Master of Science on Computational Science

Institute of Computational Science

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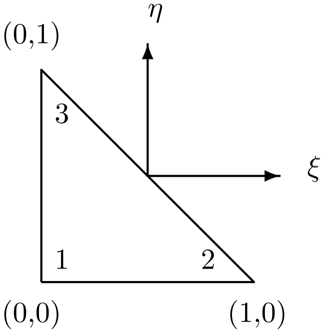
The 2D reference triangular element



$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$

The affine mapping

The mapping from the reference triangle T_0 to the current triangle T_e with coordinates $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ is

$$x = x_0 + (x_1 - x_0)\xi + (x_2 - x_0)\eta$$
$$y = y_0 + (y_1 - y_0)\xi + (y_2 - y_0)\eta.$$

Now for any function $f = f(x, y) = f(x(\xi, \eta), y(\xi, \eta))$ using the chain rule of differentiation, we have

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$
$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

From the current to the reference

or in more compact form

$$\begin{pmatrix} \frac{\partial f}{\partial \xi} \\ \\ \frac{\partial f}{\partial \eta} \end{pmatrix} = J \begin{pmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{pmatrix}, \quad J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{pmatrix} = J^{-1} \begin{pmatrix} \frac{\partial f}{\partial \xi} \\ \\ \frac{\partial f}{\partial \eta} \end{pmatrix}, \quad dxdy = |J| \, d\xi d\eta$$

Even more compact, when we differentiate we should keep in mind

$$\nabla_{\boldsymbol{\xi}} = J \; \nabla_{\mathbf{x}}$$
$$\nabla_{\mathbf{x}} = J^{-1} \; \nabla_{\boldsymbol{\xi}}$$

The entries of the matrix J

We can map a point (ξ, η) in the reference triangle to the point (x, y) in the current triangle using the equations

$$x = x_0 + (x_1 - x_0)\xi + (x_2 - x_0)\eta$$
$$y = y_0 + (y_1 - y_0)\xi + (y_2 - y_0)\eta.$$

A similar mapping cam be written for the tetrahedron in 3D. Then it is easy to see that the entries of the Jacobian of the mapping J is

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} x_1 - x_0 & y_1 - y_0 \\ \\ x_2 - x_0 & y_2 - y_0 \end{pmatrix}$$

2D integrals

Quadrature rules:

`http://lsec.cc.ac.cn/~tcui/myinfo/paper/quad.pdf`

2D integrals . . .

Integrating the Laplacian on the reference element

$$I = \int_{\Omega} \nabla N_i(x, y) \cdot \nabla N_j(x, y) d\Omega$$

$$I = \int_{\Omega_0} J^{-1} \nabla N_i(\xi, \eta) \cdot J^{-1} \nabla N_j(\xi, \eta) |J| d\Omega_0$$

$$I = \int_0^1 \int_0^{1-\eta} J^{-1} \nabla N_i(\xi, \eta) \cdot J^{-1} \nabla N_j(\xi, \eta) |J| d\xi \, d\eta$$

The 2D triangular element using barycentric coordinates

$$\begin{aligned} a_1^e x_1^e + b_1^e y_1^e + c_1^e &= 1 \\ a_1^e x_2^e + b_1^e y_2^e + c_1^e &= 0 \\ a_1^e x_3^e + b_1^e y_3^e + c_1^e &= 0 \end{aligned}$$

$$\begin{aligned} a_2^e x_1^e + b_2^e y_1^e + c_2^e &= 0 \\ a_2^e x_2^e + b_2^e y_2^e + c_2^e &= 1 \\ a_2^e x_3^e + b_2^e y_3^e + c_2^e &= 0 \end{aligned}$$

$$\begin{aligned} a_3^e x_1^e + b_3^e y_1^e + c_3^e &= 0 \\ a_3^e x_2^e + b_3^e y_2^e + c_3^e &= 0 \\ a_3^e x_3^e + b_3^e y_3^e + c_3^e &= 1 \end{aligned}$$

$$\begin{pmatrix} x_1^e & y_1^e & 1 \\ x_2^e & y_2^e & 1 \\ x_3^e & y_3^e & 1 \end{pmatrix} \begin{pmatrix} a_1^e & a_2^e & a_3^e \\ b_1^e & b_2^e & b_3^e \\ c_1^e & c_2^e & c_3^e \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 2D triangular element

$$\begin{pmatrix} a_1^e & a_2^e & a_3^e \\ b_1^e & b_2^e & b_3^e \\ c_1^e & c_2^e & c_3^e \end{pmatrix} = \begin{pmatrix} x_1^e & y_1^e & 1 \\ x_2^e & y_2^e & 1 \\ x_3^e & y_3^e & 1 \end{pmatrix}^{-1}$$

