Shape

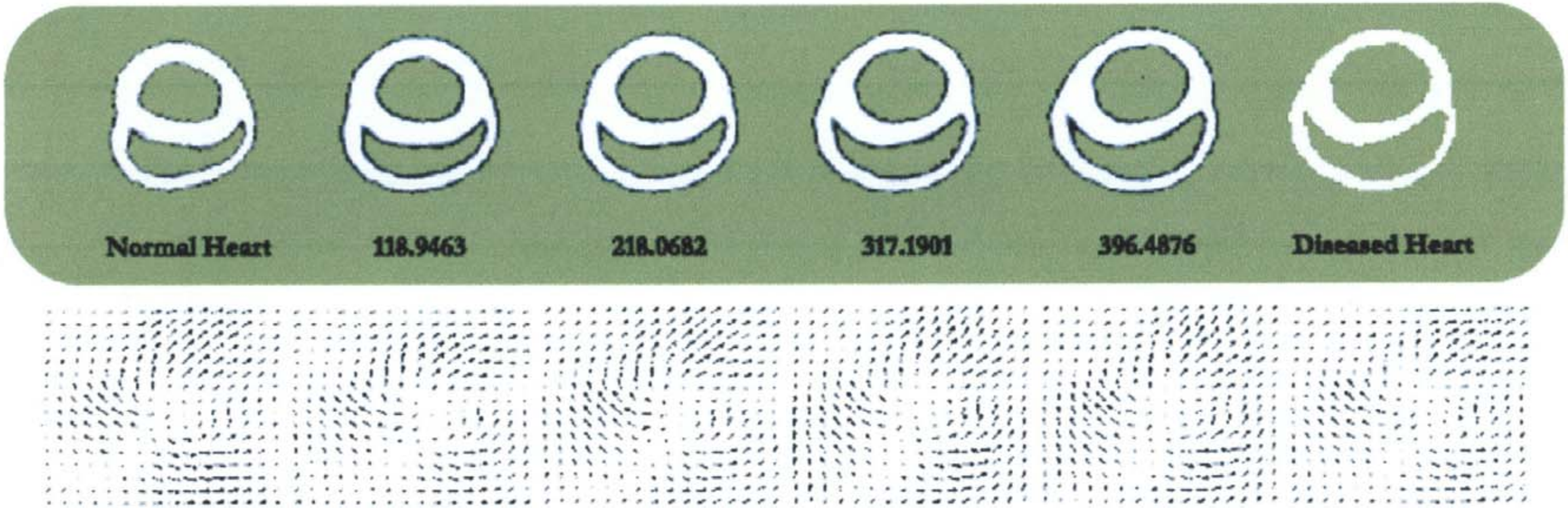
- Convergence between

  a) major computer vision applications,

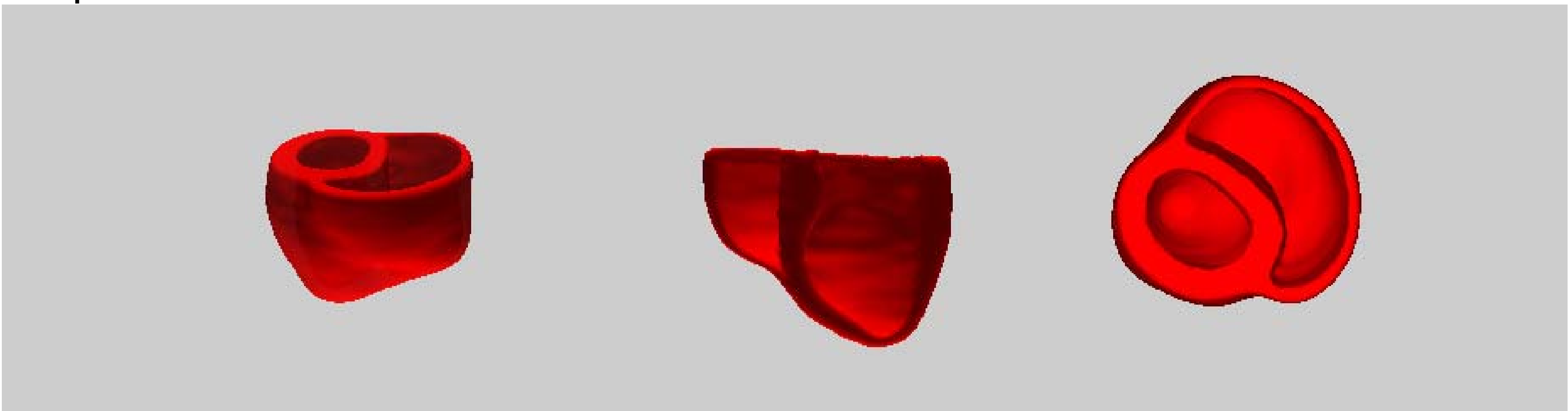
  b) non-linear statistics of shape (Kendall),
  c) need for a deeper math understanding of geometry on infinite dimensional manifolds

- Let's start with a start-of-the-art application from Miller's group at Hopkins.

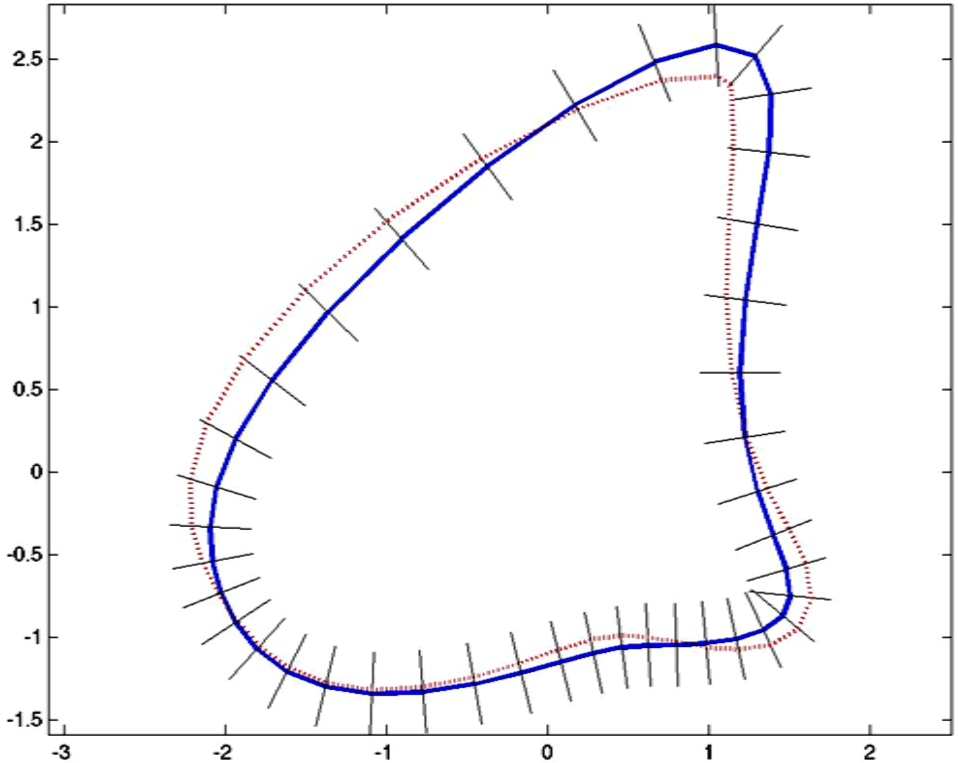
Geodesics between hearts with global Sobolev metric (Miller et al)



Above:original study; Below (video): the geodesic tracking the principal component of diseased hearts.



The set $\mathcal{S}$ of all smooth plane curves forms a manifold!



