PCA

1)

Eigenvalues of $X^T X$ as calculated from the code are:

[2.81035710e+02 -5.49536140e-15 9.64290269e-01 2.43463565e-15

 -6.88454893e-16 -4.43466032e-15]

```
C:\Users\user\Desktop\MS\Fall_2019\FML\ass3>python hw03.py
Eigen values of X.T@X are:
[ 2.81035710e+02 -5.49536140e-15  9.64290269e-01  2.43463565e-15
 -6.88454893e-16 -4.43466032e-15]
```

2)

$$X = \begin{bmatrix} 2 & 3 & 3 & 4 & 5 & 7 \\ 2 & 4 & 5 & 5 & 6 & 8 \end{bmatrix}$$

$$XX^T = \begin{bmatrix} 2 & 3 & 3 & 4 & 5 & 7 \\ 2 & 4 & 5 & 5 & 6 & 8 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \\ 3 & 5 \\ 4 & 5 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 112 & 137 \\ 137 & 170 \end{bmatrix}$$

Let the Eigenvalue of XX^T be λ and Eigenvector be V.

$$\therefore \quad (XX^T)V = \lambda V$$

$$\Rightarrow (XX^T)V - \lambda V = 0$$

$$\Rightarrow (XX^T - \lambda I)V = 0$$

Hence for a non-Zero V, $\det(XX^T - \lambda I) = 0$

$$\therefore \det\left(\begin{bmatrix} 112 & 137 \\ 137 & 170 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 112 & 137 \\ 137 & 170 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\begin{pmatrix} 112-\lambda & 137 \\ 137 & 170-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (112-\lambda)(170-\lambda) - 18769 = 0$$

$$\Rightarrow 19040 - 282\lambda + \lambda^2 - 18769 = 0$$

$$\Rightarrow \lambda^2 - 282\lambda + 271 = 0$$

For a quadratic equation of the form
$ax^2 + bx + c = 0$ the solutions are,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here we have,

$a = 1$, $b = -282$ and $c = 271$

Hence the 2 possible values of λ are:

$$\lambda = \frac{-(-282) + \sqrt{(-282)^2 - 4 \times 1 \times 271}}{2 \times 1}$$

$$= 141 + \sqrt{19610}$$

$$= 281.03570$$

and,

$$\lambda = \frac{-(-282) - \sqrt{(-282)^2 - 4 \times 1 \times 271}}{2 \times 1}$$

$$= 141 - \sqrt{19610}$$

$$= 0.96429$$

Using the eigen values we can calculate the eigenvectors as,

$$\left(XX^T - \begin{bmatrix} 0.96429 & 0 \\ 0 & 0.96429 \end{bmatrix}\right) V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[using $\lambda = 0.96429$]

$$\Rightarrow \left(\begin{bmatrix} 112 & 137 \\ 137 & 170 \end{bmatrix} - \begin{bmatrix} 0.96429 & 0 \\ 0 & 0.96429 \end{bmatrix}\right) V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 111.03571 & 137 \\ 137 & 169.03571 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad 111.03571 V_1 + 137 V_2 = 0$ ——— ①

$137 V_1 + 169.03571 V_2 = 0$ ——— ②

Let $V_1 = \alpha$.

putting $V_1 = \alpha$ in ① we get,

$$111.03571\alpha + 137 V_2 = 0$$

$$\Rightarrow \quad 137 V_2 = -111.03571\alpha$$

$$\Rightarrow \quad V_2 = -\frac{111.03571\alpha}{137}$$

$$\|V\| = \sqrt{\alpha^2 + \left(\frac{111.03571}{137}\right)^2 \alpha^2}$$

$$= \alpha\sqrt{1 + 0.65688}$$

$$= \alpha\sqrt{1.65688}$$

Hence ~~the~~ a normalized eigenvector would be,

$$\begin{bmatrix} \alpha / \alpha\sqrt{1.65688} \\ \left(-\frac{111.03571}{137}\right)\alpha / \alpha\sqrt{1.65688} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1.65688}} \\ -\frac{0.81048}{\sqrt{1.65688}} \end{bmatrix} = \begin{bmatrix} 0.77688 \\ -0.62964 \end{bmatrix}$$

using $\lambda = 281.0357$ we get,

$$\left(\begin{bmatrix} 112 & 137 \\ 137 & 170 \end{bmatrix} - \begin{bmatrix} 281.0357 & 0 \\ 0 & 281.0357 \end{bmatrix}\right) V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ then,

$$\begin{bmatrix} -169.0357 & 137 \\ 137 & -111.0357 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$

$$-169.0357 V_1 + 137 V_2 = 0 \quad \text{—} ①$$

$$137 V_1 - 111.0357 V_2 = 0 \quad \text{—} ②$$

Let $V_1 = \alpha$.

