Computable Structure Theory and Interactions

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Approximately computable isomorphisms

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Generic case complexity

- Complexity results in computable structure theory often depend on the behavior of the hardest instances of the problem.

- For problems on groups, Kapovich, Myasnikov, Schupp, and Shpilrain (2003) proposed using the notion of asymptotic density to see whether a partial algorithmic (partial computable) function could solve "almost all" instances of a problem.

- They showed that for a large class of finitely generated groups the classical decision problems, such as the word problem or the conjugacy problem, have linear time generic case complexity.

- Jockusch and Schupp (2012) introduced this topic to computability theory. They defined and investigated *generically computable* and *coarsely computable* sets of natural numbers.

- For $A \subseteq \mathbb{N}$ and $n \geq 1$, the density of a set A up to n, denoted by $\rho_n(A)$, is
$$\frac{|A \cap \{0, 1, 2, \ldots, n-1\}|}{n}$$

- The (asymptotic) *density* of A is $\rho(A) = \lim_{n\to\infty} \rho_n(A)$.

  A is (asymptotically) *dense* if $\rho(A) = 1$.

- For example, $A = \{2^n : n \in \mathbb{N}\}$ has density 0.

Generically and coarsely computable sets of natural numbers (Jockusch and Schupp)

- S is *generically computable* if there is a partial computable function $\varphi : \mathbb{N} \to \{0,1\}$ such that: $dom(\varphi)$ is asymptotically dense, and for the characteristic function c_S, we have $c_S \restriction dom(\varphi) = \varphi$.

  (There may be no answers.)

- S is *coarsely computable* if there is a (total) computable function $\tau : \mathbb{N} \to \{0,1\}$ such that $\{a : c_S(a) = \tau(a)\}$ is asymptotically dense.

  Equivalently, S is coarsely computable if there is a computable set T such that $S \triangle T = (S - T) \cup (T - S)$ has asymptotic density 0.

  (There may be wrong answers.)

Asymptotic density in $\mathbb{N}\times\mathbb{N}$

- Let $D \subseteq \mathbb{N}$. Then D has asymptotic density δ in $\mathbb{N}$ if and only if $D \times D$ has asymptotic density δ^2 in $\mathbb{N} \times \mathbb{N}$.

  Hence: D is asymptotically dense in $\mathbb{N}$ iff $D \times D$ is asymptotically dense in $\mathbb{N} \times \mathbb{N}$.

- There is a computable, dense, binary relation $R \subseteq \mathbb{N} \times \mathbb{N}$ such that for any infinite c.e. set $E \subseteq \mathbb{N}$, the product $E \times E$ is not a subset of R.

Generically computable structures

- Consider a structure $\mathcal{A}$ with a computable domain A (say $A = \mathbb{N}$): with finitely many functions $\{f_i : i \in I\}$, each f_i of arity p_i, and finitely many relations $\{R_j : j \in J\}$, each R_j of arity r_j.

- We call $\mathcal{A}$ *generically computable* if $\mathcal{A}$ has a substructure $\mathcal{D}$ with a c.e. domain D that is asymptotically dense, and partial computable functions $\{\phi_i : i \in I\}$ and $\{\psi_j : j \in J\}$ such that each ϕ_i agrees with f_i on D^{p_i} and each ψ_j agrees with c_{R_j} on the set D^{r_j}.

- $\mathcal{A}$ is *computable* if its domain A is a computable set and each $f_i : A^{p_i} \to A$ is a computable function and each R_j is a computable relation (i.e., the characteristic function $c_{R_j} : A^{r_j} \to \{0, 1\}$ is computable).

- Examples of computable structures:

  $(\mathbb{N}, \equiv_m)$

  $(\mathbb{Q}, +)$

  $\mathbb{Z}(p^n)$ the cyclic group of order p^n

  $\mathbb{Z}(p^\infty)$ the quasicyclic (Prüfer) p-group
  the set of rational numbers in $[0, 1)$ of the form $\frac{i}{p^k}$
  with addition modulo 1

- $(\mathbb{N}, R)$, where R is a computable set (unary relation),

  is a computable structure.

- $\mathcal{A} = (\mathbb{N}, S)$ is a generically computable structure iff S is a generically computable set (following Jockusch and Schupp).

