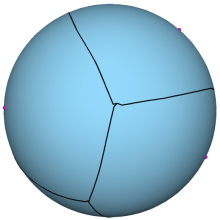

Geometrical Radii and Surface Sampling

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AGGEOM - Freie Universität Berlin

01.07.2010

Outline

Introduction

Geometry
- Injectivity Radius
- Strong Convexity
  - Strongly Convex
  - Strong Convexity Radius
- Relations

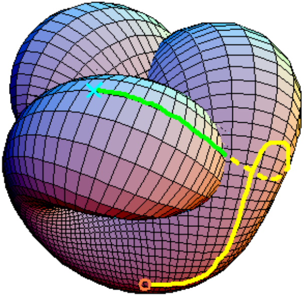
Surface (manifold) Smapling
- Previous Work

Conclusion

Outline

Motivation

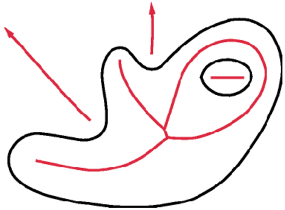
- Study the topology of manifolds.
- Consider a high dimensional data set, of a geometrical nature. E.g. robots motion planning.
- Extrinsic geometry can be very complicated due to the dimension. E.g. what is the medial axis?



Boy's surface, thanks to *AugPi, 2004*

Extrinsic Work

- A lot of work is basing on extrinsic notions
- For example the *medial axis* and the celebrated *local feature size* of [Am99].



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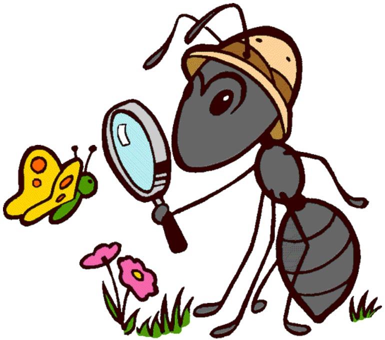


Intrinsic Work

On the other hand. . .

An ant living on a surface can learn a lot about it. . .

Consider the *intrinsic* geometry.

Study the topology of the manifold intrinsically.

Theorem

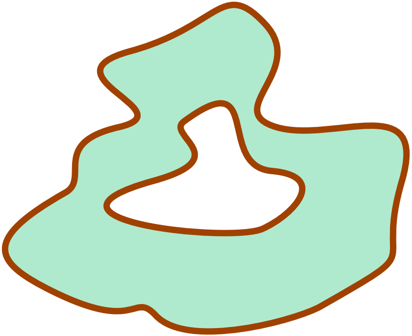
$$\int_M K dA = 2\pi\chi(M)$$

Where M is a triangulated, compact, oriented 2-Riemannian manifold.

This theorem relates an intrinsic quantatiy (*Theorema Egregium*) and a topological one (χ).

Theorem (Nerve Theorem, See [Bjo95])

If X is a triangulable space, and $\{A_i\}$ is a finite close cover such that $\bigcap A_{i_j}$ is contractible, then X and $\text{nerve}(A_i)$ are homotopic equivalent.



This was used in [Gao08] for example.

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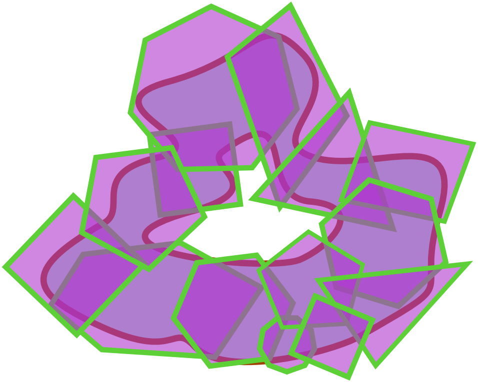

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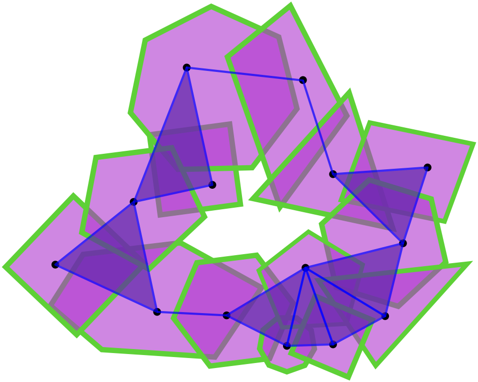

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What's the plan?

