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Loops and the Fundamental Group

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What is Seminar, 21/11/2008

Outline

Motivation

Loops

- Relations Between Loops
- Loops Concatenation

Group Structure on Loops

Examples

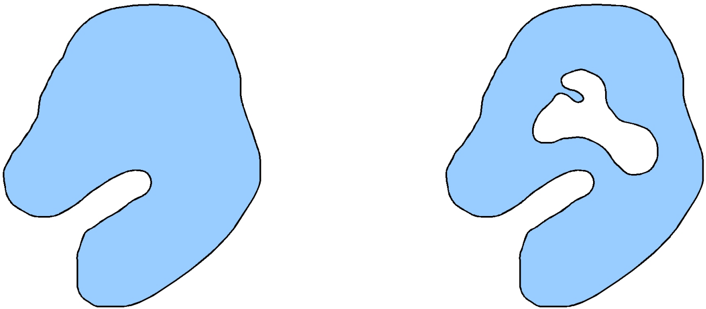
- Subset of plane
- The Circle
- The Torus

Study the Topology of Spaces

- We want to study topological spaces

Study the Topology of Spaces

- We want to study topological spaces
- What is the difference between the following two spaces?



Let’s have some loops

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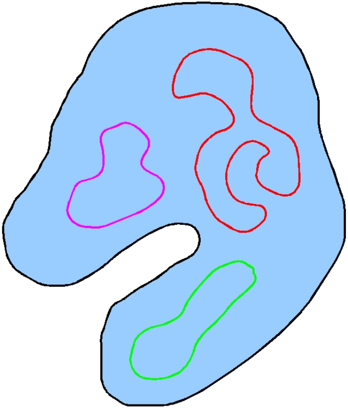

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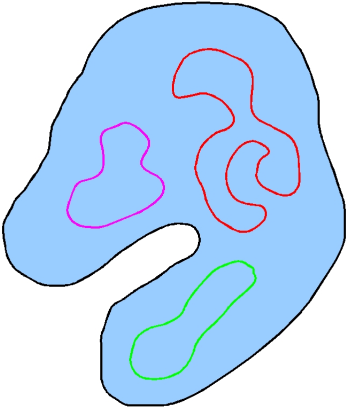
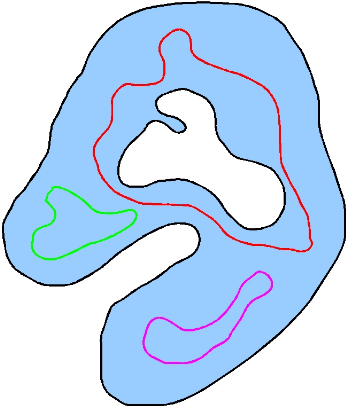


Each loop can be continuously contracted into a point.

And what about loops here?

Next, lets consider the second example:



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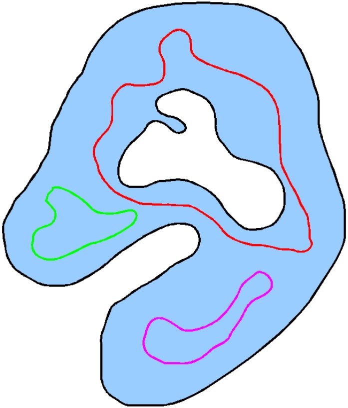

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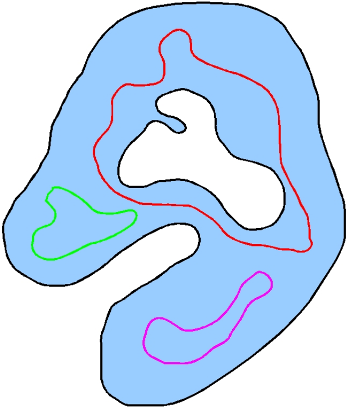


Here, the *red* loop cannot be contracted to a point.
Let's introduce some definitions!

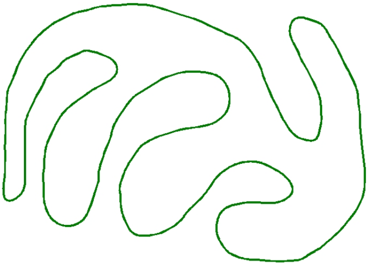
Loop

Definition (Loop)

Given a topological space X and the unit interval $I \subset \mathbb{R}$, a loop is a continuous map

$$\lambda : I \to X$$

such that $\lambda(0) = \lambda(1)$.



