Definability, continuous embeddings and learning classes of algebraic structures

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Introduction

- Classical algorithmic learning theory - deals with classes of languages (i.e. c.e. sets).
- Two of the main notions - **Inf**-learning and **Txt**-learning.
- We want to explore lerning theory in the context of computable structure theory.
- **Inf**-learning for structures - already established notion.
- **Txt**-learning for structures - new work.

Basic notions

- We work with countable structures with domains $\subseteq \omega$.
- $D(\mathcal{A})$ - the diagram of $\mathcal{A}$, consisting of *p*ositive and *n*egative atomic sentences true in $\mathcal{A}$.
- An **informant** for the structure $\mathcal{A}$ is a surjective function $\mathbb{I}_{\mathcal{A}} : \omega \to D(\mathcal{A})$.
- $D_+(\mathcal{A})$ - the positive diagram of $\mathcal{A}$, consisting only of *p*ositive atomic sentences true in $\mathcal{A}$.
- A **text** for the structure $\mathcal{A}$ is a surjective function $\mathbb{T}_{\mathcal{A}} : \omega \to D_+(\mathcal{A})$.
- Let $Atm(L)$ denote the set of (the Gödel numbers) of all *p*ositive and *n*egative atomic sentences in the language $L \cup \omega$.
- $Atm_+(L)$ - only the positive.

Let $\mathfrak{K} \subseteq Mod(L)$ be a family with countably many nonisomorphic structures for a fixed language L including $=$ and $\neq$, closed under isomorphism.

- The *learning domain* (LD) is the collection of all copies $\mathcal{S}$ of the structures from $\mathfrak{K}$ such that $\mathrm{dom}(\mathcal{S}) \subseteq \omega$, i.e.,

$$\mathrm{LD}(\mathfrak{K}) = \bigcup_{i \in \omega} \{\mathcal{S} \in Mod(L) : \mathcal{S} \cong \mathcal{A}_i\}.$$

- The *hypothesis space* (HS) contains the indices i for $\mathcal{A}_i \in \mathfrak{K}$ (an index is viewed as a conjecture about the isomorphism type of an input structure $\mathcal{S}$) and a question mark symbol:

$$\mathrm{HS}(\mathfrak{K}) = \omega \cup \{?\}.$$

- A *learner* M sees, stage by stage, **positive and negative** or **only positive** data about a given structure from $LD(\mathfrak{K})$. The learner M is required to output conjectures from $HS(\mathfrak{K})$.

Inf-learning

A *learner* is a function $M : Atm(L)^{<\omega} \to \mathrm{HS}(\mathfrak{K})$.

Definition (Bazhenov-Fokina-San Mauro)

We say that the family $\mathfrak{K}$ is **Inf**-*learnable* if there exists a learner M such that for any structure $\mathcal{S} \in \mathrm{LD}(\mathfrak{K})$ and any informant $\mathbb{I}_{\mathcal{S}}$ for $\mathcal{S}$, the learner eventually stabilizes to a correct conjecture about the isomorphism type of $\mathcal{S}$. More formally, there exists a limit

$$\lim_{k \to \omega} M(\mathbb{I}_{\mathcal{S}} \restriction_k) = i$$

belonging to ω, and $\mathcal{A}_i$ is isomorphic to $\mathcal{S}$.

Txt-learning

A *learner* is a function $M : Atm_+(L)^{<\omega} \to \mathrm{HS}(\mathfrak{K})$.

Definition (following Fokina-Kötzing-San Mauro)

We say that the family $\mathfrak{K}$ is **Txt**-*learnable* if there exists a learner M such that for any structure $\mathcal{S} \in \mathrm{LD}(\mathfrak{K})$ and any text $\mathbb{T}_{\mathcal{S}}$ for $\mathcal{S}$, the learner eventually stabilizes to a correct conjecture about the isomorphism type of $\mathcal{S}$. More formally, there exists a limit

$$\lim_{k \to \omega} M(\mathbb{T}_{\mathcal{S}} \upharpoonright_k) = i$$

belonging to ω, and $\mathcal{A}_i$ is isomorphic to $\mathcal{S}$.

