Geodesic Delaunay Triangulation and Witness Complex in the Plane

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AG Geometry Processing

Outline

Introduction

Definitions

Geodesic Delaunay Triangulation

- Homotopy Feature Size
- ϵhfs-sample
- Computing The Triangulation

Witness Complexes

- Definitions
- Sandwich Property

Discussion

Outline

Goal

Given a planar Lipschitz domain, X, extract its homological properties

Remark

- Method would be applicable on a sampled subset $W \subset X$
- This presentation is based on the paper "**Geodesic Delaunay Triangulation and Witness Complex in the Plane**" [3]

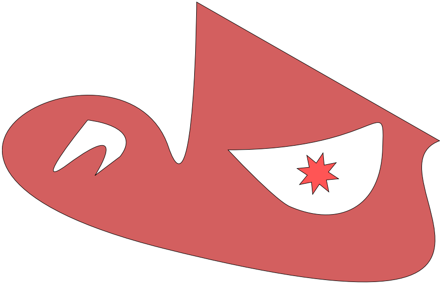
The Means

- Obtain a set of landmarks L out of X or out of $W \subseteq X$
- Two possible cases:
  - *X is known*: Construct a Delaunay triangulation of L, denoted by $\mathcal{D}_X(L)$
  - *X is unknown*: Construct the witness complex and the relaxed witness complex
- Extract the homological properties of X

Outline

Definition (Lipschitz Domain)

A Lipschitz domain in the plane is a compact embedded topological 2-submanifold of $\mathbb{R}^2$ with Lipschitz boundary



Definition (Intrinsic Metric)

For every $x, y \in X$ we define:

$$\mathrm{d}_X(x,y) = \inf\{|\gamma|, \quad \gamma : I \to X, \quad \gamma(0) = x \quad \& \quad \gamma(1) = y\}$$

Remark

- $\mathrm{d}_X(x,y) = +\infty$ if x and y belong to different connected components

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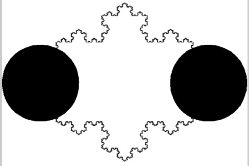
Remark

- $d_X(x,y) = +\infty$ if x and y belong to different connected components
- The converse is not always true!
- $\Rightarrow$ In general the topologies induced by d_X and by d_E are not the same



Theorem

*If X is a Lipschitz domain in the plane, then the **intrinsic topology** coincides with the **Euclidean topology** on X.*

Definition (Geodesic Voronoi Diagram)

Consider:

- The *domain* $X \subset \mathbb{R}^2$
- $L \subset X$ a subset of *Landmarks*

Then the cover:

$$\mathcal{V}_X(L) = \{V_p | \quad p \in L\}$$

where:

$$V_p = \{x \in X | \quad \mathrm{d}_X(x,p) \leq \mathrm{d}_X(x,q) \quad \forall q \in L\}$$

is called the geodesic Voronoi diagram of L in X

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The Voronoi Edges

Edges of the geodesic Voronoi diagram can be of non-zero measure!

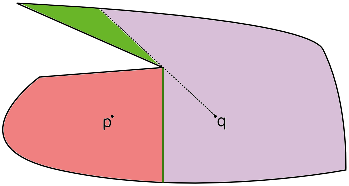
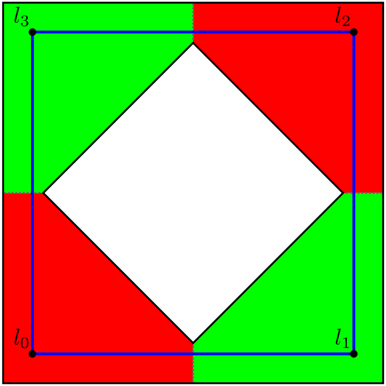


Figure: The green area is the Voronoi edge

Definition (Geodesic Delaunay Triangulation)

Given a domain X, a subset of landmarks $L \subset X$ and the Voronoi diagram $\mathcal{V}_X(L)$, then the **nerve** of $\mathcal{V}_X(L)$ is called the geodesic Delaunay Triangulation, $\mathcal{D}_X(L)$



Outline

Obtaining a sampling

Problem

How to sample the set of landmarks, L, out of the domain X?

