Master of Science on Computational Science

Institute of Computational Science

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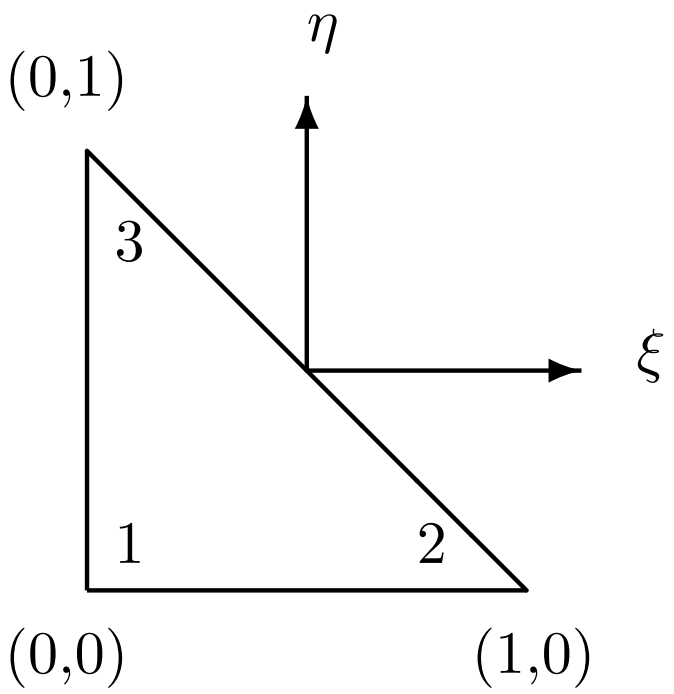
The 2D reference triangular element



$$N_1 = (1-\xi)(1-\eta)$$
$$N_2 = \xi$$
$$N_3 = \eta$$

The affine mapping

The mapping from the reference triangle T_0 to the current triangle T_e with coordinates $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ is

$$x = x_0 + (x_1 - x_0)\xi + (x_2 - x_0)\eta$$
$$y = y_0 + (y_1 - y_0)\xi + (y_2 - y_0)\eta.$$

Now for any function $f = f(x, y) = f(x(\xi, \eta), y(\xi, \eta))$ using the chain rule of differentiation, we have

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$
$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

From the current to the reference

or in more compact form

$$\begin{pmatrix} \frac{\partial f}{\partial \xi} \\ \\ \frac{\partial f}{\partial \eta} \end{pmatrix} = J \begin{pmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{pmatrix}, \quad J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{pmatrix} = J^{-1} \begin{pmatrix} \frac{\partial f}{\partial \xi} \\ \\ \frac{\partial f}{\partial \eta} \end{pmatrix}, \quad dxdy = |J| \; d\xi d\eta$$

Even more compact, when we differentiate we should keep in mind

$$\nabla_{\boldsymbol{\xi}} = J \; \nabla_{\mathbf{x}}$$
$$\nabla_{\mathbf{x}} = J^{-1} \; \nabla_{\boldsymbol{\xi}}$$

The entries of the matrix J

We can map a point (ξ, η) in the reference triangle to the point (x, y) in the current triangle using the equations

$$x = x_0 + (x_1 - x_0)\xi + (x_2 - x_0)\eta$$
$$y = y_0 + (y_1 - y_0)\xi + (y_2 - y_0)\eta.$$

A similar mapping cam be written for the tetrahedron in 3D. Then it is easy to see that the entries of the Jacobian of the mapping J is

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} x_1 - x_0 & y_1 - y_0 \\ \\ x_2 - x_0 & y_2 - y_0 \end{pmatrix}$$

2D integrals

Gaussian quadrature in 1D: n points can integrate exactly a polynomial of degree equal to $2n-1$

$$I = \int_{-1}^{1} f(\xi) d\xi$$

$$I \approx \sum_{k=1}^{n} w_k f(\xi_k)$$

Gaussian quadrature in 2D

$$I = \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) d\xi \, d\eta$$

$$I \approx \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} w_{k_1} w_{k_2} f(\xi_{k_1}, \eta_{k_2})$$

2D integrals ...

Integrating the Laplacian on the reference element

$$I = \int_{\Omega} \nabla N_i(x,y) \cdot \nabla N_j(x,y) d\Omega$$

$$I = \int_{\Omega_0} J^{-1} \nabla N_i(\xi,\eta) \cdot J^{-1} \nabla N_j(\xi,\eta) |J| d\Omega_0$$

$$I = \int_{-1}^{1} \int_{-1}^{1} J^{-1} \nabla N_i(\xi,\eta) \cdot J^{-1} \nabla N_j(\xi,\eta) |J| d\xi \; d\eta$$

$$I \approx \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} w_{k_1} w_{k_2} J^{-1}_{k_1,k_2} \nabla N_i(\xi_{k_1}, \eta_{k_2}) \cdot J^{-1}_{k_1,k_2} \nabla N_j(\xi_{k_1}, \eta_{k_2}) |J_{k_1,k_2}|$$

The 2D triangular element using barycentric coordinates

$$a_1^e x_1^e + b_1^e y_1^e + c_1^e = 1$$
$$a_1^e x_2^e + b_1^e y_2^e + c_1^e = 0$$
$$a_1^e x_3^e + b_1^e y_3^e + c_1^e = 0$$

$$a_2^e x_1^e + b_2^e y_1^e + c_2^e = 0$$
$$a_2^e x_2^e + b_2^e y_2^e + c_2^e = 1$$
$$a_2^e x_3^e + b_2^e y_3^e + c_2^e = 0$$

$$a_3^e x_1^e + b_3^e y_1^e + c_3^e = 0$$
$$a_3^e x_2^e + b_3^e y_2^e + c_3^e = 0$$
$$a_3^e x_3^e + b_3^e y_3^e + c_3^e = 1$$

$$\begin{pmatrix} x_1^e & y_1^e & 1 \\ x_2^e & y_2^e & 1 \\ x_3^e & y_3^e & 1 \end{pmatrix} \begin{pmatrix} a_1^e & a_2^e & a_3^e \\ b_1^e & b_2^e & b_3^e \\ c_1^e & c_2^e & c_3^e \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 2D triangular element

$$\begin{pmatrix} a_1^e & a_2^e & a_3^e \\ b_1^e & b_2^e & b_3^e \\ c_1^e & c_2^e & c_3^e \end{pmatrix} = \begin{pmatrix} x_1^e & y_1^e & 1 \\ x_2^e & y_2^e & 1 \\ x_3^e & y_3^e & 1 \end{pmatrix}^{-1}$$

Barycentric coordinates

$N_i^e = a_i^e x + b_i^e y + c_i^e$

Local approximation

$u^e(x,y) = u_1^e N_1^e(x,y) + u_2^e N_2^e(x,y) + u_3^e N_3^e(x,y)$

Local mass matrix

Integration on a simplex

$$\int_{V_e} N_1^i N_2^j N_3^k dV_e = \frac{i!j!k!}{(d+i+j+k)!} d! V_e$$

Local mass matrix

$$M_{ij}^e = \int_{V_e} N_i^e N_j^e dV_e = \begin{cases} \frac{V_e}{12}, & i \neq j \\ \frac{V_e}{6}, & i = j \end{cases}$$

Local Laplacian matrix

Barycentric coordinates

$$N_i^e = a_i^e x + b_i^e y + c_i^e, \quad i = 1, 2, 3$$

Laplacian

$$K_{ij}^e = \int_{V_e} \nabla N_i^e \cdot \nabla N_j^e dV_e = (a_i^e a_j^e + b_i^e b_j^e) V_e$$