Examples

We use the notation $[\alpha_1 : \beta_1, \ldots, \alpha_n : \beta_n]$, where $\alpha_i, \beta_i \leq \omega$, to denote the equivalence structure with β_i number of equivalence classes of size α_i, for all $i = 1, \ldots, n$.

- The class $\{[\omega : 1], [\omega : 2]\}$ is **Inf**-learnable, but not **Txt**-learnable.
- The class $\{[3 : \omega, 2 : 1], [3 : \omega]\}$ is not **Inf**-learnable.

For linear orderings,

- $\{\omega, \zeta\}$ is not **Inf**-learnable.
- $\{\omega, \omega^\star\}$ is **Inf**-learnable.
- $\{1 + \eta, \eta + 1\}$ is **Inf**-learnable.

Turing computable embeddings

Definition (Knight-Miller-Vanden Boom)

The class $\mathfrak{K}$ is **Turing computably embeddable** in $\mathfrak{K}'$,

$$\mathfrak{K} \leq_{tc} \mathfrak{K}',$$

if there is a Turing operator Φ_e such that

- for each $\mathcal{A} \in \mathfrak{K}$, $\varphi_e^{D(\mathcal{A})} = \chi_{D(\mathcal{B})}$, where $B \in \mathfrak{K}'$;
- Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathfrak{K}$, $\varphi_e^{D(\mathcal{A}_1)} = \chi_{D(\mathcal{B}_1)}$ and $\varphi_e^{D(\mathcal{A}_2)} = \chi_{D(\mathcal{B}_2)}$. Then $\mathcal{A}_1 \cong \mathcal{A}_2$ iff $\mathcal{B}_1 \cong \mathcal{B}_2$.

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In general, $\Psi : 2^\omega \to 2^\omega$ is Cantor-continuous iff there exists a Turing operator Φ_e and a set X such that $\Psi(A) = \Phi_e(A \oplus X)$. Then we write $\mathfrak{K} \leq_{\text{Cantor}} \mathfrak{K}'$.

The Pullback Theorem

A main tool in the syntactic characterization of **Inf**-learning.

Theorem (Knight-Miller-Vanden Boom)

Suppose that $\mathfrak{K}_1 \leq_{Cantor} \mathfrak{K}_2$ *via continuous* $\Phi : 2^\omega \to 2^\omega$*. Then for any infinitary sentence* ψ_2 *in the language of* $\mathfrak{K}_2$*, one can find an infinitary sentence* ψ_1 *in the language of* $\mathfrak{K}_1$ *such that for all* $\mathcal{A} \in \mathfrak{K}_1$*, we have*

$$\mathcal{A} \models \psi_1 \ \leftrightarrow \ \Phi(\mathcal{A}) \models \psi_2.$$

Moreover, for a non-zero $\alpha < \omega_1^{CK}$*, if* ψ_2 *is a* Σ_α^{inf} *sentence, then so is* ψ_2*.*

It can be effectivized relative to some oracle X.

Characterization of **Inf**-learning.

Consider the class $\mathfrak{K}_{\text{univ}} = \{\mathcal{A}_i : i \in \omega\}$, where $\mathcal{A}_i = [\omega : 1, i+1 : 1]$.

Theorem (Bazhenov-Fokina-San Mauro 2020)

Let $\mathfrak{K} = \{\mathcal{A}_i \mid i \in \omega\}$. The following are equivalent:

- *$\mathfrak{K}$ is* **Inf***-learnable;*
- *$\mathfrak{K} \leq_{Cantor} \mathfrak{K}_{univ}$;*
- *There are Σ_2^{inf} sentences ψ_i such that*

$$\mathcal{A}_i \models \psi_j \leftrightarrow i = j.$$

Continuous operators on $\mathcal{P}(\omega)$

- The Scott topology on $\mathcal{P}(\omega)$ is formed by taking $U_v = \{A \subseteq \omega \mid D_v \subseteq A\}$ as the basic open sets.
- We say that $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is an **enumeration operator** if there exists a c.e. set W such that

$$\Gamma(A) = \{x \mid (\exists v)[\langle x, v\rangle \in W \ \& \ D_v \subseteq A]\}.$$

