Computable categoricity relative to a degree

Computable Structure Theory and Interactions at Technische Universität Wien in Vienna, Austria

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Outline

1. Notions of categoricity
2. Categoricity relative to a degree

   Above $\mathbf{0}''$ and below $\mathbf{0}'$

   Generalizing the DHM result
3. Extensions of current work

   Embedding lattices

   Categoricity relative to a generic degree

   Focusing on structures
4. Proof sketch of the poset result

Notions of categoricity

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These notions are not equivalent in general. Gončarov [Gon77] built the first example of a structure which was computably categorical but *not* relatively computably categorical, using an enumeration result by Selivanov [Sel76].

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Categoricity relative to a degree

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We first begin with the following result from [DHTM21].

Fact (Downey, Harrison-Trainor, Melnikov [DHTM21])

If $\mathcal{A}$ is a computable structure and it is computably categorical relative to some degree $\mathbf{d} \geq \mathbf{0}''$, then $\mathcal{A}$ has a $\mathbf{0}''$-computable Σ_1^0 Scott family. In particular, $\mathcal{A}$ is computably categorical relative to all $\mathbf{d} \geq \mathbf{0}''$.

The cone above $0''$

We sketch the proof of this fact. We first need the following results.

Theorem (Ash, Knight, Manasse, and Slaman [Ash+89]; Chisholm [Chi90])

A structure is relatively computably categorical if and only if it has a formally Σ_1 Scott family.

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Theorem (Gončarov [Gon80] (relativized))

If a structure is computably categorical relative to $\mathbf{d}$ and its $\forall\exists$ theory is $\mathbf{d}$-decidable, then it has a Scott family of $\exists$-formulas that is c.e. in $\mathbf{d}$.

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The cone above $0''$: proof sketch

Proof sketch.

(1) Suppose $\mathcal{A}$ is computably categorical relative to a degree $\mathbf{d} \geq \mathbf{0}''$. Since $\mathcal{A}$ is computable, its $\forall\exists$ diagram is computable from $\mathbf{0}''$ and hence from $\mathbf{d}$.

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(3) We can use $\mathbf{0}''$ to enumerate this Scott family of $\exists$-formulas, and so this is a formally Σ_1 Scott family relative to $\mathbf{0}''$.

Using this Scott family, we can computably build isomorphisms, and so for every $\mathbf{d} \geq \mathbf{0}''$, $\mathcal{A}$ is computably categorical relative to $\mathbf{d}$. The DHM fact implies that for *any* computable structure $\mathcal{A}$, either it is computably categorical relative to *all* degrees above $\mathbf{0}''$ or to *no* degree above $\mathbf{0}''$.

Below $\mathbf{0}'$

In the c.e. degrees, being computably categorical relative to a degree is *not* monotonic.

Theorem (Downey, Harrison-Trainor, Melnikov [DHTM21])

There is a computable structure $\mathcal{A}$ *and c.e. degrees* $\mathbf{0} = Y_0 <_T X_0 <_T Y_1 <_T X_1 <_T \dots$ *such that*

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The structure they constructed to witness this was a directed graph.

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Theorem (V. [Vil24])

Let $\mathcal{P} = (P, \leq)$ be a computable partially ordered set and let $P = P_0 \sqcup P_1$ be a computable partition. Then, there exists a computable directed graph $\mathcal{G}$ and an embedding h of $\mathcal{P}$ into the c.e. degrees where

(1) *$\mathcal{G}$ is computably categorical;*

(2) *$\mathcal{G}$ is computably categorical relative to each degree in $h(P_0)$; and*

(3) *$\mathcal{G}$ is not computably categorical relative to each degree in $h(P_1)$.*

Extensions of current work

Future directions: embedding a lattice

The techniques utilized in proving the poset result can also be combined with the usual techniques to construct minimal pairs.

Theorem (V. [Vil24])

There exists a computable computably categorical directed graph $\mathcal{G}$ and c.e. sets X_0 and X_1 such that

(1) *X_0 and X_1 form a minimal pair,*

(2) *$\mathcal{G}$ is not computably categorical relative to X_0 and to X_1, and*

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Question

Can you embed bigger distributive lattices into the c.e. degrees in a manner similar to the poset result?

