homework_1

May 3, 2017

1 Social and Economic Networks | problem set 1

1.1 Problems from Barabasi (Ch 2)

1.1.1 Matrix formalism

Let A be the NxN adjacency matrix of an undirected unweighted network, without self-loops. Let $\mathbb{1}$ be a column vector of N elements, all equal to 1. In other words $\mathbb{1} = (1, 1, ..., 1)^T$, where the superscript T indicates the transpose operation.

- We can obtain the vector k whose elements are the degrees k_i of all nodes $i = 1, 2, ..., N$ by multiplying the matrix A by $\mathbb{1}$ as follows: $k = A\mathbb{1}$.
- The total number of links in the network can be obtained by pre-multiplying the column vector k by the transpose of the column vector $\mathbb{1}$ and dividing it by 2 (since each link will be counted twice otherwise): $L = \frac{1}{2}\mathbb{1}k$.
- The number of triangles T can be obtain by performing A^3, which gives us a matrix in which each cell a_{ij} gives us the number of 3-edge paths from i to j, and taking the $trace(A^3)$, since all the diagonal elements of A^3 give us the number of 3-edges paths from i to i, in other words triangles.
- The vector k_nn whose element i is the sum of the degrees of node i's neighbors can be obtained by multiplying the adjacency matrix A by itself and then by $\mathbb{1}$. So $k_nn = A^2\mathbb{1}$.
- The vector k_nn whose element i is the sum of the degrees of node i's neighbors can be obtained by multiplying the adjacency matrix A by itself twice and then by $\mathbb{1}$. So $k_nn = A^3\mathbb{1}$.

1.1.2 Bipartite networks

We consider the bipartite network in image 2.21.

First, we build its adjacency matrix. It is block-diagonal because it is a square matrix composed of two blocks aligned diagonally (which are symmetric).

```
In [1]: from IPython.display import HTML, display

        data = [[0,1,2,3,4,5,6,7,8,9,10,11],[1,0,0,0,0,0,0,1,0,0,0,0],[2,0,0,0,0,0,
               [4,0,0,0,0,0,0,0,0,1,1,0],[5,0,0,0,0,0,0,0,0,0,1,0,1],[6,0,0,0,0,0,0,0,0
               [8,0,0,1,0,0,0,0,0,0,0,0],[9,0,1,1,1,1,0,0,0,0,0,0],[10,0,0,0,1,0,0,

        display(HTML(
            '<table><tr>{}</tr></table>'.format(
```

```
                '</tr><tr>'.join(
                    '<td>{}</td>'.format('</td><td>'.join(str(_) for _ in row)) for
                )
        ))
```

```
<IPython.core.display.HTML object>
```

We now construct the adjacency matrix of its two projections:

```
In [2]: data = [[0,1,2,3,4,5,6],[1,0,0,1,0,0,0],[2,0,0,1,1,1,0],[3,1,1,0,1,1,0],[4,

        display(HTML(
            '<table><tr>{}</tr></table>'.format(
                '</tr><tr>'.join(
                    '<td>{}</td>'.format('</td><td>'.join(str(_) for _ in row)) for
                )
        ))

        data = [[0,7,8,9,10,11],[7,0,1,1,0,0],[8,1,0,1,0,0],[9,1,1,0,1,1],[10,0,0,1

        display(HTML(
            '<table><tr>{}</tr></table>'.format(
                '</tr><tr>'.join(
                    '<td>{}</td>'.format('</td><td>'.join(str(_) for _ in row)) for
                )
        ))
```

```
<IPython.core.display.HTML object>
```

```
<IPython.core.display.HTML object>
```

In the bipartite network, the average degree for the purple nodes is $\frac{10}{6}$, and for the green nodes is $\frac{10}{5}$.

In each projection, however, the average degree for purple nodes is $\frac{16}{6}$, and for green nodes is $\frac{10}{5}$. The answer is different from the first because node 9 is connected to many purple nodes, so it makes sense that purple nodes have a higher average degree.

1.1.3 Bipartite graph - general considerations

We consider a bipartite network with N_1 and N_2 nodes in the two sets.

