

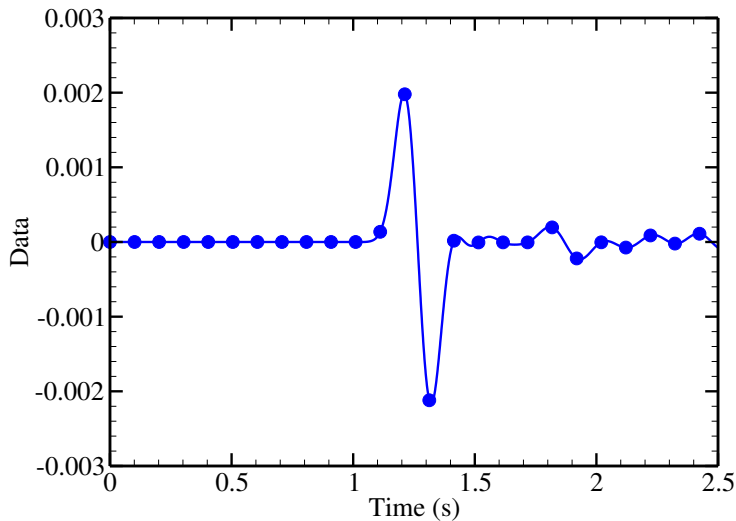
Master of Science on Computational Science

Institute of Computational Science

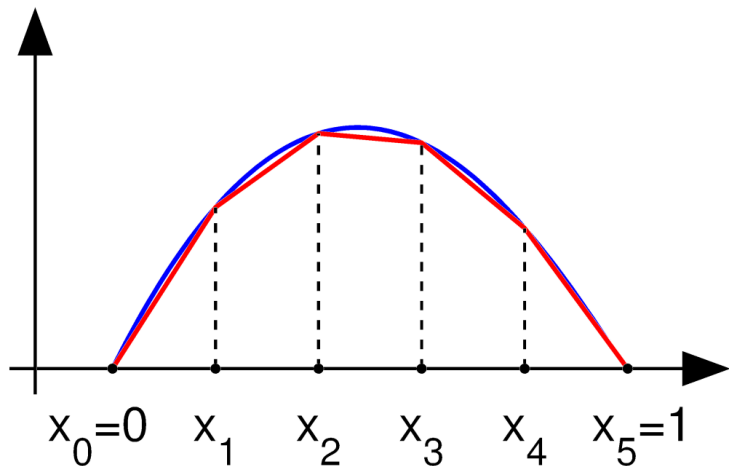
prof. Dr. Rolf Krause & Dr. Drosos Kourounis

24 Sep 2015, LAB2

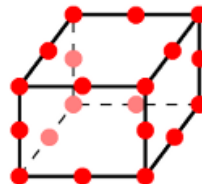
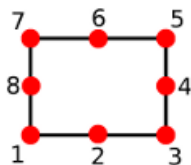
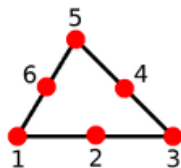
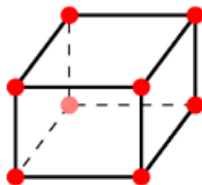
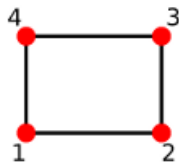
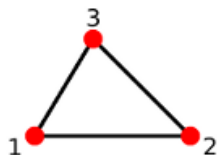
Interpolation in one dimension (1D)



Finite Elements in one dimension (1D)