Some Conventions

- Later on, we will consider topological spaces with a base point (X, x_0), where X is the topological space, and x_0 is the base point.

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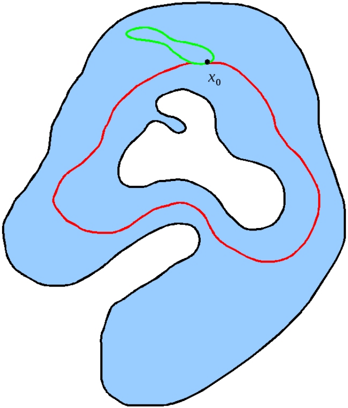
- For any loop λ we have $\lambda(0) = \lambda(1) = x_0$

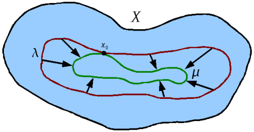
Outline

How To Distinguish Between Loops?

Consider the following picture



Homotopy



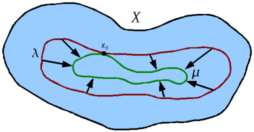
Definition (Homotopy of loops)

Two loops $\lambda, \mu : I \to (X, x_0)$ are called *homotopic* with base point held fixed, or for short:

$$\lambda \simeq \mu \quad \mathrm{rel}(0,1)$$

if there exists a continuous map $F : I \times I \to (X, x_0)$ such that the following holds:

1. $F(s,0) = \lambda(s), \quad \forall s \in I$
2. $F(s,1) = \mu(s), \quad \forall s \in I$
3. $F(0,t) = F(1,t) = x_0, \quad \forall t \in I$



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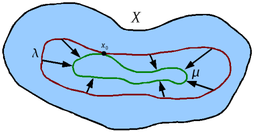

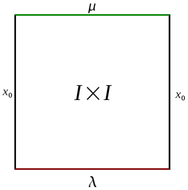

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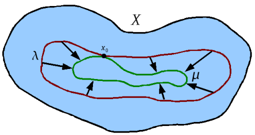
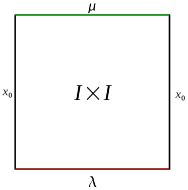
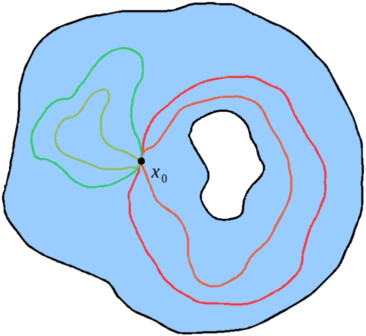
The map F is called the *homotopy* between λ and μ.





Example of Homotopy

Let’s have a look at the following figure:



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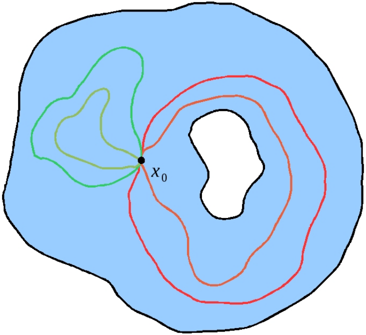

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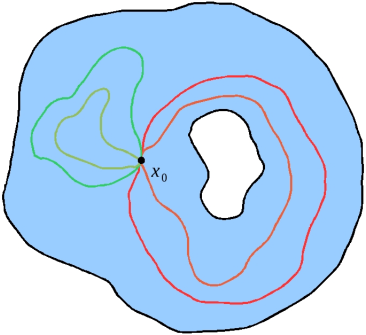


- The red loops are homotopic, so are the green loops.

- **Alert!**

  No red loop is homotopic to a green one.

