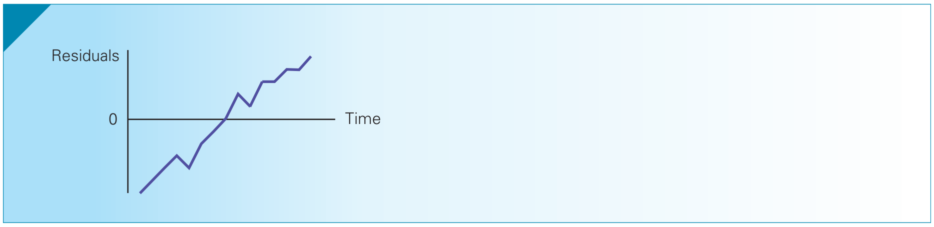
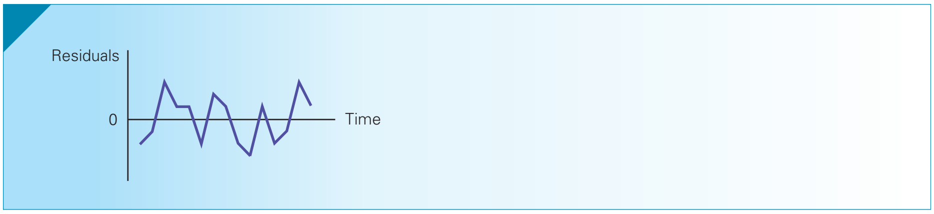


FIGURE 16.15 Plot of Residuals versus Time Indicating Independence



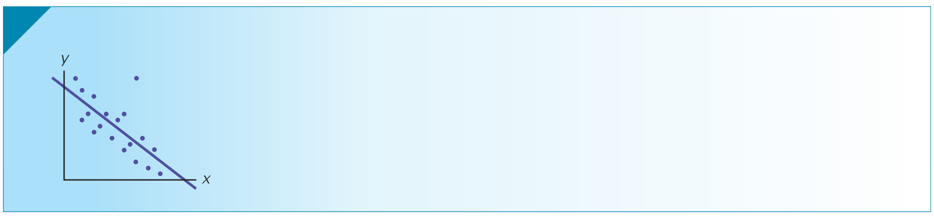
Outliers

An **outlier** is an observation that is unusually small or unusually large. To illustrate, consider Example 16.2, where the range of odometer readings was 19.1 to 49.2 thousand miles. If we had observed a value of 5,000 miles, we would identify that point as an outlier. We need to investigate several possibilities.

1. There was an error in recording the value. To detect an error, we would check the point or points in question. In Example 16.2, we could check the car's odometer to determine whether a mistake was made. If so, we would correct it before proceeding with the regression analysis.
2. The point should not have been included in the sample. Occasionally, measurements are taken from experimental units that do not belong with the sample. We can check to ensure that the car with the 5,000-mile odometer reading was actually 3 years old. We should also investigate the possibility that the odometer was rolled back. In either case, the outlier should be discarded.
3. The observation was simply an unusually large or small value that belongs to the sample and that was recorded properly. In this case, we would do nothing to the outlier. It would be judged to be valid.

Outliers can be identified from the scatter diagram. Figure 16.16 depicts a scatter diagram with one outlier. The statistics practitioner should check to determine whether the measurement was recorded accurately and whether the experimental unit should be included in the sample.

FIGURE 16.16 Scatter Diagram with One Outlier



The standardized residuals also can be helpful in identifying outliers. Large absolute values of the standardized residuals should be thoroughly investigated. Minitab automatically reports standardized residuals that are less than −2 and greater than 2.

Influential Observations

Occasionally, in a regression analysis, one or more observations have a large influence on the statistics. Figure 16.17 describes such an observation and the resulting least squares line. If the point had not been included, the least squares line in Figure 16.18 would have been produced. Obviously, one point has had an enormous influence on the results. Influential points can be identified by the scatter diagram. The point may be an outlier and as such must be investigated thoroughly. Minitab also identities influential observations.

FIGURE **16.17** **Scatter Diagram with One Influential Observation**

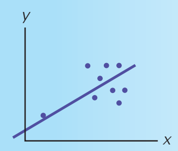
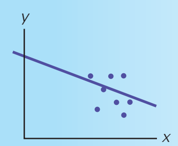


FIGURE **16.18** **Scatter Diagram without the Influential Observation**



Procedure for Regression Diagnostics

The order of the material presented in this chapter is dictated by pedagogical requirements. Consequently, we presented the least squares method of assessing the model's fit, predicting and estimating using the regression equation, coefficient of correlation, and finally, the regression diagnostics. In a practical application, the regression diagnostics would be conducted earlier in the process. It is appropriate to investigate violations of the required conditions when the model is assessed and before using the regression equation to predict and estimate. The following steps describe the entire process. (In Chapter 18, we will discuss model building, for which the following steps represent only a part of the entire procedure.)

1. Develop a model that has a theoretical basis; that is, for the dependent variable in question, find an independent variable that you believe is linearly related to it.
2. Gather data for the two variables. Ideally, conduct a controlled experiment. If that is not possible, collect observational data.

3. Draw the scatter diagram to determine whether a linear model appears to be appropriate. Identify possible outliers.

4. Determine the regression equation.

5. Calculate the residuals and check the required conditions:

Is the error variable nonnormal?
Is the variance constant?
Are the errors independent?
Check the outliers and influential observations.

6. Assess the model's fit.

Compute the standard error of estimate.
Test to determine whether there is a linear relationship. (Test β_1 or ρ.)
Compute the coefficient of determination.

7. If the model fits the data, use the regression equation to predict a particular value of the dependent variable or estimate its mean (or both).

Exercises

16.81 You are given the following six points:

x	−5	−2	0	3	4	7
y	15	9	7	6	4	1

a. Determine the regression equation.
b. Use the regression equation to determine the predicted values of y.
c. Use the predicted and actual values of y to calculate the residuals.
d. Compute the standardized residuals.
e. Identify possible outliers.

16.82 Refer to Exercise 16.2. Calculate the residuals and the predicted values of y.

16.83 Calculate the residuals and predicted values of y in Exercise 16.3.

16.84 Refer to Exercise 16.4.
a. Calculate the residuals
b. Calculate the predicted values of y.
c. Plot the residuals (on the vertical axis) and the predicted values of y.

16.85 Calculate and plot the residuals and predicted values of y for Exercise 16.5.

The following exercises require the use of a computer and software.

16.86 Refer to Exercise 16.6.
a. Determine the residuals and the standardized residuals.
b. Draw the histogram of the residuals. Does it appear that the errors are normally distributed? Explain.
c. Identify possible outliers.
d. Plot the residuals versus the predicted values of y. Does it appear that heteroscedasticity is a problem? Explain.

16.87 Refer to Exercise 16.7.
a. Does it appear that the errors are normally distributed? Explain.
b. Does it appear that heteroscedasticity is a problem? Explain.

16.88 Are the required conditions satisfied in Exercise 16.8?

16.89 Refer to Exercise 16.9.
a. Determine the residuals and the standardized residuals.
b. Draw the histogram of the residuals. Does it appear that the errors are normally distributed? Explain.
c. Identify possible outliers.
d. Plot the residuals versus the predicted values of y. Does it appear that heteroscedasticity is a problem? Explain.

16.90 Refer to Exercise 16.10. Are the required conditions satisfied?

16.91 Refer to Exercise 16.11.

a. Determine the residuals and the standardized residuals.
b. Draw the histogram of the residuals. Does it appear that the errors are normally distributed? Explain.
c. Identify possible outliers.
d. Plot the residuals versus the predicted values of y. Does it appear that heteroscedasticity is a problem? Explain.

16.92 Check the required conditions for Exercise 16.12.

16.93 Refer to Exercise 16.13. Are the required conditions satisfied?

16.94 Refer to Exercise 16.14.

a. Determine the residuals and the standardized residuals.
b. Draw the histogram of the residuals. Does it appear that the errors are normally distributed? Explain.
c. Identify possible outliers.
d. Plot the residuals versus the predicted values of y. Does it appear that heteroscedasticity is a problem? Explain.

16.95 Are the required conditions satisfied for Exercise 16.15?

16.96 Check to ensure that the required conditions for Exercise 16.16 are satisfied.

16.97 Are the required conditions satisfied for Exercise 16.17?

16.98 Perform a complete diagnostic analysis for Exercise 16.18 to determine whether the required conditions are satisfied.

CHAPTER SUMMARY

Simple linear regression and correlation are techniques for analyzing the relationship between two interval variables. Regression analysis assumes that the two variables are linearly related. The least squares method produces estimates of the intercept and the slope of the regression line. Considerable effort is expended in assessing how well the linear model fits the data. We calculate the standard error of estimate, which is an estimate of the standard deviation of the error variable. We test the slope to determine whether there is sufficient evidence of a linear relationship. The strength of the linear association is measured by the coefficient of determination. When the model provides a good fit, we can use it to predict the particular value and to estimate the expected value of the dependent variable. We can also use the Pearson correlation coefficient to measure and test the relationship between two bivariate normally distributed variables. We completed this chapter with a discussion of how to diagnose violations of the required conditions.

IMPORTANT TERMS

SYMBOLS

Symbol	Pronounced	Represents
β_0	Beta sub zero or beta zero	y-intercept coefficient
β_1	Beta sub one or beta one	Slope coefficient
ε	Epsilon	Error variable
$\hat{y}$	y hat	Fitted or calculated value of y
b_0	b sub zero or b zero	Sample y-intercept coefficient
b_1	b sub one or b one	Sample slope coefficient
σ_ε	Sigma sub epsilon or sigma epsilon	Standard deviation of error variable
s_ε	s sub epsilon or s epsilon	Standard error of estimate
s_{b_1}	s sub b sub one or s b one	Standard error of b_1
R^2	R squared	Coefficient of determination
x_g	x sub g or x g	Given value of x
ρ	Rho	Pearson coefficient of correlation
r		Sample coefficient of correlation
e_i	e sub i or e i	Residual of ith point

FORMULAS

Sample slope

$$b_1 = \frac{s_{xy}}{s_x^2}$$

Sample y-intercept

$$b_0 = \bar{y} - b_1\bar{x}$$

Sum of squares for error

$$\text{SSE} = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

Standard error of estimate

$$s_\varepsilon = \sqrt{\frac{\text{SSE}}{n-2}}$$

Test statistic for the slope

$$t = \frac{b_1 - \beta_1}{s_{b_1}}$$

Standard error of b_1

$$s_{b_1} = \frac{s_\varepsilon}{\sqrt{(n-1)s_x^2}}$$

Coefficient of determination

$$R^2 = \frac{s_{xy}^2}{s_x^2 s_y^2} = 1 - \frac{\text{SSE}}{\sum(y_i - \bar{y})^2}$$

Prediction interval

$$\hat{y} \pm t_{\alpha/2,n-2}s_\varepsilon\sqrt{1 + \frac{1}{n} + \frac{(x_g - \bar{x})^2}{(n-1)s_x^2}}$$

Confidence interval estimator of the expected value of y

$$\hat{y} \pm t_{\alpha/2,n-2}s_\varepsilon\sqrt{\frac{1}{n} + \frac{(x_g - \bar{x})^2}{(n-1)s_x^2}}$$

Sample coefficient of correlation

$$r = \frac{s_{xy}}{s_x s_y}$$

Test statistic for testing $\rho = 0$

$$t = r\sqrt{\frac{n-2}{1-r^2}}$$

COMPUTER OUTPUT AND INSTRUCTIONS

Technique	Excel	Minitab
Regression	642	643
Correlation	662	662
Prediction interval	669	669
Regression diagnostics	672	673

CHAPTER EXERCISES

The following exercises require the use of a computer and software. The answers to some of the questions may be calculated manually. See Appendix A for the sample statistics. Conduct all tests of hypotheses at the 5% significance level.

16.99 Xr16-99 The manager of Colonial Furniture has been reviewing weekly advertising expenditures. During the past 6 months, all advertisements for the store have appeared in the local newspaper. The number of ads per week has varied from one to seven. The store's sales staff has been tracking the number of customers who enter the store each week. The number of ads and the number of customers per week for the past 26 weeks were recorded.

a. Determine the sample regression line.
b. Interpret the coefficients.
c. Can the manager infer that the larger the number of ads, the larger the number of customers?
d. Find and interpret the coefficient of determination.
e. In your opinion, is it a worthwhile exercise to use the regression equation to predict the number of customers who will enter the store, given that Colonial intends to advertise five times in the newspaper? If so, find a 95% prediction interval. If not, explain why not.

16.100 Xr16-100 The president of a company that manufactures car seats has been concerned about the number and cost of machine breakdowns. The problem is that the machines are old and becoming quite unreliable. However, the cost of replacing them is quite high, and the president is not certain that the cost can be made up in today's slow economy. To help make a decision about replacement, he gathered data about last month's costs for repairs and the ages (in months) of the plant's 20 welding machines.

a. Find the sample regression line.
b. Interpret the coefficients.
c. Determine the coefficient of determination, and discuss what this statistic tells you.
d. Conduct a test to determine whether the age of a machine and its monthly cost of repair are linearly related.
e. Is the fit of the simple linear model good enough to allow the president to predict the monthly repair cost of a welding machine that is 120 months old? If so, find a 95% prediction interval. If not, explain why not.

16.101 Xr16-101 An agronomist wanted to investigate the factors that determine crop yield. Accordingly, she undertook an experiment wherein a farm was divided into 30 1-acre plots. The amount of fertilizer applied to each plot was varied. Corn was then planted, and the amount of corn harvested at the end of the season was recorded.

a. Find the sample regression line and interpret the coefficients.
b. Can the agronomist conclude that there is a linear relationship between the amount of fertilizer and the crop yield?
c. Find the coefficient of determination and interpret its value.
d. Does the simple linear model appear to be a useful tool in predicting crop yield from the amount of fertilizer applied? If so, produce a 95% prediction interval of the crop yield when 300 pounds of fertilizer are applied. If not, explain why not.

16.102 Xr16-102 Every year, the United States Federal Trade Commission rates cigarette brands according to their levels of tar and nicotine, substances that are hazardous to smokers' health. In addition, the commission includes the amount of carbon monoxide, which is a by-product of burning tobacco that seriously affects the heart. A random sample of 25 brands was taken.

a. Are the levels of tar and nicotine linearly related?
b. Are the levels of nicotine and carbon monoxide linearly related?

16.103 Xr16-103 Some critics of television complain that the amount of violence shown on television contributes to violence in our society. Others point out that television also contributes to the high level of obesity among children. We may have to add financial problems to the list. A sociologist theorized that people who watch television frequently are exposed to many commercials, which in turn leads them to buy more, finally resulting in increasing debt. To test this belief, a sample of 430 families was drawn. For each, the total debt and the number of hours the television is turned on per week were recorded. Perform a statistical procedure to help test the theory.

16.104 Xr16-104 The analysis the human resources manager performed in Exercise 16.18 indicated that the dexterity test is not a predictor of job performance. However, before discontinuing the test he decided that the problem is that the statistical analysis was flawed because it examined the relationship between test score and job performance only for those who scored well in the test. (Recall that only those who scored above 70 were hired; applicants who achieved scores below 70 were not hired.) The manager decided to perform another statistical analysis. A sample of 50 job applicants who scored above 50 were hired; as before, the workers' performance was measured. The test scores and percentages of nondefective computers produced were recorded. On the basis of these data, should the manager discontinue the dexterity tests?

16.105 Xr16-105 Mutual funds minimize risks by diversifying the investments they make. There are mutual funds that specialize in particular types of investments. For example, the TD Precious Metal Mutual Fund buys shares in gold-mining companies. The value of this mutual fund depends on a number of factors related to the companies in which the fund invests as well as on the price of gold. To investigate the relationship between the value of the fund and the price of gold, an MBA student gathered the daily fund price and the daily price of gold for a 28-day period. Can we infer from these data that there is a positive linear relationship between the value of the fund and the price of gold? (The author is grateful to Jim Wheat for writing this exercise.)

16.106 Xr03-59 (Exercise 3.59 revisited) A very large contribution to profits for a movie theater is the sale of popcorn, soft drinks, and candy. A movie theater manager speculated that the longer the time between showings of a movie, the greater the sales of concessions. To acquire more information, the manager conducted an experiment. For a month, he varied the amount of time between movie showings and calculated the sales. Can the manager conclude that when the times between movies increase so do sales?

16.107 Xr16-107* A computer dating service typically asks for various pieces of information such as height, weight, and income. One such service requested the length of index fingers. The only plausible reason for this request is to act as a proxy on height. Women have often complained that men lie about their heights. If there is a strong relationship between heights and index fingers, the information can be used to "correct" false claims about heights. To test the relationship between the two variables, researchers gathered the heights and lengths of index fingers (in centimeters) of 121 students.

a. Graph the relationship between the two variables.
b. Is there sufficient evidence to infer that height and length of index fingers are linearly related?
c. Predict with 95% confidence the height of someone whose index finger is 6.5 cm long. Is this prediction likely to be useful? Explain. (The author would like to thank Howard Waner for supplying the problem and data.)

The following exercises employ data files associated with two previous exercises.

16.108 Xr12-31* In addition to the data recorded for Exercises 12.31 and 13.153, we recorded the grade point average of the students who held down part-time jobs. Determine whether there is evidence of a linear relationship between the hours spent at part-time jobs and the grade point averages.

16.109 Xr13-19* Exercise 13.19 described a survey that asked people between 18 and 34 years of age and 35 to 50 years of age how much time they spent listening to FM radio each day. Also recorded were the amounts spent on music throughout the year. Can we infer that a linear relationship exists between listening times and amounts spent on music?

CASE 16.1 Insurance Compensation for Lost Revenues‡


© Spencer Grant/PhotoEdit

DATA C16-01

In July 1990, a rock-and-roll museum opened in Atlanta, Georgia. The museum was located in a large city block containing a variety of stores. In late July 1992, a fire that started in one of these stores burned the entire block, including the museum. Fortunately, the museum had taken out insurance to cover the cost of rebuilding as well as lost revenue. As a general rule, insurance companies base their payment on how well the company performed in the past. However, the owners of the museum argued that the revenues were increasing, and hence they were entitled to more money under their insurance plan. The argument was based on the revenues and attendance figures of an amusement park, featuring rides and other similar attractions that had opened nearby. The amusement park opened in December 1991. The two entertainment facilities were operating jointly during the last 4 weeks of 1991 and the first 28 weeks of 1992 (the point at which the fire destroyed the museum). In April 1995, the museum reopened with considerably more features than the original one.

The attendance figures for both facilities for December 1991 to October 1995 are listed in columns 1 (museum) and 2 (amusement park). During the period when the museum was closed, the data show zero attendance.

The owners of the museum argued that the weekly attendance from the 29th week of 1992 to the 16th week of 1995 should be estimated using the most current data (17th to 42nd week of 1995). The insurance company argued that the estimates should be based on the 4 weeks of 1991 and the 28 weeks of 1992, when both facilities were operating and before the museum reopened with more features than the original museum.

a. Estimate the coefficients of the simple regression model based on the insurance company's argument. In other words, use the attendance figures for the last 4 weeks in 1991 and the next 28 weeks in 1992 to estimate the coefficients. Then use the model to calculate point predictions for the museum's weekly attendance figures when the museum was closed. Calculate the predicted total attendance.

b. Repeat part (a) using the museum's argument—that is, use the attendance figures after the reopening in 1995 to estimate the regression coefficients and use the equation to predict the weekly attendance when the museum was closed. Calculate the total attendance that was lost because of the fire.

c. Write a report to the insurance company discussing this analysis and include your recommendation about how much the insurance company should award the museum?

[†] The case and the data are real. The names have been changed to preserve anonymity. The author wishes to thank Dr. Kevin Leonard for supplying the problem and the data.

CASE 16.2 Predicting University Grades from High School Grades[§]

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DATA C16-02

Ontario high school students must complete a minimum of six Ontario Academic Credits (OACs) to gain admission to a university in the province. Most students take more than six OACs because universities take the average of the best six in deciding which students to admit. Most programs at universities require high school students to select certain courses. For example, science programs require two of chemistry, biology, and physics. Students applying to engineering must complete at least two mathematics OACs as well as physics. In recent years, one business program began an examination of all aspects of its program, including the criteria used to admit students. Students are required to take English and calculus OACs, and the minimum high school average is about 85%. Strangely enough, even though students are required to complete English and calculus, the marks in these subjects are not included in the average unless they are in the top six courses in a student's transcript. To examine the issue, the registrar took a random sample of students who recently graduated with the BBA (bachelor of business administration degree). He recorded the university GPA (range 0 to 12), the high school average based on the best six courses, and the high school average using English and calculus and the four next best marks.

a. Is there a relationship between university grades and high school average using the best six OACs?

b. Is there a relationship between university grades and high school average using the best four OACs plus calculus and English?

c. Write a report to the university's academic vice president describing your statistical analysis and your recommendations.

[§] The author is grateful to Leslie Grauer for her help in gathering the data for this case.

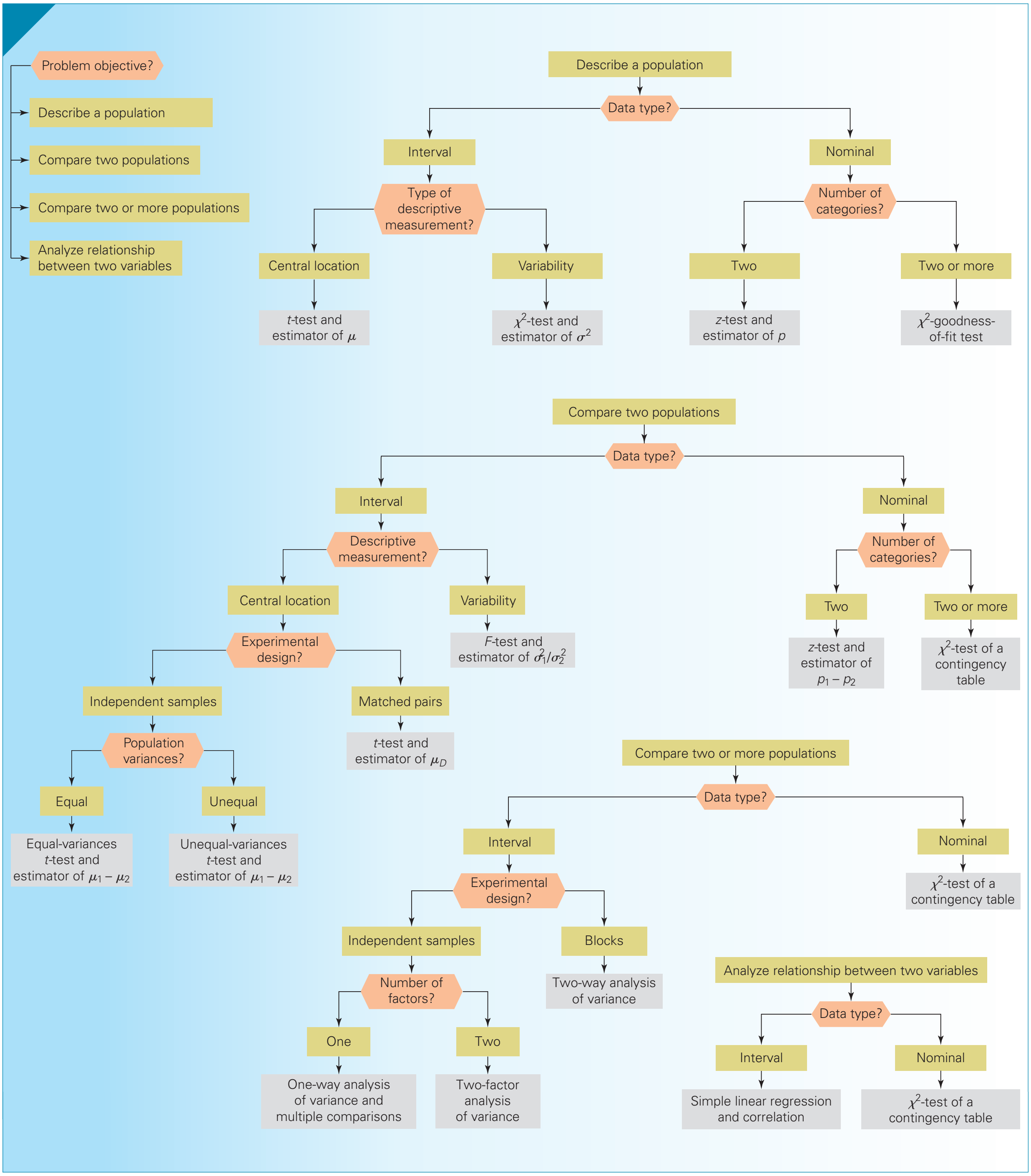
APPENDIX 16 / REVIEW OF CHAPTERS 12 TO 16

We have now presented two dozen inferential techniques. Undoubtedly, the task of choosing the appropriate technique is growing more difficult. Table A16.1 lists all the statistical inference methods covered since Chapter 12. Figure A16.1 is a flowchart to help you choose the correct technique.

TABLE **A16.1** Summary of Statistical Techniques in Chapters 12 to 16

t-test of μ
Estimator of μ (including estimator of $N\mu$)
χ^2-test of σ^2
Estimator of σ^2
z-test of p
Estimator of p (including estimator of Np)
Equal-variances t-test of $\mu_1 - \mu_2$
Equal-variances estimator of $\mu_1 - \mu_2$
Unequal-variances t-test of $\mu_1 - \mu_2$
Unequal-variances estimator of $\mu_1 - \mu_2$
t-test of μ_D
Estimator of μ_D
F-test of σ_1^2/σ_2^2
Estimator of σ_1^2/σ_2^2
z-test of $p_1 - p_2$ (Case 1)
z-test of $p_1 - p_2$ (Case 2)
Estimator of $p_1 - p_2$
One-way analysis of variance (including multiple comparisons)
Two-way (randomized blocks) analysis of variance
Two-factor analysis of variance
χ^2-goodness-of-fit test
χ^2-test of a contingency table
Simple linear regression and correlation (including t-tests of β_1 and ρ, and prediction and confidence intervals)

FIGURE **A16.1** Flowchart of Techniques in Chapters 12 to 16



EXERCISES

A16.1 XrA16-01 In the last decade, society in general and the judicial system in particular have altered their opinions on the seriousness of drunken driving. In most jurisdictions, driving an automobile with a blood alcohol level in excess of .08 is a felony. Because of a number of factors, it is difficult to provide guidelines for when it is safe for someone who has consumed alcohol to drive a car. In an experiment to examine the relationship between blood alcohol level and the weight of a drinker, 50 men of varying weights were each given three beers to drink, and 1 hour later their blood alcohol levels were measured. If we assume that the two variables are normally distributed, can we conclude that blood alcohol level and weight are related?

A16.2 XrA16-02 An article in the journal *Appetite* (December 2003) described an experiment to determine the effect that breakfast meals have on school children. A sample of 29 children was tested on four successive days, having a different breakfast each day. The breakfast meals were

1. Cereal (Cheerios)
2. Cereal (Shreddies)
3. A glucose drink
4. No breakfast

The order of breakfast meals was randomly assigned. A computerized test of working memory was conducted prior to breakfast and again 2 hours later. The decrease in scores was recorded. Do these data allow us to infer that there are differences in the decrease depending on the type of breakfast?

A16.3 Do cell phones cause cancer? This is a multibillion-dollar question. Currently, dozens of lawsuits are pending that claim cell phone use has caused cancer. To help shed light on the issue, several scientific research projects have been undertaken. One such project was conducted by Danish researchers (*Source*: Journal of the National Cancer Institute, 2001). The 13-year study examined 420,000 Danish cell phone users. The scientists determined the number of Danes who would be expected to contract various forms of cancer. The expected number and the actual number of cell phone users who developed each type of cancer are listed here.

Cancer	Expected Number	Actual Number
Brain and nervous system	143	135
Salivary glands	9	7
Leukemia	80	77
Pharynx	52	32
Esophagus	57	42
Eye	12	8
Thyroid	13	13

a. Can we infer from these data that there is a relationship between cell phone use and cancer?
b. Discuss the results, including whether the data are observational or experimental. Provide several interpretations of the statistics. In particular, indicate whether you can infer that cell phone use causes cancer.

A16.4 XrA16-04 A new antiflu vaccine designed to reduce the duration of symptoms has been developed. However, the effect of the drug varies from person to person. To examine the effect of age on the effectiveness of the drug, a sample of 140 flu sufferers was drawn. Each person reported how long the symptoms of the flu persisted and his or her age. Do these data provide sufficient evidence to infer that the older the patient, the longer it takes for the symptoms to disappear?

A16.5 XrA16-05 Several years ago we heard about the "Mommy Track," the phenomenon of women being underpaid in the corporate world because of what is seen as their divided loyalties between home and office. There may also be a "Daddy Differential," which refers to the situation where men whose wives stay at home earn more than men whose wives work. It is argued that the differential occurs because bosses reward their male employees if they come from "traditional families." Linda Stroh of Loyola University of Chicago studied a random sample of 348 male managers employed by 20 *Fortune 500* companies. Each manager reported whether his wife stayed at home to care for their children or worked outside the home, and his annual income. The incomes (in thousands of dollars) were recorded. The incomes of the managers whose wives stay at home are stored in column 1. Column 2 contains the incomes of managers whose wives work outside the home.

a. Can we conclude that men whose wives stay at home earn more than men whose wives work outside the home?
b. If your answer in part (a) is affirmative, does this establish a case for discrimination? Can you think of another cause-and-effect scenario? Explain.

A16.6 XrA16-06 There are enormous differences between health-care systems in the United States and Canada. In a study to examine one dimension of these differences, 300 heart attack victims in each country were randomly selected. (Results of the study conducted by Dr. Daniel Mark of Duke University Medical Center, Dr. David Naylor of Sunnybrook Hospital in Toronto, and Dr. Paul Armstrong of the University of Alberta were published in the *Toronto Sun*, October 27, 1994.) Each patient was asked the following questions regarding the effect of his or her treatment:

1. How many days did it take you to return to work?
2. Do you still have chest pain? (This question was asked 1 month, 6 months, and 12 months after the patients' heart attacks.)

The responses were recorded in the following way:

Column 1: Code representing nationality: 1 = U.S.; 2 = Canada
Column 2: Responses to question 1
Column 3: Responses to question 2–1 month after heart attack: 2 = yes; 1 = no
Column 4: Responses to question 2–6 months after heart attack: 2 = yes; 1 = no
Column 5: Responses to question 2–12 months after heart attack: 2 = yes; 1 = no

Can we conclude that recovery is faster in the United States?

A16.7 XrA16-07 Betting on the results of National Football League games is a popular North American activity. In some states and provinces, it is legal to do so provided that wagers are made through a government-authorized betting organization. In the province of Ontario, Pro-Line serves that function. Bettors can choose any team on which to wager, and Pro-Line sets the odds, which determine the winning payoffs. It is also possible to bet that in any game a tie will be the result. (A tie is defined as a game in which the winning margin is 3 or fewer points. A win occurs when the winning margin is greater than 3.) To assist bettors, Pro-Line lists the favorite for each game and predicts the point spread between the two teams. To judge how well Pro-Line predicts outcomes, the Creative Statistics Company tracked the results of a recent season. It recorded whether a team was favored by (1) 3 or fewer points, (2) 3.5 to 7 points, (3) 7.5 to 11 points, or (4) 11.5 or more points. It also recorded whether the favored team (1) won, (2) lost, or (3) tied. These data are recorded in columns 1 (Pro-Line's predictions) and 2 (game results). Can we conclude that Pro-Line's forecasts are useful for bettors?

A16.8 XrA16-08 As all baseball fans know, first base is the only base that the base runner may overrun. At both second and third base, the runner may be tagged out if he runs past the base. Consequently, on close plays at second and third base, the runner will slide, enabling him to stop at the base. In recent years, however, several players have chosen to slide headfirst when approaching first base, claiming that this is faster than simply running over the base. In an experiment to test this claim, 25 players on one National League team were recruited. Each player ran to first base with and without sliding, and the times to reach the base were recorded. Can we conclude that sliding is slower than not sliding?

A16.9 XrA16-09 How does mental outlook affect a person's health? The answer to this question may allow physicians to care more effectively for their patients. In an experiment to examine the relationship between attitude and physical health, Dr. Daniel Mark, a heart specialist at Duke University, studied 1,719 men and women who had recently undergone a heart catheterization, a procedure that checks for clogged arteries. Patients undergo this procedure when heart disease results in chest pain. All of the patients in the experiment were in about the same condition. In interviews, 14% of the patients doubted that they would recover sufficiently to resume their daily routines. Dr. Mark identified these individuals as pessimists; the others were (by default) optimists. After one year, Dr. Mark recorded how many patients were still alive. The data are stored in columns 1 (1 = optimist, 2 = pessimist) and 2 (2 = alive, 1 = dead). Do these data allow us to infer that pessimists are less likely to survive than optimists with similar physical ailments?

A16.10 XrA16-10 Physicians have been recommending more exercise for their patients, particularly those who are overweight. One benefit of regular exercise appears to be a reduction in cholesterol, a substance associated with heart disease. To study the relationship more carefully, a physician took a random sample of 50 patients who do not exercise and measured their cholesterol levels. He then started them on regular exercise programs. After 4 months, he asked each patient how many minutes per week (on average) he or she exercised; he also measured their cholesterol levels. Column 1 = weekly exercise in minutes, column 2 = cholesterol level before exercise program, and column 3 = cholesterol level after exercise program.

a. Do these data allow us to infer that the amount of exercise and the reduction in cholesterol levels are related?
b. Produce a 95% interval of the amount of cholesterol reduction for someone who exercises for 100 minutes per week.
c. Produce a 95% interval for the average cholesterol reduction for people who exercise for 120 minutes per week.

A16.11 XrA16-11 An economist working for a state university wanted to acquire information about salaries in publicly funded and private colleges and universities. She conducted a survey of 623 public-university faculty members and 592 private-university faculty members asking each to report his or her rank (instructor = 1, assistant professor = 2, associate professor = 3, and professor = 4) and current salary ($1,000). (Adapted from the American Association of University Professors, *AAUP Annual Report on the Economic Status of the Profession.)*

a. Conduct a test to determine whether public colleges and universities and private colleges and universities pay different salaries when all ranks are combined.
b. For each rank, determine whether there is enough evidence to infer that the private college and university salaries differ from that of publicly funded colleges and universities.
c. If the answers to parts (a) and (b) differ, suggest a cause.
d. Conduct a test to determine whether your suggested cause is valid.

A16.12 XrA16-12 Millions of people suffer from migraine headaches. The costs in work days lost, medication, and treatment are measured in the billions of dollars. A study reported in the *Journal of the American Medical Association* (2005, 203: 2118–2125) described an experiment that examined whether acupuncture is an effective procedure in treating migraines. A random sample of 302 migraine patients was selected and divided into three groups. Group 1 was treated with acupuncture; group 2 was treated with sham acupuncture (patients believed that they were being treated with acupuncture but were not); and group 3 was not treated at all. The number of headache days per month was recorded for each patient before the treatments began. The number of headache days per month after treatment was also measured.

a. Conduct a test to determine whether there are differences in the number of headache days before treatment between the three groups of patients.
b. Test to determine whether differences exist after treatment. If so, what are the differences?
c. Why was the test in part (a) conducted?

A16.13 XrA16-13 The battle between customers and car dealerships is often intense. Customers want the lowest price, and dealers want to extract as much money as possible. One source of conflict is the trade-in car. Most dealers will offer a relatively low trade-in in anticipation of negotiating the final package. In an effort to determine how dealers operate, a consumer organization undertook an experiment. Seventy-two individuals were recruited. Each solicited an offer on his or her 5-year-old Toyota Camry. The exact same car was used throughout the experiment. The only variables were the age and gender of the "owner." The ages were categorized as (1) young, (2) middle, and (3) senior. The cash offers are stored in columns 1 and 2. Column 1 stores the data for female owners, and column 2 contains the offers made to male owners. The first 12 rows in both columns represent the offers made to young people, the next 12 rows represent the middle group, and the last 12 rows represent the elderly owners.

a. Can we infer that differences exist between the six groups?
b. If differences exist, determine whether the differences are due to gender, age, or some interaction.

A16.14 XrA16-14 In the presidential elections of 2000 and 2004, the vote in the state of Florida was crucial. It is important for the political parties to track party affiliation. Surveys in Broward and Miami-Dade counties were conducted in 1990, 1996, 2000, and 2004. The numbers of Democrats, Republicans, and other voters were recorded for both counties and for all four years. Test each of the following.

a. Party affiliation changed over the four surveys in Broward.
b. Party affiliation changed over the four surveys in Miami-Dade.
c. There were differences between Broward and Miami-Dade in 2004.

A16.15 XrA16-15 Auto manufacturers are required to test their vehicles for a variety of pollutants in the exhaust. The amount of pollutant varies even among identical vehicles so that several vehicles must be tested. The engineer in charge of testing has collected data (in grams per kilometer driven) on the amounts of two pollutants—carbon monoxide and nitrous oxide—for 50 identical vehicles. The engineer believes the company can save money by testing for only one of the pollutants because the two pollutants are closely linked; that is, if a car is emitting a large amount of carbon monoxide, it will also emit a large amount of nitrous oxide. Do the data support the engineer's belief?

A16.16 XrA16-16 In 2003, there were 129,142,000 workers in the United States (*Source:* U.S. Census Bureau). The general manager for a public transportation company wanted to learn more about how workers commute to work and how long it takes them. A random sample of workers was interviewed. Each reported how he or she typically get to work and how long it takes. Estimate with 95% confidence the total amount of time spent commuting. (Data for this exercise were adapted from the *Statistical Abstract of the United States, 2006*, Table 1083.)

General Social Survey Exercises

A16.17 GSS2008* Is there sufficient evidence to conclude that less than 50% of Americans support gun laws (GUNLAW)?

A16.18 GSS2008* Can we infer from the data that Democrats and Republicans (PARTYID: 0, 1 = Democrat, 5, 6 = Republican) differ in their position on whether the government should reduce income differences between rich and poor (EQWLTH)?

A16.19 GSS2008* How does income affect a person's response to the question, Should the government improve the living conditions of poor people (HELPPOOR)? Test the relationship between income (INCOME) and (HELPPOOR) to answer the question.

A16.20 GSS2008* Do the data allow us to infer that households with at least one union member (UNION: 1 = Respondent belongs, 2 = Spouse belongs, 3 = Both belong, 4 = Neither belong) differ from households with no union members with respect to their position on whether the government should improve the standard of living of poor people (HELPPOOR)?

A16.21 GSS2008* Is there sufficient evidence to conclude that people who have taken college-level science courses (COLSCINM: 1 = Yes, 2 = No) are more likely to answer the following question correctly (HOTCORE): Is the center of Earth very hot? 1 = Yes, 2 = No. Correct answer: Yes.

A16.22 GSS2006* Do larger companies pay better than smaller companies? Answer the question by testing to determine whether there is enough evidence to infer that there is a positive linear relationship between income (INCOME) and the number of people working in the company (NUMORG).

A16.23 GSS2004* Test to determine whether people who went bankrupt in the previous year (FINAN1: 1 = Yes, 2 = No) differ in their political affiliation (PARTYID: 0, 1 = Democrat; 2, 3, 4 = Independent; 5, 6 = Republican)?

A16.24 GSS2008* It seems reasonable to assume that the more one works, the greater the income. Test this assumption by analyzing the relationship between hours worked per week (HRS) and income (INCOME).

A16.25 GSS2004* Is there enough evidence to conclude that victims of a robbery [LAW1: Were you a victim of a robbery (mugging or stick-up) in the previous year? 1 = Yes, 2 = No] are less likely to favor requiring a police permit to buy a gun (GUNLAW: 1 = Favor, 2 = Oppose)?

A16.26 GSS2008* Do you get a more prestigious occupation if you acquire more education? Analyze the relationship between occupation prestige score (PRESTG80) and education (EDUC) to answer the question.

American National Election Survey Exercises

A16.27 ANES2008* Newspaper readership is down all over North America. Newspaper publishers need to acquire more information to stop this worrying trend. Do more educated people spend more time reading newspapers? Conduct a test to determine whether there is evidence to infer that more education (EDUC) is related to more time reading newspapers (TIME3).

A16.28 ANES2008* In many cities, the network national news is broadcast at 6:30 or 7:00 P.M. In most cities, the national news is preceded by local news in the late afternoon or early evening. Do most viewers watch both news shows? To help answer this question, test to determine whether the number of days watching national news (DAYS1) is related to the number of days watching local news in the late afternoon or early evening (DAYS2).

CASE A16.1 Nutrition Education Programs*

DATA CA16-01

Nutrition education programs, which teach clients how to lose weight or reduce cholesterol levels through better eating patterns, have been growing in popularity. The nurse in charge of one such program at a local hospital wanted to know whether the programs actually work. A random sample was drawn of 33 clients who attended a nutrition education program for those with elevated cholesterol levels. The study recorded the weight, cholesterol levels, total dietary fat intake per average day, total dietary cholesterol intake per average day, and percent of daily calories from fat. These data were gathered both before and 3 months after the program. The researchers also determined the clients' genders, ages, and heights. The data are stored in the following way:

Column 1: Gender (1 = female, 2 = male)
Column 2: Age
Column 3: Height (in meters)
Columns 4 and 5: Weight, before and after (in kilograms)
Columns 6 and 7: Cholesterol level, before and after
Columns 8 and 9: Total dietary fat intake per average day, before and after (in grams)
Columns 10 and 11: Dietary cholesterol intake per average day, before and after (in milligrams)
Columns 12 and 13: Percent daily calories from fat, before and after

The nurse would like the following information:

a. In terms of each of weight, cholesterol level, fat intake, cholesterol intake, and calories from fat, is the program a success?
b. Does gender affect the amount of reduction in each of weight, cholesterol level, fat intake, cholesterol intake, and calories from fat?
c. Does age affect the amount of reduction in weight, cholesterol level, fat intake, cholesterol intake, and calories from fat cholesterol?