- The maximum number of links in a bipartite network is $L_{max} = N_1.N_2$.
- Compared to a non-bipartite graph of size $N = N_1 + N_2$, where $L_{max} = N!$, the number of links that cannot occur is of $N! - N_1.N_2$
-

- We can express the average degree in each part of the bipartite network as follows:

$$< k_1 >= \frac{L}{N_1}$$

$$< k_2 >= \frac{L}{N_2}$$

1.2 Problem from Easley and Kleinberg: Problem 6, Chapter 14

a) Concept: The Hub-Authority algorithm is a network structure that models two types of nodes: directories (hubs) and "expert" sites (authorities). Each page has 2 scores: 1) the total sum of votes of authorites pointed to hub; and, 2) the total sum of votes coming from authorities.

The 2-step hub-authority is computed by the following algorithm: * Step 1: Initialize by scoring every Hub and Authority with equal weight * Step 2: Select number of K steps * Step 3: Then perform a sequence of K Hub-Authority updates * Apply Authority Update Rule: sum all of the Hub scores of pages that point to the authority

```
* Apply Hub Update Rule: sum all of the Authority scores of all its linking pages
```

- Step 4: Normalize

When we compute the 2-step Hub-Authority algorithm to the provided network in problem 6, we find each node has the following value: A1 = 6.75, A2 = 6.75, A3 = 6.75, B1 = 2.25, B2 = 2.25, B3 = 2.25, C1 = .083, C2 = .083, C3 = .083, C4 = .083, C5 = .083, D = .83

b) A k-step application of the Hub-Authority algorithm is when the sequence of authority and hub updates is applied k times. If you run the algorithm a few times on the previous network, a pattern from the raw scores emerges and can be generalized as follows:

- auth(B1) = 3^k

- auth(B2) = 3^k
- auth(B3) = 3^k
- auth(D) = 5^k
- hub(A1) = 3^k
- hub(A2) = 3^k
- hub(A3) = 3^k
- hub(C1) = 0^k
- hub(C2) = 0^k
- hub(C3) = 0^k
- hub(C4) = 0^k
- hub(C5) = 0^k

c) As k goes to infinity the normalized values at each node will converge to a stable set of values and will remain fixed at the point of convergence. Because the hub and authority scores are proportional to each other, independent of what initial values with which you begin (provided they are positive), the approximate scores will reach an equilibrium (i.e. limit) over time. The Hub

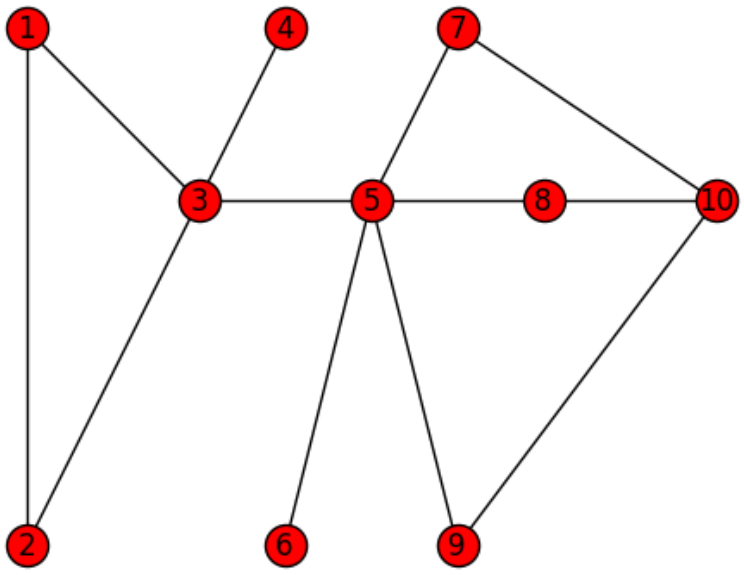
Authority ranking procedure provides an information retrieval system that takes into consideration both quality (weights based on endorsements) and quantity (weights based on the number of sites each hub/authority is pointing to and from). Thus, these scores are purely dependent on the link structure and as such sites can be powerful endorsers without themeslves being heavily endorsed. Said differently, a "good" hub would be a page that points to many "good" authorities and a "good" authority is referenced by many hubs.