- $A \leq_e B$ iff there exists an enumeration operator Γ such that $A = \Gamma(B)$.
- We say that $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is a *generalized* enumeration operator if there exists a set B such that

$$\Gamma(A) = \{x \mid (\exists v)[\langle x, v\rangle \in B \ \& \ D_v \subseteq A]\}.$$

- It is well-known that $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is Scott-continuous iff Γ is a generalized enumeration operator.

Scott-Continuous Embeddings

A mapping $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is Scott-continuous if and only if Γ is

(a) monotone, i.e., $A \subseteq B$ implies $\Gamma(A) \subseteq \Gamma(B)$, and

(b) compact, i.e., $x \in \Gamma(A)$ if and only if $x \in \Gamma(D)$ for some finite $D \subseteq A$.

Definition

A mapping Γ is a **Scott-continuous embedding** of $\mathfrak{K}_0$ into $\mathfrak{K}_1$, denoted by $\Gamma\colon \mathfrak{K}_0 \leq_{\mathrm{Scott}} \mathfrak{K}_1$, if Γ is Scott-continuous and satisfies the following:

1. For any $\mathcal{A} \in \mathfrak{K}_0$, $\Gamma(\mathcal{D}_+(\mathcal{A}))$ is the positive atomic diagram of a structure from $\mathfrak{K}_1$.
2. For any $\mathcal{A}, \mathcal{B} \in \mathfrak{K}_0$, we have $\mathcal{A} \cong \mathcal{B}$ if and only if $\Gamma(\mathcal{A}) \cong \Gamma(\mathcal{B})$.

An important remark

- When the operator Γ is an enumeration operator, then we write $\leq_{pc}$ instead of $\leq_{\text{Scott}}$.
- The relation $\leq_{pc}$ is **different** from $\leq_c$ as defined in [Knight-Miller-Vanden Boom], which is like $\leq_{tc}$ where Turing operator is replaced by enumeration operator.
- The reason for this is the same as for $K \oplus \overline{K} \not\leq_e K$.

- When $\mathfrak{K}_0 \leq_{pc} \mathfrak{K}_1$, then **any** enumeration of a copy of the given input structure must always produce an enumeration of **the same** copy of the output structure.
- In other words, the construction must not depend on the enumeration of the input.

- When $\mathfrak{K}_0 \leq_{pc} \mathfrak{K}_1$, then **any** enumeration of a copy of the given input structure must always produce an enumeration of **the same** copy of the output structure.
- In other words, the construction must not depend on the enumeration of the input.
- By monotonicity, $\{1, 2\} \leq_{tc} \{\omega, \omega^\star\}$, but $\{1, 2\} \not\leq_{pc} \{\omega, \omega^\star\}$.

Characterization of the *tc*-class of $\{\omega, \omega^\star\}$

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be non-isomorphic L-structures. T.F.A.E.

(1) $\{\omega, \omega^\star\} \equiv_{tc} \{\mathcal{A}, \mathcal{B}\}$;

(2) *$\mathcal{A}$ and $\mathcal{B}$ have computable copies, they have the same finite substructures, and they differ by Σ_2^c sentences.*

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(2) *$\mathcal{A}$ and $\mathcal{B}$ have computable copies, they have the same finite substructures, and they differ by Σ_2^c sentences.*

It follows that all pairs of the form $\{\omega \cdot k, \omega^\star \cdot k\}$, for any $k > 0$ are equivalent under Turing computable embeddings.
What about computable embeddings (enumeration operators) ?

The top pair among linear orderings

As a first step, we would like to study the pairs of linear orderings inside the *tc*-degree of $\{\omega, \omega^\star\}$ relative to $\leq_{pc}$. It turns out that we have a top pair.