*The author would like to thank Karen Cavrag for writing this case.

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17 MULTIPLE REGRESSION

General Social Survey

Variables That Affect Income

DATA
GSS2008*

In the Chapter 16 opening example, we used the General Social Survey to show that income and education are linearly related. This raises the question, What other variables affect one's income? To answer this question, we need to expand the simple linear regression technique used in the previous chapter to allow for more than one independent variable.

Here is a list of all the interval variables the General Social Survey created.

Age (AGE)

Years of education of respondent, spouse, father, and mother (EDUC, SPEDUC, PAEDUC, MAEDUC)

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Our answer appears on page 712.

Hours of work per week of respondent and of spouse (HRS and SPHRS)

Occupation prestige score of respondent, spouse, father, and mother (PRESTG80, SPPRES80, PAPRES80, MAPRES80)

Number of children (CHILDS)

Age when first child was born (AGEKDBRN)

Number of family members earning money (EARNRS)

Score on question "Should government reduce income differences between rich and poor?" (EQWLTH)

Score on question "Should government improve standard of living of poor people?" (HELPPOOR)

Score on question "Should government do more or less to solve country's problems?" (HELPNOT)

Score on question "Is it government's responsibility to help pay for doctor and hospital bills?" (HELPSICK)

Number of hours of television viewing per day (TVHOURS)

Years with current employer (CUREMPYR)

The goal is to create a regression analysis that includes all variables that you believe affect the amount of time spent watching television.

Introduction

In the previous chapter, we employed the simple linear regression model to analyze how one variable (the dependent variable y) is related to another interval variable (the independent variable x). The restriction of using only one independent variable was motivated by the need to simplify the introduction to regression analysis. Although there are a number of applications where we purposely develop a model with only one independent variable (see Section 4.6, for example), in general we prefer to include as many independent variables as are believed to affect the dependent variable. Arbitrarily limiting the number of independent variables also limits the usefulness of the model.

In this chapter, we allow for any number of independent variables. In so doing, we expect to develop models that fit the data better than would a simple linear regression model. We begin by describing the multiple regression model and listing the required conditions. We let the computer produce the required statistics and use them to assess the model's fit and diagnose violations of the required conditions. We use the model by interpreting the coefficients, predicting the particular value of the dependent variable, and estimating its expected value.

17.1 Model and Required Conditions

We now assume that k independent variables are potentially related to the dependent variable. Thus, the model is represented by the following equation:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \varepsilon$$

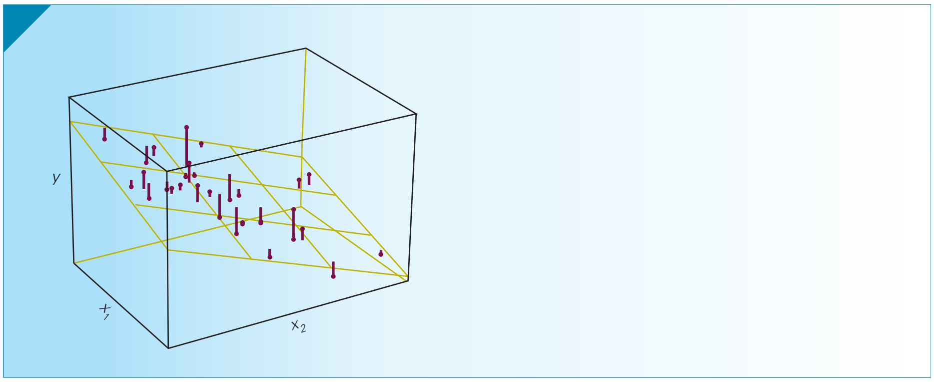
where y is the dependent variable, $x_1, x_2, \ldots, x_k$ are the independent variables, $\beta_0, \beta_1, \ldots, \beta_k$ are the coefficients, and ε is the error variable. The independent variables may actually be functions of other variables. For example, we might define some of the independent variables as follows:

$$x_2 = x_1^2$$
$$x_5 = x_3 x_4$$
$$x_7 = \log(x_6)$$

In Chapter 18, we will discuss how and under what circumstances such functions can be used in regression analysis.

The error variable is retained because, even though we have included additional independent variables, deviations between predicted values of y and actual values of y will still occur. Incidentally, when there is more than one independent variable in the regression model, we refer to the graphical depiction of the equation as a **response surface** rather than as a straight line. Figure 17.1 depicts a scatter diagram of a response surface with $k = 2$. (When $k = 2$, the regression equation creates a plane.) Of course, whenever k is greater than 2, we can only imagine the response surface; we cannot draw it.

FIGURE **17.1** Scatter Diagram and Response Surface with $k = 2$



An important part of the regression analysis comprises several statistical techniques that evaluate how well the model fits the data. These techniques require the following conditions, which we introduced in the previous chapter.

Required Conditions for Error Variable

1. The probability distribution of the error variable ε is normal.
2. The mean of the error variable is 0.
3. The standard deviation of ε is σ_ε, which is a constant.
4. The errors are independent.

In Section 16.6, we discussed how to recognize when the requirements are unsatisfied. Those same procedures can be used to detect violations of required conditions in the multiple regression model.

We now proceed as we did in Chapter 16. We discuss how the model's coefficients are estimated and how we assess the model's fit. However, there is one major difference between Chapters 16 and 17. In Chapter 16, we allowed for the possibility that some students will perform the calculations manually. The multiple regression model involves so many computations that it is virtually impossible to conduct the analysis without a computer. All analyses in this chapter will be performed by Excel and Minitab. Your job will be to interpret the output.

17.2 Estimating the Coefficients and Assessing the Model

The multiple regression equation is expressed similarly to the simple regression equation. The general form is

$$\hat{y} = b_0 + b_1x_1 + b_2x_2 + \cdots + b_kx_k$$

where k is the number of independent variables.

The procedures introduced in Chapter 16 are extended to the multiple regression model. However, in Chapter 16, we first discussed how to interpret the coefficients and then discussed how to assess the model's fit. In practice, we reverse the process—that is, the first step is to determine how well the model fits. If the model's fit is poor, there is no point in a further analysis of the coefficients of that model. A much higher priority is assigned to the task of improving the model. We will discuss the art and science of model building in Chapter 18. In this chapter, we show how a regression analysis is performed. The steps we use are as follows:

1. Select variables that you believe are linearly related to the dependent variable.
2. Use a computer and software to generate the coefficients and the statistics used to assess the model.
3. Diagnose violations of required conditions. If there are problems, attempt to remedy them.
4. Assess the model's fit. Three statistics that perform this function are the standard error of estimate, the coefficient of determination, and the F-test of the analysis of variance. The first two were introduced in Chapter 16; the third will be introduced here.
5. If we are satisfied with the model's fit and that the required conditions are met, we can interpret the coefficients and test them as we did in Chapter 16. We use the model to predict a value of the dependent variable or estimate the expected value of the dependent variable.

We'll illustrate the procedure with the chapter-opening example.

Step 1: Select the Independent Variables

Here are the variables we believe may be linearly related to income.

Age (AGE): For most people income increases with age.

Years of education (EDUC): We've already shown that education is linearly related to income.

Hours of work per week (HRS): Obviously, more hours of work should equal more income.

Spouse's hours of work (SPHRS): It is possible that if one's spouse works more and earns more, the other spouse may choose to work less and thus earn less.

Occupation prestige score (PRESTG80): Occupations with higher prestige scores tend to pay more.

Number of children (CHILDS): Children are expensive, which may encourage their parents to work harder and thus earn more.

Number of family members earning money (EARNRS): As is the case with SPHRS, if more family members earn income, there may be less pressure on the respondent to work harder.

Years with current employer (CUREMPYR): This variable could be negatively or positively related to income.

You may be wondering why we don't simply include all the interval variables that are available to us. There are three reasons. First, the objective is to determine whether our hypothesized model is valid and whether the independent variables in the model are linearly related to the dependent variable. In other words, we should screen the independent variables and include only those that, in theory, affect the dependent variable.

Second, by including large numbers of independent variables, we increase the probability of Type I errors. For example, if we include 100 independent variables, none of which are related to the dependent variable, we're likely to conclude that 5 of them are linearly related to the dependent variable. This is a problem we discussed in Chapter 14.

Third, because of a problem called *multicollinearity* (described in Section 17.3), we may conclude that none of the independent variables are linearly related to the dependent variable when in fact one or more are.

Step 2: Use a Computer to Compute All Coefficients and Other Statistics

EXCEL

	A	B	C	D	E	F
1	SUMMARY OUTPUT					
2						
3	*Regression Statistics*					
4	Multiple R	0.5809				
5	R Square	0.3374				
6	Adjusted R Square	0.3180				
7	Standard Error	33250				
8	Observations	282				
9						
10	ANOVA					
11		*df*	*SS*	*MS*	*F*	*Significance F*
12	Regression	1	153,716,984,625	19,214,623,078	17.38	7.02E-21
13	Residual	273	301,813,647,689	1,105,544,497		
14	Total	281	455,530,632,314			
15						
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	
17	Intercept	−51785	19259	−2.69	0.0076	
18	AGE	461	237	1.95	0.0527	
19	EDUC	4101	848	4.84	0.0000	
20	HRS	620	173	3.59	0.0004	
21	SPHRS	−862	185	−4.67	4.71E-06	
22	PRESTG80	641	176	3.64	0.0003	
23	CHILDS	−331	1522	−0.22	0.8279	
24	EARNRS	687	2929	0.23	0.8147	
25	CUREMPYR	330	237	1.39	0.1649	

INSTRUCTIONS

1. Type or import the data so that the independent variables are in adjacent columns. Note that all rows with blanks (missing data) must be deleted.
2. Click **Data, Data Analysis,** and **Regression**.
3. Specify the **Input *Y* Range**, the **Input *X* Range**, and a value for α (.05).

MINITAB

Regression Analysis: Income versus AGE, EDUC, . . .

The regression equation is
Income = – 51785 + 461 Age + 4101 Educ + 620 Hrs – 862 Sphrs + 641 Prestg80 – 331 Childs + 687 Earnrs + 330 Curempyr
282 cases used, 1741 cases contain missing values

Predictor	Coef	SE Coef	T	P
Constant	–51785	19259	–2.69	0.008
Age	460.9	236.9	1.95	0.053
Educ	4100.9	847.7	4.84	0.000
Hrs	620.0	172.9	3.59	0.000
Sphrs	–862.2	184.6	–4.67	0.000
Prestg80	640.5	175.9	3.64	0.000
Childs	–331	1522	–0.22	0.828
Earnrs	687	2929	0.23	0.815
Curempyr	329.8	236.8	1.39	0.165

S = 33249.7 R-Sq = 33.7% R-Sq(adj) = 31.8%

INSTRUCTIONS

1. Click **Stat, Regression,** and **Regression . . . .**
2. Specify the dependent variable in the **Response** box and the independent variables in the **Predictors** box.

INTERPRET

The regression model is estimated by

$$\hat{y}(\text{INCOME}) = -51{,}785 + 461\text{ AGE} + 4101\text{ EDUC} + 620\text{ HRS} - 862\text{ SPHRS} + 641\text{ PRESTG80} - 331\text{ CHILDS} + 687\text{ EARNRS} + 330\text{ CUREMPYR}$$

We assess the model in three ways: the standard error of estimate, the coefficient of determination (both introduced in Chapter 16) and the F-test of the analysis of variance (presented subsequently).

Standard Error of Estimate

Recall that σ_ε is the standard deviation of the error variable ε and that, because σ_ε is a population parameter, it is necessary to estimate its value by using s_ε. In multiple regression, the standard error of estimate is defined as follows.

Standard Error of Estimate

$$s_\varepsilon = \sqrt{\frac{\text{SSE}}{n - k - 1}}$$

where n is the sample size and k is the number of independent variables in the model.

As we noted in Chapter 16, each of our software packages reports the standard error of estimate in a different way.

EXCEL

	A	B
7	Standard Error	33250

MINITAB

S = 33249.7

INTERPRET

Recall that we judge the magnitude of the standard error of estimate relative to the values of the dependent variable, and particularly to the mean of y. In this example, $\bar{y} = 41{,}746$ (not shown in printouts). It appears that the standard error of estimate is quite large.

Coefficient of Determination

Recall from Chapter 16 that the coefficient of determination is defined as

$$R^2 = 1 - \frac{\text{SSE}}{\sum (y_i - \bar{y})^2}$$

EXCEL

	A	B
5	R Square	0.3374

MINITAB

R-Sq = 33.7%

INTERPRET

This means that 33.74% of the total variation in income is explained by the variation in the eight independent variables, whereas 66.26% remains unexplained.

Notice that Excel and Minitab print a second R^2 statistic, called the **coefficient of determination adjusted for degrees of freedom**, which has been adjusted to take into account the sample size and the number of independent variables. The rationale for this statistic is that, if the number of independent variables k is large relative to the sample size n, the unadjusted R^2 value may be unrealistically high. To understand this point, consider what would happen if the sample size is 2 in a simple linear regression model. The line would fit the data perfectly, resulting in $R^2 = 1$ when, in fact, there may be no linear relationship. To avoid creating a false impression, the adjusted R^2 is often calculated. Its formula follows.

Coefficient of Determination Adjusted for Degrees of Freedom

$$\text{Adjusted } R^2 = 1 - \frac{\text{SSE}/(n - k - 1)}{\sum (y_i - \bar{y})^2/(n - 1)} = 1 - \frac{\text{MSE}}{s_y^2}$$

If n is considerably larger than k, the unadjusted and adjusted R^2 values will be similar. But if SSE is quite different from 0 and k is large compared to n, the unadjusted and adjusted values of R^2 will differ substantially. If such differences exist, the analyst should be alerted to a potential problem in interpreting the coefficient of determination. In this example, the adjusted coefficient of determination is 31.80%, indicating that, no matter how we measure the coefficient of determination, the model's fit is not very good.

Testing the Validity of the Model

In the simple linear regression model, we tested the slope coefficient to determine whether sufficient evidence existed to allow us to conclude that there was a linear relationship between the independent variable and the dependent variable. However, because there is only one independent variable in that model, that same t-test was also tested to determine whether that model is valid. When there is more than one independent variable, we need another method to test the overall validity of the model. The technique is a version of the analysis of variance, which we introduced in Chapter 14.

To test the validity of the regression model, we specify the following hypotheses:

H_0: $\beta_1 = \beta_2 = \cdots = \beta_k = 0$

H_1: At least one β_i is not equal to 0

If the null hypothesis is true, none of the independent variables $x_1, x_2, \ldots, x_k$ is linearly related to y, and therefore the model is invalid. If at least one β_i is not equal to 0, the model does have some validity.

When we discussed the coefficient of determination in Chapter 16, we noted that the total variation in the dependent variable [measured by $\sum (y_i - \bar{y})^2$] can be decomposed into two parts: the explained variation (measured by SSR) and the unexplained variation (measured by SSE); that is,

Total Variation in y = SSR + SSE

Furthermore, we established that, if SSR is large relative to SSE, the coefficient of determination will be high—signifying a good model. On the other hand, if SSE is large, most of the variation will be unexplained, which indicates that the model provides a poor fit and consequently has little validity.

The test statistic is the same one we encountered in Section 14.1, where we tested for the equivalence of two or more population means. To judge whether SSR is large enough relative to SSE to allow us to infer that at least one coefficient is not equal to 0, we compute the ratio of the two mean squares. (Recall that the mean square is the sum of squares divided by its degrees of freedom; recall, too, that the ratio of two mean squares is F-distributed as long as the underlying population is normal—a required condition for this application.) The calculation of the test statistic is summarized in an analysis of variance (ANOVA) table, whose general form appears in Table 17.1. The Excel and Minitab ANOVA tables are shown next.

TABLE **17.1** Analysis of Variance Table for Regression Analysis

SOURCE OF VARIATION	DEGREES OF FREEDOM	SUMS OF SQUARES	MEAN SQUARES	F-STATISTIC
Regression	k	SSR	$\text{MSR} = \text{SSR}/k$	$F = \text{MSR}/\text{MSE}$
Residual	$n - k - 1$	SSE	$\text{MSE} = \text{SSE}/(n - k - 1)$	
Total	$n - 1$	$\sum(y_i - \bar{y})^2$		

EXCEL

	A	B	C	D	E	F
10	ANOVA					
11		*df*	*SS*	*MS*	*F*	*Significance F*
12	Regression	1	153,716,984,625	19,214,623,078	17.38	7.02E-21
13	Residual	273	301,813,647,689	1,105,544,497		
14	Total	281	455,530,632,314			

MINITAB

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	8	1.53717E+11	19,214,623,078	17.38	0.000
Residual Error	273	3.01814E+11	1,105,544,497		
Total	281	4.55531E+11			

A large value of F indicates that most of the variation in y is explained by the regression equation and that the model is valid. A small value of F indicates that most of the variation in y is unexplained. The rejection region allows us to determine whether F is large enough to justify rejecting the null hypothesis. For this test, the rejection region is

$$F > F_{\alpha,k,n-k-1}$$

In Example 17.1, the rejection region (assuming $\alpha = .05$) is

$$F > F_{\alpha,k,n-k-1} = F_{.05,8,273} \approx 1.98$$

As you can see from the printout, $F = 17.38$. The printout also includes the p-value of the test, which is 0. Obviously, there is a great deal of evidence to infer that the model is valid.

Although each assessment measurement offers a different perspective, all agree in their assessment of how well the model fits the data, because all are based on the sum of squares for error, SSE. The standard error of estimate is

$$s_\varepsilon = \sqrt{\frac{\text{SSE}}{n - k - 1}}$$

and the coefficient of determination is

$$R^2 = 1 - \frac{\text{SSE}}{\sum(y_i - \bar{y})^2}$$

When the response surface hits every single point, SSE = 0. Hence, $s_\varepsilon = 0$ and $R^2 = 1$.

If the model provides a poor fit, we know that SSE will be large [its maximum value is $\sum(y_i - \bar{y})^2$], s_ε will be large, and [because SSE is close to $\sum(y_i - \bar{y})^2$] R^2 will be close to 0.

The F-statistic also depends on SSE. Specifically,

$$F = \frac{\text{MSR}}{\text{MSE}} = \frac{\left(\sum(y_i - \bar{y})^2 - \text{SSE}\right)/k}{\text{SSE}/(n - k - 1)}$$

When SSE = 0,

$$F = \frac{\sum(y_i - \bar{y})^2/k}{0/(n - k - 1)}$$

which is infinitely large. When SSE is large, SSE is close to $\sum(y_i - \bar{y})^2$ and F is quite small.

The relationship among s_ε, R^2, and F are summarized in Table 17.2.

TABLE **17.2** Relationship among SSE, s_ε, R^2, and F

SSE	s_ε	R^2	F	ASSESSMENT OF MODEL
0	0	1	∞	Perfect
Small	Small	Close to 1	Large	Good
Large	Large	Close to 0	Small	Poor
$\sum(y_i - \bar{y})^2$	$\sqrt{\frac{\sum(y_i - \bar{y})^2}{n - k - 1}}$*	0	0	Useless

*When n is large and k is small, this quantity is approximately equal to the standard deviation of y.

If we're satisfied that the model fits the data as well as possible and that the required conditions are satisfied, we can interpret and test the individual coefficients and use the model to predict and estimate.

Interpreting the Coefficients

The coefficients $b_0, b_1, \ldots, b_k$ describe the relationship between each of the independent variables and the dependent variable in the sample. We need to use inferential methods (described below) to draw conclusions about the population. In our example, the sample consists of the 657 observations. The population is composed of all American adults.

Intercept The intercept is $b_0 = -51{,}785$. This is the average income when all the independent variables are zero. As we observed in Chapter 16, it is often misleading to try to interpret this value, particularly if 0 is outside the range of the values of the independent variables (as is the case here).

Age The relationship between income and age is described by $b_1 = 461$. From this number, we learn that for each additional year of age in this model, income increases on average by $461, assuming that the other independent variables in this model are held constant.

Education The coefficient $b_2 = 4{,}101$ specifies that in this sample for each additional year of education the income increases on average by $4,101, assuming the constancy of the other independent variables.

Hours of Work The relationship between hours of work per week is expressed by $b_3 = 620$. We interpret this number as the average increase in annual income for each additional hour of work per week, keeping the other independent variables fixed in this sample.

Spouse's Hours of Work The relationship between annual income and a spouse's hours of work per week is described in this sample by $b_4 = -862$, which we interpret to mean that for each additional hour a spouse works per week, income decreases on average by $862 when the other variables are constant.

Occupation Prestige Score In this sample, the relationship between annual income and occupation prestige score is described by $b_5 = 641$. For each additional unit increase in the occupation prestige score, annual income increases on average by $641, holding all other variables constant.

Number of Children The relationship between annual income and number of children is expressed by $b_6 = -331$, which tells us that for each additional child, annual income decreases on average by $331 in this sample.

Number of Family Members Earning Income In this dataset, the relationship between annual income and the number of family members who earn money is expressed by $b_7 = 687$, which tells us that for each additional family member earner, annual income increases on average by $687, assuming that the other independent variables are constant.

Number of Years with Current Job The coefficient of the last independent variable in this model is $b_8 = 330$. This number means that for each additional year of job tenure with the current company, annual income increases on average by $330, keeping the other independent variables constant in this sample.

Testing the Coefficients

In Chapter 16, we described how to test to determine whether there is sufficient evidence to infer that in the simple linear regression model x and y are linearly related. The null and alternative hypotheses were

$$H_0: \quad \beta_1 = 0$$
$$H_1: \quad \beta_1 \neq 0$$

The test statistic was

$$t = \frac{b_1 - \beta_1}{s_{b_1}}$$

which is Student t distributed with $\nu = n - 2$ degrees of freedom.

In the multiple regression model, we have more than one independent variable. For each such variable, we can test to determine whether there is enough evidence of a linear relationship between it and the dependent variable for the entire population when the other independent variables are included in the model.

Testing the Coefficients

$$H_0: \quad \beta_i = 0$$
$$H_1: \quad \beta_i \neq 0$$

(for $i = 1, 2, \ldots, k$); the test statistic is

$$t = \frac{b_i - \beta_i}{s_{b_i}}$$

which is Student t distributed with $\nu = n - k - 1$ degrees of freedom.

To illustrate, we test each coefficient in the multiple regression model in the chapter-opening example. The tests that follow are performed just as all other tests in this book have been performed. We set up the null and alternative hypotheses, identify the test statistic, and use the computer to calculate the value of the test statistic and its p-value. For each independent variable, we test ($i = 1, 2, 3, 4, 5, 6, 7, 8$).

$$H_0: \quad \beta_i = 0$$
$$H_1: \quad \beta_i \neq 0$$

Refer to page 696 and 697 and examine the computer output. The output includes the t-tests of β_i. The results of these tests pertain to the entire population of the United States in 2008. It is also important to add that these test results were determined when the other independent variables were included in the model. We add this statement because a simple linear regression will very likely result in different values of the test statistics and possibly the conclusion.

Test of β_1 (Coefficient of age)

Value of the test statistic: $t = 1.95$; p-value $= .0527$

Test of β_2 (Coefficient of education)

Value of the test statistic: $t = 4.84$; p-value $= 0$

Test of β_3 (Coefficient of number of hours of work per week)

Value of the test statistic: $t = 3.59$; p-value $= .0004$

Test of β_4 (Coefficient of spouse's number of hours of work per week)

Value of the test statistic: $t = -4.67$; p-value $= 0$

Test of β_5 (Coefficient of occupation prestige score)

Value of the test statistic: $t = 3.64$; p-value $= .0003$

Test of β_6 (Coefficient of number of children)

Value of the test statistic: $t = -.22$; p-value $= .8279$

Test of β_7 (Coefficient of number of earners in family)

Value of the test statistic: $t = .23$; p-value $= .8147$

Test of β_8 (Coefficient of years with current employer)

Value of the test statistic: $t = 1.39$; p-value $= .1649$

There is sufficient evidence at the 5% significance level to infer that each of the following variables is linearly related to income:

Education

Number of hours of work per week

Spouse's number of hours of work per week

Occupation prestige score

There is weak evidence to infer that income and age are linearly related.

In this model, there is not enough evidence to conclude that each of the following variables is linearly related to income:

Number of children

Number of earners in the family

Number of years with current employer

Note that this may mean that there is no evidence of a linear relationship between income and these three independent variables. However, it may also mean that there is a linear relationship between income and one or more of these variables, but because of a condition called *multicollinearity*, the t-tests revealed no linear relationship. We will discuss multicollinearity in Section 17.3.

A Cautionary Note About Interpreting the Results

Care should be taken when interpreting the results of this and other regression analyses. We might find that in one model there is enough evidence to conclude that a particular independent variable is linearly related to the dependent variable, but that no such evidence exists in another model. Consequently, whenever a particular t-test is *not* significant, we state that there is not enough evidence to infer that the independent and dependent variable are linearly related *in this model.* The implication is that another model may yield different conclusions.

Furthermore, if one or more of the required conditions are violated, the results may be invalid. In Section 16.6, we introduced the procedures that allow the statistics practitioner to examine the model's requirements. We will add to this discussion in Section 17.3. We also remind you that it is dangerous to extrapolate far outside the range of the observed values of the independent variables.

t-Tests and the Analysis of Variance

The *t*-tests of the individual coefficients allow us to determine whether $\beta_i \neq 0$ (for $i = 1, 2, \ldots, k$), which tells us whether a linear relationship exists between x_i and y. There is a *t*-test for each independent variable. Consequently, the computer automatically performs k *t*-tests. (It actually conducts $k + 1$ *t*-tests, including the one for the intercept β_0, which we usually ignore.) The *F*-test in the analysis of variance combines these *t*-tests into a single test. In other words, we test all the β_i at one time to determine whether at least one of them is not equal to 0. The question naturally arises, Why do we need the *F*-test if it is nothing more than the combination of the previously performed *t*-tests? Recall that we addressed this issue before. In Chapter 14, we pointed out that we can replace the analysis of variance by a series of *t*-tests of the difference between two means. However, by doing so, we increase the probability of making a Type I error. That means that even when there is no linear relationship between each of the independent variables and the dependent variable, multiple *t*-tests will likely show some are significant. As a result, you will conclude erroneously that, because at least one β_i is not equal to 0, the model is valid. The *F*-test, on the other hand, is performed only once. Because the probability that a Type I error will occur in a single trial is equal to α, the chance of erroneously concluding that the model is valid is substantially less with the *F*-test than with multiple *t*-tests.

There is another reason that the *F*-test is superior to multiple *t*-tests. Because of a commonly occurring problem called *multicollinearity*, the *t*-tests may indicate that some independent variables are not linearly related to the dependent variable, when in fact they are. The problem of multicollinearity does not affect the *F*-test, nor does it inhibit us from developing a model that fits the data well. Multicollinearity is discussed in Section 17.3.

The *F*-Test and the *t*-Test in the Simple Linear Regression Model

It is useful for you to know that we can use the *F*-test to test the validity of the simple linear regression model. However, this test is identical to the *t*-test of β_1. The *t*-test of β_1 in the simple linear regression model tells us whether that independent variable is linearly related to the dependent variable. However, because there is only one independent variable, the *t*-test of β_1 also tells us whether the model is valid, which is the purpose of the *F*-test.

The relationship between the *t*-test of β_i and the *F*-test can be explained mathematically. Statisticians can show that if we square a *t*-statistic with ν degrees of freedom, we produce an *F*-statistic with 1 and ν degrees of freedom. (We briefly discussed this relationship in Chapter 14.) To illustrate, consider Example 16.2 on page 641. We found the *t*-test of β_1 to be -13.44, with degrees of freedom equal to 98. The *p*-value was 5.75×10^{-24}. The output included the analysis of variance table where $F = 180.64$ and *p*-value was 5.75×10^{-24}. The *t*-statistic squared is $t^2 = (-13.44)^2 = 180.63$. (The difference is the result of rounding errors.) Notice that the degrees of freedom of the *F*-statistic are 1 and 98. Thus, we can use either test to test the validity of the simple linear regression model.

Using the Regression Equation

As was the case with simple linear regression, we can use the multiple regression equation in two ways: We can produce the prediction interval for a particular value of y, and we can produce the confidence interval estimate of the expected value of y. Like the other calculations associated with multiple regression, we call on the computer to do the work.

To illustrate, we'll predict the income of a 50-year-old, with 12 years of education, who works 40 hours per week, has a spouse who also works 40 hours per week (i.e., 2 earners in the family), has an occupation prestige score of 50, has 2 children, and has worked for the same company for 5 years.

As you discovered in the previous chapter, both Excel and Minitab output the prediction interval and interval estimate of the expected value of incomes for all people with the given variables.

EXCEL

	A	B	C
1	**Prediction Interval**		
2			
3			Margin
4			
5	Predicted value		45,168
6			
7	Prediction Interval		
8	Lower limit		−20,719
9	Upper limit		111,056
10			
11	Interval Estimate of Expected Value		
12	Lower limit		37,661
13	Upper limit		52,675

INSTRUCTIONS

See the instructions on page 669. In cells B284 to I284, we input the values 50 12 40 40 50 2 2 5, respectively. We specified 95% confidence.

MINITAB

Predicted Values for New Observations

New Obs	Fit	SE Fit	95% CI	95% PI
1	45,168	3,813	(37,661, 52,675)	(−20,719, 111,056)

Values of Predictors for New Observations

New Obs	Age	Educ	Hrs	Sphrs	Prestg80	Childs	Earnrs	Curempyr
1	50.0	12.0	40.0	40.0	50.0	2.00	2.00	5.00

INSTRUCTIONS

See the instructions on page 669. We input the values 50 12 40 40 50 2 2 5. We specified 95% confidence.

INTERPRET

The prediction interval is −20,719, 111,056. It is so wide as to be completely useless. To be useful in predicting values, the model must be considerably better. The confidence interval estimate of the expected income of a population is 37,661, 52,675.

EXERCISES

The following exercises require the use of a computer and statistical software. Exercises 17.1–17.4 can be solved manually. See Appendix A for the sample statistics. Use a 5% significance level.

17.1 Xr17-01 A developer who specializes in summer cottage properties is considering purchasing a large tract of land adjoining a lake. The current owner of the tract has already subdivided the land into separate building lots and has prepared the lots by removing some of the trees. The developer wants to forecast the value of each lot. From previous experience, she knows that the most important factors affecting the price of a lot are size, number of mature trees, and distance to the lake. From a nearby area, she gathers the relevant data for 60 recently sold lots.

a. Find the regression equation.
b. What is the standard error of estimate? Interpret its value.
c. What is the coefficient of determination? What does this statistic tell you?
d. What is the coefficient of determination, adjusted for degrees of freedom? Why does this value differ from the coefficient of determination? What does this tell you about the model?
e. Test the validity of the model. What does the *p*-value of the test statistic tell you?
f. Interpret each of the coefficients.
g. Test to determine whether each of the independent variables is linearly related to the price of the lot in this model.
h. Predict with 90% confidence the selling price of a 40,000-square-foot lot that has 50 mature trees and is 25 feet from the lake.
i. Estimate with 90% confidence the average selling price of 50,000-square-foot lots that have 10 mature trees and are 75 feet from the lake.

17.2 Xr17-02 Pat Statsdud, a student ranking near the bottom of the statistics class, decided that a certain amount of studying could actually improve final grades. However, too much studying would not be warranted because Pat's ambition (if that's what one could call it) was to ultimately graduate with the absolute minimum level of work. Pat was registered in a statistics course that had only 3 weeks to go before the final exam and for which the final grade was determined in the following way:

$$\text{Total mark} = 20\%\ (\text{Assignment}) + 30\%\ (\text{Midterm test}) + 50\%\ (\text{Final exam})$$

To determine how much work to do in the remaining 3 weeks, Pat needed to be able to predict the final exam mark on the basis of the assignment mark (worth 20 points) and the midterm mark (worth 30 points). Pat's marks on these were 12/20 and 14/30, respectively. Accordingly, Pat undertook the following analysis. The final exam mark, assignment mark, and midterm test mark for 30 students who took the statistics course last year were collected.

a. Determine the regression equation.
b. What is the standard error of estimate? Briefly describe how you interpret this statistic.
c. What is the coefficient of determination? What does this statistic tell you?
d. Test the validity of the model.
e. Interpret each of the coefficients.
f. Can Pat infer that the assignment mark is linearly related to the final grade in this model?
g. Can Pat infer that the midterm mark is linearly related to the final grade in this model?
h. Predict Pat's final exam mark with 95% confidence.
i. Predict Pat's final grade with 95% confidence.

17.3 Xr17-03 The president of a company that manufactures drywall wants to analyze the variables that affect demand for his product. Drywall is used to construct walls in houses and offices. Consequently, the president decides to develop a regression model in which the dependent variable is monthly sales of drywall (in hundreds of 4 ×8 sheets) and the independent variables are

Number of building permits issued in the county
Five-year mortgage rates (in percentage points)
Vacancy rate in apartments (in percentage points)
Vacancy rate in office buildings (in percentage points)
To estimate a multiple regression model, he took monthly observations from the past 2 years.

a. Analyze the data using multiple regression.
b. What is the standard error of estimate? Can you use this statistic to assess the model's fit? If so, how?
c. What is the coefficient of determination, and what does it tell you about the regression model?
d. Test the overall validity of the model.
e. Interpret each of the coefficients.
f. Test to determine whether each of the independent variables is linearly related to drywall demand in this model.
g. Predict next month's drywall sales with 95% confidence if the number of building permits is 50, the 5-year mortgage rate is 9.0%, and the vacancy rates are 3.6% in apartments and 14.3% in office buildings.

17.4 Xr17-04 The general manager of the Cleveland Indians baseball team is in the process of determining which minor-league players to draft. He is aware that his team needs home-run hitters and would like to find a way to predict the number of home runs a player will hit. Being an astute statistician, he gathers a random sample of players and records the number of home runs each player hit in his first two full years as a major-league player, the number of home runs he hit in his last full year in the minor leagues, his age, and the number of years of professional baseball.

a. Develop a regression model and use a software package to produce the statistics.
b. Interpret each of the coefficients.
c. How well does the model fit?
d. Test the model's validity.
e. Do each of the independent variables belong in the model?
f. Calculate the 95% interval of the number of home runs in the first two years of a player who is 25 years old, has played professional baseball for 7 years, and hit 22 home runs in his last year in the minor leagues.
g. Calculate the 95% interval of the expected number of home runs in the first two years of players who are 27 years old, have played professional baseball for 5 years, and hit 18 home runs in their last year in the minors.

APPLICATIONS in HUMAN RESOURCES MANAGEMENT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Severance Pay

In most firms, the entire issue of compensation falls into the domain of the human resources manager. The manager must ensure that the method used to determine compensation contributes to the firm's objectives. Moreover, the firm needs to ensure that discrimination or bias of any kind is not a factor. Another function of the personnel manager is to develop severance packages for employees whose services are no longer needed because of downsizing or merger. The size and nature of severance is rarely part of any working agreement and must be determined by a variety of factors. Regression analysis is often useful in this area.

17.5 Xr17-05 When one company buys another company, it is not unusual that some workers are terminated. The severance benefits offered to the laid-off workers are often the subject of dispute. Suppose that the Laurier Company recently bought the Western Company and subsequently terminated 20 of Western's employees. As part of the buyout agreement, it was promised that the severance packages offered to the former Western employees would be equivalent to those offered to Laurier employees who had been terminated in the past year. Thirty-six-year-old Bill Smith, a Western employee for the past 10 years, earning $32,000 per year, was one of those let go. His severance package included an offer of 5 weeks' severance pay. Bill complained that this offer was less than that offered to Laurier's employees when they were laid off, in contravention of the buyout agreement. A statistician was called in to settle the dispute. The statistician was told that severance is determined by three factors: age, length of service with the company, and pay. To determine how generous the severance package had been, a random sample of 50 Laurier ex-employees was taken. For each, the following variables were recorded:

Number of weeks of severance pay
Age of employee
Number of years with the company
Annual pay (in thousands of dollars)

a. Determine the regression equation.
b. Comment on how well the model fits the data.
c. Do all the independent variables belong in the equation? Explain.
d. Perform an analysis to determine whether Bill is correct in his assessment of the severance package.

17.6 Xr17-06 The admissions officer of a university is trying to develop a formal system to decide which students to admit to the university. She believes that determinants of success include the standard variables—high school grades and SAT scores. However, she also believes that students who have participated in extracurricular activities are more likely to succeed than those who have not. To investigate the issue, she randomly sampled 100 fourth-year students and recorded the following variables:

GPA for the first 3 years at the university (range: 0 to 12)
GPA from high school (range: 0 to 12)
SAT score (range: 400 to 1600)
Number of hours on average spent per week in organized extracurricular activities in the last year of high school

a. Develop a model that helps the admissions officer decide which students to admit and use the computer to generate the usual statistics.
b. What is the coefficient of determination? Interpret its value.
c. Test the overall validity of the model.
d. Test to determine whether each of the independent variables is linearly related to the dependent variable in this model.
e. Determine the 95% interval of the GPA for the first 3 years of university for a student whose high school GPA is 10, whose SAT score is 1200, and who worked an average of 2 hours per week on organized extracurricular activities in the last year of high school.
f. Find the 90% interval of the mean GPA for the first 3 years of university for all students whose high school GPA is 8, whose SAT score is 1100, and who worked an average of 10 hours per week on organized extracurricular activities in the last year of high school.

17.7 Xr17-07 The marketing manager for a chain of hardware stores needed more information about the effectiveness of the three types of advertising that the chain used. These are localized direct mailing (in which flyers describing sales and featured products are distributed to homes in the area surrounding a store), newspaper advertising, and local television advertisements. To determine which type is most effective, the manager collected 1 week's data from 100 randomly selected stores. For each store, the following variables were recorded:

Weekly gross sales
Weekly expenditures on direct mailing
Weekly expenditures on newspaper advertising
Weekly expenditures on television commercials

All variables were recorded in thousands of dollars.

a. Find the regression equation.
b. What are the coefficient of determination and the coefficient of determination adjusted for degrees of freedom? What do these statistics tell you about the regression equation?
c. What does the standard error of estimate tell you about the regression model?
d. Test the validity of the model.
e. Which independent variables are linearly related to weekly gross sales in this model? Explain.
f. Compute the 95% interval of the week's gross sales if a local store spent $800 on direct mailing, $1,200 on newspaper advertisements, and $2,000 on television commercials.
g. Calculate the 95% interval of the mean weekly gross sales for all stores that spend $800 on direct mailing, $1,200 on newspaper advertising, and $2,000 on television commercials.
h. Discuss the difference between the two intervals found in parts (f) and (g).

17.8 Xr17-08 For many cities around the world, garbage is an increasing problem. Many North American cities have virtually run out of space to dump the garbage. A consultant for a large American city decided to gather data about the problem. She took a random sample of houses and determined the following:

Y = the amount of garbage per average week (pounds)
X_1 = Size of the house (square feet)
X_2 = Number of children
X_3 = Number of adults who are usually home during the day

a. Conduct a regression analysis.
b. Is the model valid?
c. Interpret each of the coefficients.
d. Test to determine whether each of the independent variables is linearly related to the dependent variable.

17.9 Xr17-09 The administrator of a school board in a large county was analyzing the average mathematics test scores in the schools under her control. She noticed that there were dramatic differences in scores among the schools. In an attempt to improve the scores of all the schools, she attempted to determine the factors that account for the differences. Accordingly, she took a random sample of 40 schools across the county and, for each, determined the mean test score last year, the percentage of teachers in each school who have at least one university degree in mathematics, the mean age, and the mean annual income (in $1,000s) of the mathematics teachers.

a. Conduct a regression analysis to develop the equation.
b. Is the model valid?

c. Interpret and test the coefficients.
d. Predict with 95% confidence the test score at a school where 50% of the mathematics teachers have mathematics degrees, the mean age is 43, and the mean annual income is $48,300.

17.10 Xr17-10* Life insurance companies are keenly interested in predicting how long their customers will live because their premiums and profitability depend on such numbers. An actuary for one insurance company gathered data from 100 recently deceased male customers. He recorded the age at death of the customer plus the ages at death of his mother and father, the mean ages at death of his grandmothers, and the mean ages at death of his grandfathers.

a. Perform a multiple regression analysis on these data.
b. Is the model valid?
c. Interpret and test the coefficients.
d. Determine the 95% interval of the longevity of a man whose parents lived to the age of 70, whose grandmothers averaged 80 years, and whose grandfathers averaged 75 years.
e. Find the 95% interval of the mean longevity of men whose mothers lived to 75 years, whose fathers lived to 65 years, whose grandmothers averaged 85 years, and whose grandfathers averaged 75 years.

17.11 Xr17-11 University students often complain that universities reward professors for research but not for teaching, and they argue that professors react to this situation by devoting more time and energy to the publication of their findings and less time and energy to classroom activities. Professors counter that research and teaching go hand in hand: More research makes better teachers. A student organization at one university decided to investigate the issue. It randomly selected 50 economics professors who are employed by a multicampus university. The students recorded the salaries (in $1,000s) of the professors, their average teaching evaluations (on a 10-point scale), and the total number of journal articles published in their careers. Perform a complete analysis (produce the regression equation, assess it, and report your findings).