- A structure $\mathcal{D}$ is *computably enumerable* if

  its domain D is c.e., and each

  function f_i is the restriction of a partial computable function to D,

  and each relation R_j is c.e.

Equivalence structures

- A structure $\mathcal{A} = (A, E)$ is an equivalence structure if E is an equivalence relation on A.

- The *character* $\chi(\mathcal{A})$ of $\mathcal{A}$ is:
  $\{(k, n) : k, n \geq 1$ & there are $\geq n$ equivalence classes of size $k\}$

  Bounded character: there is a bound on size k.

- If $\mathcal{A}$ is c.e., then $\chi(\mathcal{A})$ is $\exists\forall$-computable.

- $K \subseteq (\mathbb{N} - \{0\}) \times (\mathbb{N} - \{0\})$ is (an abstract) *character* if for all $n \geq 1$ and k: $(k, n+1) \in K \Rightarrow (k, n) \in K$.

- For any $\exists\forall$-computable set K that is a character, there is a computable equivalence structure $\mathcal{A}$ with character $\chi(\mathcal{A}) = K$, which has infinitely many infinite equivalence classes.

- There are c.e. equivalence structures that are not isomorphic to computable structures (i.e., have no computable copies).

Generically computable equivalence structures

A surprising result:

- Every countable equivalence structure $\mathcal{A} = (A, E)$ has a generically computable (isomorphic) copy.

Abelian p-groups

- Let p be a prime number.
  A group $\mathcal{A}$ is a p-group if the order of every element is a power of p.

- Suppose that $\mathcal{A}$ is a countable Abelian p-group isomorphic to a product of quasi-cyclic and cyclic groups:
  $\oplus_\alpha \mathbb{Z}(p^\infty) \oplus \oplus_{i \in \mathbb{N}} \mathbb{Z}(p^{k_i})$.

  $\oplus_\alpha \mathbb{Z}(p^\infty)$ is the divisible part.

  $\oplus_{i \in \mathbb{N}} \mathbb{Z}(p^{k_i})$ is the reduced part.

- Then the *character* of $\mathcal{A}$ is
  $\chi(\mathcal{A}) = \{(k,n) : k,n \geq 1\&\ card(\{i : k_i = k\} \geq n\}$.

- If $\mathcal{A}$ is computable, then $\chi(\mathcal{A})$ is a $\exists\forall$-computable set.

- Let $\mathcal{A}$ be a computable equivalence structure with character K and α infinite equivalence classes. Then there exists a computable Abelian p-group isomorphic to $\oplus_\alpha \mathbb{Z}(p^\infty) \oplus G$ where G is a direct sum of cyclic p-groups with character K.

- For any $\exists\forall$-computable set K that is a character, there is a computable Abelian p-group $\mathcal{A}$ with character K and the divisible part isomorphic to $\oplus_\omega \mathbb{Z}(p^\infty)$.

- Every c.e. Abelian group is computably isomorphic to a computable group.

Generically computable Abelian groups

- Every countable Abelian p-group has a generically computable copy.

  There are countable Abelian groups without generically computable copies.

- For an Abelian group $\mathcal{A}$, let
  $\mathcal{A}[p] = \{g \in A : p^n g = 0 \text{ for some } n\}$.

- A countable infinite Abelian group has a generically computable copy if and only if:

  (i) For some prime p, $\mathcal{A}[p]$ is infinite; or

  (ii) $\{q : \mathcal{A}[q] \neq \{0\}\}$ has an infinite c.e. subset.

Σ_n generically c.e. structures

- For $n \geq 1$, a substructure $\mathcal{B}$ of $\mathcal{A}$ is a Σ_n *elementary substructure* if for every Σ_n formula $\theta(x_1, \ldots, x_m)$ and $b_1, \ldots, b_m \in B$:

  $\mathcal{A} \models \theta(b_1, \ldots, b_m)$ iff $\mathcal{B} \models \theta(b_1, \ldots, b_m)$.

- A structure $\mathcal{A}$ is Σ_n *generically c.e.* if $\mathcal{A}$ has a c.e. substructure $\mathcal{D}$ with an asymptotically dense domain D, such that
  $\mathcal{D}$ is also a Σ_n elementary substructure of $\mathcal{A}$.