Theorem

For any pair $\{\mathcal{A}, \mathcal{B}\} \equiv_{tc} \{\omega, \omega^\star\}$*, we have that*

$$\{\mathcal{A}, \mathcal{B}\} \leq_{pc} \{1 + \eta, \eta + 1\},$$

where η *is the order type of the rationals.*

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$$\{\mathcal{A}, \mathcal{B}\} \leq_{pc} \{1 + \eta, \eta + 1\},$$

where η is the order type of the rationals.

Corollary

For any natural number $k > 0$,

$$\{\omega \cdot k, \omega^\star \cdot k\} <_{pc} \{1 + \eta, \eta + 1\}.$$

(The strictness comes from monotonicity.)

Theorem (Bazhenov-Ganchev-V 2021)

$\{\omega \cdot n, \omega^\star \cdot n\} \leq_{pc} \{\omega \cdot k, \omega^\star \cdot k\} \;\leftrightarrow\; n \;|\; k.$

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$\{\omega \cdot n, \omega^\star \cdot n\} \leq_{pc} \{\omega \cdot k, \omega^\star \cdot k\} \;\leftrightarrow\; n \mid k.$

Easy to see that for all n, k,

$$\{\omega \cdot n, \omega^\star \cdot n\} \equiv_{tc} \{\omega \cdot k, \omega^\star \cdot k\}.$$

More sample results:

- $\{\omega, \omega^\star\} \leq_{pc} \{\omega^2, (\omega^2)^\star\}$.
- $\{\omega.2, \omega^\star.2\} \leq_{pc} \{\omega^2, (\omega^2)^\star\}$.
- But $\{\omega.n, \omega^\star.n\} \not\leq_{pc} \{\omega^2, (\omega^2)^\star\}$ for $n \geq 3$.

We lack good intuition what happens in general.

Examples

For a **strict** linear order L, by $\tilde{L}$ we denote the preorder such that:

- every equivalence class of the induced equivalence relation $\sim$ is countably infinite;
- the quotient poset $L/_{\sim}$ is isomorphic to L.

It is clear that

$$\{\omega, \omega^\star\} \equiv_{tc} \{\tilde{\omega}, \tilde{\omega}^\star\},$$

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It is clear that

$$\{\omega, \omega^\star\} \equiv_{tc} \{\tilde{\omega}, \tilde{\omega}^\star\},$$

The following is also clear:

$$\{\omega, \omega^\star\} \leq_{pc} \{\tilde{\omega}, \tilde{\omega}^\star\}.$$

Question (Kalimullin)

Is the following true:

$$\{\tilde{\omega}, \tilde{\omega}^\star\} \leq_{pc} \{\omega, \omega^\star\}?$$

Examples

For a linear ordering L, let L_S be the enrichment of L with the successor relation.

Theorem (Ganchev-Kalimullin-V 2018)

$$\{\tilde{\omega}_S, \tilde{\omega}^\star_S\} \not\leq_{pc} \{\omega_S, \omega^\star_S\}.$$

The original question is still open, as far as I know.

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If you consider ω and $\omega^\star$ as **non-strict** linear orderings, then we have the following:

Proposition (B-F-R-S-V 2024)

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Intuition: With $x \leq y$, it may later turn out that $x \sim y$. With $x < y$, it implies that $x \not\sim y$.

Towards a characterization of **Txt**-learning

We introduce *positive infinitary formulas* in a fashion similar to Ivan Soskov's works:

- A Σ^p_0-formula is a finite conjunction of atomic τ-formulas.
- A Π^p_0-formula is a finite disjunction of negations of atomic τ-formulas.
- A Σ^p_1-formula is a formula of the form:

$$\bigvee_{i \in I} \exists \bar{y}_i \psi_i(\bar{x}, \bar{y}_i),$$

where each ψ_i is a Σ^p_0-formula.

- A Π^p_1-formula is a formula of the form:

$$\bigwedge_{i \in I} \forall \bar{y}_i \psi_i(\bar{x}, \bar{y}_i),$$

where each ψ_i is a Π^p_0-formula.