17.12 Xr17-12* One critical factor that determines the success of a catalog store chain is the availability of products that consumers want to buy. If a store is sold out, future sales to that customer are less likely. Accordingly, delivery trucks operating from a central warehouse regularly resupply stores. In an analysis of a chain's operations, the general manager wanted to determine the factors that are related to how long it takes to unload delivery trucks. A random sample of 50 deliveries to one store was observed. The times (in minutes) to unload the truck, the total number of boxes, and the total weight (in hundreds of pounds) of the boxes were recorded.

a. Determine the multiple regression equation.
b. How well does the model fit the data? Explain.
c. Interpret and test the coefficients.
d. Produce a 95% interval of the amount of time needed to unload a truck with 100 boxes weighing 5,000 pounds.
e. Produce a 95% interval of the average amount of time needed to unload trucks with 100 boxes weighing 5,000 pounds.

17.13 Xr17-13 Lotteries have become important sources of revenue for governments. Many people have criticized lotteries, however, referring to them as a tax on the poor and uneducated. In an examination of the issue, a random sample of 100 adults was asked how much they spend on lottery tickets and was interviewed about various socioeconomic variables. The purpose of this study is to test the following beliefs:

1. Relatively uneducated people spend more on lotteries than do relatively educated people.
2. Older people buy more lottery tickets than younger people.
3. People with more children spend more on lotteries than people with fewer children.
4. Relatively poor people spend a greater proportion of their income on lotteries than relatively rich people.

The following data were recorded:

Amount spent on lottery tickets as a percentage of total household income
Number of years of education
Age
Number of children
Personal income (in thousands of dollars)

a. Develop the multiple regression equation.
b. Is the model valid?
c. Test each of the beliefs. What conclusions can you draw?

17.14 Xr17-14* The MBA program at a large university is facing a pleasant problem—too many applicants. The current admissions policy requires students to have completed at least 3 years of work experience and an undergraduate degree with a B– average or better. Until 3 years ago, the school admitted any applicant who met these requirements. However, because the program recently converted from a 2-year program (four semesters) to a 1-year program (three semesters), the number of applicants has increased substantially. The dean, who teaches statistics courses, wants to raise the admissions standards by developing a method that more accurately predicts how well an applicant will perform in the MBA program. She believes that the primary determinants of success are the following:

Undergraduate grade point average (GPA)
Graduate Management Admissions Test (GMAT) score
Number of years of work experience

She randomly sampled students who completed the MBA and recorded their MBA program GPA, as well as the three variables listed here.

a. Develop a multiple regression model.
b. Test the model's validity.
c. Test to determine which of the independent variables is linearly related to MBA GPA.

APPLICATIONS in OPERATIONS MANAGEMENT

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Location Analysis

Location analysis is one function of operations management. Deciding where to locate a plant, warehouse, or retail outlet is a critical decision for any organization. A large number of variables must be considered in this decision problem. For example, a production facility must be located close to suppliers of raw resources and supplies, skilled labor, and transportation to customers. Retail outlets must consider the type and number of potential customers. In the next example, we describe an application of regression analysis to find profitable locations for a motel chain.

17.15 Xr17-15 La Quinta Motor Inns is a moderately priced chain of motor inns located across the United States. Its market is the frequent business traveler. The chain recently launched a campaign to increase market share by building new inns. The management of the chain is aware of the difficulty in choosing locations for new motels. Moreover, making decisions without adequate information often results in poor decisions. Consequently, the chain's management acquired data on 100 randomly selected inns belonging to La Quinta. The objective was to predict which sites are likely to be profitable.

To measure profitability, La Quinta used *operating margin*, which is the ratio of the sum of profit, depreciation, and interest expenses divided by total revenue. (Although occupancy is often used as a measure of a motel's success, the company statistician concluded that occupancy was too unstable, especially during economic turbulence.) The higher the operating margin, the greater the success of the inn. La Quinta defines profitable inns as those with an operating margin in excess of 50%; unprofitable inns are those with margins of less than 30%. After a discussion with a number of experienced managers, La Quinta decided to select one or two independent variables from each of the following categories: competition, market awareness, demand generators, demographics, and physical location. To measure the degree of competition, they determined the total number of motel and hotel rooms within 3 miles of each La Quinta inn. Market awareness was measured by the number of miles to the closest competing motel. Two variables that represent sources of customers were chosen. The amount of office space and college and university enrollment in the surrounding community are demand generators. Both of these are measures of economic activity. A demographic variable that describes the community is the median household income. Finally, as a measure of the physical qualities of the location La Quinta chose the distance to the downtown core. These data are stored using the following format:

Column 1: y = operating margin, in percent
Column 2: x_1 = Total number of motel and hotel rooms within 3 miles of La Quinta inn

(Continued)

Column 3: x_2 = Number of miles to closest competition

Column 4: x_3 = Office space in thousands of square feet in surrounding community

Column 5: x_4 = College and university enrollment (in thousands) in nearby university or college

Column 6: x_5 = Median household income (in $thousands) in surrounding community

Column 7: x_6 = Distance (in miles) to the downtown core

Adapted from Sheryl E. Kimes and James A. Fitzsimmons, "Selecting Profitable Hotel Sites at La Quinta Motor Inns," *INTERFACES* 20 March–April 1990, pp. 12–20.

a. Develop a regression analysis.
b. Test to determine whether there is enough evidence to infer that the model is valid.
c. Test each of the slope coefficients.
d. Interpret the coefficients.
e. Predict with 95% confidence the operating margin of a site with the following characteristics.
   There are 3,815 rooms within 3 miles of the site, the closest other hotel or motel is .9 miles away, the amount of office space is 476,000 square feet, there is one college and one university with a total enrollment of 24,500 students, the median income in the area is $35,000, and the distance to the downtown core is 11.2 miles.
f. Refer to part (e). Estimate with 95% confidence the mean operating margin of all La Quinta inns with those characteristics.

General Social Survey Exercises

17.16 GSS2008* How does the amount of education of one's parents (PAEDUC, MAEDUC) affect your education (EDUC)? Excel users note: You must delete rows with blanks.

a. Develop a regression model.
b. Test the validity of the model.
c. Test the two slope coefficients.
d. Interpret the coefficients.

17.17 GSS2008* What determines people's opinion on the following question? Should the government reduce income differences between rich and poor (EQWLTH)? (1 = government should reduce differences, 2–7 = No government action.)

a. Develop a regression analysis using demographic variables education (EDUC), age, (AGE), number of children (CHILDS), and occupation prestige score (PRESTG80).
b. Test the model's validity.
c. Test each of the slope coefficients.
d. Interpret the coefficient of determination.

17.18 GSS2008* The Nielsen ratings estimate the numbers of televisions tuned to various channels. However, television executives need more information. The General Social Survey may be the source of this information. Respondents were asked to report the number of hours per average day of television viewing (TVHOURS). Conduct a regression analysis using the following independent variables

Education (EDUC)
Age (AGE)
Hours of work (HRS)
Number of children (CHILDS)
Number of family members earning money (EARNRS)
Occupation prestige score (PRESTG80)

a. Test the model's validity.
b. Test each slope coefficient.
c. Determine the coefficient of determination and describe what it tells you.

17.19 GSS2008* What determines people's opinion on the following question? Should the government improve the standard of living of poor people (HELPPOOR)? (1 = Government act; 2–5 = People should help themselves).

a. Develop a regression analysis using demographic variables education (EDUC), age, (AGE), number of children (CHILDS), and occupation prestige score (PRESTG80.)
b. Test the model's validity.
c. Test each of the slope coefficients.
d. Interpret the coefficient of determination.

17.20 GSS2006* Xr17-20 Use the General Social Survey of 2006 to undertake a regression analysis of income (INCOME) using the following independent variables. (Because the GSS2008 file is so large we deleted the blanks and stored the variables in Xr17-20.)

Age (AGE)
Education (EDUC)
Hours of work (HRS)
Number of children (CHILDS)
Age when first child was born (AGEKDBRN)
Years with current job (YEARSJOB)
Number of days per month working extra hours (MOREDAYS)
Number of people working for company (NUMORG)

a. Test the model's validity.
b. Test each of the slope coefficients.

American National Election Survey Exercises

17.21 ANES2008* With voter turnout during presidential elections around 50%, a vital task for politicians is to try to predict who will actually vote. Develop a regression model to predict intention to vote (DEFINITE) using the following demographic independent variables:

Age (AGE)
Education (EDUC)
Income (INCOME)

a. Determine the regression equation.
b. Test the model's validity.
c. Test to determine whether there is sufficient evidence to infer a linear relationship between the dependent variable and each independent variable.

17.22 ANES2008* Does watching news on television or reading newspapers provide indicators of who will vote? Conduct a regression analysis with intention to vote (DEFINITE) as the dependent variable and the following independent variables:

Number of days in previous week watching national news on television (DAYS1)
Number of days in previous week watching local television news in afternoon or early evening (DAYS2)
Number of days in previous week watching local television news in late evening (DAYS3)
Number of days in previous week reading a daily newspaper (DAYS4)
Number of days in previous week reading a daily newspaper on the Internet (DAYS5)
Number of days in previous week listening to news on radio (DAYS6)

a. Compute the regression equation.
b. Is there enough evidence to conclude that the model is valid?
c. Test each slope coefficient.

17.3 Regression Diagnostics—II

In Section 16.7, we discussed how to determine whether the required conditions are unsatisfied. The same procedures can be used to diagnose problems in the multiple regression model. Here is a brief summary of the diagnostic procedure we described in Chapter 16.

Calculate the residuals and check the following:

1. *Is the error variable nonnormal?* Draw the histogram of the residuals.
2. *Is the error variance constant?* Plot the residuals versus the predicted values of y.

3. *Are the errors independent (time-series data)?* Plot the residuals versus the time periods.

4. *Are there observations that are inaccurate or do not belong to the target population?* Double-check the accuracy of outliers and influential observations.

If the error is nonnormal and/or the variance is not a constant, several remedies can be attempted. These are beyond the level of this book.

Outliers and influential observations are checked by examining the data in question to ensure accuracy.

Nonindependence of a time series can sometimes be detected by graphing the residuals and the time periods and looking for evidence of autocorrelation. In Section 17.4, we introduce the Durbin–Watson test, which tests for one form of autocorrelation. We will offer a corrective measure for nonindependence.

There is another problem that is applicable to multiple regression models only. *Multicollinearity* is a condition wherein the independent variables are highly correlated. Multicollincarity distorts the t-tests of the coefficients, making it difficult to determine whether any of the independent variables are linearly related to the dependent variable. It also makes interpreting the coefficients problematic. We will discuss this condition and its remedy next.

Multicollinearity

Multicollinearity (also called *collinearity* and *intercorrelation*) is a condition that exists when the independent variables are correlated with one another. The adverse effect of multicollinearity is that the estimated regression coefficients of the independent variables that are correlated tend to have large sampling errors. There are two consequences of multicollinearity. First, because the variability of the coefficients is large, the sample coefficient may be far from the actual population parameter, including the possibility that the statistic and parameter may have opposite signs. Second, when the coefficients are tested, the t-statistics will be small, which leads to the inference that there is no linear relationship between the affected independent variables and the dependent variable. In some cases, this inference will be wrong. Fortunately, multicollinearity does not affect the F-test of the analysis of variance.

Consider the chapter-opening example where we found that age and years with current employer were not statistically significant at the 5% significance level. However, if we test the coefficient of correlation between income and age and between income and years with current employer, both will be statistically significant. The Excel printout is shown below. How do we explain the apparent contradiction between the multiple regression t-tests of the coefficients of age and of years with current employer and the results of the t-test of the correlation coefficients? The answer is multicollinearity.

	A	B
1	**Correlation (Pearson)**	
2		
3	*INCOME and AGE*	
4	Pearson Coefficient of Correlation	0.1883
5	t Stat	3.2083
6	df	280
7	P(T<=t) one tail	0.0007
8	t Critical one tail	1.6503
9	P(T<=t) two tail	0.0015
10	t Critical two tail	1.9685

	A	B
1	**Correlation (Pearson)**	
2		
3	*INCOME and CUREMPYR*	
4	Pearson Coefficient of Correlation	0.1972
5	t Stat	3.3652
6	df	280
7	P(T<=t) one tail	0.0004
8	t Critical one tail	1.6503
9	P(T<=t) two tail	0.0009
10	t Critical two tail	1.9685

There is a relatively high degree of correlation between age and years at current job. This should not be surprising because it is not likely that young people will have been at the same job for many years. As a result, multicollinearity affected the results of the multiple regression t-tests so that it appears that both age and years at current job are not significantly significant when, in fact, both variables are linearly related to income.

Another problem caused by multicollinearity is the interpretation of the coefficients. We interpret the coefficients as measuring the change in the dependent variable when the corresponding independent variable increases by one unit while all the other independent variables are held constant. This interpretation may be impossible when the independent variables are highly correlated because when the independent variable increases by one unit, some or all of the other independent variables will change.

This raises two important questions for the statistics practitioner. First, how do we recognize the problem of multicollinearity when it occurs? Second, how do we avoid or correct it?

Multicollinearity exists in virtually all multiple regression models. In fact, finding two completely uncorrelated variables is rare. The problem becomes serious, however, only when two or more independent variables are highly correlated. Unfortunately, we do not have a critical value that indicates when the correlation between two independent variables is large enough to cause problems. To complicate the issue, multicollinearity also occurs when a combination of several independent variables is correlated with another independent variable or with a combination of other independent variables. Consequently, even with access to all the correlation coefficients, determining when the multicollinearity problem has reached the serious stage may be extremely difficult. A good indicator of the problem is a large F-statistic but small t-statistics.

Minimizing the effect of multicollinearity is often easier than correcting it. The statistics practitioner must try to include independent variables that are independent of each other. Another alternative is to use a stepwise regression package. *Forward stepwise regression* brings independent variables into the equation one at a time. Only if an independent variable improves the model's fit is it included. If two variables are strongly correlated, the inclusion of one of them in the model makes the second one unnecessary. *Backward stepwise regression* starts with all the independent variables included in the equation and removes variables if they are not strongly related to the dependent variable. Because the stepwise technique excludes redundant variables, it minimizes multicollinearity. Stepwise regression is presented in Chapter 18.

EXERCISES

The following exercises require a computer and software.

17.23 Compute the residuals and the predicted values for the regression analysis in Exercise 17.1.

a. Is the normality requirement violated? Explain.

b. Is the variance of the error variable constant? Explain.

17.24 Calculate the coefficients of correlation for each pair of independent variables in Exercise 17.1. What do these statistics tell you about the independent variables and the *t*-tests of the coefficients?

17.25 Refer to Exercise 17.2.
a. Determine the residuals and predicted values.
b. Does it appear that the normality requirement is violated? Explain.
c. Is the variance of the error variable constant? Explain.
d. Determine the coefficient of correlation between the assignment mark and the midterm mark. What does this statistic tell you about the *t*-tests of the coefficients?

17.26 Compute the residuals and predicted values for the regression analysis in Exercise 17.3.
a. Does it appear that the error variable is not normally distributed?
b. Is the variance of the error variable constant?
c. Is multicollinearity a problem?

17.27 Refer to Exercise 17.4. Find the coefficients of correlation of the independent variables.
a. What do these correlations tell you about the independent variables?
b. What do they say about the *t*-tests of the coefficients?

17.28 Calculate the residuals and predicted values for the regression analysis in Exercise 17.5.
a. Does the error variable appear to be normally distributed?
b. Is the variance of the error variable constant?
c. Is multicollinearity a problem?

17.29 Are the required conditions satisfied in Exercise 17.6?

17.30 Refer to Exercise 17.7.
a. Conduct an analysis of the residuals to determine whether any of the required conditions are violated.
b. Does it appear that multicollinearity is a problem?
c. Identify any observations that should be checked for accuracy.

17.31 Are the required conditions satisfied for the regression analysis in Exercise 17.8?

17.32 Determine whether the required conditions are satisfied in Exercise 17.9

17.33 Refer to Exercise 17.10. Calculate the residuals and predicted values.
a. Is the normality requirement satisfied?
b. Is the variance of the error variable constant?
c. Is multicollinearity a problem?

17.34 Determine whether there are violations of the required conditions in the regression model used in Exercise 17.11.

17.35 Determine whether the required conditions are satisfied in Exercise 17.12.

17.36 Refer to Exercise 17.13.
a. Are the required conditions satisfied?
b. Is multicollinearity a problem? If so, explain the consequences.

17.37 Refer to Exercise 17.14. Are the required conditions satisfied?

17.38 Refer to Exercise 17.15. Check the required conditions.

17.4 Regression Diagnostics—III (Time Series)

In Chapter 16, we pointed out that, in general, we check to see whether the errors are independent when the data constitute a *times series*—data gathered sequentially over a series of time periods. In Section 16.6, we described the graphical procedure for determining whether the required condition that the errors are independent is violated. We plot the residuals versus the time periods and look for patterns. In this section, we augment that procedure with the **Durbin–Watson test**.

Durbin–Watson Test

The Durbin–Watson test allows the statistics practitioner to determine whether there is evidence of **first-order autocorrelation**—a condition in which a relationship exists between consecutive residuals e_i and e_{i-1}, where i is the time period. The Durbin–Watson statistic is defined as

$$d = \frac{\sum_{i=2}^{n}(e_i - e_{i-1})^2}{\sum_{i=1}^{n} e_i^2}$$

The range of the values of d is

$$0 \le d \le 4$$

where small values of d ($d < 2$) indicate a positive first-order autocorrelation and large values of d ($d > 2$) imply a negative first-order autocorrelation. Positive first-order autocorrelation is a common occurrence in business and economic time series. It occurs when consecutive residuals tend to be similar. In that case, $(e_i - e_{i-1})^2$ will be small, producing a small value for d. Negative first-order autocorrelation occurs when consecutive residuals differ widely. For example, if positive and negative residuals generally alternate, $(e_i - e_{i-1})^2$ will be large; as a result, d will be greater than 2. Figures 17.2 and 17.3 depict positive first-order autocorrelation, whereas Figure 17.4 illustrates negative autocorrelation. Notice that in Figure 17.2 the first residual is a small number; the second residual, also a small number, is somewhat larger; and that trend continues. In Figure 17.3, the first residual is large and, in general, succeeding residuals decrease. In both figures, consecutive residuals are similar. In Figure 17.4, the first residual is a positive number and is followed by a negative residual. The remaining residuals follow this pattern (with some exceptions). Consecutive residuals are quite different.

FIGURE **17.2** **Positive First-Order Autocorrelation**

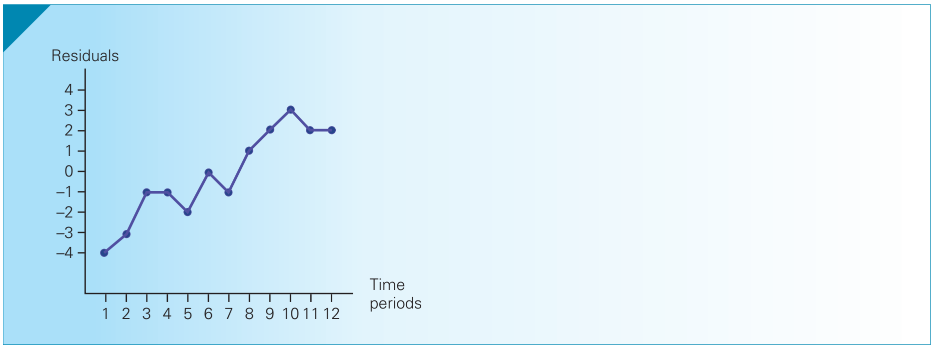


FIGURE **17.3** **Positive First-Order Autocorrelation**

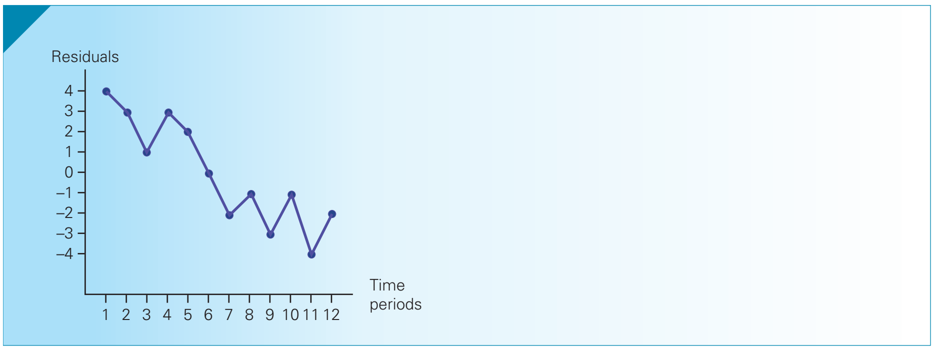


FIGURE 17.4 Negative First-Order Autocorrelation

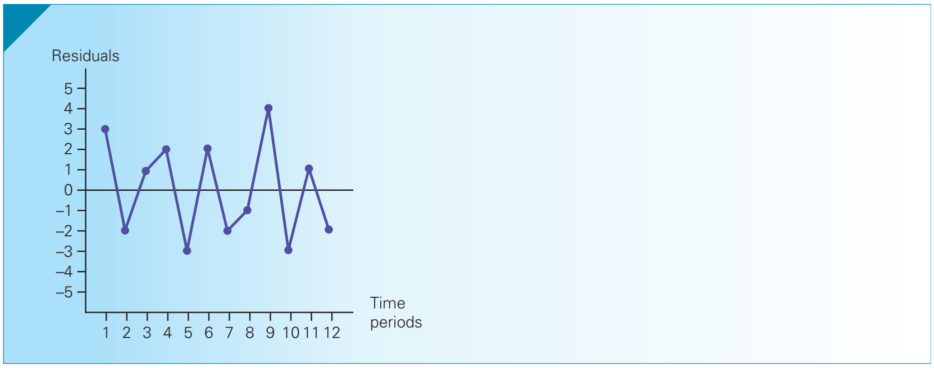


Table 8 in Appendix B is designed to test for positive first-order autocorrelation by providing values of d_L and d_U for a variety of values of n and k and for $\alpha = .01$ and .05.

The decision is made in the following way. If $d < d_L$, we conclude that there is enough evidence to show that positive first-order autocorrelation exists. If $d > d_U$, we conclude that there is not enough evidence to show that positive first-order autocorrelation exists. And if $d_L \leq d \leq d_U$, the test is inconclusive. The recommended course of action when the test is inconclusive is to continue testing with more data until a conclusive decision can be made.

For example, to test for positive first-order autocorrelation with $n = 20$, $k = 3$, and $\alpha = .05$, we test the following hypotheses:

H_0: There is no first-order autocorrelation.

H_1: There is positive first-order autocorrelation.

The decision is made as follows:

If $d < d_L = 1.00$, reject the null hypothesis in favor of the alternative hypothesis.

If $d > d_U = 1.68$, do not reject the null hypothesis.

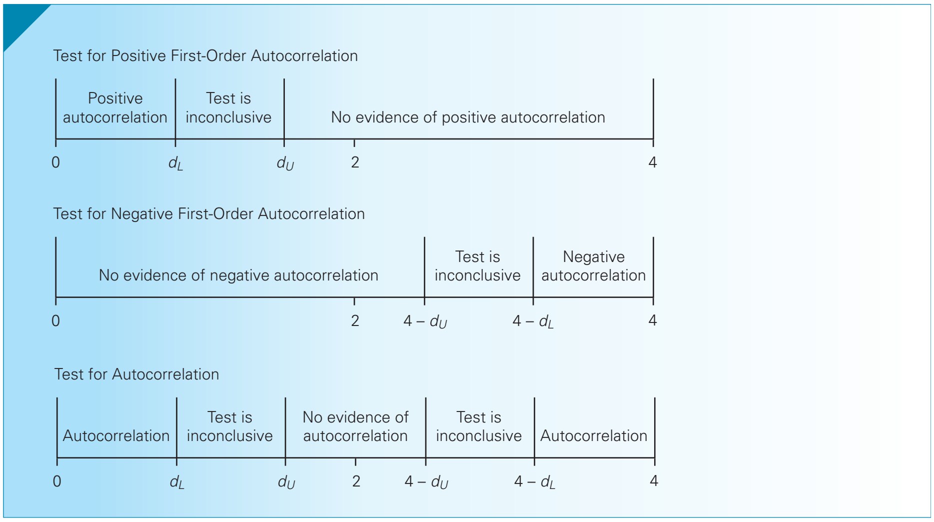
If $1.00 \leq d \leq 1.68$, the test is inconclusive.

To test for negative first-order autocorrelation, we change the critical values. If $d > 4 - d_L$, we conclude that negative first-order autocorrelation exists. If $d < 4 - d_U$, we conclude that there is not enough evidence to show that negative first-order autocorrelation exists. If $4 - d_U \leq d \leq 4 - d_L$, the test is inconclusive.

We can also test simply for first-order autocorrelation by combining the two one-tail tests. If $d < d_L$ or $d > 4 - d_L$, we conclude that autocorrelation exists. If $d_U \leq d \leq 4 - d_U$, we conclude that there is no evidence of autocorrelation. If $d_L \leq d \leq d_U$ or $4 - d_U \leq d \leq 4 - d_L$, the test is inconclusive. The significance level will be 2α (where α is the one-tail significance level). Figure 17.5 describes the range of values of d and the conclusion for each interval.

For time-series data, we add the Durbin–Watson test to our list of regression diagnostics. In other words, we determine whether the error variable is normally distributed with constant variance (as we did in Section 17.3), we identify outliers and (if our software allows it) influential observations that should be verified, and we conduct the Durbin–Watson test.

FIGURE 17.5 Durbin-Watson Test



EXAMPLE 17.1 Christmas Week Ski Lift Sales

DATA
Xm17-01

Christmas week is a critical period for most ski resorts. Because many students and adults are free from other obligations, they are able to spend several days indulging in their favorite pastime, skiing. A large proportion of gross revenue is earned during this period. A ski resort in Vermont wanted to determine the effect that weather had on its sales of lift tickets. The manager of the resort collected data on the number of lift tickets sold during Christmas week (y), the total snowfall in inches (x_1), and the average temperature in degrees Fahrenheit (x_2) for the past 20 years. Develop the multiple regression model and diagnose any violations of the required conditions.

SOLUTION

The model is

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

EXCEL

	A	B	C	D	E	F
1	SUMMARY OUTPUT					
2						
3	*Regression Statistics*					
4	Multiple R	0.3465				
5	R Square	0.1200				
6	Adjusted R Square	0.0165				
7	Standard Error	1712				
8	Observations	20				
9						
10	ANOVA					
11		*df*	*SS*	*MS*	*F*	*Significance F*
12	Regression	2	6,793,798	3,396,899	1.16	0.3373
13	Residual	17	49,807,214	2,929,836		
14	Total	19	56,601,012			
15						
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	
17	Intercept	8308	904	9.19	5.24E-08	
18	Snowfall	74.59	51.57	1.45	0.1663	
19	Temperature	–8.75	19.70	–0.44	0.6625	

MINITAB

Regression Analysis: Tickets versus Snowfall, Temperature

The regression equation is
Tickets = 8308 + 74.6 Snowfall − 8.8 Temperature

Predictor	Coef	SE Coef	T	P
Constant	8308.0	903.7	9.19	0.000
Snowfall	74.59	51.57	1.45	0.166
Temperature	−8.75	19.70	−0.44	0.662

S = 1712 R-Sq = 12.0% R-Sq(adj) = 1.7%

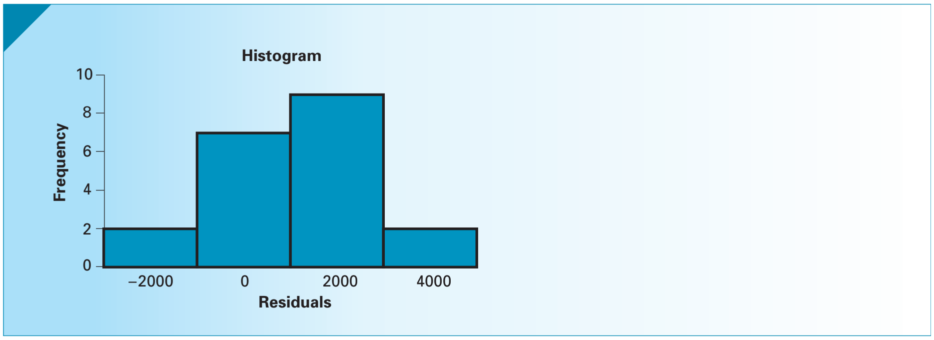
Analysis of Variance

Source	DF	SS	MS	F	P
Regression	2	6,793,798	3,396,899	1.16	0.337
Residual Error	17	49,807,214	2,929,836		
Total	19	56,601,012			

INTERPRET

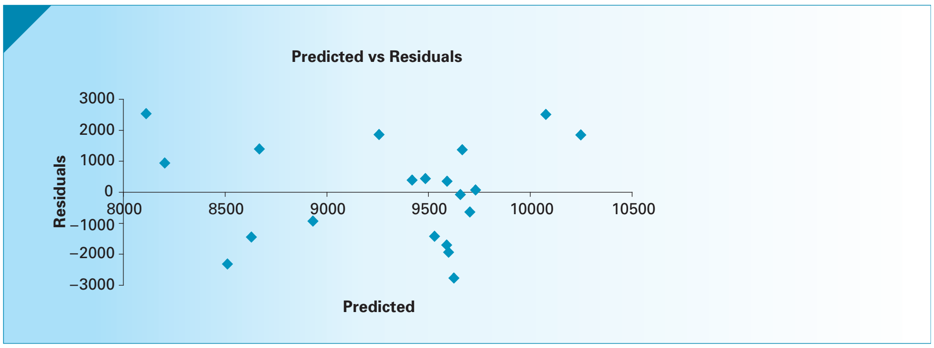
As you can see, the coefficient of determination is small ($R^2 = 12\%$) and the p-value of the F-test is .3373, both of which indicate that the model is poor. We used Excel to draw the histogram (Figure 17.6) of the residuals and plot the predicted values of y versus the residuals in Figure 17.7. Because the observations constitute a time series, we also used Excel to plot the time periods (years) versus the residuals (Figure 17.8).

FIGURE **17.6** Histogram of Residuals in Example 17.1



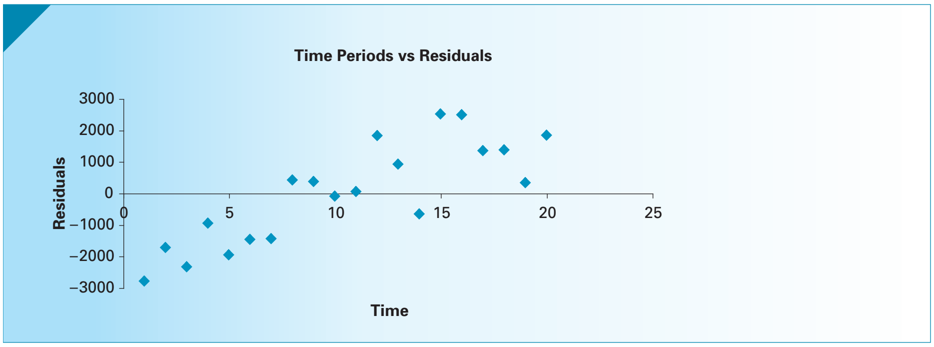
The histogram reveals that the error may be normally distributed.

FIGURE **17.7** Plot of Predicted Values versus Residuals in Example 17.1



There does not appear to be any evidence of heteroscedasticity.

FIGURE 17.8 Plot of Time Periods versus Residuals in Example 17.1



This graph reveals a serious problem. There is a strong relationship between consecutive values of the residuals, which indicates that the requirement that the errors are independent has been violated. To confirm this diagnosis, we instructed Excel and Minitab to calculate the Durbin–Watson statistic.

EXCEL

	A	B	C
1	Durbin–Watson Statistic		
2			
3	d = 0.5931		

INSTRUCTIONS

Proceed through the usual steps to conduct a regression analysis and print the residuals (see page 672). Highlight the entire list of residuals and click **Add-Ins, Data Analysis Plus**, and **Durbin–Watson Statistic**.

MINITAB

```
Durbin–Watson statistic = 0.593140
```

INSTRUCTIONS

Follow the instructions on page 673. Before clicking **OK**, click **Options . . .** and **Durbin–Watson statistic.**

The critical values are determined by noting that $n = 20$ and $k = 2$ (there are two independent variables in the model). If we wish to test for positive first-order autocorrelation with $\alpha = .05$, we find in Table 8(a) in Appendix B

$$d_L = 1.10 \quad \text{and} \quad d_U = 1.54$$

The null and alternative hypotheses are

H_0: There is no first-order autocorrelation.

H_1: There is positive first-order autocorrelation.

The rejection region is $d < d_L = 1.10$. Because $d = .59$, we reject the null hypothesis and conclude that there is enough evidence to infer that positive first-order autocorrelation exists.

Autocorrelation usually indicates that the model needs to include an independent variable that has a time-ordered effect on the dependent variable. The simplest such independent variable represents the time periods. To illustrate, we included a third independent variable that records the number of years since the year the data were gathered. Thus, $x_3 = 1, 2, \ldots, 20$. The new model is

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon$$

EXCEL

	A	B	C	D	E	F
1	SUMMARY OUTPUT					
2						
3	*Regression Statistics*					
4	Multiple R	0.8608				
5	R Square	0.7410				
6	Adjusted R Square	0.6924				
7	Standard Error	957				
8	Observations	20				
9						
10	ANOVA					
11		*df*	*SS*	*MS*	*F*	*Significance F*
12	Regression	3	41,940,217	13,980,072	15.26	0.0001
13	Residual	16	14,660,795	916,300		
14	Total	19	56,601,012			
15						
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	
17	Intercept	5966	631.3	9.45	6.00E-08	
18	Snowfall	70.18	28.85	2.43	0.0271	
19	Temperature	−9.23	11.02	−0.84	0.4145	
20	Time	230.0	37.13	6.19	1.29E-05	

MINITAB

Regression Analysis: Tickets versus Snowfall, Temperature, Time

The regression equation is
Tickets = 5966 + 70.2 Snowfall − 9.2 Temperature + 230 Time

Predictor	Coef	SE Coef	T	P
Constant	5965.6	631.3	9.45	0.000
Snowfall	70.18	28.85	2.43	0.027
Temperature	−9.23	11.02	−0.84	0.414
Time	229.97	37.13	6.19	0.000

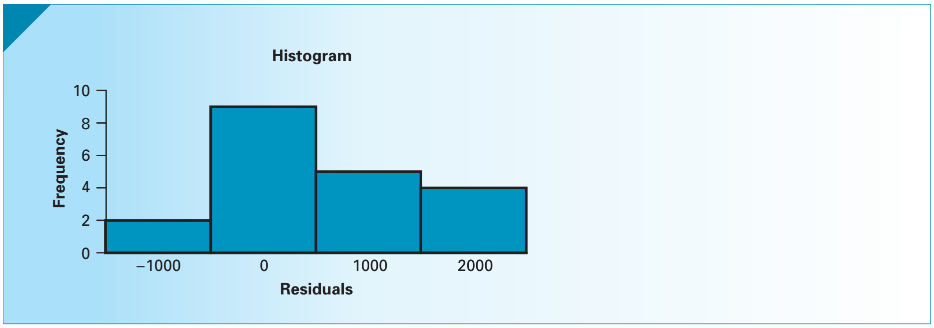
S = 957.2 R-Sq = 74.1% R-Sq(adj) = 69.2%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	3	41,940,217	13,980,072	15.26	0.000
Residual Error	16	14,660,795	916,300		
Total	19	56,601,012			

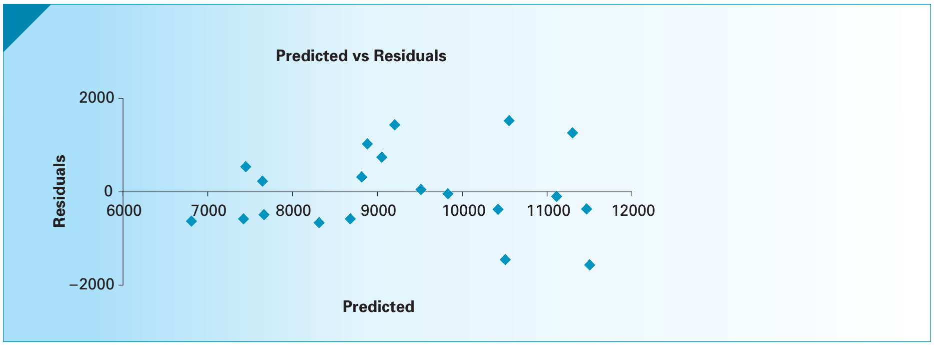
As we did before, we calculate the residuals and conduct regression diagnostics using Excel. The results are shown in Figures 17.9–17.11.

FIGURE **17.9** Histogram of Residuals in Example 17.1 (Time variable included)



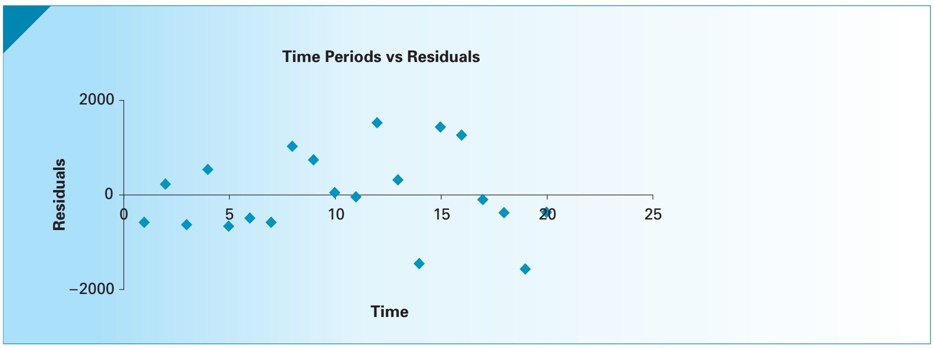
The histogram reveals that the error may be normally distributed.

FIGURE **17.10** Plot of Predicted Values versus Residuals in Example 17.1 (Time variable included)



The error variable variance appears to be constant.

FIGURE **17.11** Plot of Time Periods versus Residuals in Example 17.1 (Time variable included)



There is no sign of autocorrelation. To confirm our diagnosis, we conducted the Durbin–Watson test.

EXCEL

	A	B	C
1	**Durbin–Watson Statistic**		
2			
3	d = 1.885		

MINITAB

```
Durbin–Watson statistic = 1.88499
```

From Table 8(a) in Appendix B, we find the critical values of the Durbin–Watson test. With $k = 3$ and $n = 20$, we find

$$d_L = 1.00 \quad \text{and} \quad d_U = 1.68$$

Because $d > 1.68$, we conclude that there is not enough evidence to infer the presence of positive first-order autocorrelation.

Notice that the model is improved dramatically. The F-test tells us that the model is valid. The t-tests tell us that both the amount of snowfall and time are significantly linearly related to the number of lift tickets. This information could prove useful in advertising for the resort. For example, the resort could emphasize any recent snowfall in its advertising. If no new snow has fallen, the resort might emphasize its snow-making facilities.

Developing an Understanding of Statistical Concepts

Notice that the addition of the time variable explained a large proportion of the variation in the number of lift tickets sold; that is, the resort experienced a relatively steady increase in sales over the past 20 years. Once this variable was included in the model, the amount of snowfall became significant because it was able to explain some of the remaining variation in lift ticket sales. Without the time variable, the amount of snowfall and the temperature were unable to explain a significant proportion of the variation in ticket sales. The graph of the residuals versus the time periods and the Durbin–Watson test enabled us to identify the problem and correct it. In overcoming the autocorrelation problem, we improved the model so that we identified the amount of snowfall as an important variable in determining ticket sales. This result is quite common. Correcting a violation of a required condition will frequently improve the model.

Exercises

17.39 Perform the Durbin–Watson test at the 5% significance level to determine whether positive first-order autocorrelation exists when $d = 1.10$, $n = 25$, and $k = 3$.

17.40 Determine whether negative first-order autocorrelation exists when $d = 2.85$, $n = 50$, and $k = 5$. (Use a 1% significance level.)

17.41 Given the following information, perform the Durbin–Watson test to determine whether first-order autocorrelation exists.

$n = 25 \quad k = 5 \quad \alpha = .10 \quad d = .90$

17.42 Test the following hypotheses with $\alpha = .05$.

H_0: There is no first-order autocorrelation.
H_1: There is positive first-order autocorrelation.
$n = 50 \quad k = 2 \quad d = 1.38$

17.43 Test the following hypotheses with $\alpha = .02$.

H_0: There is no first-order autocorrelation.
H_1: There is first-order autocorrelation.
$n = 90 \quad k = 5 \quad d = 1.60$

17.44 Test the following hypotheses with $\alpha = .05$.

H_0: There is no first-order autocorrelation.
H_1: There is negative first-order autocorrelation.
$n = 33 \quad k = 4 \quad d = 2.25$

The following exercises require a computer and software.

17.45 Xr17-45 Observations of variables y, x_1, and x_2 were taken over 100 consecutive time periods.

a. Conduct a regression analysis of these data.
b. Plot the residuals versus the time periods. Describe the graph.
c. Perform the Durbin–Watson test. Is there evidence of autocorrelation? Use $\alpha = .10$.
d. If autocorrelation was detected in part (c), propose an alternative regression model to remedy the problem. Use the computer to generate the statistics associated with this model.
e. Redo parts (b) and (c). Compare the two models.