2D and 3D Finite Elements



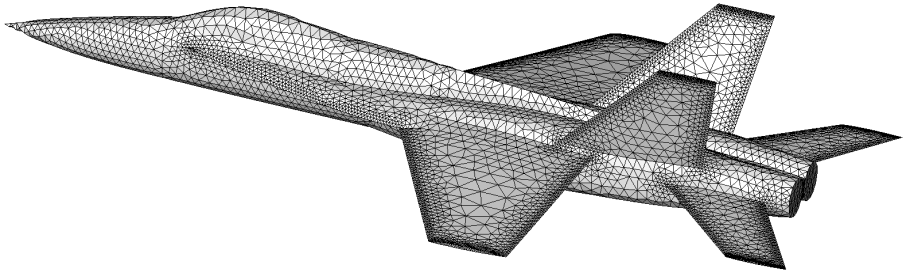
Triangle element
with 3 and 6 nodes

Rectangle element
with 4 and 8 nodes

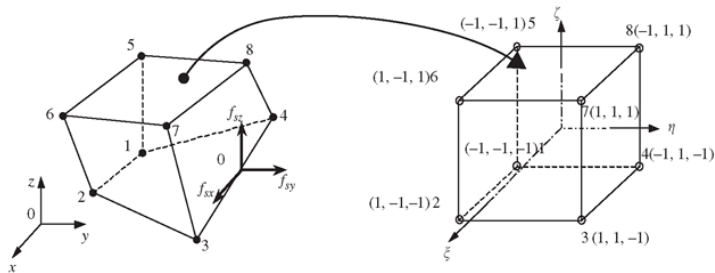
Box element (for 3D)
with 8 and 20 nodes

Sample of some simple element shapes and standard node placement. By convention nodes are numbered anti-clockwise.

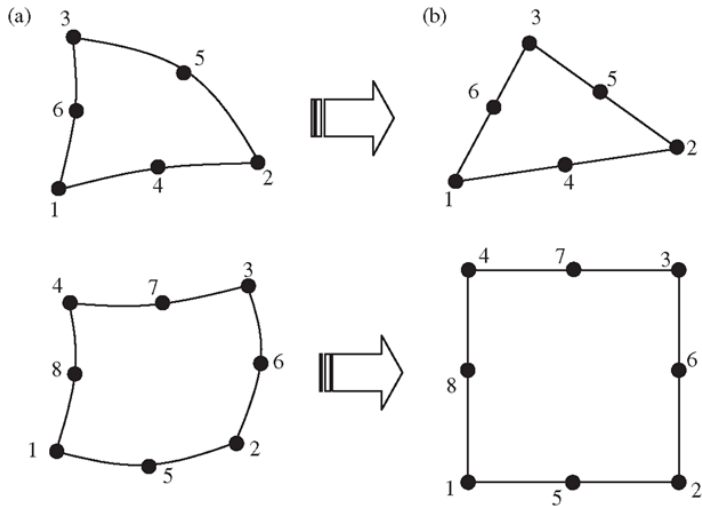
The 3D case can be very challenging



Mapping to the reference element 3D

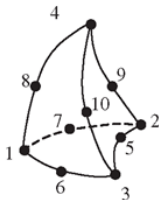


Mapping to the reference in 2D

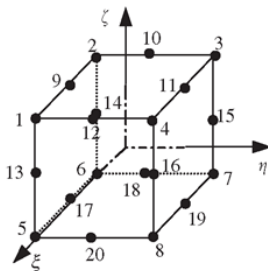
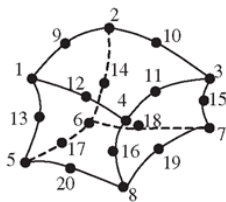
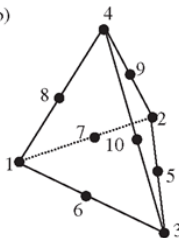


Mapping to the reference in 3D

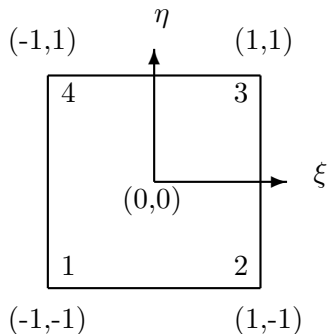
(a)



(b)



The 2D quadrilateral element



$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)$$

$$N_2 = \frac{1}{4}(1 + \xi)(1 - \eta)$$

$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta)$$

$$N_4 = \frac{1}{4}(1 - \xi)(1 + \eta)$$

$$N_i = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta)$$

The 2D quadrilateral element ...

