COMPUTABLE SCOTT SENTENCES AND THE WEAK WITHEAD PROBLEM FOR FINITELY PRESENTED GROUPS

VIENNA, 15/07/24

We steal a bit from the excellent survey:

M. Harrison-Trainor. *An introduction to the Scott complexity of countable structures and a survey of recent results*. Bulletin of Symbolic Logic, **28** (2022), no. 01, 71-103.

Definition 1.2. Let $\mathcal{A}$ be a countable structure. A *Scott sentence* for $\mathcal{A}$ is a sentence φ of $\mathcal{L}_{\omega_1\omega}$ such that, up to isomorphism, $\mathcal{A}$ is the only countably model of φ.

<u>Fact (Scott)</u> Every countable str. A has a Scott sentence φ_A.

- But how complex can φ_A be chosen?
- And "complex" in what sense?

Traditional notion of complexity

Scott rank

Least $\alpha < \omega_1$ s.t. $\forall n < \omega$ and $\forall \bar{a}, \bar{b} \in A^n$

we have $(A, \bar{a}) \equiv_\alpha (B, \bar{b})$ (α-back-and-forth)

implies that $\bar{a}$ and $\bar{b}$ are AUTOMORPHIC

"New" way of measuring complexity

$\Updownarrow$

Smallest $\alpha < \omega_1$ s.t. A has a Scott sentence which is $\Pi_{\alpha+1}$ in the following sense:

(1) $\varphi(\bar{x})$ is computable Π_0 and computable Σ_0 if it is finitary quantifier-free;

(2) For an ordinal (resp. a computable ordinal) $\alpha > 0$:

(2.1) $\varphi(\bar{x})$ is Σ_α (resp. computable Σ_α) if it is a disjunction (resp. a computably enumerable disjunction) of formulas of the form $\exists\bar{y}\psi(\bar{x},\bar{y})$, where ψ is a Π_β-formula (resp. computable Π_β-formula) for some $\beta < \alpha$;

(2.2) $\varphi(\bar{x})$ is Π_α (resp. computable Π_α) if it is a conjunction (resp. a computably enumerable conjunction) of formulas of the form $\forall\bar{y}\psi(\bar{x},\bar{y})$, where ψ is a Σ_β-formula (resp. computable Σ_β-formula) for some $\beta < \alpha$;

(2.3) $\varphi(\bar{x})$ is d-Σ_α if it is a conjunction of a Σ_α-formula and a Π_α-formula.

$$
\begin{tikzcd}[column sep=small]
 & \Sigma^0_1 \arrow[dr] & & & \Sigma^0_2 \arrow[dr] & & & \Sigma^0_3 \arrow[dr] & & \\
\Sigma^0_1 \cap \Pi^0_1 \arrow[ur] \arrow[dr] & & \text{d-}\Sigma^0_1 \arrow[r] & \Sigma^0_2 \cap \Pi^0_2 \arrow[ur] \arrow[dr] & & \text{d-}\Sigma^0_2 \arrow[r] & \Sigma^0_3 \cap \Pi^0_3 \arrow[ur] \arrow[dr] & & \text{d-}\Sigma^0_3 \arrow[r] & \cdots \\
 & \Pi^0_1 \arrow[ur] & & & \Pi^0_2 \arrow[ur] & & & \Pi^0_3 \arrow[ur] & &
\end{tikzcd}
$$

<u>Fact (Montalbán)</u> Let A be countable. Then A has a $\Pi_{\alpha+1}$ Scott sentence iff the orbits of all finite tuples of A are defined by Σ_α form.

Antonio Montalbán. A robuster Scott rank. *Proc. Amer. Math. Soc.*, 143(12):5427–5436, 2015.

(computable)

<u>Program</u> : look at optimal Scott complexity for finitely generated structures.

<u>Fact</u> Every finitely generated str. has a Σ_3 Scott sentence

computable computable

<u>Fact</u> Every finitely generated str. has a Σ_3 Scott sentence

But actually most naturally occurring fin. gen. structures have not only a Σ_3 Scott sentence but a d-Σ_2 Scott sentence.

J. F. Knight and V. Saraph. *Scott sentences for certain groups.* Arch. Math. Logic **57** (2018), no. 03-04, 453-472.

the same paper it was proved that several familiar finitely generated structures (e.g. free groups of finite rank, free abelian groups of finite rank, the infinite dihedral group) actually have a d-Σ_2 Scott sentence, and in fact a computable one. This

<u>Question</u> Is there a fin. gen. group A which does not have a d-Σ_2 Scott sentence?

