Math 4 Data Science- Exam

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Instructions

1. Consulting:
    1. You may not discuss the contents of this exam with any living being, until you have submitted.
    2. You may write me an email requesting clarifications. I will answer within that day.
    3. You may consult any reference you like. In particular, this library I have compiled:
       https://bit.ly/3239joD
       I recommend indexing it before you start. I use DocFetcher.
2. Submission:
    1. By email to johnros@bgu.ac.il
    2. No later than 24 Feb, 2020 at 21:00 Israel time.
    3. You may submit in Hebrew or English.
    4. PDF format only. PDF may include images if you prefer to do the math manually and take a picture. Make sure it is readable.
3. I reserve myself the right to call you to my office, and explain your solutions in person.

Questions

1. (40 pts) In the following, supply the code, and a plot, of the iterations in the x_1, x_2 plane, starting at $x = (1, 1)$. Define $Ax = b$ where
$$A := \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}; b := (13, 21).$$
    1. Find x using Gauss-Seidel iterations.
    2. Find x using Jacobi iterations.
    3. Find x using Steepest Descent (a.k.a. Gradient Decent), with exact line-searches.
    4. Find x using coordinate-decent, i.e., optimizing a single coordinate per iteration. You are free to chose the type of descent you perform at each coordinate.
2. (20 pts) Generate 100 samples from $Binom(10, 0.5)$. You are allowed to use your software's $Unif[0, 1]$ generator. I want the code, and a histogram of 1,000 samples.
    1. Use the inverse probability transform ($F^{-1}(t)$). You are allowed to use your software's quantile functions.
    2. Use accept reject (a.k.a. rejection sampling), with proposals from $Unif\{0, 10\}$. Use only your software's $Unif[0, 1]$ generator.
    3. Use Metropolis-Hastings with a proposal distribution of your choice.
3. (10 pts) What are the first 10 numbers in the sequence of a linear congruent generator with: $a = 1664525, c = 0, m = 2^{32}, x_0 = 3$? Provide sequence and code.
4. (10pts) Let N_t be a simple birth process, i.e., a Poisson point process with rate $\lambda_i = \lambda i$. Let X_i be the times between event, so that $N_t = \max\{n \quad s.t. \quad \sum_{i=0}^{n} X_i \leq t\}$. Write the likelihood of λ, given $X_1, \ldots, X_N$. Is it convex in λ?
5. (10pts) Prove that the leading eigenvalue of Markov Chain's transition matrix, is 1.
6. (10pts) Show that the regression's "Hat Matrix" ($H = (X'X)^{-1}X'$) is the Moore-Penrose Pseudo-Inverse of the matrix X.

7. (10pts) Prove that the QR decomposition of a matrix may be found with a series of Householder Transformations. How many floating point operations (FLOPS) are required (explain)?