Another question you can ask is the following.

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Given an arbitrary noncomputable c.e. set D, can you always build a computable graph $\mathcal{G}$ where

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Conjecture

Given an arbitrary noncomputable c.e. set D, there is a computable graph $\mathcal{G}$ which is computably categorical and not computably categorical relative to D, and vice-versa.

Future directions: in the generic degrees

Definition

A degree $\mathbf{d}$ is **low for isomorphism** if for every pair of computable structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cong_{\mathbf{d}} \mathcal{B}$ if and only if $\mathcal{A} \cong_{\Delta^0_1} \mathcal{B}$.

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Every 2-generic degree is low for isomorphism.

This means that there *cannot* be a computable structure $\mathcal{A}$ which is not computably categorical but is computably categorical relative to $\mathbf{d}$ for a 2-generic degree $\mathbf{d}$.

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Conjecture

There exists a (properly) 1-generic G such that there is a computable directed graph $\mathcal{A}$ where $\mathcal{A}$ is not computably categorical but is computably categorical relative to G.

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Theorem (Bazhenov [Baz14])

For every degree $\mathbf{d} < \mathbf{0}'$, a computable Boolean algebra is $\mathbf{d}$-computably categorical if and only if it is computably categorical.

Future directions: identifying pathological behavior in classes of structures

Corollary (from results in [Hir+02] and [Mil+18])

For the following classes of structures, there exists a computable example in each class which witnesses the pathological behavior in the poset result:

(1) *symmetric, irreflexive graphs; partial orderings; lattices; rings with zero-divisors; integral domains of arbitrary characteristic; commutative semigroups; and* 2*-step nilpotent groups (Theorem* 1.22 *of [Hir+02])*

(2) *countable fields (Theorem* 1.8 *of [Mil+18])*

Proof sketch of the poset result

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We also have the following notation for convenience.

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$$\overline{D_p} := \bigoplus_{q \neq p} A_q.$$

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- $\mathcal{M}_e$ is the eth (partial) computable graph with domain ω where $E(x,y) \iff \Phi_e(x,y) = 1$ and $\neg E(x,y) \iff \Phi_e(x,y) = 0$.

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- $\mathcal{M}_i^{D_p}$ is the ith (partial) D_p-computable graph with domain ω where $E(x, y) \iff \Phi_i^{D_p}(x, y) = 1$ and $\neg E(x, y) \iff \Phi_i^{D_p}(x, y) = 0$.

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- for $q \in P_1$, R_e^q : $\Phi_e^{D_q} : \mathcal{G} \to \mathcal{B}_q$ is not an isomorphism where $\mathcal{B}_q$ is a D_q-computable copy of $\mathcal{G}$ we build.

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Building $\mathcal{G}$ in stages

