

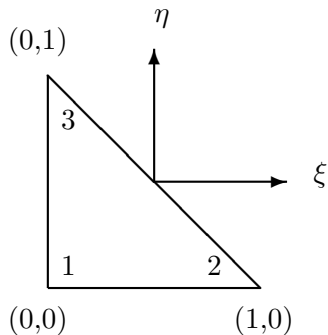
Master of Science on Computational Science

Institute of Computational Science

Dr. Drosos Kourounis & Hardik Kothari (TA)

17 Oct 2016, LAB3

The 2D reference triangular element



$$N_1 = (1 - \xi)(1 - \eta)$$

$$N_2 = \xi$$

$$N_3 = \eta$$

The affine mapping

The mapping from the reference triangle T_0 to the current triangle T_e with coordinates $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ is

$$\begin{aligned}x &= x_0 + (x_1 - x_0)\xi + (x_2 - x_0)\eta \\y &= y_0 + (y_1 - y_0)\xi + (y_2 - y_0)\eta.\end{aligned}$$

Now for any function $f = f(x, y) = f(x(\xi, \eta), y(\xi, \eta))$ using the chain rule of differentiation, we have

$$\begin{aligned}\frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \xi} \\ \frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \eta}\end{aligned}$$

From the current to the reference

or in more compact form

$$\begin{pmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{pmatrix} = J \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}, \quad J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = J^{-1} \begin{pmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{pmatrix}, \quad dx dy = |J| d\xi d\eta$$

Even more compact, when we differentiate we should keep in mind

$$\nabla_{\xi} = J \nabla_{\mathbf{x}}$$

$$\nabla_{\mathbf{x}} = J^{-1} \nabla_{\xi}$$

The entries of the matrix J

We can map a point (ξ, η) in the reference triangle to the point (x, y) in the current triangle using the equations

$$\begin{aligned}x &= x_0 + (x_1 - x_0)\xi + (x_2 - x_0)\eta \\y &= y_0 + (y_1 - y_0)\xi + (y_2 - y_0)\eta.\end{aligned}$$

A similar mapping can be written for the tetrahedron in 3D. Then it is easy to see that the entries of the Jacobian of the mapping J is

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{pmatrix}$$

Quadrature rules:

<http://lsec.cc.ac.cn/~tcui/myinfo/paper/quad.pdf>

2D integrals ...

Integrating the Laplacian on the reference element

$$I = \int_{\Omega} \nabla N_i(x, y) \cdot \nabla N_j(x, y) d\Omega$$

$$I = \int_{\Omega_0} J^{-1} \nabla N_i(\xi, \eta) \cdot J^{-1} \nabla N_j(\xi, \eta) |J| d\Omega_0$$

$$I = \int_{-1}^1 \int_{-1}^1 J^{-1} \nabla N_i(\xi, \eta) \cdot J^{-1} \nabla N_j(\xi, \eta) |J| d\xi d\eta$$

$$I \approx \sum_{k_1=1}^n \sum_{k_2=1}^n w_{k_1} w_{k_2} J_{k_1, k_2}^{-1} \nabla N_i(\xi_{k_1}, \eta_{k_2}) \cdot J_{k_1, k_2}^{-1} \nabla N_j(\xi_{k_1}, \eta_{k_2}) |J_{k_1, k_2}|$$

The 2D triangular element using barycentric coordinates

$$a_1^e x_1^e + b_1^e y_1^e + c_1^e = 1$$

$$a_2^e x_1^e + b_2^e y_1^e + c_2^e = 0$$

$$a_1^e x_2^e + b_1^e y_2^e + c_1^e = 0$$

$$a_2^e x_2^e + b_2^e y_2^e + c_2^e = 1$$

$$a_1^e x_3^e + b_1^e y_3^e + c_1^e = 0$$

$$a_2^e x_3^e + b_2^e y_3^e + c_2^e = 0$$

$$a_3^e x_1^e + b_3^e y_1^e + c_3^e = 0$$

$$a_3^e x_2^e + b_3^e y_2^e + c_3^e = 0$$

$$a_3^e x_3^e + b_3^e y_3^e + c_3^e = 1$$

$$\begin{pmatrix} x_1^e & y_1^e & 1 \\ x_2^e & y_2^e & 1 \\ x_3^e & y_3^e & 1 \end{pmatrix} \begin{pmatrix} a_1^e & a_2^e & a_3^e \\ b_1^e & b_2^e & b_3^e \\ c_1^e & c_2^e & c_3^e \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 2D triangular element

$$\begin{pmatrix} a_1^e & a_2^e & a_3^e \\ b_1^e & b_2^e & b_3^e \\ c_1^e & c_2^e & c_3^e \end{pmatrix} = \begin{pmatrix} x_1^e & y_1^e & 1 \\ x_2^e & y_2^e & 1 \\ x_3^e & y_3^e & 1 \end{pmatrix}^{-1}$$

Barycentric coordinates

$$N_i^e = a_i^e x + b_i^e y + c_i^e$$

Local approximation

$$u^e(x, y) = u_1^e N_1^e(x, y) + u_2^e N_2^e(x, y) + u_3^e N_3^e(x, y)$$

Local mass matrix

Integration on a simplex

$$\int_{V_e} N_1^i N_2^j N_3^k dV_e = \frac{i!j!k!}{(d+i+j+k)!} dV_e$$

Local mass matrix

$$M_{ij}^e = \int_{V_e} N_i^e N_j^e dV_e = \begin{cases} \frac{V_e}{12}, & i \neq j \\ \frac{V_e}{6}, & i = j \end{cases}$$

Local Laplacian matrix

Barycentric coordinates

$$N_i^e = a_i^e x + b_i^e y + c_i^e, \quad i = 1, 2, 3$$

Laplacian

$$K_{ij}^e = \int_{V_e} \nabla N_i^e \cdot \nabla N_j^e dV_e = (a_i^e a_j^e + b_i^e b_j^e) V_e$$