1.3 Problem 1

1.3.1 a)

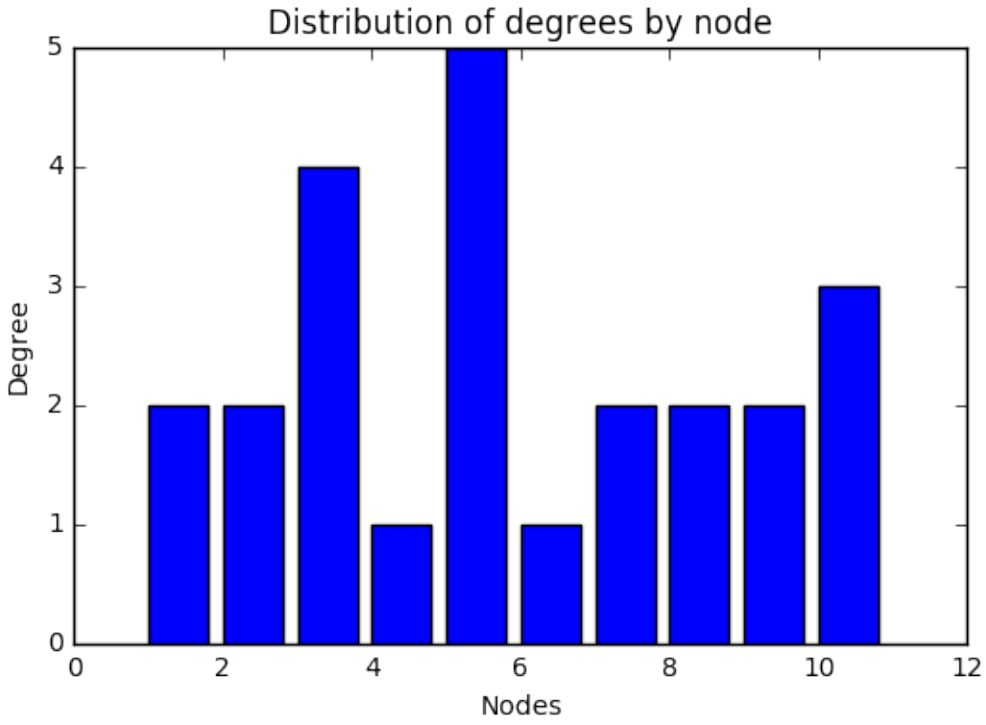
We label the nodes of the graph as follows:

```
In [4]: import matplotlib.pyplot as plt
        import networkx as nx

        G = nx.Graph()

        G.add_nodes_from(range(1,10))

        G.add_edges_from([(1,2),(1,3),(2,3),(3,4),(3,5),(5,6),(5,7),(5,8),(5,9),(8,

        pos_dict = {1:(1,4), 2:(1,1), 3:(2,3), 4:(2.5,4), 5:(3,3), 6:(2.5,1), 7:(3.

        nx.draw(G, pos = pos_dict)
        nx.draw_networkx_labels(G, pos = pos_dict)
        plt.show()
```



We then plot the degree distribution in the following bar chart:

```
In [5]: x = G.nodes()
        y = [len(G.neighbors(x)) for x in G.nodes()]

        plt.bar(x, y)
        plt.title("Distribution of degrees by node")
        plt.xlabel("Nodes")
        plt.ylabel("Degree")
        plt.show()
```



1.3.2 b)

To find the clustering coefficient CC(v) of each node, we first compute its neighbor K(v) (the degree of the node) and the number of links between its neighbors N(v). We then compute CC(v) through the following formula $\frac{2N(v)}{K(v)(K(v)-1)}$. Thus, * CC(1) = $\frac{2(1)}{2(2-1)}$ = 1 * CC(2) = $\frac{2(1)}{2(2-1)}$ = 1 * CC(3) = $\frac{2(1)}{4(4-1)}$ = 1/6 * CC(4) = $\frac{2(0)}{1(1-1)}$ = 0 * CC(5) = $\frac{2(0)}{5(5-1)}$ = 0 * CC(6) = $\frac{2(0)}{1(1-1)}$ = 0 * CC(7) = $\frac{2(0)}{2(2-1)}$ = 0 * CC(8) = $\frac{2(0)}{2(2-1)}$ = 0 * CC(9) = $\frac{2(0)}{2(2-1)}$ = 0 * CC(10) = $\frac{2(1)}{3(3-1)}$ = 1/3