Σ_{n} generically c.e. equivalence structures and Abelian p-groups

- Σ_{n+1} generically c.e. structure $\Rightarrow \Sigma_n$ generically c.e. structure

- Every c.e. structure is Σ_n generically c.e. for any n.

- A function $h : \mathbb{N}^2 \to \mathbb{N}$ is a Khisamiev s_1-*function* if

  $h(i,t) \leq h(i,t+1)$ for all i, t

  $m_i = \lim_{t\to\infty} h(i,t)$ exists for each i, and

  $m_0 < m_1 < \cdots < m_i < \cdots$

- Let $\mathcal{A} = (A, E)$ be a c.e. equivalence structure with no infinite equivalence classes and an unbounded character.

  Then there is a computable s_1-function h such that $\mathcal{A}$ contains an equivalence class of size m_i for each $i \in \mathbb{N}$.

- We say that a character K has an s_1-function h (as above) if $(m_i, 1) \in K$ for each i.

- For every $\exists\forall$-computable character K that is either bounded or has a computable s_1-function, there is a computable equivalence structure $\mathcal{A}$ with character K and no infinite equivalence classes.

- An equivalence structure $\mathcal{A} = (\mathbb{N}, E)$ has a Σ_1 *generically c.e. copy* iff at least one of the following conditions holds:

  1. $\chi(\mathcal{A})$ is bounded;

  2. $\chi(\mathcal{A})$ has a $\exists\forall$-computable sub-character K with a computable s_1-function;

  3. $\chi(\mathcal{A})$ has a $\exists\forall$-computable sub-character H, and $\mathcal{A}$ has an infinite class;

  4. $\mathcal{A}$ has infinitely many infinite classes.

- An equivalence structure A has a Σ_2 *generically c.e. copy* iff

  (i) $\mathcal{A}$ has a c.e. copy iff

  (ii) $\mathcal{A}$ has a Σ_3 generically c.e. copy.

- Let $\mathcal{A}$ be an Abelian p-group that is a product of quasi-cyclic and cyclic groups. Then $\mathcal{A}$ has a Σ_1 *generically c.e. copy* iff at least one of the following conditions holds:

  1. $\chi(\mathcal{A})$ is bounded;

  2. $\chi(\mathcal{A})$ has a $\exists\forall$-computable subset K with a computable s_1-function;

  3. $\mathcal{A}$ has a divisible component.

- $\mathcal{A}$ is Σ_2 generically c.e. if and only if $\mathcal{A}$ has a computable copy.

Computable isomorphisms

- A computable structure $\mathcal{A}$ is *computably categorical* if for any computable isomorphic copy of $\mathcal{A}$ there is a computable isomorphism.

- A computable equivalence structure $\mathcal{A}$ is *computably categorical* iff

  $\mathcal{A}$ has finitely many finite equivalence classes, or

  $\mathcal{A}$ has finitely many infinite classes, bounded character, and at most one finite $k \geq 1$ with infinitely many classes of size k.

- (Goncharov; Smith)

  A computable Abelian p-group $\mathcal{A}$ is *computably categorical* iff $\mathcal{A}$ is isomorphic to:

  $\oplus_\alpha \mathbb{Z}(p^\infty) \oplus G$ where $\alpha \leq \omega$ and G is finite, or

  $\oplus_n \mathbb{Z}(p^\infty) \oplus G \oplus \oplus_\omega \mathbb{Z}(p^k)$ where $n, k \in \mathbb{N}$ and G is finite.

- We have a number of examples of structures that are not computably categorical but are generically computably categorical.

Generically computable isomorphisms

- An isomorphism $F : \mathcal{A} \to \mathcal{B}$ is *generically computable* if there are

  a c.e. set C of asymptotic density 1 and

  a partial computable function ψ with $C = dom(\psi)$ such that:

  (1) C is the domain of a substructure $\mathcal{C}$ of $\mathcal{A}$,

  (2) $F(x) = \psi(x)$ for all $x \in C$,

  (3) the image $F[C]$ has asymptotic density 1.

- A simple example of a computable equivalence structure that is not computably categorical is one which consists only of infinitely many classes of size 1 and infinitely many classes of size 2.

  Call such an equivalence structure a $(1,2)$-equivalence structure.

- For an equivalence structure $\mathcal{A}$, denote by $\mathcal{A}(k)$ the set of elements belonging to classes of size k.