17.46 Xr17-46 Weekly sales of a company's product (y) and those of its main competitor (x) were recorded for one year.

a. Conduct a regression analysis of these data.
b. Plot the residuals versus the time periods. Does there appear to be autocorrelation?
c. Perform the Durbin–Watson test. Is there evidence of autocorrelation? Use $\alpha = .10$.
d. If autocorrelation was detected in part (c), propose an alternative regression model to remedy the problem. Use the computer to generate the statistics associated with this model.
e. Redo parts (b) and (c). Compare the two models.

17.47 Refer to Exercise 17.3. Is there evidence of positive first-order autocorrelation?

17.48 Refer to Exercise 16.99. Determine whether there is evidence of first-order autocorrelation.

17.49 Xr17-49 The manager of a tire store in Minneapolis has been concerned with the high cost of inventory. The current policy is to stock all the snow tires that are predicted to sell over the entire winter at the beginning of the season (end of October). The manager can reduce inventory costs by having suppliers deliver snow tires regularly from October to February. However, he needs to be able to predict weekly sales to avoid stockouts that will ultimately lose sales. To help develop a forecasting model, he records the number of snow tires sold weekly during the last winter and the amount of snowfall (in inches) in each week.

a. Develop a regression model and use a software package to produce the statistics.
b. Perform a complete diagnostic analysis to determine whether the required conditions are satisfied.
c. If one or more conditions are unsatisfied, attempt to remedy the problem.
d. Use whatever procedures you wish to assess how well the new model fits the data.
e. Interpret and test each of the coefficients.

Chapter Summary

The multiple regression model extends the model introduced in Chapter 16. The statistical concepts and techniques are similar to those presented in simple linear regression. We assess the model in three ways: standard error of estimate, the coefficient of determination (and the coefficient of determination adjusted for degrees of freedom), and the F-test of the analysis of variance. We can use the t-tests of the coefficients to determine whether each of the independent variables is linearly related to the dependent variable. As we did in Chapter 16, we showed how to diagnose violations of the required conditions and to identify other problems. We introduced multicollinearity and demonstrated its effect and its remedy. Finally, we presented the Durbin–Watson test to detect first-order autocorrelation.

IMPORTANT TERMS

Response surface 694
Coefficient of determination adjusted for degrees of freedom 698
Multicollinearity 714
Durbin–Watson test 716
First-order autocorrelation 716

SYMBOLS

Symbol	Pronounced	Represents
β_i	Beta sub *i* or beta *i*	Coefficient of *i*th independent variable
b_i	*b* sub *i* or *b i*	Sample coefficient

FORMULAS

Standard error of estimate

$$s_\varepsilon = \sqrt{\frac{\text{SSE}}{n-k-1}}$$

Test statistic for β_i

$$t = \frac{b_i - \beta_i}{s_{b_i}}$$

Coefficient of determination

$$R^2 = \frac{s_{xy}^2}{s_x^2 s_y^2} = 1 - \frac{\text{SSE}}{\sum (y_i - \bar{y})^2}$$

Adjusted coefficient of determination

$$\text{Adjusted } R^2 = 1 - \frac{\text{SSE}/(n-k-1)}{\sum (y_i - \bar{y})^2/(n-1)}$$

Mean square for error

$$\text{MSE} = \text{SSE}/k$$

Mean square for regression

$$\text{MSR} = \text{SSR}/(n-k-1)$$

F-statistic

$$F = \text{MSR}/\text{MSE}$$

Durbin–Watson statistic

$$d = \frac{\sum_{i=2}^{n} (e_i - e_{i-1})^2}{\sum_{i=1}^{n} e_i^2}$$

COMPUTER OUTPUT AND INSTRUCTIONS

Technique	Excel	Minitab
Regression	696	697
Prediction interval	706	706
Durbin–Watson statistic	721	721

CHAPTER EXERCISES

The following exercises require the use of a computer and statistical software. Use a 5% significance level.

17.50 Xr17-50 The agronomist referred to in Exercise 16.101 believed that the amount of rainfall as well as the amount of fertilizer used would affect the crop yield. She redid the experiment in the following way. Thirty greenhouses were rented. In each, the amount of fertilizer and the amount of water were varied. At the end of the growing season, the amount of corn was recorded.

a. Determine the sample regression line, and interpret the coefficients.
b. Do these data allow us to infer that there is a linear relationship between the amount of fertilizer and the crop yield?
c. Do these data allow us to infer that there is a linear relationship between the amount of water and the crop yield?
d. What can you say about the fit of the multiple regression model?

e. Is it reasonable to believe that the error variable is normally distributed with constant variance?
f. Predict the crop yield when 100 kilograms of fertilizer and 1,000 liters of water are applied. Use a confidence level of 95%.

17.51 Xr16-12* Exercise 16.12 addressed the problem of determining the relationship between the price of apartment buildings and number of square feet. Hoping to improve the predictive capability of the model the real estate agent also recorded the number of apartments, the age, and the number of floors.
a. Calculate the regression equation.
b. Is the model valid?
c. Compare your answer with that of Exercise 16.12.

17.52 Xr16-16* In Exercise 16.16, a statistics practitioner examined the relationship between office rents and the city's office vacancy rate. The model appears to be quite poor. It was decided to add another variable that measures the state of the economy. The city's unemployment rate was chosen for this purpose.
a. Determine the regression equation.
b. Determine the coefficient of determination and describe what this value means.
c. Test the model's validity in explaining office rent.
d. Determine which of the two independent variables is linearly related to rents.
e. Determine whether the error is normally distributed with a constant variance.
f. Determine whether there is evidence of autocorrelation.
g. Predict with 95% confidence the office rent in a city whose vacancy rate is 10% and whose unemployment rate is 7%.

CASE 17.1 An Analysis of Mutual Fund Managers, Part 1*

© Vicki Beaver

DATA
C17-01a

There are thousands of mutual funds available (see page 181 for a brief introduction to mutual funds). There is no shortage of sources of information about them. Newspapers regularly report the value of each unit, mutual fund companies and brokers advertise extensively, and there are books on the subject. Many of the advertisements imply that individuals should invest in the advertiser's mutual fund because it has performed well in the past. Unfortunately, there is little evidence to infer that past performance is a predictor of the future. However, it may be possible to acquire useful information by examining the managers of mutual funds. Several researchers have studied the issue. One project gathered data concerning the performance of 2,029 funds.

The performance of each fund was measured by its risk-adjusted excess return, which is the difference between the return on investment of the fund and a return that is considered a standard. The standard is based on a variety of variables, including the risk-free rate.

Four variables describe the fund manager: age, tenure (how many years the manager has been in charge), whether the manager had an MBA (1 = yes, 0 = no), and a measure of the quality of the manager's education [the average Scholastic Achievement Test (SAT) score of students at the university where the manager received his or her undergraduate degree].

Conduct an analysis of the data. Discuss how the average SAT score of the manager's alma mater, whether he or she has an MBA, and his or her age and tenure are related to the performance of the fund.

*This case is based on "Are Some Mutual Fund Managers Better Than Others? Cross-Sectional Patterns in Behavior and Performance," Judith Chevalier and Glenn Ellison, Working Paper 5852, National Bureau of Economic Research.

CASE 17.2 An Analysis of Mutual Fund Managers, Part 2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DATA
C17-02a
C17-02b

In addition to analyzing the relationship between the managers' characteristic and the performance of the fund, researchers wanted to determine whether the same characteristics are related to the behavior of the fund. In particular, they wanted to know whether the risk of the fund and its management expense ratio (MER) were related to the manager's age, tenure, university SAT score, and whether he or she had an MBA.

In Section 4.6, we introduced the market model wherein we measure the systematic risk of stocks by the stock's beta. The beta of a portfolio is the average of the betas of the stocks that make up the portfolio. File C17-02a stores the same managers' characteristics as those in file C17-01. However, the first column contains the betas of the mutual funds.

To analyze the management expense ratios, it was decided to include a measure of the size of the fund. The logarithm of the funds' assets (in \$millions) was recorded with the MER. These data are stored in file C17-02b.

Analyze both sets of data and write a brief report of your findings.

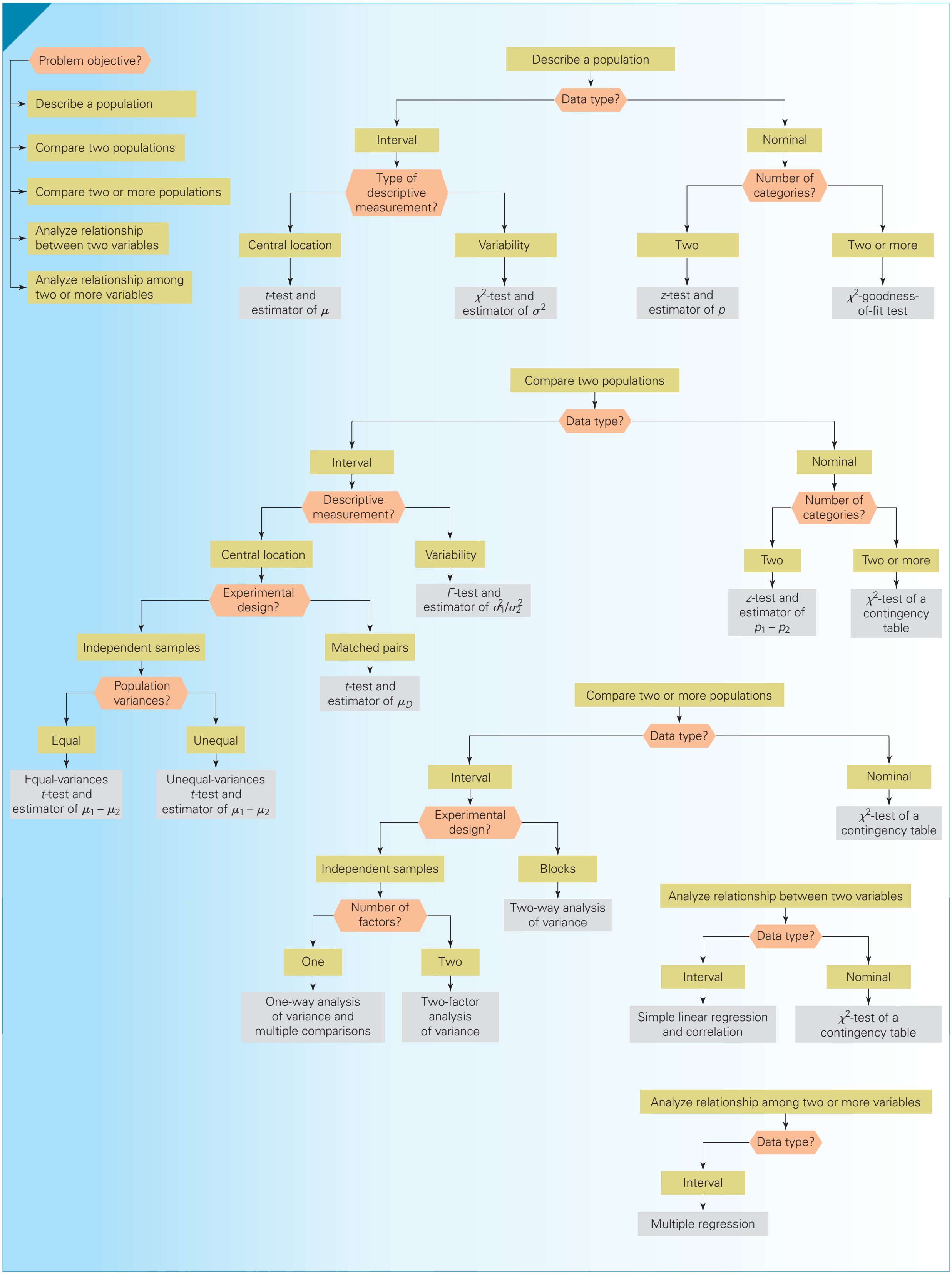
APPENDIX 17 / REVIEW OF CHAPTERS 12 TO 17

Table A17.1 presents a list of inferential methods presented thus far, and Figure A17.1 depicts a flowchart designed to help students identify the correct statistical technique.

TABLE **A17.1** Summary of Statistical Techniques in Chapters 12 to 17

t-test of μ
Estimator of μ (including estimator of $N\mu$)
χ^2 test of σ^2
Estimator of σ^2
z-test of p
Estimator of p (including estimator of Np)
Equal-variances t-test of $\mu_1 - \mu_2$
Equal-variances estimator of $\mu_1 - \mu_2$
Unequal-variances t-test of $\mu_1 - \mu_2$
Unequal-variances estimator of $\mu_1 - \mu_2$
t-test of μ_D
Estimator of μ_D
F-test of σ_1^2/σ_2^2
Estimator of σ_1^2/σ_2^2
z-test of $p_1 - p_2$ (Case 1)
z-test of $p_1 - p_2$ (Case 2)
Estimator of $p_1 - p_2$
One-way analysis of variance (including multiple comparisons)
Two-way (randomized blocks) analysis of variance
Two-factor analysis of variance
χ^2-goodness-of-fit test
χ^2-test of a contingency table
Simple linear regression and correlation (including t-tests of β_1 and ρ, and prediction and confidence intervals)
Multiple regression (including t-tests of β_i, F-test, and prediction and confidence intervals)

FIGURE **A17.1** Flowchart of Techniques in Chapters 12 to 17



EXERCISES

A17.1 XrA17-01 Garlic has long been considered a remedy to ward off the common cold. A British researcher organized an experiment to see if this generally held belief is true. A random sample of 146 volunteers was recruited. Half the sample took one capsule of an allicin-containing garlic supplement each day. The others took a placebo. The results for each volunteer after the winter months were recorded in the following way.

Column
1. Identification number
2. 1 = allicin-containing capsule; 2 = placebo
3. Suffered a cold (1 = no, 2 = yes)
4. If individual caught a cold, the number of days until recovery (999 was recorded if no cold)

a. Can the researcher conclude that garlic does help prevent colds?
b. Does garlic reduce the number of days until recovery if a cold was caught?

A17.2 XrA17-02 Because shelf space is a limited resource for a retail store, product selection, shelf-space allocation, and shelf-space placement decisions must be made according to a careful analysis of profitability and inventory turnover. The manager of a chain of variety stores wishes to see whether shelf location affects the sales of a canned soup. She believes that placing the product at eye level will result in greater sales than will placing the product on a lower shelf. She observed the number of sales of the product in 40 different stores. Sales were observed over 2 weeks, with product placement at eye level one week and on a lower shelf the other week. Can we conclude that placement of the product at eye level significantly increases sales?

A17.3 XrA17-03 In an effort to explain the results of Exercise A15.9, a researcher recorded the distances for the random sample of British and American golf courses. Can we infer that British courses are shorter than American courses?

A17.4 XrA17-04 It is generally assumed that alcohol consumption tends to make drinkers more impulsive. However, a recent study in the journal *Alcohol and Alcoholism* may contradict this assumption. The study took a random sample of 76 male undergraduate students and divided them into three groups. One group remained sober; the second group was given flavored drinks with not enough alcohol to intoxicate; and the students in third group were intoxicated. Each student was offered a chance of receiving $15 at the end of the session or double that amount later. The results were recorded using the following format:

Column 1: Group number
Column 2: Code 1 = chose $15, 2 = chose $30 later

Do the data allow us to infer that there is a relationship between the choices students make and their level of intoxication?

A17.5 XrA17-05 Refer to Exercise 13.35. The executive did a further analysis by taking another random sample. This time she tracked the number of customers who have had an accident in the last 5 years. For each she recorded the total amount of repairs and the credit score. Do these data allow the executive to conclude that the higher the credit score the lower the cost of repairs will be?

A17.6 XrA17-06 The U.S. National Endowment for the Arts conducts surveys of American adults to determine, among other things, their participation in various arts activities. A recent survey asked a random sample of American adults whether they participate in photography. The responses are 1 = yes and 2 = no. There were 205.8 million American adults. Estimate with 95% confidence the number of American adults are participate in photography. (Adapted from the *Statistical Abstract of the United States, 2006*, Table 1228.)

A17.7 XrA17-07 Mouth-to-mouth resuscitation has long been considered better than chest compression for people who have suffered a heart attack. To determine if this indeed is the better way, Japanese researchers analyzed 4,068 adult patients who had cardiac arrest witnessed by bystanders. Of those, 439 received only chest compressions from bystanders and 712 received conventional CPR compressions and breaths. The results for each group was recorded where 1 = did not survive with good neurological function and 2 = did survive with good neurological function. What conclusions can be drawn from these data?

A17.8 XrA17-08 Refer to Exercise A15.6. The financial analyst undertook another project wherein respondents were also asked the age of the head of the household. The choices are

1. Younger than 25
2. 25 to 34
3. 35 to 44
4. 45 to 54
5. 55 to 64
6. 65 and older

The responses to questions about ownership of mutual funds is No = 1 and Yes = 2. Do these data allow us to infer that the age of the head of the household is related to whether he or she owns mutual funds? (*Source:* Adapted from the *Statistical Abstract of the United States, 2006*, Table 1200.)

A17.9 XrA17-09 Over one decade (1995–2005), the number of hip and knee replacement surgeries increased by 87%. Because the costs of hip and knee replacements are so expensive, private health-insurance and government-operated health-care plans have become more concerned. To get more information, random samples of people who had hip replacements in 1995 and in 2005 were drawn. From the files, the ages of the patients were recorded. Is there enough evidence to infer that the ages of people who require hip replacements are getting smaller? (*Source:* Canadian Joint Replacement Registry.)

A17.10 XrA17-10 Refer to Exercise A17.9. Weight is a major factor that determines whether a person will need a hip or knee replacement and at what age. To learn more about the topic, a medical researcher randomly sampled individuals who had hip replacement (code = 1) and knee replacement (code = 2) and one of the following categories:

1. Underweight 2. Normal range
3. Overweight but not obese 4. Obese

Do the data allow the researcher to conclude that weight and the joint needing replacement are related?

A17.11 XrA17-11 Television shows with large amounts of sex or violence tend to attract more viewers. Advertisers want large audiences, but they also want viewers to remember the brand names of their products. A study was undertaken to determine the effect that shows with sex and violence have on their viewers. A random sample of 328 adults was divided into three groups. Group 1 watched violent programs, group 2 watched sexually explicit shows, and group 3 watched neutral shows. The researchers spliced nine 30-second commercials for a wide range of products. After the show, the subjects were quizzed to see if they could recall the brand name of the products. They were also asked to name the brands 24 hours later. The number of correct answers was recorded. Conduct a test to determine whether differences exist between the three groups of viewers and which type of program does best in brand recall. Results were published in the *Journal of Applied Psychology* (*National Post*, August 16, 2004).

A17.12 XrA17-12 In an effort to explain to customers why their electricity bills have been so high lately, and how customers could save money by reducing the thermostat settings on both space heaters and water heaters, a public utility commission has collected total kilowatt consumption figures for last year's winter months, as well as thermostat settings on space and water heaters, for 100 homes.

a. Determine the regression equation.
b. Determine the coefficient of determination and describe what it tells you.
c. Test the validity of the model.
d. Find the 95% interval of the electricity consumption of a house whose space heater thermostat is set at 70 and whose water heater thermostat is set at 130.
e. Calculate the 95% interval of the average electricity consumption for houses whose space heater thermostat is set at 70 and whose water heater thermostat is set at 130.

A17.13 XrA17-13 An economist wanted to learn more about total compensation packages. She conducted a survey of 858 workers and asked all to report their hourly wages or salaries, their total benefits, and whether the companies they worked for produced goods or services. Determine whether differences exist between goods-producing and services-producing firms in terms of hourly wages and total benefits. (Adapted from the *Statistical Abstract of the United States, 2006*, Table 637.)

A17.14 XrA17-14 Professional athletes in North America are paid very well for their ability to play games that amateurs play for fun. To determine the factors that influence a team to pay a hockey player's salary, an MBA student randomly selected 50 hockey players who played in the 1992–1993 and 1993–1994 seasons. He recorded their salaries at the end of the 1993–1994 season as well as a number of performance measures in the previous two seasons. The following data were recorded.

Columns 1 and 2: Games played in 1992–1993 and 1993–1994
Columns 3 and 4: Goals scored in 1992–1993 and 1993–1994
Columns 5 and 6: Assists recorded in 1992–1993 and 1993–1994
Columns 7 and 8: Plus/minus score in 1992–1993 and 1993–1994
Columns 9 and 10: Penalty minutes served in 1992–1993 and 1993–1994
Column 11: Salary in U.S. dollars

(Plus/minus is the number of goals scored by the player's team minus the number of goals scored by the opposing team while the player is on the ice.) Develop a model that analyzes the relationship between salary and the performance measures. Describe your findings. (The author wishes to thank Gordon Barnett for writing this exercise.)

A17.15 XrA17-15 The risks associated with smoking are well known. Virtually all physicals recommend that their patients quit. This raises the question, What are the risks for people who quit smoking compared to continuing smokers and those who have never smoked? In a study described in the *Journal of Internal Medicine* [Feb. 2004, 255(2): 266–272], researchers took samples of each of the following groups.

Group 1: Never smokers
Group 2: Continuing smokers
Group 3: Smokers who quit

At the beginning of the 10-year research project, there were 238 people who had never smoked and 155 smokers. Over the year, 39 smokers quit. The weight gain, increase in systolic (SBP) blood pressure, and increase in diastolic (DBP) blood pressure were measured and recorded. Determine whether differences exist between the three groups in terms of weight gain, increases in systolic blood pressure, and increases in diastolic blood pressure and which groups differ.

A17.16 XrA17-16 A survey was conducted among Canadian farmers, who were each asked to report the number of acres in his or her farm. There were a total of 229,373 farms in Canada in 2006 (*Source:* Statistics Canada). Estimate with 95% confidence the total amount of area (in acres) that was farmed in Canada in 2006.

GENERAL SOCIAL SURVEY EXERCISES

A17.17 GSS2008* Do the data allow us to infer that households with at least one union member (UNION: 1 = Respondent belongs, 2 = Spouse belongs, 3 = Both belong, 4 = Neither belong) differ from households with no union members with respect to their position on whether the government should do more or less to solve the country's problems (HELPNOT: 1 = Government should do more; 2, 3, 4, 5 = Government does too much)?

A17.18 GSS2008* Estimate with 95% confidence the proportion of Americans who are divorced (DIVORCE: 1 = Yes, 2 = No).

A17.19 GSS2008* Is there sufficient evidence to conclude that people who have taken college-level science courses (COLSCINM: 1 = Yes, 2 = No) are more likely to answer the following question correctly (ODDS1): A doctor tells a couple that there is one chance in four that their child will have an inherited disease. Does this mean that if the first child has the illness, the next three will not? 1 = Yes, 2 = No. Correct answer: No.

A17.20 GSS2008* Estimate with 95% confidence the mean job tenure (CUREMPYR).

A17.21 GSS2008* Is there sufficient evidence to infer that the three groups of conservatives (POLVIEWS: 5 = slightly conservative, 6 = conservative, 7 = extremely conservative) differ in support for capital punishment (CAPPUN: 1 = Favor, 2 = Oppose)?

A17.22 GSS2008* Do older people watch more television? To answer the question, analyze the relationship between age (AGE) and the amount of time spent watching television (TVHOURS).

A17.23 GSS2008* If a person has a higher income, is he or she more likely to believe that the government should do more to solve the country's problems? Conduct a test of the relationship between income (INCOME) and HELPNOT to answer the question.

A17.24 GSS2006* Is there enough evidence to conclude that people with vision problems (DISABLD2: Do you have a vision problem that prevents you from reading a newspaper even when wearing glasses or contacts: 1 = Yes, 2 = No) are more likely to believe that it is the government's responsibility to help pay for doctor and hospital bills (1 = Government should help; 2, 3 ,4, 5 = People should help themselves)?

A17.25 GSS2008* Do the children of men with prestigious occupations have prestigious occupations themselves? Conduct a test to determine whether there is a positive linear relationship between PRESTG80 and PAPRES80.

A17.26 GSS2008* Is there a relationship between the number of hours a husband works and the number of hours his wife works? Answer the question by conducting a test of the two variables (HRS and SPHRS).

American National Election Survey Exercises

A17.27 ANES2008* Are older people more likely to vote? One way to answer this question is to conduct a test to determine whether there is enough evidence to conclude that age (AGE) and intention to vote (DEFINITE) are positively related.

A17.28 ANES2008* Is the amount of time a person watches television news per day affected by his or her education? Test to determine whether TIME2 and EDUC are linearly related.

CASE A17.1 Testing a More Effective Device to Keep Arteries Open

DATA CA17-01

A stent is a metal mesh cylinder that holds a coronary artery open after a blockage has been removed. However, in many patients the stents, which are made from bare metal, become blocked as well. One cause of the reoccurrence of blockages is the body's rejection of the foreign object. In a study published in the *New England Journal of Medicine* (January 2004), a new polymer-based stent was tested. After insertion, the new stents slowly release a drug (paclitaxel) to prevent the rejection problem. A sample was recruited of 1,314 patients who were receiving a stent in a single, previously untreated coronary artery blockage. A total of 652 were randomly assigned to receive a bare-metal stent, and 662 to receive an identical-looking polymer drug-releasing stent. The results were recorded in the following way:

Column 1: Patient identification number

Column 2: Stent type (1 = bare metal, 2 = polymer based)

Column 3: Reference-vessel diameter (the diameter of the artery that is blocked, in millimeters)

Column 4: Lesion length (the length of the blockage, in millimeters)

Reference-vessel diameters and lesion lengths were measured before the stents were inserted.

The following data were recorded 12 months after the stents were inserted.

Column 5: Blockage reoccurrence after 9 months (2 = yes, 1 = no)

Column 6: Blockage that needed to be reopened (2 = yes, 1 = no)

Column 7: Death from cardiac causes (2 = yes, 1 = no)

Column 8: Stroke caused by stent (2 = yes, 1 = no)

a. Using the variables stored in columns 3 through 8, determine whether there is enough evidence to infer that the polymer-based stent is superior to the bare-metal stent.

b. As a laboratory researcher in the pharmaceutical company write a report that describes this experiment and the results.

DATA
CA17-02

CASE A17.2 Automobile Crashes and the Ages of Drivers*

Setting premiums for insurance is a complex task. If the premium is too high, the insurance company will lose customers; if it is too low, the company will lose money. Statistics plays a critical role in almost all aspects of the insurance business. As part of a statistical analysis, an insurance company in Florida studied the relationship between the severity of car crashes and the ages of the drivers. A random sample of crashes in 2002 in the state of Florida was drawn. For each crash, the age category of the driver was recorded as well as whether the driver was injured or killed. The data were stored as follows:

Column 1: Crash number

Column 2: Age category

1. 5 to 34
2. 35 to 44
3. 45 to 54
4. 55 to 64
5. 65 and over

Column 3: Medical status of driver

1 = Uninjured
2 = Injured (but not killed)
3 = Killed

a. Is there enough evidence to conclude that age and medical status of the driver in car crashes are related?
b. Estimate with 95% confidence the proportion of all Florida drivers in crashes in 2002 who were uninjured.

*Adapted from Florida Department of Highway Safety and Vehicles as reported in the *Miami Herald* January 1, 2004, p. 2B.

APPENDIX A

DATA FILE SAMPLE STATISTICS

Chapter 10

10.30 $\bar{x} = 252.38$
10.31 $\bar{x} = 1{,}810.16$
10.32 $\bar{x} = 12.10$
10.33 $\bar{x} = 10.21$
10.34 $\bar{x} = .510$
10.35 $\bar{x} = 26.81$
10.36 $\bar{x} = 19.28$
10.37 $\bar{x} = 15.00$
10.38 $\bar{x} = 585{,}063$
10.39 $\bar{x} = 14.98$
10.40 $\bar{x} = 27.19$

Chapter 11

11.35 $\bar{x} = 5{,}065$
11.36 $\bar{x} = 29{,}120$
11.37 $\bar{x} = 569$
11.38 $\bar{x} = 19.13$
11.39 $\bar{x} = -1.20$
11.40 $\bar{x} = 55.8$
11.41 $\bar{x} = 5.04$
11.42 $\bar{x} = 19.39$
11.43 $\bar{x} = 105.7$
11.44 $\bar{x} = 4.84$
11.45 $\bar{x} = 5.64$
11.46 $\bar{x} = 29.92$
11.47 $\bar{x} = 231.56$

Chapter 12

12.31 $\bar{x} = 7.15$, $s = 1.65$, $n = 200$
12.32 $\bar{x} = 4.66$, $s = 2.37$, $n = 240$
12.33 $\bar{x} = 17.00$, $s = 4.31$, $n = 162$
12.34 $\bar{x} = 15{,}137$, $s = 5{,}263$, $n = 306$
12.35 $\bar{x} = 59.04$, $s = 20.62$, $n = 122$
12.36 $\bar{x} = 2.67$, $s = 2.50$, $n = 188$
12.37 $\bar{x} = 34.49$, $s = 7.82$, $n = 900$
12.38 $\bar{x} = 422.36$, $s = 122.77$, $n = 176$
12.39 $\bar{x} = 13.94$, $s = 2.16$, $n = 212$
12.40 $\bar{x} = 15.27$, $s = 5.72$, $n = 116$
12.41 $\bar{x} = 3.79$, $s = 4.25$, $n = 564$
12.42 $\bar{x} = 89.27$, $s = 17.30$, $n = 85$
12.43 $\bar{x} = 15.02$, $s = 8.31$, $n = 83$
12.44 $\bar{x} = 96{,}100$, $s = 34{,}468$, $n = 473$
12.45 $\bar{x} = 1.507$, $s = .640$, $n = 473$
12.63 $s^2 = 270.58$, $n = 25$
12.64 $s^2 = 22.56$, $n = 245$
12.65 $s^2 = 4.72$, $n = 90$
12.66 $s^2 = 174.47$, $n = 100$
12.67 $s^2 = 19.68$, $n = 25$
12.91 $n(1) = 466$, $n(2) = 55$
12.93 $n(1) = 140$, $n(2) = 59$, $n(3) = 39$, $n(4) = 106$, $n(5) = 47$
12.94 $n(1) = 153$, $n(2) = 24$
12.95 $n(1) = 92$, $n(2) = 28$
12.96 $n(1) = 603$, $n(2) = 905$
12.97 $n(1) = 92$, $n(2) = 334$
12.98 $n(1) = 57$, $n(2) = 35$, $n(3) = 4$, $n(4) = 4$
12.100 $n(1) = 245$, $n(2) = 745$, $n(3) = 238$, $n(4) = 1319$, $n(5) = 2453$
12.101 $n(1) = 786$, $n(2) = 254$
12.102 $n(1) = 518$, $n(2) = 132$
12.124 $n(1) = 81$, $n(2) = 47$, $n(3) = 167$, $n(4) = 146$, $n(5) = 34$
12.125 $n(1) = 63$, $n(2) = 125$, $n(3) = 45$, $n(4) = 87$
12.126 $n(1) = 418$, $n(2) = 536$, $n(3) = 882$
12.127 $n(1) = 290$, $n(2) = 35$
12.128 $n(1) = 72$, $n(2) = 77$, $n(3) = 37$, $n(4) = 50$, $n(5) = 176$
12.129 $n(1) = 289$, $n(2) = 51$

Chapter 13

13.17 Tastee: $\bar{x}_1 = 36.93$, $s_1 = 4.23$, $n_1 = 15$
Competitor: $\bar{x}_2 = 31.36$, $s_2 = 3.35$, $n_2 = 25$
13.18 Oat bran: $\bar{x}_1 = 10.01$, $s_1 = 4.43$, $n_1 = 120$
Other: $\bar{x}_2 = 9.12$, $s_2 = 4.45$, $n_2 = 120$
13.19 18-to-34: $\bar{x}_1 = 58.99$, $s_1 = 30.77$, $n_1 = 250$
35-to-50: $\bar{x}_2 = 52.96$, $s_2 = 43.32$, $n_2 = 250$
13.20 2 yrs ago: $\bar{x}_1 = 59.81$, $s_1 = 7.02$, $n_1 = 125$
This year: $\bar{x}_2 = 57.40$, $s_2 = 6.99$, $n_2 = 159$
13.21 Male: $\bar{x}_1 = 10.23$, $s_1 = 2.87$, $n_1 = 100$
Female: $\bar{x}_2 = 9.66$, $s_2 = 2.90$, $n_2 = 100$
13.22 A: $\bar{x}_1 = 115.50$, $s_1 = 21.69$, $n_1 = 30$
B: $\bar{x}_2 = 110.20$, $s_2 = 21.93$, $n_2 = 30$
13.23 Men: $\bar{x}_1 = 5.56$, $s_1 = 5.36$, $n_1 = 306$
Women: $\bar{x}_2 = 5.49$, $s_2 = 5.58$, $n_2 = 290$
13.24 A: $\bar{x}_1 = 70.42$, $s_1 = 20.54$, $n_1 = 24$
B: $\bar{x}_2 = 56.44$, $s_2 = 9.03$, $n_2 = 16$
13.25 Successful: $\bar{x}_1 = 5.02$, $s_1 = 1.39$, $n_1 = 200$
Unsuccessful: $\bar{x}_2 = 7.80$, $s_2 = 3.09$, $n_2 = 200$
13.26 Phone: $\bar{x}_1 = .646$, $s_1 = .045$, $n_1 = 125$
Not: $\bar{x}_2 = .601$, $s_2 = .053$, $n_2 = 145$
13.27 Chitchat: $\bar{x}_1 = .654$, $s_1 = .048$, $n_1 = 95$
Political: $\bar{x}_2 = .662$, $s_2 = .045$, $n_2 = 90$
13.28 Planner: $\bar{x}_1 = 6.18$, $s_1 = 1.59$, $n_1 = 64$
Broker: $\bar{x}_2 = 5.94$, $s_2 = 1.61$, $n_2 = 81$
13.29 Textbook: $\bar{x}_1 = 63.71$, $s_1 = 5.90$, $n_1 = 173$
No book: $\bar{x}_2 = 66.80$, $s_2 = 6.85$, $n_2 = 202$
13.30 Wendy's: $\bar{x}_1 = 149.85$, $s_1 = 21.82$, $n_1 = 213$
McDonald's: $\bar{x}_2 = 154.43$, $s_2 = 23.64$, $n_2 = 202$
13.31 Men: $\bar{x}_1 = 488.4$, $s_1 = 19.6$, $n_1 = 124$
Women: $\bar{x}_2 = 498.1$, $s_2 = 21.9$, $n_2 = 187$
13.32 Applied: $\bar{x}_1 = 130.93$, $s_1 = 31.99$, $n_1 = 100$
Contacted: $\bar{x}_2 = 126.14$, $s_2 = 26.00$, $n_2 = 100$
13.33 New: $\bar{x}_1 = 73.60$, $s_1 = 15.60$, $n_1 = 20$
Existing: $\bar{x}_2 = 69.20$, $s_2 = 15.06$, $n_2 = 20$
13.34 Fixed: $\bar{x}_1 = 60{,}245$, $s_1 = 10{,}506$, $n_1 = 90$
Commission: $\bar{x}_2 = 63{,}563$, $s_2 = 10{,}755$, $n_2 = 90$
13.35 Accident: $\bar{x}_1 = 633.97$, $s_1 = 49.45$, $n_1 = 93$
No accident: $\bar{x}_2 = 661.86$, $s_2 = 52.69$, $n_2 = 338$
13.36 Cork: $\bar{x}_1 = 14.20$, $s_1 = 2.84$, $n_1 = 130$
Metal: $\bar{x}_2 = 11.27$, $s_2 = 4.42$, $n_2 = 130$
13.37 Before: $\bar{x}_1 = 496.9$, $s_1 = 73.8$, $n_1 = 355$
After: $\bar{x}_2 = 511.3$, $s_2 = 69.1$, $n_2 = 288$
13.57 $D = X$[This year] $- X$[5 years ago]: $\bar{x}_D = 12.4$, $s_D = 99.1$, $n_D = 150$
13.58 $D = X$[Waiter] $- X$[Waitress]: $\bar{x}_D = -1.16$, $s_D = 2.22$, $n_D = 50$
13.59 $D = X$[This year] $- X$[Last year]: $\bar{x}_D = 19.75$, $n_D = 30.63$, $n_D = 40$
13.60 $D = X$[Uninsulated] $- X$[Insulated]: $\bar{x}_D = 57.40$, $s_D = 13.14$, $n_D = 15$
13.61 $D = X$[Men] $- X$[Women]: $\bar{x}_D = -42.94$, $s_D = 317.16$, $n_D = 45$
13.62 $D = X$[Last year] $- X$[Previous year]: $\bar{x}_D = -183.35$, $s_D = 1{,}568.94$, $n_D = 170$
13.63 $D = X$[This year] $- X$[Last year]: $\bar{x}_D = .0422$, $s_D = .1634$, $n_D = 38$
13.64 $D = X$[Company 1] $- X$[Company 2]: $\bar{x}_D = 520.85$, $s_D = 1{,}854.92$, $n_D = 55$

13.65 $D = X$[New] $- X$[Existing]:
$\bar{x}_D = 4.55$, $s_D = 7.22$, $n_D = 20$

13.67 $D = X$[Finance] $- X$[Marketing]:
$\bar{x}_D = 4{,}587$, $s_D = 22{,}851$, $n_D = 25$

13.69 **a.** $D = X$[After] $- X$[Before]:
$\bar{x}_D = -.10$, $s_D = 1.95$, $n_D = 42$

b. $D = X$[After] $- X$[Before]:
$\bar{x}_D = 1.24$, $s_D = 2.83$, $n_D = 98$

13.81 Week 1: $s_1^2 = 19.38$, $n_1 = 100$
Week 2: $s_2^2 = 12.70$, $n_2 = 100$

13.82 A: $s_1^2 = 41{,}309$, $n_1 = 100$
B: $s_2^2 = 19{,}850$, $n_2 = 100$

13.83 Portfolio 1: $s_1^2 = .0261$, $n_1 = 52$
Portfolio 2: $s_2^2 = .0875$, $n_2 = 52$

13.84 Teller 1: $s_1^2 = 3.35$, $n_1 = 100$
Teller 2: $s_2^2 = 10.95$, $n_2 = 100$

13.101 Lexus: $n(1) = 33$, $n(2) = 317$
Acura: $n(1) = 33$, $n(2) = 261$

13.102 Smokers: $n_1(1) = 28$; $n_1(2) = 10$
Nonsmokers: $n_2(1) = 150$;
$n_2(2) = 12$

13.103 This year: $n_1(1) = 306$; $n_1(2) = 171$
10 years ago: $n_2(1) = 304$;
$n_2(2) = 158$

13.104 Canada: $n_1(1) = 230$; $n_1(2) = 215$
U.S.: $n_2(1) = 165$; $n_2(2) = 275$

13.105 A: $n_1(1) = 189$; $n_1(2) = 11$
B: $n_2(1) = 178$; $n_2(2) = 22$

13.106 High school: $n_1(1) = 27$;
$n_1(2) = 167$
Postsecondary: $n_2(1) = 17$;
$n_2(2) = 63$

13.107 2008: $n_1(1) = 63$; $n_1(2) = 41$
2011: $n_2(1) = 81$; $n_2(2) = 44$

13.108 Canada: Nov $n_1(1) = 244$; $n_1(2) = 62$;
$n_1(3) = 62$; $n_1(4) = 19$
Canada: Dec: $n_2(1) = 162$;
$n_2(2) = 53$; $n_2(3) = 53$; $n_2(4) = 41$
U.S.: Nov: $n_3(1) = 232$; $n_3(2) = 95$;
$n_3(3) = 90$; $n_3(4) = 52$
U.S.: Dec: $n_4(1) = 185$; $n_4(2) = 92$;
$n_4(3) = 84$; $n_4(4) = 40$
Britain: Nov: $n_5(1) = 160$; $n_5(2) = 85$;
$n_5(3) = 72$; $n_5(4) = 24$
Britain: Dec: $n_6(1) = 129$; $n_6(2) = 84$;
$n_6(3) = 60$; $n_6(4) = 27$

13.109 Canada: 2008 $n_1(1) = 192$;
$n_1(2) = 373$
Canada: 2009: $n_2(1) = 154$;
$n_2(2) = 438$
U.S.: 2008: $n_3(1) = 157$; $n_3(2) = 446$
U.S.: 2009: $n_4(1) = 106$; $n_4(2) = 480$
Britain: 2008: $n_5(1) = 117$; $n_5(2) = 332$
Britain: 2009: $n_6(1) = 72$; $n_6(2) = 405$

13.110 Health conscious: $n_1(1) = 199$;
$n_1(2) = 32$
Not health conscious: $n_2(1) = 563$;
$n_2(2) = 56$

13.111 Segment 1: $n(1) = 68$, $n(2) = 95$
Segment 2: $n(1) = 20$, $n(2) = 34$
Segment 3: $n(1) = 10$, $n(2) = 13$
Segment 4: $n(1) = 29$, $n(2) = 79$

13.112 Source 1: $n_1(1) = 344$, $n_1(2) = 38$
Source 2: $n_2(1) = 275$, $n_2(2) = 41$

Chapter 14

14.9

Sample	$\bar{x}_i$	s_i^2	n_i
1	68.83	52.28	20
2	65.08	37.38	26
3	62.01	63.46	16
4	64.64	56.88	19

14.10

Sample	$\bar{x}_i$	s_i^2	n_i
1	90.17	991.5	30
2	95.77	900.9	30
3	106.8	928.7	30
4	111.2	1,023	30

14.11

Sample	$\bar{x}_i$	s_i^2	n_i
1	196.8	914.1	41
2	207.8	861.1	73
3	223.4	1,195	86
4	232.7	1,080	79

14.12

Sample	$\bar{x}_i$	s_i^2	n_i
1	164.6	1,164	25
2	185.6	1,719	25
3	154.8	1,113	25
4	182.6	1,657	25
5	178.9	841.8	25

14.13

Sample	$\bar{x}_i$	s_i^2	n_i
1	22.21	121.6	39
2	18.46	90.39	114
3	15.49	85.25	81
4	9.31	65.40	67

14.14

Sample	$\bar{x}_i$	s_i^2	n_i
1	551.5	2,742	20
2	576.8	2,641	20
3	559.5	3,129	20

14.15

Sample	$\bar{x}_i$	s_i^2	n_i
1	5.81	6.22	100
2	5.30	4.05	100
3	5.33	3.90	100

14.16

Sample	$\bar{x}_i$	s_i^2	n_i
1	74.10	250.0	30
2	75.67	184.2	30
3	78.50	233.4	30
4	81.30	242.9	30

14.17

	Size		
Sample	$\bar{x}_i$	s_i^2	n_i
1	24.97	48.23	50
2	21.65	54.54	50
3	17.84	33.85	50

	Nicotine		
Sample	$\bar{x}_i$	s_i^2	n_i
1	15.52	3.72	50
2	13.39	3.59	50
3	10.08	3.83	50

14.18 a.