Yes!

M. Harrison-Trainor and Meng-Che Ho. *On optimal Scott sentences of finitely generated algebraic structures*. Proc. Amer. Math. Soc. **146** (2018), no. 10, 4473-4485.

Methods of small cancellation theory

All structures that we consider from now on are FINITELY GENERATED in a language L without relational symbols

<u>Def</u> For $A = \langle \bar{a} \rangle$ fin. gen., A is said to be COMPUTABLE if the word problem for A is decidable, i.e. we can decide whether $w(\bar{a}) = e$ $\forall$ words (terms) $w(\bar{x})$.

two main questions left open:

Question 1.1. *Is there a finitely presented group with no d-Σ_2 Scott sentence?*

Question 1.2. *Is there a computable finitely presented group with a d-Σ_2 Scott sentence but no computable d-Σ_2 Scott sentence?*

this is the subject of the present talk

But first a few words on Question 1.1

Concerning the first question, in [7] Harrison-Trainor gave a purely group theoretic characterization of the finitely presented groups without a d-Σ_2 Scott sentence. In the definition below we will reformulate his condition using the terminology of P.M. Neumann [14], in the hope to make this more familiar to group theorists.

M. Harrison-Trainor. *An introduction to the Scott complexity of countable structures and a survey of recent results*. Bulletin of Symbolic Logic, **28** (2022), no. 01, 71-103.

To explain this we borrow some terminology from:

P. M. Neumann. *Endomorphisms of infinite soluble groups*. Bull. Lond. Math. Soc. **12** (1980), no. 01, 13-16.

A is Hopfian if $\forall \alpha \in \mathrm{End}(A)$ if α is surjective, then it is injective.

Sup. A Hopfian and $\alpha \in \mathrm{End}(A)$ has a left inverse β, then:

$$\beta \circ \alpha = \mathrm{id}_A \Rightarrow \beta \text{ is surjective}$$

$$\Rightarrow \beta \text{ injective}$$

(Hopfianity!)

$$\Rightarrow \beta \in \mathrm{Aut}(A)$$

$$\Rightarrow \cancel{\beta}^{-1} \circ \cancel{\beta} \circ \alpha = \beta^{-1}$$

$$\Rightarrow \alpha \in \mathrm{Aut}(A)$$

(2) *We say that A is weakly Hopfian if when $\alpha \in \mathrm{End}(A)$ has a left inverse (i.e., there exists $\beta \in \mathrm{End}(A)$ such that $\beta \circ \alpha = \mathrm{id}_A$) we have that $\alpha \in \mathrm{Aut}(A)$.*

(3) *We say that A is very badly non-Hopfian if there exists $\alpha \in \mathrm{End}(A) \setminus \mathrm{Aut}(A)$ such that for every finite $X \subseteq A$ there is $\beta \in \mathrm{End}(A)$ such that $\beta \circ \alpha = \mathrm{id}_A$ and β is injective on X (notice that β injective on A would imply $\alpha \in \mathrm{Aut}(A)$).*

(as β has a right inverse and so β is surjective)

This means that A fails weak Hopfianity very badly as $\alpha \notin \mathrm{Aut}(A)$ BUT it has left inverses which are closer and closer to be AUTOS!!!

Fact 1.4. [7] *A finitely presented group does not have a d-Σ_2-Scott sentence if and only if it is very badly non-Hopfian.*

Much is known on finitely presented Hopfian groups, in fact this is a main dividing line in the area. Notice also that many finitely presented groups of interest are Hopfian, most notably the residually finite ones. Less familiar is instead the notion of weak Hopfianity, but this should not come as a surprise, as indeed it appears to be open if there exists a finitely presented non weakly Hopfian group (and, to the best of our knowledge, more generally, if there exists a finitely presented non weakly Hopfian structure). Although being not very badly non-Hopfian is a priori a much stronger condition than being non weakly Hopfian, this makes the problem more tractable only under the assumption that such finitely presented objects do not exist; as if it would be the case that there exists a very badly non-Hopfian finitely presented group we would be in serious trouble, as with the current technology we are not even able to construct a finitely presented non weakly Hopfian one!

Conclusion: Question 1.1 MIGHT be INTRACTABLE

So we go back to question

Question 1.2. *Is there a computable finitely presented group with a d-Σ_2 Scott sentence but no computable d-Σ_2 Scott sentence?*

Partial results on this appeared in my paper:

G. Paolini. *Computable Scott sentences for quasi-Hopfian finitely presented structures.* Arch. Math. Logic. **62** (2023), 55-65.

