Punctual presentability of trees

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Computable presentability

- a countable structure with domain $\mathbb{N}$ is computable if its relations and functions are uniformly Turing computable [Mal'tsev, 1961, Rabin, 1960, Tennenbaum, 1959]
- $\mathscr{A}$ is computably presentable if $\mathscr{A}$ is isomorphic to some computable structure
- computable presentability of familiar classes of structures (e.g., linear orders, Boolean algebras, and so on) has been studied extensively
- we want to go below computable

Subrecursive model theory

- algebraic structures presented by finite state automata [Khoussainov and Nerode, 1995],
- polynomial-time algebra [Cenzer and Remmel, 1991].
- Kalimullin, Melnikov, and Ng [Kalimullin et al., 2017] initiated the systematic study of punctual presentations, that lies somewhere in the between of computability and complexity theory:

Definition ([Kalimullin et al., 2017])

A structure with domain $\mathbb{N}$ is *punctual* (or, fully primitive recursive) if its relations and functions are uniformly primitive recursive.

- we forbid unbounded loops
- theoretical underpinning of online computation [Bazhenov et al., 2019]

Punctual robustness

Definition

A class of structures $\mathfrak{K}$ is *punctually robust* if every computable member of $\mathfrak{K}$ is punctually presentable.

In [Kalimullin et al., 2017]:

- punctually robust:
  - torsion-free abelian groups
  - Boolean algebras
  - equivalence structures
  - abelian p-groups
  - linear orders
- not punctually robust:
  - undirected graphs
  - Archimedean ordered abelian groups
  - torsion abelian groups

Trees

- the abstract concept of a tree is consistent across disciplines
- different representations of trees
- how to turn a given representation into a structure?

Trees represented as:

1. prefix-closed subsets of $\mathbb{N}^{<\mathbb{N}}$ **sequential trees**
2. (un)directed graphs **graph-theoretic trees**
3. partially ordered sets **partially-ordered trees**
4. predecessor functions **predecessor trees**

SEQUENTIAL TREES

From sequential trees to structures

Definition (sequential tree)

$T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a tree if for all $\alpha, \beta \in \mathbb{N}^{<\mathbb{N}}$: $\alpha \in T \wedge \beta \subset \alpha \implies \beta \in T$.

$\downarrow$

Definition (tentative)

Consider τ-structures, where $\tau = \{R_1, R_2, \dots\}$ is a relational vocabulary with R_n being n-ary, for $n > 0$. We say $\mathscr{A}$ is a sequential tree structure if for every n, for every $x_1, \dots, x_n \in \mathscr{A}$, $R_n(x_1, \dots, x_n)$ implies $R_k(x_1, \dots, x_k)$ for $k < n$.

GRAPH-THEORETIC TREES

Definition (graph-theoretic tree)

(V, E), where E is a binary relation, is a graph-theoretic tree if (V, E) is a connected acyclic (un)directed graph.

Theorem (essentially [Cenzer and Remmel, 1998])

The class of graph-theoretic trees is punctually robust.

Case 1: there is a node with infinitely many children

Case 2: no node has infinitely many children

PARTIALLY-ORDERED TREES

Definition (partially ordered tree)

