Almost periodic Functions and the Scott Analysis of Linear Orderings

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July 2024
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Outline

- The definitions and history of Scott analysis
- Classification for linear orderings
- A recent construction using almost periodic functions

$\mathcal{L}_{\omega_1,\omega}$ formulas and their complexity

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- For $\alpha \in \omega_1$, φ is Π^{in}_α if $\varphi = \bigwedge\!\!\!\bigwedge_i \forall \bar{x} \psi_i(\bar{x})$ for $\psi_i \in \Sigma^{\text{in}}_\beta$ with $\beta < \alpha$.

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- For $\alpha \in \omega_1$, φ is Π_α^{in} if $\varphi = \bigwedge\mkern-15mu\bigwedge_i \forall \bar{x} \psi_i(\bar{x})$ for $\psi_i \in \Sigma_\beta^{\mathrm{in}}$ with $\beta < \alpha$.
- For $\alpha \in \omega_1$, φ is d-$\Sigma_\alpha^{\mathrm{in}}$ if $\varphi = \psi \wedge \chi$ for $\psi \in \Sigma_\alpha^{\mathrm{in}}$ and $\chi \in \Pi_\alpha^{\mathrm{in}}$

$\mathcal{L}_{\omega_1,\omega}$ formulas and their complexity

- For two models M, N we say $M \leq_\alpha N$ if $\Pi^{\mathrm{in}}_\alpha - \mathrm{Th}(M) \subseteq \Pi^{\mathrm{in}}_\alpha - \mathrm{Th}(N)$.
- Note that $M \geq_\alpha N$ if and only if $\Sigma^{\mathrm{in}}_\alpha - \mathrm{Th}(M) \subseteq \Sigma^{\mathrm{in}}_\alpha - \mathrm{Th}(N)$.
- We put $M \equiv_\alpha N$ if both of the above hold.

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Definition: *The Scott complexity* (SC) of M is the least among $\{\Sigma^{\mathrm{in}}_\alpha, \Pi^{\mathrm{in}}_\alpha, d\text{-}\Sigma^{\mathrm{in}}_\alpha\}_{\alpha\in\omega_1}$ such that M has a Scott sentence of said complexity.

$$\begin{tikzcd}
 & \Sigma^{\mathrm{in}}_1 \arrow[dr] & & \Sigma^{\mathrm{in}}_2 \arrow[dr] & & \cdots \\
d\text{-}\Sigma^{\mathrm{in}}_0 \arrow[ur] \arrow[dr] & & d\text{-}\Sigma^{\mathrm{in}}_1 \arrow[ur] \arrow[dr] & & d\text{-}\Sigma^{\mathrm{in}}_2 \arrow[ur] \arrow[dr] & \\
 & \Pi^{\mathrm{in}}_1 \arrow[ur] & & \Pi^{\mathrm{in}}_2 \arrow[ur] & & \cdots
\end{tikzcd}$$

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Theorem: [Montalbán] The (un)parameterized Scott rank of M is the $\alpha \in \omega_1$ such that M the automorphism orbits of all tuples in M are definable in a $\Sigma^{\mathrm{in}}_{\alpha}$ way with(out) parameters. There are also *many* other equivalent statements.

Scott analysis

For $T \in \mathcal{L}_{\omega_1,\omega}$ let

$$I_{SC}(T,\Gamma) = |\{M : M \models T \wedge \mathrm{SC}(M) = \Gamma\}|,$$

$$I_u(T,\alpha) = |\{M : M \models T \wedge \mathrm{uSR}(M) = \alpha\}|,$$

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Scott analysis generally refers to any inquiry into the behavior of the above functions.

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4. $I_{SC}(LO, \Sigma_4) = \aleph_0$
5. $I_{SC}(LO, \Pi_3) = \aleph_0$
6. $I_{SC}(LO, \Gamma) = 2^{\aleph_0}$ otherwise

The Relationship of the Concepts

Alvir, Greenberg, Harrison-Trainor and Turetsky (AGHTT) showed that Scott sentence complexity is related to Montalbán's Scott ranks.

SC	pSR	uSR	complexity of parameters
$\Sigma^{\mathrm{in}}_{\alpha+2}$	α	$\alpha+2$	$\Pi^{\mathrm{in}}_{\alpha+1}$
d-$\Sigma^{\mathrm{in}}_{\alpha+1}$	α	$\alpha+1$	$\Pi^{\mathrm{in}}_{\alpha}$
$\Pi^{\mathrm{in}}_{\alpha+1}$	α	α	none
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For limit α, $\Sigma^{\mathrm{in}}_{\alpha}$ and d-$\Sigma^{\mathrm{in}}_{\alpha}$ are not possible.
Notice the limit case is left ambiguous in their work.

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- 38 years later AGHTT gave an example of Scott complexity $\Sigma_{\lambda+1}$.
- The example is very pretty, but quite complex.