The average cluster = C$\frac{1}{n}\sum_{i=1}^{n} C_i = \frac{1}{4}$

1.3.3 c)

The distance from u to v, d(u,v), is the length of the shortest path from u to v in G. The average distance can be found by finding the shortest path between all pairs of nodes, adding them and then dividing by the total number of pairs: * d(1,2)= 1, d(1,3)= 1, d(1,4)= 2, d(1,5)= 2, d(1,6)= 3, d(1,7)= 3, d(1,8)= 3, d(1,9)= 3, d(1,10)= 4 * d(2,3)= 1, d(2,4)= 2, d(2,5)= 2, d(2,6)= 3, d(2,7)= 3, d(2,8)= 3, d(2,9)= 3, d(2,10)= 4 * d(3,4)= 1, d(3,5)= 1, d(3,6)= 2, d(3,7)= 2, d(3,8)= 2, d(3,9)= 2, d(3,10)= 3 * d(4,5)= 2, d(4,6)= 3, d(4,7)= 3, d(4,8)= 3, d(4,9)= 3, d(4,10)= 4 * d(5,6)= 1, d(5,7)= 1, d(5,8)= 1, d(5,9)= 1, d(5,10)= 2 * d(6,7)= 2, d(6,8)= 2, d(6,9)= 2, d(6,10)= 3 * d(7,8)= 2, d(7,9)= 2, d(7,10)= 1 * d(8,9)= 2, d(8,10)= 1 * d(9,10)= 1

Given the pair, the total average distance = (22 + 21 + 13 + 18 + 6 + 9 + 5 + 3 + 1)/45 = 98/45 = 2.17

The diameter of a network can be found by finding the longest path of all the shortest paths. To begin, we first find the eccentricity of a vertex v $\in V(G)$, which can be defined as e(v) = max{d, u, v} | v $\in V(G)$. * e(v_1) = 4 * e(v_2) = 4 * e(v_3) = 3 * e(v_4) = 4 * e(v_5) = 2 * e(v_6) = 3 * e(v_7) = 3 * e(v_8) = 3 * e(v_9) = 3 * e(v_10) = 4 Then, the diameter dim(G) = max {e(v) | v $\in V(G)$} = 4

1.4 Problem 2

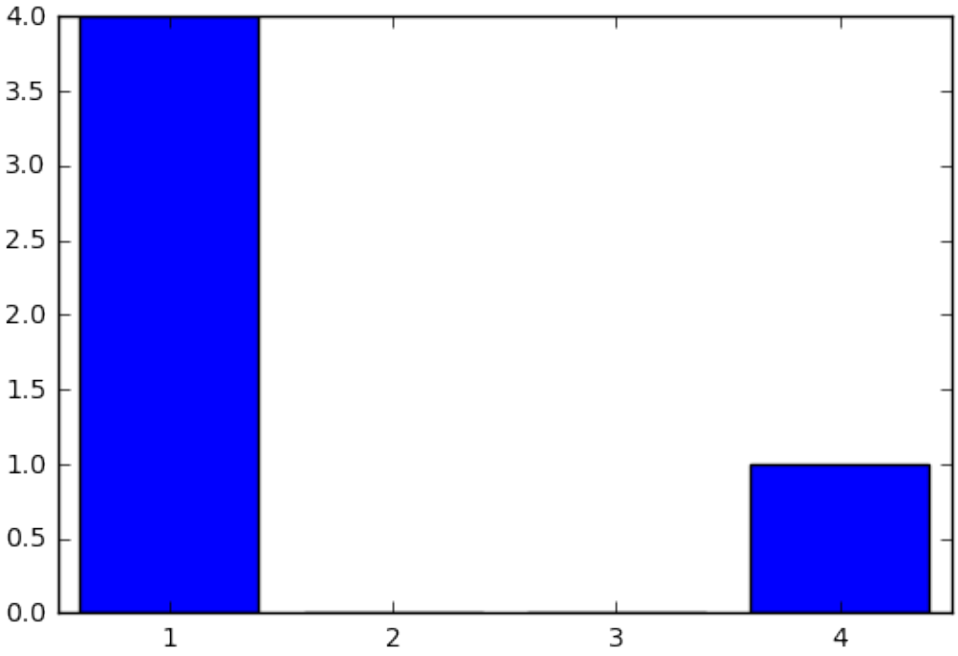
1.4.1 (a) Star networks

(a.1) Plot the degree distribution of the network

In a star network with n spokes, the distribution of degrees will be such that there will be n nodes with degree 1, and 1 node with degree n.