Putting $V_1 = \alpha$ in ① we get,

$$-169.0357\alpha + 137 V_2 = 0$$

$$\Rightarrow V_2 = \frac{169.0357}{137}\alpha$$

$$= 1.23384\alpha$$

$$\|v\| = \sqrt{\alpha^2 + (1.23384)^2\alpha^2}$$

$$= \alpha\sqrt{1+(1.23384)^2}$$

$$= \alpha\sqrt{2.52236}$$

Hence a normalized eigenvector would be

$$= \begin{bmatrix} \alpha / \alpha\sqrt{2.52236} \\ 1.23384\alpha / \alpha\sqrt{2.52236} \end{bmatrix}$$

$$= \begin{bmatrix} 1 / \sqrt{2.52236} \\ 1.23384 / \sqrt{2.52236} \end{bmatrix}$$

$$= \begin{bmatrix} 0.62964 \\ 0.77688 \end{bmatrix}$$

Results from code:

```
Eigenvalues of X@X.T are:
[  0.96429027 281.03570973]


Eigenvectors of X@X.T are:
[[-0.7768816  -0.62964671]
 [ 0.62964671 -0.7768816 ]]
```

On verifying the solution with the results from code, we see that the ratio (v_1/v_2) for both the eigenvectors found is same as the ratio (v_1/v_2) found from the code. Also, the eigenvalues found match with the results from code.

Thus, we can conclude that the solution is correct.

3)

Yes, the eigenvalues of XX^T :

λ = 0.96429027 and

λ = 281.03570973

are also eigenvalues of X^TX.

The eigenvalues are same because,

for the non-zero eigenvalues λ and eigenvector v (v ≠ 0) of XX^T we will have,

XX^Tv = λv,

then,

X^TXu = $X^TX(X^T$v) = λX^Tv = λu.

Thus, we can say u = X^Tv ≠ 0, otherwise XX^Tv = 0 as well, and hence λ is an eigenvalue of X^TX also.

4)

Data $X = \begin{bmatrix} 2 & 3 & 3 & 4 & 5 & 7 \\ 2 & 4 & 5 & 5 & 6 & 8 \end{bmatrix}$

$$\text{mean}(X) = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Subtracting the mean from the Data we get,

$$X = \begin{bmatrix} -2 & -1 & -1 & 0 & 1 & 3 \\ -3 & -1 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\text{Standard deviation}(X) = \begin{bmatrix} 1.633 \\ 1.826 \end{bmatrix}$$

Dividing the resultant matrix after subtracting the mean, by the standard deviation (X), we get normalized data as,

$$X = \begin{bmatrix} -1.225 & -0.612 & -0.612 & 0 & 0.612 & 1.837 \\ -1.643 & -0.548 & 0 & 0 & 0.548 & 1.643 \end{bmatrix}$$

The Covariance Matrix for the normalized data is

$$M = \begin{bmatrix} 1.2 & 1.14 \\ 1.14 & 1.2 \end{bmatrix}$$

Let eigenvalue of M be λ and eigenvector V.

$$\therefore \quad MV = \lambda V$$

$$\Rightarrow (M - \lambda I)V = 0$$

Hence, for a non-zero v, $\det(M - \lambda I) = 0$

Now, $M - \lambda I$

$$= \begin{bmatrix} 1.2 & 1.14 \\ 1.14 & 1.2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.2 & 1.14 \\ 1.14 & 1.2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1.2-\lambda & 1.14 \\ 1.14 & 1.2-\lambda \end{bmatrix}$$

$$\therefore \begin{vmatrix} 1.2-\lambda & 1.14 \\ 1.14 & 1.2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1.2-\lambda)^2 - (1.14)^2 = 0$$

$$\Rightarrow (1.2-\lambda)^2 = 1.299$$

$$\Rightarrow \lambda^2 - 2.4\lambda + 0.141 = 0$$

Using quadratic formula with

$a = 1$

$b = -2.4$ and

$c = 0.141$ we get,

$$\lambda = \frac{-(-2.4) \pm \sqrt{(-2.4)^2 - 4(1)(0.141)}}{2(1)}$$

$$= \frac{2.4 \pm \sqrt{5.196}}{2}$$

$\therefore \lambda = 2.34$

or, $\lambda = 0.06$

Now, let us find the eigenvector corresponding to the larger eigenvalue $\lambda = 2.34$.