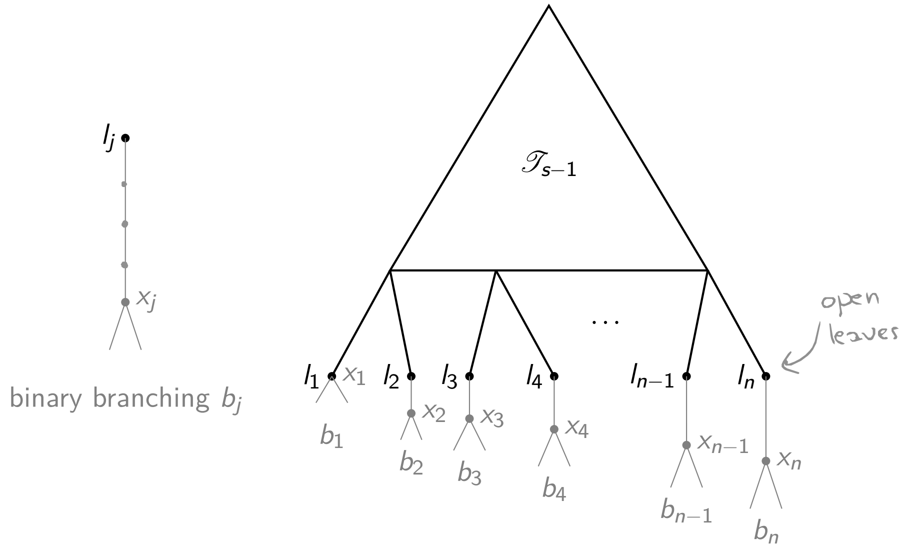
A partial order $(P, \leq)$ is a tree if $(P, \leq)$ has a unique maximal element and, for every $x \in P$, $P_{\geq x}$ is a finite linear order.

Theorem (K.)

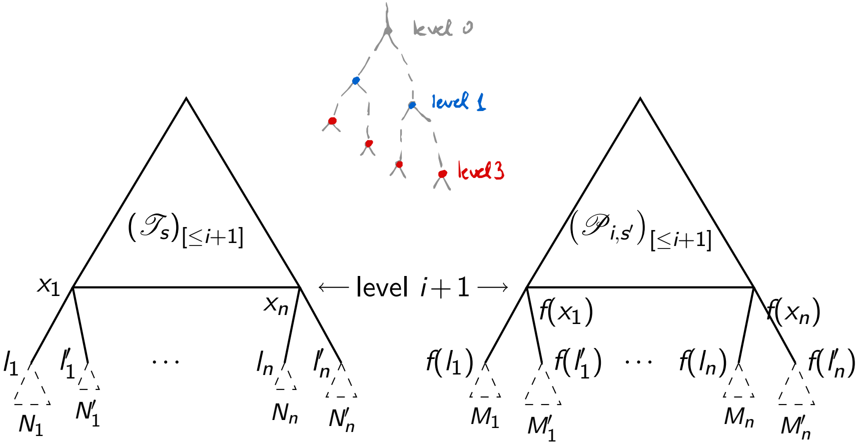
The class of partially-ordered trees is not punctually robust.

- build a computable p.o. tree $\mathscr{T} = (\mathbb{N}, \leq_T)$ such that for every $i \in \mathbb{N}$: $\mathscr{T} \ncong \mathscr{P}_i = (\mathbb{N}, p_i)$, where p_i is an i-th p.r. binary relation.
- $\mathscr{T}$ will be binary and uniquely branching.
- maintain a dynamic set $F \subset \mathscr{T}$
- a node x is **closed** if for some $y \geq x$, y is in F (stage-sensitive); otherwise x is **open**
- at odd stages the tree grows through open leaves
- at even stages, the strategies for $\mathscr{P}_i$ monitor the relationship between $\mathscr{T}$ and $\mathscr{P}_i$ and change open vs closed nodes to satisfy R_i

Growing the tree



Changing open vs closed nodes



PREDECESSOR TREES

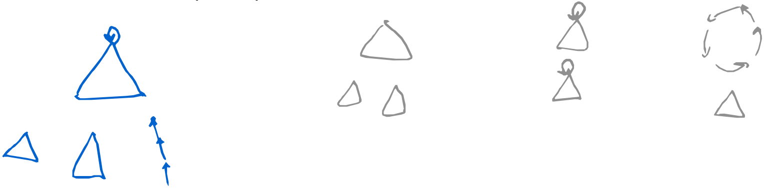
Definition (predecessor tree)

Let A be a set and let $T : A \to A$. (A, T) is a *predecessor tree*, if there is a unique $r \in A$ such that $T(r) = r$, and for every $x \in A$ there exists $i \in \mathbb{N}$ such that $T^i(x) = r$. The unique r is called the root.

