Open Problems
Computable Structure Theory and Interactions 2024
TU Wien July 15–17 2024

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Question 1 by Dino Rossegger

Suppose $R \subseteq \mathbb{N}^k$ is definable by a $\Sigma_1(X)$ formula in $\mathcal{L}_{\omega_1\omega}$ for some comeager set X.

Questions

- Is R $\Sigma_1(\emptyset)$-definable?
- What about the same question as above replacing "comeager" by "co-null"?

Answer (Aguilera, Miller, Rossegger, Villano)

By the relativization of the easy direction of the Ash-Knight-Manasse-Slaman and Chisholm theorem for $\alpha = 1$, if a relation R on a computable structure $\mathcal{A}$ is $\Sigma_1^{in}(X)$ definable, then $R^{\mathcal{B}}$ is X-c.e. for every computable copy $\mathcal{B}$ of $\mathcal{A}$. By the folklore (?) result that a set $A \subseteq \omega$ is computably enumerable if and only if the set $\{X \subseteq \omega : A \text{ is } X\text{-c.e.}\}$ is comeagre we obtain that $R^{\mathcal{B}}$ is c.e. for every computable $\mathcal{B} \cong \mathcal{A}$. Applying the Ash-Knight-Manasse-Slaman and Chisholm theorem we obtain that R is $\Sigma_1^{in}(\emptyset)$-definable.

The same argument works with conull instead of comeagre by replacing the folklore result in the above argument by an argument of de Leeuw-Moore-Shannon-Shapiro (see [DH, Theorem 8.12.1]).

[DH] Downey, Rodney G., and Denis R. Hirschfeldt. 2010. Algorithmic Randomness and Complexity. Springer Science & Business Media.

Further questions (added post-hoc)

These results lead to the following questions:

- If R is Σ_α^{in}-definable for a comeagre (conull) set of X, is R $\Sigma_\alpha^{in}(\emptyset)$?
- In the above we implicitly allowed parameters in our formulas. If we disallow parameters we obtain the following version of Ash-Knight-Manasse-Slaman and Chisholm for Σ_1^{in}-definability: There is a computable function f such that for $\Phi_e \cong \mathcal{A}$, $W_{f(e)} = R^{\Phi_e}$ if and only if R is Σ_1^{in}-definable without parameters. There are many ways one could try to uniformize the above question. For example, what if on a comeagre set of reals we have Σ_1^{in}-definability without parameters but always with different formulas?

Question 2 by Russell Miller

Notations:

- TFAB_n denotes the class of torsion free abelian group of rank n;
- FTD_n denotes the class of fields of trascendence degree n;
- TD_n denotes the class of fields of characteristic 0 and transcendence degree n.

Known results:

- (Hjorth, Thomas) $\mathsf{TFAB}_n <_B \mathsf{TFAB}_{n+1}$ (here $<_B$ denotes Borel reducibility). In contrast,
- (Thomas, Velikovic) For every n, $\mathsf{FTD}_{13} \equiv_B \mathsf{FTD}_n$.

The following is known in the context of Turing computable embeddings (denoted by $\leq_{tc}$).

- (Ho, Knight, Miller) For every n, $\mathsf{TFAB}_n \leq_{tc} \mathsf{FTD}_n$.

Questions:

- $\mathsf{TD}_n \geq_B \mathsf{FTD}_{n+1}$?
- $\mathsf{TD}_n \leq_B \mathsf{TFAB}_n$? The answer is no for $n \geq 13$.
- Same question as above but replacing $\leq_B$ with $\leq_{tc}$. Again, the answer is no for $n \geq 13$.

Question 3 by Stefan Vatev

It is known that for all limit ordinals α, β, $(\alpha, \alpha^*) \equiv_{tc} (\beta, \beta^*)$.
Question:

- What happens if we replace Turing computable embedding with enumeration reducibility?

Definition 1 (Enumeration reducibility) *Let $A, B \subseteq \mathbb{N}$. We say that A is enumeration reducible to B (notation, $A \leq_e B$) if:*

$$(\exists e)(x \in A \iff (\exists \text{ finite } D \subseteq B)(\langle x, D\rangle \in W_e)).$$

For example, $(\omega \cdot n, \omega^* \cdot n) \leq_c (\omega \cdot k, \omega^* \cdot k)$ if and only if $n|k$ (here $\leq_c$ denotes computable embedding).

Question 4 by Gianluca Paolini

1. Is there a computable first-order theory T such that the isomorphism relation for models of T with domain ω is as complicated as graph isomorphism with respect to Borel reducibility?
2. Consider classes where we know that embeddability is not an analytic complete quasi-order such as linear orderings or Boolean algebras. Is elementary embeddability for this classes analytic complete?

Answer to (1)

Yes! We can obtain a reduction $f : \mathit{Graphs} \to \mathit{Graphs}$ such that for all graphs x, y, $f(x) \equiv f(y)$. Essentially, we just have to use the pairs-of-structure theorem of Ash and Knight and do a clever jump inversion. For example the construction given in [Ros] already has the above property. The proof of Lemma 7 in [Ros] already shows mutatis mutandis that $f(x) \equiv f(y)$ for the reduction provided there.

[Ros] Rossegger, Dino. 2022. "Degree Spectra of Analytic Complete Equivalence Relations." The Journal of Symbolic Logic 87 (4): 1663–76. https://doi.org/10.1017/jsl.2021.82.

Question 5 by Mateusz Łełik

Let the Scott function of a theory be the function defined as

$$S_T(\alpha) := |\{\mathcal{M} \models T : SR(\mathcal{M}) = \alpha\}|.$$

Question:

- Is there a non-trivial first-order theory T such that S_T is not monotone or has interesting behavior?

Question 6 by David Gonzalez

What are the possible Scott spectra of first-order theories?