- A Σ^p_{n+2}-formula is a countable disjunction

$$\bigvee_{i \in I} \exists \bar{y}_i (\psi_i(\bar{x}, \bar{y}_i) \,\&\, \xi_i(\bar{x}, \bar{y}_i)),$$

where ψ_i is a Σ^p_{n+1}-formula and ξ_i is a Π^p_{n+1}-formula.

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- A Π^p_{n+2}-formula is a countable conjunction

$$\bigwedge_{i \in I} \forall \bar{y}_i (\psi_i(\bar{x}, \bar{y}_i) \vee \xi_i(\bar{x}, \bar{y}_i)),$$

where ψ_i is a Σ^p_{n+1}-formula and ξ_i is a Π^p_{n+1}-formula.

In a similar way, one could define Σ^p_α-formulas for every countable ordinal α.

Relatively intrisically relations

- Recall that a relation R on $\mathcal{A}$ is relatively intrisically Σ^0_α if in any copy $\mathcal{B}$ of $\mathcal{A}$, $R^{\mathcal{B}}$ is c.e. in $D(\mathcal{A})^{(\alpha)}$.

Theorem (AKMS-C)

A relation R on $\mathcal{A}$ is relatively intrisically Σ^0_α iff R is definable in $\mathcal{A}$ by a Σ^c_α.

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Theorem (AKMS-C)

A relation R on $\mathcal{A}$ is relatively intrisically Σ^0_α iff R is definable in $\mathcal{A}$ by a Σ^c_α.

- For a set A, define $K_A = \{e \mid e \in \Gamma_e(A)\}$, where Γ_e is an enumeration operator.
- Then the enumeration jump of A, $J_e(A) = K_A \oplus \overline{K}_A$.
- A relation R on $\mathcal{A}$ is relatively α-intrinsically reducible to $\mathcal{A}$ if in any copy $\mathcal{B}$ of $\mathcal{A}$, $R^{\mathcal{B}} \leq_e J_e^{(\alpha)}(D_+(\mathcal{A}))$.

Theorem (Soskov)

A relation R on $\mathcal{A}$ is relatively α-intrinsically reducible to $\mathcal{A}$ iff R is Σ^{pc}_α-definable in $\mathcal{A}$.

A Borel hierarchy

Let $\mathcal{X}$ be an **arbitrary** topological space.

Definition (Selivanov, De Brecht)

- The class $\mathbf{\Sigma}^0_1$ contains precisely the open sets in $\mathcal{X}$.
- For a countable $\alpha > 1$, the class $\mathbf{\Sigma}^0_\alpha$ contains the sets $A \subseteq \mathcal{X}$ which can be expressed in the form

$$A = \bigcup_{i \in \omega} (B_i \setminus C_i),$$

where B_i and C_i belong to $\mathbf{\Sigma}^0_{\beta_i}$ for some $\beta_i < \alpha$.

- As usual, a set A belongs to $\mathbf{\Pi}^0_\alpha$ if its complement $\overline{A}$ lies in $\mathbf{\Sigma}^0_\alpha$.

For Polish spaces, this definition is equivalent to the classical one.
$P(\omega)$ **is not Polish, but quasi-Polish** (after de Brecht).

Recall the following classical result which connects $L_{\omega_1,\omega}$-formulas with Borel compexity:

Theorem (Lopez-Escobar; Vaught)

Let $\mathcal{K}$ be a subclass of $Mod(L)$ which is closed under isomorphisms. Let $\alpha > 0$ be a countable ordinal. Then $\mathcal{K}$ is $\mathbf{\Sigma}^0_\alpha$ in $Mod(L)$ if and only if $\mathcal{K}$ is axiomatizable by a Σ^{inf}_α-sentence.

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We prove a new version of this result:

Theorem (B-F-R-S-V 2024)

Let $\mathcal{K}$ be a subclass of $Mod_p(L)$ which is closed under isomorphisms. Let $\alpha > 0$ be a countable ordinal. Then $\mathcal{K}$ is $\mathbf{\Sigma}^0_\alpha$ in the space $Mod_p(L)$ if and only if $\mathcal{K}$ is axiomatizable by a Σ^p_α-sentence.

