Complexity of cohesive powers

Paul Shafer
University of Leeds
p.e.shafer@leeds.ac.uk
https://peshafer.github.io

Computable Structure Theory and Interactions
Technische Universität Wien
17 July 2024

Part II joint with David Gonzalez

Part I: Survey of cohesive powers

Reminder: cohesive sets

Let

$$\vec{A} = (A_0, A_1, A_2, \dots)$$

be a countable sequence of subsets of $\mathbb{N}$.

Then there is an **infinite** set $C \subseteq \mathbb{N}$ such that for every i:

$$\begin{aligned} &\text{either } C \subseteq^* A_i \\ &\quad\text{or } C \subseteq^* \mathbb{N} \setminus A_i. \end{aligned}$$

C is called **cohesive** for $\vec{A}$, or simply $\vec{A}$-**cohesive**.

If $\vec{A}$ is the sequence of recursive sets, then C is called **r-cohesive**.

If $\vec{A}$ is the sequence of r.e. sets, then C is called **cohesive**.

Skolem's countable non-standard model of true arithmetic

Skolem (1934):
Let C be cohesive for the sequence of arithmetical sets.

Consider arithmetical functions $f, g\colon \mathbb{N} \to \mathbb{N}$. Define:

$$\begin{aligned} f =_C g \quad & \text{if} \quad C \subseteq^* \{n : f(n) = g(n)\} \\ f < g \quad & \text{if} \quad C \subseteq^* \{n : f(n) < g(n)\} \\ (f+g)(n) \quad & = \quad f(n) + g(n) \\ (f \times g)(n) \quad & = \quad f(n) \times g(n) \end{aligned}$$

Let $[f] = \{g : g =_C f\}$ denote the $=_C$-equivalence class of f.

Form a structure $\mathcal{M}$ with domain $\{[f] : f \text{ arithmetical}\}$ and

$$[f] < [g] \text{ if } f < g; \qquad [f] + [g] = [f+g]; \qquad [f] \times [g] = [f \times g].$$

Then $\mathcal{M}$ models true arithmetic!

- The **standard** elements of $\mathcal{M}$ are those represented by constant functions.
- The **non-standard** part of $\mathcal{M}$ is everything else. Note $[\mathrm{id}]$ is non-standard.

Effectivizing Skolem's construction

Tennenbaum wanted to know:
What if we did Skolem's construction, but

- used recursive functions $f\colon \mathbb{N} \to \mathbb{N}$ in place of arithmetical functions;
- only assumed that C is r-cohesive?

Do we still get models of true arithmetic?

Feferman-Scott-Tennenbaum (1959):
It is not even possible to get models of Peano arithmetic in this way.

Let $\prod_C(\mathbb{N}; <, +, \times)$ denote the **cohesive power** of the standard model of arithmetic over the (r-)cohesive set C.

Theorem (Lerman 1970)

Let C and D be Π_1 cohesive sets. Then $\prod_C(\mathbb{N}; <, +, \times)$ and $\prod_D(\mathbb{N}; <, +, \times)$ are elementary equivalent if and only if $C \equiv_{\mathrm{m}} D$.

Effective powers of $(\mathbb{N}; <, +, \times)$ and rigidity

A long line of work studies various flavors of effective ultrapowers of $(\mathbb{N}; <, +, \times)$.

Two popular flavors:

- **Recursive ultrapower** (aka **Nerode semi-ring**): Built from total recursive functions via an ultrafilter in the Boolean algebra of recursive sets.
- **R.e. ultrapower**: Built from partial recursive functions via a maximal filter in the lattice of r.e. sets.

Much of the focus is on questions of **rigidity** and **total rigidity**.

- **Rigid**: No non-identity automorphism.
- **Totally rigid**: No non-identity self-embedding.

