Scott ranks of models of arithmetic

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Remark

PA *is biinterpretable with* $\mathsf{ZF} \setminus \{\mathsf{Infty}\} + \neg\mathsf{Infty} + \mathsf{TC}$.

Some salient classes of models of PA

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- Finitely generated models. Let $K(\mathcal{M}, a)$ be the submodel of $\mathcal{M}$ generated by $\mathrm{Def}(\mathcal{M}, a)$. Then $a \in K(\mathcal{M}, a) \preceq \mathcal{M}$.

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- Recursively saturated models. $\mathcal{M}$ is recursively saturated if every recursive set of formulae with finitely many parameters which is consistent with $\mathrm{Th}((\mathcal{M}, a)_{a \in M})$ is realized in $\mathcal{M}$.

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Remark

Prime models and countable recursively saturated models of PA *are homogeneous.*

The structural complexity of models of PA

Theorem (Montalbán and Rossegger, 2023)

1. *For every* $\alpha \geq \omega$ *there is a model of* PA *of Scott rank exactly* α*.*
2. *Assume that* $\mathcal{M} \models \mathrm{PA}$.
   - *If* $\mathcal{M}$ *is standard, then* $\mathrm{SR}(\mathcal{M}) = 1$*. If* $\mathcal{M}$ *is nonstandard, then* $\mathrm{SR}(\mathcal{M}) \geq \omega$*.*
   - *If* $\mathcal{M} = K(\mathcal{M})$ *is nonstandard, then* $\mathrm{SR}(\mathcal{M}) = \omega$*.*
   - *If* $\mathcal{M}$ *is homogeneous, then* $\mathrm{SR}(\mathcal{M}) \leq \omega + 1$*.*

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   - *If $\mathcal{M}$ is homogeneous, then* $\mathrm{SR}(\mathcal{M}) \leq \omega + 1$.

Question

What are the models of PA *of rank ω?*

Models of rank ω

Theorem (Ł., Szlufik)

Assume $\mathcal{M} \models \mathrm{PA}$ and $a \in M$ is undefinable. Then the automorphism orbit of a is not Σ_{ω}^{inf} definable.

Key ingredients: Σ_n-recursive saturation.

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Lemma

Let $p(x, a)$ be a Σ_n-type over $\mathcal{M}$. Then

1. *If $p(x, a)$ is coded in $\mathcal{M}$, then $p(x, a)$ is realized in $\mathcal{M}$.*
2. *If for some b, $p(x, a) = \{\phi(x, a) \in \Sigma_n : \mathcal{M} \models \phi(b, a)\}$, then it is coded.*

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Proof (1). Assume that c codes $p(x, a)$ and consider a formula $\psi(y)$

$$\exists x \forall \phi < y \big(\phi \in c \rightarrow \mathrm{Sat}_n(\phi, \langle x, a \rangle)\big).$$

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Proof (2). Assume d realizes $p(x, a)$ and consider the type $p'(x, a, d)$

$$\{\phi(d, a) \equiv \ulcorner \phi(v, w) \urcorner \in x : \phi \in \Sigma_n\}$$

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1. *If $p(x, a)$ is coded in $\mathcal{M}$, then $p(x, a)$ is realized in $\mathcal{M}$.*
2. *If $p(x, a)$ is Σ_n-complete and realized in $\mathcal{M}$, then it is coded.*

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2. *If $p(x, a)$ is Σ_n-complete and realized in $\mathcal{M}$, then it is coded.*

Lemma

Suppose $\mathcal{M} \models \mathrm{PA}$, $a, b \in M$. Then, for every n, the following are equivalent:

- *Σ_n^{inf}-types of a, b are the same.*
- *Σ_n-types of a, b are the same.*

Key ingredients: The Ehrenfeucht Lemma

Lemma (Ehrenfeucht)

If $\mathcal{M} \models \mathrm{PA}$, $a \neq b \in M$ and b is definable from a, then the types of a and b are different.

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Then $(\mathcal{M}, R)$ is a forest and we have bRa. Now a, b differ on the following formula

"The shortest path between me and the least element of my connected component is of even length".

Proof sketch

Assume that $a \in M$ is undefinable but the automorphism orbit of a is Σ^{inf}_{ω}-definable.

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$$p(x) := \{\phi(x) \wedge x \neq a : K(\mathcal{M}, a) \models \phi(a), \phi \in \Sigma_n\}.$$

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By induction and the undefinability of a, it follows that $p(x)$ is realized in $K(\mathcal{M}, a)$ and we are done.

Consequences

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If $\mathcal{M} \models \mathrm{Th}(\mathbb{N})$ *is nonstandard, then* $\mathrm{SR}(\mathcal{M}) > \omega$.

Corollary

If $\mathcal{M}$ *is recursively saturated, then* $\mathrm{SR}(\mathcal{M}) = \omega + 1$.

Adding parameters

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Proposition

The models of PA *with the parametrized Scott rank* ω *are exactly the finitely generated models of* PA.

Fragments of PA

Recall that

- the quantifier $\forall x\phi$ ($\exists x\phi$) is *bounded* iff ϕ is of the form $x < t \to \psi$ ($x < t \wedge \psi$) and t does not mention x.
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$$\begin{aligned} \Sigma_0 &= \Pi_0 = \Delta_0 \\ \Sigma_{n+1} &= \exists x \Pi_n \\ \Pi_{n+1} &= \forall x \Sigma_n \end{aligned}$$

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Remark

$\mathrm{I}\Sigma_1$ *proves MRDP theorem, so actually over* $\mathrm{I}\Sigma_1$ Σ_n*-definable sets are* $\exists_n$ *-definable.*

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Corollary

$\mathrm{I}\Sigma_1$ *has SMP.*

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Corollary

$\mathrm{I}\Sigma_1$ *has SMP. But there are nonstandard models of* Π_2*-Cons*$(\mathrm{I}\Sigma_1)$ *of rank* 1.

Extensions

Lemma (Essentially Ehrenfeucht)

Suppose that $\mathcal{M} \models \mathrm{I}\Sigma_{n+1}$ and $a \neq b \in M$ are such that b is Σ_n definable from a. Then Σ_{n+2}-types of a, b are different.

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Proposition

Suppose $\mathcal{M} \models \mathrm{I}\Sigma_{n+1}$ is nonstandard. Then $\mathrm{SR}(\mathcal{M}) \geq n$.