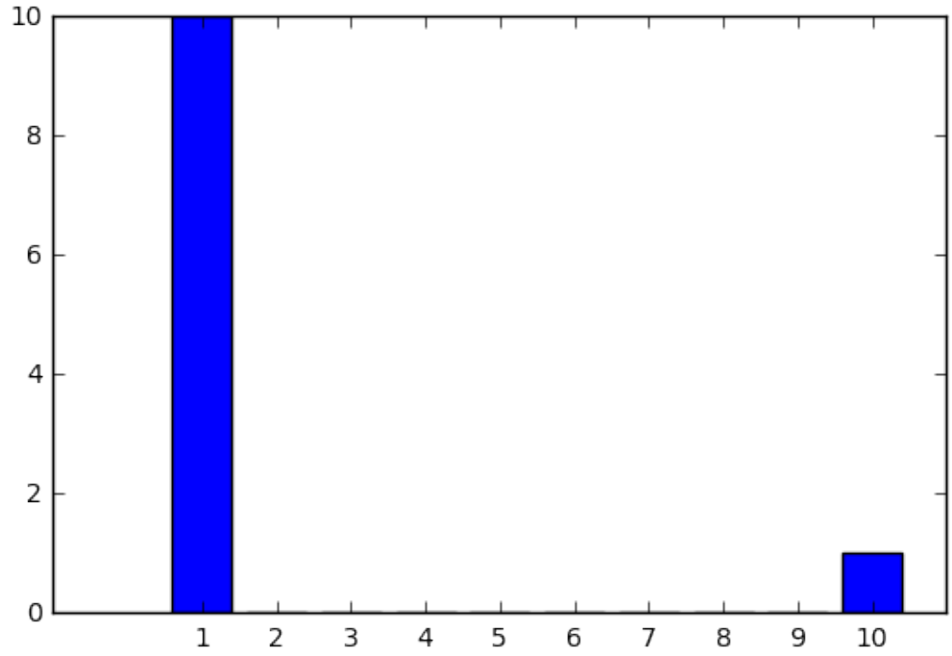
We illustrate this by plotting the degree distribution of a star network with 4 spokes:

```
In [6]: import matplotlib.pyplot as plt


        D = {1:4, 2:0, 3:0, 4:1}

        plt.bar(range(len(D)), D.values(), align='center')
        plt.xticks(range(len(D)), D.keys())

        plt.show()
```



And the degree distribution of a star network with 10 spokes:

```
In [7]: D = {1:10, 2:0, 3:0, 4:0, 5:0, 6:0, 7:0, 8:0, 9:0, 10:1}

        plt.bar(range(len(D)), D.values(), align='center')
        plt.xticks(range(len(D)), D.keys())

        plt.show()
```



(a.2) The average distance is the sum of the distance between all pairs of points divided by the number of pairs of points. So, given a distance d_ij between a pair of points i and j, we can write the average distance as $\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} d_ij}{2N!}$.

The diameter of a network is its maximum shortest path, so in the case of a start network, the diameter is 2.

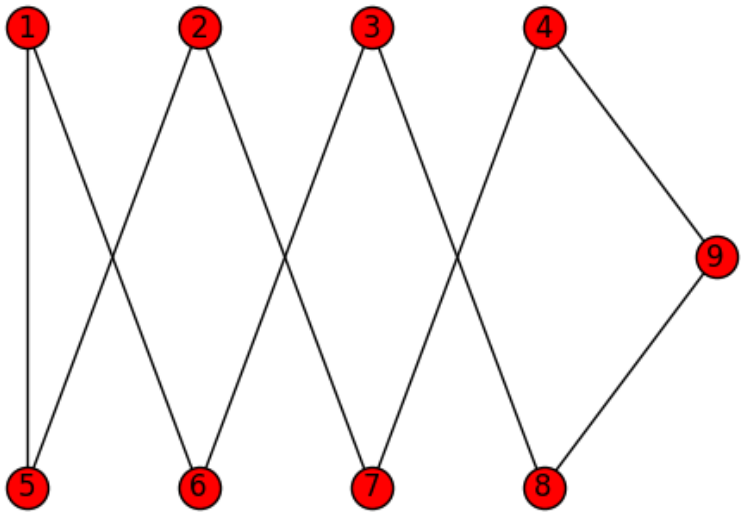
(a.3) There are $N-1$ ways to go from hub to hub (by leaving the hub, reaching one of the spokes, and returning to the hub). There is only one way to go from the hub to each of the respective spokes.