- High number of sampling points ⇒ Space costly
- Low number of sampling points ⇒ Not accurate

Goal

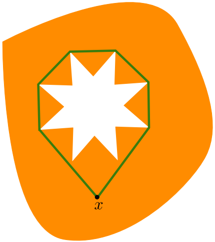
Obtain an accurate *AND* small sampling of *X*

Homotopy Feature Size

Definition (Homotopy Feature Size [3])

Let X be a Lipschitz planar domain, then $\forall p \in X$:

$$\mathrm{hfs}(p) = \frac{1}{2} \inf\{|\gamma|, \quad \gamma : (S^1, 1) \to (X, p) \text{ non null-homotopic}\}$$



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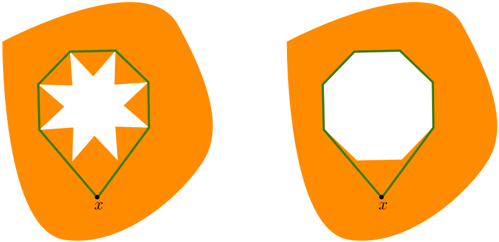
Other types of feature size

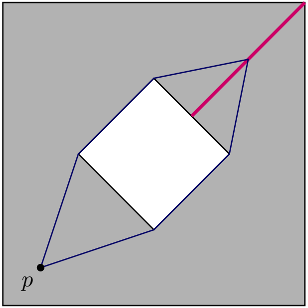
- **Local feature size** [1]
- **Weak feature size** [2]



Advantage of hfs

In contrast to other feature sizes hfs is insensitive to the local geometry

The homotopy feature size is closely related to the concept of **cut-locus**



Definition (Cut-Locus)

The cut-locus of a point $p \in X$, $\mathrm{CL}_X(p)$, is the locus of all points $y \in X$ such that there exist two distinct shortest pathes in X connecting p and y

Theorem

For a Lipschitz domain X and a point $p \in X$, there exists:

$$\mathrm{hfs}(p) = \mathrm{d}_X(p, \mathrm{CL}_X(p))$$

Computing hfs(p) - Cont.

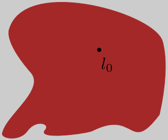
The last theorem give rise to an easy method for evaluating hfs(p)

1. Initialize a geodesic ball of radius 0 centered at p, $B_X(p)$
2. Increase the radius in constant speed
3. As long as the ball's boundary has no self intersections continue to expand it
4. Termination occurs in one of the following cases:
   - Self intersection was detected $\Rightarrow$ Return the radius
   - The ball covers the connected component containing p $\Rightarrow$ Return ∞

Definition (ϵhfs-sample)

L is a ϵhfs-sample of X if every point $p \in X$ is at finite distance to L and

$$\mathrm{d}_X(p, L) \leq \epsilon\mathrm{hfs}(p)$$



Remark

If L is an ϵhfs-sample of X then in every connected component of X there exists a vertex of L

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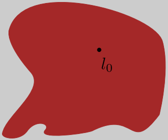

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Theorem

If X is a Lipschitz domain, and L a geodesic ϵhfs-sample of X for some $\epsilon < \frac{1}{3}$, then $\mathcal{D}_X(L)$ and X have the same homotopy type

Generating ϵhfs-Sample

1. Set $L = \emptyset$
2. Add $p \in X$ to L and record $B_p = B(p, \frac{\epsilon}{1+\epsilon}\mathrm{hfs}(p))$
   If $\mathrm{hfs}(p) = +\infty$ then B_p coincides with X_p
3. While $X \setminus \bigcup_{p \in L} B_p \neq \emptyset$:
   Pick a new point p' which is not covered by any geodesic ball

Remarks

- The Algorithm terminates for all $\epsilon > 0$
- Pick $\epsilon < \frac{1}{3}$
- L is an ϵhfs-sample of X

Compute $\mathcal{D}_X(L)$

- At this point we have the domain X and a subset of landmarks L
- Grow geodesic balls around the vertices of L at constant speed
- Report the intersections between the fronts

Result

Extract the homological properties of X out of $\mathcal{D}_X(L)$ - They have the same type!

Outline

Motivation

- Consider the case when only a sample $W \subset X$ is given
- Fronts intersections turn to be hard to detect
- Try to encode the information in a different structure rather then $\mathcal{D}_X(L)$

Notations

- W is called the set of *witnesses*
- L is called the set of *landmarks*

Definition (Witness)

Consider $X \subset \mathbb{R}^2$ and two subsets: $W, L \subset X$. A point $w \in W$ is a witness of a simplex $\sigma = [p_0, \ldots, p_l]$ with vertices in L if:

$$+\infty > \mathrm{d}_X(w, p_i) \leq \mathrm{d}_X(w, q) \quad \forall i \in \{0, \ldots, l\} \quad \forall q \in L \setminus \{p_0, \ldots, p_l\}$$

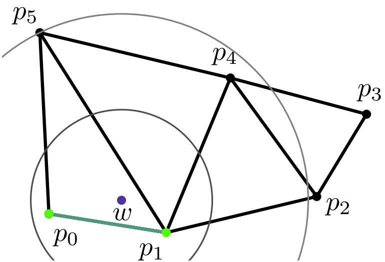


Figure: $L = \{p_i\}$ and w witnesses $[p_0, p_1]$

For the sake of completeness:

Definition (Strong Witness)

A point $w \in W$ is a strong witness of σ if it is a witness of σ, and in addition $\mathrm{d}_X(w, p_i)$ is constant

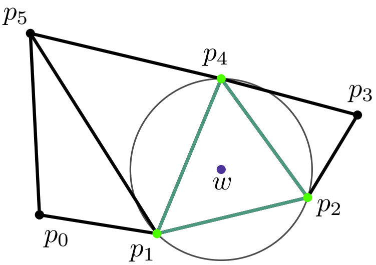
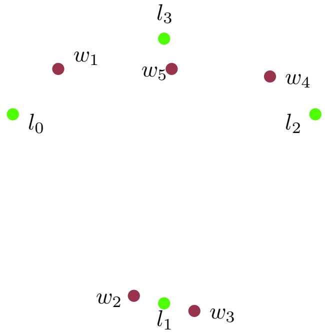


Figure: w witnesses $[p_1, p_2, p_4]$

Definition (Geodesic Witness Complex)

The geodesic witness complex of L relative to W, $\mathcal{C}_X^W(L)$, is the maximal abstract simplicial complex with vertices in L, whose faces are witnessed by points of W



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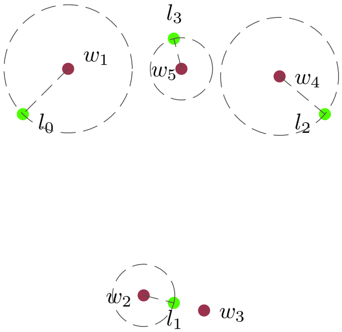

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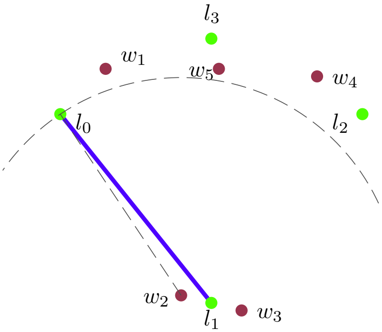

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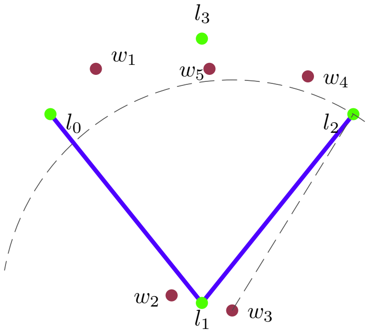

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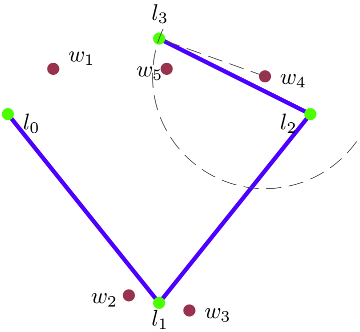

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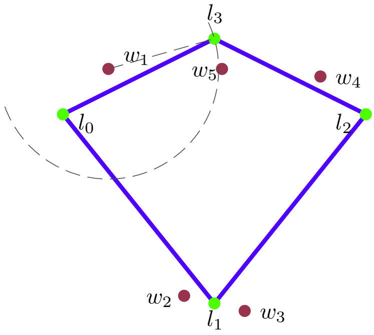

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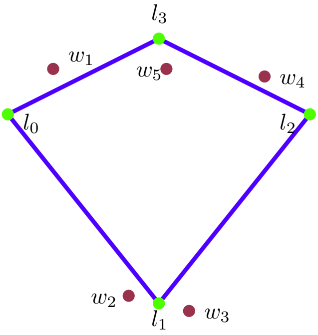


Definition (ν-Witness)

Given an integer $\nu \geq 0$, then a simplex σ with vertices in L is ν-witnessed by $w \in W$ if the vertices of σ belong to the $\nu + 1$ landmarks closest to w in the intrinsic metric

Definition (ν-Witness Complex)

The geodesic ν-witness complex of L relative to W, $\mathcal{C}^{W}_{X,\nu}(L)$, is the maximal abstract simplicial complex with vertices in L, whose faces are ν-witnessed by points of W

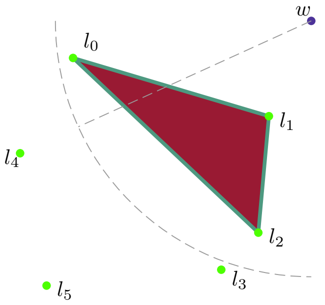


Figure: Simplicial complex 2-witnessed by w

The Sandwich Property – First Part

Theorem ($\mathcal{C}_X^W(L) \subset \mathcal{D}_X(L)$)

Let X be a Lipschitz domain in the plane, and L a geodesic ϵhfs-sample of X. If $\epsilon \leq \frac{1}{4^{k+1}}$, for some integer $k \geq 0$, then the k-skeleton of $\mathcal{C}_X^W(L)$ is included in $\mathcal{D}_X(L)$ for all $W \subset X$.

