

POSITIVE ENUMERABLE FUNCTORS

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ABSTRACT. We study an effectivization of functors well suited to compare structures and classes of structures with respect to properties based on enumeration reducibility. Our main results show that positive enumerable functors, an effectivization of functors based on enumeration reducibility that has access to the positive information of a structure, are equivalent to the syntactic notion of Σ_1^c interpretability. We also study the relationship between positive enumerable functors and computable functors. We show that that positive enumerable functors and computable functors are independent while enumerable functors and computable functors yield equivalent notions of reduction.

1. INTRODUCTION

In this article we study notions of reductions that let us compare classes of structures with respect to their computability theoretic properties. Computability theoretic reductions between classes of structures can be formalized using effective versions of the category theoretic notion of a functor. While computable functors have already been used in the 80's by Goncharov [Gon80], the formal investigation of this notion was only started recently after R. Miller, Poonen, Schoutens, and Shlapentokh [Mil+18] explicitly used a computable functor to obtain a reduction from the class of graphs to the class of fields. Their result shows that fields are universal with respect to many properties studied in computable structure theory.

Harrison-Trainor, Melnikov, R. Miller, and Montalbán [Har+17] investigated notions of reducibility based on computable functors and showed that they are equivalent to variations of interpretability using computable infinitary formulas. In [HMM18] these results were generalized to show that if a functor between two classes \mathfrak{C} and \mathfrak{D} is given by Δ_α^0 operators then there is an interpretation of \mathfrak{D} in \mathfrak{C} using Δ_α^0 infinitary formulas. A similar correspondence was obtained for Δ_α^0 operators and computable Δ_α^0 formulas.

In [Ros17] the third author studied effective versions of functors based on enumeration reducibility and their relation to notions of interpretability. There, it was shown that the existence of a computable functor implies the existence of an enumerable functor effectively isomorphic to it. In that article there also appeared an unfortunately incorrect claim that enumerable functors are equivalent to a variation of effective interpretability. Indeed, it was later shown in Rossegger's thesis [Ros19], that the existence of a computable functor implies the existence of an enumerable functor and thus the two notions are equivalent. We provide in this paper a simple

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proof of the latter result. It is not very surprising that enumerable and computable functors are equivalent, as the enumeration operators witnessing the effectiveness of an enumerable functor are given access to the atomic diagrams of structures, which are total sets.

The main objective of this article is the study of positive enumerable functors, an effectivization of functors that grants the involved enumeration operators access to the positive diagrams of structures instead of the atomic diagrams. While computable functors are well suited to compare structures with respect to properties related to relative computability and the Turing degrees, positive enumerable functors provide the right framework to compare structures with respect to their enumerations and properties related to the enumeration degrees.

The paper is organized as follows. In Section 2, we first show that computable functors and enumerable functors are equivalent, and then begin the study of positive enumerable functors and reductions based on them. We show that reductions by positive enumerable bi-transformations preserve enumeration degree spectra, a generalization of degree spectra considering all enumerations of a structure. We then exhibit an example consisting of two structures which are computably bi-transformable but whose enumeration degree spectra are different. This implies that positive enumerable functors and computable functors are independent notions. Towards the end of the section we show that our definition of positive enumerable functor is robust in the sense that other possible definitions are equivalent to ours, and we extend our results to reductions between arbitrary classes of structures.

In Section 4, we prove that the existence of a positive enumerable functor from \mathcal{A} to \mathcal{B} is equivalent to \mathcal{B} being Σ_1^c interpretable in \mathcal{A} . Σ_1^c interpretability is an adaptation of effective interpretability, a notion introduced by Montalbán [Mon12]. In [Har+17] it was shown that effective interpretability is equivalent to computable functors. In [Har+17] an analogous result was shown for computa

2. COMPUTABLE AND ENUMERABLE FUNCTORS

In this article we assume that our structures are in a relational language $(R_i)_{i \in \omega}$ where each R_i has arity a_i and the map $i \mapsto a_i$ is computable. We furthermore only consider countable structures with universe ω . We view classes of structures as categories where the objects are structures in a given language \mathcal{L} and the morphisms are isomorphisms between them.

Definition 2.1. Let $\mathfrak{C}, \mathfrak{D}$ be classes of structures. A *functor* F from \mathfrak{C} to \mathfrak{D} is a map of objects and morphisms. Given $\mathcal{A} \in \mathfrak{C}$, $F(\mathcal{A}) \in \mathfrak{D}$, and given a morphism $f \in \text{Hom}(\mathcal{A}, \mathcal{B})$ for structures $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$, $F(f) \in \text{Hom}(F(\mathcal{A}), F(\mathcal{B}))$. Moreover, for any $\mathcal{A} \in \mathfrak{C}$ and compatible morphisms f, g , we have

- (1) $F(\text{Id}_{\mathcal{A}}) = \text{Id}_{F(\mathcal{A})}$,
- (2) $F(f \circ g) = F(f) \circ F(g)$.

The smallest classes are isomorphism classes of a single structure \mathcal{A} ,

$$\text{Iso}(\mathcal{A}) = \{\mathcal{B} : \mathcal{B} \cong \mathcal{A}\}.$$

We will often talk about a functor from \mathcal{A} to \mathcal{B} , $F : \mathcal{A} \rightarrow \mathcal{B}$ when we mean a functor $F : Iso(\mathcal{A}) \rightarrow Iso(\mathcal{B})$. Depending on the properties that we want our functor to preserve we may use different effectivizations, but they will all be of the following form. Generally, an effectivization of a functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ will consist of a pair of operators (Φ, Φ_*) and a suitable coding C such that

- (1) for all $\mathcal{A} \in \mathfrak{C}$, $\Phi(C(\mathcal{A})) = C(F(\mathcal{A}))$,
- (2) for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$ and $f \in Hom(\mathcal{A}, \mathcal{B})$, $\Phi(C(\mathcal{A}), f, C(\mathcal{B})) = F(f)$.

In this article the operators will either be enumeration or Turing operators. If the coding is clear from context we will omit the coding function, i.e., we write $\Phi(\mathcal{A})$ instead of $\Phi(C(\mathcal{A}))$. The most common coding in computable structure theory is the following.

Definition 2.2. Let \mathcal{A} be a structure in relational language $(R_i)_{i \in \omega}$. Then the *atomic diagram* $D(\mathcal{A})$ of \mathcal{A} is the set

$$\bigoplus_{i \in \omega} R_i^{\mathcal{A}} \oplus \bigoplus_{i \in \omega} \neg R_i^{\mathcal{A}}.$$

In the literature one can often find different definitions of the atomic diagram. It is easy to show that all of these notions are Turing and enumeration equivalent. The reason why we chose this definition is that it is conceptually easier to define the positive diagram and deal with enumerations of structures like this. We are now ready to define various effectivizations of functors.

Definition 2.3 ([Mil+18],[Har+17]). A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is *computable* if there is a pair of Turing operators (Φ, Φ_*) such that for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$

- (1) $\Phi^{D(\mathcal{A})} = D(F(\mathcal{A}))$,
- (2) for all $f \in Hom(\mathcal{A}, \mathcal{B})$, $\Phi^{D(\mathcal{A}) \oplus f \oplus D(\mathcal{B})} = F(f)$.

Definition 2.4 ([Ros17]). A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is *enumerable* if there is a pair (Ψ, Φ_*) where Ψ is an enumeration operator and Φ_* a Turing operator such that for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$

- (1) $\Psi^{D(\mathcal{A})} = D(F(\mathcal{A}))$,
- (2) for all $f \in hom(\mathcal{A}, \mathcal{B})$, $\Phi_*^{D(\mathcal{A}) \oplus f \oplus D(\mathcal{B})} = F(f)$.

One might ask why this definition of enumerable functor was chosen and not one where the second operator is also an enumeration operator. We will see at the end of this section that this definition is equivalent to Definition 2.4.

In [Ros17] it was shown that the existence of an enumerable functor implies the existence of a computable functor. We now give a simple proof of the converse. This result first appeared in Rossegger's thesis.

Theorem 2.5. *If $F : \mathcal{A} \rightarrow \mathcal{B}$ is a computable functor, then it is enumerable.*

Proof. Let $D(L_{\mathcal{A}})$ denote the collection of finite atomic diagrams in the language of \mathcal{A} . To every $p \in D(L_{\mathcal{A}})$ we associate a finite string α_p in the alphabet $\{0, 1, \uparrow\}$

where $\alpha_p(x) = 1$ if $x \in p$, $\alpha_p(x) = 0$ if $x = 2\langle i, u \rangle$ and $2\langle i, u \rangle + 1 \in p$ or $x = 2\langle i, u \rangle + 1$ and $2\langle i, u \rangle \in p^1$, and $\alpha_p(x) = \uparrow$ if x is less than the largest element of p and none of the other cases fits. We furthermore associate a string $\tilde{\alpha}_p \in 2^{|\alpha_p|}$ with p where $\tilde{\alpha}_p(x) = 1$ if and only if $\alpha_p(x) = 1$ and $\tilde{\alpha}_p(x) = 0$ if and only if $\alpha_p(x) = 0$ or $\alpha_p(x) = \uparrow$.