Outline

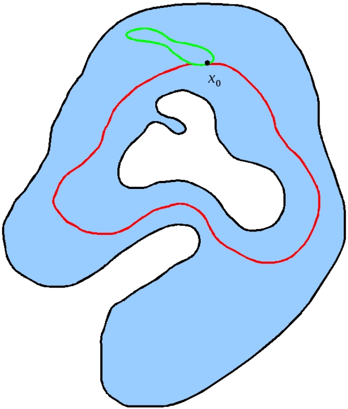
Concatenating Loops

Definition

Given two loops $\lambda, \mu : I \to X$ with a base point x_0, we define the concatenation of them as follows:

$$\lambda * \mu(t) = \begin{cases} \lambda(2t) & 0 \leq t \leq \frac{1}{2} \\ \mu(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Example



We first traverse along the red loop and then along the green one.

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Equivalence Classes

So we can consider the set of equivalence classes of loops $[\lambda]$ over a topological space X with base point x_0.

The Group Operation

In order to have a group structure on the set of equivalence classes, we have to define a group operation.

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Definition (Group Operation)

Given two loop classes $[\lambda]$ and $[\mu]$ we define:

1. $[\lambda] * [\mu] := [\lambda * \mu]$
2. The inverse of $[\lambda]$ is given by $[\lambda^{-1}]$ that is $[\lambda]^{-1} = [\lambda^{-1}]$, where $\lambda^{-1}(t) = \lambda(1-t)$.

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Definition

Given a topological space X with a base point x_0, the *fundamental group*, $\pi_1(X, x_0)$, is the set of equivalence classes of loops along with the operation $*$

Two More Definitions

Before we go on, for the sake of completeness!

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The constant loop in a pointed topological space (X, x_0) is the the map:

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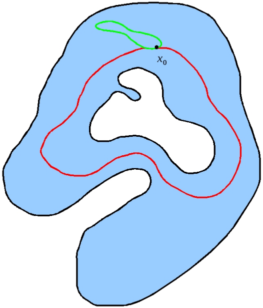
A loop λ is called *null-homotopic* if it is homotopic to the constant loop.

The Group's Unit Element

Note that the class of null-homotopic loops is the unit element of the fundamental group.

Example

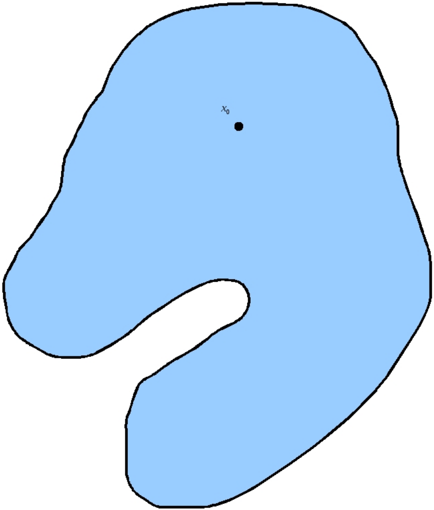
- Here, concatenating the green loop to the red one does NOT change the homotopy type of the red loop
- In other words the green loop's class is the identity of the fundamental group



Outline

Set With No Holes

Consider the following pointed space



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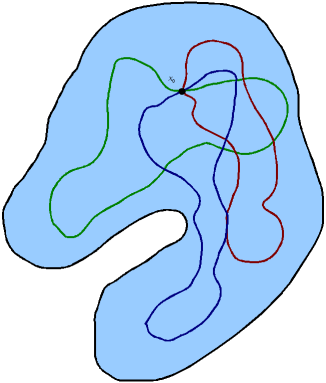

- Here all loops are null-homotopic, i.e. all are homotopic to the constant loop

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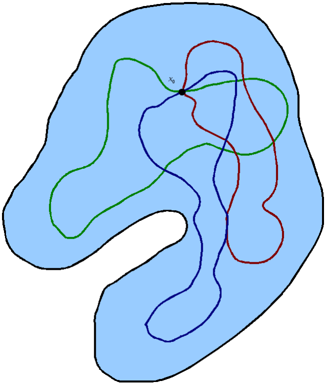
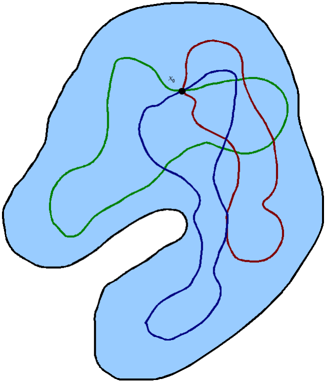