More Definitions

Definition (μhfs-Sparse)

Let L be a geodesic ϵhfs-sample of X, then, it is μhfs-sparse, if $\forall p, q \in L$:

$$\mathrm{d}_X(p, q) \geq \mu \min\{\mathrm{hfs}(p), \mathrm{hfs}(q)\}$$

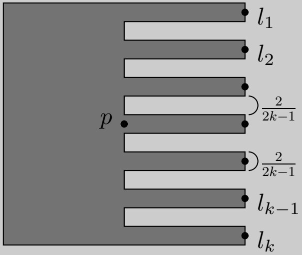
Definition (Doubling Dimension)

The doubling dimension of (X, d_X), is the smallest integer d such that every geodesic closed ball can be covered by a union of 2^d geodesic closed balls of half its radius

- The doubling dimension measures the shape complexity of X

Example

- 1×2 rectangle with k branches
- $B_X(p, 2)$ coveres the whole domain
- $B_X(l_i, 1)$'s are disjoint
- At least k balls of radius 1 are needed to cover $B_X(p, 2)$
- $\Rightarrow \quad d \geq \log_2(k)$
- The doubling dimension can be arbitrarily large!



Theorem ($\mathcal{D}_X(L) \subset \mathcal{C}^W_{X,\nu}(L)$)

Let X be a Lipschitz domain in the plane, of doubling dimension d. Let W be a geodesic δhfs-sample of X, and L a geodesic ϵhfs-sample of X that is also $\frac{\epsilon}{1+\epsilon}$hfs-sparse. If $\epsilon + 2\delta < 1$, then, for any integer $\nu \geq 2^{ld} - 1$, where $l = \lceil \log_2 \frac{2(1+\delta/\epsilon)(1+\epsilon)}{1-\epsilon-2\delta} \rceil$, $\mathcal{D}_X(L)$ is included in $\mathcal{C}^W_{X,\nu}(L)$.

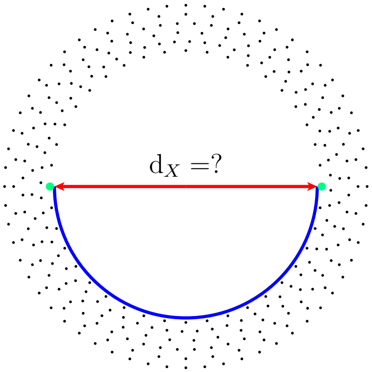
- Under the above conditions we have:

$$\mathcal{C}_X^W(L) \subseteq \mathcal{D}_X(L) \subseteq \mathcal{C}_{X,\nu}^W(L)$$

- Persistent homology computation will yield the properties of $\mathcal{D}_X(L)$
- This, in turn, has the same type as the one of X

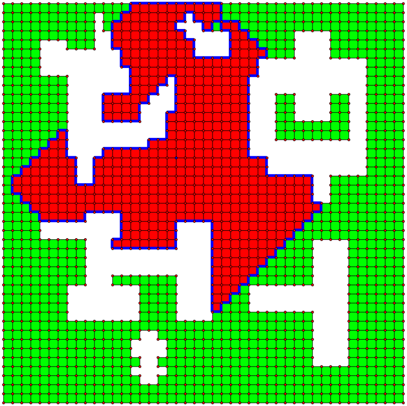
Outline

- If only a subset $W \subset X$ is known then evaluating geodesic distances is not trivial
- Properties of W should be given to enable distance measuring



Computing hfs(x)

- The algorithm bases on two oracles:
  - Detecting when $B_X(x)$ covers X_x
  - Detecting when $\partial B_X(x)$ has self intersection for the first time
- X is known, algorithm can be used
- X is unknown
  - Growing the geodesic balls is not trivial
  - Problematic when trying to obtain the set of landmarks L



$\mathcal{C}_X^W(L)$ and $\mathcal{C}_{X,\nu}^W(L)$

- Obtaining L using the algorithm is not possible, it should be given
- Constructing the complexes requires distances evaluation
- The conditions for the sandwich property
  - Depends on the *doubling dimension*
  - Nevertheless, small ν seems to be enough

References

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Thank you!

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