Theorem (Nerve Theorem, See [Bjo95])

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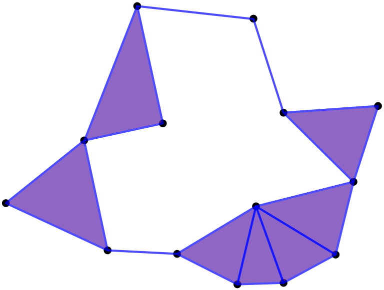

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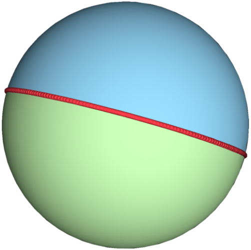
Another Example



- Intersection is *not* contractible
- Nerve is *not* homotopic equivalent to S^2

Outline

Outline

- Given $p \in M$ and $\xi \in T_pM$ ($|\xi| = 1$), then there exsits γ_ξ.
- *Locally* γ_ξ is a shortest path.
- *And globally?*

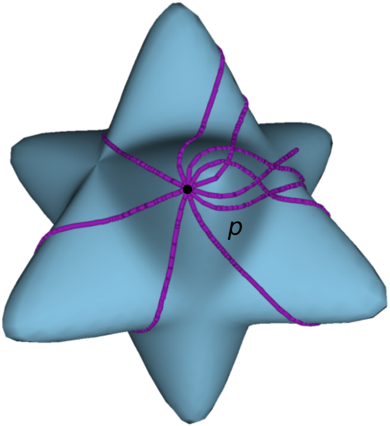


Figure: Fresnel's wave-surface

Straight or Short?

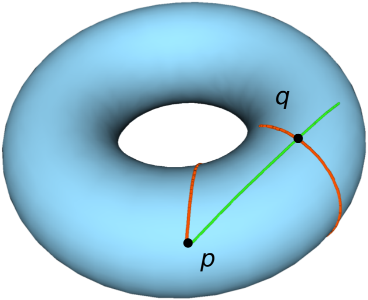


Figure: Two intersecting geodesics on the torus

Definition (Injectivity Radius)

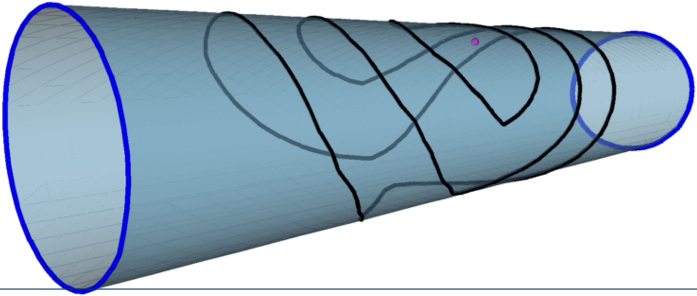
Given p in then

$$\operatorname{inj}(p) = \inf\left\{c(\xi) \,\middle|\, \quad \xi \in T_pM, \quad |\xi| = 1\right\}$$

