Let $h(x) = \sum_i^n e^{-x_i}$ then $h(x) = exp(\underbrace{log - sum - expr(-x)}_{convex})$, so h is a convex function, becouse $e^{g(x)}$ is convex if $g(x)$ is convex.
Thus $\{x : x \in \sum_i^n e^{-x_i} \leq 1\}$ is a convex set as it is a 1-sublevel set of a convex function h.
$f(x) = \prod_{i=1}^n (1 - e^{-x_i})$ $g(x) = -\prod_{i=1}^n (1 - e^{-x_i})$

$$\nabla^2 f(x)_{i,j} = \begin{cases} e^{-x_i} e^{x_j} \prod_{k \neq i,j}^k (1 - e^{-x_k}), & \text{when } i \neq j; \\ -e^{-x_i} \prod_{k \neq i}^k (1 - e^{-x_k}), & \text{when } i = j. \end{cases}$$

Thus:

$$\nabla^2 g(x)_{i,j} = \begin{cases} -e^{-x_i} e^{x_j} \prod_{k \neq i,j}^k (1 - e^{-x_k}), & \text{when } i \neq j; \\ e^{-x_i} \prod_{k \neq i}^k (1 - e^{-x_k}), & \text{when } i = j. \end{cases}$$

So $\nabla^2 g(x) = M \circ (diag(e^{x_i} - J_n)$ where:

$$M_{i,j} = \begin{cases} e^{-x_i} e^{x_j} \prod_{k \neq i,j}^k (1 - e^{-x_k}), & \text{when } i \neq j; \\ \frac{e^{-2x_i}}{(1-e^{x_i})} \prod_{k \neq i}^k (1 - e^{-x_k}), & \text{when } i = j. \end{cases}$$

Matrix M is PSD becouse $M = m * m^T$ where m is a vector such that $m_i = \frac{e^{-x_i}}{(1-e^{-x_i})} \sqrt{\prod_{k=1}^n (1 - e^{-x_i})}$.
The assumptions that $\sum_i^n e^{-x_i} \leq 1$ and all $x_i > 0$ implies that the value of $z(diag(e^{x_i}) - J_n) z^T$ is minimal when $\sum_i^n e^{-x_i} = 1$ and $\forall_i e^{-x_i} = \frac{1}{n}$, so:
$z(diag(e^{x_i}) - J_n) z^T \geq z(diag(n) - J_n) z^T$ and as $diag(n) - J_n$ is PSD (its only eugenvalues are 1 and $n+1$) the $diag(e^{x_i}) - J_n$ is PSD as well. So the domain of $g(x)$ is convex and its hessian is PSD as the coordinate-wise matrix product of PSD matricies and thus $g(x)$ is a convex function, so $f(x)$ is concave QED.