Effective powers of $(\mathbb{N}; <, +, \times)$ and rigidity

Theorem (Hirschfeld & Wheeler 1975)

R.e. ultrapowers of $(\mathbb{N}; <, +, \times)$ *are rigid.*

Theorem (McLaughlin 2007)

Recursive ultrapowers of $(\mathbb{N}; <, +, \times)$ *are totally rigid.*

Theorem (Shavrukov 2020)

R.e. ultrapowers of $(\mathbb{N}; <, +, \times)$ *are totally rigid.*

Cohesive powers

Dimitrov (2009):
Let $\mathcal{A}$ be a computable structure.
(i.e., $\mathcal{A}$ has domain $\mathbb{N}$ and recursive functions and relations.)

Let C be cohesive. Form the **cohesive power** $\prod_C \mathcal{A}$ of $\mathcal{A}$ over C:

Consider partial recursive $\varphi, \psi \colon \mathbb{N} \to \mathbb{N}$ with $C \subseteq^* \mathrm{dom}(\varphi)$. Define:

$$\begin{array}{lcl} \varphi =_C \psi & \text{if} & C \subseteq^* \{n : \varphi(n) = \psi(n)\} \\ R(\psi_0, \dots, \psi_{k-1}) & \text{if} & C \subseteq^* \{n : R(\psi_0(n), \dots, \psi_{k-1}(n))\} \\ F(\psi_0, \dots, \psi_{k-1})(n) & = & F(\psi_0(n), \dots, \psi_{k-1}(n)) \end{array}$$

Let $[\varphi]$ denote the $=_C$-equivalence class of φ.

Let $\prod_C \mathcal{A}$ be the structure with domain $\{[\varphi] : C \subseteq^* \mathrm{dom}(\varphi)\}$ and

$$\begin{array}{lcl} R([\psi_0], \dots, [\psi_{k-1}]) & \text{if} & R(\psi_0, \dots, \psi_{k-1}) \\ F([\psi_0], \dots, [\psi_{k-1}]) & = & [F(\psi_0, \dots, \psi_{k-1})]. \end{array}$$

Decidability, n-decidability, and a little Łoś

A computable structure $\mathcal{A}$ is:

- **decidable** if its elementary diagram is recursive
- **n-decidable** if its Σ_n-elementary diagram is recursive.

The following is due to Dimitrov, Harizanov, Morozov, (S), A. Soskova, and Vatev, building on work of Dimitrov.

Theorem

Let $\mathcal{A}$ be a computable structure, C be cohesive, $\Phi(v)$ a first-order formula, and $[\varphi]$ an element of $\prod_C \mathcal{A}$.

- *If $\mathcal{A}$ is n-decidable and Φ is Π_{n+2}, then*

$$\forall^\infty i \in C \;\; \mathcal{A} \models \Phi(\varphi(i)) \quad \textit{implies} \quad \prod_C \mathcal{A} \models \Phi([\varphi]).$$

- *If $\mathcal{A}$ is decidable, then*

$$\forall^\infty i \in C \;\; \mathcal{A} \models \Phi(\varphi(i)) \quad \textit{if and only if} \quad \prod_C \mathcal{A} \models \Phi([\varphi]).$$

Cohesive powers and saturation

A structure is:

- **recursively saturated** if it realizes every recursive type
- **Σ_n-recursively saturated** if it realizes every recursive type of Σ_n formulas.

Let $\mathcal{A}$ be a computable structure and C be cohesive.

- If $\mathcal{A}$ is decidable, then $\prod_C \mathcal{A}$ is recursively saturated. (Essentially Nelson).
- If $\mathcal{A}$ is n-decidable for $n \geq 1$, then $\prod_C \mathcal{A}$ is Σ_n-recursively saturated. (Dimitrov, Harizanov, Morozov, (S), A. Soskova, and Vatev).
- If $\mathcal{A}$ is n-decidable and C is Π_1, then $\prod_C \mathcal{A}$ is Σ_{n+1}-recursively saturated. (Dimitrov, Harizanov, Morozov, (S), A. Soskova, and Vatev).
- **If $\mathcal{A}$ is n-decidable and C is Δ_2, then $\prod_C \mathcal{A}$ is Σ_{n+1}-recursively saturated.** ((S), building on the above).