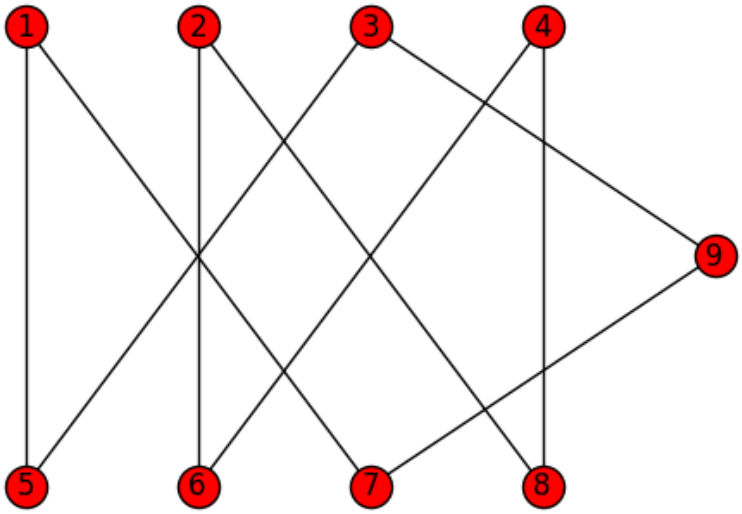
1.4.2 (b) Regular networks

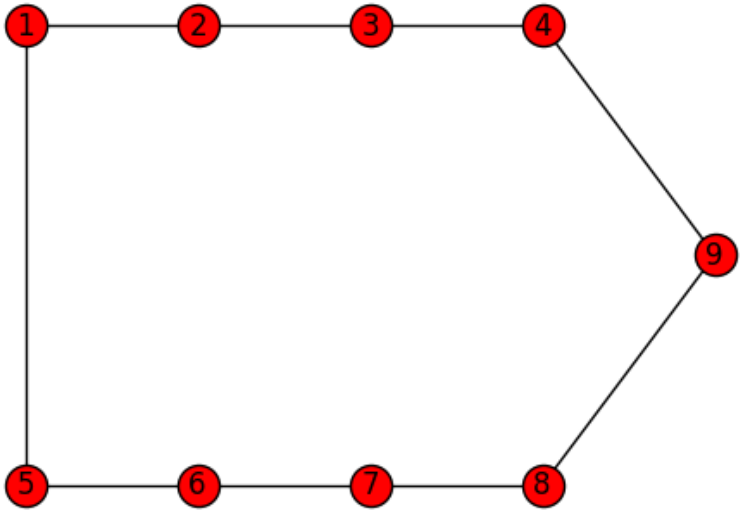
(b.1) We plot 3 different regular networks with n = 9 and k = 2. However, although the ordering of the nodes is different in each case, each of these networks can also be represented as a circle (or in the second case, as two circles).

```
In [8]: import networkx as nx

        G = nx.Graph()

        G.add_nodes_from(range(1,9))

        G.add_edges_from([(1,5),(1,6),(2,5),(2,7),(3,6),(3,8),(4,7),(4,9),(8,9)])

        pos_dict = {1:(1,4), 2:(2,4), 3:(3,4), 4:(4,4), 5:(1,2), 6:(2,2), 7:(3,2),
```

```
nx.draw(G, pos = pos_dict)
nx.draw_networkx_labels(G, pos = pos_dict)
plt.show()

G = nx.Graph()

G.add_nodes_from(range(1,9))

G.add_edges_from([(1,5),(1,7),(2,6),(2,8),(3,5),(3,9),(4,6),(4,8),(7,9)])

nx.draw(G, pos = pos_dict)
nx.draw_networkx_labels(G, pos = pos_dict)
plt.show()


G = nx.Graph()

G.add_nodes_from(range(1,9))

G.add_edges_from([(1,2),(2,3),(3,4),(4,9),(9,8),(8,7),(7,6),(6,5),(5,1)])

nx.draw(G, pos = pos_dict)
nx.draw_networkx_labels(G, pos = pos_dict)
plt.show()
```




1
2
3
4
9
5
6
7
8


1
2
3
4
9
5
6
7
8

1.4.3 (c) Bipartite networks

(c.1) The first network is not bipartite, since we cannot divide its nodes into two disjoint sets. The second graph is bipartite, where we can divide the nodes into two disjoint sets: one set with the nodes in the middle row, and one set including the nodes in the upper and lower rows.