We have, $(M - \lambda I)V = 0$

$$\Rightarrow \left(\begin{bmatrix} 1.2 & 1.14 \\ 1.14 & 1.2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1.2-2.34) & 1.14 \\ 1.14 & (1.2-2.34) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[using $\lambda = 2.34$]

$$\Rightarrow \begin{bmatrix} -1.14V_1 + 1.14V_2 \\ 1.14V_1 + (-1.14V_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ $-1.14V_1 + 1.14V_2 = 0$ — ①

$1.14V_1 - 1.14V_2 = 0$ — ②.

from ①,

$$1.14V_1 = 1.14V_2$$

$$\Rightarrow V_1 = V_2$$

Let $V_1 = V_2 = \alpha$

$$\therefore \|V\| = \sqrt{\alpha^2 + \alpha^2}$$

$$= \sqrt{2\alpha^2}$$

$$= \alpha\sqrt{2}$$

$\therefore$ The normalized eigenvector,

$$V = \begin{bmatrix} \alpha/\alpha\sqrt{2} \\ \alpha/\alpha\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

∴ The reduced 1-dimension data will be,

$$\left(\left(\text{Normalized Data}\right)^T V\right)^T$$

$$= \left[\begin{bmatrix} -1.225 & -1.643 \\ -0.612 & -0.548 \\ -0.612 & 0 \\ 0 & 0 \\ 0.612 & 0.548 \\ 1.837 & 1.643 \end{bmatrix} \times \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}\right]^T$$

$$= \begin{bmatrix} -2.028 \\ -0.820 \\ -0.433 \\ 0 \\ 0.820 \\ 2.461 \end{bmatrix}^T$$

$$= \begin{bmatrix} -2.028 & -0.820 & -0.433 & 0 & 0.820 & 2.461 \end{bmatrix}$$ Ans

The solution is same as the result from Code. Thus we can say that the solution is correct.

Results from the code:

Shape of transformed data:

(1, 6)

Transformed data after dimension reduction:

[[-2.02792041 -0.82031104 -0.4330127   0.   0.82031104 2.46093311]]

```
Question 4

Shape of transformed data:
 (1, 6)

Transformed data after dimension reduction:
 [[-2.02792041 -0.82031104 -0.4330127   0.          0.82031104  2.46093311]]
```

On verifying the solution with the results from code, we see that the results match. Thus, we can say that the solution is correct.

EM

2.1)

No, Eq.3 can not be solved directly. The data has unobserved (hidden) latent variables. We need to use the latent data and EM algorithm to solve the equation.

2.2)

$$\theta^{(0)} = (\pi^{(0)}, p^{(0)}, q^{(0)}) = (0.5, 0.5, 0.5)$$

For $y_j = 0$,
Putting $i=0$ and using the $\theta^{(0)}$ values in (4) we get,

$$\mu^{(1)} = \frac{(0.5)(0.5)^0(1-0.5)^{1-0}}{(0.5)(0.5)^0(1-0.5)^{1-0} + (1-0.5)(0.5)^0(1-0.5)^{1-0}}$$

$$= \frac{(0.5)(0.5)}{(0.5)(0.5)+(0.5)(0.5)}$$

$$= \frac{0.25}{0.25+0.25}$$

$$= \frac{0.25}{0.50}$$

$$= 0.5$$

For $y_j = 1$,

putting $i=0$ and ~~$\theta^{(0)}$~~ using the $\theta^{(0)}$ values in (4) we get,

$$\mu^{(1)} = \frac{(0.5)(0.5)^1(1-0.5)^{1-1}}{(0.5)(0.5)^1(1-0.5)^{1-1} + (1-0.5)(0.5)^1(1-0.5)^{1-1}}$$

$$= \frac{(0.5)(0.5)(0.5)^0}{(0.5)(0.5)(0.5)^0 + (0.5)(0.5)(0.5)^0}$$

$$= \frac{0.25}{0.25+0.25}$$

$$= \frac{0.25}{0.50}$$

$$= 0.5$$

$\therefore$ The mean $\mu^{(1)}$ value ~~for each~~ of the 10 samples $= \frac{0.5\times 6 + 0.5\times 4}{10}$ [$\because$ we have 6 occurrences of $y_j = 1$ and 4 occurrences of $y_j = 0$]

$= 0.5$

2.3)

2.3)

$$\theta^{(0)} = (\pi^{(0)}, p^{(0)}, q^{(0)}) = (0.5, 0.5, 0.5)$$

$\mu^{(1)} = 0.5$ (calculated in question 2.2, for both $y_j = 0$ and $y_j = 1$)