Barycentric coordinates

$N_i^e = a_i^e x + b_i^e y + c_i^e$

Local approximation

$u^e(x,y) = u_1^e N_1^e(x,y) + u_2^e N_2^e(x,y) + u_3^e N_3^e(x,y)$

Local mass matrix

Integration on a simplex

$$\int_{V_e} N_1^i N_2^j N_3^k dV_e = \frac{i!j!k!}{(d+i+j+k)!} d! V_e$$

Local mass matrix

$$M_{ij}^e = \int_{V_e} N_i^e N_j^e dV_e = \begin{cases} \frac{V_e}{12}, & i \neq j \\ \frac{V_e}{6}, & i = j \end{cases}$$

Local Laplacian matrix

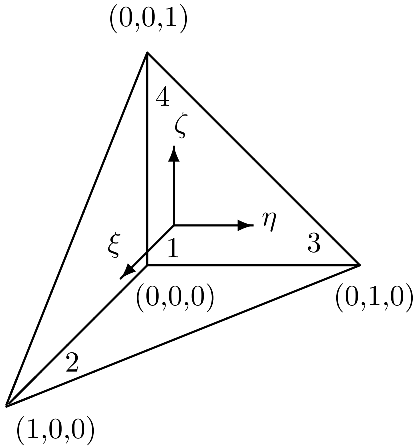
Barycentric coordinates

$N_i^e = a_i^e x + b_i^e y + c_i^e, \quad i = 1, 2, 3$

Laplacian

$$K_{ij}^e = \int_{V_e} \nabla N_i^e \cdot \nabla N_j^e dV_e = (a_i^e a_j^e + b_i^e b_j^e) V_e$$

The 3D reference tetrahedral element



$$N_1 = (1 - \xi - \eta - \zeta)$$
$$N_2 = \xi$$
$$N_3 = \eta$$
$$N_4 = \zeta$$

The affine mapping

The mapping from the reference triangle T_0 to the current triangle T_e with coordinates $(x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ is

$$\begin{aligned} x &= x_0 + (x_1 - x_0)\xi + (x_2 - x_0)\eta + (x_3 - x_0)\zeta \\ y &= y_0 + (y_1 - y_0)\xi + (y_2 - y_0)\eta + (y_3 - y_0)\zeta \\ z &= z_0 + (z_1 - z_0)\xi + (z_2 - z_0)\eta + (z_3 - z_0)\zeta. \end{aligned}$$

Now for any function $f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta))$ using the chain rule of differentiation, we have

$$\begin{aligned} \frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \xi} \\ \frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \eta} \\ \frac{\partial f}{\partial \zeta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \zeta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \zeta} \end{aligned}$$

From the current to the reference

or in more compact form

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{pmatrix} \quad dxdydz = |J| \, d\xi d\eta d\zeta$$

Even more compact, when we differentiate we should keep in mind

$$\nabla_{\boldsymbol{\xi}} = J \ \nabla_{\mathbf{x}}$$
$$\nabla_{\mathbf{x}} = J^{-1} \ \nabla_{\boldsymbol{\xi}}$$

The entries of the matrix J

We can map a point (ξ, η, ζ) in the reference tetrahedron to the point (x, y, z) in the current tetrahedron using the equations

$$\begin{aligned} x &= x_0 + (x_1 - x_0)\xi + (x_2 - x_0)\eta + (x_3 - x_0)\zeta \\ y &= y_0 + (y_1 - y_0)\xi + (y_2 - y_0)\eta + (y_3 - y_0)\zeta \\ z &= z_0 + (z_1 - z_0)\xi + (z_2 - z_0)\eta + (z_3 - z_0)\zeta. \end{aligned}$$

