A forcing proof of the Silver dichotomy

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Our goal is to prove the following theorem about coanalytic equivalence relations due to Silver.

Theorem 1 (Silver)**.** *Suppose X is a Polish space and E is a $\mathbf{\Pi}^1_1$ equivalence relation on X. Then either E has countably many equivalence classes or a perfect set of pairwise inequivalent elements.*

We will prove the effective version of the Theorem. The boldface version follows by relativizing. It will be convenient to assume that $X = \omega^\omega$. The proof we give is a forcing proof due to Harrington. The presentation follows Arnold Miller's "Descriptive Set Theory and Forcing". The forcing used is often referred to as *Gandy forcing* or *Gandy-Harrington forcing*. Our forcing notion $\mathbb{P}$ consists of the non-empty $\mathbf{\Sigma}^1_1$ subsets of ω^ω ordered by inclusion.

Let us first prove a general observation about Gandy forcing. Gandy forcing is often used and always written in the language of forcing, it is called Gandy forcing, because it appeared in the proof of Gandy's Basis Theorem. Of course, it was not written in forcing language but as a classical construction of a real in ω many steps. After all, Gandy proved his basis theorem before Cohen invented the forcing method. In proofs using this method one often refers to *generic reals*. This is warranted as the following lemma says.

Lemma 2. *Let G be a $\mathbb{P}$-generic filter, then there exists $a \in \omega^\omega$ such that $G = \{p \in \mathbb{P} : a \in p\}$ and $\{a\} = \bigcap G$.*

Proof. Given $s \in \omega^n$ let $[s] = \{x \in \omega^\omega : x \supset s\}$, then $[s]$ is clearly Σ^1_1 and for every forcing condition p and every n, there is $[s]$ compatible with p. However, if s and t are two incompatible strings in $\omega^{<\omega}$ and $[s] \in G$, then $[t] \notin G$. Thus there is a unique $a \in \omega^\omega$ with $[a \restriction n] \in G$ for each n and clearly, $\bigcap G \subseteq \{a\}$. Also note that $G \subseteq \{p \in \mathbb{P} : a \in p\}$.

Now, let $B \in G$, we need to show that $a \in B$. As B is Σ^1_1 it is defined by a formula of the form

$$x \in B \Leftrightarrow \exists (y \in \omega^\omega) \forall n \theta(x \restriction n, y \restriction n, n)$$

where θ is a recursive predicate. We can associate to θ a tree T in $\omega^\omega \times \omega^\omega$ such that $B = \{x : \exists y \langle y, x\rangle \in [T]\}$.

We show that $a \in B$ by induction. Let $[T^{y\restriction n, a\restriction n}] = [T] \cap [\langle y \restriction n, a \restriction n\rangle]$ and $p[T^{y\restriction n, a\restriction n}]$ be its right projection. Assume that $p[T^{y\restriction n, a\restriction n}] \in G$. First, e that

$$p[T^{y\restriction n, a\restriction n+1}] = [a \restriction n+1] \cap p[T^{y\restriction n, a\restriction n}]$$

and as the two latter sets are in G, $p[T^{y\restriction n, a\restriction n+1}] \in G$. Notice that

$$p[T^{y\restriction n, a\restriction n+1}] \in G = \bigcup_{k\in\omega} p[T^{(x\restriction n)^\frown k, a\restriction n+1}]$$

The set $\{q : \exists k p[T^{(x\restriction n)^\frown k, a\restriction n+1}] \cap q \neq \emptyset\}$ is dense below $p[T^{x\restriction n, a\restriction n}]$ and thus for some k, $p[T^{(x\restriction n)^\frown k, a\restriction n+1}] \in G$. Let $x \restriction n+1 = (x \restriction n)^\frown k$. So $\langle x, a\rangle \in [T]$ and thus $a \in B$. Hence, $\bigcap G = \{a\}$.

At last we need to verify that $G \supseteq \{p \in \mathbb{P} : a \in p\}$. Towards a contradiction assume there is $p \notin G$ with $a \in p$. As the set $\{q \in \mathbb{P} : q \leq p \vee q \cap p = \emptyset\}$ is dense there must be $q \in G$ disjoint from p. But as for every $q \in G$, $a \in q$, $p \cap q \neq \emptyset$, a contradiction. □

Lemma 3. *Say a is $\mathbb{P}$-generic and $a = \langle a_0, a_1\rangle$, then a_0 and a_1 are both $\mathbb{P}$-generic.*

Proof. Suppose $D \subseteq \mathbb{P}$ is dense open and let

$$E = \{p \in \mathbb{P} : \{x_0 : x \in p\} \in D\}.$$

We claim that E is dense. For arbitrary $q \in \mathbb{P}$ let $q_0 = \{x_0 : x \in q\}$, e that $q_0 \in \mathbb{P}$ as it is Σ^1_1 and non-empty. As D is dense there is $r_0 \leq q_0$ with $r_0 \in D$. Let $r = \{x \in q : x_0 \in r_0\}$. Then $r \in E$ and $r \leq q$, so E is dense. Thus, there is $p \in E$ with $a \in p$ and hence $a_0 \in p_0 = \{x_0 : x \in p\} \in D$. Thus a_0 is generic. The proof for a_1 is symmetric. □