$\therefore$ Putting $i = 0$ and using the $\mu^{(1)}$ values in (5) we get,

$$\pi^{(1)} = \frac{1}{10} \sum_{j=1}^{10} \mu_j^{(1)}$$

$$= \frac{1}{10}(10 \times 0.5) \quad [\because \mu_j^{(1)} = 0.5 \text{ for all values of } j]$$

$$= \frac{1}{10}(5) = 0.5$$

Similarly, putting $i = 0$ and using the $\mu^{(1)}$ values in (6),

$$p^{(1)} = \frac{\sum_{j=1}^{10} \mu_j^{(1)} y_j}{\sum_{j=1}^{10} \mu_j^{(1)}}$$

$$= \frac{0.5(6)}{10 \times 0.5} \quad [\because \mu_j^{(1)} = 0.5 \text{ for all values of } j]$$

$$= 0.6$$

and putting $i=0$ and using the $\mu^{(1)}$ values in (7) we get,

$$q^{(1)} = \frac{\sum_{j=1}^{10} (1-\mu_j^{(1)}) y_j}{\sum_{j=1}^{10} (1-\mu_j^{(1)})}$$

$$= \frac{(1-0.5)\sum_{j=1}^{10} y_j}{(1-0.5)\times 10} \quad \left[\because \mu_j^{(1)} = 0.5 \text{ for all values of } j\right]$$

$$= \frac{0.5\times 6}{0.5\times 10}$$

$$= 0.6$$

$$\therefore \theta^{(1)} = (\pi^{(1)}, p^{(1)}, q^{(1)})$$

$$= (0.5, 0.6, 0.6)$$

2.4)

2.4) For all $y_j = 0$ and when $i=1$, we will have,

$$\mu^{(2)} = \frac{\pi^{(1)}(p^{(1)})^0(1-p^{(1)})^{1-0}}{\pi^{(1)}(p^{(1)})^0(1-p^{(1)})^{1-0} + (1-\pi^{(1)})(q^{(1)})^0(1-q^{(1)})^{1-0}}$$

$$= \frac{(0.5)(0.4)}{(0.5)(0.4) + (0.5)(0.4)}$$

[Using $\theta^{(1)}$ Values Calculated in 2.3)] and given $\theta^{(0)}$

$$= 0.5$$

For all $y_j = 1$ and when $i=1$, we will have

$$\mu^{(2)} = \frac{\pi^{(1)}(p^{(1)})^1(1-p^{(1)})^{1-1}}{\pi^{(1)}(p^{(1)})^1(1-p^{(1)})^{1-1} + (1-\pi^{(1)})(q^{(1)})^1(1-q^{(1)})^{1-1}}$$

$$= \frac{(0.5)(0.6)}{(0.5)(0.6) + (0.5)(0.6)}$$

[Using $\theta^{(1)}$ Values Calculated in 2.3) and given $\theta^{(0)}$ Value]

$$= 0.5$$

$\therefore$ The mean $\mu^{(2)}$ value of the 10 samples

$$= \frac{0.5\times 6 + 0.5\times 4}{10}$$

[$\because$ we have 6 occurances of $y_j = 1$ and 4 occurances of $y_j = 0$]

$$= 0.5$$

2.5)

2.5)

$$\theta^{(0)} = (\pi^{(0)}, p^{(0)}, q^{(0)}) = (0.5, 0.5, 0.5)$$

$\mu^{(2)} = 0.5$ (Calculated in question (2.4), for both $y_j=0$ and $y_j=1$)

$\therefore$ Putting $i=1$ and using $\mu^{(2)}$ values in (5) we get,

$$\pi^{(2)} = \frac{1}{10}\sum_{j=1}^{10} \mu_j^{(2)}$$

$$= \frac{1}{10} \times 10 \times 0.5 \quad [\because \mu_j^{(2)} = 0.5 \text{ for all values of } j]$$

$$= 0.5$$

Similarly, putting $i=1$ and using $\mu^{(2)}$ values in (6) we get,

$$p^{(2)} = \frac{\sum_{j=1}^{10} \mu_j^{(2)} y_j}{\sum_{j=1}^{10} \mu_j^{(2)}}$$

$$= \frac{0.5 \times 6}{0.5 \times 10} \quad [\because \mu_j^{(2)} = 0.5 \text{ for all values of } j \text{ and } \sum_{j=1}^{10} y_j = 6]$$