- For a finite set $H \subseteq \mathbb{N} - \{0\}$, we say that $\mathcal{A}$ has generic character H if for each $k \in H$, the set $\mathcal{A}(k)$ has positive asymptotic density, and the union $\bigcup_{k \in H} \mathcal{A}(k)$ has asymptotic density 1.

- If $\mathcal{A}$ and $\mathcal{B}$ are computable $(1,2)$-equivalence structures, each having generic character $\{2\}$, then $\mathcal{A}$ and $\mathcal{B}$ are *generically computably isomorphic*.

- If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic computable equivalence structures with finitely many infinite classes, such that the set of elements that belong to infinite classes has asymptotic density 1 in each structure, then $\mathcal{A}$ and $\mathcal{B}$ are *generically computably isomorphic*.

- A computable Abelian p-group $\mathcal{G}$

$$\oplus_{i \in \mathbb{N}} \mathbb{Z}(p) \oplus \oplus_{i \in \mathbb{N}} \mathbb{Z}(p^2)$$

  is not computably categorical.

- If $\mathcal{A}$ and $\mathcal{B}$ are computable groups isomorphic to $\mathcal{G}$ such that the elements of height 1 are asymptotically dense in each of them, then $\mathcal{A}$ and $\mathcal{B}$ are generically computably isomorphic.

- The *height* (p-*height*) of a group element x is the largest n such that $p^n \mid x$.

- If F is a generically computable isomorphism, then F^{-1} also is.

- For transitivity we need to preserve density.

- A function F mapping A_1 to A_2 is *density preserving* if for any subset X of A_1 of density p, the image $F[X]$ has density p.

- If $F_1 : \mathcal{A}_1 \to \mathcal{A}_2$ and $F_2 : \mathcal{A}_2 \to \mathcal{A}_3$ are generically computable and density preserving isomorphisms, then the composition $F_2 \circ F_1 : \mathcal{A}_1 \to \mathcal{A}_3$ also is.

Coarsely computable structures

- A structure $\mathcal{A}$ is *coarsely computable* if there are a computable structure $\mathcal{E}$ and a dense set D such that the structure $\mathcal{D}$ with domain D is a substructure of both $\mathcal{A}$ and of $\mathcal{E}$, and all relations and functions agree on D:

$$\mathcal{D} \subseteq \begin{matrix} \mathcal{A} \\ \mathcal{E} \end{matrix}$$

- $\mathcal{A} = (\mathbb{N}, S)$ is a coarsely computable structure iff S is a coarsely computable set.
  There is a generically computable structure that is not coarsely computable, and vice versa.

  There is an equivalence structure with no coarsely computable copy.

Coarsely computable groups

- Every countable Abelian p-group has a coarsely computable copy.

- There is a Σ_1 coarsely c.e. Abelian group with no Σ_1 generically c.e. copy.

- Let $\mathcal{A}$ be an Abelian p-group with no elements of infinite height and with an unbounded character $\chi(\mathcal{A})$ that does not have a computable s_1-function. Then the following are equivalent:

  1. $\mathcal{A}$ has a Σ_2 coarsely c.e. copy.

  2. $\chi(\mathcal{A})$ is $\exists\forall$-computable, and for some k, $\mathcal{A}$ has infinitely many components of type $\mathbb{Z}(p^k)$.

- Let $\mathcal{A}$ be an Abelian p-group with no elements of infinite height such that its character $\chi(\mathcal{A})$ is either bounded or has a computable s_1-function. Then the following are equivalent:

  1. $\mathcal{A}$ has a Σ_2 coarsely c.e. copy.

  2. $\chi(\mathcal{A})$ is $\exists\forall$-computable.

  3. $\mathcal{A}$ has a computable copy.

Coarsely computable isomorphisms

- An isomorphism $F : \mathcal{A} \to \mathcal{B}$ is *coarsely computable* if there are

  a set C of asymptotic density 1 and

  a (total) computable function θ such that:

  (1) C is the domain of a substructure $\mathcal{C}$ of $\mathcal{A}$,

  (2) $F(x) = \theta(x)$ for all $x \in C$,

  (3) the image $F[C]$ has asymptotic density 1.

- *Weakly coarsely computable:* if F is just a bijection while $F[C]$ is still the domain of a substructure $\mathcal{C}_1$ of $\mathcal{B}$.