Today we present MORE ROBUST RESULTS from:

33. **Computable Scott sentences and the weak Whitehead problem for finitely presented groups**.

Ann. Pure Appl. Logic **175** (2024), no. 07, 103441

Recall

All structures that we consider are FINITELY GENERATED.

Fact 2.13 ([2, 4.5]). *Let A be a computable structure. Then TFAE:*

(1) A has a d-Σ_2 Scott sentence;

(2) $\exists\, \bar{a}$ s.t. $\langle \bar{a} \rangle_A = A$ and the $\mathrm{Aut}(A)$-orbit of $\bar{a}$ is defined by a computable Π_1 fmla;

(3) $\forall\, \bar{a}$ s.t. $\langle \bar{a} \rangle_A = A$, the $\mathrm{Aut}(A)$-orbit of $\bar{a}$ is defined by a computable Π_1 fmla.

[2] R. Alvir, J. F. Knight, and C. F. D. McCoy. *Complexity of Scott sentences.* Preprint, available on the ArXiv.

~~important conceptual result is~~ in [9], in this paper Harrison-Trainor constructs a finitely generated module with a d-Σ_2 Scott sentence but no computable d-Σ_2 Scott sentence. From this, using the universality among finitely generated structures of finitely generated groups [10], Harrison-Trainor infers the abstract existence of a finitely generated group with a d-Σ_2 Scott sentence but no computable d-Σ_2 Scott sentence, but no explicit example of such a group can be exhibited using his argument. Furthermore, the argument of Harrison-Trainor from [9] says nothing on finite presentability. ~~The only known general results that we are aware of come from~~

[9] M. Harrison-Trainor. *Describing finitely presented algebraic structures*. Preprint, available on the ArXiv.

Our main interest is on FINITELY PRESENTED (f.p.) structures, and in particular GROUPS

Two algorithmic problems of relevance.

Def. Let A be f.p. (finitely presentable) structure. We say that A has decidable Whitehead problem if $\exists$ an algorithm which decides whether any two tuples from A are AUTOMORPHIC.

Definition 1.3. *Given a f.p. computable structure A we say that A has decidable weak Whitehead problem if for every finite generating tuple $\bar{a}$ of A we have that the set $\{\alpha(\bar{a}) : \alpha \in \mathrm{Aut}(A)\}$ is a computable subset of A^n, where n is the length of $\bar{a}$.*

WHITHEAD PROBLEM

$\Downarrow$

WEAK WHITHED PROBLEM

(Just check $\bar{a}$ s.t. $A = \langle \bar{a} \rangle$ against any tuple of the same lenght)

Recall that A is Hopfian if any surjective $\alpha \in \mathrm{End}(A)$ is an AUTOMORPHISM

Def. G is residually finite if $\forall e_G \neq g \in G$ $\exists$ a finite F and hom. $f : G \to F$ s.t. $f(g) \neq e_F$.

Def. G is linear if $\exists$ a field K and $n < \omega$ s.t. $G \hookrightarrow GL_n(K)$ (invertible matrices).

LINEAR $\Rightarrow$ RESIDUALLY FINITE $\Rightarrow$ HOPFIAN

MAIN THEOREM

Theorem 1.4. *Let A be a computable Hopfian f.p. structure. Then A has a computable d-Σ_2 Scott sentence iff the weak Whitehead problem for A is decidable.*

Conjecture 1.7. *There exists a Hopfian finitely presented group with decidable word problem (so a computable group) but undecidable weak Whitehead problem.*

If we restrict to Hopfian etc. then the problem reduces to this problem/conjecture

F. Dahmani and V. Guirardel. *The isomorphism problem for all hyperbolic groups*. Geom. Funct. Anal. **21** (2011), no. 02, 223-300.

$$\Downarrow$$

Hyperbolic groups have decidable Whitehead pr.

$$\Downarrow$$

Corollary 1.5. *Hyperbolic groups have a computable d-Σ_2 Scott sentence.*

<u>Def</u> G is HYPERBOLIC if $\exists\, S \subseteq G$ finite generating set s.t. the resulting Cayley graph endowed with its graph metric is a HYPERBOLIC SPACE.