Start with a fixed curve $C \in \mathcal{S}$

parametrized by $s \mapsto \phi(s)$

Define a local chart near ϕ:

$\psi_a(s) = \phi(s) + a(s).\vec{n}(s)$,

$\vec{n}(s) =$ unit normal to C,

$C_a =$ image of ψ_a

$U_\phi = \left\{ a \middle| \psi_a \text{ smooth} \right\}$

$\quad \subset \left(\text{v.sp.of fcns. } a \right)$

$a \mapsto C_a$ is the chart,

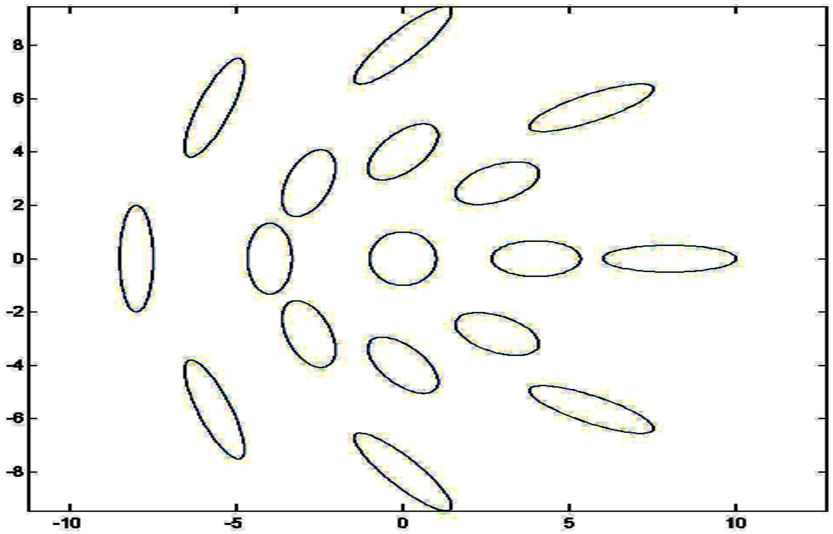
$a(s)$ the local linear coord.

$\mathcal{S} = \bigcup_\phi U_\phi,$ gives the atlas

A *topological* Banach manifold BUT not a *differentiable* Banach manifold (e.g. the map taking *f* to its inverse has no Frechet derivative).

Think of $\mathcal{S}$ geometrically

- A curve on $\mathcal{S}$ is a warping of one shape to another.
- On $\mathcal{S}$, the set of ellipses forms a surface:



- The geometric heat equation:

is a vector field on $\mathcal{S}$

$$\boxed{\frac{\partial C_t}{\partial t} = \kappa_{C_t} \vec{n}_{C_t},}$$

Advantages of L^2 metrics

- Can define gradient flows of a function (example below).
- Have a beautiful theory of locally unique geodesics, thus a warping of one shape to another (examples below).
- Can define the Riemannian curvature tensor. If non-positive, have a good theory of *means.*
- Can expect a theory of diffusion, of Brownian motion, hence Gaussian-type measures and their mixtures.

WHERE DO THEY COME FROM?

1. *Local, boundary based*:e.g. $\|a \cdot \vec{n}\|_R^2 = \int_{\partial R} |a(s)|^2 .ds$ (DM, Michor)

2. *Global, extending match to interior:* use $\mathcal{G}_n$ = gp. of diffeomorphisms of $\mathbf{R}^n$ and $\mathcal{S}_n \approx \mathcal{G}_n$ / subgp fixing unit ball, take quotient of metric on $\mathcal{G}_n$ (Miller, Younes, Trouvé).

3. *Conformal (n=2 only):* use $\mathcal{S}_2$ ~ diffeos of S^1 (DM, Sharon)

Ex: using L^2 metric, the geometric heat eqn is the gradient of curve length!

- $\boxed{C \mapsto \ell(C), \text{ the length of } C}$ is a function on $\mathcal{S}$
- To form a gradient, we need an inner prod:

$$\boxed{\langle \nabla f, v \rangle = D_v(f),\ D_v = \text{directional.deriv., } \forall \text{vectors } v}$$

- Use the simplest inner product of 2 vectors:

$$\boxed{\langle a(s), b(s) \rangle = \int_C a(s).b(s)\,ds}$$

- What makes it work:

$$\boxed{D_{a\vec{n}}(\ell) = \frac{\partial}{\partial t}\ell(C_{ta})\bigg|_{t=0} = \int_C \kappa_C(s)a(s)\,ds = \langle \kappa_C \vec{n}, a\vec{n} \rangle}$$

Geodesics come from differential equations

Start with the variational principle:

$$\delta\left(\text{path length } = \int_0^1 \left\| \frac{dx}{dt} \right\| dt \right) = 0$$

On a manifold with coordinates $x^1,\ldots,x^n$, get:

$$\frac{d^2 x^i}{dt^2}(t) = \sum_{j,k} \Gamma^i_{jk}(x) \cdot x^j(t) \cdot x^k(t)$$

Analogs on the infinite dim'l space of shapes

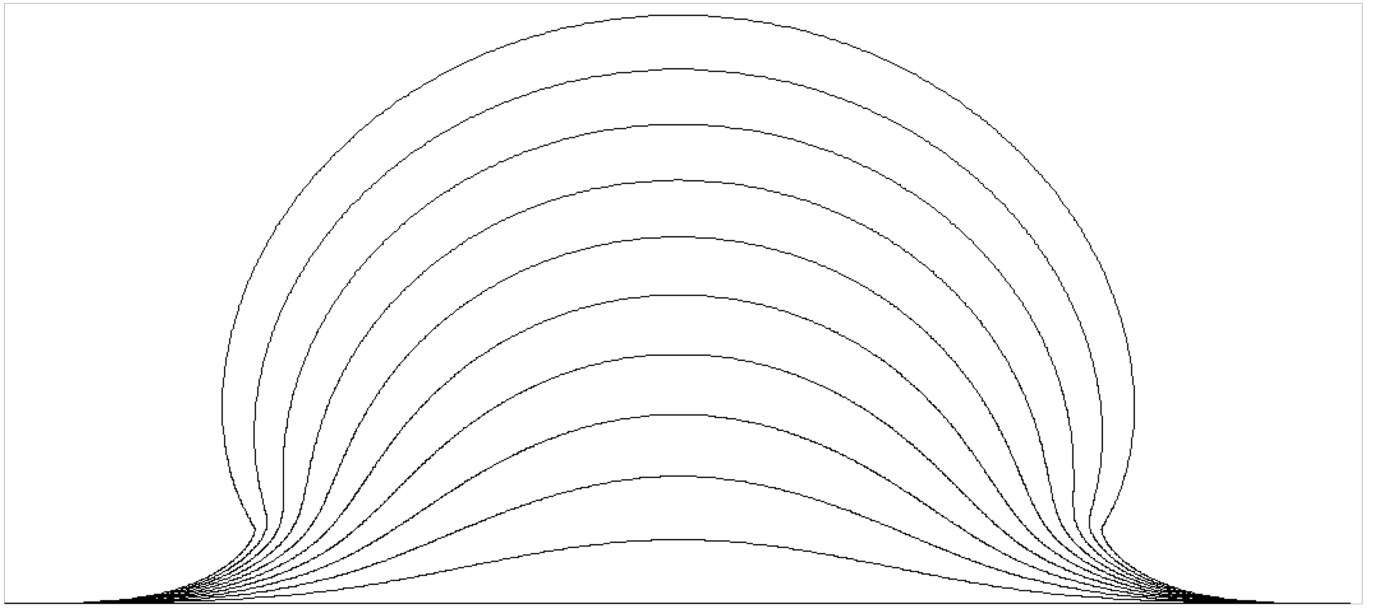
$$a(s,t) \;\leftrightarrow\; \frac{dx^i}{dt}, \text{ velocity}$$

$$\frac{\partial a}{\partial t} = \text{Quadratic expr in } a \;\leftrightarrow\; \text{above geod.eqn}$$

$$= -\frac{1}{2}\kappa_{C_t}(s).a^2, \text{ for the simplest } L^2 \text{ metric}$$

A geodesic in the simple L^2 metric

(Like Burger's eqn, develops singularities)