Let the computability of F be witnessed by (Φ, Φ_*) . We build the enumeration operator Ψ as follows. For every $p \in D(L_{\mathcal{A}})$ and every x if $\Phi^{\tilde{\alpha}_p}(x) \downarrow = 1$ and every call to the oracle during the computation is on an element z such that $\alpha_p(z) \neq \uparrow$, then enumerate (p, x) into Ψ . This finishes the construction of Ψ .

Now, let $\hat{\mathcal{A}} \cong \mathcal{A}$. We have that $x \in \Psi^{\hat{\mathcal{A}}}(x)$ if and only if there exists $p \in D(L_{\mathcal{A}})$ such that $p \subseteq D(\hat{\mathcal{A}})$ and $(p, x) \in \Psi$. We further have that $(p, x) \in \Psi$ if and only if $\Phi^{\tilde{\alpha}_p}(x) \downarrow = 1$ if and only if $\Phi^{\hat{\mathcal{A}}}(x) = 1$. Thus F is enumerable, witnessed by (Ψ, Φ_*) . \square

Combining Theorem 2.5 with the results from [Ros17] we obtain that enumerable functors and computable functors defined using the atomic diagram of a structure as input are equivalent notions. This is not surprising. After all, the atomic diagram of a structure always has total enumeration degree and there is a canonical isomorphism between the total enumeration degrees and the Turing degrees. In order to make this equivalence precise we need another definition.

Definition 2.6 ([Har+17]). A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is *effectively isomorphic* to a functor $G : \mathfrak{C} \rightarrow \mathfrak{D}$ if there is a Turing functional Λ such that for any $\mathcal{A} \in \mathfrak{C}$, $\Lambda^{\mathcal{A}} : F(\mathcal{A}) \rightarrow G(\mathcal{A})$ is an isomorphism. Moreover, for any morphism $h \in \text{Hom}(\mathcal{A}, \mathcal{B})$ in \mathfrak{C} , $\Lambda^{\mathcal{B}} \circ F(h) = G(h) \circ \Lambda^{\mathcal{A}}$. That is, the diagram below commutes.

$$\begin{array}{ccc} F(\mathcal{A}) & \xrightarrow{\Lambda^{\mathcal{A}}} & G(\mathcal{A}) \\ F(h) \downarrow & & \downarrow G(h) \\ F(\mathcal{B}) & \xrightarrow{\Lambda^{\mathcal{B}}} & G(\mathcal{B}) \end{array}$$

The following is now an immediate corollary of Theorem 2.5 and [Ros17, Theorem 2].

Theorem 2.7. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor. Then F is computable if and only if there is an enumerable functor $G : \mathcal{A} \rightarrow \mathcal{B}$ effectively isomorphic to F .*

We now turn our attention to notions of reduction based on effective functors that preserve many computability theoretic properties. In order to give the definitions, we need another definition.

¹That is if p specifies that R_i holds on elements coded by u , then we set that $\neg R_i$ does not hold on these elements.

Definition 2.8 ([Har+17]). Suppose $F : \mathfrak{C} \rightarrow \mathfrak{D}$, $G : \mathfrak{D} \rightarrow \mathfrak{C}$ are functors such that $G \circ F$ is effectively isomorphic to $Id_{\mathfrak{C}}$ via the Turing functional $\Lambda_{\mathfrak{C}}$ and $F \circ G$ is effectively isomorphic to $Id_{\mathfrak{D}}$ via the Turing functional $\Lambda_{\mathfrak{D}}$. If furthermore, for any $\mathcal{A} \in \mathfrak{C}$ and $\mathcal{B} \in \mathfrak{D}$, $\Lambda_{\mathfrak{D}}^{F(\mathcal{A})} = F(\Lambda_{\mathfrak{C}}^{\mathcal{A}}) : F(\mathcal{A}) \rightarrow F(G(F(\mathcal{A})))$ and $\Lambda_{\mathfrak{C}}^{G(\mathcal{B})} = G(\Lambda_{\mathfrak{D}}^{\mathcal{B}}) : G(\mathcal{B}) \rightarrow G(F(G(\mathcal{B})))$, then F and G are said to be *pseudo inverses*.

Definition 2.9 ([Har+17]). Two structures \mathcal{A} and \mathcal{B} are *computably bi-transformable* if there are computable pseudo-inverse functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$.

If the functors in Definition 2.9 are enumerable instead of computable then we say that \mathcal{A} and \mathcal{B} are *enumerably bi-transformable*. As an immediate corollary of Theorem 2.7 we obtain the following.

Corollary 2.10. *Two structures \mathcal{A} and \mathcal{B} are enumerably bi-transformable if and only if they are computably bi-transformable.*

3. EFFECTIVIZATIONS USING POSITIVE DIAGRAMS

We now turn our attention to the setting where we only have positive information about the structure available and related properties of structures. We follow Soskov [Sos04] in our definitions. See also the survey paper by Soskova and Soskova [SS17] on computable structure theory and enumeration degrees.

Definition 3.1. Let \mathcal{A} be a structure in relational language $(R_i)_{i \in \omega}$. The *positive diagram* of \mathcal{A} , denoted by $P(\mathcal{A})$, is the set

$$= \oplus \neq \oplus \bigoplus_{i \in \omega} R_0^{\mathcal{A}}.$$

We are interested in the degrees of enumerations of $P(\mathcal{A})$. To be more precise let f be an enumeration of ω and for $X \subseteq \omega^n$ let

$$f^{-1}(X) = \{\langle x_1, \dots, x_n \rangle : \langle f(x_1), \dots, f(x_n) \rangle \in X\}.$$

Given \mathcal{A} let $f^{-1}(\mathcal{A}) = f^{-1}(=) \oplus f^{-1}(\neq) \oplus f^{-1}(R_0^{\mathcal{A}}) \oplus \dots$. Notice that if $f = id$, then f^{-1} is just the positive diagram of \mathcal{A} .

Definition 3.2. The *enumeration degree spectrum* of \mathcal{A} is the set

$$eSp(\mathcal{A}) = \{d_e(f^{-1}(\mathcal{A})) : f \text{ is an enumeration of } \omega\}.$$

If \mathbf{a} is the least element of $eSp(\mathcal{A})$, then \mathbf{a} is called the *enumeration degree* of \mathcal{A} .

In order to obtain a notion of reduction that preserves enumeration spectra we need an effectivization of functors where we use positive diagrams of structures as coding. It is clear that for computable functors this makes no difference as $P(\mathcal{A}) \equiv_T D(\mathcal{A})$. For enumerable functors it does make a difference. We prepend the word positive to our notions to avoid confusion.

Proposition 3.3. *Let \mathcal{A} and \mathcal{B} be positively enumerably bi-transformable. Then $eSp(\mathcal{A}) = eSp(\mathcal{B})$.*

Proof. Say \mathcal{A} and \mathcal{B} are positively enumerably bi-transformable by $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$. Let f be an arbitrary enumeration of \mathcal{A} , then, viewing $f^{-1}(\mathcal{A})/=$ as a structure on ω by pulling back a canonic enumeration of the least elements in its $=$ -equivalence classes, we have that there is $\hat{\mathcal{A}} \cong \mathcal{A}$ such that $P(\hat{\mathcal{A}}) = f^{-1}(\mathcal{A})/=$ and $P(\hat{\mathcal{A}}) \leq_e f^{-1}(\mathcal{A})$. As F is positive enumerable we have that $f^{-1}(\mathcal{A}) \geq_e P(F(\hat{\mathcal{A}}))$. Furthermore, there is an enumeration g such that $f^{-1}(\mathcal{A}) \geq_e g^{-1}(F(\hat{\mathcal{A}}))$, g has the same multiplicity as f , i.e., if f enumerates $n \in \omega$ $\alpha \leq \omega$ times, then so does g , and $P(F(\hat{\mathcal{A}})) = g^{-1}(F(\hat{\mathcal{A}}))/=$. To see this first notice that from the $=$ -part of an enumeration of $P(F(\hat{\mathcal{A}}))$ we can obtain a permutation \hat{g} of ω . To get g with the desired properties we construct it in stages. At stage s we let $g(2s) = \hat{g}(\mu x[\hat{g}(x) \notin \text{range}(g)])$ and if there is a least $n < s$ such that f enumerates $f(n)$ more often up to $f(s)$ than g enumerates $\hat{g}(n)$, then let $g(2s+1) = \hat{g}(n)$, otherwise let $g(2s+1) = \hat{g}(\mu x[\hat{g}(x) \notin \text{range}(g)])$. It is not hard to check that g so defined satisfies the desired properties.