$\square$

A New version of the Pullback Theorem

A main tool in the syntactic characterization of **Txt**-learning.

Theorem (B-F-R-S-V 2024)

Suppose that $\mathfrak{K}_1 \leq_{Scott} \mathfrak{K}_2$ *via continuous* $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$*.*
Then for any **positive** *infinitary sentence* ψ_2 *in the language of* $\mathfrak{K}_2$*, one can find an* **positive** *infinitary sentence* ψ_1 *in the language of* $\mathfrak{K}_1$ *such that for all* $\mathcal{A} \in \mathfrak{K}_1$*, we have*

$$\mathcal{A} \models \psi_1 \ \leftrightarrow \ \Phi(\mathcal{A}) \models \psi_2.$$

Moreover, for a non-zero $\alpha < \omega_1^{CK}$*, if* ψ_2 *is a* Σ^p_α *sentence, then so is* ψ_2*.*

Characterization of **Txt**-learning

Theorem (B-F-R-S-V CiE2024)

Let $\mathfrak{K} = \{\mathcal{A}_i \mid i \in \omega\}$. The following are equivalent:

- $\mathfrak{K}$ *is* **Txt**-*learnable;*
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- *There are* Σ_2^p *sentences* ψ_i *such that*

$$\mathcal{A}_i \models \psi_j \leftrightarrow i = j.$$

Interestingly, it turns out that the syntactical characterization was obtained by Eric Martin and Daniel Osherson (2000) who work in Algorithmic Learning Theory.

Towards a descriptive set-theoretic characterization for **Txt**-learnability?

For any two $\alpha, \beta \in 2^\omega$,

$$\alpha \, E_0 \, \beta \;\Leftrightarrow\; (\exists n)(\forall m \geq n)(\alpha(m) = \beta(m)).$$

Theorem (Bazhenov-Cipriani-San Mauro)

Let $\mathcal{K} = \{\mathcal{A}_i : i \in \omega\}$. T.F.A.E.

(a) *The family $\mathcal{K}$ is* **Inf**-*learnable.*

(b) *There is a continuous function $\Phi \colon 2^\omega \to 2^\omega$ such that for all $\mathcal{A}, \mathcal{B} \in \mathrm{LD}(\mathcal{K})$, we have:*

$$\mathcal{A} \cong \mathcal{B} \;\Leftrightarrow\; \Phi(\mathcal{A}) \, E_0 \, \Phi(\mathcal{B}).$$

Towards a descriptive set-theoretic characterization for **Txt**-learnability?

For $X \in P(\omega)$ and $m \in \omega$, by $X^{[m]}$ we denote the m-th column of X— i.e., $\{y : \langle m, y\rangle \in X\}$.
The equivalence relation E_{set} is defined as follows:

$$X \, E_{set} \, Y \;\Leftrightarrow\; \{X^{[m]} : m \in \omega\} = \{Y^{[m]} : m \in \omega\}.$$

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Proposition (B-F-R-S-V unpublished)

Let $\mathcal{K} = \{\mathcal{A}_i : i \in \omega\}$ be a family of countable structures. Then the following conditions are equivalent:

(a) The family $\mathcal{K}$ is **Txt**-learnable.

(b) There is a continuous function $\Gamma \colon P(\omega) \to P(\omega)$ such that for all $\mathcal{A}, \mathcal{B} \in \mathrm{LD}(\mathcal{K})$:

  (b.1) $\mathcal{A} \cong \mathcal{B} \;\Leftrightarrow\; \Gamma(\mathcal{A}) \, E_{set} \, \Gamma(\mathcal{B})$;

  (b.2) for each $i \in \omega$, we have $\{\Gamma(\mathcal{A}_i)^{[m]} : m \in \omega\} = \{\omega, \{i\}\}$.

The end

Thank you for your attention!