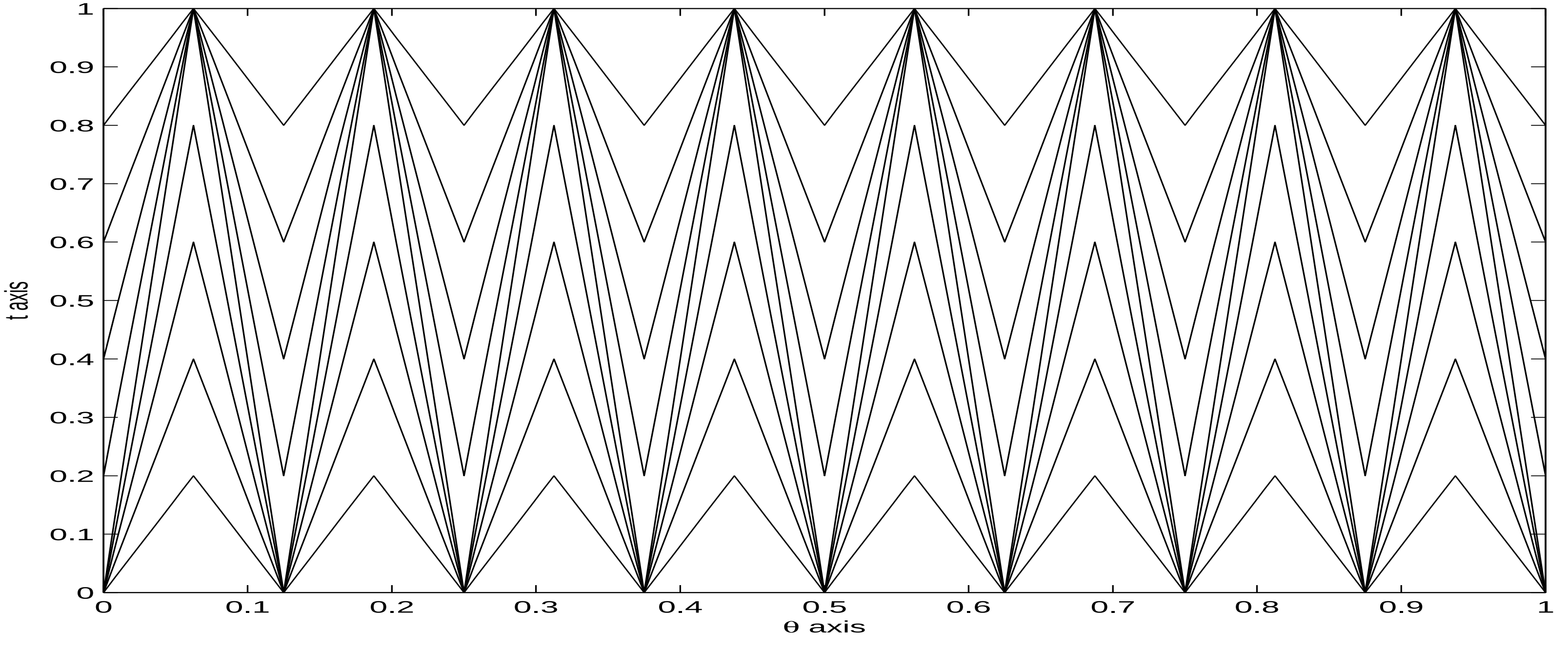


$$\frac{\partial a_t}{\partial t} = -\kappa_{C_t} . a_t^2 \big/ 2, \ C_0 = \text{straight line}, \ a_0 = \text{blip. Why hyperbolic?}$$

$a_t \sim \partial f / \partial t, \ \kappa \sim \partial^2 f / \partial s^2$ Incredibly nice formula for

sectional curvature: $$R(a,b,a,b) = \int_C \left(a.\frac{db}{ds} - b.\frac{da}{ds} \right)^2 ds \geq 0$$

But distances collapse in this metric: positive sectional curvatures – you can “cut corners” by adding higher frequencies – hyperbolic geodesic equation



The line on the bottom is moved to the line on the top by growing “teeth” upwards and then shrinking them again.

Dichotomy: pos curved, global geod bad, geod eqn hyperbolic vs.
neg. curved, global geod good, geod eqn elliptic

Conformally equiv. metrics are the simplest good Riemannian metric
(Michor, DM, Yezzi, Menucci, Shah)

- Infinitesimally:
- Idea is to show some fcn

of length is Lipschitz and use:

$$\boxed{\|a\|_\Phi^2 = \int_C \Phi(\ell_C, \kappa_C(s)) \cdot a(s)^2 \cdot ds}$$

$$\left.\begin{array}{c}\text{area}\\ \text{swept}\\ \text{over}\end{array}\right\} = \int_0^1 \int_{C_t} |a_t|\, ds dt \le \int_0^1 dt \cdot \left(\int_{C_t} 1.ds\right)^{1/2} . \left(\int_{C_t} |a_t|^2\, ds\right)^{1/2} \le \max_t \ell(C_t) \cdot \int_0^1 dt. \left(\int_{C_t} |a_t|^2\, ds\right)^{1/2}$$

Case 1: $\Phi = \ell$, then prairie fire is a geodesic, other geodesics go crazy and path length=area swept over! (Shah); $\Phi = \Phi(\ell) \geq c.e^{a\ell}$, some geodesics are stable (Yezzi, Menucchi).

Case 2: $\Phi = 1+\kappa^2$, then numerically get good geodesics (Michor, DM)

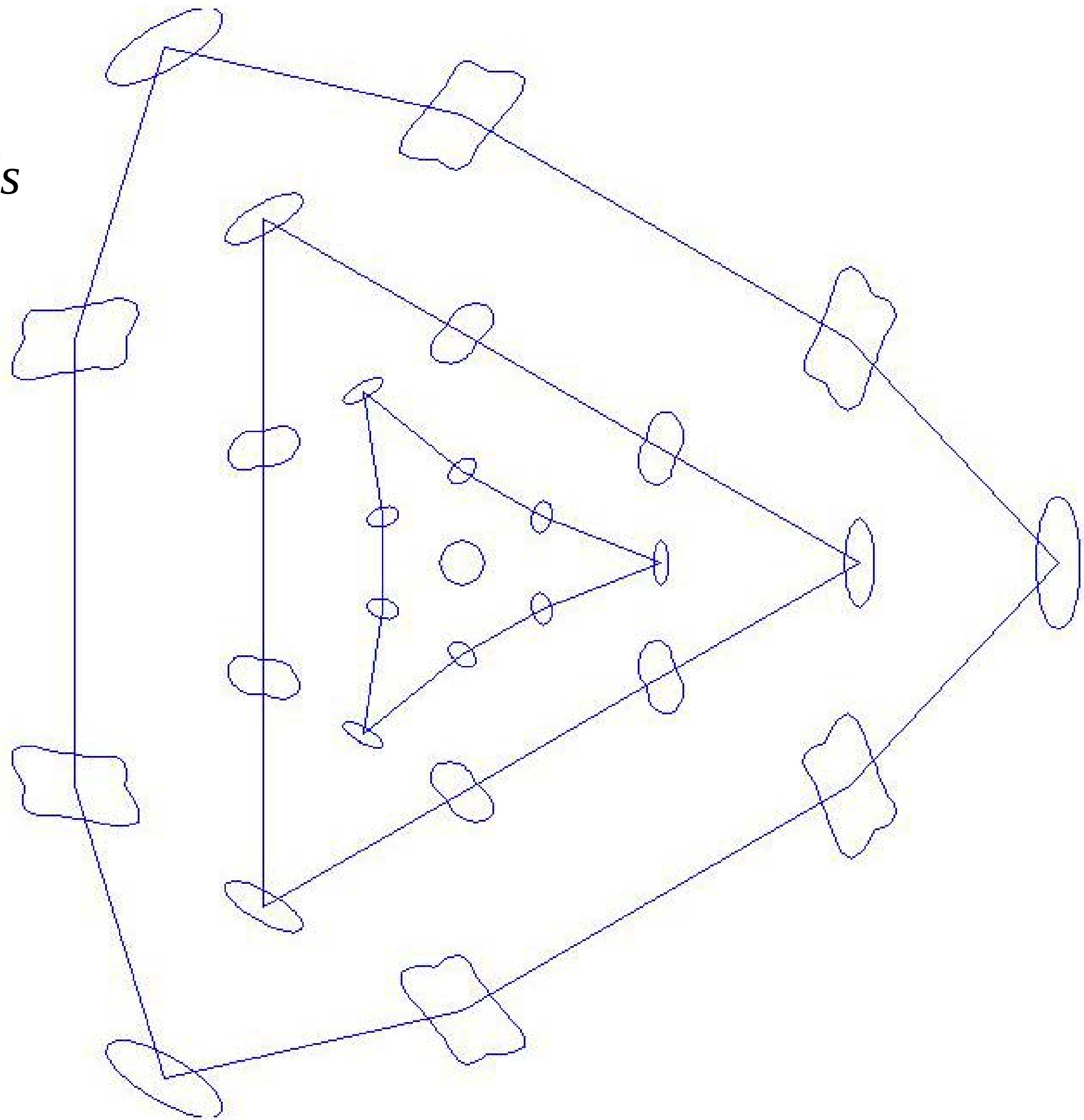
Case 3: $\Phi = \ell^{-3}.(1+(\ell\kappa)^2)$ also has lower bnd on path length and is scale invariant.