D. Segal. *Decidable properties of polycyclic groups.* Proc. Lond. Math. Soc. (3) **61** (1990), no. 03, 497-528.

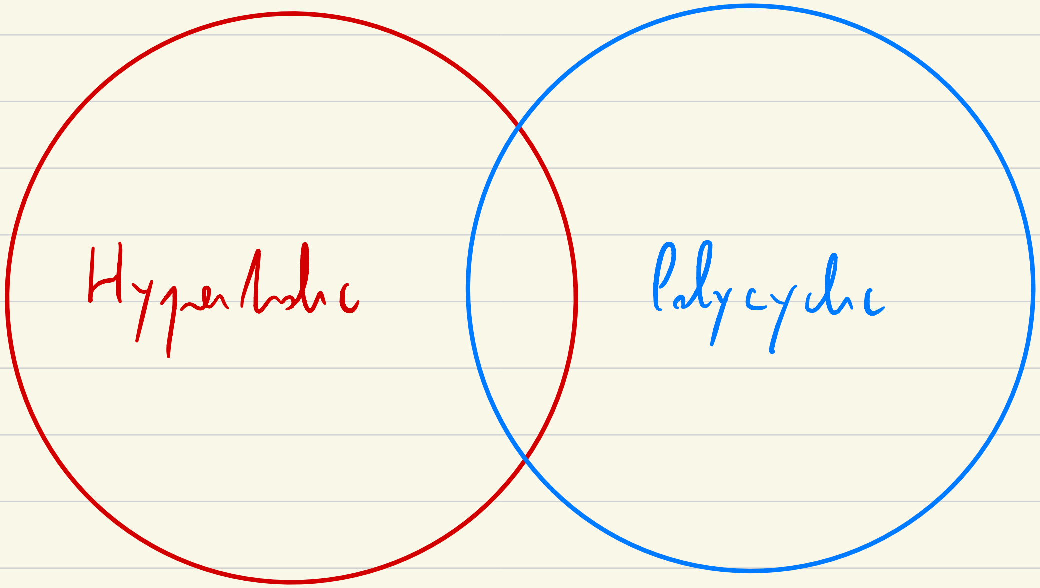
$\Downarrow$

Polycyclic-by-finite grps. have decidable Whitehead pr.

$\Downarrow$

Corollary 1.6. *Polycyclic-by-finite groups have a computable d-Σ_2 Scott sentence.*

In mathematics, a **polycyclic group** is a solvable group that satisfies the maximal condition on subgroups (that is, every subgroup is finitely generated). Polycyclic groups are finitely presented, which makes them interesting from a computational point of view.



Conjecture 1.7. *There exists a Hopfian finitely presented group with decidable word problem (so a computable group) but undecidable weak Whitehead problem.*

I asked many experts about this, including M. Bridson, we conjecture this is true.

Let A be fin. pres. computable + Hopf-ian

Whitehead problem

Generation problem
(for $\bar{a} \in A^{<\omega}$ decide if $\langle \bar{a} \rangle = A$)

Weak Whitehead problem

the group generates the whole group. It is known that in $F_2 \times F_2$ the Whitehead problem is decidable but the generation problem is not (cf. [16, Theorem 4.4]). The difficulty about finding an example as in Conjecture 1.7 is that both the decidability of the Whitehead problem and the decidability of the generation problem imply the decidability of the weak Whitehead problem, and so establishing undecidability of this latter problem is a harder task.

An unexpected solution to a problem from:

A. G. Myasnikov and N. S. Romanovskii. *Characterization of finitely generated groups by types*. Internat. J. Algebra Comput. **28** (2018), no. 08, 1613-1632.

Theorem 1.8. *Every weakly Hopfian f.p. group is strongly defined by its $\exists^+$-types.*

Definition 3.1. *Let $\mathcal{F}$ be a set of first-order formulas in the variables $\{x_i : i < \omega\}$ such that $\mathcal{F}$ contains all the positive quantifier-free formulas in the group theory language, $\mathcal{F}$ is closed under $\wedge$ and $\vee$, and $\mathcal{F}$ is closed under substitution of variables in words in $\{x_i : i < \omega\}$. We say that $f : A \to B$ is an $\mathcal{F}$-embedding if, for every $\varphi(\bar{a}) \in \mathcal{F}$ and $\bar{a} \in A^{\lg(\bar{x})}$, we have that $A \models \varphi(\bar{x})$ if and only if $B \models \varphi(f(\bar{a}))$. We say that a finitely generated group A (or, more generally, structure) is strongly defined by its $\mathcal{F}$-types iff any $\mathcal{F}$-embedding $f : A \to A$ is an automorphism of A.*

Notation 3.2. *If $\mathcal{F}$ consists of the positive quantifier-free formulas, then an $\mathcal{F}$-embedding is referred to as an $\exists^+$-embedding (as usual in model theory literature).*