Sample	$\bar{x}_i$	s_i^2	n_i
1	31.30	28.34	63
2	34.42	23.20	81
3	37.38	31.16	40
4	39.93	72.03	111

b.

Sample	$\bar{x}_i$	s_i^2	n_i
1	37.22	39.82	63
2	38.91	40.85	81
3	41.48	61.38	40
4	41.75	46.59	111

c.

Sample	$\bar{x}_i$	s_i^2	n_i
1	11.75	3.93	63
2	12.41	3.39	81
3	11.73	4.26	40
4	11.89	4.30	111

14.19

Sample	$\bar{x}_i$	s_i^2	n_i
1	153.6	654.3	20
2	151.5	924.0	20
3	133.3	626.8	20

14.20

Sample	$\bar{x}_i$	s_i^2	n_i
1	18.54	178.0	61
2	19.34	171.4	83
3	20.29	297.5	91

14.39

Sample	$\bar{x}_i$	s_i^2	n_i
1	61.60	80.49	10
2	57.30	70.46	10
3	61.80	22.18	10
4	51.80	75.29	10

14.41

Sample	$\bar{x}_i$	s_i^2	n_i
1	53.17	194.6	30
2	49.37	152.6	30
3	44.33	129.9	30

14.59 $k = 3$, $b = 12$, SST $= 204.2$,
SSB $= 1{,}150.2$, SSE $= 495.1$

14.60 $k = 3$, $b = 20$, SST $= 7{,}131$,
SSB $= 177{,}465$, SSE $= 1{,}098$

14.61 $k = 3$, $b = 20$, SST $= 10.26$,
SSB $= 3{,}020.30$, SSE $= 226.71$

14.62 $k = 4$, $b = 30$, SST $= 4{,}206$,
SSB $= 126{,}843$, SSE $= 5{,}764$

14.63 $k = 7$, $b = 200$, SST $= 28{,}674$,
SSB $= 209{,}835$, SSE $= 479{,}125$

14.64 $k = 5$, $b = 36$, SST $= 1{,}406.4$,
SSB $= 7{,}309.7$, SSE $= 4{,}593.9$

14.65 $k = 4$, $b = 21$, SST $= 563.82$,
SSB $= 1{,}327.33$, SSE $= 748.70$

Chapter 15

15.7 $n(1) = 28$, $n(2) = 17$, $n(3) = 19$,
$n(4) = 17$, $n(5) = 19$

15.8 $n(1) = 41$, $n(2) = 107$, $n(3) = 66$,
$n(4) = 19$

15.9 $n(1) = 114$, $n(2) = 92$, $n(3) = 84$,
$n(4) = 101$, $n(5) = 107$, $n(6) = 102$

15.10 $n(1) = 11$, $n(2) = 32$, $n(3) = 62$,
$n(4) = 29$, $n(5) = 16$

15.11 $n(1) = 8$, $n(2) = 4$, $n(3) = 3$, $n(4) = 8$,
$n(5) = 2$

15.12 $n(1) = 159$, $n(2) = 28$, $n(3) = 47$,
$n(4) = 16$

15.13 $n(1) = 36$, $n(2) = 58$, $n(3) = 74$,
$n(4) = 29$

15.14 $n(1) = 408$, $n(2) = 571$, $n(3) = 221$

15.15 $n(1) = 19$, $n(2) = 23$, $n(3) = 14$,
$n(4) = 194$

15.16 $n(1) = 63$, $n(2) = 125$, $n(3) = 45$,
$n(4) = 87$

15.31

	Newspaper			
Occupation	G&M	Post	Star	Sun
Blue collar	27	18	38	37
White collar	29	43	21	15
Professional	33	51	22	20

15.32

	Actual	
Predicted	Positive	Negative
Positive	65	64
Negative	39	48

15.33

	Last			
Second-last	1	2	3	4
1	39	36	51	23
2	36	32	46	20
3	54	46	65	29
4	24	20	28	10

15.34

Education	Continuing	Quitter
1	34	23
2	251	212
3	159	248
4	16	57

15.35

	Heartburn Condition			
Source	1	2	3	4
ABC	60	23	13	25
CBS	65	19	14	28
NBC	73	26	9	24
Newspaper	67	11	10	7
Radio	57	16	9	14
None	47	21	10	10

15.36

	Degree			
University	B.A.	B.Eng.	B.B.A.	Other
1	44	11	34	11
2	52	14	27	7
3	31	27	18	24
4	40	12	42	6

15.37

	Financial Ties	
Results	Yes	No
Favorable	29	1
Neutral	10	7
Critical	9	14

15.38

	Degree			
Approach	1	2	3	4
1	51	8	5	11
2	24	14	12	8
3	26	9	19	8

Chapter 16

16.6 Lengths: $\bar{x} = 38.00$, $s_x^2 = 193.90$
Test: $\bar{y} = 13.80$, $s_y^2 = 47.96$;
$n = 60$, $s_{xy} = 51.86$

16.7 Floors: $\bar{x} = 13.68$, $s_x^2 = 59.32$
Price: $\bar{y} = 210.42$, $s_y^2 = 496.41$;
$n = 50$, $s_{xy} = 86.93$

16.8 Age: $\bar{x} = 45.49$, $s_x^2 = 107.51$
Time: $\bar{y} = 11.55$, $s_y^2 = 42.54$;
$n = 229$, $s_{xy} = 9.67$

16.9 Age: $\bar{x} = 37.28$, $s_x^2 = 55.11$
Employment: $\bar{y} = 26.28$, $s_y^2 = 4.00$;
$n = 80$, $s_{xy} = -6.44$

16.10 Cigarettes: $\bar{x} = 37.64$, $s_x^2 = 108.3$
Days: $\bar{y} = 14.43$, $s_y^2 = 19.80$;
$n = 231$, $s_{xy} = 20.55$

16.11 Distance: $\bar{x} = 4.88$, $s_x^2 = 4.27$
Percent: $\bar{y} = 49.22$, $s_y^2 = 243.94$;
$n = 85$, $s_{xy} = 22.83$

16.12 Size: $\bar{x} = 53.93$, $s_x^2 = 688.18$
Price: $\bar{y} = 6{,}465$, $s_y^2 = 11{,}918{,}489$;
$n = 40$, $s_{xy} = 30{,}945$

16.13 Hours: $\bar{x} = 1{,}199$, $s_x^2 = 59{,}153$
Price: $\bar{y} = 27.73$, $s_y^2 = 3.62$;
$n = 60$, $s_{xy} = -81.78$

16.14 Occupants: $\bar{x} = 4.75$, $s_x^2 = 4.84$
Electricity: $\bar{y} = 762.6$, $s_y^2 = 56{,}725$;
$n = 200$, $s_{xy} = 310.0$

16.15 Income: $\bar{x} = 59.42$, $s_x^2 = 115.24$
Food: $\bar{y} = 270.3$, $s_y^2 = 1{,}797.25$;
$n = 150$, $s_{xy} = 225.66$

16.16 Vacancy: $\bar{x} = 11.33$, $s_x^2 = 35.47$
Rent: $\bar{y} = 17.20$, $s_y^2 = 11.24$;
$n = 30$, $s_{xy} = -10.78$

16.17 Height: $\bar{x} = 68.95$, $s_x^2 = 9.97$
Income: $\bar{y} = 59.59$, $s_y^2 = 71.95$;
$n = 250$, $s_{xy} = 6.02$

16.18 Test: $\bar{x} = 79.47$, $s_x^2 = 16.07$
Nondefective: $\bar{y} = 93.89$, $s_y^2 = 1.28$;
$n = 45$, $s_{xy} = .83$

16.99 Ads: $\bar{x} = 4.12$, $s_x^2 = 3.47$
Customers: $\bar{y} = 384.81$, $s_y^2 = 18{,}552$;
$n = 26$, $s_{xy} = 74.02$

16.100 Age: $\bar{x} = 113.35$, $s_x^2 = 378.77$
Repairs: $\bar{y} = 395.21$, $s_y^2 = 4{,}094.79$;
$n = 20$, $s_{xy} = 936.82$

16.101 Fertilizer: $\bar{x} = 300$, $s_x^2 = 20{,}690$
Yield: $\bar{y} = 318.60$, $s_y^2 = 5{,}230$;
$n = 30$, $s_{xy} = 2{,}538$

16.102 Tar: $\bar{x} = 12.22$, $s_x^2 = 32.10$
Nicotine: $\bar{y} = .88$, $s_y^2 = .13$;
$n = 25$, $s_{xy} = 1.96$

16.103 Television: $\bar{x} = 30.43$, $s_x^2 = 99.11$,
Debt: $\bar{y} = 126{,}604$,
$s_y^2 = 2{,}152{,}602{,}614$; $n = 430$,
$s_{xy} = 255{,}877$

16.104 Test: $\bar{x} = 71.92$, $s_x^2 = 90.97$
Nondefective: $\bar{y} = 94.44$,
$s_y^2 = 11.84$; $n = 50$, $s_{xy} = 13.08$

Chapter 17

17.1 $R^2 = .2425$, $R^2(adjusted) = .2019$,
$s_\varepsilon = 40.24$, $F = 5.97$, p-value $= .0013$

	Coefficients	Standard Error	t-Statistic	p-Value
Intercept	51.39	23.52	2.19	.0331
Lot size	.700	.559	1.25	.2156
Trees	.679	.229	2.96	.0045
Distance	−.378	.195	−1.94	.0577

17.2 $R^2 = .7629$, $R^2(adjusted) = .7453$,
$s_\varepsilon = 3.75$, $F = 43.43$, p-value $= 0$

	Coefficients	Standard Error	t-Statistic	p-Value
Intercept	13.01	3.53	3.69	.0010
Assignment	.194	.200	.97	.3417
Midterm	1.11	.122	9.12	0

17.3 $R^2 = .8935$, $R^2(adjusted) = .8711$,
$s_\varepsilon = 40.13$, $F = 39.86$, p-value $= 0$

	Coefficients	Standard Error	t-Statistic	p-Value
Intercept	−111.83	134.34	−.83	.4155
Permits	4.76	.395	12.06	0
Mortgage	16.99	15.16	1.12	.2764
Apartment vacancy	−10.53	6.39	−1.65	.1161
Office vacancy	1.31	2.79	.47	.6446

17.4 $R^2 = .3511$, $R^2(adjusted) = .3352$,
$s_\varepsilon = 6.99$, $F = 22.01$, p-value $= 0$

	Coefficients	Standard Error	t-Statistic	p-Value
Intercept	−1.97	9.55	−.21	.8369
Minor HR	.666	.087	7.64	0
Age	.136	.524	.26	.7961
Years pro	1.18	.671	1.75	.0819

APPENDIX B

TABLES

TABLE 1 Binomial Probabilities

Tabulated values are $P(X \leq k) = \sum_{x=0}^{k} p(x_i)$. (Values are rounded to four decimal places.)

$n = 5$

	p														
k	**0.01**	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
0	0.9510	0.7738	0.5905	0.3277	0.2373	0.1681	0.0778	0.0313	0.0102	0.0024	0.0010	0.0003	0.0000	0.0000	0.0000
1	0.9990	0.9774	0.9185	0.7373	0.6328	0.5282	0.3370	0.1875	0.0870	0.0308	0.0156	0.0067	0.0005	0.0000	0.0000
2	1.0000	0.9988	0.9914	0.9421	0.8965	0.8369	0.6826	0.5000	0.3174	0.1631	0.1035	0.0579	0.0086	0.0012	0.0000
3	1.0000	1.0000	0.9995	0.9933	0.9844	0.9692	0.9130	0.8125	0.6630	0.4718	0.3672	0.2627	0.0815	0.0226	0.0010
4	1.0000	1.0000	1.0000	0.9997	0.9990	0.9976	0.9898	0.9688	0.9222	0.8319	0.7627	0.6723	0.4095	0.2262	0.0490

$n = 6$

	p														
k	**0.01**	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
0	0.9415	0.7351	0.5314	0.2621	0.1780	0.1176	0.0467	0.0156	0.0041	0.0007	0.0002	0.0001	0.0000	0.0000	0.0000
1	0.9985	0.9672	0.8857	0.6554	0.5339	0.4202	0.2333	0.1094	0.0410	0.0109	0.0046	0.0016	0.0001	0.0000	0.0000
2	1.0000	0.9978	0.9842	0.9011	0.8306	0.7443	0.5443	0.3438	0.1792	0.0705	0.0376	0.0170	0.0013	0.0001	0.0000
3	1.0000	0.9999	0.9987	0.9830	0.9624	0.9295	0.8208	0.6563	0.4557	0.2557	0.1694	0.0989	0.0159	0.0022	0.0000
4	1.0000	1.0000	0.9999	0.9984	0.9954	0.9891	0.9590	0.8906	0.7667	0.5798	0.4661	0.3446	0.1143	0.0328	0.0015
5	1.0000	1.0000	1.0000	0.9999	0.9998	0.9993	0.9959	0.9844	0.9533	0.8824	0.8220	0.7379	0.4686	0.2649	0.0585

$n = 7$

	p														
k	**0.01**	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
0	0.9321	0.6983	0.4783	0.2097	0.1335	0.0824	0.0280	0.0078	0.0016	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
1	0.9980	0.9556	0.8503	0.5767	0.4449	0.3294	0.1586	0.0625	0.0188	0.0038	0.0013	0.0004	0.0000	0.0000	0.0000
2	1.0000	0.9962	0.9743	0.8520	0.7564	0.6471	0.4199	0.2266	0.0963	0.0288	0.0129	0.0047	0.0002	0.0000	0.0000
3	1.0000	0.9998	0.9973	0.9667	0.9294	0.8740	0.7102	0.5000	0.2898	0.1260	0.0706	0.0333	0.0027	0.0002	0.0000
4	1.0000	1.0000	0.9998	0.9953	0.9871	0.9712	0.9037	0.7734	0.5801	0.3529	0.2436	0.1480	0.0257	0.0038	0.0000
5	1.0000	1.0000	1.0000	0.9996	0.9987	0.9962	0.9812	0.9375	0.8414	0.6706	0.5551	0.4233	0.1497	0.0444	0.0020
6	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9984	0.9922	0.9720	0.9176	0.8665	0.7903	0.5217	0.3017	0.0679

TABLE 1 (*Continued*)

$n = 8$

	p														
k	**0.01**	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
0	0.9227	0.6634	0.4305	0.1678	0.1001	0.0576	0.0168	0.0039	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.9973	0.9428	0.8131	0.5033	0.3671	0.2553	0.1064	0.0352	0.0085	0.0013	0.0004	0.0001	0.0000	0.0000	0.0000
2	0.9999	0.9942	0.9619	0.7969	0.6785	0.5518	0.3154	0.1445	0.0498	0.0113	0.0042	0.0012	0.0000	0.0000	0.0000
3	1.0000	0.9996	0.9950	0.9437	0.8862	0.8059	0.5941	0.3633	0.1737	0.0580	0.0273	0.0104	0.0004	0.0000	0.0000
4	1.0000	1.0000	0.9996	0.9896	0.9727	0.9420	0.8263	0.6367	0.4059	0.1941	0.1138	0.0563	0.0050	0.0004	0.0000
5	1.0000	1.0000	1.0000	0.9988	0.9958	0.9887	0.9502	0.8555	0.6846	0.4482	0.3215	0.2031	0.0381	0.0058	0.0001
6	1.0000	1.0000	1.0000	0.9999	0.9996	0.9987	0.9915	0.9648	0.8936	0.7447	0.6329	0.4967	0.1869	0.0572	0.0027
7	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9961	0.9832	0.9424	0.8999	0.8322	0.5695	0.3366	0.0773

$n = 9$

	p														
k	**0.01**	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
0	0.9135	0.6302	0.3874	0.1342	0.0751	0.0404	0.0101	0.0020	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.9966	0.9288	0.7748	0.4362	0.3003	0.1960	0.0705	0.0195	0.0038	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000
2	0.9999	0.9916	0.9470	0.7382	0.6007	0.4628	0.2318	0.0898	0.0250	0.0043	0.0013	0.0003	0.0000	0.0000	0.0000
3	1.0000	0.9994	0.9917	0.9144	0.8343	0.7297	0.4826	0.2539	0.0994	0.0253	0.0100	0.0031	0.0001	0.0000	0.0000
4	1.0000	1.0000	0.9991	0.9804	0.9511	0.9012	0.7334	0.5000	0.2666	0.0988	0.0489	0.0196	0.0009	0.0000	0.0000
5	1.0000	1.0000	0.9999	0.9969	0.9900	0.9747	0.9006	0.7461	0.5174	0.2703	0.1657	0.0856	0.0083	0.0006	0.0000
6	1.0000	1.0000	1.0000	0.9997	0.9987	0.9957	0.9750	0.9102	0.7682	0.5372	0.3993	0.2618	0.0530	0.0084	0.0001
7	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.9962	0.9805	0.9295	0.8040	0.6997	0.5638	0.2252	0.0712	0.0034
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9980	0.9899	0.9596	0.9249	0.8658	0.6126	0.3698	0.0865

TABLE 1 (*Continued*)

n = 10

	p														
k	**0.01**	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
0	0.9044	0.5987	0.3487	0.1074	0.0563	0.0282	0.0060	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.9957	0.9139	0.7361	0.3758	0.2440	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.9999	0.9885	0.9298	0.6778	0.5256	0.3828	0.1673	0.0547	0.0123	0.0016	0.0004	0.0001	0.0000	0.0000	0.0000
3	1.0000	0.9990	0.9872	0.8791	0.7759	0.6496	0.3823	0.1719	0.0548	0.0106	0.0035	0.0009	0.0000	0.0000	0.0000
4	1.0000	0.9999	0.9984	0.9672	0.9219	0.8497	0.6331	0.3770	0.1662	0.0473	0.0197	0.0064	0.0001	0.0000	0.0000
5	1.0000	1.0000	0.9999	0.9936	0.9803	0.9527	0.8338	0.6230	0.3669	0.1503	0.0781	0.0328	0.0016	0.0001	0.0000
6	1.0000	1.0000	1.0000	0.9991	0.9965	0.9894	0.9452	0.8281	0.6177	0.3504	0.2241	0.1209	0.0128	0.0010	0.0000
7	1.0000	1.0000	1.0000	0.9999	0.9996	0.9984	0.9877	0.9453	0.8327	0.6172	0.4744	0.3222	0.0702	0.0115	0.0001
8	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.7560	0.6242	0.2639	0.0861	0.0043
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9990	0.9940	0.9718	0.9437	0.8926	0.6513	0.4013	0.0956

n = 15

	p														
k	**0.01**	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
0	0.8601	0.4633	0.2059	0.0352	0.0134	0.0047	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.9904	0.8290	0.5490	0.1671	0.0802	0.0353	0.0052	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.9996	0.9638	0.8159	0.3980	0.2361	0.1268	0.0271	0.0037	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	1.0000	0.9945	0.9444	0.6482	0.4613	0.2969	0.0905	0.0176	0.0019	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
4	1.0000	0.9994	0.9873	0.8358	0.6865	0.5155	0.2173	0.0592	0.0093	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000
5	1.0000	0.9999	0.9978	0.9389	0.8516	0.7216	0.4032	0.1509	0.0338	0.0037	0.0008	0.0001	0.0000	0.0000	0.0000
6	1.0000	1.0000	0.9997	0.9819	0.9434	0.8689	0.6098	0.3036	0.0950	0.0152	0.0042	0.0008	0.0000	0.0000	0.0000
7	1.0000	1.0000	1.0000	0.9958	0.9827	0.9500	0.7869	0.5000	0.2131	0.0500	0.0173	0.0042	0.0000	0.0000	0.0000
8	1.0000	1.0000	1.0000	0.9992	0.9958	0.9848	0.9050	0.6964	0.3902	0.1311	0.0566	0.0181	0.0003	0.0000	0.0000
9	1.0000	1.0000	1.0000	0.9999	0.9992	0.9963	0.9662	0.8491	0.5968	0.2784	0.1484	0.0611	0.0022	0.0001	0.0000
10	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9907	0.9408	0.7827	0.4845	0.3135	0.1642	0.0127	0.0006	0.0000
11	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.5387	0.3518	0.0556	0.0055	0.0000
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9963	0.9729	0.8732	0.7639	0.6020	0.1841	0.0362	0.0004
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9948	0.9647	0.9198	0.8329	0.4510	0.1710	0.0096
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9953	0.9866	0.9648	0.7941	0.5367	0.1399

TABLE 1 (*Continued*)

n = 20

k	0.01	0.05	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.90	0.95	0.99
0	0.8179	0.3585	0.1216	0.0115	0.0032	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.9831	0.7358	0.3917	0.0692	0.0243	0.0076	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.9990	0.9245	0.6769	0.2061	0.0913	0.0355	0.0036	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	1.0000	0.9841	0.8670	0.4114	0.2252	0.1071	0.0160	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	1.0000	0.9974	0.9568	0.6296	0.4148	0.2375	0.0510	0.0059	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	1.0000	0.9997	0.9887	0.8042	0.6172	0.4164	0.1256	0.0207	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	1.0000	1.0000	0.9976	0.9133	0.7858	0.6080	0.2500	0.0577	0.0065	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
7	1.0000	1.0000	0.9996	0.9679	0.8982	0.7723	0.4159	0.1316	0.0210	0.0013	0.0002	0.0000	0.0000	0.0000	0.0000
8	1.0000	1.0000	0.9999	0.9900	0.9591	0.8867	0.5956	0.2517	0.0565	0.0051	0.0009	0.0001	0.0000	0.0000	0.0000
9	1.0000	1.0000	1.0000	0.9974	0.9861	0.9520	0.7553	0.4119	0.1275	0.0171	0.0039	0.0006	0.0000	0.0000	0.0000
10	1.0000	1.0000	1.0000	0.9994	0.9961	0.9829	0.8725	0.5881	0.2447	0.0480	0.0139	0.0026	0.0000	0.0000	0.0000
11	1.0000	1.0000	1.0000	0.9999	0.9991	0.9949	0.9435	0.7483	0.4044	0.1133	0.0409	0.0100	0.0001	0.0000	0.0000
12	1.0000	1.0000	1.0000	1.0000	0.9998	0.9987	0.9790	0.8684	0.5841	0.2277	0.1018	0.0321	0.0004	0.0000	0.0000
13	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9935	0.9423	0.7500	0.3920	0.2142	0.0867	0.0024	0.0000	0.0000
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9984	0.9793	0.8744	0.5836	0.3828	0.1958	0.0113	0.0003	0.0000
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9941	0.9490	0.7625	0.5852	0.3704	0.0432	0.0026	0.0000
16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9987	0.9840	0.8929	0.7748	0.5886	0.1330	0.0159	0.0000
17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9964	0.9645	0.9087	0.7939	0.3231	0.0755	0.0010
18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9924	0.9757	0.9308	0.6083	0.2642	0.0169
19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9968	0.9885	0.8784	0.6415	0.1821

TABLE 1 (*Continued*)

n = 25

	p														
k	**0.01**	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
0	0.7778	0.2774	0.0718	0.0038	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.9742	0.6424	0.2712	0.0274	0.0070	0.0016	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.9980	0.8729	0.5371	0.0982	0.0321	0.0090	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.9999	0.9659	0.7636	0.2340	0.0962	0.0332	0.0024	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	1.0000	0.9928	0.9020	0.4207	0.2137	0.0905	0.0095	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	1.0000	0.9988	0.9666	0.6167	0.3783	0.1935	0.0294	0.0020	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	1.0000	0.9998	0.9905	0.7800	0.5611	0.3407	0.0736	0.0073	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	1.0000	1.0000	0.9977	0.8909	0.7265	0.5118	0.1536	0.0216	0.0012	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	1.0000	1.0000	0.9995	0.9532	0.8506	0.6769	0.2735	0.0539	0.0043	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
9	1.0000	1.0000	0.9999	0.9827	0.9287	0.8106	0.4246	0.1148	0.0132	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000
10	1.0000	1.0000	1.0000	0.9944	0.9703	0.9022	0.5858	0.2122	0.0344	0.0018	0.0002	0.0000	0.0000	0.0000	0.0000
11	1.0000	1.0000	1.0000	0.9985	0.9893	0.9558	0.7323	0.3450	0.0778	0.0060	0.0009	0.0001	0.0000	0.0000	0.0000
12	1.0000	1.0000	1.0000	0.9996	0.9966	0.9825	0.8462	0.5000	0.1538	0.0175	0.0034	0.0004	0.0000	0.0000	0.0000
13	1.0000	1.0000	1.0000	0.9999	0.9991	0.9940	0.9222	0.6550	0.2677	0.0442	0.0107	0.0015	0.0000	0.0000	0.0000
14	1.0000	1.0000	1.0000	1.0000	0.9998	0.9982	0.9656	0.7878	0.4142	0.0978	0.0297	0.0056	0.0000	0.0000	0.0000
15	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9868	0.8852	0.5754	0.1894	0.0713	0.0173	0.0001	0.0000	0.0000
16	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9957	0.9461	0.7265	0.3231	0.1494	0.0468	0.0005	0.0000	0.0000
17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9988	0.9784	0.8464	0.4882	0.2735	0.1091	0.0023	0.0000	0.0000
18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9927	0.9264	0.6593	0.4389	0.2200	0.0095	0.0002	0.0000
19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9980	0.9706	0.8065	0.6217	0.3833	0.0334	0.0012	0.0000
20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9905	0.9095	0.7863	0.5793	0.0980	0.0072	0.0000
21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9976	0.9668	0.9038	0.7660	0.2364	0.0341	0.0001
22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9910	0.9679	0.9018	0.4629	0.1271	0.0020
23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9984	0.9930	0.9726	0.7288	0.3576	0.0258
24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9992	0.9962	0.9282	0.7226	0.2222

TABLE 2 Poisson Probabilities

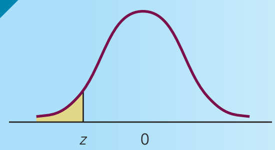
Tabulated values are $P(X \leq k) = \sum_{x=0}^{k} p(xi)$. (Values are rounded to four decimal places.)

	μ															
k	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**	**4.5**	**5.0**	**5.5**	**6.0**
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.3679	0.2231	0.1353	0.0821	0.0498	0.0302	0.0183	0.0111	0.0067	0.0041	0.0025
1	0.9953	0.9825	0.9631	0.9384	0.9098	0.7358	0.5578	0.4060	0.2873	0.1991	0.1359	0.0916	0.0611	0.0404	0.0266	0.0174
2	0.9998	0.9989	0.9964	0.9921	0.9856	0.9197	0.8088	0.6767	0.5438	0.4232	0.3208	0.2381	0.1736	0.1247	0.0884	0.0620
3	1.0000	0.9999	0.9997	0.9992	0.9982	0.9810	0.9344	0.8571	0.7576	0.6472	0.5366	0.4335	0.3423	0.2650	0.2017	0.1512
4		1.0000	1.0000	0.9999	0.9998	0.9963	0.9814	0.9473	0.8912	0.8153	0.7254	0.6288	0.5321	0.4405	0.3575	0.2851
5				1.0000	1.0000	0.9994	0.9955	0.9834	0.9580	0.9161	0.8576	0.7851	0.7029	0.6160	0.5289	0.4457
6						0.9999	0.9991	0.9955	0.9858	0.9665	0.9347	0.8893	0.8311	0.7622	0.6860	0.6063
7						1.0000	0.9998	0.9989	0.9958	0.9881	0.9733	0.9489	0.9134	0.8666	0.8095	0.7440
8							1.0000	0.9998	0.9989	0.9962	0.9901	0.9786	0.9597	0.9319	0.8944	0.8472
9								1.0000	0.9997	0.9989	0.9967	0.9919	0.9829	0.9682	0.9462	0.9161
10									0.9999	0.9997	0.9990	0.9972	0.9933	0.9863	0.9747	0.9574
11									1.0000	0.9999	0.9997	0.9991	0.9976	0.9945	0.9890	0.9799
12										1.0000	0.9999	0.9997	0.9992	0.9980	0.9955	0.9912
13											1.0000	0.9999	0.9997	0.9993	0.9983	0.9964
14												1.0000	0.9999	0.9998	0.9994	0.9986
15													1.0000	0.9999	0.9998	0.9995
16														1.0000	0.9999	0.9998
17															1.0000	0.9999
18																1.0000
19																
20																

TABLE **2** (*Continued*)

	μ												
k	**6.50**	**7.00**	**7.50**	**8.00**	**8.50**	**9.00**	**9.50**	**10**	**11**	**12**	**13**	**14**	**15**
0	0.0015	0.0009	0.0006	0.0003	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.0113	0.0073	0.0047	0.0030	0.0019	0.0012	0.0008	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000
2	0.0430	0.0296	0.0203	0.0138	0.0093	0.0062	0.0042	0.0028	0.0012	0.0005	0.0002	0.0001	0.0000
3	0.1118	0.0818	0.0591	0.0424	0.0301	0.0212	0.0149	0.0103	0.0049	0.0023	0.0011	0.0005	0.0002
4	0.2237	0.1730	0.1321	0.0996	0.0744	0.0550	0.0403	0.0293	0.0151	0.0076	0.0037	0.0018	0.0009
5	0.3690	0.3007	0.2414	0.1912	0.1496	0.1157	0.0885	0.0671	0.0375	0.0203	0.0107	0.0055	0.0028
6	0.5265	0.4497	0.3782	0.3134	0.2562	0.2068	0.1649	0.1301	0.0786	0.0458	0.0259	0.0142	0.0076
7	0.6728	0.5987	0.5246	0.4530	0.3856	0.3239	0.2687	0.2202	0.1432	0.0895	0.0540	0.0316	0.0180
8	0.7916	0.7291	0.6620	0.5925	0.5231	0.4557	0.3918	0.3328	0.2320	0.1550	0.0998	0.0621	0.0374
9	0.8774	0.8305	0.7764	0.7166	0.6530	0.5874	0.5218	0.4579	0.3405	0.2424	0.1658	0.1094	0.0699
10	0.9332	0.9015	0.8622	0.8159	0.7634	0.7060	0.6453	0.5830	0.4599	0.3472	0.2517	0.1757	0.1185
11	0.9661	0.9467	0.9208	0.8881	0.8487	0.8030	0.7520	0.6968	0.5793	0.4616	0.3532	0.2600	0.1848
12	0.9840	0.9730	0.9573	0.9362	0.9091	0.8758	0.8364	0.7916	0.6887	0.5760	0.4631	0.3585	0.2676
13	0.9929	0.9872	0.9784	0.9658	0.9486	0.9261	0.8981	0.8645	0.7813	0.6815	0.5730	0.4644	0.3632
14	0.9970	0.9943	0.9897	0.9827	0.9726	0.9585	0.9400	0.9165	0.8540	0.7720	0.6751	0.5704	0.4657
15	0.9988	0.9976	0.9954	0.9918	0.9862	0.9780	0.9665	0.9513	0.9074	0.8444	0.7636	0.6694	0.5681
16	0.9996	0.9990	0.9980	0.9963	0.9934	0.9889	0.9823	0.9730	0.9441	0.8987	0.8355	0.7559	0.6641
17	0.9998	0.9996	0.9992	0.9984	0.9970	0.9947	0.9911	0.9857	0.9678	0.9370	0.8905	0.8272	0.7489
18	0.9999	0.9999	0.9997	0.9993	0.9987	0.9976	0.9957	0.9928	0.9823	0.9626	0.9302	0.8826	0.8195
19	1.0000	1.0000	0.9999	0.9997	0.9995	0.9989	0.9980	0.9965	0.9907	0.9787	0.9573	0.9235	0.8752
20			1.0000	0.9999	0.9998	0.9996	0.9991	0.9984	0.9953	0.9884	0.9750	0.9521	0.9170
21				1.0000	0.9999	0.9998	0.9996	0.9993	0.9977	0.9939	0.9859	0.9712	0.9469
22					1.0000	0.9999	0.9999	0.9997	0.9990	0.9970	0.9924	0.9833	0.9673
23						1.0000	0.9999	0.9999	0.9995	0.9985	0.9960	0.9907	0.9805
24							1.0000	1.0000	0.9998	0.9993	0.9980	0.9950	0.9888
25									0.9999	0.9997	0.9990	0.9974	0.9938
26									1.0000	0.9999	0.9995	0.9987	0.9967
27										0.9999	0.9998	0.9994	0.9983
28										1.0000	0.9999	0.9997	0.9991
29											1.0000	0.9999	0.9996
30												0.9999	0.9998
31												1.0000	0.9999
32													1.0000

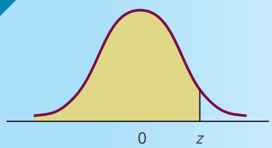
TABLE 3 Cumulative Standardized Normal Probabilities



$P(-\infty < Z < z)$

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

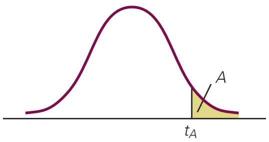
TABLE 3 (*Continued*)



$P(-\infty < Z < z)$.

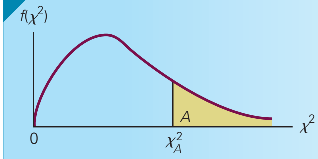
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

TABLE 4
Critical Values of the Student *t* Distribution



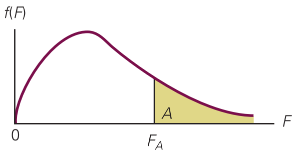
Degrees of Freedom	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
35	1.306	1.690	2.030	2.438	2.724
40	1.303	1.684	2.021	2.423	2.704
45	1.301	1.679	2.014	2.412	2.690
50	1.299	1.676	2.009	2.403	2.678
55	1.297	1.673	2.004	2.396	2.668
60	1.296	1.671	2.000	2.390	2.660
65	1.295	1.669	1.997	2.385	2.654
70	1.294	1.667	1.994	2.381	2.648
75	1.293	1.665	1.992	2.377	2.643
80	1.292	1.664	1.990	2.374	2.639
85	1.292	1.663	1.988	2.371	2.635
90	1.291	1.662	1.987	2.368	2.632
95	1.291	1.661	1.985	2.366	2.629
100	1.290	1.660	1.984	2.364	2.626
110	1.289	1.659	1.982	2.361	2.621
120	1.289	1.658	1.980	2.358	2.617
130	1.288	1.657	1.978	2.355	2.614
140	1.288	1.656	1.977	2.353	2.611
150	1.287	1.655	1.976	2.351	2.609
160	1.287	1.654	1.975	2.350	2.607
170	1.287	1.654	1.974	2.348	2.605
180	1.286	1.653	1.973	2.347	2.603
190	1.286	1.653	1.973	2.346	2.602
200	1.286	1.653	1.972	2.345	2.601
∞	1.282	1.645	1.960	2.326	2.576

TABLE 5 Critical Values of the χ^2 Distribution



Degrees of Freedom	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000039	0.000157	0.000982	0.00393	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.103	0.211	4.61	5.99	7.38	9.21	10.6
3	0.072	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.3	12.8
4	0.207	0.297	0.484	0.711	1.06	7.78	9.49	11.1	13.3	14.9
5	0.412	0.554	0.831	1.15	1.61	9.24	11.1	12.8	15.1	16.7
6	0.676	0.872	1.24	1.64	2.20	10.6	12.6	14.4	16.8	18.5
7	0.989	1.24	1.69	2.17	2.83	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	40.3	43.8	47.0	50.9	53.7
40	20.7	22.2	24.4	26.5	29.1	51.8	55.8	59.3	63.7	66.8
50	28.0	29.7	32.4	34.8	37.7	63.2	67.5	71.4	76.2	79.5
60	35.5	37.5	40.5	43.2	46.5	74.4	79.1	83.3	88.4	92.0
70	43.3	45.4	48.8	51.7	55.3	85.5	90.5	95.0	100	104
80	51.2	53.5	57.2	60.4	64.3	96.6	102	107	112	116
90	59.2	61.8	65.6	69.1	73.3	108	113	118	124	128
100	67.3	70.1	74.2	77.9	82.4	118	124	130	136	140

TABLE 6(a) Critical Values of the F-Distribution: $A = .05$



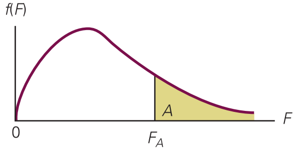
NUMERATOR DEGREES OF FREEDOM (ν_1); DENOMINATOR DEGREES OF FREEDOM (ν_2)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	161	199	216	225	230	234	237	239	241	242	243	244	245	245	246	246	247	247	248	248
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74	8.73	8.71	8.70	8.69	8.68	8.67	8.67	8.66
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91	5.89	5.87	5.86	5.84	5.83	5.82	5.81	5.80
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70	4.68	4.66	4.64	4.62	4.60	4.59	4.58	4.57	4.56
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.98	3.96	3.94	3.92	3.91	3.90	3.88	3.87
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57	3.55	3.53	3.51	3.49	3.48	3.47	3.46	3.44
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	3.26	3.24	3.22	3.20	3.19	3.17	3.16	3.15
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	3.05	3.03	3.01	2.99	2.97	2.96	2.95	2.94
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91	2.89	2.86	2.85	2.83	2.81	2.80	2.79	2.77
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.69	2.67	2.66	2.65
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69	2.66	2.64	2.62	2.60	2.58	2.57	2.56	2.54
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60	2.58	2.55	2.53	2.51	2.50	2.48	2.47	2.46
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53	2.51	2.48	2.46	2.44	2.43	2.41	2.40	2.39
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48	2.45	2.42	2.40	2.38	2.37	2.35	2.34	2.33
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42	2.40	2.37	2.35	2.33	2.32	2.30	2.29	2.28
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38	2.35	2.33	2.31	2.29	2.27	2.26	2.24	2.23
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.31	2.29	2.27	2.25	2.23	2.22	2.20	2.19
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31	2.28	2.26	2.23	2.21	2.20	2.18	2.17	2.16
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28	2.25	2.22	2.20	2.18	2.17	2.15	2.14	2.12
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23	2.20	2.17	2.15	2.13	2.11	2.10	2.08	2.07
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.22	2.18	2.15	2.13	2.11	2.09	2.07	2.05	2.04	2.03
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15	2.12	2.09	2.07	2.05	2.03	2.02	2.00	1.99
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12	2.09	2.06	2.04	2.02	2.00	1.99	1.97	1.96
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09	2.06	2.04	2.01	1.99	1.98	1.96	1.95	1.93
35	4.12	3.27	2.87	2.64	2.49	2.37	2.29	2.22	2.16	2.11	2.07	2.04	2.01	1.99	1.96	1.94	1.92	1.91	1.89	1.88
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00	1.97	1.95	1.92	1.90	1.89	1.87	1.85	1.84
45	4.06	3.20	2.81	2.58	2.42	2.31	2.22	2.15	2.10	2.05	2.01	1.97	1.94	1.92	1.89	1.87	1.86	1.84	1.82	1.81
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.99	1.95	1.92	1.89	1.87	1.85	1.83	1.81	1.80	1.78
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92	1.89	1.86	1.84	1.82	1.80	1.78	1.76	1.75
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.93	1.89	1.86	1.84	1.81	1.79	1.77	1.75	1.74	1.72
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.91	1.88	1.84	1.82	1.79	1.77	1.75	1.73	1.72	1.70
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.90	1.86	1.83	1.80	1.78	1.76	1.74	1.72	1.70	1.69
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.89	1.85	1.82	1.79	1.77	1.75	1.73	1.71	1.69	1.68
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.87	1.83	1.80	1.78	1.75	1.73	1.71	1.69	1.67	1.66
140	3.91	3.06	2.67	2.44	2.28	2.16	2.08	2.01	1.95	1.90	1.86	1.82	1.79	1.76	1.74	1.72	1.70	1.68	1.66	1.65
160	3.90	3.05	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.85	1.81	1.78	1.75	1.73	1.71	1.69	1.67	1.65	1.64
180	3.89	3.05	2.65	2.42	2.26	2.15	2.06	1.99	1.93	1.88	1.84	1.81	1.77	1.75	1.72	1.70	1.68	1.66	1.64	1.63
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84	1.80	1.77	1.74	1.72	1.69	1.67	1.66	1.64	1.62
∞	3.84	3.00	2.61	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75	1.72	1.69	1.67	1.64	1.62	1.60	1.59	1.57