By a similar argument we get h such that $P(G(F(\hat{\mathcal{A}}))) = h^{-1}(G(F(\hat{\mathcal{A}})))/=$ and

$$h^{-1}(G(F(\hat{\mathcal{A}}))) \leq_e g^{-1}(F(\hat{\mathcal{A}})) \leq_e f^{-1}(\mathcal{A}).$$

At last, as $G(F(\hat{\mathcal{A}}))$ is computably isomorphic to $\hat{\mathcal{A}}$ we get that

$$f^{-1}(\mathcal{A}) \leq_e h^{-1}(G(F(\hat{\mathcal{A}}))) \leq_e g^{-1}(F(\hat{\mathcal{A}})) \leq_e f^{-1}(\mathcal{A})$$

and thus $eSp(\mathcal{A}) \subseteq eSp(\mathcal{B})$. The proof that $eSp(\mathcal{B}) \subseteq eSp(\mathcal{A})$ is analogous. \square

Proposition 3.4. *There are computably bi-transformable structures \mathcal{A} and \mathcal{B} , such that $eSp(\mathcal{A}) \neq eSp(\mathcal{B})$.*

Proof. Let $\mathcal{A} = (\omega, \underline{0}, s, K)$ where s is the successor relation on ω , $\underline{0}$ the first element, and K the membership relation of the halting set. Assume $\mathcal{B} = (\omega, \underline{0}, s, \overline{K})$ is defined as \mathcal{A} except that $\overline{K}(x)$ if and only if $\neg K(x)$. There is a computable functor $F : \mathcal{A} \rightarrow \mathcal{B}$ taking $\hat{\mathcal{A}} = (\omega, \underline{0}^{\hat{\mathcal{A}}}, s^{\hat{\mathcal{A}}}, K^{\hat{\mathcal{A}}}) \cong \mathcal{A}$ to $F(\hat{\mathcal{A}}) = (\omega, \underline{0}^{\hat{\mathcal{A}}}, s^{\hat{\mathcal{A}}}, \neg K^{\hat{\mathcal{A}}})$ and acting as the identity on isomorphisms. Furthermore, G has a computable inverse and thus \mathcal{A} is computably bi-transformable to \mathcal{B} .

However, \mathcal{A} has enumeration degree $\mathbf{0}'_e$ and \mathcal{B} has enumeration degree $\overline{\mathbf{0}}'_e$. Thus there can not be a positive enumerable functor from \mathcal{B} to \mathcal{A} . \square

The following shows that computable functors and positive enumerable functors are independent notions.

Proposition 3.5. *There are structures \mathcal{A} and \mathcal{B} such that \mathcal{A} is positively enumerably bi-transformable with \mathcal{B} but \mathcal{A} is not computably bi-transformable with \mathcal{B} .*

Proof. Let \mathcal{A} be as in Proposition 3.4, i.e., $\mathcal{A} = (\omega, \underline{0}, s, K)$ and $\mathcal{B} = (\omega, \underline{0}, s)$. Then it is not hard to see that \mathcal{A} is positively enumerably bi-transformable with \mathcal{B} . However, there can not be a computable functor from \mathcal{B} to \mathcal{A} as \mathcal{B} has Turing degree $\mathbf{0}$ and \mathcal{A} has Turing degree $\mathbf{0}'$. \square

3.1. Alternative definitions. That an enumerable functor is given by an enumeration and Turing operator might appear forced and one might be tempted to define an enumerable functor with a pair of enumeration operators as in the following definition.

Definition 3.6. A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is \star -enumerable if there is a pair of enumeration operators (Ψ, Ψ_*) such that for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$

- (1) $\Psi^{D(\mathcal{A})} = F(\mathcal{A})$,
- (2) for all $f \in \text{hom}(\mathcal{A}, \mathcal{B})$, $\Psi_*^{D(\mathcal{A}) \oplus f \oplus D(\mathcal{B})} = \text{Graph}(F(f))$.

It turns out that the two definitions are equivalent.

Proposition 3.7. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is (positive) enumerable if and only if it is (positive) \star -enumerable.

Proof. That any (positive) enumerable functor is (positive) \star -enumerable is trivial. To see the converse, say we have a (positive) \star -enumerable functor given by (Ψ, Ψ_*) and an isomorphism $f : \mathcal{A} \rightarrow \hat{\mathcal{A}}$. We can compute the isomorphism $F(f)$ by enumerating $\text{Graph}(F(f))$ using $\Psi_*^{A \oplus f \oplus \hat{A}}$. For every x we are guaranteed to enumerate $(x, y) \in \text{Graph}(F(f))$ for some y as the domain of \mathcal{A} is ω . This is uniform in \mathcal{A} , f and $\hat{\mathcal{A}}$. Thus there is a Turing operator Φ_* such that (Ψ, Φ_*) witnesses that F is (positive) enumerable. \square

3.2. Reductions between classes of structures. Using effectivizations of functors we can get strong notions of reductions between classes of structures.

Definition 3.8 ([Har+17]). Let \mathfrak{C} and \mathfrak{D} be classes of structures. The class \mathfrak{C} is *uniformly computably transformably reducible*, short u.c.t. reducible, to \mathfrak{D} if there are a subclass $\mathfrak{D}' \subseteq \mathfrak{D}$ and computable functors $F : \mathfrak{C} \rightarrow \mathfrak{D}' \subseteq \mathfrak{D}$ and $G : \mathfrak{D}' \rightarrow \mathfrak{C}$ such that F and G are pseudo-inverses.

Definition 3.9. Let \mathfrak{C} and \mathfrak{D} be classes of structures. The class \mathfrak{C} is *uniformly (positively) enumerably transformably reducible*, short u.e.t., (u.p.e.t.) reducible, to \mathfrak{D} if there are a subclass $\mathfrak{D}' \subseteq \mathfrak{D}$ and (positive) enumerable functors $F : \mathfrak{C} \rightarrow \mathfrak{D}' \subseteq \mathfrak{D}$ and $G : \mathfrak{D}' \rightarrow \mathfrak{C}$ such that F and G are pseudo-inverses.

Propositions 3.4 and 3.5 show that u.p.e.t. and u.c.t reductions are independent notions.

Corollary 3.10. There are classes of structures $\mathfrak{C}_1, \mathfrak{C}_2$ and $\mathfrak{D}_1, \mathfrak{D}_2$ such that

- (1) \mathfrak{C}_1 is u.c.t. reducible to \mathfrak{D}_1 but \mathfrak{C}_1 is not u.p.e.t. reducible to \mathfrak{D}_1 .
- (2) \mathfrak{C}_2 is u.p.e.t. reducible to \mathfrak{D}_2 but \mathfrak{C}_2 is not u.c.t. reducible to \mathfrak{D}_2 .

Similar to Corollary 2.10 we obtain the equivalence of u.e.t. and u.c.t reductions.

Corollary 3.11. Let \mathfrak{C} and \mathfrak{D} be arbitrary classes of countable structures. Then \mathfrak{C} is u.e.t. reducible to \mathfrak{D} if and only if it is u.c.t. reducible to \mathfrak{D} .

4. A SYNTACTIC EQUIVALENT

In this section we show that the existence of a positive enumerable functor from a structure \mathcal{A} to a structure \mathcal{B} implies that \mathcal{B} is interpretable in \mathcal{A} via an interpretation that uses infinitary formulas. We start by giving necessary definitions.

Definition 4.1. Let \mathcal{A} be a structure. A relation $R \subseteq \omega \times A^{<\omega}$ of \mathcal{A} is *uniformly relatively intrinsically computably enumerable*, short *u.r.i.c.e.*, in \mathcal{A} if there is a c.e. operator W such that $(\mathcal{A}, R) \cong (\mathcal{B}, W^{D(\mathcal{B})})$ for all $\mathcal{B} \cong \mathcal{A}$.

A relation $R \subseteq A^{<\omega}$ is *uniformly relatively intrinsically computable* if both R and \bar{R} are u.r.i.c.e. A sequence of relations $(R_i)_{i \in \omega}$ is u.r.i.c.e. if there is a computable function f such that $W_{f(i)}$ witnesses that R_i is a u.r.i.c.e. relation.

Recall that a $L_{\omega_1, \omega}$ formula φ in a language \mathcal{L} is Σ_1^c if it is equivalent to a computable disjunction of existential \mathcal{L} -formulas. A relation $R \subseteq \omega \times A^{<\omega}$ is Σ_1^c definable in \mathcal{A} if there is a computable sequence of Σ_1^c formulas $\varphi_{i,j}$ such that

$$R = \{\langle i, \bar{a} \rangle : \mathcal{A} \models \varphi_{i, |\bar{a}|}(\bar{p}, \bar{a})\}$$

where \bar{p} is a possibly empty tuple of parameters from \mathcal{A} .

The following theorem for relations of finite arity is due to Ash, Knight, Manasse, Slaman [Ash+89] and independently Chisholm [Chi90]. The proof for relations of unbounded arity is not much different, see [Mon18].

Theorem 4.2. *Let \mathcal{A} be a structure and $R \subseteq \omega \times A^{<\omega}$. The following are equivalent:*

- (1) *R is u.r.i.c.e.*
- (2) *R is Σ_1^c definable in \mathcal{A} without parameters.*

In [Mon12], Montalbán defined the notion of effective interpretability, a version of interpretability using computable infinitary formulas. It was shown in [Har+17] that there is a computable functor from \mathcal{A} to \mathcal{B} if and only if \mathcal{B} is effectively interpretable in \mathcal{A} . The main goal of this section is to obtain a similar result for positive enumerable functors. We will show that positive enumerable functors are equivalent to the following version of interpretability. The main difference between our definition and effective interpretability is that we only require the relations to be u.r.i.c.e. while effective interpretability requires them to be u.r.i. computable.