An easy fix:

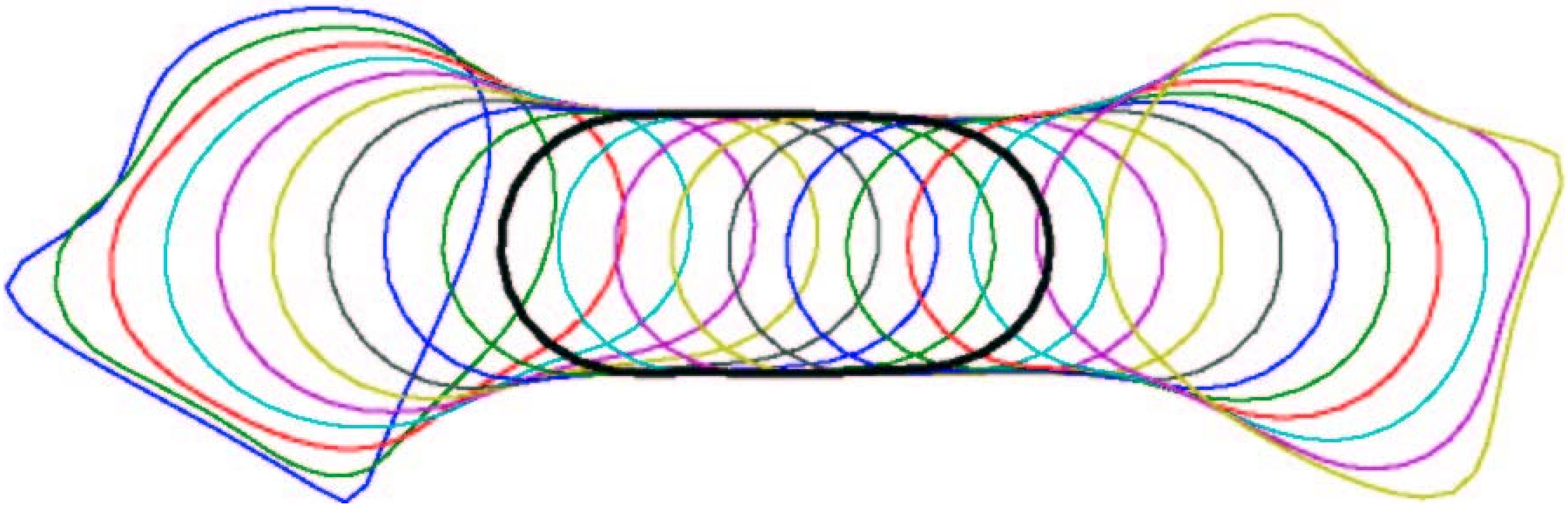
$$\|a\|^2 = \int (1 + A\kappa^2)|a|^2\, ds$$

For small shapes, curvature is negative and the path nearly goes back to the circle (= the ‘origin’). Angle sum = 102 degrees.

For large shapes, curvature is positive, 2 protrusions grow while 2 shrink. Angle sum = 207 degrees.



<u>Same metric</u>: a reflection of its negative curvature for small shapes: to get from any shape to any other *which is far* away, go via ‘cigars’ (in neg. curved space, to get from one city to another, everyone takes the same highway)



Requiring derivatives to match gives more stable metrics

We generalize Sobolev spaces W_n^2 of functions with n L^2 –derivs. to the space S of curves:

$$\|a\|^2_{H_n,\mathrm{loc}} = \inf_{\text{v.fld. } \vec{b} \text{ along } C, \langle \vec{b},\vec{n}\rangle = a} \sum_{0\le k\le n} \int_C \left\| D_s^k \vec{b} \right\|^2 ds$$

$$\|a\|^2_{H_n,\mathrm{glob}} = \inf_{\text{v.fld. } \vec{b} \text{ on } \mathbb{R}^2, \langle \vec{b},\vec{n}\rangle = a} \sum_{0\le k\le n} \int_{\mathbb{R}^2} \left\| D_s^k \vec{b} \right\|^2 dxdy$$

The resulting geodesic eqn for the global metric comes directly from work of Arnold. It is an integro-differential fluid eqn – named '**EPDiff**' by Holm and Marsden:

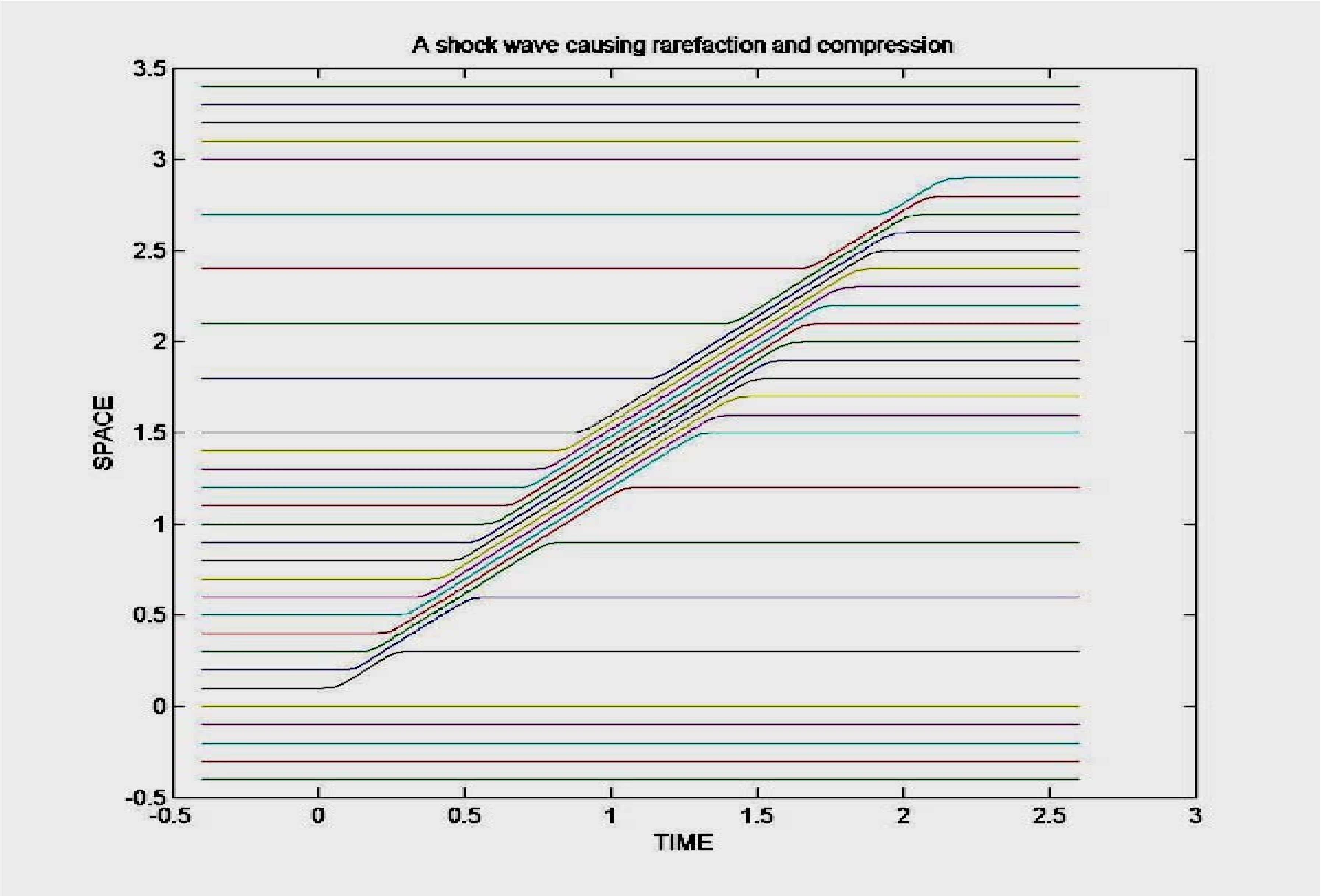
$$\vec{v}(x,y,t) = \text{velocity},$$
$$\vec{u}(x,y,t) = (I-\Delta)^n \vec{v} = \text{momentum}$$

$$\frac{\partial \vec{u}}{\partial t} = -\left((\vec{v}\bullet\nabla)\vec{u} + (\mathrm{div}\vec{v}).\vec{u} + u_x \nabla v_x + u_y \nabla v_y \right)$$