Theorem (MFCS '24 paper with San Mauro & Wrocławski)

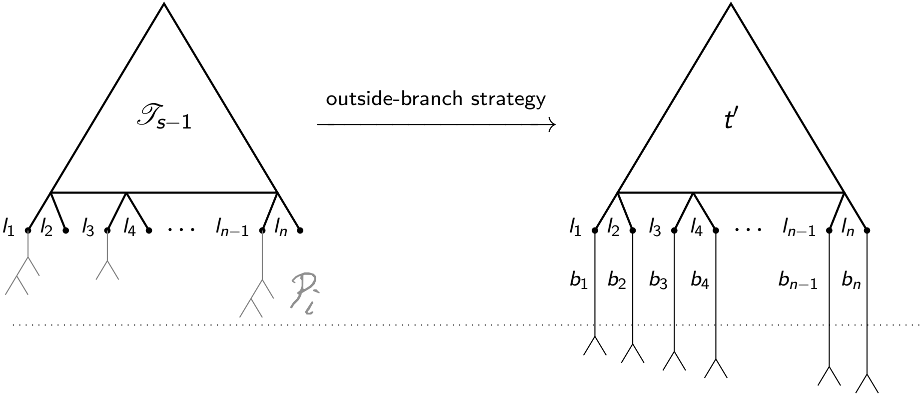
The class of predecessor trees is not punctually robust.

- build a computable predecessor tree $\mathscr{T} = (\mathbb{N}, T)$ such that $\mathscr{T} \ncong \mathscr{P}_i = (\mathbb{N}, p_i)$, where p_i is the i-th p.r. unary function.
- $\mathscr{T}$ will be binary and uniquely branching (explain)
- approximating $(\mathbb{N}, p_i)$



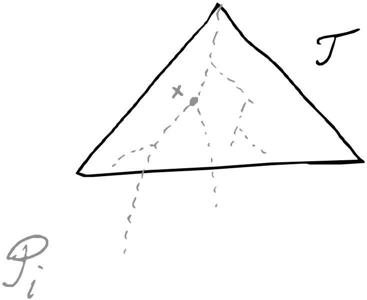
Outside-branch strategy (very roughly)

- an approximation of $\mathscr{P}_i$ branches outside the current $\mathscr{T}$
- avoid this branching length in $\mathscr{T}$ by attaching to the leaves of $\mathscr{T}$ very long binary branchings
- this kills $\mathscr{T} \cong \mathscr{P}_i$



Inside-branch strategy (very roughly)

- an approximation of $\mathscr{P}_i$ branches inside the current $\mathscr{T}$ at a node x
- forbid growing $\mathscr{T}$ from x from level $i+1$
- this kills $\mathscr{T} \cong \mathscr{P}_i$



Implications for (semi)lattices

Corollary

The following classes of structures are not punctually robust:

1. *join semilattices,*
2. *meet semilattices,*
3. *(complemented) lattices.*

This remains valid under order-theoretic and algebraic interpretation.

Add a least element to the computable tree from the result on partially-ordered trees.

Recap and conclusions

- punctual robustness of trees under various structural definitions
- interaction between structure and p.r. computatation
- intital (invalid) suspicion: punctual robustness depends on whether a signature is relational or functional
- punctual robustness is contingent on the mode of representation
- (semi)lattices are not punctually robust (previous slide)

Further directions

1.
   - Boolean algebras are punctually robust [Kalimullin et al., 2017].
   - Complemented lattices are not (this talk)
   - Since Boolean algebras are complemented distributive lattices:

> Determine whether the class of distributive lattices is punctually robust.

2. Consider a more general concept of a partially ordered tree:

Definition (from set theory, e.g. [Jech, 2007])

A partially ordered set $(P, \leq)$ is a set-theoretic tree if it has a least element and for every $x \in P$, $P_{\leq x}$ is well-ordered by $\leq$.

> Investigate punctual robustness of set-theoretic trees.

Thank you for your attention

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