(c.2) Star networks are bipartite since we can divide them into two disjoint sets: one with the hub, and the other with the spokes. Circles are only bipartite if they contain an even number of nodes, in which case each disjoint set will include every other node.

1.5 Problem 3

1.5.1 Degree centrality

We calculate basic degree centrality to get an idea of centrality, additionally we normilise it using different approaches, to test if it will change the outcome in any way.

$$C_d(v_i) = d_i C_d^{max}(v_i) = \frac{d_i}{max_j d_j} C_d^{sum}(v_i) = \frac{d_i}{\sum_j d_j} c_d^{norm} = \frac{d_i}{n-1}$$

```
In [11]: cols = ["node","C_d","C_d_norm","C_d_max","C_d_sum"]
         C_d = [1,2,1,4,1,1]
         C_d_norm = np.array(C_d)/5
         C_d_max = np.array(C_d)/4
         C_d_sum = np.array(C_d)/10
         C_table = pd.DataFrame(
                 {'node' : index,
                  'C_d' : C_d,
                  'C_d_norm': C_d_norm,
                  'C_d_max' : C_d_max,
                  'C_d_sum' : C_d_sum
                  })
         C_table
```

Out[11]:

	C_d	C_d_max	C_d_norm	C_d_sum	node
0	1	0.25	0.2	0.1	A
1	2	0.50	0.4	0.2	B
2	1	0.25	0.2	0.1	C
3	4	1.00	0.8	0.4	D
4	1	0.25	0.2	0.1	E
5	1	0.25	0.2	0.1	F

1.5.2 Eigenvalue Centrality

```
In [2]: import numpy as np
        import pandas as pd
```

```
#define adjecency matrix
A = [[0,1,0,0,0,0],[1,0,0,1,0,0],[0,0,0,1,0,0],[0,1,1,0,1,1],[0,0,0,1,0,0],

graph = { "A":["B"],
          "B":["A","D"],
          "C":["D"],
          "D":["B,C,E,F"],
          "E":["D"],
          "F":["D"],
        }

#convert to mumpy array, calculate it's eigenvectors and eignevalues
A = np.matrix(A)
lam = np.linalg.eig(A)

#find max eigenvalue and it's corresponding vecotor
eigenvalue = lam[0].max()
eigenvector = lam[1][0] #results not ordered, 1st is only all-positive

#turn matrix into an array
C = np.squeeze(np.asarray(eigenvector))
index = ["A","B","C","D","E","F"] #index the nodes of the matrix
C = dict(zip(index,C))

#We can see that some of the centrality values are 0, and in order to impro
beta = 0.2
alpha = 1/eigenvalue - 0.1

#C_katz = beta *(I - alpha*A.T)^-1 * 1
C_katz = (beta * np.linalg.inv(np.identity(6) - alpha * np.transpose(A))).c
C_katz = np.squeeze(np.asarray(C_katz))
C_katz = dict(zip(index,C_katz))
C_katz
```

```
Out[2]: {'A': 0.56536135916031816,
         'B': 0.956224933051394,
         'C': 0.74020756364398432,
         'D': 1.4138329859690022,
         'E': 0.74020756364398443,
         'F': 0.74020756364398432}
```

D is still most central node, followed by B, the single-connected C,E,F and A as least central. This measures centrality more in line with our expectation from the visual analysis. Additionally, we don't have and 0 values.