In our case:

$$\vec{u}(\cdot,\cdot,t) = c(s,t).\vec{n}_{C_t}(s).\delta_{C_t}, \vec{v} = G_n * \vec{u}$$

Simple L^2 metric on Diff(R^n) also leads to inf path length = 0



Some geodesics in H^2-metric on $Diff(\boldsymbol{R}^2)$ (Mario Micheli)

- OPEN QUICK-TIME HERE BECAUSE MICROSOFT IS STILL FIGHTING APPLE

Shape via complex analysis

- In dimension 2 *only*, can replace the real coordinates x,y by a single complex coordinate $z=x+iy$. A basic construction from complex analysis puts nearly unique global coordinates on any shape:

$$\boxed{\begin{array}{l}\forall R \subset \mathbb{C},\ \exists \phi : \Delta \xrightarrow{\approx} R,\ \textit{conformal} \\ \text{and unique up to } \phi \circ A, \\ A(z) = \dfrac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \ \text{ a Mobius map of } \Delta\end{array}}$$

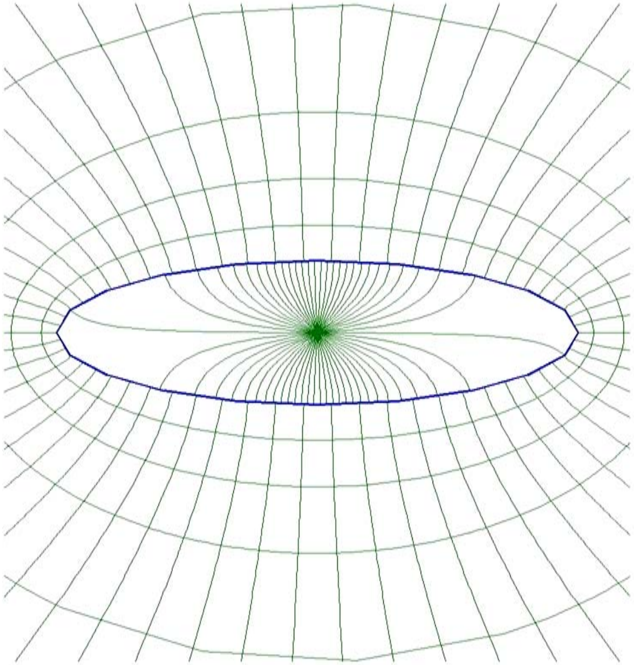
- Apply this *twice*, to the inside and outside of a shape:

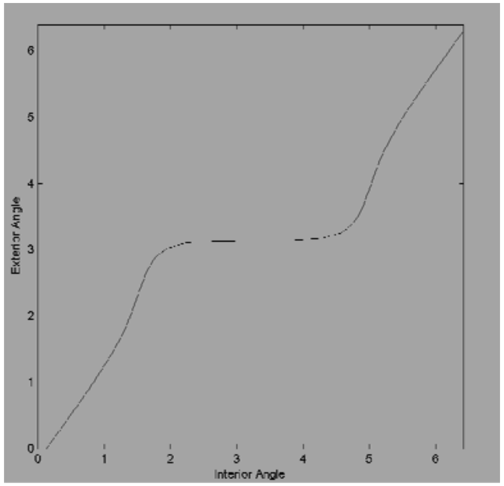
$$\boxed{\begin{array}{l}\phi_0 : \Delta \xrightarrow{\approx} R, \\ \phi_\infty : \left(\mathbb{C} - \Delta\right) \cup \{\infty\} \xrightarrow{\approx} \left(\mathbb{C} - R\right) \cup \{\infty\}, \\ \text{with } \phi_\infty(\infty) = \infty,\ \phi_\infty'(\infty) = \ \text{pos.real}\end{array}}$$

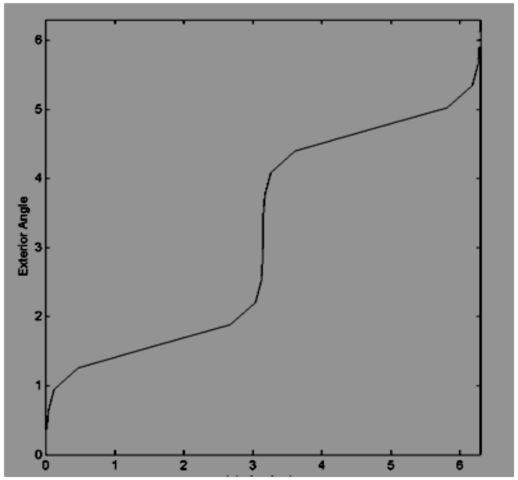
- The *fingerprint* of the shape is:

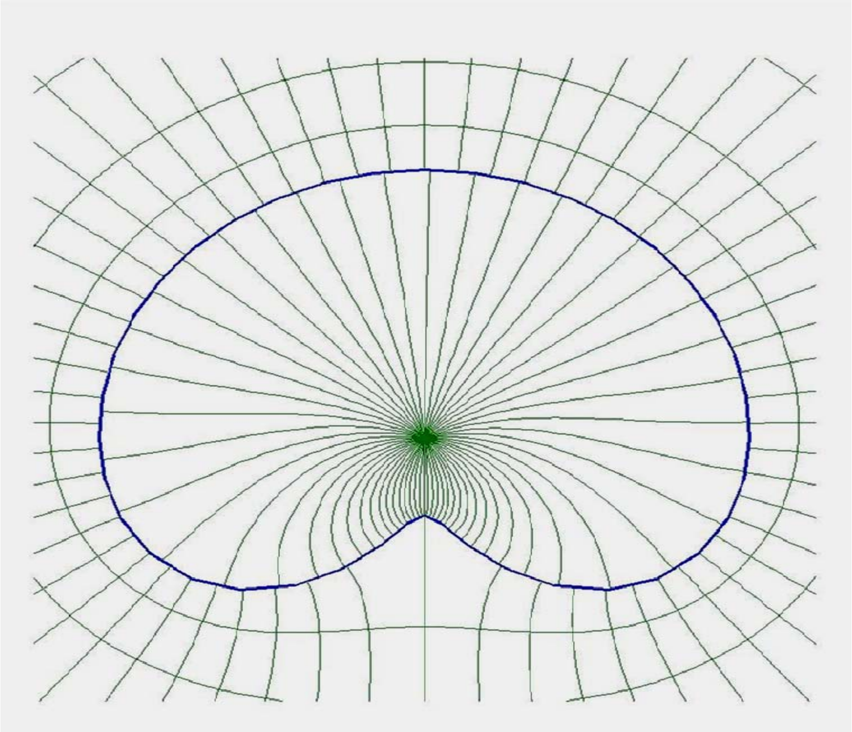
$$\boxed{\psi(z) = \phi_\infty^{(-1)}\left(\phi_0(z)\right),\ z \in \text{the circle } S^1}$$

Examples of the conformal approach









The conformal approach makes $\mathcal{S}$ almost into a group!

- The fingerprint determines the shape up to translation and scaling, i.e. there is a bijection:

$$\boxed{\mathbf{Diff}(S^1)\big/\big(\text{Mobius maps}\big) \leftrightarrow \mathcal{S}\big/\big(\text{transl.+scalings}\big) = \bar{\mathcal{S}}}$$

- We get an action of the group **Diff**(S^1) on the space of shapes, hence can approximate shapes via *words* in elementary diffeomorphisms.