The point: If C is Δ_2, then we get one more level of saturation (and also $n = 0$).

Cohesive powers of recursive linear orders

We investigated cohesive powers of recursive linear orders to show how different presentations of the same structure can produce different cohesive powers.

For example:
(S), building on Dimitrov, Harizanov, Morozov, (S), A. Soskova, and Vatev

There are recursive linear orders $\mathcal{L}$ and $\mathcal{M}$, both of order-type ω, such that:

- $\prod_C \mathcal{L} \;\cong\; \omega + \zeta\eta$ for every cohesive set C
- $\prod_C \mathcal{M} \;\cong\; \omega + \eta$ for every $\boldsymbol{\Delta_2}$ cohesive set C.

Here $\omega = \mathrm{otype}(\mathbb{N}; <)$, $\zeta = \mathrm{otype}(\mathbb{Z}; <)$, and $\eta = \mathrm{otype}(\mathbb{Q}; <)$.

The orders $\mathcal{L}$ and $\mathcal{M}$ are **isomorphic** but not **recursively isomorphic**.

What other cohesive powers of copies of ω are possible?

Order-types $\boldsymbol{\omega + \zeta\eta}$ and $\boldsymbol{\omega + \eta}$ appear as cohesive powers of recursive copies of ω.

What other order-types are possible?

Order-types $\boldsymbol{\omega +}$ **certain shuffles** are possible!

For example, the order-type $\boldsymbol{\omega + \sigma(\{2, 3\})}$ is possible.

Building on Dimitrov, Harizanov, Morozov, (S), A. Soskova, and Vatev:

Theorem (S)

Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Boolean combination of Σ_2 sets. There is a recursive copy $\mathcal{M}$ of ω such that for every Δ_2 cohesive C,

$$\prod\nolimits_C \mathcal{M} \;\cong\; \omega + \boldsymbol{\sigma}\big(X \cup \{\omega + \zeta\eta + \omega^*\}\big).$$

Moreover, if X is finite, then $\omega + \zeta\eta + \omega^*$ can be removed.

Part II: Complexity of cohesive powers

Joint with David Gonzalez

Are non-recursive order-types possible?

We saw that if $X \subseteq \mathbb{N} \setminus \{0\}$ is a Boolean combination of Σ_2 sets, then

$$\omega + \boldsymbol{\sigma}\big(X \cup \{\omega + \zeta\eta + \omega^*\}\big)$$

appears as a cohesive power of a recursive copy of ω.

However, by **Ash, Jockusch, Knight**:

If $X \subseteq \mathbb{N} \setminus \{0\}$ is a Σ_3 set, then there is a recursive linear order of type $\omega + \boldsymbol{\sigma}\big(X \cup \{\omega + \zeta\eta + \omega^*\}\big)$.

On the other hand:

- If $\mathcal{L}$ is a recursive linear order, then $\{n : \mathcal{L} \text{ contains a block of size } n\}$ is Σ_3.
- Hence if X is (say) Π_3 but not Σ_3, then $\omega + \boldsymbol{\sigma}\big(X \cup \{\omega + \zeta\eta + \omega^*\}\big)$ is **not** a recursive order-type.

Question:
If $X \subseteq \mathbb{N} \setminus \{0\}$ is Π_3, does $\omega + \boldsymbol{\sigma}\big(X \cup \{\omega + \zeta\eta + \omega^*\}\big)$ appear as a cohesive power of a recursive copy of ω?

Are non-recursive order-types possible?

Indeed, is it possible to achieve non-recursive order-types at all?