- Here all loops are null-homotopic, i.e. all are homotopic to the constant loop
- This means that the fundamental group is trivial

Contractible Space - A side Remark

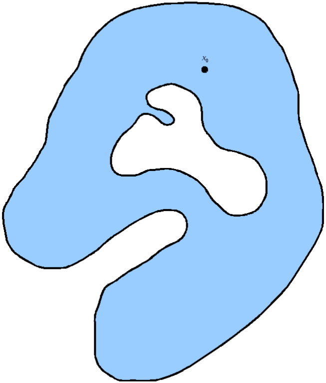
- If all loops are null-homotopic then the space is called *contractible*
- A contractible space is one which is *homotopy equivalent* to a one-point space



Definition (Homotopy Equivalent)

Two topological spaces X and Y are called homotopically equivalent, or of the same homotopy type, if there exists two maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_X$ and $g \circ f \simeq id_Y$.

Now Consider the following space



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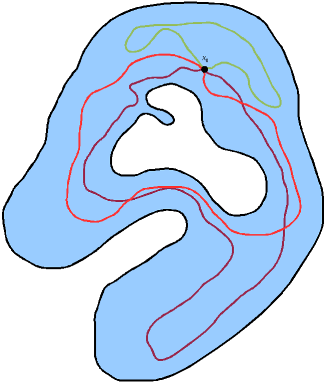

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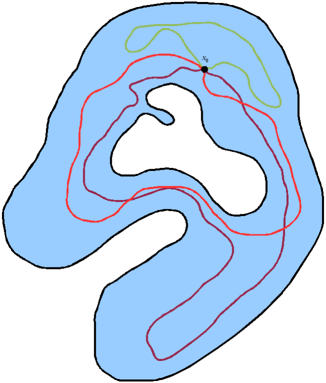

Here we have two types of loops:

- Those homotopically equivalent to the constant loop
- Those which enclose the hole

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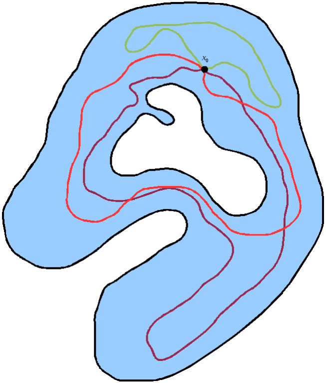

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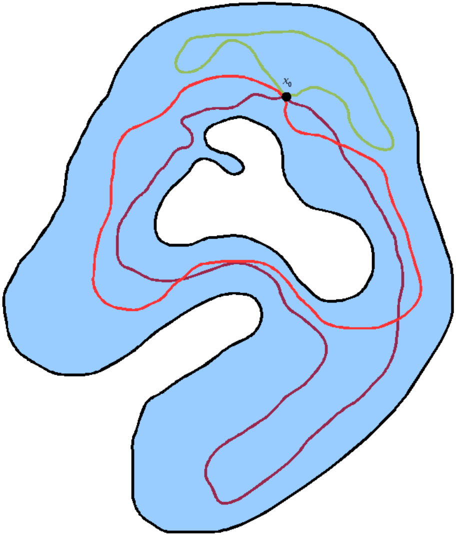
The Fundamental Group

What is the fundamental group in this case?

Outline

The Circle



The Fundamental Group of the Circle

Theorem

We have for any point $x_0 \in S^1$

$$\pi_1(S^1, x_0) = \mathbb{Z}$$

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Proof Outline.

- Up to rotations, all loops in S^1 are characterized by the number of times they wind around the origin
- A negative integer i is isomorphic to a loop winding i times clockwise
- Concatenation of loops in $\pi_1(S^1, x_0)$ is equivalent to addition of integers in $\mathbb{Z}$

□

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The Fundamental Group of The Torus

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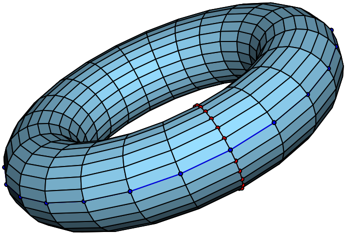

For Further Reading I

Marvin J. Greenberg
Lectures on Algebraic Topology.

Allen Hatcher
Algebraic Topology.

Thank you!

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