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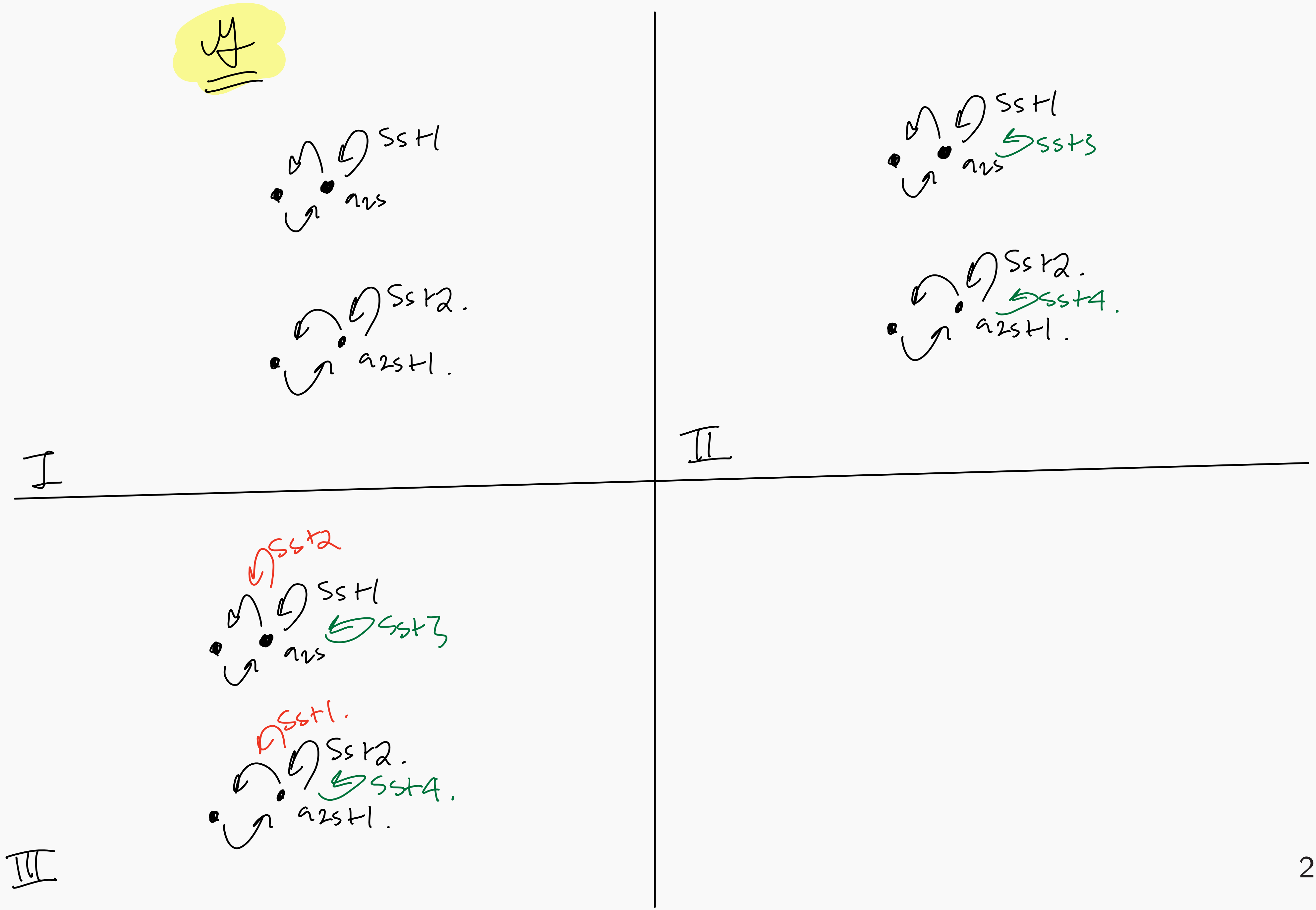
Definition

The root node a_{2s} in our graph $\mathcal{G}$ with its loops is the $2s$**th connected component** or just the $2s$th component of $\mathcal{G}$.

Configuration of loops in $\mathcal{G}$



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2. If n_α is defined and $f_\alpha[s-1]$ is defined for all $m < n_\alpha$, α looks for copies of the $2n_\alpha$th and $(2n_\alpha + 1)$st components of $\mathcal{G}[s]$.

Basic strategies: S_e

3. If no copies of the $2n_\alpha$th and $(2n_\alpha + 1)$st components are found, α takes no additional action at stage s, retains the value of n_α, and sets $f_\alpha[s] = f_\alpha[s-1]$.

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3. If no copies of the $2n_\alpha$th and $(2n_\alpha + 1)$st components are found, α takes no additional action at stage s, retains the value of n_α, and sets $f_\alpha[s] = f_\alpha[s-1]$. If copies are found, α extends $f_\alpha[s-1]$ to $f_\alpha[s]$ by matching the components in $\mathcal{G}[s]$ to the copies found in $\mathcal{M}_e[s]$ and increments n_α by 1.

Basic strategies: T_i^p

Let $p \in P_0$. Our basic strategy to satisfy all T_i^p requirements to make $\mathcal{G}$ computably categorical relative to D_p is similar to our S_e-strategy. Let α be a T_i^p-strategy.