ν_2 \ ν_1	22	24	26	28	30	35	40	45	50	60	70	80	90	100	120	140	160	180	200	∞
	NUMERATOR DEGREES OF FREEDOM																			
1	249	249	249	250	250	251	251	251	252	252	252	253	253	253	253	253	254	254	254	254
2	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5
3	8.65	8.64	8.63	8.62	8.62	8.60	8.59	8.59	8.58	8.57	8.57	8.56	8.56	8.55	8.55	8.55	8.54	8.54	8.54	8.53
4	5.79	5.77	5.76	5.75	5.75	5.73	5.72	5.71	5.70	5.69	5.68	5.67	5.67	5.66	5.66	5.65	5.65	5.65	5.65	5.63
5	4.54	4.53	4.52	4.50	4.50	4.48	4.46	4.45	4.44	4.43	4.42	4.41	4.41	4.41	4.40	4.39	4.39	4.39	4.39	4.37
6	3.86	3.84	3.83	3.82	3.81	3.79	3.77	3.76	3.75	3.74	3.73	3.72	3.72	3.71	3.70	3.70	3.70	3.69	3.69	3.67
7	3.43	3.41	3.40	3.39	3.38	3.36	3.34	3.33	3.32	3.30	3.29	3.29	3.28	3.27	3.27	3.26	3.26	3.25	3.25	3.23
8	3.13	3.12	3.10	3.09	3.08	3.06	3.04	3.03	3.02	3.01	2.99	2.99	2.98	2.97	2.97	2.96	2.96	2.95	2.95	2.93
9	2.92	2.90	2.89	2.87	2.86	2.84	2.83	2.81	2.80	2.79	2.78	2.77	2.76	2.76	2.75	2.74	2.74	2.73	2.73	2.71
10	2.75	2.74	2.72	2.71	2.70	2.68	2.66	2.65	2.64	2.62	2.61	2.60	2.59	2.59	2.58	2.57	2.57	2.57	2.56	2.54
11	2.63	2.61	2.59	2.58	2.57	2.55	2.53	2.52	2.51	2.49	2.48	2.47	2.46	2.46	2.45	2.44	2.44	2.43	2.43	2.41
12	2.52	2.51	2.49	2.48	2.47	2.44	2.43	2.41	2.40	2.38	2.37	2.36	2.36	2.35	2.34	2.33	2.33	2.33	2.32	2.30
13	2.44	2.42	2.41	2.39	2.38	2.36	2.34	2.33	2.31	2.30	2.28	2.27	2.27	2.26	2.25	2.25	2.24	2.24	2.23	2.21
14	2.37	2.35	2.33	2.32	2.31	2.28	2.27	2.25	2.24	2.22	2.21	2.20	2.19	2.19	2.18	2.17	2.17	2.16	2.16	2.13
15	2.31	2.29	2.27	2.26	2.25	2.22	2.20	2.19	2.18	2.16	2.15	2.14	2.13	2.12	2.11	2.11	2.10	2.10	2.10	2.07
16	2.25	2.24	2.22	2.21	2.19	2.17	2.15	2.14	2.12	2.11	2.09	2.08	2.07	2.07	2.06	2.05	2.05	2.04	2.04	2.01
17	2.21	2.19	2.17	2.16	2.15	2.12	2.10	2.09	2.08	2.06	2.05	2.03	2.03	2.02	2.01	2.00	2.00	1.99	1.99	1.96
18	2.17	2.15	2.13	2.12	2.11	2.08	2.06	2.05	2.04	2.02	2.00	1.99	1.98	1.98	1.97	1.96	1.96	1.95	1.95	1.92
19	2.13	2.11	2.10	2.08	2.07	2.05	2.03	2.01	2.00	1.98	1.97	1.96	1.95	1.94	1.93	1.92	1.92	1.91	1.91	1.88
20	2.10	2.08	2.07	2.05	2.04	2.01	1.99	1.98	1.97	1.95	1.93	1.92	1.91	1.91	1.90	1.89	1.88	1.88	1.88	1.84
22	2.05	2.03	2.01	2.00	1.98	1.96	1.94	1.92	1.91	1.89	1.88	1.86	1.86	1.85	1.84	1.83	1.82	1.82	1.82	1.78
24	2.00	1.98	1.97	1.95	1.94	1.91	1.89	1.88	1.86	1.84	1.83	1.82	1.81	1.80	1.79	1.78	1.78	1.77	1.77	1.73
26	1.97	1.95	1.93	1.91	1.90	1.87	1.85	1.84	1.82	1.80	1.79	1.78	1.77	1.76	1.75	1.74	1.73	1.73	1.73	1.69
28	1.93	1.91	1.90	1.88	1.87	1.84	1.82	1.80	1.79	1.77	1.75	1.74	1.73	1.73	1.71	1.71	1.70	1.69	1.69	1.65
30	1.91	1.89	1.87	1.85	1.84	1.81	1.79	1.77	1.76	1.74	1.72	1.71	1.70	1.70	1.68	1.68	1.67	1.66	1.66	1.62
35	1.85	1.83	1.82	1.80	1.79	1.76	1.74	1.72	1.70	1.68	1.66	1.65	1.64	1.63	1.62	1.61	1.61	1.60	1.60	1.56
40	1.81	1.79	1.77	1.76	1.74	1.72	1.69	1.67	1.66	1.64	1.62	1.61	1.60	1.59	1.58	1.57	1.56	1.55	1.55	1.51
45	1.78	1.76	1.74	1.73	1.71	1.68	1.66	1.64	1.63	1.60	1.59	1.57	1.56	1.55	1.54	1.53	1.52	1.52	1.51	1.47
50	1.76	1.74	1.72	1.70	1.69	1.66	1.63	1.61	1.60	1.58	1.56	1.54	1.53	1.52	1.51	1.50	1.49	1.49	1.48	1.44
60	1.72	1.70	1.68	1.66	1.65	1.62	1.59	1.57	1.56	1.53	1.52	1.50	1.49	1.48	1.47	1.46	1.45	1.44	1.44	1.39
70	1.70	1.67	1.65	1.64	1.62	1.59	1.57	1.55	1.53	1.50	1.49	1.47	1.46	1.45	1.44	1.42	1.42	1.41	1.40	1.35
80	1.68	1.65	1.63	1.62	1.60	1.57	1.54	1.52	1.51	1.48	1.46	1.45	1.44	1.43	1.41	1.40	1.39	1.38	1.38	1.33
90	1.66	1.64	1.62	1.60	1.59	1.55	1.53	1.51	1.49	1.46	1.44	1.43	1.42	1.41	1.39	1.38	1.37	1.36	1.36	1.30
100	1.65	1.63	1.61	1.59	1.57	1.54	1.52	1.49	1.48	1.45	1.43	1.41	1.40	1.39	1.38	1.36	1.35	1.35	1.34	1.28
120	1.63	1.61	1.59	1.57	1.55	1.52	1.50	1.47	1.46	1.43	1.41	1.39	1.38	1.37	1.35	1.34	1.33	1.32	1.32	1.26
140	1.62	1.60	1.57	1.56	1.54	1.51	1.48	1.46	1.44	1.41	1.39	1.38	1.36	1.35	1.33	1.32	1.31	1.30	1.30	1.23
160	1.61	1.59	1.57	1.55	1.53	1.50	1.47	1.45	1.43	1.40	1.38	1.36	1.35	1.34	1.32	1.31	1.30	1.29	1.28	1.22
180	1.60	1.58	1.56	1.54	1.52	1.49	1.46	1.44	1.42	1.39	1.37	1.35	1.34	1.33	1.31	1.30	1.29	1.28	1.27	1.20
200	1.60	1.57	1.55	1.53	1.52	1.48	1.46	1.43	1.41	1.39	1.36	1.35	1.33	1.32	1.30	1.29	1.28	1.27	1.26	1.19
∞	1.54	1.52	1.50	1.48	1.46	1.42	1.40	1.37	1.35	1.32	1.29	1.28	1.26	1.25	1.22	1.21	1.19	1.18	1.17	1.00

DENOMINATOR DEGREES OF FREEDOM

TABLE 6(b) Values of the F-Distribution: $A = .025$

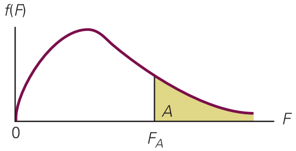


NUMERATOR DEGREES OF FREEDOM (ν_1); DENOMINATOR DEGREES OF FREEDOM (ν_2)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	648	799	864	900	922	937	948	957	963	969	973	977	980	983	985	987	989	990	992	993
2	38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4
3	17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5	14.4	14.4	14.3	14.3	14.3	14.3	14.2	14.2	14.2	14.2	14.2
4	12.2	10.6	10.0	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.79	8.75	8.71	8.68	8.66	8.63	8.61	8.59	8.58	8.56
5	10.0	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.57	6.52	6.49	6.46	6.43	6.40	6.38	6.36	6.34	6.33
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.41	5.37	5.33	5.30	5.27	5.24	5.22	5.20	5.18	5.17
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.71	4.67	4.63	4.60	4.57	4.54	4.52	4.50	4.48	4.47
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.24	4.20	4.16	4.13	4.10	4.08	4.05	4.03	4.02	4.00
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.91	3.87	3.83	3.80	3.77	3.74	3.72	3.70	3.68	3.67
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.66	3.62	3.58	3.55	3.52	3.50	3.47	3.45	3.44	3.42
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.47	3.43	3.39	3.36	3.33	3.30	3.28	3.26	3.24	3.23
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.32	3.28	3.24	3.21	3.18	3.15	3.13	3.11	3.09	3.07
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.20	3.15	3.12	3.08	3.05	3.03	3.00	2.98	2.96	2.95
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.09	3.05	3.01	2.98	2.95	2.92	2.90	2.88	2.86	2.84
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	3.01	2.96	2.92	2.89	2.86	2.84	2.81	2.79	2.77	2.76
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.93	2.89	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.68
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.87	2.82	2.79	2.75	2.72	2.70	2.67	2.65	2.63	2.62
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.81	2.77	2.73	2.70	2.67	2.64	2.62	2.60	2.58	2.56
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.76	2.72	2.68	2.65	2.62	2.59	2.57	2.55	2.53	2.51
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.72	2.68	2.64	2.60	2.57	2.55	2.52	2.50	2.48	2.46
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.65	2.60	2.56	2.53	2.50	2.47	2.45	2.43	2.41	2.39
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.59	2.54	2.50	2.47	2.44	2.41	2.39	2.36	2.35	2.33
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.54	2.49	2.45	2.42	2.39	2.36	2.34	2.31	2.29	2.28
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.49	2.45	2.41	2.37	2.34	2.32	2.29	2.27	2.25	2.23
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.46	2.41	2.37	2.34	2.31	2.28	2.26	2.23	2.21	2.20
35	5.48	4.11	3.52	3.18	2.96	2.80	2.68	2.58	2.50	2.44	2.39	2.34	2.30	2.27	2.23	2.21	2.18	2.16	2.14	2.12
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.33	2.29	2.25	2.21	2.18	2.15	2.13	2.11	2.09	2.07
45	5.38	4.01	3.42	3.09	2.86	2.70	2.58	2.49	2.41	2.35	2.29	2.25	2.21	2.17	2.14	2.11	2.09	2.07	2.04	2.03
50	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38	2.32	2.26	2.22	2.18	2.14	2.11	2.08	2.06	2.03	2.01	1.99
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.22	2.17	2.13	2.09	2.06	2.03	2.01	1.98	1.96	1.94
70	5.25	3.89	3.31	2.97	2.75	2.59	2.47	2.38	2.30	2.24	2.18	2.14	2.10	2.06	2.03	2.00	1.97	1.95	1.93	1.91
80	5.22	3.86	3.28	2.95	2.73	2.57	2.45	2.35	2.28	2.21	2.16	2.11	2.07	2.03	2.00	1.97	1.95	1.92	1.90	1.88
90	5.20	3.84	3.26	2.93	2.71	2.55	2.43	2.34	2.26	2.19	2.14	2.09	2.05	2.02	1.98	1.95	1.93	1.91	1.88	1.86
100	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24	2.18	2.12	2.08	2.04	2.00	1.97	1.94	1.91	1.89	1.87	1.85
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.10	2.05	2.01	1.98	1.94	1.92	1.89	1.87	1.84	1.82
140	5.13	3.79	3.21	2.88	2.66	2.50	2.38	2.28	2.21	2.14	2.09	2.04	2.00	1.96	1.93	1.90	1.87	1.85	1.83	1.81
160	5.12	3.78	3.20	2.87	2.65	2.49	2.37	2.27	2.19	2.13	2.07	2.03	1.99	1.95	1.92	1.89	1.86	1.84	1.82	1.80
180	5.11	3.77	3.19	2.86	2.64	2.48	2.36	2.26	2.19	2.12	2.07	2.02	1.98	1.94	1.91	1.88	1.85	1.83	1.81	1.79
200	5.10	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18	2.11	2.06	2.01	1.97	1.93	1.90	1.87	1.84	1.82	1.80	1.78
∞	5.03	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.99	1.95	1.90	1.87	1.83	1.80	1.78	1.75	1.73	1.71

ν_2 \ ν_1	NUMERATOR DEGREES OF FREEDOM																			
DENOMINATOR DEGREES OF FREEDOM	22	24	26	28	30	35	40	45	50	60	70	80	90	100	120	140	160	180	200	∞
1	995	997	999	1000	1001	1004	1006	1007	1008	1010	1011	1012	1013	1013	1014	1015	1015	1015	1016	1018
2	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5
3	14.1	14.1	14.1	14.1	14.1	14.1	14.0	14.0	14.0	14.0	14.0	14.0	14.0	14.0	13.9	13.9	13.9	13.9	13.9	13.9
4	8.53	8.51	8.49	8.48	8.46	8.43	8.41	8.39	8.38	8.36	8.35	8.33	8.33	8.32	8.31	8.30	8.30	8.29	8.29	8.26
5	6.30	6.28	6.26	6.24	6.23	6.20	6.18	6.16	6.14	6.12	6.11	6.10	6.09	6.08	6.07	6.06	6.06	6.05	6.05	6.02
6	5.14	5.12	5.10	5.08	5.07	5.04	5.01	4.99	4.98	4.96	4.94	4.93	4.92	4.92	4.90	4.90	4.89	4.89	4.88	4.85
7	4.44	4.41	4.39	4.38	4.36	4.33	4.31	4.29	4.28	4.25	4.24	4.23	4.22	4.21	4.20	4.19	4.18	4.18	4.18	4.14
8	3.97	3.95	3.93	3.91	3.89	3.86	3.84	3.82	3.81	3.78	3.77	3.76	3.75	3.74	3.73	3.72	3.71	3.71	3.70	3.67
9	3.64	3.61	3.59	3.58	3.56	3.53	3.51	3.49	3.47	3.45	3.43	3.42	3.41	3.40	3.39	3.38	3.38	3.37	3.37	3.33
10	3.39	3.37	3.34	3.33	3.31	3.28	3.26	3.24	3.22	3.20	3.18	3.17	3.16	3.15	3.14	3.13	3.13	3.12	3.12	3.08
11	3.20	3.17	3.15	3.13	3.12	3.09	3.06	3.04	3.03	3.00	2.99	2.97	2.96	2.96	2.94	2.94	2.93	2.92	2.92	2.88
12	3.04	3.02	3.00	2.98	2.96	2.93	2.91	2.89	2.87	2.85	2.83	2.82	2.81	2.80	2.79	2.78	2.77	2.77	2.76	2.73
13	2.92	2.89	2.87	2.85	2.84	2.80	2.78	2.76	2.74	2.72	2.70	2.69	2.68	2.67	2.66	2.65	2.64	2.64	2.63	2.60
14	2.81	2.79	2.77	2.75	2.73	2.70	2.67	2.65	2.64	2.61	2.60	2.58	2.57	2.56	2.55	2.54	2.54	2.53	2.53	2.49
15	2.73	2.70	2.68	2.66	2.64	2.61	2.59	2.56	2.55	2.52	2.51	2.49	2.48	2.47	2.46	2.45	2.44	2.44	2.44	2.40
16	2.65	2.63	2.60	2.58	2.57	2.53	2.51	2.49	2.47	2.45	2.43	2.42	2.40	2.40	2.38	2.37	2.37	2.36	2.36	2.32
17	2.59	2.56	2.54	2.52	2.50	2.47	2.44	2.42	2.41	2.38	2.36	2.35	2.34	2.33	2.32	2.31	2.30	2.29	2.29	2.25
18	2.53	2.50	2.48	2.46	2.44	2.41	2.38	2.36	2.35	2.32	2.30	2.29	2.28	2.27	2.26	2.25	2.24	2.23	2.23	2.19
19	2.48	2.45	2.43	2.41	2.39	2.36	2.33	2.31	2.30	2.27	2.25	2.24	2.23	2.22	2.20	2.19	2.19	2.18	2.18	2.13
20	2.43	2.41	2.39	2.37	2.35	2.31	2.29	2.27	2.25	2.22	2.20	2.19	2.18	2.17	2.16	2.15	2.14	2.13	2.13	2.09
22	2.36	2.33	2.31	2.29	2.27	2.24	2.21	2.19	2.17	2.14	2.13	2.11	2.10	2.09	2.08	2.07	2.06	2.05	2.05	2.00
24	2.30	2.27	2.25	2.23	2.21	2.17	2.15	2.12	2.11	2.08	2.06	2.05	2.03	2.02	2.01	2.00	1.99	1.99	1.98	1.94
26	2.24	2.22	2.19	2.17	2.16	2.12	2.09	2.07	2.05	2.03	2.01	1.99	1.98	1.97	1.95	1.94	1.94	1.93	1.92	1.88
28	2.20	2.17	2.15	2.13	2.11	2.08	2.05	2.03	2.01	1.98	1.96	1.94	1.93	1.92	1.91	1.90	1.89	1.88	1.88	1.83
30	2.16	2.14	2.11	2.09	2.07	2.04	2.01	1.99	1.97	1.94	1.92	1.90	1.89	1.88	1.87	1.86	1.85	1.84	1.84	1.79
35	2.09	2.06	2.04	2.02	2.00	1.96	1.93	1.91	1.89	1.86	1.84	1.82	1.81	1.80	1.79	1.77	1.77	1.76	1.75	1.70
40	2.03	2.01	1.98	1.96	1.94	1.90	1.88	1.85	1.83	1.80	1.78	1.76	1.75	1.74	1.72	1.71	1.70	1.70	1.69	1.64
45	1.99	1.96	1.94	1.92	1.90	1.86	1.83	1.81	1.79	1.76	1.74	1.72	1.70	1.69	1.68	1.66	1.66	1.65	1.64	1.59
50	1.96	1.93	1.91	1.89	1.87	1.83	1.80	1.77	1.75	1.72	1.70	1.68	1.67	1.66	1.64	1.63	1.62	1.61	1.60	1.55
60	1.91	1.88	1.86	1.83	1.82	1.78	1.74	1.72	1.70	1.67	1.64	1.63	1.61	1.60	1.58	1.57	1.56	1.55	1.54	1.48
70	1.88	1.85	1.82	1.80	1.78	1.74	1.71	1.68	1.66	1.63	1.60	1.59	1.57	1.56	1.54	1.53	1.52	1.51	1.50	1.44
80	1.85	1.82	1.79	1.77	1.75	1.71	1.68	1.65	1.63	1.60	1.57	1.55	1.54	1.53	1.51	1.49	1.48	1.47	1.47	1.40
90	1.83	1.80	1.77	1.75	1.73	1.69	1.66	1.63	1.61	1.58	1.55	1.53	1.52	1.50	1.48	1.47	1.46	1.45	1.44	1.37
100	1.81	1.78	1.76	1.74	1.71	1.67	1.64	1.61	1.59	1.56	1.53	1.51	1.50	1.48	1.46	1.45	1.44	1.43	1.42	1.35
120	1.79	1.76	1.73	1.71	1.69	1.65	1.61	1.59	1.56	1.53	1.50	1.48	1.47	1.45	1.43	1.42	1.41	1.40	1.39	1.31
140	1.77	1.74	1.72	1.69	1.67	1.63	1.60	1.57	1.55	1.51	1.48	1.46	1.45	1.43	1.41	1.39	1.38	1.37	1.36	1.28
160	1.76	1.73	1.70	1.68	1.66	1.62	1.58	1.55	1.53	1.50	1.47	1.45	1.43	1.42	1.39	1.38	1.36	1.35	1.35	1.26
180	1.75	1.72	1.69	1.67	1.65	1.61	1.57	1.54	1.52	1.48	1.46	1.43	1.42	1.40	1.38	1.36	1.35	1.34	1.33	1.25
200	1.74	1.71	1.68	1.66	1.64	1.60	1.56	1.53	1.51	1.47	1.45	1.42	1.41	1.39	1.37	1.35	1.34	1.33	1.32	1.23
∞	1.67	1.64	1.61	1.59	1.57	1.52	1.49	1.46	1.43	1.39	1.36	1.33	1.31	1.30	1.27	1.25	1.23	1.22	1.21	1.00

TABLE 6(c) Values of the F-Distribution: $A = .01$

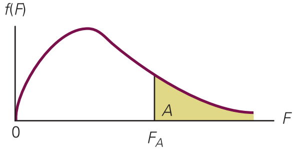


ν_2 \ ν_1 (DENOMINATOR / NUMERATOR DEGREES OF FREEDOM)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	4052	4999	5403	5625	5764	5859	5928	5981	6022	6056	6083	6106	6126	6143	6157	6170	6181	6192	6201	6209
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	27.1	27.0	26.9	26.9	26.8	26.8	26.8	26.7	26.7
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.5	14.4	14.3	14.2	14.2	14.2	14.1	14.1	14.0	14.0
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.96	9.89	9.82	9.77	9.72	9.68	9.64	9.61	9.58	9.55
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.79	7.72	7.66	7.60	7.56	7.52	7.48	7.45	7.42	7.40
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54	6.47	6.41	6.36	6.31	6.28	6.24	6.21	6.18	6.16
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73	5.67	5.61	5.56	5.52	5.48	5.44	5.41	5.38	5.36
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18	5.11	5.05	5.01	4.96	4.92	4.89	4.86	4.83	4.81
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.77	4.71	4.65	4.60	4.56	4.52	4.49	4.46	4.43	4.41
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.46	4.40	4.34	4.29	4.25	4.21	4.18	4.15	4.12	4.10
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.22	4.16	4.10	4.05	4.01	3.97	3.94	3.91	3.88	3.86
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	4.02	3.96	3.91	3.86	3.82	3.78	3.75	3.72	3.69	3.66
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.86	3.80	3.75	3.70	3.66	3.62	3.59	3.56	3.53	3.51
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.73	3.67	3.61	3.56	3.52	3.49	3.45	3.42	3.40	3.37
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.62	3.55	3.50	3.45	3.41	3.37	3.34	3.31	3.28	3.26
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.52	3.46	3.40	3.35	3.31	3.27	3.24	3.21	3.19	3.16
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.43	3.37	3.32	3.27	3.23	3.19	3.16	3.13	3.10	3.08
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.36	3.30	3.24	3.19	3.15	3.12	3.08	3.05	3.03	3.00
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.29	3.23	3.18	3.13	3.09	3.05	3.02	2.99	2.96	2.94
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12	3.07	3.02	2.98	2.94	2.91	2.88	2.85	2.83
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03	2.98	2.93	2.89	2.85	2.82	2.79	2.76	2.74
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	3.02	2.96	2.90	2.86	2.81	2.78	2.75	2.72	2.69	2.66
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.96	2.90	2.84	2.79	2.75	2.72	2.68	2.65	2.63	2.60
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84	2.79	2.74	2.70	2.66	2.63	2.60	2.57	2.55
35	7.42	5.27	4.40	3.91	3.59	3.37	3.20	3.07	2.96	2.88	2.80	2.74	2.69	2.64	2.60	2.56	2.53	2.50	2.47	2.44
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73	2.66	2.61	2.56	2.52	2.48	2.45	2.42	2.39	2.37
45	7.23	5.11	4.25	3.77	3.45	3.23	3.07	2.94	2.83	2.74	2.67	2.61	2.55	2.51	2.46	2.43	2.39	2.36	2.34	2.31
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78	2.70	2.63	2.56	2.51	2.46	2.42	2.38	2.35	2.32	2.29	2.27
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56	2.50	2.44	2.39	2.35	2.31	2.28	2.25	2.22	2.20
70	7.01	4.92	4.07	3.60	3.29	3.07	2.91	2.78	2.67	2.59	2.51	2.45	2.40	2.35	2.31	2.27	2.23	2.20	2.18	2.15
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55	2.48	2.42	2.36	2.31	2.27	2.23	2.20	2.17	2.14	2.12
90	6.93	4.85	4.01	3.53	3.23	3.01	2.84	2.72	2.61	2.52	2.45	2.39	2.33	2.29	2.24	2.21	2.17	2.14	2.11	2.09
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.43	2.37	2.31	2.27	2.22	2.19	2.15	2.12	2.09	2.07
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.40	2.34	2.28	2.23	2.19	2.15	2.12	2.09	2.06	2.03
140	6.82	4.76	3.92	3.46	3.15	2.93	2.77	2.64	2.54	2.45	2.38	2.31	2.26	2.21	2.17	2.13	2.10	2.07	2.04	2.01
160	6.80	4.74	3.91	3.44	3.13	2.92	2.75	2.62	2.52	2.43	2.36	2.30	2.24	2.20	2.15	2.11	2.08	2.05	2.02	1.99
180	6.78	4.73	3.89	3.43	3.12	2.90	2.74	2.61	2.51	2.42	2.35	2.28	2.23	2.18	2.14	2.10	2.07	2.04	2.01	1.98
200	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.34	2.27	2.22	2.17	2.13	2.09	2.06	2.03	2.00	1.97
∞	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25	2.19	2.13	2.08	2.04	2.00	1.97	1.94	1.91	1.88

ν_2 \ ν_1	22	24	26	28	30	35	40	45	50	60	70	80	90	100	120	140	160	180	200	∞
	NUMERATOR DEGREES OF FREEDOM																			
1	6223	6235	6245	6253	6261	6276	6287	6296	6303	6313	6321	6326	6331	6334	6339	6343	6346	6348	6350	6366
2	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5
3	26.6	26.6	26.6	26.5	26.5	26.5	26.4	26.4	26.4	26.3	26.3	26.3	26.3	26.2	26.2	26.2	26.2	26.2	26.2	26.1
4	14.0	13.9	13.9	13.9	13.8	13.8	13.7	13.7	13.7	13.7	13.6	13.6	13.6	13.6	13.6	13.5	13.5	13.5	13.5	13.5
5	9.51	9.47	9.43	9.40	9.38	9.33	9.29	9.26	9.24	9.20	9.18	9.16	9.14	9.13	9.11	9.10	9.09	9.08	9.08	9.02
6	7.35	7.31	7.28	7.25	7.23	7.18	7.14	7.11	7.09	7.06	7.03	7.01	7.00	6.99	6.97	6.96	6.95	6.94	6.93	6.88
7	6.11	6.07	6.04	6.02	5.99	5.94	5.91	5.88	5.86	5.82	5.80	5.78	5.77	5.75	5.74	5.72	5.72	5.71	5.70	5.65
8	5.32	5.28	5.25	5.22	5.20	5.15	5.12	5.09	5.07	5.03	5.01	4.99	4.97	4.96	4.95	4.93	4.92	4.92	4.91	4.86
9	4.77	4.73	4.70	4.67	4.65	4.60	4.57	4.54	4.52	4.48	4.46	4.44	4.43	4.41	4.40	4.39	4.38	4.37	4.36	4.31
10	4.36	4.33	4.30	4.27	4.25	4.20	4.17	4.14	4.12	4.08	4.06	4.04	4.03	4.01	4.00	3.98	3.97	3.97	3.96	3.91
11	4.06	4.02	3.99	3.96	3.94	3.89	3.86	3.83	3.81	3.78	3.75	3.73	3.72	3.71	3.69	3.68	3.67	3.66	3.66	3.60
12	3.82	3.78	3.75	3.72	3.70	3.65	3.62	3.59	3.57	3.54	3.51	3.49	3.48	3.47	3.45	3.44	3.43	3.42	3.41	3.36
13	3.62	3.59	3.56	3.53	3.51	3.46	3.43	3.40	3.38	3.34	3.32	3.30	3.28	3.27	3.25	3.24	3.23	3.23	3.22	3.17
14	3.46	3.43	3.40	3.37	3.35	3.30	3.27	3.24	3.22	3.18	3.16	3.14	3.12	3.11	3.09	3.08	3.07	3.06	3.06	3.01
15	3.33	3.29	3.26	3.24	3.21	3.17	3.13	3.10	3.08	3.05	3.02	3.00	2.99	2.98	2.96	2.95	2.94	2.93	2.92	2.87
16	3.22	3.18	3.15	3.12	3.10	3.05	3.02	2.99	2.97	2.93	2.91	2.89	2.87	2.86	2.84	2.83	2.82	2.81	2.81	2.75
17	3.12	3.08	3.05	3.03	3.00	2.96	2.92	2.89	2.87	2.83	2.81	2.79	2.78	2.76	2.75	2.73	2.72	2.72	2.71	2.65
18	3.03	3.00	2.97	2.94	2.92	2.87	2.84	2.81	2.78	2.75	2.72	2.70	2.69	2.68	2.66	2.65	2.64	2.63	2.62	2.57
19	2.96	2.92	2.89	2.87	2.84	2.80	2.76	2.73	2.71	2.67	2.65	2.63	2.61	2.60	2.58	2.57	2.56	2.55	2.55	2.49
20	2.90	2.86	2.83	2.80	2.78	2.73	2.69	2.67	2.64	2.61	2.58	2.56	2.55	2.54	2.52	2.50	2.49	2.49	2.48	2.42
22	2.78	2.75	2.72	2.69	2.67	2.62	2.58	2.55	2.53	2.50	2.47	2.45	2.43	2.42	2.40	2.39	2.38	2.37	2.36	2.31
24	2.70	2.66	2.63	2.60	2.58	2.53	2.49	2.46	2.44	2.40	2.38	2.36	2.34	2.33	2.31	2.30	2.29	2.28	2.27	2.21
26	2.62	2.58	2.55	2.53	2.50	2.45	2.42	2.39	2.36	2.33	2.30	2.28	2.26	2.25	2.23	2.22	2.21	2.20	2.19	2.13
28	2.56	2.52	2.49	2.46	2.44	2.39	2.35	2.32	2.30	2.26	2.24	2.22	2.20	2.19	2.17	2.15	2.14	2.13	2.13	2.07
30	2.51	2.47	2.44	2.41	2.39	2.34	2.30	2.27	2.25	2.21	2.18	2.16	2.14	2.13	2.11	2.10	2.09	2.08	2.07	2.01
35	2.40	2.36	2.33	2.30	2.28	2.23	2.19	2.16	2.14	2.10	2.07	2.05	2.03	2.02	2.00	1.98	1.97	1.96	1.96	1.89
40	2.33	2.29	2.26	2.23	2.20	2.15	2.11	2.08	2.06	2.02	1.99	1.97	1.95	1.94	1.92	1.90	1.89	1.88	1.87	1.81
45	2.27	2.23	2.20	2.17	2.14	2.09	2.05	2.02	2.00	1.96	1.93	1.91	1.89	1.88	1.85	1.84	1.83	1.82	1.81	1.74
50	2.22	2.18	2.15	2.12	2.10	2.05	2.01	1.97	1.95	1.91	1.88	1.86	1.84	1.82	1.80	1.79	1.77	1.76	1.76	1.68
60	2.15	2.12	2.08	2.05	2.03	1.98	1.94	1.90	1.88	1.84	1.81	1.78	1.76	1.75	1.73	1.71	1.70	1.69	1.68	1.60
70	2.11	2.07	2.03	2.01	1.98	1.93	1.89	1.85	1.83	1.78	1.75	1.73	1.71	1.70	1.67	1.65	1.64	1.63	1.62	1.54
80	2.07	2.03	2.00	1.97	1.94	1.89	1.85	1.82	1.79	1.75	1.71	1.69	1.67	1.65	1.63	1.61	1.60	1.59	1.58	1.50
90	2.04	2.00	1.97	1.94	1.92	1.86	1.82	1.79	1.76	1.72	1.68	1.66	1.64	1.62	1.60	1.58	1.57	1.55	1.55	1.46
100	2.02	1.98	1.95	1.92	1.89	1.84	1.80	1.76	1.74	1.69	1.66	1.63	1.61	1.60	1.57	1.55	1.54	1.53	1.52	1.43
120	1.99	1.95	1.92	1.89	1.86	1.81	1.76	1.73	1.70	1.66	1.62	1.60	1.58	1.56	1.53	1.51	1.50	1.49	1.48	1.38
140	1.97	1.93	1.89	1.86	1.84	1.78	1.74	1.70	1.67	1.63	1.60	1.57	1.55	1.53	1.50	1.48	1.47	1.46	1.45	1.35
160	1.95	1.91	1.88	1.85	1.82	1.76	1.72	1.68	1.66	1.61	1.58	1.55	1.53	1.51	1.48	1.46	1.45	1.43	1.42	1.32
180	1.94	1.90	1.86	1.83	1.81	1.75	1.71	1.67	1.64	1.60	1.56	1.53	1.51	1.49	1.47	1.45	1.43	1.42	1.41	1.30
200	1.93	1.89	1.85	1.82	1.79	1.74	1.69	1.66	1.63	1.58	1.55	1.52	1.50	1.48	1.45	1.43	1.42	1.40	1.39	1.28
∞	1.83	1.79	1.76	1.73	1.70	1.64	1.59	1.56	1.53	1.48	1.44	1.41	1.38	1.36	1.33	1.30	1.28	1.26	1.25	1.00

DENOMINATOR DEGREES OF FREEDOM

TABLE **6(d)** Values of the *F*-Distribution: $A = .005$



ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	NUMERATOR DEGREES OF FREEDOM																			
1	16211	19999	21615	22500	23056	23437	23715	23925	24091	24224	24334	24426	24505	24572	24630	24681	24727	24767	24803	24836
2	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199
3	55.6	49.8	47.5	46.2	45.4	44.8	44.4	44.1	43.9	43.7	43.5	43.4	43.3	43.2	43.1	43.0	42.9	42.9	42.8	42.8
4	31.3	26.3	24.3	23.2	22.5	22.0	21.6	21.4	21.1	21.0	20.8	20.7	20.6	20.5	20.4	20.4	20.3	20.3	20.2	20.2
5	22.8	18.3	16.5	15.6	14.9	14.5	14.2	14.0	13.8	13.6	13.5	13.4	13.3	13.2	13.1	13.1	13.0	13.0	12.9	12.9
6	18.6	14.5	12.9	12.0	11.5	11.1	10.8	10.6	10.4	10.3	10.1	10.0	9.95	9.88	9.81	9.76	9.71	9.66	9.62	9.59
7	16.2	12.4	10.9	10.1	9.52	9.16	8.89	8.68	8.51	8.38	8.27	8.18	8.10	8.03	7.97	7.91	7.87	7.83	7.79	7.75
8	14.7	11.0	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.10	7.01	6.94	6.87	6.81	6.76	6.72	6.68	6.64	6.61
9	13.6	10.1	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.31	6.23	6.15	6.09	6.03	5.98	5.94	5.90	5.86	5.83
10	12.8	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.75	5.66	5.59	5.53	5.47	5.42	5.38	5.34	5.31	5.27
11	12.2	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.32	5.24	5.16	5.10	5.05	5.00	4.96	4.92	4.89	4.86
12	11.8	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.99	4.91	4.84	4.77	4.72	4.67	4.63	4.59	4.56	4.53
13	11.4	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.72	4.64	4.57	4.51	4.46	4.41	4.37	4.33	4.30	4.27
14	11.1	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.51	4.43	4.36	4.30	4.25	4.20	4.16	4.12	4.09	4.06
15	10.8	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.33	4.25	4.18	4.12	4.07	4.02	3.98	3.95	3.91	3.88
16	10.6	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.18	4.10	4.03	3.97	3.92	3.87	3.83	3.80	3.76	3.73
17	10.4	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	4.05	3.97	3.90	3.84	3.79	3.75	3.71	3.67	3.64	3.61
18	10.2	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.94	3.86	3.79	3.73	3.68	3.64	3.60	3.56	3.53	3.50
19	10.1	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.84	3.76	3.70	3.64	3.59	3.54	3.50	3.46	3.43	3.40
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.76	3.68	3.61	3.55	3.50	3.46	3.42	3.38	3.35	3.32
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.61	3.54	3.47	3.41	3.36	3.31	3.27	3.24	3.21	3.18
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.50	3.42	3.35	3.30	3.25	3.20	3.16	3.12	3.09	3.06
26	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60	3.49	3.40	3.33	3.26	3.20	3.15	3.11	3.07	3.03	3.00	2.97
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.32	3.25	3.18	3.12	3.07	3.03	2.99	2.95	2.92	2.89
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.25	3.18	3.11	3.06	3.01	2.96	2.92	2.89	2.85	2.82
35	8.98	6.19	5.09	4.48	4.09	3.81	3.61	3.45	3.32	3.21	3.12	3.05	2.98	2.93	2.88	2.83	2.79	2.76	2.72	2.69
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	3.03	2.95	2.89	2.83	2.78	2.74	2.70	2.66	2.63	2.60
45	8.71	5.97	4.89	4.29	3.91	3.64	3.43	3.28	3.15	3.04	2.96	2.88	2.82	2.76	2.71	2.66	2.62	2.59	2.56	2.53
50	8.63	5.90	4.83	4.23	3.85	3.58	3.38	3.22	3.09	2.99	2.90	2.82	2.76	2.70	2.65	2.61	2.57	2.53	2.50	2.47
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.82	2.74	2.68	2.62	2.57	2.53	2.49	2.45	2.42	2.39
70	8.40	5.72	4.66	4.08	3.70	3.43	3.23	3.08	2.95	2.85	2.76	2.68	2.62	2.56	2.51	2.47	2.43	2.39	2.36	2.33
80	8.33	5.67	4.61	4.03	3.65	3.39	3.19	3.03	2.91	2.80	2.72	2.64	2.58	2.52	2.47	2.43	2.39	2.35	2.32	2.29
90	8.28	5.62	4.57	3.99	3.62	3.35	3.15	3.00	2.87	2.77	2.68	2.61	2.54	2.49	2.44	2.39	2.35	2.32	2.28	2.25
100	8.24	5.59	4.54	3.96	3.59	3.33	3.13	2.97	2.85	2.74	2.66	2.58	2.52	2.46	2.41	2.37	2.33	2.29	2.26	2.23
120	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.62	2.54	2.48	2.42	2.37	2.33	2.29	2.25	2.22	2.19
140	8.14	5.50	4.47	3.89	3.52	3.26	3.06	2.91	2.78	2.68	2.59	2.52	2.45	2.40	2.35	2.30	2.26	2.22	2.19	2.16
160	8.10	5.48	4.44	3.87	3.50	3.24	3.04	2.88	2.76	2.66	2.57	2.50	2.43	2.38	2.33	2.28	2.24	2.20	2.17	2.14
180	8.08	5.46	4.42	3.85	3.48	3.22	3.02	2.87	2.74	2.64	2.56	2.48	2.42	2.36	2.31	2.26	2.22	2.19	2.15	2.12
200	8.06	5.44	4.41	3.84	3.47	3.21	3.01	2.86	2.73	2.63	2.54	2.47	2.40	2.35	2.30	2.25	2.21	2.18	2.14	2.11
∞	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.75	2.62	2.52	2.43	2.36	2.30	2.24	2.19	2.14	2.10	2.07	2.03	2.00