Any unknown field u as well as the coordinates (x, y) may be expressed as functions of (ξ, η) as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4.$$

Now suppose $f = f(x, y) = f(x(\xi, \eta), y(\xi, \eta))$. Using the chain rule of differentiation, we have

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \eta}$$

The 2D quadrilateral element ...

$$\begin{pmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{pmatrix} = J \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}, \quad J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = J^{-1} \begin{pmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{pmatrix}, \quad dxdy = |J| d\xi d\eta$$

The 2D quadrilateral element ...

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

$$J_{11} = \frac{1}{4} (-(1 - \eta)x_1 + (1 - \eta)x_2 + (1 + \eta)x_3 - (1 + \eta)x_4)$$

$$J_{12} = \frac{1}{4} (-(1 - \eta)y_1 + (1 - \eta)y_2 + (1 + \eta)y_3 - (1 + \eta)y_4)$$

$$J_{21} = \frac{1}{4} (-(1 - \xi)x_1 - (1 + \xi)x_2 + (1 + \xi)x_3 + (1 - \xi)x_4)$$

$$J_{22} = \frac{1}{4} (-(1 - \xi)y_1 - (1 + \xi)y_2 + (1 + \xi)y_3 + (1 - \xi)y_4)$$

2D integrals

Gaussian quadrature in 1D: n points can integrate exactly a polynomial of degree equal to $2n - 1$

$$I = \int_{-1}^1 f(\xi) d\xi$$

$$I \approx \sum_{k=1}^n w_k f(\xi_k)$$

Gaussian quadrature in 2D

$$I = \int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta$$

$$I \approx \sum_{k_1=1}^n \sum_{k_2=1}^n w_{k_1} w_{k_2} f(\xi_{k_1}, \eta_{k_2})$$

2D integrals ...

Integrating the Laplacian on the reference element

$$I = \int_{\Omega} \nabla N_i(x, y) \cdot \nabla N_j(x, y) d\Omega$$

$$I = \int_{\Omega_0} J^{-1} \nabla N_i(\xi, \eta) \cdot J^{-1} \nabla N_j(\xi, \eta) |J| d\Omega_0$$

$$I = \int_{-1}^1 \int_{-1}^1 J^{-1} \nabla N_i(\xi, \eta) \cdot J^{-1} \nabla N_j(\xi, \eta) |J| d\xi d\eta$$

$$I \approx \sum_{k_1=1}^n \sum_{k_2=1}^n w_{k_1} w_{k_2} J_{k_1, k_2}^{-1} \nabla N_i(\xi_{k_1}, \eta_{k_2}) \cdot J_{k_1, k_2}^{-1} \nabla N_j(\xi_{k_1}, \eta_{k_2}) |J_{k_1, k_2}|$$

$$M = \int_V N_i N_j dV$$

```
function I = Mass2DSymbolic()
    xi = sym('xi', 'real'); eta = sym('eta', 'real');
    dx = sym('dx', 'real'); dy = sym('dy', 'real');

    c=[-1 -1; 1 -1; 1 1; -1 1];

    J(1,1) = 0.5*dx; J(2,2) = 0.5*dy;

    for i=1:4
        N(i) = 1/4*( 1+c(i,1)*xi )*( 1+c(i,2)*eta );
    end

    F = det(J)*N'*N;
    M = int(int(F, 'xi', -1, 1), 'eta', -1, 1);
end
```

$$K = \int_V \nabla N_i \cdot \nabla N_j dV$$

```
function I = Laplace2DSymbolic()
    xi = sym('xi', 'real'); eta = sym('eta', 'real');
    dx = sym('dx', 'real'); dy = sym('dy', 'real');

    c=[-1 -1; 1 -1; 1 1; -1 1];
    J(1,1) = 0.5*dx; J(2,2) = 0.5*dy;

    for i=1:4
        N(i) = 1/4*( 1+c(i,1)*xi )*( 1+c(i,2)*eta );
    end

    Nx = diff(N, 'xi'); Ny = diff(N, 'eta');
    dN = [Nx; Ny];
    Jd = inv(J) * dN;
    F = det(J)*Jd'*Jd;
    I = int(int(F, 'xi', -1, 1), 'eta', -1, 1);
end
```