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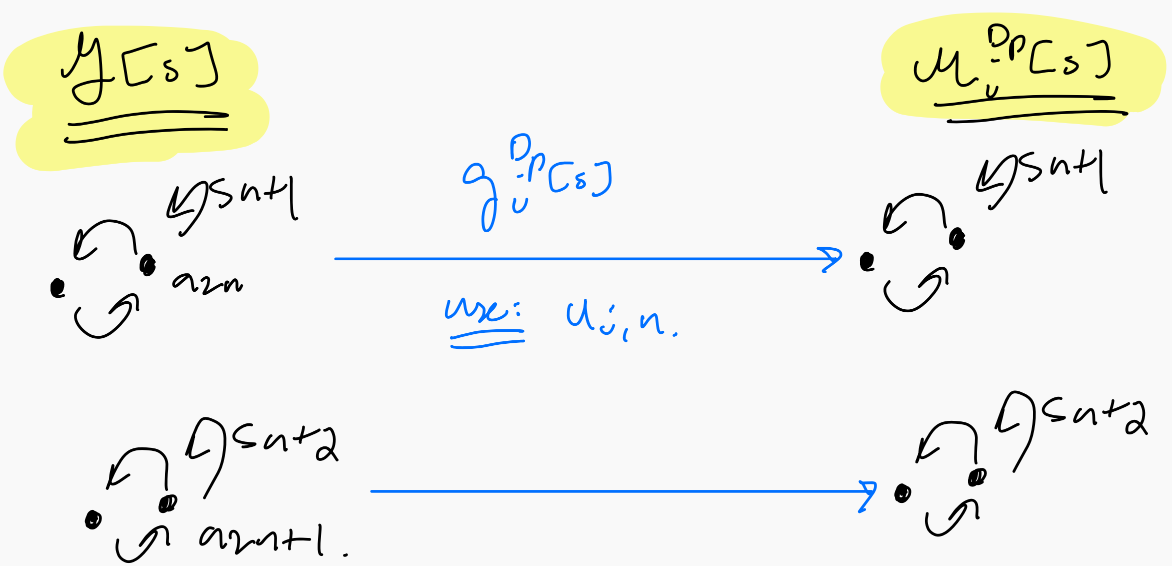

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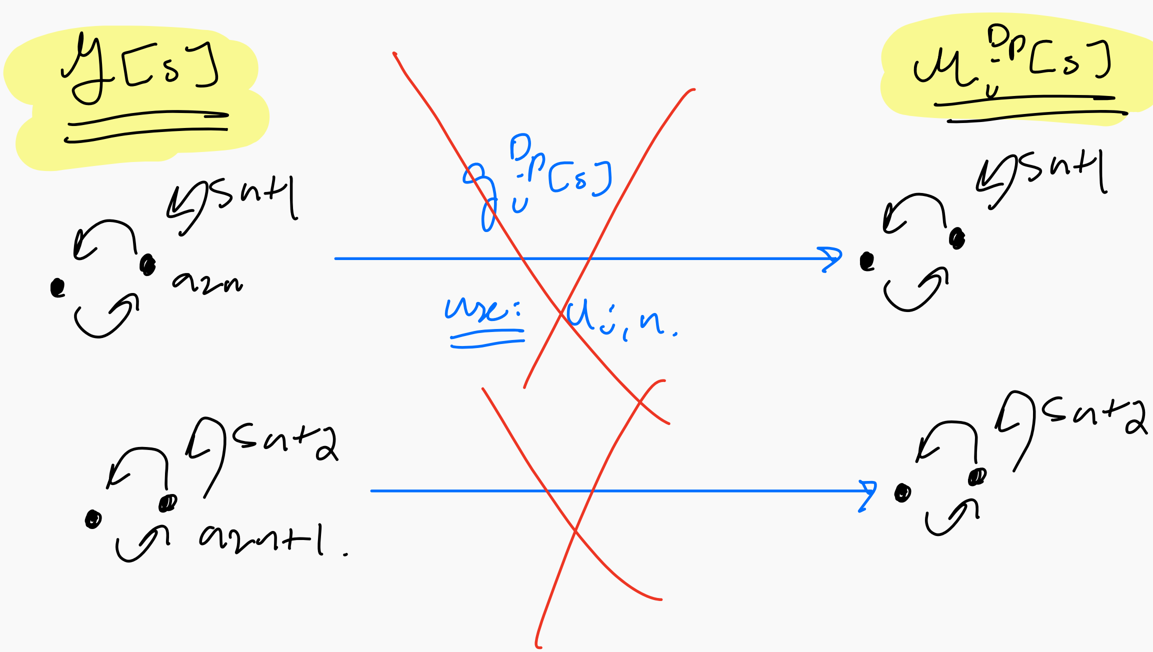

* if $\exists k$ s.t. (1) $k < u_{i,n}$ and (2) $k \hookrightarrow D_p[t]$ for $t > s$, then $g_i^{D_p}$ on these components disappear.