A few words on the proof of ↓

Theorem 1.4. *Let A be a computable Hopfian f.p. structure. Then A has a computable d-Σ_2 Scott sentence iff the weak Whitehead problem for A is decidable.*

Notation 2.1. *In what follows we only consider finitely presented structures. Also, below we fix one such structure A and one finite presentation $A = \langle \bar{a} \mid \psi(\bar{x}) \rangle$.*

Remark 2.2. *In the context of Notation 2.1, if $A \models \psi(\bar{b})$, then the map $\bar{a} \mapsto \bar{b}$ extends uniquely to an endomorphism of A.*

Definition 2.3. *In the context of Notation 2.1, given $\bar{b} \in A^n$, we let:*

$$T(\bar{b}) = \{(t_1, ..., t_n) \in \mathrm{Term}^n : \exists \bar{c} \in A^n \ s.t. \ A \models \psi(\bar{c}) \wedge \bigwedge_{i \in [1,n]} b_i = t_i(\bar{c})\};$$

Lemma 2.4. *Given* $\bar{b}, \bar{c} \in A^n$ *we have that the following are equivalent:*

(1) $T(\bar{b}) \subseteq T(\bar{c})$*;*

(2) there is $\alpha \in \mathrm{End}(A)$ *such that* $\alpha(b_i) = c_i$*, for all* $i \in [1, n]$*.*

Idea from

A. Nies. *Aspects of free groups.* J. Algebra **263** (2003), no. 01, 119-125.

Lemma 2.6. *Suppose that the structure* A *is weakly Hopfian. Then, given* $\bar{b} \in A^n$*, the following are equivalent:*

(1) $T(\bar{a}) = T(\bar{b})$*;*

(2) there is $\beta \in \mathrm{Aut}(A)$ *such that* $\beta(\bar{a}) = \bar{b}$*.*

(2) We say that A *is weakly Hopfian if when* $\alpha \in \mathrm{End}(A)$ *has a left inverse (i.e., there exists* $\beta \in \mathrm{End}(A)$ *such that* $\beta \circ \alpha = \mathrm{id}_A$*) we have that* $\alpha \in \mathrm{Aut}(A)$*.*

Recall, already introduced

Lemma 2.7. *Suppose that the structure A is weakly Hopfian and let $\Theta(\bar{x})$ be the following Π_1-formula:*

$$\psi(\bar{x}) \wedge \bigwedge_{\bar{t} \in \widehat{T}(\bar{a})} \forall \bar{y} \neg \Big(\psi(\bar{y}) \wedge \bigwedge_{i \in [1,n]} x_i = t_i(\bar{y})\Big).$$

Then, given $\bar{b} \in A^n$, the following are equivalent:

(1) $A \models \Theta(\bar{b})$;

(2) *there is* $\alpha \in \mathrm{Aut}(A)$ *such that* $\alpha(\bar{a}) = \bar{b}$.

$$\hat{T}(\bar{a}) = \mathrm{Term}^n \setminus T(\bar{a})$$

It is the tuples of terms in which $\bar{a}$ CANNOT be written

$$\Theta(\bar{x}) \doteq \psi(\bar{x}) \wedge \bigwedge_{\bar{t} \in \widehat{T}(\bar{a})} \forall \bar{y} \neg (\psi(\bar{y}) \wedge \bigwedge_{i \in [1,n]} x_i = t_i(\bar{y})).$$

Lemma 2.9. *Suppose that A is computable and Hopfian. Then the following are equivalent:*

(1) $\widehat{T}(\bar{a})$ (cf. 2.3) is a computably enumerable subset of Term^n;
(2) the formula $\Theta(\bar{x})$ from Lemma 2.7 is a computable Π_1-formula;
(3) the set of tuples $\bar{b} \in A^n$ not in the $\mathrm{Aut}(A)$-orbit of $\bar{a}$ is computably enumerable.

Lemma 2.11. *Suppose that A is computable and Hopfian. Then the following are equivalent:*

(1) the $\mathrm{Aut}(A)$-orbit of $\bar{a}$ is definable by a computable Π_1-formula;
(2) the weak Whitehead problem for $(A, \bar{a})$ is decidable.

$\Downarrow$

Theorem 1.4. *Let A be a computable Hopfian f.p. structure. Then A has a computable d-Σ_2 Scott sentence iff the weak Whitehead problem for A is decidable.*

THANK YOU FOR YOUR

ATTENTION