DENOMINATOR DEGREES OF FREEDOM

ν_2 \ ν_1	22	24	26	28	30	35	40	45	50	60	70	80	90	100	120	140	160	180	200	∞
	NUMERATOR DEGREES OF FREEDOM																			
1	24892	24940	24980	25014	25044	25103	25148	25183	25211	25253	25283	25306	25323	25337	25359	25374	25385	25394	25401	25464
2	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199
3	42.7	42.6	42.6	42.5	42.5	42.4	42.3	42.3	42.2	42.1	42.1	42.1	42.0	42.0	42.0	42.0	41.9	41.9	41.9	41.8
4	20.1	20.0	20.0	19.9	19.9	19.8	19.8	19.7	19.7	19.6	19.6	19.5	19.5	19.5	19.5	19.4	19.4	19.4	19.4	19.3
5	12.8	12.8	12.7	12.7	12.7	12.6	12.5	12.5	12.5	12.4	12.4	12.3	12.3	12.3	12.3	12.3	12.2	12.2	12.2	12.1
6	9.53	9.47	9.43	9.39	9.36	9.29	9.24	9.20	9.17	9.12	9.09	9.06	9.04	9.03	9.00	8.98	8.97	8.96	8.95	8.88
7	7.69	7.64	7.60	7.57	7.53	7.47	7.42	7.38	7.35	7.31	7.28	7.25	7.23	7.22	7.19	7.18	7.16	7.15	7.15	7.08
8	6.55	6.50	6.46	6.43	6.40	6.33	6.29	6.25	6.22	6.18	6.15	6.12	6.10	6.09	6.06	6.05	6.04	6.03	6.02	5.95
9	5.78	5.73	5.69	5.65	5.62	5.56	5.52	5.48	5.45	5.41	5.38	5.36	5.34	5.32	5.30	5.28	5.27	5.26	5.26	5.19
10	5.22	5.17	5.13	5.10	5.07	5.01	4.97	4.93	4.90	4.86	4.83	4.80	4.79	4.77	4.75	4.73	4.72	4.71	4.71	4.64
11	4.80	4.76	4.72	4.68	4.65	4.60	4.55	4.52	4.49	4.45	4.41	4.39	4.37	4.36	4.34	4.32	4.31	4.30	4.29	4.23
12	4.48	4.43	4.39	4.36	4.33	4.27	4.23	4.19	4.17	4.12	4.09	4.07	4.05	4.04	4.01	4.00	3.99	3.98	3.97	3.91
13	4.22	4.17	4.13	4.10	4.07	4.01	3.97	3.94	3.91	3.87	3.84	3.81	3.79	3.78	3.76	3.74	3.73	3.72	3.71	3.65
14	4.01	3.96	3.92	3.89	3.86	3.80	3.76	3.73	3.70	3.66	3.62	3.60	3.58	3.57	3.55	3.53	3.52	3.51	3.50	3.44
15	3.83	3.79	3.75	3.72	3.69	3.63	3.58	3.55	3.52	3.48	3.45	3.43	3.41	3.39	3.37	3.36	3.34	3.34	3.33	3.26
16	3.68	3.64	3.60	3.57	3.54	3.48	3.44	3.40	3.37	3.33	3.30	3.28	3.26	3.25	3.22	3.21	3.20	3.19	3.18	3.11
17	3.56	3.51	3.47	3.44	3.41	3.35	3.31	3.28	3.25	3.21	3.18	3.15	3.13	3.12	3.10	3.08	3.07	3.06	3.05	2.99
18	3.45	3.40	3.36	3.33	3.30	3.25	3.20	3.17	3.14	3.10	3.07	3.04	3.02	3.01	2.99	2.97	2.96	2.95	2.94	2.87
19	3.35	3.31	3.27	3.24	3.21	3.15	3.11	3.07	3.04	3.00	2.97	2.95	2.93	2.91	2.89	2.87	2.86	2.85	2.85	2.78
20	3.27	3.22	3.18	3.15	3.12	3.07	3.02	2.99	2.96	2.92	2.88	2.86	2.84	2.83	2.81	2.79	2.78	2.77	2.76	2.69
22	3.12	3.08	3.04	3.01	2.98	2.92	2.88	2.84	2.82	2.77	2.74	2.72	2.70	2.69	2.66	2.65	2.63	2.62	2.62	2.55
24	3.01	2.97	2.93	2.90	2.87	2.81	2.77	2.73	2.70	2.66	2.63	2.60	2.58	2.57	2.55	2.53	2.52	2.51	2.50	2.43
26	2.92	2.87	2.84	2.80	2.77	2.72	2.67	2.64	2.61	2.56	2.53	2.51	2.49	2.47	2.45	2.43	2.42	2.41	2.40	2.33
28	2.84	2.79	2.76	2.72	2.69	2.64	2.59	2.56	2.53	2.48	2.45	2.43	2.41	2.39	2.37	2.35	2.34	2.33	2.32	2.25
30	2.77	2.73	2.69	2.66	2.63	2.57	2.52	2.49	2.46	2.42	2.38	2.36	2.34	2.32	2.30	2.28	2.27	2.26	2.25	2.18
35	2.64	2.60	2.56	2.53	2.50	2.44	2.39	2.36	2.33	2.28	2.25	2.22	2.20	2.19	2.16	2.15	2.13	2.12	2.11	2.04
40	2.55	2.50	2.46	2.43	2.40	2.34	2.30	2.26	2.23	2.18	2.15	2.12	2.10	2.09	2.06	2.05	2.03	2.02	2.01	1.93
45	2.47	2.43	2.39	2.36	2.33	2.27	2.22	2.19	2.16	2.11	2.08	2.05	2.03	2.01	1.99	1.97	1.95	1.94	1.93	1.85
50	2.42	2.37	2.33	2.30	2.27	2.21	2.16	2.13	2.10	2.05	2.02	1.99	1.97	1.95	1.93	1.91	1.89	1.88	1.87	1.79
60	2.33	2.29	2.25	2.22	2.19	2.13	2.08	2.04	2.01	1.96	1.93	1.90	1.88	1.86	1.83	1.81	1.80	1.79	1.78	1.69
70	2.28	2.23	2.19	2.16	2.13	2.07	2.02	1.98	1.95	1.90	1.86	1.84	1.81	1.80	1.77	1.75	1.73	1.72	1.71	1.62
80	2.23	2.19	2.15	2.11	2.08	2.02	1.97	1.94	1.90	1.85	1.82	1.79	1.77	1.75	1.72	1.70	1.68	1.67	1.66	1.57
90	2.20	2.15	2.12	2.08	2.05	1.99	1.94	1.90	1.87	1.82	1.78	1.75	1.73	1.71	1.68	1.66	1.64	1.63	1.62	1.52
100	2.17	2.13	2.09	2.05	2.02	1.96	1.91	1.87	1.84	1.79	1.75	1.72	1.70	1.68	1.65	1.63	1.61	1.60	1.59	1.49
120	2.13	2.09	2.05	2.01	1.98	1.92	1.87	1.83	1.80	1.75	1.71	1.68	1.66	1.64	1.61	1.58	1.57	1.55	1.54	1.43
140	2.11	2.06	2.02	1.99	1.96	1.89	1.84	1.80	1.77	1.72	1.68	1.65	1.62	1.60	1.57	1.55	1.53	1.52	1.51	1.39
160	2.09	2.04	2.00	1.97	1.93	1.87	1.82	1.78	1.75	1.69	1.65	1.62	1.60	1.58	1.55	1.52	1.51	1.49	1.48	1.36
180	2.07	2.02	1.98	1.95	1.92	1.85	1.80	1.76	1.73	1.68	1.64	1.61	1.58	1.56	1.53	1.50	1.49	1.47	1.46	1.34
200	2.06	2.01	1.97	1.94	1.91	1.84	1.79	1.75	1.71	1.66	1.62	1.59	1.56	1.54	1.51	1.49	1.47	1.45	1.44	1.32
∞	1.95	1.90	1.86	1.82	1.79	1.72	1.67	1.63	1.59	1.54	1.49	1.46	1.43	1.40	1.37	1.34	1.31	1.30	1.28	1.00

DENOMINATOR DEGREES OF FREEDOM

TABLE 7(a) Critical Values of the Studentized Range, $\alpha = .05$

ν	k=2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	18.0	27.0	32.8	37.1	40.4	43.1	45.4	47.4	49.1	50.6	52.0	53.2	54.3	55.4	56.3	57.2	58.0	58.8	59.6
2	6.08	8.33	9.80	10.9	11.7	12.4	13.0	13.5	14.0	14.4	14.7	15.1	15.4	15.7	15.9	16.1	16.4	16.6	16.8
3	4.50	5.91	6.82	7.50	8.04	8.48	8.85	9.18	9.46	9.72	9.95	10.2	10.3	10.5	10.7	10.8	11.0	11.1	11.2
4	3.93	5.04	5.76	6.29	6.71	7.05	7.35	7.60	7.83	8.03	8.21	8.37	8.52	8.66	8.79	8.91	9.03	9.13	9.23
5	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99	7.17	7.32	7.47	7.60	7.72	7.83	7.93	8.03	8.12	8.21
6	3.46	4.34	4.90	5.30	5.63	5.90	6.12	6.32	6.49	6.65	6.79	6.92	7.03	7.14	7.24	7.34	7.43	7.51	7.59
7	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16	6.30	6.43	6.55	6.66	6.76	6.85	6.94	7.02	7.10	7.17
8	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92	6.05	6.18	6.29	6.39	6.48	6.57	6.65	6.73	6.80	6.87
9	3.20	3.95	4.41	4.76	5.02	5.24	5.43	5.59	5.74	5.87	5.98	6.09	6.19	6.28	6.36	6.44	6.51	6.58	6.64
10	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60	5.72	5.83	5.93	6.03	6.11	6.19	6.27	6.34	6.40	6.47
11	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.49	5.61	5.71	5.81	5.90	5.98	6.06	6.13	6.20	6.27	6.33
12	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.39	5.51	5.61	5.71	5.80	5.88	5.95	6.02	6.09	6.15	6.21
13	3.06	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32	5.43	5.53	5.63	5.71	5.79	5.86	5.93	5.99	6.05	6.11
14	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25	5.36	5.46	5.55	5.64	5.71	5.79	5.85	5.91	5.97	6.03
15	3.01	3.67	4.08	4.37	4.59	4.78	4.94	5.08	5.20	5.31	5.40	5.49	5.57	5.65	5.72	5.78	5.85	5.90	5.96
16	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15	5.26	5.35	5.44	5.52	5.59	5.66	5.73	5.79	5.84	5.90
17	2.98	3.63	4.02	4.30	4.52	4.70	4.86	4.99	5.11	5.21	5.31	5.39	5.47	5.54	5.61	5.67	5.73	5.79	5.84
18	2.97	3.61	4.00	4.28	4.49	4.67	4.82	4.96	5.07	5.17	5.27	5.35	5.43	5.50	5.57	5.63	5.69	5.74	5.79
19	2.96	3.59	3.98	4.25	4.47	4.65	4.79	4.92	5.04	5.14	5.23	5.31	5.39	5.46	5.53	5.59	5.65	5.70	5.75
20	2.95	3.58	3.96	4.23	4.45	4.62	4.77	4.90	5.01	5.11	5.20	5.28	5.36	5.43	5.49	5.55	5.61	5.66	5.71
24	2.92	3.53	3.90	4.17	4.37	4.54	4.68	4.81	4.92	5.01	5.10	5.18	5.25	5.32	5.38	5.44	5.49	5.55	5.59
30	2.89	3.49	3.85	4.10	4.30	4.46	4.60	4.72	4.82	4.92	5.00	5.08	5.15	5.21	5.27	5.33	5.38	5.43	5.47
40	2.86	3.44	3.79	4.04	4.23	4.39	4.52	4.63	4.73	4.82	4.90	4.98	5.04	5.11	5.16	5.22	5.27	5.31	5.36
60	2.83	3.40	3.74	3.98	4.16	4.31	4.44	4.55	4.65	4.73	4.81	4.88	4.94	5.00	5.06	5.11	5.15	5.20	5.24
120	2.80	3.36	3.68	3.92	4.10	4.24	4.36	4.47	4.56	4.64	4.71	4.78	4.84	4.90	4.95	5.00	5.04	5.09	5.13
∞	2.77	3.31	3.63	3.86	4.03	4.17	4.29	4.39	4.47	4.55	4.62	4.68	4.74	4.80	4.85	4.89	4.93	4.97	5.01

TABLE **7(b)** Critical Values of the Studentized Range, $\alpha = .01$

ν	k = 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	90.0	135	164	186	202	216	227	237	246	253	260	266	272	277	282	286	290	294	298
2	14.0	19.0	22.3	24.7	26.6	28.2	29.5	30.7	31.7	32.6	33.4	34.1	34.8	35.4	36.0	36.5	37.0	37.5	37.9
3	8.26	10.6	12.2	13.3	14.2	15.0	15.6	16.2	16.7	17.1	17.5	17.9	18.2	18.5	18.8	19.1	19.3	19.5	19.8
4	6.51	8.12	9.17	9.96	10.6	11.1	11.5	11.9	12.3	12.6	12.8	13.1	13.3	13.5	13.7	13.9	14.1	14.2	14.4
5	5.70	6.97	7.80	8.42	8.91	9.32	9.67	9.97	10.2	10.5	10.7	10.9	11.1	11.2	11.4	11.6	11.7	11.8	11.9
6	5.24	6.33	7.03	7.56	7.97	8.32	8.61	8.87	9.10	9.30	9.49	9.65	9.81	9.95	10.1	10.2	10.3	10.4	10.5
7	4.95	5.92	6.54	7.01	7.37	7.68	7.94	8.17	8.37	8.55	8.71	8.86	9.00	9.12	9.24	9.35	9.46	9.55	9.65
8	4.74	5.63	6.20	6.63	6.96	7.24	7.47	7.68	7.87	8.03	8.18	8.31	8.44	8.55	8.66	8.76	8.85	8.94	9.03
9	4.60	5.43	5.96	6.35	6.66	6.91	7.13	7.32	7.49	7.65	7.78	7.91	8.03	8.13	8.23	8.32	8.41	8.49	8.57
10	4.48	5.27	5.77	6.14	6.43	6.67	6.87	7.05	7.21	7.36	7.48	7.60	7.71	7.81	7.91	7.99	8.07	8.15	8.22
11	4.39	5.14	5.62	5.97	6.25	6.48	6.67	6.84	6.99	7.13	7.25	7.36	7.46	7.56	7.65	7.73	7.81	7.88	7.95
12	4.32	5.04	5.50	5.84	6.10	6.32	6.51	6.67	6.81	6.94	7.06	7.17	7.26	7.36	7.44	7.52	7.59	7.66	7.73
13	4.26	4.96	5.40	5.73	5.98	6.19	6.37	6.53	6.67	6.79	6.90	7.01	7.10	7.19	7.27	7.34	7.42	7.48	7.55
14	4.21	4.89	5.32	5.63	5.88	6.08	6.26	6.41	6.54	6.66	6.77	6.87	6.96	7.05	7.12	7.20	7.27	7.33	7.39
15	4.17	4.83	5.25	5.56	5.80	5.99	6.16	6.31	6.44	6.55	6.66	6.76	6.84	6.93	7.00	7.07	7.14	7.20	7.26
16	4.13	4.78	5.19	5.49	5.72	5.92	6.08	6.22	6.35	6.46	6.56	6.66	6.74	6.82	6.90	6.97	7.03	7.09	7.15
17	4.10	4.74	5.14	5.43	5.66	5.85	6.01	6.15	6.27	6.38	6.48	6.57	6.66	6.73	6.80	6.87	6.94	7.00	7.05
18	4.07	4.70	5.09	5.38	5.60	5.79	5.94	6.08	6.20	6.31	6.41	6.50	6.58	6.65	6.72	6.79	6.85	6.91	6.96
19	4.05	4.67	5.05	5.33	5.55	5.73	5.89	6.02	6.14	6.25	6.34	6.43	6.51	6.58	6.65	6.72	6.78	6.84	6.89
20	4.02	4.64	5.02	5.29	5.51	5.69	5.84	5.97	6.09	6.19	6.29	6.37	6.45	6.52	6.59	6.65	6.71	6.76	6.82
24	3.96	4.54	4.91	5.17	5.37	5.54	5.69	5.81	5.92	6.02	6.11	6.19	6.26	6.33	6.39	6.45	6.51	6.56	6.61
30	3.89	4.45	4.80	5.05	5.24	5.40	5.54	5.65	5.76	5.85	5.93	6.01	6.08	6.14	6.20	6.26	6.31	6.36	6.41
40	3.82	4.37	4.70	4.93	5.11	5.27	5.39	5.50	5.60	5.69	5.77	5.84	5.90	5.96	6.02	6.07	6.12	6.17	6.21
60	3.76	4.28	4.60	4.82	4.99	5.13	5.25	5.36	5.45	5.53	5.60	5.67	5.73	5.79	5.84	5.89	5.93	5.98	6.02
120	3.70	4.20	4.50	4.71	4.87	5.01	5.12	5.21	5.30	5.38	5.44	5.51	5.56	5.61	5.66	5.71	5.75	5.79	5.83
∞	3.64	4.12	4.40	4.60	4.76	4.88	4.99	5.08	5.16	5.23	5.29	5.35	5.40	5.45	5.49	5.54	5.57	5.61	5.65

Source: From E. S. Pearson and H. O. Hartley, *Biometrika Tables for Statisticians*, 1: 176–77. Reproduced by permission of the Biometrika Trustees.

TABLE **8(a)** Critical Values for the Durbin-Watson Statistic, $\alpha = .05$

	$k = 1$		$k = 2$		$k = 3$		$k = 4$		$k = 5$	
n	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U
15	1.08	1.36	.95	1.54	.82	1.75	.69	1.97	.56	2.21
16	1.10	1.37	.98	1.54	.86	1.73	.74	1.93	.62	2.15
17	1.13	1.38	1.02	1.54	.90	1.71	.78	1.90	.67	2.10
18	1.16	1.39	1.05	1.53	.93	1.69	.82	1.87	.71	2.06
19	1.18	1.40	1.08	1.53	.97	1.68	.86	1.85	.75	2.02
20	1.20	1.41	1.10	1.54	1.00	1.68	.90	1.83	.79	1.99
21	1.22	1.42	1.13	1.54	1.03	1.67	.93	1.81	.83	1.96
22	1.24	1.43	1.15	1.54	1.05	1.66	.96	1.80	.86	1.94
23	1.26	1.44	1.17	1.54	1.08	1.66	.99	1.79	.90	1.92
24	1.27	1.45	1.19	1.55	1.10	1.66	1.01	1.78	.93	1.90
25	1.29	1.45	1.21	1.55	1.12	1.66	1.04	1.77	.95	1.89
26	1.30	1.46	1.22	1.55	1.14	1.65	1.06	1.76	.98	1.88
27	1.32	1.47	1.24	1.56	1.16	1.65	1.08	1.76	1.01	1.86
28	1.33	1.48	1.26	1.56	1.18	1.65	1.10	1.75	1.03	1.85
29	1.34	1.48	1.27	1.56	1.20	1.65	1.12	1.74	1.05	1.84
30	1.35	1.49	1.28	1.57	1.21	1.65	1.14	1.74	1.07	1.83
31	1.36	1.50	1.30	1.57	1.23	1.65	1.16	1.74	1.09	1.83
32	1.37	1.50	1.31	1.57	1.24	1.65	1.18	1.73	1.11	1.82
33	1.38	1.51	1.32	1.58	1.26	1.65	1.19	1.73	1.13	1.81
34	1.39	1.51	1.33	1.58	1.27	1.65	1.21	1.73	1.15	1.81
35	1.40	1.52	1.34	1.58	1.28	1.65	1.22	1.73	1.16	1.80
36	1.41	1.52	1.35	1.59	1.29	1.65	1.24	1.73	1.18	1.80
37	1.42	1.53	1.36	1.59	1.31	1.66	1.25	1.72	1.19	1.80
38	1.43	1.54	1.37	1.59	1.32	1.66	1.26	1.72	1.21	1.79
39	1.43	1.54	1.38	1.60	1.33	1.66	1.27	1.72	1.22	1.79
40	1.44	1.54	1.39	1.60	1.34	1.66	1.29	1.72	1.23	1.79
45	1.48	1.57	1.43	1.62	1.38	1.67	1.34	1.72	1.29	1.78
50	1.50	1.59	1.46	1.63	1.42	1.67	1.38	1.72	1.34	1.77
55	1.53	1.60	1.49	1.64	1.45	1.68	1.41	1.72	1.38	1.77
60	1.55	1.62	1.51	1.65	1.48	1.69	1.44	1.73	1.41	1.77
65	1.57	1.63	1.54	1.66	1.50	1.70	1.47	1.73	1.44	1.77
70	1.58	1.64	1.55	1.67	1.52	1.70	1.49	1.74	1.46	1.77
75	1.60	1.65	1.57	1.68	1.54	1.71	1.51	1.74	1.49	1.77
80	1.61	1.66	1.59	1.69	1.56	1.72	1.53	1.74	1.51	1.77
85	1.62	1.67	1.60	1.70	1.57	1.72	1.55	1.75	1.52	1.77
90	1.63	1.68	1.61	1.70	1.59	1.73	1.57	1.75	1.54	1.78
95	1.64	1.69	1.62	1.71	1.60	1.73	1.58	1.75	1.56	1.78
100	1.65	1.69	1.63	1.72	1.61	1.74	1.59	1.76	1.57	1.78

Source: From J. Durbin and G. S. Watson, "Testing for Serial Correlation in Least Squares Regression, II," *Biometrika* 30 (1951): 159–78. Reproduced by permission of the Biometrika Trustees.

TABLE **8(b)** Critical Values for the Durbin-Watson Statistic, $\alpha = .01$

	$k = 1$		$k = 2$		$k = 3$		$k = 4$		$k = 5$	
n	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U
15	.81	1.07	.70	1.25	.59	1.46	.49	1.70	.39	1.96
16	.84	1.09	.74	1.25	.63	1.44	.53	1.66	.44	1.90
17	.87	1.10	.77	1.25	.67	1.43	.57	1.63	.48	1.85
18	.90	1.12	.80	1.26	.71	1.42	.61	1.60	.52	1.80
19	.93	1.13	.83	1.26	.74	1.41	.65	1.58	.56	1.77
20	.95	1.15	.86	1.27	.77	1.41	.68	1.57	.60	1.74
21	.97	1.16	.89	1.27	.80	1.41	.72	1.55	.63	1.71
22	1.00	1.17	.91	1.28	.83	1.40	.75	1.54	.66	1.69
23	1.02	1.19	.94	1.29	.86	1.40	.77	1.53	.70	1.67
24	1.04	1.20	.96	1.30	.88	1.41	.80	1.53	.72	1.66
25	1.05	1.21	.98	1.30	.90	1.41	.83	1.52	.75	1.65
26	1.07	1.22	1.00	1.31	.93	1.41	.85	1.52	.78	1.64
27	1.09	1.23	1.02	1.32	.95	1.41	.88	1.51	.81	1.63
28	1.10	1.24	1.04	1.32	.97	1.41	.90	1.51	.83	1.62
29	1.12	1.25	1.05	1.33	.99	1.42	.92	1.51	.85	1.61
30	1.13	1.26	1.07	1.34	1.01	1.42	.94	1.51	.88	1.61
31	1.15	1.27	1.08	1.34	1.02	1.42	.96	1.51	.90	1.60
32	1.16	1.28	1.10	1.35	1.04	1.43	.98	1.51	.92	1.60
33	1.17	1.29	1.11	1.36	1.05	1.43	1.00	1.51	.94	1.59
34	1.18	1.30	1.13	1.36	1.07	1.43	1.01	1.51	.95	1.59
35	1.19	1.31	1.14	1.37	1.08	1.44	1.03	1.51	.97	1.59
36	1.21	1.32	1.15	1.38	1.10	1.44	1.04	1.51	.99	1.59
37	1.22	1.32	1.16	1.38	1.11	1.45	1.06	1.51	1.00	1.59
38	1.23	1.33	1.18	1.39	1.12	1.45	1.07	1.52	1.02	1.58
39	1.24	1.34	1.19	1.39	1.14	1.45	1.09	1.52	1.03	1.58
40	1.25	1.34	1.20	1.40	1.15	1.46	1.10	1.52	1.05	1.58
45	1.29	1.38	1.24	1.42	1.20	1.48	1.16	1.53	1.11	1.58
50	1.32	1.40	1.28	1.45	1.24	1.49	1.20	1.54	1.16	1.59
55	1.36	1.43	1.32	1.47	1.28	1.51	1.25	1.55	1.21	1.59
60	1.38	1.45	1.35	1.48	1.32	1.52	1.28	1.56	1.25	1.60
65	1.41	1.47	1.38	1.50	1.35	1.53	1.31	1.57	1.28	1.61
70	1.43	1.49	1.40	1.52	1.37	1.55	1.34	1.58	1.31	1.61
75	1.45	1.50	1.42	1.53	1.39	1.56	1.37	1.59	1.34	1.62
80	1.47	1.52	1.44	1.54	1.42	1.57	1.39	1.60	1.36	1.62
85	1.48	1.53	1.46	1.55	1.43	1.58	1.41	1.60	1.39	1.63
90	1.50	1.54	1.47	1.56	1.45	1.59	1.43	1.61	1.41	1.64
95	1.51	1.55	1.49	1.57	1.47	1.60	1.45	1.62	1.42	1.64
100	1.52	1.56	1.50	1.58	1.48	1.60	1.46	1.63	1.44	1.65

Source: From J. Durbin and G. S. Watson, "Testing for Serial Correlation in Least Squares Regression, II," *Biometrika* 30 (1951): 159–78. Reproduced by permission of the Biometrika Trustees.

TABLE 9 Critical Values for the Wilcoxon Rank Sum Test

(a) $\alpha = .025$ one-tail; $\alpha = .05$ two-tail

n_2 \ n_1	3		4		5		6		7		8		9		10	
	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U
4	6	18	11	25	17	33	23	43	31	53	40	64	50	76	61	89
5	6	11	12	28	18	37	25	47	33	58	42	70	52	83	64	96
6	7	23	12	32	19	41	26	52	35	63	44	76	55	89	66	104
7	7	26	13	35	20	45	28	56	37	68	47	81	58	95	70	110
8	8	28	14	38	21	49	29	61	39	63	49	87	60	102	73	117
9	8	31	15	41	22	53	31	65	41	78	51	93	63	108	76	124
10	9	33	16	44	24	56	32	70	43	83	54	98	66	114	79	131

(b) $\alpha = .05$ one-tail; $\alpha = .10$ two-tail

n_2 \ n_1	3		4		5		6		7		8		9		10	
	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U
3	6	15	11	21	16	29	23	37	31	46	39	57	49	68	60	80
4	7	17	12	24	18	32	25	41	33	51	42	62	52	74	63	87
5	7	20	13	27	19	37	26	46	35	56	45	67	55	80	66	94
6	8	22	14	30	20	40	28	50	37	61	47	73	57	87	69	101
7	9	24	15	33	22	43	30	54	39	66	49	79	60	93	73	107
8	9	27	16	36	24	46	32	58	41	71	52	84	63	99	76	114
9	10	29	17	39	25	50	33	63	43	76	54	90	66	105	79	121
10	11	31	18	42	26	54	35	67	46	80	57	95	69	111	83	127

Source: From F. Wilcoxon and R. A. Wilcox, "Some Rapid Approximate Statistical Procedures" (1964), p. 28. Reproduced with the permission of American Cyanamid Company.

TABLE 10
Critical Values for the Wilcoxon Signed Rank Sum Test

	(a) $\alpha = .025$ one-tail; $\alpha = .05$ two-tail		(b) $\alpha = .05$ one-tail; $\alpha = .10$ two-tail	
n	T_L	T_U	T_L	T_U
6	1	20	2	19
7	2	26	4	24
8	4	32	6	30
9	6	39	8	37
10	8	47	11	44
11	11	55	14	52
12	14	64	17	61
13	17	74	21	70
14	21	84	26	79
15	25	95	30	90
16	30	106	36	100
17	35	118	41	112
18	40	131	47	124
19	46	144	54	136
20	52	158	60	150
21	59	172	68	163
22	66	187	75	178
23	73	203	83	193
24	81	219	92	208
25	90	235	101	224
26	98	253	110	241
27	107	271	120	258
28	117	289	130	276
29	127	308	141	294
30	137	328	152	313

Source: From F. Wilcoxon and R. A. Wilcox, "Some Rapid Approximate Statistical Procedures" (1964), p. 28. Reproduced with the permission of American Cyanamid Company.

TABLE 11 Critical Values for the Spearman Rank Correlation Coefficient

The α values correspond to a one-tail test of H_0: $\rho_s = 0$.
The value should be doubled for two-tail tests.

n	$\alpha = .05$	$\alpha = .025$	$\alpha = .01$
5	.900	–	–
6	.829	.886	.943
7	.714	.786	.893
8	.643	.738	.833
9	.600	.683	.783
10	.564	.648	.745
11	.523	.623	.736
12	.497	.591	.703
13	.475	.566	.673
14	.457	.545	.646
15	.441	.525	.623
16	.425	.507	.601
17	.412	.490	.582
18	.399	.476	.564
19	.388	.462	.549
20	.377	.450	.534
21	.368	.438	.521
22	.359	.428	.508
23	.351	.418	.496
24	.343	.409	.485
25	.336	.400	.475
26	.329	.392	.465
27	.323	.385	.456
28	.317	.377	.448
29	.311	.370	.440
30	.305	.364	.432

Source: From E. G. Olds, "Distribution of Sums of Squares of Rank Differences for Small Samples," *Annals of Mathematical Statistics* 9 (1938). Reproduced with the permission of the Institute of Mathematical Statistics.

TABLE 12 Control Chart Constants

SAMPLE SIZE n	A_2	d_2	d_3	D_3	D_4
2	1.880	1.128	.853	.000	3.267
3	1.023	1.693	.888	.000	2.575
4	.729	2.059	.880	.000	2.282
5	.577	2.326	.864	.000	2.115
6	.483	2.534	.848	.000	2.004
7	.419	2.704	.833	.076	1.924
8	.373	2.847	.820	.136	1.864
9	.337	2.970	.808	.184	1.816
10	.308	3.078	.797	.223	1.777
11	.285	3.173	.787	.256	1.744
12	.266	3.258	.778	.284	1.716
13	.249	3.336	.770	.308	1.692
14	.235	3.407	.762	.329	1.671
15	.223	3.472	.755	.348	1.652
16	.212	3.532	.749	.364	1.636
17	.203	3.588	.743	.379	1.621
18	.194	3.640	.738	.392	1.608
19	.187	3.689	.733	.404	1.596
20	.180	3.735	.729	.414	1.586
21	.173	3.778	.724	.425	1.575
22	.167	3.819	.720	.434	1.566
23	.162	3.858	.716	.443	1.557
24	.157	3.895	.712	.452	1.548
25	.153	3.931	.709	.459	1.541

Source: From E. S. Pearson, "The Percentage Limits for the Distribution of Range in Samples from a Normal Population," *Biometrika* 24 (1932): 416. Reproduced by permission of the Biometrika Trustees.

APPENDIX C

Answers to Selected Even-Numbered Exercises

All answers have been double-checked for accuracy. However, we cannot be absolutely certain that there are no errors. Students should not automatically assume that answers that don't match ours are wrong. When and if we discover mistakes we will post corrected answers on our web page. (See page 10 for the address.) If you find any errors, please email the author (address on web page). We will be happy to acknowledge you with the discovery.

Chapter 1

1.2 Descriptive statistics summarizes a set of data. Inferential statistics makes inferences about populations from samples.

1.4 **a.** The complete production run
b. 1,000 chips
c. Proportion defective
d. Proportion of sample chips that are defective (7.5%)
e. Parameter
f. Statistic
g. Because the sample proportion is less than 10%, we can conclude that the claim is true.

1.6 **a.** Flip the coin 100 times and count the number of heads and tails.
b. Outcomes of flips
c. Outcomes of the 100 flips
d. Proportion of heads
e. Proportion of heads in the 100 flips

1.8 **a.** Fuel mileage of all the taxis in the fleet.
b. Mean mileage
c. The 50 observations
d. Mean of the 50 observations
e. The statistic would be used to estimate the parameter from which the owner can calculate total costs.

We computed the sample mean to be 19.8 mpg.

Chapter 2

2.2 **a.** Interval **b.** Interval
c. Nominal **d.** Ordinal

2.4 **a.** Nominal **b.** Interval
c. Nominal **d.** Interval **e.** Ordinal

2.6 **a.** Interval **b.** Interval
c. Nominal **d.** Ordinal **e.** Interval

2.8 **a.** Interval **b.** Ordinal
c. Nominal **d.** Ordinal

2.10 **a.** Ordinal **b.** Ordinal **c.** Ordinal

2.34 Three out of four Americans are White. Note that the survey did not separate Hispanics.

2.36 Almost half the sample is married and about one out of four were never married.

2.38 The "Less than high school" category has remained constant, while the number of college graduates has increased.

2.40 The dominant source in Australia is coal. In New Zealand it is oil.

2.42 Universities 1 and 2 are similar and quite dissimilar from universities 3 and 4, which also differ. The two nominal variables appear to be related.

2.44 The two variables are related.

2.46 The number of prescriptions filled by independent drug stores has decreased while the others remained constant or increased slightly.

2.48 More than 40% rate the food as less than good.

2.50 There are considerable differences between the two countries.

2.52 Customers with children rated the restaurant more highly than did customers with no children.

2.54 **a.** Males and females differ in their areas of employment. Females tend to choose accounting, marketing, or sales and males opt for finance.
b. Area and job satisfaction are related. Graduates who work in finance and general management appear to be more satisfied than those in accounting, marketing, sales, and others.

Chapter 3

3.2 10 or 11

3.4 **a.** 7 to 9
b. 5.25, 5.40, 5.55, 5.70, 5.85, 6.00, 6.15

3.6 **c.** The number of pages is bimodal and slightly positively skewed.

3.8 The histogram is bimodal.

3.10 **c.** The number of stores is bimodal and positively skewed.

3.12 **d.** The histogram is symmetric (approximately) and bimodal.

3.14 **d.** The histogram is slightly positively skewed, unimodal, and not bell-shaped.

3.16 **a.** The histogram should contain 9 or 10 bins.
c. The histogram is positively skewed.
d. The histogram is not bell shaped.

3.18 The histogram is unimodal, bell shaped, and roughly symmetric. Most of the lengths lie between 18 and 23 inches.

3.20 The histogram is unimodal, symmetric, and bell shaped. Most tomatoes weigh between 2 and 7 ounces with a small fraction weighing less than 2 ounces or more than 7 ounces.

3.22 The histogram of the number of books shipped daily is negatively skewed. It appears that there is a maximum number that the company can ship.

3.24 **c. and d.** This scorecard is a much better predictor.

3.26 The histogram is highly positively skewed indicating that most people watch 4 or less hours per day with some watching considerably more.

3.28 Many people work more than 40 hours per week.

3.32 The numbers of females and males are both increasing with the number of females increasing faster.

3.34 The per capita number of property crimes decreased faster than did the absolute number of property crimes.

3.36 Consumption is increasing and production is falling.

3.38 **c.** Over the last 28 years, both receipts and outlays increased rapidly. There was a 5-year period where receipts were higher than outlays. Between 2004 and 2007, the deficit has decreased.

3.40 The inflation adjusted deficits are not large.

3.42 Imports from Canada has greatly exceeded exports to Canada.

3.44 In the early 1970s, the Canadian dollar was worth more than the U.S. dollar. By the late 1970s, the Canadian dollar lost ground but has recently recovered.

3.46 The index grew slowly until month 400 and then grew quickly until month 600. It then fell sharply and recently recovered.

3.48 There does not appear to be a linear relationship between the two variables.

3.50 **b.** There is a positive linear relationship between calculus and statistics marks.

3.52 **b.** There is a moderately strong positive linear relationship. In general, those with more education use the Internet more frequently.

3.54 **b.** There is a moderately strong positive linear relationship.

3.56 **b.** There is a very weak positive linear relationship.

3.58 There is a moderately strong positive linear relationship.

3.60 There is moderately strong positive linear relationship.
3.62 There does not appear to be any relationship between the two variables.
3.64 There does not appear to be a linear relationship.
3.66 There does not appear to be a linear relationship between the two variables.
3.68 There is a moderately strong positive linear relationship between the education levels of spouses.
3.70 There is a weak positive linear relationship between the amount of education of mothers and their children.
3.76 **c.** The accident rate generally decreases as the ages increase. The fatal accident rate decreases until the age of 64.
3.84 There has been a long-term decline in the value of the Australian dollar.
3.86 There is a very strong positive linear relationship.
3.88 **b.** The slope is positive.
c. There is a moderately strong linear relationship.
3.90 The value of the British pound has fluctuated quite a bit but the current exchange rate is close to the value in 1987.
3.92 **d.** The United States imports more products from Mexico than it exports to Mexico. Moreover, the trade imbalance is worsening (only interrupted by the recession in 2008–2009).
3.96 The number of fatal accidents and the number of deaths have been decreasing.
3.98 The histogram tells us that about 70% of gallery visitors stay for 60 minutes or less, and most of the remainder leave within 120 minutes.
3.100 The relationship between midterm marks and final marks appear to be similar for both statistics courses; that is, there is a weak positive linear relationship.

Chapter 4

4.2 $\bar{x} = 6$, median $= 5$, mode $= 5$
4.4 **a.** $\bar{x} = 39.3$, median $= 38$, mode $=$ all
4.6 $R_g = .19$
4.8 **a.** $\bar{x} = .106$, median $= .10$
b. $R_g = .102$ **c.** Geometric mean
4.10 **a.** .20, 0, .25, .33
b. $\bar{x} = .195$, median $= .225$
c. $R_g = .188$
d. Geometric mean
4.12 **a.** $\bar{x} = 75{,}750$, median $= 76{,}410$
4.14 **a.** $\bar{x} = 117.08$; median $= 124.00$
4.16 **a.** $\bar{x} = .81$; median $= .83$
4.18 **a.** $\bar{x} = 592.04$; median $= 591.00$
4.20 $s^2 = 1.14$
4.22 $s^2 = 15.12$, $s = 3.89$
4.24 **a.** $s^2 = 51.5$ **b.** $s^2 = 6.5$
c. $s^2 = 174.5$
4.26 6, 6, 6, 6, 6
4.28 **a.** 16% **b.** 97.5% **c.** 16%
4.30 **a.** Nothing
b. At least 75% lie between 60 and 180
c. At least 88.9% lie between 30 and 210
4.32 $s^2 = 40.73$ mph^2, and $s = 6.38$ mph; at least 75% of the speeds lie within 12.76 mph of the mean; at least 88.9% of the speeds lie within 19.14 mph of the mean.
4.34 $s^2 = .0858$ cm^2, and $s = .2929$cm; at least 75% of the lengths lie within .5858 of the mean; at least 88.9% of the rods will lie within .8787 cm of the mean.
4.36 **a.** $s = 15.01$
4.38 **a.** $\bar{x} = 77.86$ and $s = 85.35$
c. The histogram is positively skewed. At least 75% of American adults watch between 0 and 249 minutes of television.
4.40 3, 5, 7
4.42 44.6, 55.2
4.44 6.6, 17.6
4.46 4
4.50 **a.** 2, 4, 8
b. Most executives spend little time reading resumes. Keep it short.
4.52 50, 125, 260. The amounts are positively skewed.
4.54 **b.** 145.11, 164.17, 175.18
c. There are no outliers.
d. The data are positively skewed. One-quarter of the times are below 145.11, and one-quarter are above 175.18.
4.56 **a.** 26, 28.5, 32
b. the times are positively skewed.
4.58 Americans spend more time watching news on television than reading news on the Internet.
4.60 The two sets of numbers are quite similar.
4.62 1, 2, 4; The number of hours of television watching is highly positively skewed.
4.64 **a.** $-.7813$; there is a moderately strong negative linear relationship.
b. 61.04% of the variation in y is explained by the variation in x.
4.66 **a.** 98.52 **b.** .8811 **c.** .7763
d. $\hat{y} = 5.917 + 1.705x$
e. There is a strong positive linear relationship between marks and study time. For each additional hour of study time, marks increased on average by 1.705.
4.68 40.09% of the variation in the employment rate is explained by the variation in the unemployment rate.
4.70 Only 5.93% of the variation in the number of houses sold is explained by the variation in interest rates.
4.72 $R^2 = .0069$. There is a very weak positive relationship between the two variables.
4.74 $\hat{y} = 263.4 + 71.65x$. Estimated fixed costs $=$ \$263.40, estimated variable costs $=$ \$71.65.
4.76 **a.** $R^2 = .0915$; there is a very weak relationship between the two variables.
b. The slope coefficient is 58.59; away attendance increases on average by 58.59 for each win. However, the relationship is very weak.
4.78 **a.** The slope coefficient is .0428; for each million dollars in payroll, the number of wins increases on average by .0428. Thus, the cost of winning one additional game is 1/.0428 million $=$ \$23.364 million.
b. The coefficient of determination $= .0866$, which reveals that the linear relationship is very weak.
4.80 **a.** For each additional win, home attendance increases on average by 84.391. The coefficient of determination is .2468; there is a weak relationship between the number of wins and home attendance.
b. For each additional win, away attendance increases on average by 31.151. The coefficient of determination is .4407; there is a moderately strong relationship between the number of wins and away attendance.
4.82 For each additional win, home attendance increases on average by 947.38. The coefficient of determination is .1108; there is a very weak linear relationship between the number of wins and home attendance.
For each additional win, away attendance increases on average by 216.74. The coefficient of determination is .0322; there is a very weak linear relationship between the number of wins and away attendance.
4.84 **a.** There is a weak negative linear relationship between education and television watching.
b. $R^2 = .0572$; 5.72% of the variation in the amount of television is explained by the variation in education.
4.86 $r = .2107$; there is a weak positive linear relationship between the two variables.
4.90 **b.** We can see that among those who repaid the mean score is larger than that of those who did not and the standard deviation is smaller. This information is similar but more precise than that obtained in Exercise 3.23.
4.92 46.03% of the variation in statistics marks is explained by the variation in calculus marks. The coefficient of determination provides a more precise indication of the strength of the linear relationship.
4.94 **a.** $\hat{y} = 17.933 + .6041x$
b. The coefficient of determination is .0505, which indicates that

only 5.05% of the variation in incomes is explained by the variation in heights.

4.96 **a.** $\hat{y} = 103.44 + .07x$

b. The slope coefficient is .07. For each additional square foot, the price increases an average of \$.07 thousand. More simply, for each additional square foot the price increases on average by \$70.

c. From the least squares line, we can more precisely measure the relationship between the two variables.

4.100 **a.** $\bar{x} = 29{,}913$, median $= 30{,}660$

b. $s^2 = 148{,}213{,}791$; $s = 12{,}174$

d. The number of coffees sold varies considerably.

4.102 **a. & b.** $R^2 = .5489$ and the least squares line is $\hat{y} = 49{,}337 - 553.7x$.

c. 54.8% of the variation in the number of coffees sold is explained by the variation in temperature. For each additional degree of temperature, the number of coffees sold decreases on average by 553.7 cups. Alternatively for each 1-degree drop in temperature, the number of coffees increases, on average, by 553.7 cups.

d. We can measure the strength of the linear relationship accurately, and the slope coefficient gives information about how temperature and the number of coffees sold are related.

4.104 **a.** $\bar{x} = 26.32$ and median $= 26$

b. $s^2 = 88.57$, $s = 9.41$

d. The times are positively skewed. Half the times are above 26 hours.

4.106 **a. & b.** $R^2 = .412$, and the least squares line is $\hat{y} = -8.2897 + 3.146x$.

c. 41.2% of the variation in Internet use is explained by the variation in education. For each additional year of education, Internet use increases on average by 3.146 hours.

d. We can measure the strength of the linear relationship accurately and the slope coefficient gives information about how education and Internet use are related.