Theorem 4 (Π^1_1 reduction)**.** *Let A_0 and A_1 be Π^1_1 sets, then there exist disjoint Π^1_1 sets $B_i \subseteq A_i$ such that $A_0 \cup A_1 = B_0 \cup B_1$.*

Proof. As the A_i are Π^1_1, there are recursive functions p_{A_i}, taking $x \to T^{i,x}$ such that $x \in A_i$ if and only if $T_{i,x}$ is well founded. Construct the sets B_i as follows:

1. $x \in B_0$ iff $x \in A_0$ and $T_{1,x} \not\prec T_{0,x}$
2. $x \in B_1$ iff $x \in A_1$ and $T_{0,x} \not\preceq T_{1,x}$

Then the $B_i \subseteq A_i$ are Π^1_1 as $\preceq$ and $\prec$ are Σ^1_1. If $x \in A_0$ and $x \notin A_1$, then $x \in\in B_0$, as $T_{1,x}$ is ill-founded and $T_{0,x}$ is well-founded. Similarly if $x \in A_1$ and not in A_0. On the other hand if $x \in A_0 \cap A_1$, then either $T_{0,x} \preceq T_{1,x}$ or $T_{1,x} \prec T_{0,x}$. In either case x is in at most one of the B_i. □

Corollary 5 (Σ^1_1 separation)**.** *Suppose $A, B \subseteq \omega^\omega$ are disjoint Σ^1_1. Then there exists a Δ^1_1 separator for A and B, i.e., a Δ^1_1 set C such that $A \subseteq C$ and $C \cap B = \emptyset$.*

Proof. As A, B are disjoint $A^c \cup B^c = \omega^\omega$. Let A_0 and B_0 be Π^1_1 sets reducing A^c, B^c, then $A_0^c = B_0$, so they are Δ^1_1. Let $C = B_0$, then $C \supseteq A$. But also $C \subseteq B^c$ and thus $C \cap B = \emptyset$. □

Lemma 6. *Suppose $B \subseteq \omega^\omega$ is Σ^1_1 and for every $x, y \in B$ xEy. Then there exists Δ^1_1 $D \supseteq B$ such that for all $x, y \in D$, xEy.*

Proof. Let $A = \{x \in \omega^\omega : \forall y y \in B \to xEy\}$. Then A is Π^1_1 and $A \supseteq B$. Thus $B \cap A^c = \emptyset$ and hence by Σ^1_1 separation there is Δ^1_1 C separating B and A^c. We have that $B \subseteq C \subseteq A$ and by transitivity of E, all elements of C are E equivalent. □

Theorem 7. *There are Π^1_1 sets $P^+, P^- \subseteq \omega \times x$ and $D \subseteq \omega$ such that*

1. *for any $n \in D$, $P^+_n = (P^-_n)^c$,*
2. *and for any Δ^1_1 set A there is $n \in D$ such that $A = P^+_n$.*

Proof. Let $P \subseteq \omega \times \omega^\omega$ be a universal Π^1_1 set and define

$$Q^+(\langle m, k\rangle, x) \Leftrightarrow P(m, x)$$
$$Q^-(\langle m, k\rangle x) \Leftrightarrow P(k, x)$$

Then Q^+ and Q^- are Π^1_1. By Π^1_1 reduction let P^+ and P^- be disjoint Π^1_1 subsets such that $P^+ \cup P^- = Q^+ \cup Q^-$. Define

$$n \in D \Leftrightarrow P^+_n \cup P^-_n = X$$

Then D is Π^1_1 and the properties 1 and 2 are satisfied. □

We now come to the heart of Harrington's proof. Define

$$H = \{x \in \omega^\omega : \text{ for no } \Delta^1_1 U \text{ s.t. } x \in U, U \subseteq [x]_E\}.$$

First assume that $H = \emptyset$, then every equivalence class contains a Δ^1_1 set. As there are only countably many Δ^1_1 sets, this implies that E contains only countably many equivalence classes. We proceed to show that if $H \neq \emptyset$, then E has perfectly many equivalence classes.