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If $D_p[t] \neq D_p[s]$, then α will update its parameter n_α accordingly depending on what type of injury occurred. Otherwise, it will proceed to try and match the $2n_\alpha$th and $(2n_\alpha + 1)$st components of $\mathcal{G}$ for the n_α parameter it had at the beginning of stage s.

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Finally, for $q \in P_1$, we do the following to satisfy all R_e^q requirements to make $\mathcal{G}$ not computably categorical relative to D_q.

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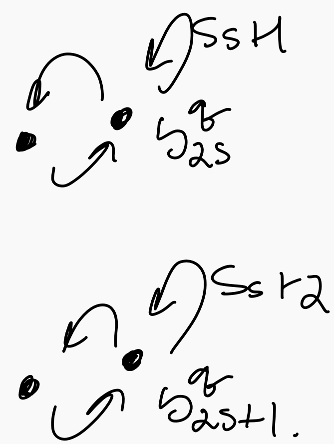
$\underline{\mathcal{B}_q}$



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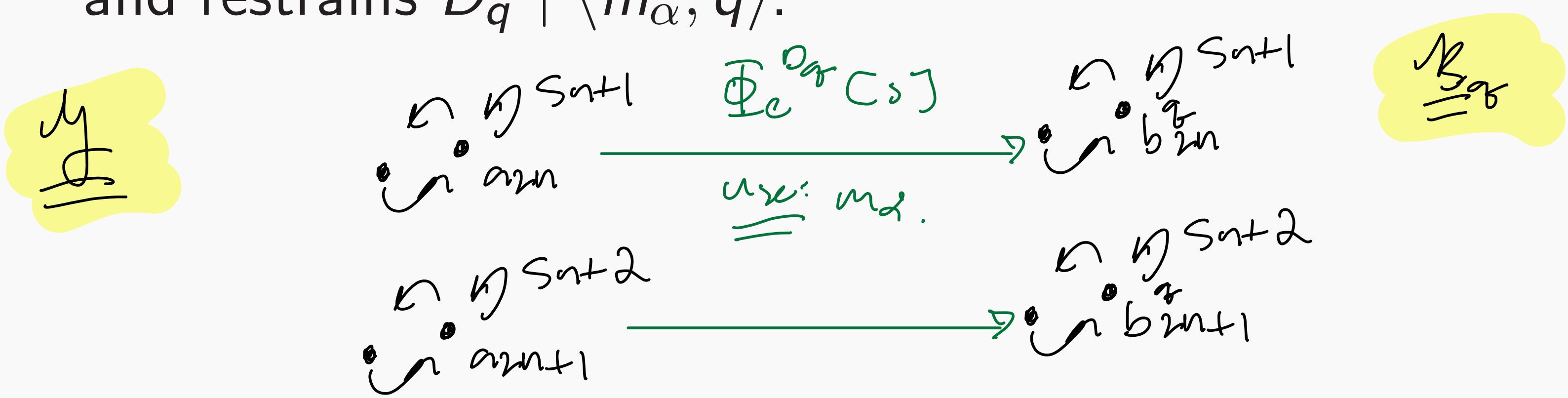