Definition 4.3. Let \mathcal{A} be a structure in the language $(P_i)_{i \in \omega}$ where P_i is of arity a_i . We say that a structure \mathcal{A} is Σ_1^c *interpretable* in \mathcal{B} if there exists u.r.i. computable relations $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}, \sim^{\mathcal{B}}$ in \mathcal{B} and a u.r.i.c.e. sequence of relations $(R_i^{\mathcal{B}})_{i \in \omega}$ in \mathcal{B} such that

- (1) $\sim^{\mathcal{B}}$ is an equivalence relation on $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$.
- (2) for each $i \in \omega$, $R_i^{\mathcal{B}} \subseteq (\text{Dom}_{\mathcal{A}}^{\mathcal{B}})^{a_i}$ is closed under $\sim^{\mathcal{B}}$.

and there is a function $f_{\mathcal{A}}^{\mathcal{B}} : \text{Dom}_{\mathcal{A}}^{\mathcal{B}} \rightarrow A$, the Σ_1^c *interpretation* of \mathcal{A} in \mathcal{B} , which induces an isomorphism:

$$f_{\mathcal{A}}^{\mathcal{B}} : (\text{Dom}_{\mathcal{A}}^{\mathcal{B}}, R_0^{\mathcal{B}}, R_1^{\mathcal{B}}, \dots) / \sim^{\mathcal{B}} \cong \mathcal{A}$$

Notice that while we use u.r.i. computable and u.r.i.c.e. in Definition 4.3, Theorem 4.2 shows that it is in essence syntactic. The following theorem is the main result of this section.

Theorem 4.4. *Let \mathcal{A} and \mathcal{B} be countable structures in relational languages. Then there is a positive enumerable functor $F : \mathcal{A} \rightarrow \mathcal{B}$ if and only if \mathcal{B} is Σ_1^c interpretable in \mathcal{A} .*

Theorem 4.4 is constructive in the sense that given a Σ_1^c interpretation of \mathcal{B} in \mathcal{A} we can compute indices for a positive enumerable functor $I : \mathcal{A} \rightarrow \mathcal{B}$. Likewise, given an enumerable functor $F : \mathcal{A} \rightarrow \mathcal{B}$ we can obtain indices for the sequence of relations giving the Σ_1^c interpretation of \mathcal{B} in \mathcal{A} and from this interpretation we get a functor I^F . We can obtain the following.

Theorem 4.5. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a positive enumerable functor. Then I^F and F are effectively isomorphic.*

We prove the two directions of Theorem 4.5 separately in Lemmas 4.6 and 4.7.

Lemma 4.6. *Say \mathcal{A} is Σ_1^c interpretable in \mathcal{B} , then there is a positive enumerable functor $F : \mathcal{B} \rightarrow \mathcal{A}$.*

Proof. Suppose \mathcal{A} is Σ_1^c interpretable in \mathcal{B} and let $\mathcal{I}_{\mathcal{A}}^{\mathcal{B}} = (Dom_{\mathcal{A}}^{\mathcal{B}}, R_0^{\mathcal{B}}, R_1^{\mathcal{B}}, \dots) / \sim^{\mathcal{B}}$ be the interpretation of \mathcal{A} inside \mathcal{B} , where $Dom_{\mathcal{A}}^{\mathcal{B}}$ and $\sim^{\mathcal{B}}$ are u.r.i. computable in \mathcal{B} and $(R_i)_{i \in \omega}$ is u.r.i.c.e. in \mathcal{B} . Let Φ_{Dom}, Φ_{\sim} be the Turing operators witnessing the u.r.i. computability of relations $Dom_{\mathcal{A}}^{\mathcal{B}}$, respectively, $\sim^{\mathcal{B}}$, and let W_0, W_1, \dots be c.e. operators witnessing the u.r.i.c.e.-ness of relations $R_0^{\mathcal{B}}, R_1^{\mathcal{B}}, \dots$. We will define a positive enumerable functor $F = (\Psi, \Psi_*)$ from $Iso(\mathcal{B})$ to $Iso(\mathcal{A})$. Let δ be the function mapping atomic formulas or their negations φ with variables from $\{x_i : i \in \omega\}$ to the code of $\varphi[x_i \mapsto i]$ in atomic diagrams in the language $L_{\mathcal{B}}$ and let ρ do the same but for positive diagrams, i.e., ρ maps atomic formulas or negations of equality φ with variables from $\{x_i : i \in \omega\}$ to their codes in positive diagrams in the language $L_{\mathcal{B}}$. Let $P(L_{\mathcal{B}})$ denote the collection of finite positive diagrams in the language of \mathcal{B} . To every $p \in P(L_{\mathcal{B}})$ we associate a finite string α_p in the alphabet $\{0, 1, \uparrow\}$ and a string $\tilde{\alpha}_p \in 2^{<\omega}$ as follows. We let $\alpha_p(x) = 1$ if $\rho(\delta^{-1}(x)) \in p$, $\alpha_p(x) = 0$ if $\delta^{-1}(x) = \neg\varphi$ and $\rho(\varphi) \in p$, and we let $\alpha_p(x) = \uparrow$ if $\rho(\delta^{-1}(x))$ is less than the largest element of p and none of the other cases fits. We furthermore associate a string $\tilde{\alpha}_p \in 2^{|\alpha_p|}$ with p where $\tilde{\alpha}_p(x) = 1$ if and only if $\alpha_p(x) = 1$ and $\tilde{\alpha}_p(x) = 0$ if and only if $\alpha_p(x) = 0$ or $\alpha_p(x) = \uparrow$.

Now, for each $p \in P(L_{\mathcal{B}})$ define the set Dom^{α_p} as $\bar{x} \in Dom^{\alpha_p}$ if and only if $\Phi_{Dom}^{\tilde{\alpha}_p}(\bar{x}) \downarrow = 1$ and the computation never requests oracle information at x such that $\alpha_p(x) = \uparrow$. We write $\bar{x} \downarrow \notin Dom^{\alpha_p}$ if and only if $\Phi_{Dom}^{\tilde{\alpha}_p}(\bar{x}) \downarrow = 0$ and the computation never requests oracle information at x such that $\alpha_p(x) = \uparrow$. The relation \sim^{α_p} and the notation $\bar{x} \downarrow \not\sim^{\alpha_p} \bar{y}$ are defined similarly.

We want the output of Ψ to be a structure with domain ω . Towards this we give an injective enumeration of representatives of the \sim -equivalence classes as follows.

Consider $\omega^{<\omega}$ under the lexicographical ordering and for $p \in P(L_{\mathcal{B}})$ define the partial function $m_{\alpha_p} : \omega \rightarrow \omega^{<\omega}$ recursively by

$$\begin{aligned} m_{\alpha_p}(0) &= \mu \bar{x} \in \omega^{<\omega} [\bar{x} \in \text{Dom}^{\alpha_p} \wedge \forall (\bar{y} <_{lex} \bar{x}) \bar{y} \downarrow \notin \text{Dom}^{\alpha_p}] \\ m_{\alpha_p}(n+1) &= \mu \bar{x} \in \omega^{<\omega} [\bar{x} \in \text{Dom}^{\alpha_p} \wedge \forall (\bar{y} <_{lex} \bar{x}) (\exists (m \leq n) \bar{y} \sim^{\alpha_p} m_{\alpha_p}(m) \vee \bar{y} \downarrow \notin \text{Dom}^{\alpha_p})] \end{aligned}$$

By definition m_{α_p} is partial computable uniformly in α . It is furthermore easy to see that if $p \subseteq q \in P(L_{\mathcal{B}})$, then $m_{\alpha_p} \subseteq m_{\alpha_q}$. Thus, given a structure $\hat{\mathcal{B}} \cong \mathcal{B}$ we can define $m_{\hat{\mathcal{B}}} : \omega \rightarrow \omega^{<\omega}$ as

$$m_{\hat{\mathcal{B}}} = \bigcup_{p \preceq P(\hat{\mathcal{B}})} m_{\alpha_p}.$$

We are now ready to give the definition of Ψ as follows. For any $p \in P(L_{\mathcal{B}})$ and $i, x_0, x_1, \dots \in \omega$

$$(p, x_0 = x_1) \in \Psi \Leftrightarrow x_0 = x_1$$

$$(p, x_0 \neq x_1) \in \Psi \Leftrightarrow x_0 \neq x_1$$

$$(p, P_i(x_0, \dots, x_n)) \in \Psi \Leftrightarrow R_i(m_{\alpha_p}(x_0), \dots, m_{\alpha_p}(x_n)) \in W_i^{\tilde{\alpha}_p} \wedge \alpha_p \in 2^{<\omega}$$