4.108 **a. & b.** $R^2 = .369$, and the least squares line is $\hat{y} = 89.543 + .128$ rainfall.

c. 36.92% of the variation in yield is explained by the variation in rainfall. For each additional inch of rainfall, yield increases on average by .128 bushels.

d. We can measure the strength of the linear relationship accurately, and the slope coefficient gives information about how rainfall and crop yield are related.

4.110 **b.** The mean debt is \$12,067. Half the sample incurred debts below \$12,047 and half incurred debts above. The mode is \$11,621.

Chapter 6

6.2 **a.** Subjective approach

b. If all the teams in major league baseball have exactly the same players, the New York Yankees will win 25% of all World Series.

6.4 **a.** Subjective approach

b. The Dow Jones Industrial Index will increase on 60% of the days if economic conditions remain unchanged.

6.6 {Adams wins. Brown wins, Collins wins, Dalton wins}

6.8 **a.** {0, 1, 2, 3, 4, 5} **b.** {4, 5} **c.** .10 **d.** .65 **e.** 0

6.10 2/6, 3/6, 1/6

6.12 **a.** .40 **b.** .90

6.14 **a.** P(single) = .15, P(married) = .50, P(divorced) = .25, P(widowed) = .10

b. Relative frequency approach

6.16 $P(A_1) = .3$, $P(A_2) = .4$, $P(A_3) = .3$. $P(B_1) = .6$, $P(B_2) = .4$.

6.18 **a.** .57 **b.** .43

c. It is not a coincidence.

6.20 The events are not independent.

6.22 The events are independent.

6.24 $P(A_1) = .40$, $P(A_2) = .45$, $P(A_3) = .15$. $P(B_1) = .45$, $P(B_2) = .55$.

6.26 **a.** .85. **b.** .75 **c.** .50

6.28 **a.** .36 **b.** .49 **c.** .83

6.30 **a.** .31 **b.** .85 **c.** .387 **d.** .043

6.32 **a.** .390 **b.** .66 **c.** No

6.34 **a.** .11 **b.** .043 **c.** .091 **d.** .909

6.36 **a.** .33 **b.** 30

c. Yes, the events are dependent.

6.38 **a.** .778 **b.** .128 **c.** .385

6.40 **a.** .636 **b.** .205

6.42 **a.** .848 **b.** .277 **c.** .077

6.44 No

6.46 **a.** .201 **b.** .199 **c.** .364 **d.** .636

6.52 **a.** .81 **b.** .01 **c.** .18 **d.** .99

6.54 **b.** .8091 **c.** .0091 **d.** .1818 **e.** .9909

6.56 **a.** .28 **b.** .30 **c.** .42

6.58 .038

6.60 .335

6.62 .698

6.64 .2520

6.66 .033

6.68 .00000001

6.70 .6125

6.72 **a.** .696 **b.** .304 **c.** .889 **d.** .111

6.74 .526

6.76 .327

6.78 .661

6.80 .593

6.82 .843

6.84 .920, .973, .1460, .9996

6.86 **a.** .290 **b.** .290 **c.** Yes

6.88 **a.** .19 **b.** .517 **c.** No

6.90 .295

6.92 .825

6.94 **a.** .3285 **b.** .2403

6.96 .9710

6.98 2/3

6.100 .2214

6.102 .3333

Chapter 7

7.2 **a.** any value between 0 and several hundred miles

b. No **c.** No **d.** continuous

7.4 **a.** 0, 1, 2, . . . , 100 **b.** Yes

c. Yes, 101 values **d.** discrete.

7.6 $P(x) = 1/6$, for $x = 1, 2, \ldots, 6$

7.8 **a.** .950 .020 .680

b. 3.066

c. 1.085

7.10 **a.** .8 **b.** .8 **c.** .8 **d.** .3

7.12 .0156

7.14 **a.** .25 **b.** .25 **c.** .25 **d.** .25

7.18 **a.** 1.40, 17.04 **c.** 7.00, 426.00

d. 7.00, 426.00

7.20 **a.** .6 **b.** 1.7, .81

7.22 **a.** .40 **b.** .95

7.24 1.025, .168

7.26 **a.** .06 **b.** 0 **c.** .35 **d.** .65

7.28 **a.** .21 **b.** .31 **c.** .26

7.30 2.76, 1.517

7.32 3.86, 2.60

7.34 E(value of coin) = \$460; take the \$500

7.36 \$18

7.38 4.00, 2.40

7.40 1.85

7.42 3,409

7.44 .14, .58

7.46 **b.** 2.8, .76

7.48 0, 0

7.50 **b.** 2.9, .45, **c.** yes

7.54 **c.** 1.07, .505 **d.** .93, .605

e. −.045, −.081

7.56 **a.** .412 **b.** .286 **c.** .148

7.58 145, 31

7.60 168, 574

7.62 **a.** .211, .1081 **b.** .211, .1064

c. .211 .1052

7.64 .1060, .1456

7.68 Coca-Cola and McDonalds: .01180, .04469

7.70 .00720, .04355

7.72 .00884, .07593

7.74 Fortis and RIM: .01895, .08421

7.78 .00913, .05313

7.84 **a.** .2668 **b.** .1029 **c.** .0014

7.86 **a.** .26683 **b.** .10292 **c.** .00145

7.88 **a.** .2457 **b.** .0819 **c.** .0015

7.90 **a.** .1711 **b.** .0916 **c.** .9095

d. .8106

7.92 **a.** .4219 **b.** .3114 **c.** .25810

7.94 **a.** .0646 **b.** .9666 **c.** .9282

d. 22.5

7.96 .0081

7.98 .1244

7.100 .00317

7.102 **a.** .3369 **b.** .75763

7.104 **a.** .2990 **b.** .91967

7.106 **a.** .69185 **b.** .12519 **c.** .44069

7.108 **a.** .05692 **b.** .47015

7.110 **a.** .1353 **b.** .1804 **c.** .0361

7.112 **a.** .0302 **b.** .2746 **c.** .3033

7.114 **a.** .1353 **b.** .0663

7.116 a. .20269 b. .26761
7.118 .6703
7.120 a. .4422 b. .1512
7.122 a. .2231 b. .7029 c. .5768
7.124 a. .8 b. .4457
7.126 a. .0993 b. .8088 c. .8881
7.128 .0473
7.130 .0064
7.132 a. .00793 b. 56 c. 4.10
7.134 a. .1612 b. .0095 c. .0132
7.136 a. 1.46, 1.49 b. 2.22, 1.45
7.138 .08755
7.140 .95099, .04803, .00097, .00001, 0, 0

Chapter 8

8.2 a. .1200 b. .4800 c. .6667 d. .1867
8.4 b. 0 c. .25 d. 005
8.6 a. .1667 b. .3333 c. 0
8.8 57 minutes
8.10 123 tons
8.12 b. .5 c. .25
8.14 b. .25 c. .33
8.16 .9345
8.18 .0559
8.20 .0107
8.22 .9251
8.24 .0475
8.26 .1196
8.28 .0010
8.30 0
8.32 1.70
8.34 .0122
8.36 .4435
8.38 a. .6759 b. .3745 c. .1469
8.40 .6915
8.42 a. .2023 b. .3372
8.44 a. .1056 b. .1056 c. .8882
8.46 Top 5%: 34.4675. Bottom 5%: 29.5325
8.48 .1151
8.50 a. .1170 b. .3559 c. .0162 d. 4.05 hours
8.52 9,636 pages
8.54 a. .3336 b. .0314 c. .0436 d. $32.88
8.56 a. .0099 b. $12.88
8.58 132.80 (rounded to 133)
8.60 .5948
8.62 .0465
8.64 171
8.66 873
8.68 .8159
8.70 a. .2327 b. .2578
8.74 a. .5488 b. .6988 c. .1920 d. 0
8.76 .1353
8.78 .8647
8.80 .4857
8.82 .1889
8.84 a. 2.750 b. 1.282 c. 2.132 d. 2.528
8.86 a. 1.6556 b. 2.6810 c. 1.9600 d. 1.6602
8.88 a. .1744 b. .0231 c. .0251 d. .0267
8.90 a. 17.3 b. 50.9 c. 2.71 d. 53.5
8.92 a. 33.5705 b. 866.911 c. 24.3976 d. 261.058
8.94 a. .4881 b. .9158 c. .9988 d. .9077
8.96 a. 2.84 b. 1.93 c. 3.60 d. 3.37
8.98 a. 1.5204 b. 1.5943 c. 2.8397 d. 1.1670
8.100 a. .1050 b. .1576 c. .0001 d. .0044

Chapter 9

9.2 a. 1/36 b. 1/36
9.4 The variance of $\bar{X}$ is smaller than the variance of X.
9.6 No, because the sample mean is approximately normally distributed.
9.8 a. .1056 b. .1587 c. .0062
9.10 a. .4435 b. .7333 c. .8185
9.12 a. .1191 b. .2347 c. .2902
9.14 a. 15.00 b. 21.80 c. 49.75
9.18 a. .0918 b. .0104 c. .00077
9.20 a. .3085 b. 0
9.22 a. .0038 b. It appears to be false.
9.26 .1170
9.28 .9319
9.30 a. 0 b. .0409 c. .5
9.32 .1056
9.34 .0035
9.36 a. .1151 b. .0287
9.38 .0096; the commercial is dishonest.
9.40 a. .0071 b. The claim appears to be false.
9.42 .0066
9.44 The claim appears to be false.
9.46 .0033
9.48 .8413
9.50 .8413
9.52 .3050
9.54 1

Chapter 10

10.10 a. 200 ± 19.60 b. 200 ± 9.80 c. 200 ± 3.92 d. The interval narrows.
10.12 a. 500 ± 3.95 b. 500 ± 3.33 c. 500 ± 2.79 d. The interval narrows.
10.14 a. 10 ± .82 b. 10 ± 1.64 c. 10 ± 2.60 d. The interval widens.
10.16 a. 400 ± 1.29 b. 200 ± 1.29 c. 100 ± 1.29 d. The width of the interval is unchanged.
10.18 Yes, because the variance decreases as the sample size increases.
10.20 a. 500 ± 3.50
10.22 LCL = 36.82, UCL = 50.68
10.24 LCL = 6.91, UCL = 12.79
10.26 LCL = 12.83, UCL = 20.97
10.28 LCL = 10.41, UCL = 15.89
10.30 LCL = 249.44, UCL = 255.32
10.32 LCL = 11.86, UCL = 12.34
10.34 LCL = .494, UCL = .526
10.36 LCL = 18.66, UCL = 19.90
10.38 LCL = 579,545, UCL = 590,581
10.40 LCL = 25.62, UCL = 28.76
10.48 a. 1,537 b. 500 ± 10
10.52 2,149
10.54 1,083
10.56 217

Chapter 11

11.2 H_0: I will complete the Ph.D.
H_1: I will not be able to complete the Ph.D.
11.4 H_0: Risky investment is more successful
H_1: Risky investment is not more successful
11.6 O. J. Simpson

All p-values and probabilities of Type II errors were calculated manually using Table 3 in Appendix B.

11.8 $z = .60$; rejection region: $z > 1.88$; p-value = .2743; not enough evidence that $\mu > 50$.
11.10 $z = 0$; rejection region: $z < -1.96$ or $z > 1.96$; p-value = 1.0; not enough evidence that $\mu \neq 100$.
11.12 $z = -1.33$; rejection region: $z < -1.645$; p-value = .0918; not enough evidence that $\mu < 50$
11.14 a. .2743 b. .1587 c. .0013 d. The test statistics decreases and the p-value decreases.
11.16 a. .2112 b. .3768 c. .5764 d. The test statistic increases and the p-value increases.
11.18 a. .0013 b. .0228 c. .1587 d. The test statistic decreases and the p-value increases.
11.20 a. $z = 4.57$, p-value = 0 b. $z = 1.60$, p-value = .0548.
11.22 a. $z = -.62$, p-value = .2676 b. $z = -1.38$, p-value = .0838
11.24 p-values: .5, .3121, .1611, .0694, .0239, .0062, .0015, 0, 0
11.26 a. $z = 2.30$, p-value = .0214 b. $z = .46$, p-value = .6456
11.28 $z = 2.11$, p-value = .0174; yes
11.30 $z = -1.29$, p-value = .0985; yes
11.32 $z = .95$, p-value = .1711; no
11.34 $z = 1.85$, p-value = .0322; no
11.36 $z = -2.06$, p-value = .0197; yes
11.38 a. $z = 1.65$, p-value = .0495; yes
11.40 $z = 2.26$, p-value = .0119; no
11.42 $z = -1.22$, p-value = .1112; no
11.44 $z = 3.33$, p-value = 0; yes
11.46 $z = -2.73$, p-value = .0032; yes
11.48 .1492
11.50 .6480
11.52 a. .6103 b. .8554 c. β increases.
11.56 a. .4404 b. .6736 c. β increases.
11.60 p-value = .9931; no evidence that the new system will not be cost effective.
11.62 .1170
11.64 .1635 (with $\alpha = .05$)

The answers for the exercises in Chapters 12 through 19 were produced in the following way. In exercises where the statistics are provided in the question or in Appendix A, the solutions were produced manually. The solutions to exercises requiring the use of a computer were produced using Excel. When the test result is calculated manually and the test statistic is normally distributed (z statistic) the p-value was computed manually using the normal table (Table 3 in Appendix B). The p-value for all other test statistics was determined using Excel.

Chapter 12

12.4 **a.** 1500 ± 59.52 **b.** 1500 ± 39.68 **c.** 1500 ± 19.84 **d.** Interval narrows
12.6 **a.** $10 \pm .20$ **b.** $10 \pm .79$ **c.** 10 ± 1.98 **d.** Interval widens
12.8 **a.** 63 ± 1.77 **b.** 63 ± 2.00 **c.** 63 ± 2.71 **d.** Interval widens
12.10 **a.** $t = -3.21$, p-value = .0015 **b.** $t = -1.57$, p-value = .1177 **c.** $t = -1.18$, p-value = .2400 **d.** t decreases and p-value increases
12.12 **a.** $t = .67$, p-value = .5113 **b.** $t = .52$, p-value = .6136 **c.** $t = .30$, p-value = .7804 **d.** t decreases and p-value increases
12.14 **a.** $t = 1.71$, p-value = .0448 **b.** $t = 2.40$, p-value = .0091 **c.** $t = 4.00$, p-value = .0001 **d.** t increase and p-value decreases
12.16 **a.** 175 ± 28.60 **b.** 175 ± 22.07 **c.** Because the distribution of Z is narrower than that of the Student t
12.18 **a.** 350 ± 11.56 **b.** 350 ± 11.52 **c.** When n is large the distribution of Z is virtually identical to that of the Student t
12.20 **a.** $t = -1.30$, p-value = .1126 **b.** $z = -1.30$, p-value = .0968 **c.** Because the distribution of Z is narrower than that of the Student t
12.22 **a.** $t = 1.58$, p-value = .0569 **b.** $z = 1.58$, p-value = .0571 **c.** When n is large the distribution of Z is virtually identical to that of the Student t
12.24 LCL = 14,422, UCL = 33,680
12.26 $t = -4.49$, p-value = .0002; yes
12.28 LCL = 18.11, UCL = 35.23
12.30 $t = -2.45$, p-value = .0185; yes
12.32 LCL = 427 million, UCL = 505 million
12.34 LCL = \$727,350 million, UCL = \$786,350 million
12.36 LCL = 2.31, UCL = 3.03
12.38 LCL = \$51,725 million, UCL = \$56,399 million
12.40 $t = .51$, p-value = .3061; no
12.42 $t = 2.28$, p-value = .0127; yes
12.44 LCL = 650,958 million, UCL = 694,442 million
12.46 $t = 20.89$, p-value = 0; yes
12.48 $t = 4.80$, p-value = 0; yes
12.50 LCL = 2.85, UCL = 3.02
12.52 LCL = 4.80, UCL = 5.12
12.56 **a.** $X^2 = 72.60$, p-value = .0427 **b.** $X^2 = 35.93$, p-value = .1643
12.58 **a.** LCL = 7.09, UCL = 25.57 **b.** LCL = 8.17, UCL = 19.66
12.60 $\chi^2 = 7.57$, p-value = .4218; no
12.62 LCL = 7.31, UCL = 51.43
12.64 $\chi^2 = 305.81$, p-value = .0044; yes
12.66 $\chi^2 = 86.36$, p-value = .1863; no
12.70 **a.** $.48 \pm .0438$ **b.** $.48 \pm .0692$ **c.** $.48 \pm .0310$
12.72 **a.** $z = .61$, p-value = .2709 **b.** $z = .87$, p-value = .1922 **c.** $z = 1.22$, p-value = .1112
12.74 752
12.76 **a.** $.75 \pm .0260$
12.78 **a.** $.75 \pm .03$
12.80 **a.** $.5 \pm .0346$
12.82 $z = -1.47$, p-value = .0708; yes
12.84 $z = .33$, p-value = .3707; no
12.86 LCL = .1332, UCL = .2068
12.88 LCL = 0, UCL = .0312
12.90 LCL = 0, UCL = .0191
12.92 LCL = 5,940, UCL = 9,900
12.94 $z = 1.58$, p-value = .0571; no
12.96 LCL = 3.45 million, UCL = 3.75 million
12.98 $z = 1.40$, p-value = .0808; yes
12.100 LCL = 4.945 million, UCL = 6.325 million
12.102 LCL = .861 million, UCL = 1.17 million
12.104 **a.** LCL = .4780, UCL = .5146 **b.** LCL = .0284, UCL = .0448
12.106 LCL = .1647, UCL = .1935
12.108 $z = 6.00$, p-value = 0; yes
12.110 $z = 3.87$, p-value = 0; yes
12.112 $z = 5.63$, p-value = 0; yes
12.114 $z = 15.08$, p-value = 0; yes
12.116 $z = 7.27$, p-value = 0; yes
12.118 $z = 5.05$, p-value = 0; yes
12.120 LCL = 35,121,043, UCL = 43,130,297
12.122 $z = -.539$, p-value = .5898.
12.124 LCL = 13,195,985, UCL = 14,720,803
12.126 **a.** LCL = .2711, UCL = .3127 **b.** LCL = 29,060,293, UCL = 33,519,564
12.128 LCL = 26.928 million, UCL = 38.447 million
12.130 **a.** $t = 3.04$, p-value = .0015; yes **b.** LCL = 30.68, UCL = 33.23 **c.** The costs are required to be normally distributed.
12.132 $\chi^2 = 30.71$, p-value = .0435; yes
12.134 **a.** LCL = 69.03, UCL = 74.73 **b.** $t = 2.74$, p-value = .0043; yes
12.136 LCL = .582, UCL = .682
12.138 LCL = 6.05, UCL = 6.65
12.140 LCL = .558, UCL = .776
12.142 $z = -1{,}33$, p-value = .0912; yes
12.144 **a.** $t = -2.97$, p-value = .0018; yes **b.** $\chi^2 = 101.58$, p-value = .0011; yes
12.146 LCL = 49,800, UCL = 72,880
12.148 **a.** LCL = −5.54%, UCL = 29.61% **b.** $t = -.47$, p-value = .3210; no
12.150 $t = .908$, p-value = .1823; no
12.152 $t = .959$, p-value = .1693; no
12.154 $t = 2.44$, p-value = .0083; yes

For all exercises in Chapter 13 and all chapter appendixes, we employed the F-test of two variances at the 5% significance level to decide which one of the equal-variances or unequal-variances t-test and estimator of the difference between two means to use to solve the problem. In addition, for exercises that compare two populations and are accompanied by data files, our answers were derived by defining the sample from population 1 as the data stored in the first column (often column A). The data stored in the second column represent the sample from population 2. Paired differences were defined as the difference between the variable in the first column minus the variable in the second column.

Chapter 13

13.6 **a.** $t = .43$, p-value = .6703; no **b.** $t = .04$, p-value = .9716; no **c.** The t-statistic decreases and the p-value increases. **d.** $t = 1.53$, p-value = .1282; no **e.** The t-statistic increases and the p-value decreases. **f.** $t = .72$, p-value = .4796; no **g.** The t-statistic increases and the p-value decreases.
13.8 **a.** $t = .62$, p-value = .2689; no **b.** $t = 2.46$, p-value = .0074; yes **c.** The t-statistic increases and the p-value decreases. **d.** $t = .23$, p-value = .4118 **e.** The t-statistic decreases and the p-value increases. **f.** $t = .35$, p-value = .3624 **g.** The t-statistic decreases and the p-value increases.
13.12 $t = -2.04$, p-value = .0283; yes
13.14 $t = -1.59$, p-value = .1368; no
13.16 $t = 1.12$, p-value = .2761; no
13.18 $t = 1.55$, p-value = .1204; no
13.20 **a.** $t = 2.88$ p-value = .0021; yes **b.** LCL = .25, UCL = 4.57
13.22 $t = .94$, p-value = .1753; switch to supplier B.
13.24 **a.** $t = 2.94$, p-value = .0060; yes **b.** LCL = 4.31, UCL = 23.65 **c.** The times are required to be normally distributed.
13.26 $t = 7.54$, p-value = 0; yes
13.28 $t = .90$, p-value = .1858; no
13.30 $t = -2.05$, p-value = .0412; yes
13.32 $t = 1.16$, p-value = .2467; no
13.34 $t = -2.09$, p-value = .0189; yes
13.36 $t = 6.28$, p-value = 0; yes
13.38 LCL = 13,282, UCL = 21,823
13.42 $t = -4.65$, p-value = 0; yes
13.44 $t = 9.20$, p-value = 0; yes
13.46 Experimental
13.52 $t = -3.22$, p-value = .0073; yes
13.54 $t = 1.98$, p-value = .0473; yes
13.56 **a.** $t = 1.82$, p-value = .0484; yes **b.** LCL = −.66, UCL = 6.82
13.58 $t = -3.70$, p-value = .0006; yes
13.60 **a.** $t = 16.92$, p-value = 0; yes **b.** LCL = 50.12, UCL = 64.48 **c.** Differences are required to be normally distributed.

13.62 $t = -1.52$, p-value $= .0647$; no
13.64 $t = 2.08$, p-value $= .0210$; yes
13.70 $t = 23.35$, p-value $= 0$; yes
13.72 $t = 2.22$, p-value $= .0132$; yes
13.76 **a.** $F = .50$, p-value $= .0669$; yes
b. $F = .50$, p-value $= .2071$; no
c. The value of the test statistic is unchanged but the conclusion did change.
13.78 $F = .50$, p-value $= .3179$; no
13.80 $F = 3.23$, p-value $= .0784$; no
13.82 $F = 2.08$, p-value $= .0003$; yes
13.84 $F = .31$, p-value $= 0$; yes
13.88 **a.** $z = 1.07$, p-value $= .2846$
b. $z = 2.01$, p-value $= .0444$
c. The p-value decreases.
13.90 $z = 1.70$, p-value $= .0446$; yes
13.92 $z = 1.74$, p-value $= .0409$; yes.
13.94 $z = -2.85$, p-value $= .0022$; yes
13.96 **a.** $z = -4.04$, p-value $= 0$; yes
13.98 $z = 2.00$, p-value $= .0228$; yes
13.100 $z = -1.19$, p-value $= .1170$; no
13.102 **a.** $z = 3.35$, p-value $= 0$; yes
b. LCL $= .0668$, UCL $= .3114$
13.104 $z = -4.24$, p-value $= 0$; yes
13.106 $z = 1.50$, p-value $= .0664$; no
13.108 Canada: $z = 2.82$, p-value $= .0024$; yes. United States: $z = .98$, p-value $= .1634$; no. Britain: $z = 1.00$, p-value $= .1587$; no
13.110 $z = 2.04$, p-value $= .0207$; yes
13.112 $z = -1.25$, p-value $= .2112$; no
13.114 $z = 4.61$, p-value $= 0$; yes
13.116 $z = 1.45$, p-value $= .1478$; no
13.118 $z = 5.13$, p-value $= 0$; yes
13.120 $z = .40$, p-value $= .6894$; no
13.122 2002: $z = 2.40$, p-value $= .0164$; yes.
2004: $z = .29$, p-value $= .7716$; no.
2006: $z = 2.24$, p-value $= .0250$.
2008: $z = .99$, p-value $= .3202$
13.124 $z = -3.69$, p-value $= .0002$; yes
13.126 **a.** $z = 2.49$, p-value $= .0065$; yes
b. $z = .89$, p-value $= .1859$; no
13.128 $t = .88$, p-value $= .1931$; no
13.130 $t = -6.09$, p-value $= 0$; yes
13.132 $z = -2.30$, p-value $= .0106$; yes
13.134 **a.** $t = -1.06$, p-value $= .2980$; no
b. $t = -2.87$, p-value $= .0040$; yes
13.136 $z = 2.26$, p-value $= .0119$; yes
13.138 $z = -4.28$, p-value $= 0$; yes
13.140 $t = -4.53$, p-value $= 0$; yes
13.142 **a.** $t = 4.14$, p-value $= .0001$; yes
b. LCL $= 1.84$, UCL $= 5.36$
13.144 $t = -2.40$, p-value $= .0100$; yes
13.146 $z = 1.20$, p-value $= .1141$; no
13.148 $t = 14.07$, p-value $= 0$; yes
13.150 $t = -2.40$, p-value $= .0092$; yes
13.152 F-Test: $F = 1.43$, p-value $= 0$. t-Test: $t = .71$, p-value $= .4763$
13.154 $t = 2.85$, p-value $= .0025$; yes
13.156 $z = -3.54$, p-value $= .0002$; yes
13.158 $t = -2.13$, p-value $= .0171$; yes
13.160 $z = -.45$, p-value $= .6512$; no

Chapter 14

14.4 $F = 4.82$, p-value $= .0377$; yes
14.6 $F = 3.91$, p-value $= .0493$; yes
14.8 $F = .81$, p-value $= .5224$; no
14.10 **a.** $F = 2.94$, p-value $= .0363$; evidence of differences
14.12 $F = 3.32$, p-value $= .0129$; yes
14.14 $F = 1.17$, p-value $= .3162$; no
14.16 $F = 1.33$, p-value $= .2675$; no
14.18 **a.** $F = 25.60$, p-value $= 0$; yes
b. $F = 7.37$, p-value $= .0001$; yes
c. $F = 1.82$, p-value $= .1428$; no
14.20 $F = .26$, p-value $= .7730$; no
14.22 $F = 31.86$, p-value $= 0$; yes
14.24 $F = .33$, p-value $= .8005$; no
14.26 $F = .50$, p-value $= .6852$; no
14.28 $F = 11.59$, p-value $= 0$; yes
14.30 $F = 17.10$, p-value $= 0$; yes
14.32 $F = 37.47$, p-value $= 0$; yes
14.34 **a.** μ_1 and μ_2, μ_1 and μ_4, μ_1 and μ_5, μ_2 and μ_4, μ_3 and μ_4, μ_3 and μ_5, and μ_4 and μ_5 differ.
b. μ_1 and μ_5, μ_2 and μ_4, μ_3 and μ_4, and μ_4 and μ_5 differ.
c. μ_1 and μ_2, μ_1 and μ_5, μ_2 and μ_4, μ_3 and μ_4, and μ_4 and μ_5 differ.
14.36 **a.** BA and BBA differ.
b. BA and BBA differ.
14.38 **a.** The means for Forms 1 and 4 differ.
b. No means differ.
14.40 **a.** Lacquers 2 and 3 differ.
b. Lacquers 2 and 3 differ.
14.42 No fertilizers differ.
14.44 Blacks differ from Whites and others.
14.46 Married and separated, married and never married, and divorced and single differ.
14.48 Democrats and Republicans and Republicans and Independents differ.
14.50 All three groups differ.
14.52 **a.** $F = 16.50$, p-value $= 0$; treatment means differ
b. $F = 4.00$, p-value $= .0005$; block means differ
14.54 **a.** $F = 7.00$, p-value $= .0078$; treatment means differ
b. $F = 10.50$, p-value $= .0016$; treatment means differ
c. $F = 21.00$, p-value $= .0001$; treatment means differ
d. F-statistic increases and p-value decreases.
14.56 **a.** SS(Total) 14.9, SST $= 8.9$, SSB $= 4.2$, SSE $= 1.8$
b. SS(Total) 14.9, SST $= 8.9$, SSE $= 6.0$
14.58 $F = 1.65$, p-value $= .2296$; no
14.60 **a.** $F = 123.36$, p-value $= 0$; yes
b. $F = 323.16$, p-value $= 0$; yes
14.62 **a.** $F = 21.16$, p-value $= 0$; yes
b. $F = 66.02$, p-value $= 0$; randomized block design is best
14.64 **a.** $F = 10.72$, p-value $= 0$; yes
b. $F = 6.36$, p-value $= 0$; yes
14.66 $F = 44.74$, p-value $= 0$; yes
14.68 **b.** $F = 8.23$; Treatment means differ
c. $F = 9.53$; evidence that factors A and B interact
14.70 **a.** $F = .31$, p-value $= .5943$; no evidence that factors A and B interact.
b. $F = 1.23$, p-value $= .2995$; no evidence of differences between the levels of factor A.
c. $F = 13.00$, p-value $= .0069$; evidence of differences between the levels of factor B.
14.72 $F = .21$, p-value $= .8915$; no evidence that educational level and gender interact. $F = 4.49$, p-value $= .0060$; evidence of differences between educational levels. $F = 15.00$, p-value $= .0002$; evidence of a difference between men and women.
14.74 **d.** $F = 4.11$, p-value $= .0190$; yes
e. $F = 1.04$, p-value $= .4030$; no
f. $F = 2.56$, p-value $= .0586$; no
14.76 **d.** $F = 7.27$, p-value $= .0007$; evidence that the schedules and drug mixtures interact.
14.78 Both machines and alloys are sources of variation.
14.80 The only source of variation is skill level.
14.82 **a.** $F = 7.67$, p-value $= .0001$; yes
14.84 $F = 13.79$, p-value $= 0$; use the typeface that was read the most quickly.
14.86 $F = 7.72$, p-value $= 0.0070$; yes
14.88 **a.** $F = 136.58$, p-value $= 0$; yes
b. All three means differ from one another. Pure method is best.
14.90 $F = 14.47$, p-value $= 0$; yes
14.92 $F = 13.84$, p-value $= 0$; yes
14.94 $F = 1.62$, p-value $= .2022$; no
14.96 $F = 45.49$, p-value $= 0$; yes
14.98 $F = 211.61$, p-value $= 0$; yes

Chapter 15

15.2 $\chi^2 = 2.26$, p-value $= .6868$; no evidence that at least one p_i is not equal to its specified value.
15.6 $\chi^2 = 9.96$, p-value $= .0189$; evidence that at least one p_i is not equal to its specified value.
15.8 $\chi^2 = 6.85$, p-value $= .0769$; not enough evidence that at least one p_i is not equal to its specified value.
15.10 $\chi^2 = 14.07$, p-value $= .0071$; yes
15.12 $\chi^2 = 33.85$, p-value $= 0$; yes
15.14 $\chi^2 = 6.35$, p-value $= .0419$; yes
15.16 $\chi^2 = 5.70$, p-value $= .1272$; no
15.18 $\chi^2 = 4.97$, p-value $= .0833$; no
15.20 $\chi^2 = 46.36$, p-value $= 0$; yes
15.22 $\chi^2 = 19.10$, p-value $= 0$; yes
15.24 $\chi^2 = 4.77$, p-value $= .0289$; yes
15.26 $\chi^2 = 4.41$, p-value $= .1110$; no
15.28 $\chi^2 = 2.36$, p-value $= .3087$; no
15.30 $\chi^2 = 19.71$, p-value $= .0001$; yes
15.32 **a.** $\chi^2 = .64$, p-value $= .4225$; no
15.34 $\chi^2 = 41.77$, p-value $= 0$; yes
15.36 $\chi^2 = 43.36$, p-value $= 0$; yes
15.38 $\chi^2 = 20.89$, p-value $= .0019$; yes
15.40 $\chi^2 = 36.57$, p-value $= .0003$; yes
15.42 $\chi^2 = 110.3$, p-value $= 0$; yes
15.44 $\chi^2 = 5.89$, p-value $= .0525$; no
15.46 $\chi^2 = 35.21$, p-value $= 0$; yes
15.48 $\chi^2 = 9.87$, p-value $= .0017$; yes
15.50 $\chi^2 = 506.76$, p-value $= 0$; yes
15.52 Phone: $\chi^2 = .2351$, p-value $= .8891$; no.
Not on phone: $\chi^2 = 3.18$, p-value $= .2044$; no
15.54 $\chi^2 = 3.20$, p-value $= .2019$; no

15.56 $\chi^2 = 5.41$, p-value $= .2465$; no
15.58 $\chi^2 = 20.38$, p-value $= .0004$; yes
15.60 $\chi^2 = 86.62$, p-value $= 0$; yes
15.62 $\chi^2 = 4.13$, p-value $= .5310$; no
15.64 $\chi^2 = 9.73$, p-value $= .0452$; yes
15.66 $\chi^2 = 4.57$, p-value $= .1016$; no
15.68 **a.** $\chi^2 = .648$, p-value $= .4207$; no
b. $\chi^2 = 7.72$, p-value $= .0521$; no
c. $\chi^2 = 23.11$, p-value $= 0$; yes
15.70 $\chi^2 = 4.51$, p-value $= .3411$; no

Chapter 16

16.2 $\hat{y} = 9.107 + .0582x$
16.4 **b.** $\hat{y} = -24.72 + .9675x$
16.6 **b.** $\hat{y} = 3.635 + .2675x$
16.8 $\hat{y} = 7.460 + 0.899x$
16.10 $\hat{y} = 7.286 + .1898x$
16.12 $\hat{y} = 4{,}040 + 44.97x$
16.14 $\hat{y} = 458.4 + 64.05x$
16.16 $\hat{y} = 20.64 - .3039x$
16.18 $\hat{y} = 89.81 + .0514x$
16.22 $t = 10.09$, p-value $= 0$; evidence of linear relationship
16.24 **a.** 1.347
b. $t = 3.93$, p-value $= .0028$; yes
c. LCL $= .0252$, UCL $= .0912$
d. .6067
16.26 $t = 6.55$, p-value $= 0$; yes
16.28 **a.** 5.888 **b.** .2892
c. $t = 4.86$, p-value $= 0$; yes
d. LCL $= .1756$, UCL $= .3594$
16.30 $t = 2.17$, p-value $= .0305$; yes
16.32 $t = 7.50$, p-value $= 0$; yes
16.34 **a.** 3,287 **b.** $t = 2.24$, p-value $= .0309$ **c.** .1167
16.36 $s_\varepsilon = 191.1$; $R^2 = .3500$; $t = 10.39$, p-value $= 0$
16.38 $t = -3.39$, p-value $= .0021$; yes
16.40 **a.** .0331
b. $t = 1.21$, p-value $= .2319$; no
16.42 $t = 4.86$, p-value $= 0$; yes
16.44 $t = 7.49$, p-value $= 0$; yes
16.46 $\hat{y} = -29{,}984 + 4905x$; $t = 15.37$, p-value $= 0$.
16.48 $t = 6.58$, p-value $= 0$; yes
16.50 $t = 7.80$, p-value $= 0$; yes
16.52 $t = -8.95$, p-value $= 0$; yes
16.56 141.8, 181.8
16.58 13,516, 27,260
16.60 **a.** 186.8, 267.2 **b.** 200.5, 215.5
16.62 24.01, 31.43
16.64 **a.** 27.62, 72.06 **b.** 29.66, 37.92
16.66 23.30, 34.10
16.68 190.4, 313.4
16.70 **a.** 60.00, 62.86 **b.** 41.51, 74.09
16.72 92.01, 95.83
16.74 16,466, 21,657
16.76 0 (increased from −83.98), 204.8
16.78 3.15, 3.40
16.80 0(increased from −.15), 8.38
16.100 **a.** $\hat{y} = 115.24 + 2.47x$ **c.** .5659
d. $t = 4.84$, p-value $= .0001$; yes
e. Lower prediction limit $= 318.1$, upper prediction limit $= 505.2$
16.102 **a.** $t = 21.78$, p-value $= 0$; yes
b. $t = 11.76$, p-value $= 0$; yes
16.104 $t = 3.01$, p-value $= .0042$; yes
16.106 $t = 1.67$, p-value $= .0522$; no
16.108 $r = t = -9.88$, p-value $= 0$; yes

Chapter 17

17.2 **a.** $\hat{y} = 13.01 + .194x_1 + 1.11x_2$
b. 3.75 **c.** .7629
d. $F = 43.43$, p-value $= 0$; evidence that the model is valid.
f. $t = .97$, p-value $= .3417$; no
g. $t = 9.12$, p-value $= 0$; yes
h. 23, 39 **i.** 49, 65
17.4 **c.** $s_\varepsilon = 6.99$, $R^2 = .3511$; model is not very good.
d. $F = 22.01$, p-value $= 0$; evidence that the model is valid.
e. Minor league home runs: $t = 7.64$, p-value $= 0$; Age: $t = .26$, p-value $= .7961$
Years professional: $t = 1.75$, p-value $= .0819$
Only the number of minor league home runs is linearly related to the number of major league home runs.
f. 9.86 (rounded to 10), 38.76 (rounded to 39)
g. 14.66, 24.47
17.6 **b.** .2882
c. .$F = 12.96$, p-value $= 0$; evidence that the model is valid.
d. High school GPA: $t = 6.06$, p-value $= 0$; SAT: $t = .94$, p-value $= .3485$
Activities: $t = .72$, p-value $= .4720$
e. 4.45, 12.00 (actual value = 12.65; 12 is the maximum)
f. 6.90, 8.22
17.8 **b.** $F = 29.80$, p-value $= 0$; evidence to conclude that the model is valid.
d. House size : $t = 3.21$, p-value $= .0006$; Number of children: $t = 7.84$ p-value $= 0$
Number of adults at home: $t = 4.48$, p-value $= 0$
17.10 **b.** $F = 67.97$, p-value $= 0$; evidence that the model is valid.
d. 65.54, 77.31
e. 68.75, 74.66
17.12 **a.** $\hat{y} = -28.43 + .604x_1 + .374x_2$
b. $s_\varepsilon = 7.07$ and $R^2 = .8072$; the model fits well.
d. 35.16, 66.24 **e.** 44.43, 56.96
17.14 **b.** $F = 24.48$, p-value $= 0$; yes
c.

Variable	t	p-value
UnderGPA	.52	.6017
GMAT	8.16	0
Work	3.00	.0036

17.16 **a.** 9.09 + .219 PAEDUC + .197 MAEDUC
b. $F = 234.9$, p-value $= 0$
c. PAEDUC: $t = 9.73$, p-value $= 0$
MAEDUC: $t = 7.69$, p-value $= 0$
17.18 **a.** $F = 9.09$, p-value $= 0$
b.

Variable	t	p-value
AGE	2.34	.0194
EDUC	−3.11	.0019
HRS	−2.35	.0189
PRESTG80	−3.47	.0005
CHILDS	−.84	.4021
EARNRS	−.98	.3299

c. $R^2 = 0659$
17.20 **a.** $F = 35.06$, p-value $= 0$
b.

Variable	t	p-value
AGE	.40	.6864
EDUC	7.89	0
HRS	7.10	0
CHILDS	1.61	.1084
AGEKDBRN	4.90	0
YEARSJOB	5.85	0
MOREDAYS	1.36	.1754
NUMORG	1.37	.1713

17.22 **a.** $\hat{y} = 6.36 + .135$ DAYS1 + .036 DAYS2 + .060 DAYS3 + .107 DAYS4 + .142 DAYS5 + .134 DAYS6
b. $F = 11.72$, p-value $= 0$
c.

Variable	t	p-value
DAYS1	3.33	.0009
DAYS2	.81	.4183
DAYS3	1.41	.1582
DAYS4	3.00	.0027
DAYS5	3.05	.0024
DAYS6	3.71	.0002

17.40 $d_L = 1.16$, $d_U = 1.59$; $4 - d_L = 2.84$, $4 - d_U = 2.41$; evidence of negative first-order autocorrelation.
17.42 $d_L = 1.46$, $d_U = 1.63$. There is evidence of positive first-order autocorrelation.
17.44 $4 - d_U = 4 - 1.73 = 2.27$, $4 - d_L = 4 - 1.19 = 2.81$. There is no evidence of negative first-order autocorrelation.
17.46 **a.** The regression equation is $\hat{y} = 2260 + .423x$
c. $d = .7859$. There is evidence of first-order autocorrelation.
17.48 $d = 2.2003$; $d_L = 1.30$, $d_U = 1.46$, $4 - d_U = 2.70$, $4 - d_L = 2.54$. There is no evidence of first-order autocorrelation.
17.50 **a.** $\hat{y} = 164.01 + .140x_1 + .0313x_2$
b. $t = 1.72$, p-value $= .0974$; no
c. $t = 4.64$, p-value $= .0001$; yes
d. $s_\varepsilon = 63.08$ and $R^2 = .4752$; the model fits moderately well.
f. 69.2, 349.3
17.52 **a.** $\hat{y} = 29.60 - .309x_1 - 1.11x_2$
b. $R_2 = .6123$; the model fits moderately well.
c. $F = 21.32$, p-value $= 0$; evidence to conclude that the model is valid.
d. Vacancy rate: $t = -4.58$, p-value $= .0001$; yes
Unemployment rate: $t = -4.73$, p-value $= .0001$; yes
e. The error is approximately normally distributed with a constant variance.
f. d = 2.0687; no evidence of first-order autocorrelation.
g. $14.18, $23.27

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APPLICATION BOXES

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