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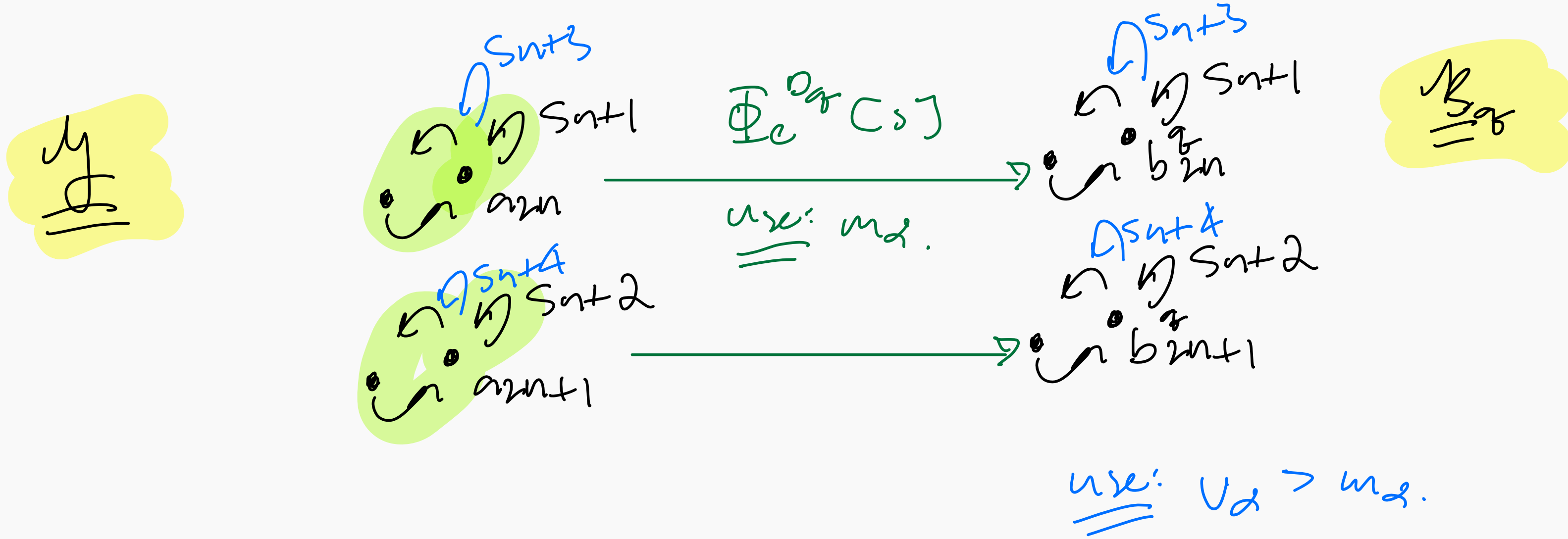
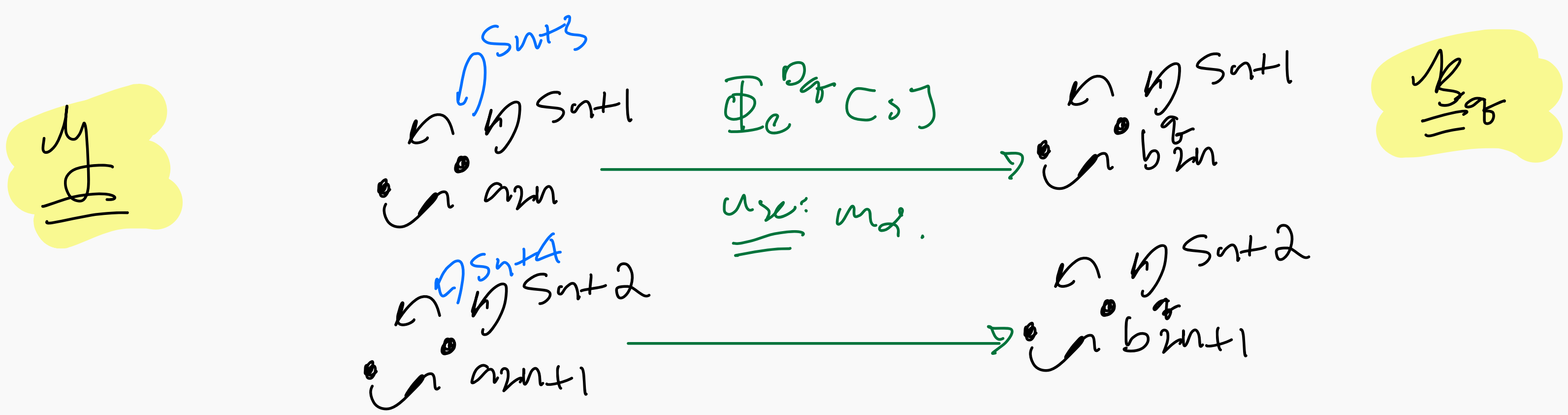
4. α now issues a challenge to all higher priority requirements which are S_e and T_i^p: they must now extend their embeddings, if possible, to include these new loops.