The operator Ψ so defined is clearly an enumeration operator. It remains to verify that given $\hat{\mathcal{B}} \cong \mathcal{B}$, $\Psi^{\hat{\mathcal{B}}}$ is the positive diagram of a structure isomorphic to $\mathcal{I}_{\mathcal{A}}^{\hat{\mathcal{B}}}$. Clearly $\Psi^{\mathcal{B}}$ produces a subset of a positive diagram of some structure in the language of \mathcal{A} . Furthermore, by construction $m_{\hat{\mathcal{B}}}$ is injective and hits all equivalence classes in $\text{Dom}_{\mathcal{A}}^{\hat{\mathcal{B}}} / \sim^{\hat{\mathcal{B}}}$. We argue that it is an isomorphism between $\mathcal{I}_{\mathcal{A}}^{\hat{\mathcal{B}}}$ and $\Psi^{\hat{\mathcal{B}}}$. For any P_i and $x_1, \dots, x_{a_i} \in \omega$ we have that

$$\begin{aligned} R_i(x_1, \dots, x_{a_i}) \in \Psi^{\hat{\mathcal{B}}} &\Leftrightarrow \text{there is } p \text{ an initial segment of } P(\hat{\mathcal{B}}) \\ &\text{s.t. } R_i(m_{\alpha_p}(x_0), \dots, m_{\alpha_p}(x_{a_i})) \in W_i^{\tilde{\alpha}_p} \\ &\Leftrightarrow \mathcal{I}_{\mathcal{A}}^{\hat{\mathcal{B}}} \models R_i([m_{\alpha_p}(x_1)], \dots, [m_{\alpha_p}(x_{a_i})]) \end{aligned}$$

where $[m_{\alpha_p}(x_i)]$ denotes the $\sim^{\hat{\mathcal{B}}}$ -equivalence class of $m_{\alpha_p}(x_i)$.

We now define Φ_* . Let $\hat{\mathcal{B}}, \tilde{\mathcal{B}} \in \text{Iso}(\mathcal{B})$ and $f : \hat{\mathcal{B}} \cong \tilde{\mathcal{B}}$. We can lift the isomorphism to a function on tuples $\bar{f} : \hat{\mathcal{B}}^{<\omega} \rightarrow \tilde{\mathcal{B}}^{<\omega}$. The function \bar{f} then induces an isomorphism between $\mathcal{I}_{\mathcal{A}}^{\hat{\mathcal{B}}}$ and $\mathcal{I}_{\mathcal{A}}^{\tilde{\mathcal{B}}}$. Recall that $m_{\hat{\mathcal{B}}}$ and $m_{\tilde{\mathcal{B}}}$ are computable from $P(\hat{\mathcal{B}})$, respectively $P(\tilde{\mathcal{B}})$. We can define $m_{\hat{\mathcal{B}}}^{-1}$ on $\text{Dom}_{\mathcal{A}}^{\hat{\mathcal{B}}}$ by

$$m_{\hat{\mathcal{B}}}^{-1}(\bar{x}) = \mu y [m_{\hat{\mathcal{B}}}(y) \sim^{\hat{\mathcal{B}}} \bar{x}]$$

and define $m_{\tilde{\mathcal{B}}}^{-1}$ likewise. It is easy to see that $m_{\tilde{\mathcal{B}}}^{-1} \circ \bar{f} \circ m_{\hat{\mathcal{B}}}$ is an isomorphism from $F(\hat{\mathcal{B}})$ to $F(\tilde{\mathcal{B}})$ computable uniformly in $\hat{\mathcal{B}} \oplus f \oplus \tilde{\mathcal{B}}$.

Finally, we show that F is indeed a functor by checking the two properties in Definition 2.1. Clearly, $F(\text{Id}_{\mathcal{B}}) = \tilde{m}_{\mathcal{B}}^{-1} \circ \bar{\text{Id}} \circ \tilde{m}_{\mathcal{B}} = \text{Id}_{F(\mathcal{B})}$. For isomorphisms $f : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ and $g : \tilde{\mathcal{B}} \rightarrow \bar{\mathcal{B}}$, $F(g) \circ F(f) = \tilde{m}_{\bar{\mathcal{B}}}^{-1} \circ \bar{g} \circ \tilde{m}_{\tilde{\mathcal{B}}} \circ \tilde{m}_{\tilde{\mathcal{B}}}^{-1} \circ \bar{f} \circ \tilde{m}_{\mathcal{B}} = \tilde{m}_{\bar{\mathcal{B}}}^{-1} \circ (\bar{f} \circ \bar{g}) \circ \tilde{m}_{\mathcal{B}} = \tilde{m}_{\bar{\mathcal{B}}}^{-1} \circ \overline{(f \circ g)} \circ \tilde{m}_{\mathcal{B}} = F(g \circ f)$. This shows that F is indeed a positive enumerable functor from $\text{Iso}(\mathcal{B})$ to $\text{Iso}(\mathcal{A})$. \square

For the following lemma we are able to relax our assumption that all structures have universe ω . Our proof goes through for arbitrary countable structures with universe a subset of ω .

The idea of the proof is similar to the one given in [Har+17] but the details differ.

Lemma 4.7. *Let $F : \mathcal{B} \rightarrow \mathcal{A}$ be a positive enumerable functor. Then \mathcal{A} is Σ_1^c interpretable in \mathcal{B} .*

Proof. Suppose F is given by (Ψ, Φ_*) . We will build a Σ_1^c interpretation of \mathcal{A} in \mathcal{B} .

The first obstacle in building this interpretation is the requirement that $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ is a subset of $B^{<\omega}$, but $F(\mathcal{B}) \subseteq \omega$. As a solution, we code ω into $B^{<\omega}$ using the computable functions

$$\eta_b : \omega \rightarrow B^{<\omega}, i \mapsto \underbrace{bb \dots b}_{i \text{ times}}$$

for all $b \in B$. Given any isomorphism $f : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$, we can extend f to a natural bijection \bar{f} from $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ to $\text{Dom}_{\mathcal{A}}^{\tilde{\mathcal{B}}}$. This extension is done component wise, i.e., $\bar{f} : B^{<\omega} \rightarrow B'^{<\omega}$ is defined by $\bar{f}(\bar{b}) = f(b_1) \dots f(b_n)$. To simplify notation, we will not distinguish f from \bar{f} , as which one we mean should be clear from context. The second obstacle comes from the observation that the encodings η_b do not ensure that $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ is u.r.i. computable in \mathcal{B} . Note that $\bar{f}(\eta_b(i)) = \eta_{f(b)}(i)$. The isomorphism f fixes i as the underlying information, but it is possible that $i \in \text{Dom}(F(\mathcal{B}))$ while $i \notin \text{Dom}(F(\tilde{\mathcal{B}}))$. To resolve this problem, we will define the elements of the domain to be $i \in \text{Dom}(F(\mathcal{B}))$, together with a finite substructure of $F(\mathcal{B})$ that has enough information to conclude that i is part of the domain.

Let B^* be the set of disjoint tuples of B . For $\bar{b}, \bar{c} \in B^*$, we let $\bar{b} - \bar{c} \in B^*$ be the tuple with elements occurring in \bar{b} but not in \bar{c} , in the order given by \bar{b} . Note that \bar{b} can be considered an initial segment of an injective function $f : \omega \rightarrow B$. If $\bar{b} \subseteq f$, and f is a bijection from ω (or \mathbf{n} if B is finite of size n) to B , we let $\mathcal{B}_f \cong \mathcal{B}$ be the pull-back of \mathcal{B} along f . Let $P(\bar{b})_s$ denote the finite partial positive diagram of \mathcal{B}_f mentioning the first $|\bar{b}|$ elements and the first s relation symbols. Clearly, we have $P(\bar{b})_s \subseteq P(\mathcal{B}_f)$. If f, g are injective enumerations of \mathcal{B} , then $g^{-1} \circ f : \mathcal{B}_f \cong \mathcal{B}_g$. Let σ be a permutation of $\{0, \dots, |\bar{b}| - 1\}$. We say that σ takes $\bar{b} \in B^*$ to $\bar{c} \in B^*$ if $\bar{b} = b_0 \dots b_{|\bar{b}|-1} = c_{\sigma(0)} \dots c_{\sigma(|\bar{b}|-1)}$. Note that if $\bar{b} \subseteq f$, $\bar{c} \subseteq g$, then σ being a permutation taking \bar{b} to \bar{c} implies $\sigma \subseteq g^{-1} \circ f$.

We will think of $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ as a subset of $B^* \times \omega$. Formally, however, $\text{Dom}_{\mathcal{A}}^{\mathcal{B}} \subseteq B^{<\omega}$. We accomplish this by coding an element $\bar{b}i \in B^* \times \omega$ as $\bar{b}\eta_{b_0}(i) \in B^{<\omega}$. The fact that \bar{b} is disjoint lets us recover i effectively. Moreover, to make the domain u.r.i. computable, we will need approximations to the enumeration operator Ψ . We will consider Ψ_s^X where $(\Psi_s)_{s \in \omega}$ is a computable approximation to Ψ and consider each Ψ_s as an enumeration operator on its own. We are now ready to define the interpretation.