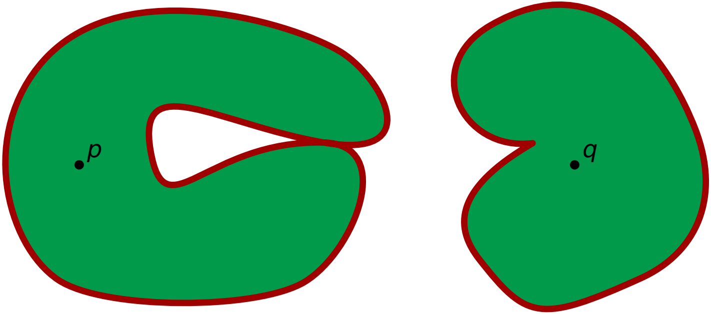
and

$$\operatorname{inj}(M) = \inf\left\{\operatorname{inj}(p) \,\middle|\, \quad p \in M\right\},$$

where $c(\xi) = \sup\left\{t > 0 \,\middle|\, \quad t\xi \in \mathcal{T}M, \mathrm{d}(p, \gamma_\xi(t)) = t\right\}$.

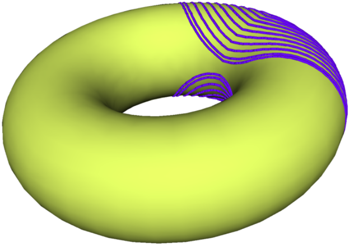


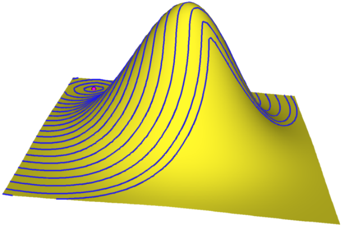
This can happen in one of the two following cases:



The Exponential Map Fails

This can happen in one of the two following cases:





Outline

Here, every *ball* is convex

Figure: A solid soccer ball

Problem

What happens on a surface? on a Riemannian manifold?

The Problem on a Surface

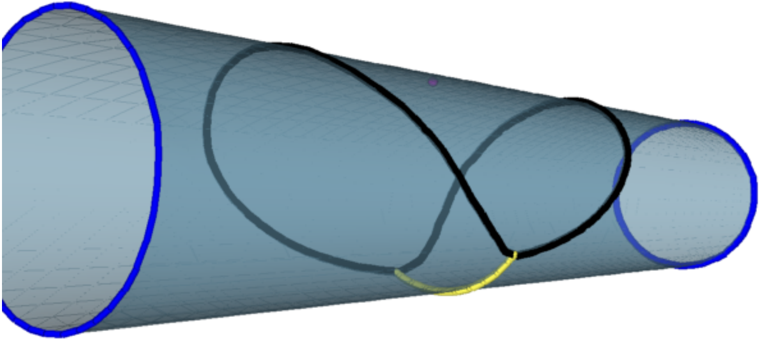
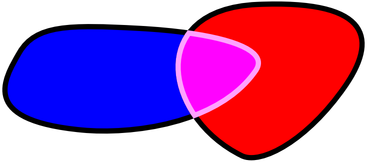


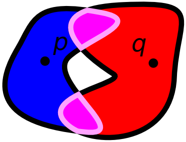
Figure: A non convex disc on the cylinder

Convex Intersection

In the plane, $A \cap B$ is convex if both A and B are.



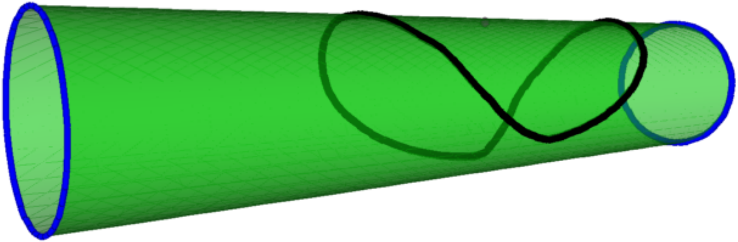
In the curved case, balls intersection must not be even connected



Definition (Weakly Convex)

$C \subset M$ is *weakly convex* if for every pair $x, y \in C$ there exists a geodesic $\gamma_{xy} \subset C$ which is the unique minimizer in C. See [Ch06, §IX.6]