$$= 0.6$$

and putting $i=1$ and $\mu_j^{(2)}$ values in (7) we get,

$$q^{(2)} = \frac{\sum_{j=1}^{10} (1-\mu_j^{(2)}) y_j}{\sum_{j=1}^{10} (1-\mu_j^{(2)})}$$

$$= \frac{(1-0.5)\sum_{j=1}^{10} y_j}{(1-0.5)\times 10} \quad \left[\because \mu_j^{(2)} = 0.5 \text{ for all values of } j\right]$$

$$= \frac{0.5 \times 6}{0.5 \times 10} \quad \left[\because \sum_{j=1}^{10} y_j = 6\right]$$

$$= 0.6$$

$$\therefore \theta^{(2)} = (\pi^{(2)}, p^{(2)}, q^{(2)}) = (0.5, 0.6, 0.6)$$

2.6)

2.6) If $\theta^{(0)} = (0.4, 0.6, 0.7)$, for $i=0$ and all $y_j = 0$ we get from (4),

$$\mu^{(1)} = \frac{\pi^{(0)}(p^{(0)})^0(1-p^{(0)})^{1-0}}{\pi^{(0)}(p^{(0)})^0(1-p^{(0)})^{1-0} + (1-\pi^{(0)})(q^{(0)})^0(1-q^{(0)})^{1-0}}$$

$$= \frac{(0.4)(1-0.6)}{(0.4)(1-0.6) + (0.6)(0.3)}$$

$$= \frac{0.16}{0.16 + 0.18}$$

$$= 0.471$$

and for $y_j = 1$ we will have,

$$\mu^{(1)} = \frac{\pi^{(0)}(p^{(0)})^1(1-p^{(0)})^{1-1}}{\pi^{(0)}(p^{(0)})^1(1-p^{(0)})^{1-1} + (1-\pi^{(0)})(q^{(0)})^1(1-q^{(0)})^{1-1}}$$

$$= \frac{(0.4)(0.6)}{(0.4)(0.6) + (0.6)(0.7)}$$ [using $\theta^{(0)}$ values]

$$= \frac{0.24}{0.24 + 0.42}$$

$$= 0.364$$

Now, putting $i = 0$ and using $\mu^{(1)}$ Values calculated above in (5) we get,

$$\pi^{(1)} = \frac{1}{10} \sum_{j=1}^{10} \mu_j^{(1)}$$

$$= \frac{1}{10} (6 \times 0.364 + 4 \times 0.471)$$

$$= \frac{1}{10} (2.184 + 1.884)$$

$$= 0.407$$

Similarly, from (6) we get,

$$p^{(1)} = \frac{\sum_{j=1}^{10} \mu_j^{(1)} y_j}{\sum_{j=1}^{10} \mu_j^{(1)}}$$

$$= \frac{6 \times 0.364}{6 \times 0.364 + 4 \times 0.471}$$

$$= \frac{2.184}{4.07}$$

$$= 0.537$$

and from (7) we get,

$$q^{(1)} = \frac{\sum_{j=1}^{10} (1-\mu_j^{(1)}) y_j}{\sum_{j=1}^{10} (1-\mu_j^{(1)})}$$

$$= \frac{6 \times (1-0.364)}{6 \times (1-0.364) + 4(1-0.471)}$$

$$= \frac{3.816}{5.932}$$

$$= 0.643$$

$$\therefore \theta^{(1)} = (0.407, 0.537, 0.643)$$

For $y_j = 0$, $i = 1$ and using the $\theta^{(1)}$ values in (4)

we get,

$$\mu^{(2)} = \frac{\pi^{(1)} (p^{(1)})^0 (1-p^{(1)})^{1-0}}{\pi^{(1)} (p^{(1)})^0 (1-p^{(1)})^{1-0} + (1-\pi^{(1)}) (q^{(1)})^0 (1-q^{(1)})^{1-0}}$$

$$= \frac{(0.407)(1-0.537)}{(0.407)(1-0.537) + (1-0.407)(1-0.643)}$$

$$= \frac{0.188}{0.188 + 0.212} = 0.47$$

when $y_j = 1$,

$$\mu^{(2)} = \frac{\pi^{(1)}(p^{(1)})^1(1-p^{(1)})^{1-1}}{\pi^{(1)}(p^{(1)})^1(1-p^{(1)})^{1-1} + (1-\pi^{(1)})(q^{(1)})^1(1-q^{(1)})^{1-1}}$$

$$= \frac{(0.407)(0.537)(1-0.537)^0}{(0.407)(0.537) + (1-0.407)(0.643)}$$

$$= \frac{0.219}{0.219 + 0.381}$$

$$= 0.365$$

Now, putting $i=1$ and using the $\mu^{(2)}$ values calculated above in (5) we get,