Define $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ to be the set of tuples $(\bar{b}, \langle s, i \rangle) \in B^* \times \omega$ where $i \in \Psi_s^{P(\bar{b})_s}$ and

$$(1) \quad \Phi_*^{P(\bar{b})_s \oplus Id \upharpoonright |\bar{b}| \oplus P(\bar{b})_s}(i) \downarrow = i,$$

where Id is the identity function.

Define $\sim^{\mathcal{B}}$ to be the binary relation on $Dom_{\mathcal{A}}^{\mathcal{B}}$ such that $(\bar{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\bar{c}, \langle t, j \rangle)$ if and only if there is an $r \in \omega$ and $\bar{d} \in B^*$ such that \bar{d} doesn't contain any elements from \bar{b} and \bar{c} , and if $\bar{b}' = \bar{b} - \bar{c}$, $\bar{c}' = \bar{c} - \bar{b}$, and σ is a finite permutation taking $\overline{bc'd}$ to $\overline{cb'd}$, we have

$$(2) \quad \Phi_*^{P(\overline{bc'd})_r \oplus \sigma \oplus P(\overline{cb'd})_r}(i) \downarrow = j \text{ and } \Phi_*^{P(\overline{cb'd})_r \oplus \sigma^{-1} \oplus P(\overline{bc'd})_r}(j) \downarrow = i.$$

We remark that the oracles in Eq. (1) and Eq. (2) specify a bound on their use, like in Lemma 4.6. We say that $\Phi_*^{P(\bar{b})_s \oplus Id \upharpoonright |\bar{b}| \oplus P(\bar{b})_s}(i)$ *converges* if the computation is not asking for information not in $P(\bar{b})_s \oplus Id \upharpoonright |\bar{b}| \oplus P(\bar{b})_s$. If it does ask for more information, the computation *exceeds the use bound*.

We also remark that s serves two purposes. First, s is used to give an approximation to the enumeration reduction in an attempt to make the domain u.r.i. computable. Furthermore, s is being attached to the positive diagram in case \mathcal{B} is a finite structure and the language is infinite. If \mathcal{B} is a finite structure, then B^* contains elements of length at most the size of B . Therefore s is required as an alternate tracker for the number of relations to consider in the finite diagrams. We now have the following consequences.

Claim 4.7.1. $Dom_{\mathcal{A}}^{\mathcal{B}}$ is u.r.i. computable in \mathcal{B} .

Proof. We will prove that $Dom_{\mathcal{A}}^{\mathcal{B}}$ is u.r.i. computable in \mathcal{B} by defining a Turing operator Φ_{Dom} computing it. Suppose $(\tilde{\mathcal{B}}, R^{\tilde{\mathcal{B}}})$ is a structure isomorphic to $(\mathcal{B}, Dom_{\mathcal{A}}^{\mathcal{B}})$ via h . Given oracle $\tilde{\mathcal{B}}$, for $(\bar{b}, \langle s, i \rangle) \in \omega^{<\omega} \times \omega$, we define $\Phi_{Dom}^{\tilde{\mathcal{B}}}$ to accept $(\bar{b}, \langle s, i \rangle)$ if and only if $\bar{b} \in \tilde{B}^*$, $i \in \Psi_s^{P(\bar{b})_s}$ and Eq. (1) holds. The interesting part of the previous statement is that we have to check that Eq. (1) is decidable. Note that $P(\bar{b})_s \oplus Id \upharpoonright |\bar{b}| \oplus P(\bar{b})_s$ is a subset of $P(\mathcal{B}_f) \oplus Id \oplus P(\mathcal{B}_f)$ for some $\mathcal{B}_f \cong \tilde{\mathcal{B}}$. $\Phi_*^{P(\mathcal{B}_f) \oplus Id \oplus P(\mathcal{B}_f)}$ is an automorphism, therefore it is defined on $\Psi_s^{P(\bar{b})_s} \subset \Psi^{P(\mathcal{B}_f)}$. So $\Phi_*^{P(\bar{b})_s \oplus Id \upharpoonright |\bar{b}| \oplus P(\bar{b})_s}$ will either converge on i , or exceed the use bound. Thus Eq. (1) is decidable.

To show that $R^{\tilde{\mathcal{B}}} = \Phi_{Dom}^{\tilde{\mathcal{B}}}$, we have to observe that isomorphisms preserve pull-backs. That is, if $h(\bar{b}') = \bar{b}$, then $P(h(\bar{b}'))_s = P(\bar{b})_s$. From the definition of the Turing operator Φ_{Dom} , we can easily check that $Dom_{\mathcal{A}}^{\mathcal{B}} = \Phi_{Dom}^{\mathcal{B}}$ and h induces a bijection $h' : \Phi_{Dom}^{\tilde{\mathcal{B}}} \rightarrow \Phi_{Dom}^{\mathcal{B}}$, $(\bar{b}, \langle s, i \rangle) \mapsto (h(\bar{b}), \langle s, i \rangle)$. Therefore

$$(h')^{-1} \circ h : R^{\tilde{\mathcal{B}}} \rightarrow Dom_{\mathcal{A}}^{\mathcal{B}} (= \Phi_{Dom}^{\mathcal{B}}) \rightarrow \Phi_{Dom}^{\tilde{\mathcal{B}}}, (\bar{b}, \langle s, i \rangle) \mapsto (h(\bar{b}), \langle s, i \rangle) \mapsto (\bar{b}, \langle s, i \rangle)$$

is a bijection. Giving us $R^{\tilde{\mathcal{B}}} = \Phi_{Dom}^{\tilde{\mathcal{B}}}$. \square

Claim 4.7.2. $\sim^{\mathcal{B}}$ is an equivalence relation on $Dom_{\mathcal{A}}^{\mathcal{B}}$.

Proof. The relation $\sim^{\mathcal{B}}$ is reflexive because when $(\bar{b}, \langle s, i \rangle) = (\bar{c}, \langle t, j \rangle) \in Dom_{\mathcal{A}}^{\mathcal{B}}$, then Eq. (2) is equal to Eq. (1). By definition of $Dom_{\mathcal{A}}^{\mathcal{B}}$, $(\bar{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\bar{b}, \langle s, i \rangle)$. $\sim^{\mathcal{B}}$ is symmetric since if $(\bar{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\bar{c}, \langle t, j \rangle)$ by r and σ , then $(\bar{c}, \langle t, j \rangle) \sim^{\mathcal{B}} (\bar{b}, \langle s, i \rangle)$ by r and σ^{-1} .

To show that it is transitive, suppose $(\bar{b}, \langle s, i \rangle), (\bar{c}, \langle t, j \rangle), (\bar{a}, \langle u, k \rangle) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ and that $(\bar{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\bar{c}, \langle t, j \rangle)$ via \bar{d}_1, r and σ , and $(\bar{c}, \langle t, j \rangle) \sim^{\mathcal{B}} (\bar{a}, \langle u, k \rangle)$ via \bar{d}_2, p and τ . Let

$$\bar{b}_1 = \bar{b} - \bar{c} \quad \bar{c}_1 = \bar{c} - \bar{b} \quad \bar{c}_2 = \bar{c} - \bar{a} \quad \bar{a}_2 = \bar{a} - \bar{c} \quad \bar{b}_3 = \bar{b} - \bar{a} \quad \bar{a}_3 = \bar{a} - \bar{b}$$

and choose bijections

$$(3) \quad f_1 \supset \bar{b}\bar{c}_1\bar{d}_1, \quad f_2 \supset \bar{c}\bar{b}_1\bar{d}_1, \quad g_1 \supset \bar{c}\bar{a}_2\bar{d}_2, \quad g_2 \supset \bar{a}\bar{c}_2\bar{d}_2, \quad h_1 \supset \bar{b}\bar{a}_3, \quad h_2 \supset \bar{a}\bar{b}_3$$

such that $h_1(x) = h_2(x)$ for all $x > |\bar{b}\bar{a}_3| = |\bar{a}\bar{b}_3|$. Consider the isomorphism $h_2^{-1} \circ h_1 : \mathcal{B}_{h_1} \rightarrow \mathcal{B}_{h_2}$. We have

$$F(h_2^{-1} \circ h_1) = F(h_2^{-1} \circ g_2) \circ F(g_2^{-1} \circ g_1) \circ F(g_1^{-1} \circ f_2) \circ F(f_2^{-1} \circ f_1) \circ F(f_1^{-1} \circ h_1)$$

We observe from Eq. (3) that both f_1 and h_1 extend \bar{b} , so \mathcal{B}_{f_1} and \mathcal{B}_{h_1} both contain the $P(\bar{b})_s$ and $f_1^{-1} \circ h_1 : \mathcal{B}_{h_1} \rightarrow \mathcal{B}_{f_1}$ fixes the first $|\bar{b}|$ elements. Therefore $P(\bar{b})_s \oplus Id \upharpoonright |\bar{b}| \oplus P(\bar{b})_s \subseteq \mathcal{B}_{h_1} \oplus f_1^{-1} \circ h_1 \oplus \mathcal{B}_{f_1}$. Eq. (1) gives us that $F(f_1^{-1} \circ h_1)(i) \downarrow = i$. Similarly, we can argue that $F(g_1^{-1} \circ f_2)(j) \downarrow = j$ and $F(h_2^{-1} \circ g_2)(k) \downarrow = k$.