Note that H is Σ^1_1 as we have

$$x \in H \Leftrightarrow \forall U \in \Delta^1_1(x \in U \to \exists y(y \in U \wedge \neg xEy))$$

and can rewrite this as:

$$x \in H \Leftrightarrow \forall n((n \in D \wedge x \in P^+_n) \to \exists y(y \notin P^-_n \wedge \neg xEy)).$$

Lemma 8. *Suppose $c \in \omega^\omega$. Then $H \Vdash \neg \check{c} E \dot{a}$.*

Proof. Assume , and let $C \subseteq H$ such that $C \Vdash cEa$. There must be two reals $c_0, c_1 \in C$ such that $\neg c_0 E c_1$, otherwise we would have that there is a Δ^1_1 superset of C that meets only one equivalence class. But these are disjoint from H. Let

$$Q = \{c : c_0 \in C, c_1 \in C, \ \& \ \neg c_0 E c_1\}$$

and let $a \in Q$ be generic. Then both a_0 and a_1 are generic, $\neg a_0 E a_1$, $a_0, a_1 \in C$. But then $a_0 E c$, $a_1 E c$ and $\neg a_0 E a_1$ and as being an equivalence relation is absolute we obtain a contradiction. $\square$

Notice that the above lemma implies that if (a_0, a_1) is $\mathbb{P} \times \mathbb{P}$ generic over V and $a_1 \in H$, then $\neg a_0 E a_1$ as a_1 is $\mathbb{P}$-generic over $V[a_0]$. To finish the proof it remains to show that there is a perfect set of product generics.

Lemma 9. *Suppose M is a countable transitive model of ZFC and $\mathbb{P}$ is partially ordered in M. Then there exists $\{G : x \in 2^\omega\}$, a perfect set of $\mathbb{P}$-filters, such that for every $x \neq y$, (G_x, G_y) is $\mathbb{P} \times \mathbb{P}$-generic over M.*

Proof. Let D_n list all dense open subsets of $\mathbb{P} \times \mathbb{P}$ in M and construct $\langle p_s : s \in 2^{<\omega} \rangle$ by induction on the length of s so that

1. $s \subseteq t$ implies $p_t \leq p_s$ and
2. $|s| = |t| = n + 1$ and s and t are distinct, then $(p_s, p_t) \in D_n$. Define for any $x \in 2^\omega$
$$G_x = \{p \in \mathbb{P} : \exists n p_{x \restriction n} \leq p\}.$$

$\square$

To finish the proof let M be a countable transitive set isomorphic to an elementary substructure of V_κ for some sufficiently large κ. Let $\{G_x : x \in 2^\omega\}$ be the generic filter given by the above lemma and let $P = \{a_x : x \in 2^\omega\}$ be the corresponding generic reals. By Lemma 8 we have that for $x \neq y \in 2^\omega$, $a_x \ E a_y$. Furthermore P is perfect, as the map $x \to a_x$ is continuous. This is because for any $n \in \omega$ there exists $m < \omega$ such that every p_s with $s \in 2^m$ decides $a \restriction n$.

Corollary 10. *Every Σ^1_1 set which contains a real which is Δ^1_1 contains a perfect subset.*

Proof. Let $A \subseteq \omega^\omega$ be a Σ^1_1 set. Define xEy iff $x, y \notin A$ or $x = y$. Then E is Π^1_1. As A contains a real that is not Δ^1_1, H is non-empty, so there is a perfect set P of non-equivalent elements which is a subset of A. $\square$

Corollary 11. *Every uncountable $\boldsymbol{\Sigma}^1_1$ set contains a perfect subset.*

Analytic equivalence relations

What about analytic equivalence relations? Unfortunately, the Silver dichotomy fails for analytic equivalence relations. Consider the following example:

$$x \sim y \Leftrightarrow \exists f : x \cong y \vee x, y \notin WO$$

Then $\sim$ contains isomorphism classes of well orders and one class containing all non-well ordered reals.

Lemma 12 ($\boldsymbol{\Sigma}^1_1$ boundedness)**.** *If $A \subseteq WO$ is $\boldsymbol{\Sigma}^1_1$, then there is α such that $A \subseteq WO_{<\alpha}$.*

Proof. Assume not, then

$$x \in WO \Leftrightarrow \exists (y \in A) x \hookrightarrow y$$

and thus WO would be $\boldsymbol{\Sigma}^1_1$, a contradiction. □

So if X was a perfect subset of non-equivalent elements, then it would be $\boldsymbol{\Sigma}^1_1$ and so would be $X \cap WO$ which is off by at most 1 in size of X. Then, by the lemma, X would be countable, a contradiction.

One dichotomy for $\boldsymbol{\Sigma}^1_1$ equivalence relation is due to Burgess.

Theorem 13. *Let E be a $\boldsymbol{\Sigma}^1_1$ equivalence relation. Then there are Borel equivalence relations E_α for $\alpha \leq \omega_1$ such that $E = \bigcap_{\alpha < \omega_1} E_\alpha$.*

Proof idea. Assume that E is $\boldsymbol{\Sigma}^1_1$, then there is a continuous map $(x, y) \mapsto T_{xy}$ such that

$$xEy \Leftrightarrow T_{xy} \notin WF$$

Let $xE_\alpha y$ if and only if T_{xy} has rank greater than α. E_α is Borel but an equivalence relation for all α. The theorem is proven by showing that there exists a club C such that every relation in $\{E_\alpha : \alpha \in C\}$ is an equivalence relation. □

A corollary of this theorem is that E has either a perfect set of non-equivalent elements or $\leq \omega_1$ many equivalence classes.