Different Definitions



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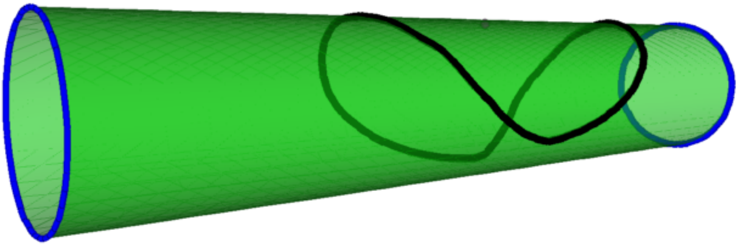
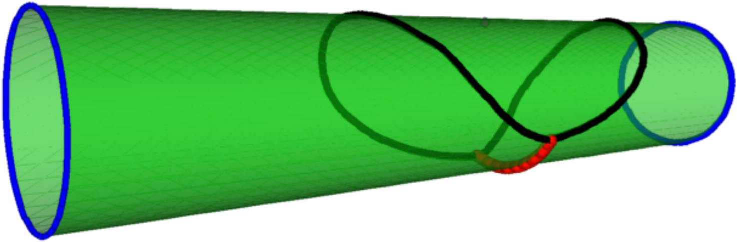


Weakly Convex



But *not* convex. . . .

Different Definitions

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Definition (Convex)

$C \subset M$ is *convex* if for every pair $x, y \in C$ there exists a geodesic $\gamma_{xy} \subset C$ which is the unique minimizer in M.

Definition (Strongly Convex)

$C \subset M$ is *strongly convex* if for every pair $x, y \in C$ there exists a unique geodesic $\gamma_{xy} \subset C$ which is also a unique minimizer in M.

Definition (Totally Convex)

$C \subset M$ is *(totally) convex* if for every pair $x, y \in C$ *all* minimizing geodesics γ_{xy} connecting x and y, are contained in C, i.e $\gamma_{xy} \subset C$. See [Br03].

Some Remarks

Remark

All the above definition coincide with the standard convexity definition in Euclidean space.

Other Definitions

Be careful! There other definitions wandering around!

What do we want?

Relate discs and convexity

Strongly Convex

Lemma

If $\{A_i\}_{i=1}^k$ are strongly convex, then $\bigcap_{i=1}^k A_i$ is as well.

Lemma

A strongly convex set is contractible.

Strongly Convex

Lemma

If $\{A_i\}_{i=1}^k$ are strongly convex, then $\bigcap_{i=1}^k A_i$ is as well.

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A strongly convex set is contractible.

Lemma

A non-empty finite intersection of strongly convex sets is contractible.

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A strongly convex set is contractible.

Lemma

A non-empty finite intersection of strongly convex sets is contractible.

All in all...

- Strongly convex disc is nice
- Apply the *nerve theorem*

Strong Convexity Radius

Definition (Strong Convexity Radius)

For a point $x \in M$:

$$\mathrm{scr}(x) = \sup\{\rho \mid \quad B_r(x) \text{ is strongly convex } \forall r < \rho\}$$

This definition helps us to have expected properties for intrinsic discs.

Outline

Some Relations

- In general $\mathrm{scr}(x) \le \mathrm{inj}(x)$
  What about

$$\frac{1}{2}\mathrm{inj}(x) \left\{ \begin{array}{c} < \\ > \\ = \end{array} \right\} \mathrm{scr}(x)?$$

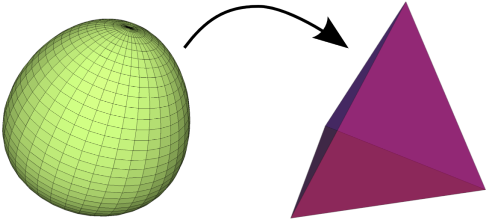
- The following holds for M:

$$\mathrm{cr}(M) \le \frac{\mathrm{inj}(M)}{2}$$

Outline

Surface Sampling's Goal

- Get a sample which encodes the topology.
- Use only intrinsic information.
- Assuming $d(x, y)$ is known for every x and y.