Questions:

- Is there a recursive copy $\mathcal{L}$ of ω such that $\prod_C \mathcal{L}$ has non-recursive order-type for every cohesive C? For every Δ_2 cohesive C? For some cohesive C?
- Is there a recursive linear order $\mathcal{L}$ such that $\prod_C \mathcal{L}$ has non-recursive order-type for every cohesive C? For every Δ_2 cohesive C? For some cohesive C?
- Is there uniformly recursive sequence of linear orders $(\mathcal{L}_n)_n$ such that $\prod_C \mathcal{L}_n$ has non-recursive order-type for every cohesive C? For every Δ_2 cohesive C? For some cohesive C?

It looks like the answer to the last question is **yes** at least.

We think there is a uniformly recursive sequence of linear orders $(\mathcal{L}_n)_n$ such that the **cohesive product** $\prod_C \mathcal{L}_n$ has non-recursive order type for every cohesive C.

The idea is to use Jockusch & Soare-style techniques for diagonalizing against recursive linear orders.

How complicated are cohesive powers anyway?

Put a pin in the questions about cohesive powers of recursive linear orders for now.

Instead ask:
If $\mathcal{A}$ is a computable structure and C is cohesive, then how complicated is $\prod_C \mathcal{A}$?

Potentially this depends on the complexity of C.

So we stick to Δ_2 cohesive sets.

Note that there are differences between **powers over $\mathbf{\Delta_2}$ cohesive sets** and **powers over $\mathbf{\Pi_2}$ cohesive sets**:

Example (S):

- There is a computable copy $\mathcal{L}$ of ω such that $\prod_C \mathcal{L} \cong \omega + \eta$ for every Δ_2 cohesive C.
- For every computable copy $\mathcal{L}$ of ω, there is a Π_2 cohesive C such that $\prod_C \mathcal{L} \ncong \omega + \eta$.

Presenting cohesive powers over Δ_2 cohesive sets

Fact:
If $\mathcal{A}$ is a computable structure and C is a Δ_2 cohesive set, then $\prod_C \mathcal{A}$ has a Δ_3 presentation.

Represent elements of $\prod_C \mathcal{A}$ by pairs $\langle e, N \rangle$ where

$$\underbrace{\forall n > N \left(n \in C \;\rightarrow\; \varphi_e(n)\downarrow\right)}_{\Pi_2 \text{ formula } D(\langle e, N\rangle)}$$

We need to identify when $\langle e, N \rangle$ and $\langle i, M \rangle$ represent the same element. Define:

$$\langle e, N \rangle \sim \langle i, M \rangle \;\Leftrightarrow$$
$$\underbrace{D(\langle e, N\rangle) \;\wedge\; D(\langle i, M\rangle) \;\wedge\; \exists K\, \forall n > K \left(n \in C \;\rightarrow\; \varphi_e(n) = \varphi_i(n)\right)}_{\Sigma_3 \text{ property}}$$

By cohesiveness:

$$\langle e, N \rangle \nsim \langle i, M \rangle \;\Leftrightarrow$$
$$\underbrace{\neg D(\langle e, N\rangle) \;\vee\; \neg D(\langle i, M\rangle) \;\vee\; \exists K\, \forall n > K \left(n \in C \;\rightarrow\; \varphi_e(n) \neq \varphi_i(n)\right)}_{\Sigma_3 \text{ property}}$$

Presenting cohesive powers over Δ_2 cohesive sets

Thus $\langle e, N\rangle \sim \langle i, M\rangle$ is a Δ_3 relation. So the set X of least representatives is Δ_3:

$$X = \big\{\langle e, N\rangle \ : \ D(\langle e, N\rangle) \ \wedge \ \forall \langle i, M\rangle < \langle e, N\rangle \ \big(\langle e, N\rangle \nsim \langle i, M\rangle\big)\big\}.$$

For simplicity, let's say $\mathcal{A}$ has one binary relation R.

By reasoning as above, the following relation S is Δ_3:

$$S(\langle e, N\rangle, \langle i, M\rangle) \ \Leftrightarrow$$
$$D(\langle e, N\rangle) \ \wedge \ D(\langle i, M\rangle) \ \wedge \ \exists K \, \forall n > K \ \big(n \in C \ \rightarrow \ R(\varphi_e(n), \varphi_i(n))\big).$$

Then $(X, X^2 \cap S)$ is a Δ_3 presentation of $\prod_C \mathcal{A}$.