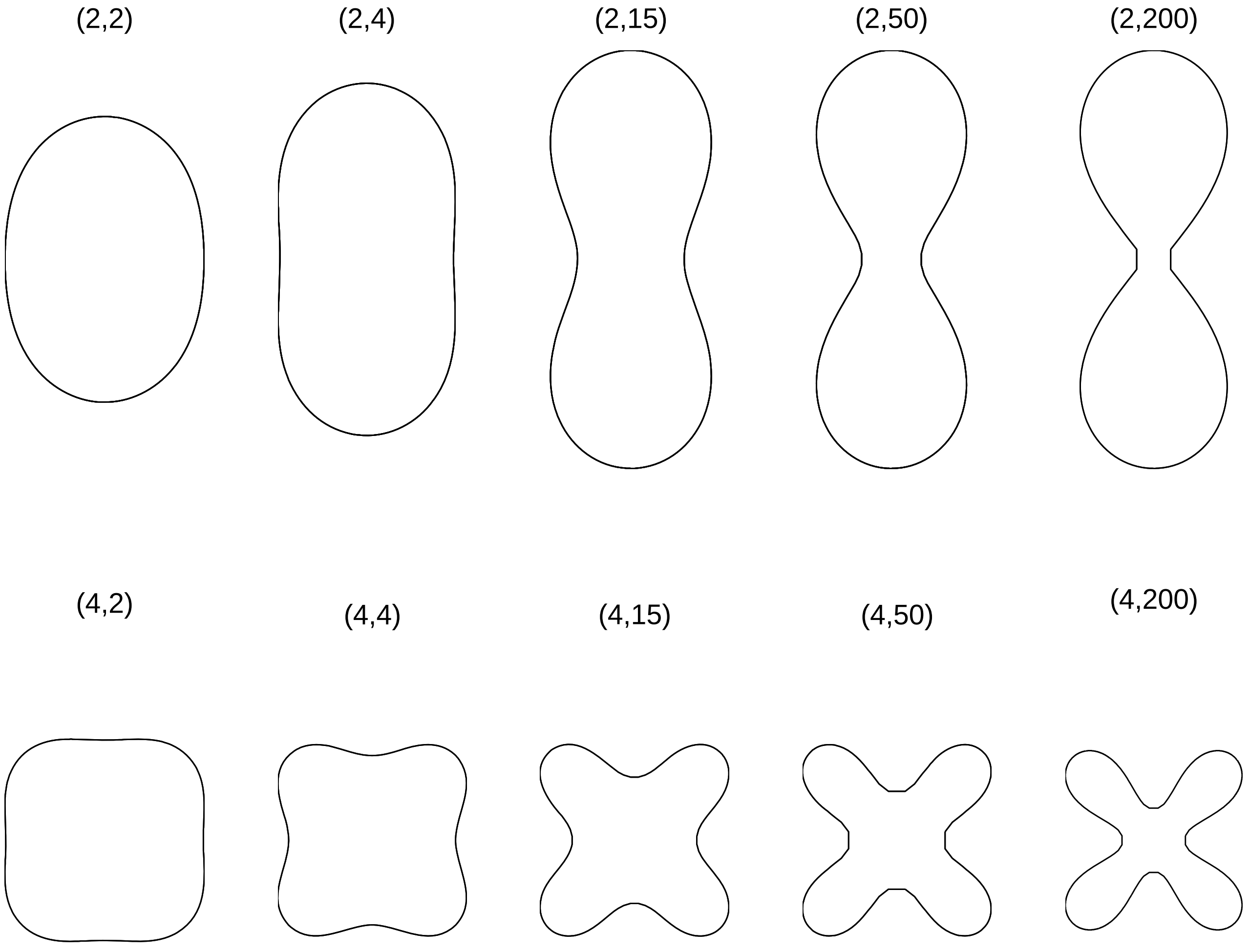
- In a group, we have 1-parameter subgroups $g(t)$ which, like geodesics, give an exponential map

- We can build up diffeos/shapes by *words* $g_1.g_2.\ldots.g_n$ (Cayley graph), using the simplest elements, e.g. Mobius maps and the ‘protrusion’ diffeos.

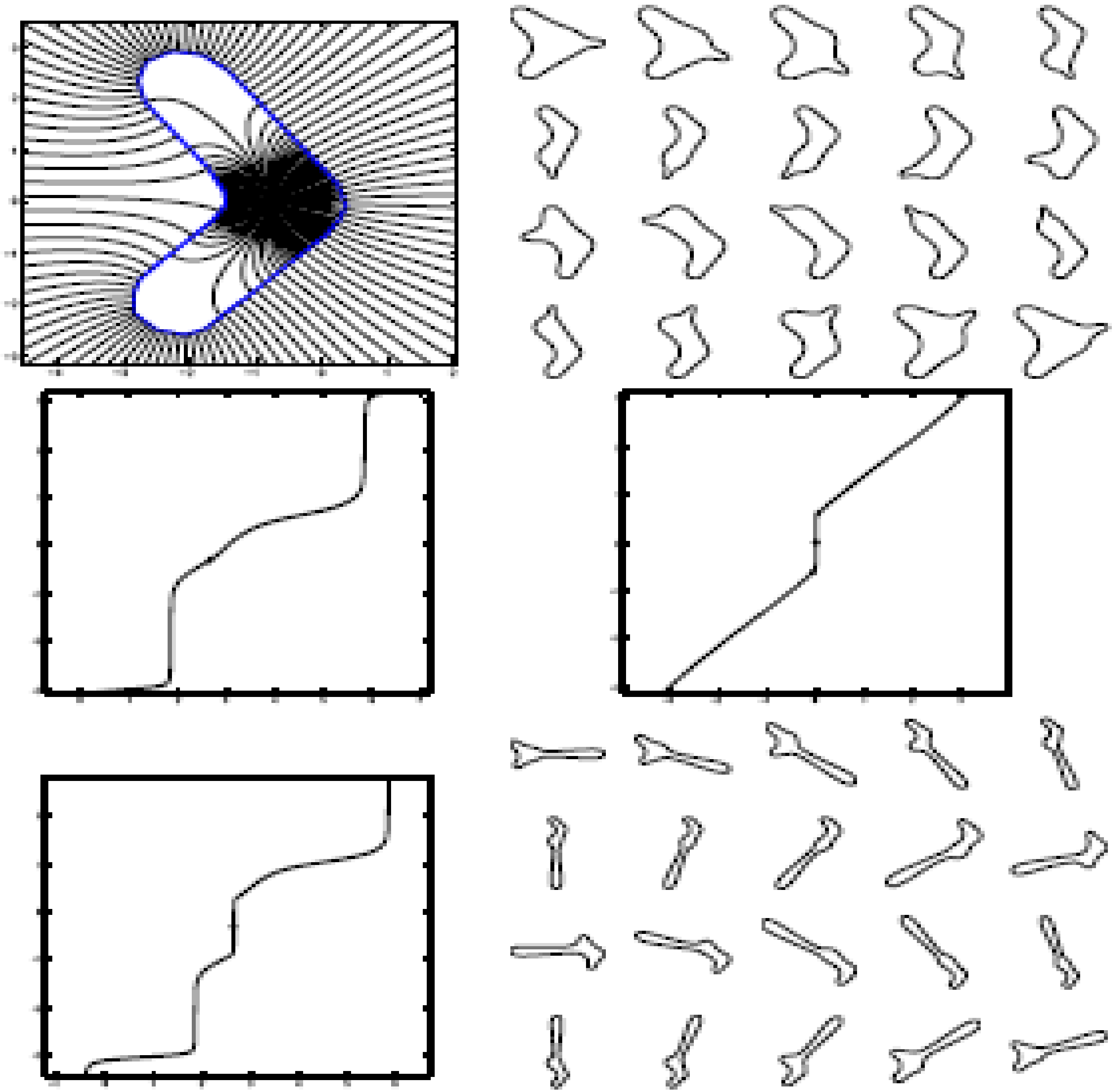
Computing the fingerprint

- Stephenson’s circle packing method
- Marshall’s ‘zipper’ algorithm
- Driscoll’s Schwarz-Christoffel method
- Clustering problem: best method may be Penner’s wavelets
- Welding is simple! Here is MatLab code:

```
n = size(phi1,1);
% interpolate to half grid points
phi1x = [(phi1(n)-2*pi); phi1; (phi1(1)+2*pi); (phi1(2)+2*pi)];
phi1mid = (-phi1x(1:n) + 9*phi1 + 9*phi1x(3:n+2) - phi1x(4:n+3))/16;
phi2x = [(phi2(n)-2*pi); phi2; (phi2(1)+2*pi); (phi2(2)+2*pi)];
phi2mid = (-phi2x(1:n) + 9*phi2 + 9*phi2x(3:n+2) - phi2x(4:n+3))/16;
% Set up the integral equation
L1 = abs(sin((phi1*ones(1,n)-ones(n,1)*phi1mid')/2));
L2 = abs(sin((phi2*ones(1,n)-ones(n,1)*phi2mid')/2));
K = log((L1(:,[n 1:n-1]).*L2) ./ (L1.*L2(:,[n 1:n-1])));
% Solve it!
f = (eye(n)+i*K/(2*pi))\(exp(i*phi1));
```



$$\psi(\theta) = \frac{2}{n}\,\mathrm{atan}\left(c \tan\left(\frac{n}{2}\theta \right) \right)$$