From the definition of f_1, f_2 , we have $P(\bar{b}\bar{c}_1\bar{d}_1)_r \oplus \sigma \oplus P(\bar{c}\bar{b}_1\bar{d}_1)_r \subseteq \mathcal{B}_{f_1} \oplus f_2^{-1} \circ f_1 \oplus \mathcal{B}_{f_2}$. Therefore Eq. (2) gives $F(f_2^{-1} \circ f_1)(i) \downarrow = j$. Similarly, we can argue that $F(g_2^{-1} \circ g_1)(j) \downarrow = k$. Composing the morphisms, we obtain $F(h_2^{-1} \circ h_1)(i) = \Phi_{*}^{\mathcal{B}_{h_1} \oplus h_2^{-1} \circ h_1 \oplus \mathcal{B}_{h_2}}(i) \downarrow = k$.

Since h_1 and h_2 agree on the extended elements, by looking for the use, we can find a $q \in \omega$ and \bar{d}_3 that doesn't mention any element from that of \bar{a} and \bar{b} such that if ρ is the finite permutation taking $\bar{b}\bar{a}_3\bar{d}_3$ to $\bar{a}\bar{b}_3\bar{d}_3$, then $P(\bar{b}\bar{a}_3\bar{d}_3)_q \oplus \rho \oplus P(\bar{a}\bar{b}_3\bar{d}_3)_q \subseteq \mathcal{B}_{h_1} \oplus h_2^{-1} \circ h_1 \oplus \mathcal{B}_{h_2}$ while $\Phi_{*}^{P(\bar{b}\bar{a}_3\bar{d}_3)_q \oplus \rho \oplus P(\bar{a}\bar{b}_3\bar{d}_3)_q}(i) \downarrow = k$. The other direction of Eq. (2) is established similarly, giving us $(\bar{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\bar{a}, \langle u, k \rangle)$. \square

Claim 4.7.3. $\sim^{\mathcal{B}}$ is u.r.i. computable in \mathcal{B} .

Proof. We begin by giving a characterization of the complement of $\sim^{\mathcal{B}}$. Let $\not\sim^{\mathcal{B}}$ be the binary relation on $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ defined by $(\bar{b}, \langle s, i \rangle) \not\sim^{\mathcal{B}} (\bar{c}, \langle t, j \rangle)$ if and only if there is $r \in \omega$, $\bar{d} \in B^*$ that is disjoint from \bar{b} and \bar{c} , and a permutation σ as in the definition of $\sim^{\mathcal{B}}$ such that

$$(4) \quad \Phi_{*}^{P(\bar{b}\bar{c}'\bar{d})_r \oplus \sigma \oplus P(\bar{c}\bar{b}'\bar{d})_r}(i) \downarrow \neq j \quad \text{or} \quad \Phi_{*}^{P(\bar{c}\bar{b}'\bar{d})_r \oplus \sigma^{-1} \oplus P(\bar{b}\bar{c}'\bar{d})_r}(j) \downarrow \neq i$$

We claim that $\not\sim^{\mathcal{B}}$ is the complement of $\sim^{\mathcal{B}}$.

Given $(\bar{b}, \langle s, i \rangle), (\bar{c}, \langle t, j \rangle) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ as arguments, we first show that at least one of $\sim^{\mathcal{B}}$ and $\not\sim^{\mathcal{B}}$ holds. Let $\bar{b}' = \bar{b} - \bar{c}$ and $\bar{c}' = \bar{c} - \bar{b}$, and take bijections $f \supset \bar{b}\bar{c}'$, $g \supset \bar{c}\bar{b}'$ that agree on their extensions. The functions f and g induce pull-backs \mathcal{B}_f and \mathcal{B}_g such that $g^{-1} \circ f : \mathcal{B}_f \cong \mathcal{B}_g$. Observe that when we restrict $g^{-1} \circ f$ to the first $|\bar{b}\bar{c}'|$ elements, we obtain the finite permutation taking $\bar{b}\bar{c}'$ to $\bar{c}\bar{b}'$. By Eq. (2),

$$(5) \quad \Phi_{*}^{\mathcal{B}_f \oplus g^{-1} \circ f \oplus \mathcal{B}_g}(i) \downarrow = j' \quad \text{and} \quad \Phi_{*}^{\mathcal{B}_g \oplus f^{-1} \circ g \oplus \mathcal{B}_f}(j) \downarrow = i'$$

for some $i', j' \in \omega$. By searching for the use of the computations Eq. (5), we can find finite subsets $P(\bar{b}\bar{c}'\bar{d})_r, \sigma, P(\bar{c}\bar{b}'\bar{d})_r$ of $\mathcal{B}_f, g^{-1} \circ f, \mathcal{B}_g$ such that Eq. (5) holds with these subsets as oracles. If $i = i', j = j'$, then \bar{d}, σ and r witness $(\bar{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\bar{c}, \langle t, j \rangle)$, otherwise they witness $(\bar{b}, \langle s, i \rangle) \not\sim^{\mathcal{B}} (\bar{c}, \langle t, j \rangle)$.

We claim that $(\bar{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\bar{c}, \langle t, j \rangle)$ and $(\bar{b}, \langle s, i \rangle) \not\sim^{\mathcal{B}} (\bar{c}, \langle t, j \rangle)$ cannot occur together. Suppose otherwise, then without loss of generality, there exist $\bar{d}_1, \bar{d}_2 \in B^*$, permutations σ, τ , and $r, p \in \omega$ such that

$$(6) \quad \Phi_*^{P(\bar{b}c'd_1)_r \oplus \sigma \oplus P(\bar{c}b'd_1)_r}(i) \downarrow = j \text{ but } \Phi_*^{P(\bar{b}c'd_2)_p \oplus \tau \oplus P(\bar{c}b'd_2)_p}(j) \downarrow \neq i$$

Choose bijections

$$(7) \quad f_1 \supset \bar{b}c'd_1, \quad g_1 \supset \bar{c}b'd_1, \quad f_2 \supset \bar{b}c'd_2, \quad g_2 \supset \bar{c}b'd_2.$$

Note that $F(g_2^{-1} \circ f_2) = F(g_2^{-1} \circ g_1) \circ F(g_1^{-1} \circ f_1) \circ F(f_1^{-1} \circ f_2)$. Using Eq. (6), Eq. (7) and an argument analogous to the one presented in Claim 4.7.2, we can deduce that

$$F(g_2^{-1} \circ f_2)(i) \neq j, F(g_2^{-1} \circ g_1)(j) = j, F(g_1^{-1} \circ f_1)(i) = j, \text{ and } F(f_1^{-1} \circ f_2)(i) = i$$

Composing the equations yields a contradiction. Therefore, $\not\sim^{\mathcal{B}}$ is the complement of $\sim^{\mathcal{B}}$.

Finally, we will show that $\sim^{\mathcal{B}}$ is u.r.i. computable in \mathcal{B} by defining a Turing operator Φ_{\sim} computing it. Suppose $(\tilde{\mathcal{B}}, R^{\tilde{\mathcal{B}}})$ is a structure isomorphic to $(\mathcal{B}, \sim^{\mathcal{B}})$ via h . Given oracle $\tilde{\mathcal{B}}$, for $(\bar{b}, \langle s, i \rangle), (\bar{c}, \langle t, j \rangle) \in \text{Dom}_{\tilde{\mathcal{A}}}^{\tilde{\mathcal{B}}}$, start enumerating $r \in \omega$ and \bar{d} as in the definition of $\sim^{\mathcal{B}}$ until one of Eq. (2) or Eq. (4) holds. If Eq. (2) holds, we ask $\Phi_{\sim}^{\tilde{\mathcal{B}}}$ to accept $\{(\bar{b}, \langle s, i \rangle), (\bar{c}, \langle t, j \rangle)\}$, otherwise, we ask $\Phi_{\sim}^{\tilde{\mathcal{B}}}$ to reject it.