Achieving the maximum complexity

Theorem (Gonzalez & S)

There is a recursive graph $\mathcal{G}$ such that for every cohesive set C, every presentation of $\prod_C \mathcal{G}$ computes $0''$.

So if we restrict to Δ_2 cohesive sets C:

- every $\prod_C \mathcal{G}$ has a $0''$-recursive presentation, and
- every presentation of every $\prod_C \mathcal{G}$ computes $0''$.

Idea:

- Code Σ_3 facts about arithmetic into Σ_1 facts about $\prod_C \mathcal{G}$.
- Then both $k \in 0''$ and $k \notin 0''$ can be coded into Σ_1 facts about $\prod_C \mathcal{G}$. That is, $0'' \oplus \overline{0''}$ becomes r.e. in all presentations of $\prod_C \mathcal{G}$.
- So every presentation of $\prod_C \mathcal{G}$ computes $0''$.

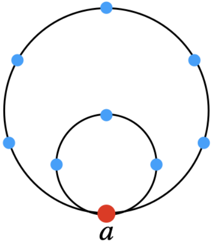
Achieving the maximum complexity

The plan:

Let $\Phi(k)$ be a Σ_3 formula.

Arrange for $\prod_C \mathcal{G}$ to have a vertex $\boldsymbol{a}$ such that for all k:

$$\Phi(k) \quad \Leftrightarrow \quad \boldsymbol{a} \text{ lies on a } (\langle k, \ell \rangle + 3)\text{-cycle for some } \ell.$$



The lengths of the cycles at $\boldsymbol{a}$ determine the k for which $\Phi(k)$ holds.

Achieving the maximum complexity

Quick reminder of the Łoś theorem for cohesive powers

Let:

- $\mathcal{A}$ be a computable structure,
- C be cohesive,
- $\Phi(v)$ be a $\mathbf{\Pi_2}$ **formula**, and
- $[\varphi]$ be an element of $\prod_C \mathcal{A}$.

Then:

$$\forall^\infty i \in C \;\; \mathcal{A} \models \Phi(\varphi(i)) \quad \text{implies} \quad \prod\nolimits_C \mathcal{A} \models \Phi([\varphi]).$$

Achieving the maximum complexity

To compute $\mathcal{G}$:

Let $\Phi(k)$ be $\exists \ell \, \forall i \, \exists j \, \Psi(k, \ell, i, j)$, where Ψ is Σ_0.

$\mathcal{G}$ has distinguished vertices $a_0, a_1, a_2, \dots$.

In $\prod_C \mathcal{G}$, $\boldsymbol{a}$ will be the element represented by $n \mapsto a_n$.

Add a cycle of length $\langle k, \ell \rangle + 3$ at a_n whenever we notice that:

$$\forall i \leq n \, \exists j \; \Psi(k, \ell, i, j).$$

This works because:

- If $\forall i \, \exists j \, \Psi(k, \ell, i, j)$ holds, then there is a cycle of length $\langle k, \ell \rangle + 3$ at every a_n in $\mathcal{G}$. Thus there is a cycle of length $\langle k, \ell \rangle + 3$ at $\boldsymbol{a}$ in $\prod_C \mathcal{G}$ by Łoś.
- If $\forall i \, \exists j \, \Psi(k, \ell, i, j)$ fails, then **for almost every $\boldsymbol{n}$** there is **no** cycle of length $\langle k, \ell \rangle + 3$ at a_n in $\mathcal{G}$. Thus there is **no** cycle of length $\langle k, \ell \rangle + 3$ at $\boldsymbol{a}$ in $\prod_C \mathcal{G}$ by Łoś.

Danke!

Thank you for coming to my talk!
Do you have a question about it?