A miraculous metric appears

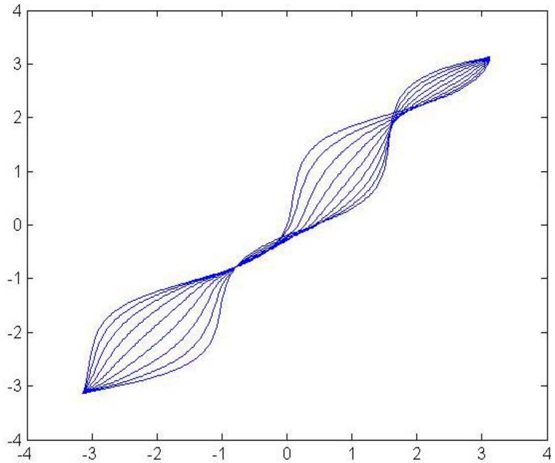
- One can define a norm on vector fields on S^1 which is invariant under conjugation by the Mobius subgroup:

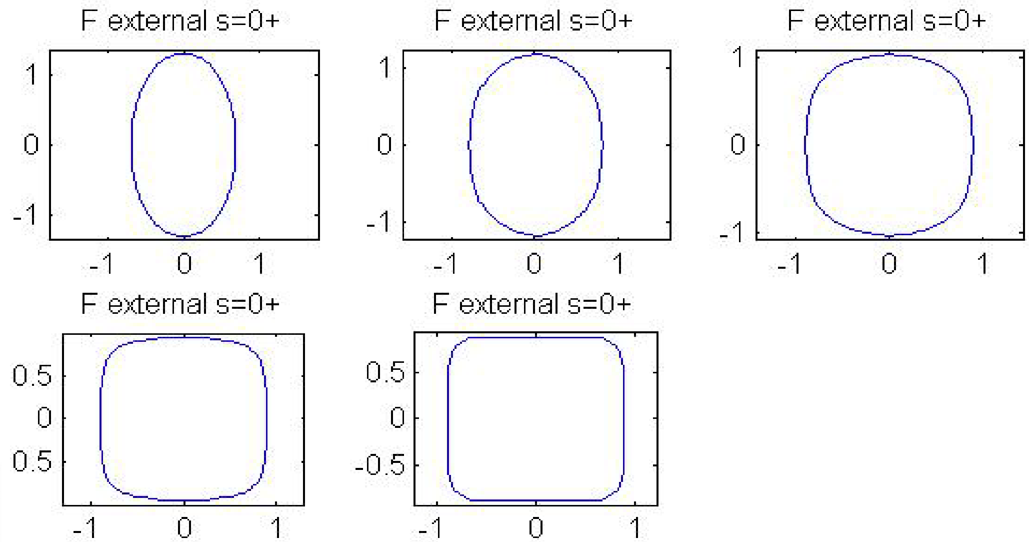
$$\boxed{\left\| \sum_{n\in\mathbb{Z}} a_n e^{in\theta} \cdot \frac{\partial}{\partial\theta} \right\|^2 = \sum_{n\geq 0} \left(n^3 - n\right) \left|a_n\right|^2}$$

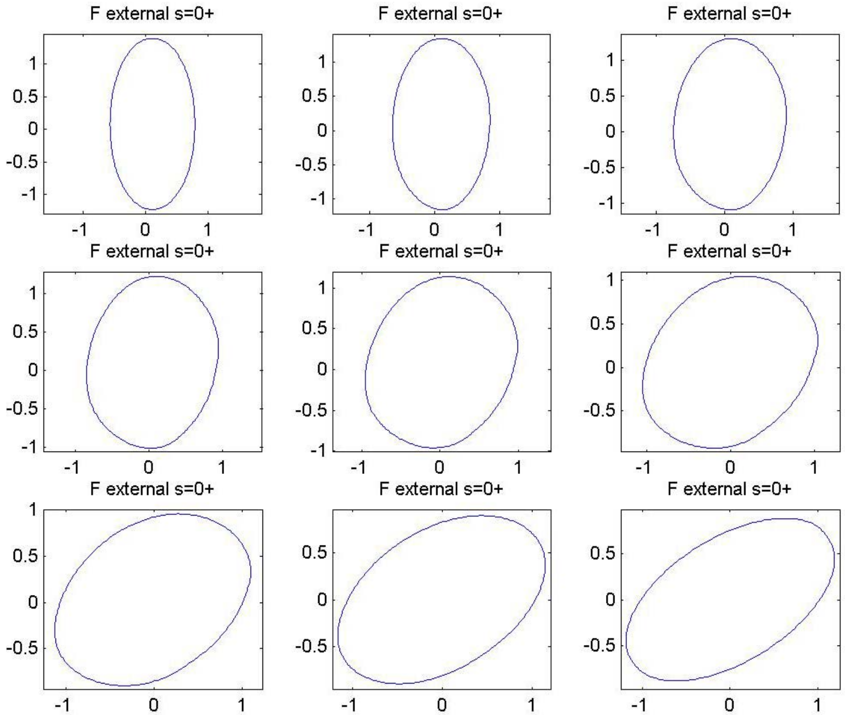
- This gives a Riemannian metric on $\mathcal{S}_2$ for which the group action is made of isometries. $\mathcal{S}_2$ is then a *homogeneous space*. (Note analogy with ordinary distances on $\mathbb{R}^n$.)

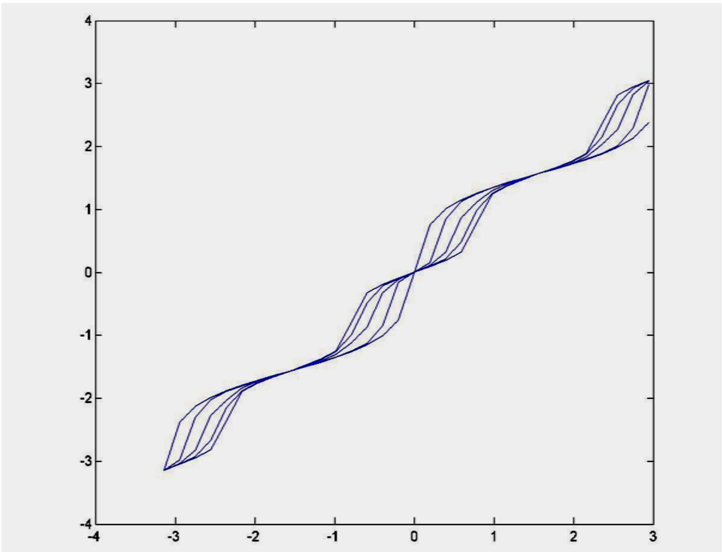
- The curvature of this metric is non-positive, so we have unique geodesics, means, etc.