Then it is easy to see that the entries of the Jacobian of J are

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{pmatrix} = \begin{pmatrix} x_1 - x_0 & y_1 - y_0 & z_1 - z_0 \\ x_2 - x_0 & y_2 - y_0 & z_2 - z_0 \\ x_3 - x_0 & y_3 - y_0 & z_3 - z_0 \end{pmatrix}$$

The tetrahedral element using barycentric coordinates

$$\begin{aligned}
a_1^e x_1^e + b_1^e y_1^e + c_1^e z_1^e + d_1^e &= 1\\
a_1^e x_2^e + b_1^e y_2^e + c_1^e z_2^e + d_1^e &= 0\\
a_1^e x_3^e + b_1^e y_3^e + c_1^e z_3^e + d_1^e &= 0\\
a_1^e x_4^e + b_1^e y_4^e + c_1^e z_4^e + d_1^e &= 0
\end{aligned}
\qquad
\begin{aligned}
a_2^e x_1^e + b_2^e y_1^e + c_2^e z_1^e + d_2^e &= 0\\
a_2^e x_2^e + b_2^e y_2^e + c_2^e z_2^e + d_2^e &= 1\\
a_2^e x_3^e + b_2^e y_3^e + c_2^e z_3^e + d_2^e &= 0\\
a_2^e x_4^e + b_2^e y_4^e + c_2^e z_4^e + d_2^e &= 0
\end{aligned}$$

$$\begin{aligned}
a_3^e x_1^e + b_3^e y_1^e + c_3^e z_1^e + d_3^e &= 0\\
a_3^e x_2^e + b_3^e y_2^e + c_3^e z_2^e + d_3^e &= 0\\
a_3^e x_3^e + b_3^e y_3^e + c_3^e z_3^e + d_3^e &= 1\\
a_3^e x_4^e + b_3^e y_4^e + c_3^e z_4^e + d_3^e &= 0
\end{aligned}
\qquad
\begin{aligned}
a_4^e x_1^e + b_4^e y_1^e + c_4^e z_1^e + d_4^e &= 0\\
a_4^e x_2^e + b_4^e y_2^e + c_4^e z_2^e + d_4^e &= 0\\
a_4^e x_3^e + b_4^e y_3^e + c_4^e z_3^e + d_4^e &= 0\\
a_4^e x_4^e + b_4^e y_4^e + c_4^e z_4^e + d_4^e &= 1
\end{aligned}$$

$$\begin{pmatrix} x_1^e & y_1^e & z_1^e & 1\\ x_2^e & y_2^e & z_2^e & 1\\ x_3^e & y_3^e & z_3^e & 1\\ x_4^e & y_4^e & z_4^e & 1 \end{pmatrix}
\begin{pmatrix} a_1^e & a_2^e & a_3^e & a_4^e\\ b_1^e & b_2^e & b_3^e & b_4^e\\ c_1^e & c_2^e & c_3^e & c_4^e\\ d_1^e & d_2^e & d_3^e & d_4^e \end{pmatrix}
= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The 3D tetrahedral element

$$\begin{pmatrix} a_1^e & a_2^e & a_3^e & a_4^e \\ b_1^e & b_2^e & b_3^e & b_4^e \\ c_1^e & c_2^e & c_3^e & c_4^e \\ d_1^e & d_2^e & d_3^e & d_4^e \end{pmatrix} = \begin{pmatrix} x_1^e & y_1^e & z_1^e & 1 \\ x_2^e & y_2^e & z_2^e & 1 \\ x_3^e & y_3^e & z_3^e & 1 \\ x_4^e & y_4^e & z_4^e & 1 \end{pmatrix}^{-1}$$

Barycentric coordinates

$$N_i^e = a_i^e x + b_i^e y + c_i^e z + d_i^e$$

Local approximation

$$u^e(x, y, z) = u_1^e N_1^e(x, y, z) + u_2^e N_2^e(x, y, z) + u_3^e N_3^e(x, y, z) + u_4^e N_4^e(x, y, z)$$

Local mass matrix

Integration on a simplex

$$\int_{V_e} N_1^i N_2^j N_3^k N_4^l dV_e = \frac{i!j!k!l!}{(d+i+j+k+l!)!} d! V_e$$