- If there is a *weakly coarsely computable* isomorphism from $\mathcal{A}$ to a *computable* structure, then $\mathcal{A}$ is coarsely computable.

- If a structure $\mathcal{A}$ is coarsely computable, then there is a density preserving *weakly coarsely computable* isomorphism from $\mathcal{A}$ to a *computable* structure.

- Let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic equivalence structures with generic character $\{1\}$.

  Then there is a (density preserving) *coarsely computable* isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

  Not necessarily true for generically computable isomorphism.

THANK YOU!

- *Proof Sketch* for: If $\mathcal{A} = (A, +^{\mathcal{A}})$ is a c.e. Abelian group, then there is a computable group $\mathcal{C}$ with universe $\mathbb{N}$ and a computable isomorphism $f : \mathcal{C} \cong \mathcal{A}$.

- Let $f : \mathbb{N} \to A$ be a computable bijection ($1-1$ computable enumeration of A).

  Define $+^{\mathcal{C}}$ so that f is an isomorphism: $i +^{\mathcal{C}} j = f^{-1}(f(i) +^{\mathcal{A}} f(j))$.

- *Proof Sketch* for: Any countable Abelian p-group $\mathcal{A}$ has a subgroup $\mathcal{B}$ which is isomorphic to a computable group.

- Case 1: Every element of $\mathcal{A}$ has finite height.
  Every $\mathbb{Z}(p^k)$ has a subgroup of type $\mathbb{Z}(p)$.
  So, $\mathcal{A}$ has a subgroup $\mathcal{B}$ isomorphic to $\oplus_{i\in\mathbb{N}}\mathbb{Z}(p)$.

- Case 2: $\mathcal{A}$ has a divisible subgroup $\mathcal{B}$.

- Case 3: $\mathcal{A}$ has no divisible subgroup, but has an element a of infinite height.
  Can prove that there is an element b such that $C = \{x : px = b\}$ is infinite.
  C generates an infinite subgroup $\mathcal{B}$ with elements of bounded order.

- *Proof Sketch* for: Every countable Abelian p-group $\mathcal{A}$ has a generically computable copy $\mathcal{C}$.

- $\mathcal{A}$ has a subgroup $\mathcal{B}$ isomorphic to a computable group.

  Obtain a computable group $\mathcal{D}$ isomorphic to $\mathcal{B}$ with dense and co-infinite domain D.

  Extend $\mathcal{D}$ to a generically computable $\mathcal{C}$ isomorphic to $\mathcal{A}$.

- *Proof Sketch* for: $\mathcal{A}$ is Σ_2 generically c.e. $\Rightarrow$ $\mathcal{A}$ has a computable copy

  Let D be a dense c.e. set with $\mathcal{D} = (D, +^{\mathcal{A}}) \prec_2 \mathcal{A}$.

  Let $K = \chi(\mathcal{D})$. K is $\exists\forall$-computable since $\mathcal{D}$ is c.e.

  $\chi(\mathcal{D}) = \chi(\mathcal{A})$ since $\mathcal{D} \prec_2 \mathcal{A}$.

- Case 1: K is bounded. Then $\mathcal{A}$ has a computable copy.

- Case 2: K is unbounded and $\mathcal{D}$ has no divisible component. Then K has a computable s_1-function, so $\mathcal{A}$ has a computable copy.

- Case 3: K is unbounded and $\mathcal{D}$ has a divisible component. Then $\mathcal{A}$ also has a divisible component. Hence $\mathcal{A}$ has a computable copy.

- Consider

  $\oplus_{i \in \mathbb{N}} \mathbb{Z}(p) \oplus \oplus_{i \in \mathbb{N}} \mathbb{Z}(p^2).$

- Every element of the first factor has order 0 or p.

  The set of elements of order p^2 is computable.

- Every element of height 1 must belong to the second factor.

  The set of elements of height 1 is c.e.

- Example: Look at $\mathbb{Z}(2) \oplus \mathbb{Z}(4) = \{0, 1\} \times \{0, 1, 2, 3\}$.

  Elements of order 2: $(0, 2), (1, 0), (1, 2)$.

  Elements of order 4: $(0, 1), (0, 3), (1, 1), (1, 3)$.

  Elements of height 1: $(0, 2)$