Definition (Voronoi Diagram)

If $L \subset \mathbb{E}^n$, then for $p \in L$

$$\mathcal{V}(p) = \{x \in \mathbb{E}^n \mid \quad |x-p| \leq |x-q|, \forall q \in L\}$$

is the *Voronoi cell* of p. The *Voronoi diagram* of L, $\mathcal{V}(L)$, is the collection of Voronoi cells and all their intersections.

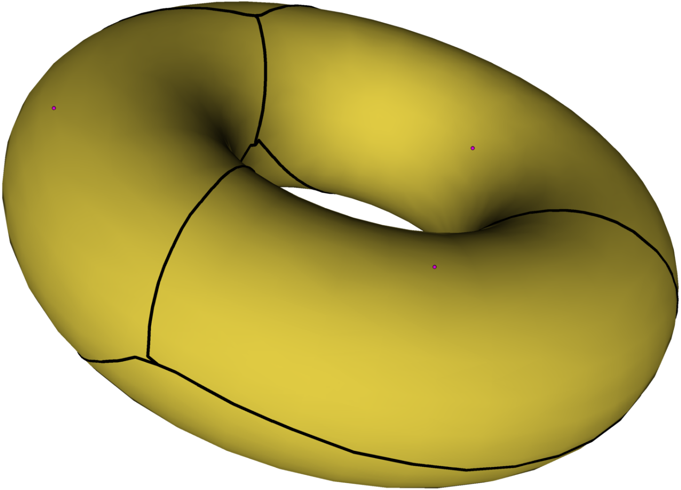
Remark

The $\mathcal{V}(p)$'s are convex $\Rightarrow$ contractible.

In general?

What about Voronoi diagrams on surfaces?

General Voronoi Diagrams and Delaunay Triangulations

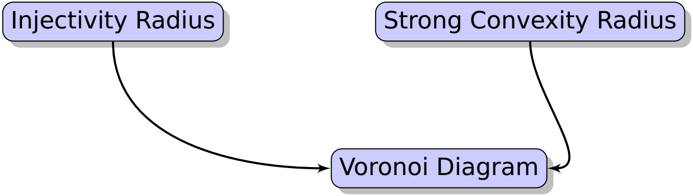


Connect the Dots

Injectivity Radius

Strong Convexity Radius

Connect the Dots



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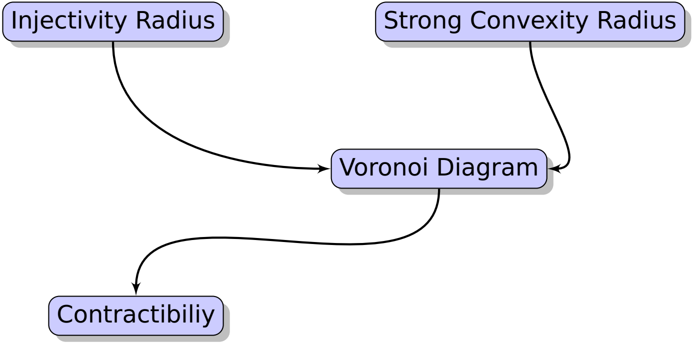

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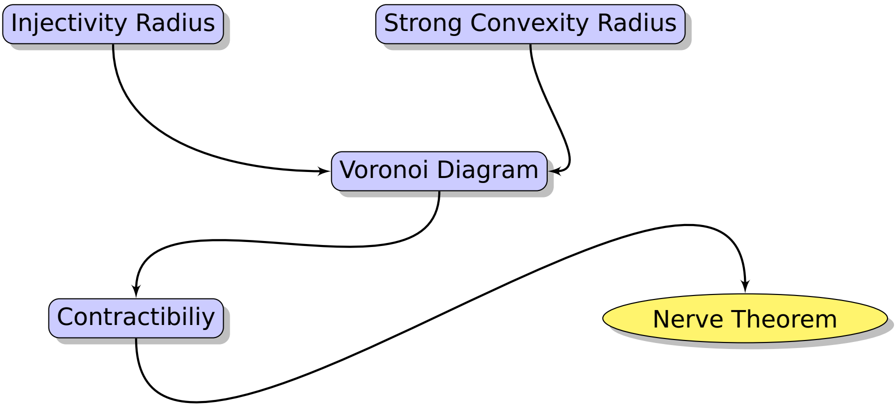