Local mass matrix

$$M_{ij}^e = \int_{V_e} N_i^e N_j^e dV_e = \begin{cases} \frac{V_e}{20}, & i \neq j \\ \frac{V_e}{10}, & i = j \end{cases}$$

Local Laplacian matrix

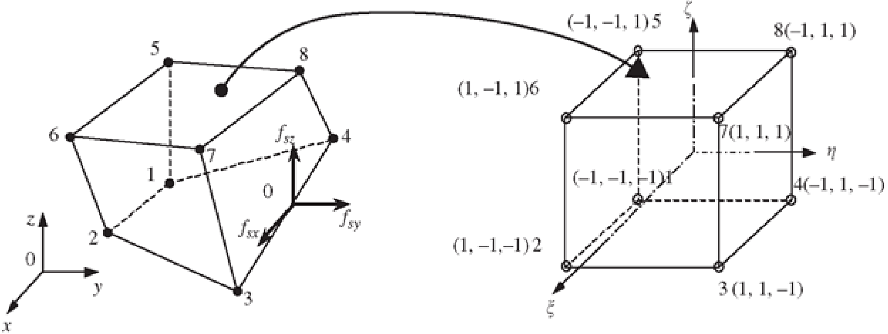
Barycentric coordinates

$$N_i^e = a_i^e x + b_i^e y + c_i^e z + d_i^e, \quad i = 1, 2, 3, 4$$

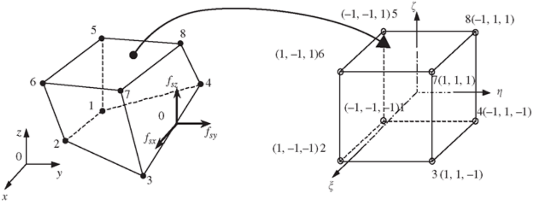
Laplacian

$$K_{ij}^e = \int_{V_e} \nabla N_i^e \cdot \nabla N_j^e dV_e = (a_i^e a_j^e + b_i^e b_j^e + c_i^e c_j^e) V_e$$

Mapping to the reference element 3D



Trilinear hexahedral element



$$N_1 = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)$$

$$N_2 = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)$$

$$N_3 = \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)$$

$$N_4 = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$$

$$N_5 = \frac{1}{8}(1-\xi)(1-\eta)(1+\zeta)$$

$$N_6 = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta)$$

$$N_7 = \frac{1}{8}(1+\xi)(1+\eta)(1+\zeta)$$

$$N_8 = \frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)$$

$$N_i = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta)$$

3D integrals

Gaussian quadrature in 1D: n points can integrate exactly a polynomial of degree equal to $2n-1$

$$I = \int_{-1}^{1} f(\xi) d\xi$$

$$I \approx \sum_{k=1}^{n} w_k f(\xi_k)$$

Gaussian cubature

$$I = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta$$

$$I \approx \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} w_{k_1} w_{k_2} w_{k_3} f(\xi_{k_1}, \eta_{k_2}, \zeta_{k_3})$$

3D integrals ...

Integrating the Laplacian on the reference element

$$I = \int_\Omega \nabla N_i(x,y,z) \cdot \nabla N_j(x,y,z) d\Omega$$

$$I = \int_{\Omega_0} J^{-1} \nabla N_i(\xi,\eta,\zeta) \cdot J^{-1} \nabla N_j(\xi,\eta,\zeta) |J| d\Omega_0$$

$$I = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} J^{-1} \nabla N_i(\xi,\eta,\zeta) \cdot J^{-1} \nabla N_j(\xi,\eta,\zeta) |J| d\xi \, d\eta \, d\zeta$$

$$I \approx \sum_{k_1,k_2,k_3=1}^{n} (w_{k_1} w_{k_2} w_{k_3} J^{-1}_{k_1,k_2,k_3} \nabla N_i(\xi_{k_1}, \eta_{k_2}, \zeta_{k_3})$$
$$\cdot J_{k_1,k_2,k_3} \nabla N_j(\xi_{k_1}, \eta_{k_2}, \zeta_{k_3}) |J_{k_1,k_2,k_3}|)$$