Pagerank

```
In [3]: a = np.asarray([1,2,1,4,1,1])
        D = np.zeros((6, 6), int)
```

```
np.fill_diagonal(D, a)

C_pagerank = (beta * np.linalg.inv((np.identity(6) - alpha * np.transpose(A

C_pagerank = np.squeeze(np.asarray(C_pagerank))
C_pagerank = dict(zip(index,C_pagerank))
C_pagerank
```

```
Out[3]: {'A': 0.56536135916031816,
 'B': 0.478112466525697,
 'C': 0.74020756364398432,
 'D': 0.35345824649225055,
 'E': 0.74020756364398443,
 'F': 0.74020756364398432}
```

This produces some interesting results, indicating that E,F and C has similar centralities (as expected), however node D has smallest centralities, as we specified that it has 4 outgoing links this might've had an effect. This raises an issue of working with non-directional graphs.

The information about the distance allows us to measure betweenety as a centrality measure, identifying most "in-between" node as an optimum node for the office.

```
In [5]: #we design matrix G, which will indicate the distance to every other node b
        G = [[0,14,50,26,43,41],[14,0,36,12,29,27],[50,36,0,24,41,39],[26,12,24,0,1
        G = np.matrix(G)
        G
```

```
Out[5]: matrix([[ 0, 14, 50, 26, 43, 41],
                [14,  0, 36, 12, 29, 27],
                [50, 36,  0, 24, 41, 39],
                [26, 12, 24,  0, 17, 15],
                [43, 29, 41, 17,  0, 32],
                [41, 27, 39, 15, 32,  0]])
```

Closeness Centrality

```
In [6]: C_C = 82/G.sum(axis=1)
        C_C_df = pd.DataFrame(C_C)

        #turn to dict
        C_C = np.squeeze(np.asarray(C_C_df))
        C_C = dict(zip(index,C_C))
        print(C_C)


        #ranking order, in reverse order?
        ab = C_C_df.rank().values
        c = []
        for i in range(6):
            c.append(int(ab[i][0]))
        dict(zip(index,c))
```

```
{'F': 0.53246753246753242, 'A': 0.47126436781609193, 'B': 0.69491525423728817, 'C':
```

```
Out[6]: {'A': 2, 'B': 5, 'C': 1, 'D': 6, 'E': 3, 'F': 4}
```

From this centrality measurment we can determine that D is the optimal centrality, followed by B with the rest of nodes having roughly the same values ##### Population

Given the population, intuitevly node B would be optimal, as if consider clustering the neighbours, cluster on the left is valued at 90k, and right cluster valued at 29k

In order to have final conclusion, we should consider both, the population and the distance, by incorporating it into the calulations.

```
In [9]: population = {"A":90000,"B":10000,"C":8500,"D":15000,"E":1200,"F":5000}

        pop = np.asarray([90000,10000,8500,15000,12000,5000])
        #normalize
        pop = pop/max(pop)
        P = np.zeros((6,6), int)
        np.fill_diagonal(P,pop)

        #Weight the pagerank centrality by population.
        C_pop_df = (beta * np.linalg.inv((np.identity(6) - alpha * np.transpose(A))
        #convert to dict
        C_pop = np.squeeze(np.asarray(C_pop_df))
        C_pop = dict(zip(index,C_pop))
        C_pop
```

```
Out[9]: {'A': 0.28745725445163312,
         'B': 0.11444670486647687,
         'C': 0.11572632331832812,
         'D': 0.063360811791216096,
         'E': 0.12350410109610591,
         'F': 0.10794854554055033}
```

This indicates the heaviest weight on node A, as previously expected. However, the range of the values is smaller then for centrality measure, indicating that they can't be weghted correspondingly , therefore using appropriate constatnt to standardize the values.

Depending on the circumstances such as cost of travel, frequency of calls, average number of specialists, ect, we can use this formula to determine the optimum location of the office, by contributing a significance factor which sums up to 1.

In our example we contribute 50/50 proportion of significance to both factors (constant a and b).

Constant "const" is selected by getting a ratio of the top centralities in both population centrality and distance centrality.

```
In [ ]: const = 0.32

        a = 0.5 #distance factor
        b = 0.5 #population factor
```

```
C_C_df = np.matrix(C_C_df)
C_opt = a * C_C_df + np.transpose(b * C_pop_df/ const)
C_opt
```

To conclude with, node A seems most appropriate for the office, despite node D being most connected. This conclusion is reached after considering distance to and population of each city, giving first preference to A and second to D.

As pointed up above, by adjusting the factors significance, optimal decision could be found.