$$\pi^{(2)} = \frac{1}{10}\sum_{j=1}^{10} \mu_j^{(2)}$$

$$= \frac{1}{10}(6 \times 0.365 + 4 \times 0.470)$$

$$= \frac{1}{10}(2.19 + 1.88)$$

$$= 0.407$$

Similarly, from (6) we get,

$$p^{(2)} = \frac{\sum_{j=1}^{10} \mu_j^{(2)} y_j}{\sum_{j=1}^{10} \mu_j^{(2)}}$$

$$= \frac{6 \times 0.365}{6 \times 0.365 + 4 \times 0.470}$$

$$= \frac{2.19}{4.07} = 0.538$$

and from (7) we get,

$$q^{(2)} = \frac{\sum_{j=1}^{10} (1-\mu_j^{(2)}) y_j}{\sum_{j=1}^{10} (1-\mu_j^{(2)})}$$

$$= \frac{(6) \times (1-0.365)}{6(1-0.365) + 4(1-0.470)}$$

$$= \frac{3.81}{3.81 + 2.12} = 0.643$$

$$\therefore \theta^{(2)} = (0.407, 0.538, 0.643)$$

Thus, we see that $\theta^{(2)}$ is NOT the same as that in Question 2.5

2.7)

No, the results of parameters estimation are different with different initialization.

Here are the results from the code implementation of the EM algorithm:

*****EM*******

Theta(0) values are:

Pie(0)= 0.4

p(0)= 0.6

q(0)= 0.7

For all values of observable data = 0:

mu(1)= 0.47058823529411764

For all values of observable data = 1:

mu(1)= 0.36363636363636365

Mean mu(1)= 0.40641711229946526

Iteration number: 1

Pie(1)= 0.40641711229946526

p(1)= 0.5368421052631579

q(1)= 0.6432432432432431

For all values of observable data = 0:

mu(2)= 0.47058823529411764

For all values of observable data = 1:

mu(2)= 0.36363636363636376

Mean mu(2)= 0.40641711229946537

Iteration number: 2

Pie(2)= 0.40641711229946537

p(2)= 0.5368421052631579

q(2)= 0.6432432432432431

For all values of observable data = 0:

mu(3)= 0.47058823529411764

For all values of observable data = 1:

mu(3)= 0.3636363636363638

Mean mu(3)= 0.40641711229946537

Iteration number: 3

Pie(3)= 0.40641711229946537

p(3)= 0.536842105263158

q(3)= 0.6432432432432431

For all values of observable data = 0:

mu(4)= 0.47058823529411764

For all values of observable data = 1:

mu(4)= 0.3636363636363638

Mean mu(4)= 0.40641711229946537

Iteration number: 4

Pie(4)= 0.40641711229946537

p(4)= 0.536842105263158

q(4)= 0.6432432432432431

For all values of observable data = 0:

mu(5)= 0.47058823529411764

For all values of observable data = 1:

mu(5)= 0.3636363636363638

Mean mu(5)= 0.40641711229946537

Iteration number: 5

Pie(5)= 0.40641711229946537

p(5)= 0.536842105263158

q(5)= 0.6432432432432431

For all values of observable data = 0:

mu(6)= 0.47058823529411764

For all values of observable data = 1:

mu(6)= 0.3636363636363638

Mean mu(6)= 0.40641711229946537

Iteration number: 6

Pie(6)= 0.40641711229946537

p(6)= 0.536842105263158

q(6)= 0.6432432432432431

For all values of observable data = 0:

mu(7)= 0.47058823529411764

For all values of observable data = 1:

mu(7)= 0.3636363636363638

Mean mu(7)= 0.40641711229946537

Iteration number: 7

Pie(7)= 0.40641711229946537

p(7)= 0.536842105263158

q(7)= 0.6432432432432431

For all values of observable data = 0:

mu(8)= 0.47058823529411764

For all values of observable data = 1:

mu(8)= 0.3636363636363638

Mean mu(8)= 0.40641711229946537

Iteration number: 8

Pie(8)= 0.40641711229946537

p(8)= 0.536842105263158

q(8)= 0.6432432432432431

For all values of observable data = 0:

mu(9)= 0.47058823529411764

For all values of observable data = 1:

mu(9)= 0.3636363636363638

Mean mu(9)= 0.40641711229946537

Iteration number: 9

Pie(9)= 0.40641711229946537

p(9)= 0.536842105263158

q(9)= 0.6432432432432431

For all values of observable data = 0:

mu(10)= 0.47058823529411764

For all values of observable data = 1:

mu(10)= 0.3636363636363638

Mean mu(10)= 0.40641711229946537

Iteration number: 10

Pie(10)= 0.40641711229946537

p(10)= 0.536842105263158

q(10)= 0.6432432432432431

For all values of observable data = 0:

mu(11)= 0.47058823529411764

For all values of observable data = 1:

mu(11)= 0.3636363636363638

Mean mu(11)= 0.40641711229946537

Iteration number: 11

Pie(11)= 0.40641711229946537

p(11)= 0.536842105263158

q(11)= 0.6432432432432431

For all values of observable data = 0:

mu(12)= 0.47058823529411764

For all values of observable data = 1:

mu(12)= 0.3636363636363638

Mean mu(12)= 0.40641711229946537

Iteration number: 12

Pie(12)= 0.40641711229946537

p(12)= 0.536842105263158

q(12)= 0.6432432432432431

For all values of observable data = 0:

mu(13)= 0.47058823529411764

For all values of observable data = 1:

mu(13)= 0.3636363636363638

Mean mu(13)= 0.40641711229946537

Iteration number: 13

Pie(13)= 0.40641711229946537

p(13)= 0.536842105263158

q(13)= 0.6432432432432431

For all values of observable data = 0:

mu(14)= 0.47058823529411764

For all values of observable data = 1:

mu(14)= 0.3636363636363638

Mean mu(14)= 0.40641711229946537

Iteration number: 14

Pie(14)= 0.40641711229946537

p(14)= 0.536842105263158

q(14)= 0.6432432432432431

For all values of observable data = 0:

mu(15)= 0.47058823529411764

For all values of observable data = 1:

mu(15)= 0.3636363636363638

Mean mu(15)= 0.40641711229946537

Iteration number: 15

Pie(15)= 0.40641711229946537

p(15)= 0.536842105263158

q(15)= 0.6432432432432431

For all values of observable data = 0:

mu(16)= 0.47058823529411764

For all values of observable data = 1:

mu(16)= 0.3636363636363638

Mean mu(16)= 0.40641711229946537

Iteration number: 16

Pie(16)= 0.40641711229946537

p(16)= 0.536842105263158

q(16)= 0.6432432432432431

For all values of observable data = 0:

mu(17)= 0.47058823529411764

For all values of observable data = 1:

mu(17)= 0.3636363636363638

Mean mu(17)= 0.40641711229946537

Iteration number: 17

Pie(17)= 0.40641711229946537

p(17)= 0.536842105263158

q(17)= 0.6432432432432431

For all values of observable data = 0:

mu(18)= 0.47058823529411764

For all values of observable data = 1:

mu(18)= 0.3636363636363638

Mean mu(18)= 0.40641711229946537

Iteration number: 18

Pie(18)= 0.40641711229946537

p(18)= 0.536842105263158

q(18)= 0.6432432432432431

For all values of observable data = 0:

mu(19)= 0.47058823529411764

For all values of observable data = 1:

mu(19)= 0.3636363636363638

Mean mu(19)= 0.40641711229946537

Iteration number: 19

Pie(19)= 0.40641711229946537

p(19)= 0.536842105263158

q(19)= 0.6432432432432431

For all values of observable data = 0:

mu(20)= 0.47058823529411764

For all values of observable data = 1:

mu(20)= 0.3636363636363638

Mean mu(20)= 0.40641711229946537

Iteration number: 20

Pie(20)= 0.40641711229946537

p(20)= 0.536842105263158

q(20)= 0.6432432432432431

For all values of observable data = 0:

mu(21)= 0.47058823529411764

For all values of observable data = 1:

mu(21)= 0.3636363636363638

Mean mu(21)= 0.40641711229946537

Theta(0) values are:

Pie(0)= 0.5

p(0)= 0.5

q(0)= 0.5

For all values of observable data = 0:

mu(1)= 0.5

For all values of observable data = 1:

mu(1)= 0.5

Mean mu(1)= 0.5

Iteration number: 1

Pie(1)= 0.5

p(1)= 0.6

q(1)= 0.6

For all values of observable data = 0:

mu(2)= 0.5

For all values of observable data = 1:

mu(2)= 0.5

Mean mu(2)= 0.5

Iteration number: 2

Pie(2)= 0.5

p(2)= 0.6

q(2)= 0.6

For all values of observable data = 0:

mu(3)= 0.5

For all values of observable data = 1:

mu(3)= 0.5

Mean mu(3)= 0.5

Iteration number: 3

Pie(3)= 0.5

p(3)= 0.6

q(3)= 0.6

For all values of observable data = 0:

mu(4)= 0.5

For all values of observable data = 1:

mu(4)= 0.5

Mean mu(4)= 0.5

Iteration number: 4

Pie(4)= 0.5

p(4)= 0.6

q(4)= 0.6

For all values of observable data = 0:

mu(5)= 0.5

For all values of observable data = 1:

mu(5)= 0.5

Mean mu(5)= 0.5

Iteration number: 5

Pie(5)= 0.5

p(5)= 0.6

q(5)= 0.6

For all values of observable data = 0:

mu(6)= 0.5

For all values of observable data = 1:

mu(6)= 0.5

Mean mu(6)= 0.5

Iteration number: 6

Pie(6)= 0.5

p(6)= 0.6

q(6)= 0.6

For all values of observable data = 0:

mu(7)= 0.5

For all values of observable data = 1:

mu(7)= 0.5

Mean mu(7)= 0.5

Iteration number: 7

Pie(7)= 0.5

p(7)= 0.6

q(7)= 0.6

For all values of observable data = 0:

mu(8)= 0.5

For all values of observable data = 1:

mu(8)= 0.5

Mean mu(8)= 0.5

Iteration number: 8

Pie(8)= 0.5

p(8)= 0.6

q(8)= 0.6

For all values of observable data = 0:

mu(9)= 0.5

For all values of observable data = 1:

mu(9)= 0.5

Mean mu(9)= 0.5

Iteration number: 9

Pie(9)= 0.5

p(9)= 0.6

q(9)= 0.6

For all values of observable data = 0:

mu(10)= 0.5

For all values of observable data = 1:

mu(10)= 0.5

Mean mu(10)= 0.5

Iteration number: 10

Pie(10)= 0.5

p(10)= 0.6

q(10)= 0.6

For all values of observable data = 0:

mu(11)= 0.5

For all values of observable data = 1:

mu(11)= 0.5

Mean mu(11)= 0.5

Iteration number: 11

Pie(11)= 0.5

p(11)= 0.6

q(11)= 0.6

For all values of observable data = 0:

mu(12)= 0.5

For all values of observable data = 1:

mu(12)= 0.5

Mean mu(12)= 0.5

Iteration number: 12

Pie(12)= 0.5

p(12)= 0.6

q(12)= 0.6

For all values of observable data = 0:

mu(13)= 0.5

For all values of observable data = 1:

mu(13)= 0.5

Mean mu(13)= 0.5

Iteration number: 13

Pie(13)= 0.5

p(13)= 0.6

q(13)= 0.6

For all values of observable data = 0:

mu(14)= 0.5

For all values of observable data = 1:

mu(14)= 0.5

Mean mu(14)= 0.5

Iteration number: 14

Pie(14)= 0.5

p(14)= 0.6

q(14)= 0.6

For all values of observable data = 0:

mu(15)= 0.5

For all values of observable data = 1:

mu(15)= 0.5

Mean mu(15)= 0.5

Iteration number: 15

Pie(15)= 0.5

p(15)= 0.6

q(15)= 0.6

For all values of observable data = 0:

mu(16)= 0.5

For all values of observable data = 1:

mu(16)= 0.5

Mean mu(16)= 0.5

Iteration number: 16

Pie(16)= 0.5

p(16)= 0.6

q(16)= 0.6

For all values of observable data = 0:

mu(17)= 0.5

For all values of observable data = 1:

mu(17)= 0.5

Mean mu(17)= 0.5

Iteration number: 17

Pie(17)= 0.5

p(17)= 0.6

q(17)= 0.6

For all values of observable data = 0:

mu(18)= 0.5

For all values of observable data = 1:

mu(18)= 0.5

Mean mu(18)= 0.5

Iteration number: 18

Pie(18)= 0.5

p(18)= 0.6

q(18)= 0.6

For all values of observable data = 0:

mu(19)= 0.5

For all values of observable data = 1:

mu(19)= 0.5

Mean mu(19)= 0.5

Iteration number: 19

Pie(19)= 0.5

p(19)= 0.6

q(19)= 0.6

For all values of observable data = 0:

mu(20)= 0.5

For all values of observable data = 1:

mu(20)= 0.5

Mean mu(20)= 0.5

Iteration number: 20

Pie(20)= 0.5

p(20)= 0.6

q(20)= 0.6

For all values of observable data = 0:

mu(21)= 0.5

For all values of observable data = 1:

mu(21)= 0.5

Mean mu(21)= 0.5