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Definition: A λ-*unstable sequence* in $\mathcal{A}$ is a fundamental sequence $\delta_i \to \lambda$ along with tuple $a_i \in \mathcal{A}$ with $a_i \equiv_{\delta_i} a_{i+1}$ and $a_i \not\equiv_{\delta_{i+1}} a_{i+1}$.

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Theorem: $\mathcal{A}$ with $\mathrm{uSR}(\mathcal{A}) = \mathrm{pSR}(\mathcal{A}) = \lambda$ has $\mathrm{SC}(\mathcal{A}) = \Pi_{\lambda+1}$ if and only if it has some λ-unstable sequence.

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Theorem: $\mathcal{A}$ with $\mathrm{uSR}(\mathcal{A}) = \lambda$ and $\mathrm{pSR}(\mathcal{A}) = \lambda + 1$ has $\mathrm{SC}(\mathcal{A}) = \Sigma_{\lambda+1}$ if and only if it has no λ-unstable sequence over some parameters.

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- G., Harrison-Trainor and Ho showed there is a linear ordering of Scott complexity $\Sigma_{\lambda+1}$.
- We will now focus on this construction and its surprising life.

The Construction: almost periodic Functions

Definition: A function $f : \mathbb{Z} \to \mathbb{N}$ is *almost periodic* if for all n $f_n := \min(f, n)$ is periodic but f itself is not periodic.

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Example: Let $\sigma_0 = 0$ and $\sigma_{i+1} = \sigma_i^\frown(i+1)^\frown\sigma_i$. Limits of these finite strings produce almost periodic function.

$\cdots$,4,0, 1, 0, 2, 0, 1, 0, 3,0, 1,0, 2, 0, 1, 0, 5, 0, 1, 0, 2, 0, 1, 0, $\cdots$

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The Construction: λ-mixable pairs

Definition

An ordered pair $(\{L_i\}_{i\in\omega}, K)$ of a sequence of linear orderings $\{L_i\}_{i\in\omega}$ and a single linear ordering K is called a λ-*mixable pair* if the following properties hold for some non-zero fundamental sequence $\delta_i \to \lambda$:

1. $\mathrm{uSR}(K) < \lambda$
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3. $L_i \equiv_{\delta_i} L_{i+1}$
4. $L_i \not\equiv_{\delta_{i+1}} L_{i+1}$
5. any finite alternating sum $1 + L_{a_0} + 1 + K + 1 + L_{a_1} + 1 + K + \cdots + 1 + L_{a_n}$ has intervals isomorphic to K only within the written K blocks (or as the entire written K block).

The Construction: λ-mixable pair examples

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- $L_i = \sum_{n<i}(n + \zeta^{\delta_n}) + \sum_{n \geq i}(n + \zeta^{\delta_i})$ and $K = \omega$.

The Construction

Given any λ-mixable pair $(\{L_i\}_{i\in\omega}, K)$ and almost periodic function $f : \mathbb{Z} \to \mathbb{N}$,

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Example:

$$\cdots + \omega + \zeta + \omega^2 + \zeta + \omega + \zeta + \omega^3 + \zeta + \omega + \zeta + \omega^2 + \zeta + \omega + \cdots$$

Going Further: The Structure of almost periodic Functions

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Translation: $fE_{\mathbb{Z}}g$ if and only if $L_f \cong L_g$. $fE_{fin}g$ if and only if $L_f \equiv_\lambda L_g$.

Going Further: Similar and Simple

There were no known previous examples of a pair $\mathcal{A}, \mathcal{B}$ with $\mathsf{SC}(\mathcal{A}) = \mathsf{SC}(\mathcal{B}) = \Sigma_{\alpha+1}$ with $\mathcal{A} \equiv_\alpha \mathcal{B}$ (even away from the limit).

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Combinatorial solution: There are continuum many E_{fin} classes in a single $E_{\mathbb{Z}}$ class. In fact, $E_0 \leq_B E_{\mathbb{Z}}|_{[b]_{E_{fin}}}$.

Thank you!

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- In his thesis, Matthew Harrison-Trainor carefully produced a Π_2 formula with only models of Scott rank α for any α.
- This was the only known example

Theorem: There is a Scott complexity $\Sigma_{\lambda+1}$ linear ordering L such that if $M \equiv_\lambda L$ then $\mathrm{SC}(M) = \Sigma_{\lambda+1}$.

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Everything works out using a λ-mixable pair that has a unique (i.e. Scott complexity Π_λ) limit structure like $L_i = \sum_{n<i}(n + \zeta^{\delta_n}) + \sum_{n \geq i}(n + \zeta^{\delta_i})$.

A Dual Surprise

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A new paradigm: $\Sigma_{\lambda+1}$ was considered rare for sociological reasons, $\Pi_{\lambda+1}$ is actually rare even if it has very simple examples like ω^λ.