Feature Sample

Definition

L is called an ϵ*feature sample* of M if for all $x \in M$ there exists $p \in L$ such that $\mathrm{d}(x, p) \leq \epsilon\,\mathrm{feature}(x)$.

Lemma

If L is a feautre sample of M, and $\mathcal{V}(L)$ is the intrinsic Voronoi diagram, then for all $p \in L$ and $x \in \mathcal{V}(p)$

$$d(p, x) \leq \epsilon feature(x).$$

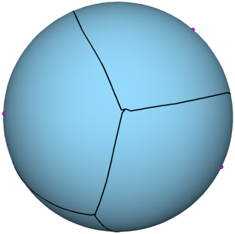
Theorem

Assume $\mathcal{K} \leq \delta$ for a manifold M, and denote

$$r_{sc} = \min\left\{\frac{inj(M)}{2}, \frac{\pi}{2\sqrt{\delta}}\right\}$$

then, $B_x(r_{sc})$ is strongly convex.

- For S^2, $\mathrm{inj}(M) = \pi$ and $\mathcal{K} = 1$
- thus, $r_{sc} = \pi/2$.
- An ϵiRadius sample captures the topology for $\epsilon < 0.5$.



Outline

Definition (Intrinsic Sampling Radius)

$$\rho_m(x) = \min\left\{\mathrm{scr}(x), \frac{1}{2}\mathrm{inj}(x)\right\}$$

Theorem

If L is an intrinsic sample, then the intrinsic Delaunay triangulation exists.

Remark

The *strong convexity radius* is hard to work with!

Surface Sampling - Some more results [Dy08]

Local feature size

$\rho_\kappa(x) = 1/\max\{|\kappa_1(x)|, |\kappa_2(x)|\}$

$$\mathrm{lfs}(x) \leq \mathrm{lr}(x) \leq \rho_\kappa(x) \leq \rho_G(x)$$

Local reach at x

$\rho_G(x) = 1/\sqrt{K(x)}$

Outline

Summary

- Intrinsic analysis
- Injectivity radius
- Notion of convexity
- Nerve theorem

Current and future Work

- Relations between inj and scr
- Relation between *curvature* and scr
- Obtain a criteria depends only on inj

References

Nina Amenta and Marshall Bern.
Surface reconstruction by voronoi filtering.
In *SCG '98: Proceedings of the fourteenth annual symposium on Computational geometry*, pages 39–48, New York, NY, USA, 1998. ACM.

A. Björner.
Topological methods.
Handbook of Combinatorics, 2:1819–1872, 1995.

Marcel Berger.
A Panoramic View of Riemannian Geometry.
Springer, 2003.

Isaac Chavel.
Riemannian Geometry: A Modern Introduction.
Cambridge University Press, 2 edition, 2006.

Ramsay Dyer, Hao Zhang, and Torsten Möller.
Surface sampling and the intrinsic voronoi diagram.
Computer Graphics Forum (Special Issue of Symposium on Geometry Processing 2008), 27(5):1393–1402, 2008.

Leonidas J. Guibas, Jie Gao, Steve Y. Oudot, and Yue Wang.
Geodesic delaunay triangulation and witness complex in the plane.
In *SODA '08: Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 571–580, Philadelphia, PA, USA, 2008. Society for Industrial and Applied Mathematics.

Thank you!

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