Sectional curvatures are *negative*: geodesics unique

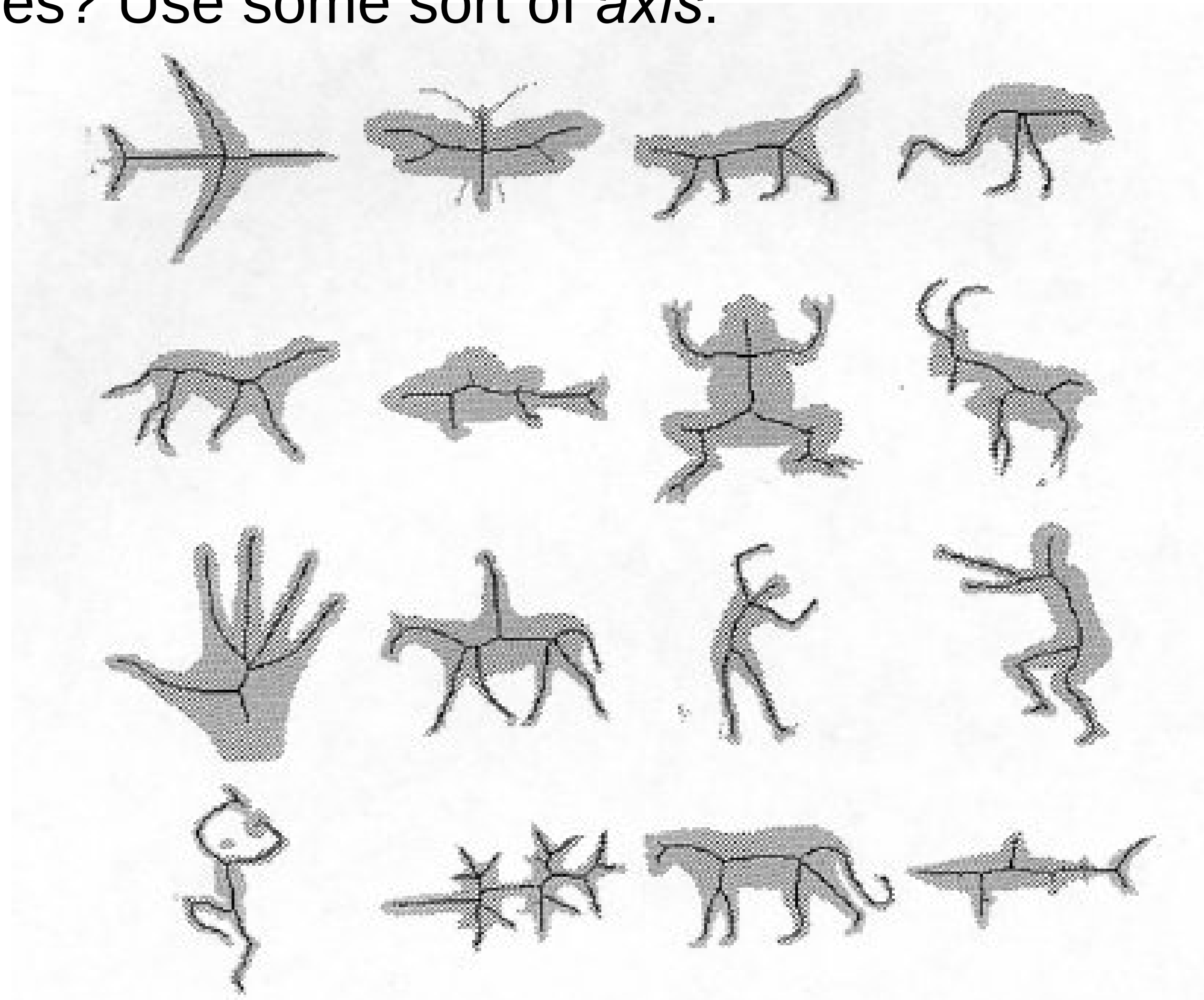








A second issue: perceptually, shapes form categories. Is there is a natural cell decomposition of the space of shapes? Use some sort of *axis*:

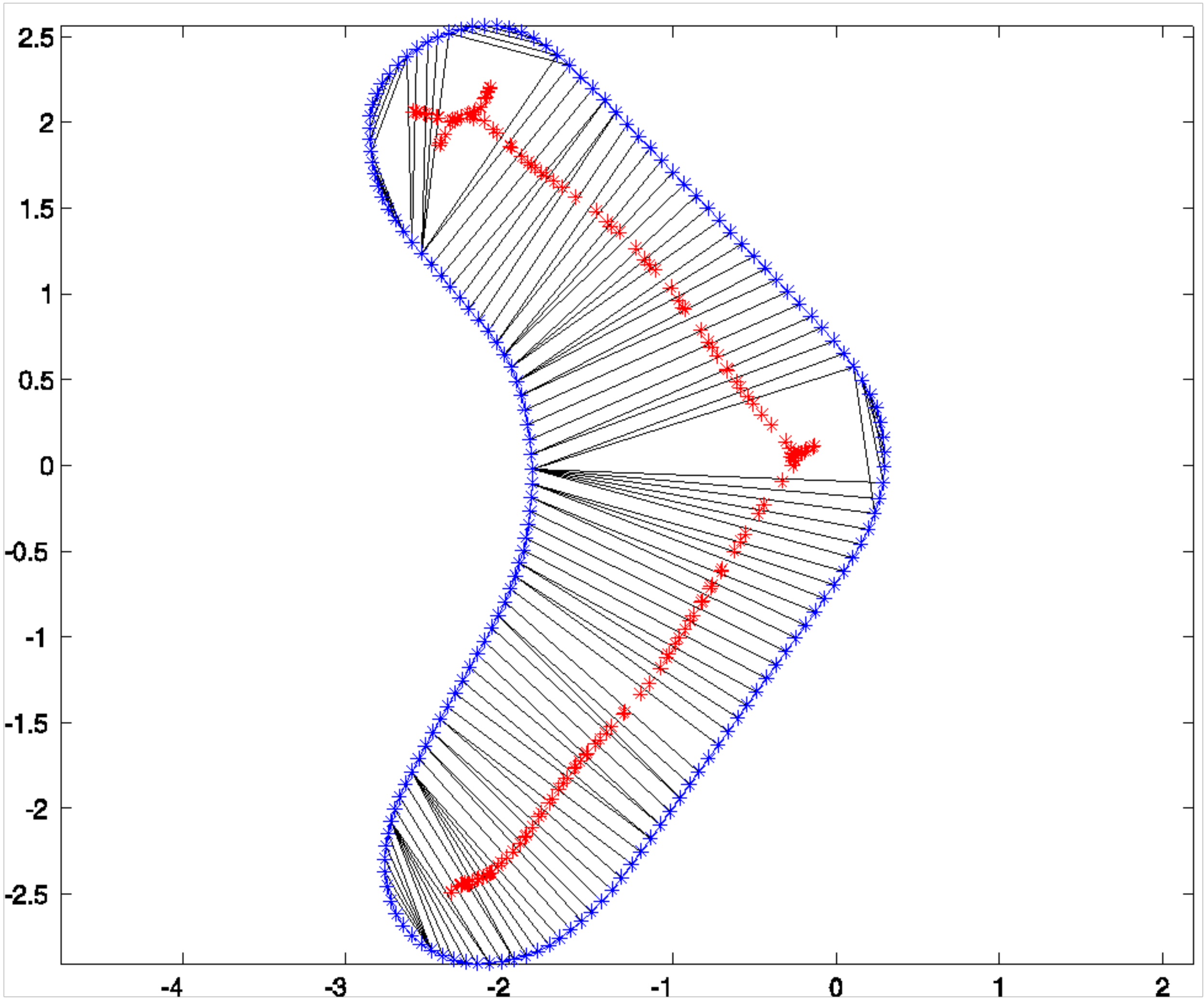


Minima of ψ' correspond (roughly) to points on C nearest to $\phi_0(0)$.

$$M_A = \arg\min\left(\left(\psi \circ A\right)'\right)$$

$$\text{complex axis}(C) \overset{def}{=} \left\{\phi_0(A(0)) \middle| \# M_A > 1\right\}$$

Combinatorial structure of the axis leads to a natural *cell decomposition* of $\mathcal{S}_2$.



A third, wide open issue

- The party line in statistical pattern recognition is to use Bayes's theorem
- Suppose we want to distinguish 2 categories of shapes, given by data forming point clouds in $\mathcal{S}$
- Want 2 probability measures and their 'ratio': but measures in these inf.diml. spaces are usually mutually singular.
- Must define stochastic shape models carefully! Not clear how to pass to the limit from e.g. Zhu's polygonal animal models on next slide.

Zhu’s ‘animals’: an exponential model was trained on curvature and medial axis statistics. Below are random samples (hatching from the medial axis):

Outlook

- Fun new area
- Tons of new mathematical problems (has anyone really thought about geometry of nonlinear infinite dimensional manifolds?)
- Maybe even benefits to medicine!