5. If all higher S_e and T_i^p requirements can meet this challenge and α becomes eligible to act again at a later stage, it enumerates v_α into A_q. This makes the $(5n+3)$- and $(5n+4)$-loops in $\mathcal{B}_q$ disappear.



6. α reattaches a $(5n+3)$-loop to b^q_{2n+1} and a $(5n+4)$-loop to b^q_{2n}. It also attaches a $(5n+1)$-loop to a_{2n+1} and to b^q_{2n+1}, and a $(5n+2)$-loop to a_{2n} and to b^q_{2n}.

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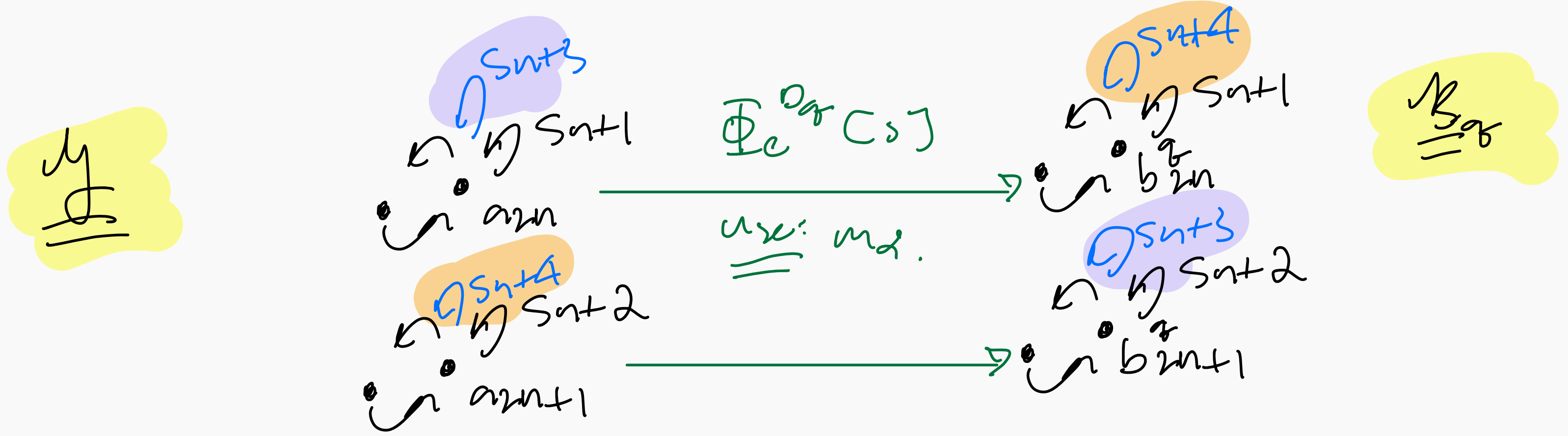
Our final configuration of loops in $\mathcal{B}_q$ is now:



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The N_e^p-strategy must enumerate numbers into A_p to achieve independence of degrees: this is resolved on a tree of strategies and by letting T_i^p check for any changes in D_p up to a finite part each stage.

Interactions between strategies

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An R_e^q-strategy β and a T_i^p-strategy α when $q < p$ in P and T_i^p is of higher priority than R_e^q:

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An R_e^q-strategy β and a T_i^p-strategy α when $q < p$ in P and T_i^p is of higher priority than R_e^q: the T_i^p-strategy needs an additional step for when it is challenged to enumerate any uses associated to the $2n_\beta$th and $(2n_\beta + 1)$st components of $\mathcal{G}$ into A_p. This lets us lift uses for T_i^p so it can succeed.

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Thanks

Thank you for your attention!

I'd be happy to answer any questions.