To show that $R^{\tilde{\mathcal{B}}} = \Phi_{\sim}^{\tilde{\mathcal{B}}}$ on $\text{Dom}_{\tilde{\mathcal{A}}}^{\tilde{\mathcal{B}}}$, we proceed like in Claim 4.7.1. Since isomorphisms h preserve pull-backs, the computations in Eq. (2) and Eq. (4) will be preserved after $(\bar{b}, \langle s, i \rangle), (\bar{c}, \langle t, j \rangle)$ are transformed by h . Therefore we can check from the definition of Φ_{\sim} that $\Phi_{\sim}^{\tilde{\mathcal{B}}} = \sim^{\mathcal{B}}$ and h induces a bijection $h' : \Phi_{\sim}^{\tilde{\mathcal{B}}} \rightarrow \Phi_{\sim}^{\mathcal{B}}, \{(\bar{b}, \langle s, i \rangle), (\bar{c}, \langle t, j \rangle)\} \mapsto \{(h(\bar{b}), \langle s, i \rangle), (h(\bar{c}), \langle t, j \rangle)\}$. Therefore

$$(h')^{-1} \circ h : R^{\tilde{\mathcal{B}}} \rightarrow \sim^{\mathcal{B}} (= \Phi_{\sim}^{\mathcal{B}}) \rightarrow \Phi_{\sim}^{\tilde{\mathcal{B}}}, \{(\bar{b}, \langle s, i \rangle), (\bar{c}, \langle t, j \rangle)\} \mapsto \{(\bar{b}, \langle s, i \rangle), (\bar{c}, \langle t, j \rangle)\}$$

is a bijection. Giving us $R^{\tilde{\mathcal{B}}} = \Phi_{\sim}^{\tilde{\mathcal{B}}}$. \square

Define the relations $R_k^{\mathcal{B}}$ for $k \in \omega$ on $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ by $((\bar{b}_1, \langle s_1, i_1 \rangle), \dots, (\bar{b}_{a_k}, \langle s_{a_k}, i_{a_k} \rangle)) \in R_k^{\mathcal{B}}$ if and only if there exists $(\bar{c}_1, \langle t_1, j_1 \rangle), \dots, (\bar{c}_{a_k}, \langle t_{a_k}, j_{a_k} \rangle) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ such that $(\bar{b}_l, \langle s_l, i_l \rangle) \sim^{\mathcal{B}} (\bar{c}_l, \langle t_l, j_l \rangle)$ for all $l < a_k$ and $P_k(j_1, \dots, j_{a_k}) \in \Psi^{\mathcal{B}}$. The fact that $\sim^{\mathcal{B}}$ and $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ are u.r.i. computable guarantee that $R_k^{\mathcal{B}}$ is u.r.i.c.e. uniformly in k . By definition it is closed under $\sim^{\mathcal{B}}$.

To show that \mathcal{A} and $\mathcal{I}_{\mathcal{A}}^{\mathcal{B}}$ are isomorphic, we consider the pull-back \mathcal{B}_p of \mathcal{B} , where p is the principal function of B . Given set B , we say that $\bar{b} \in B^*$ is an initial segment of B if \bar{b} lists the elements of $B \cap \mathbf{n}$ in increasing order for some $n \in \omega$. In particular, every initial segment of p is an initial segment of B . The following two claims are used to define the isomorphism.

Claim 4.7.4. *If $(\bar{b}, \langle s, i \rangle), (\bar{c}, \langle t, j \rangle) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ and \bar{b} is an initial segment of \bar{c} , then $(\bar{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\bar{c}, \langle t, j \rangle)$ if and only if $i = j$.*

Proof. Let $\bar{c}' = \bar{c} - \bar{b}$, since \bar{b} is an initial segment of \bar{c} , $\bar{b}\bar{c}' = \bar{c}$. Suppose $(\bar{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\bar{c}, \langle t, j \rangle)$, then for some \bar{d} and $r > s$, $\Phi_*^{P(\bar{b}c'd)_r \oplus \sigma \oplus P(\bar{c}d)_r}(i) = j$. Since $\bar{c} = \bar{b}\bar{c}'$, this

implies $\Phi_*^{P(\overline{bc'd})_r \oplus Id \upharpoonright |\overline{bc'd}| \oplus P(\overline{bc'd})_r}(i) = j$. By Eq. (1), since $P(\overline{b})_s \oplus Id \upharpoonright |\overline{b}| \oplus P(\overline{b})_s \subseteq P(\overline{bc'd})_r \oplus Id \upharpoonright |\overline{bc'd}| \oplus P(\overline{bc'd})_r$, we have $i = j$.

Conversely, suppose $i = j$. Take \overline{d} to be the empty tuple, $r = t$, then

$$\Phi_*^{P(\overline{bc'd})_r \oplus \sigma \oplus P(\overline{cd})_r}(i) = \Phi_*^{P(\overline{c})_t \oplus Id \upharpoonright |\overline{c}| \oplus P(\overline{c})_t}(j) = j$$

by (Eq. (2)). For the same reason, we have $\Phi_*^{P(\overline{cd})_r \oplus \sigma^{-1} \oplus P(\overline{bc'd})_r}(j) = i$. Therefore $(\overline{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\overline{c}, \langle t, j \rangle)$. \square

Claim 4.7.5. *For each $(\overline{b}, \langle s, i \rangle) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$, there is an initial segment $\overline{c} \in B^*$ of B and $t, j \in \omega$ such that $(\overline{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\overline{c}, \langle t, j \rangle)$.*

Proof. Given $(\overline{b}, \langle s, i \rangle) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$, let $f \supset \overline{b}$ be a bijection onto B that is principal on the extended elements, then there is a $\overline{c} \subset f$, every component of \overline{b} is a component of \overline{c} . Now, suppose $\Phi_*^{\mathcal{B}_f \oplus p^{-1} \circ f \oplus \mathcal{B}_p}(i) \downarrow = j$ for some $j \in \omega$, then $j \in \text{Dom}(F(\mathcal{B}_p))$ since $i \in \text{Dom}(F(\mathcal{B}_f))$. Moreover, we can choose $t \in \omega$ such that $(\overline{c}, \langle t, j \rangle) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ by looking for the use of the computation $\Phi_*^{\mathcal{B}_p \oplus Id \oplus \mathcal{B}_p}(j) \downarrow = j$. Finally, we claim $(\overline{b}, \langle s, i \rangle) \sim^{\mathcal{B}} (\overline{c}, \langle t, j \rangle)$. Let $\overline{c'} = \overline{c} - \overline{b}$, then looking for the use of the computation $\Phi_*^{\mathcal{B}_f \oplus p^{-1} \circ f \oplus \mathcal{B}_p}(i) \downarrow = j$ and $\Phi_*^{\mathcal{B}_p \oplus f^{-1} \circ p \oplus \mathcal{B}_f}(j) \downarrow = i$, we can find $\overline{d} \in B^*$ and $r \in \omega$ such that

$$\Phi_*^{P(\overline{ba'd})_r \oplus \sigma \oplus P(\overline{ad})_r}(i) \downarrow = j \text{ and } \Phi_*^{P(\overline{ad})_r \oplus \sigma^{-1} \oplus P(\overline{ba'd})_r}(j) \downarrow = i$$

This completes the proof of the claim. \square

Finally, to show that $\mathcal{I}_{\mathcal{A}}^{\mathcal{B}}$ is isomorphic to \mathcal{A} , we will define an isomorphism $f^{\mathcal{B}} : F(\mathcal{B}_p) \rightarrow \mathcal{I}_{\mathcal{A}}^{\mathcal{B}}$. For $i \in \text{Dom}(F(\mathcal{B}_p))$, since $i \in \Psi^{\mathcal{B}_p}$ and $\Phi_*^{\mathcal{B}_p \oplus Id \oplus \mathcal{B}_p}(i) = i$, by searching the use, there is an initial segment \overline{c} of B and $t \in \omega$ such that $(\overline{c}, \langle t, i \rangle) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$. Define $f^{\mathcal{B}}(i) = [(\overline{c}, \langle t, i \rangle)]_{\sim^{\mathcal{B}}}$. Although, it is not required by the definition, it can be easily checked that $f^{\mathcal{B}}$ is computable in \mathcal{B} . Claim 4.7.4 shows that $f^{\mathcal{B}}$ is injective and well defined, Claim 4.7.5 shows that $f^{\mathcal{B}}$ is surjective, therefore $f^{\mathcal{B}}$ is a bijection. From the definition of relations, it is easy check $P_k(j_i, \dots, j_{a_k}) \in \Psi^{\mathcal{B}_p}$ if and only if $(f^{\mathcal{B}}(j_1), \dots, f^{\mathcal{B}}(j_{a_k})) \in R_k^{\mathcal{B}} / \sim^{\mathcal{B}}$. Therefore $f^{\mathcal{B}}$ defines an isomorphism. This completes the proof of this theorem and the proof of Theorem 4.4. \square

Proof of Theorem 4.5. To show that F and I^F are effectively isomorphic, we will build a Turing functional Λ , such that for every $\tilde{\mathcal{B}} \in \text{Iso}(\mathcal{B})$, $\Lambda^{\tilde{\mathcal{B}}}$ is an isomorphism from $F(\tilde{\mathcal{B}})$ to $I^F(\tilde{\mathcal{B}})$.

Let $p_{\tilde{\mathcal{B}}}$ be the principal function for $\tilde{\mathcal{B}}$. In the proof of Lemma 4.7 we constructed the isomorphism $f^{\mathcal{B}} : F(\mathcal{B}_{p_{\tilde{\mathcal{B}}}}) \rightarrow \mathcal{I}_{\mathcal{A}}^{\mathcal{B}}$, which is uniformly computable in $\tilde{\mathcal{B}}$. Since we need to compute isomorphisms from every copy of \mathcal{B} and not just those with universe ω . We thus precompose $f^{\mathcal{B}}$ with $F(p_{\tilde{\mathcal{B}}}^{-1}) : F(\tilde{\mathcal{B}}) \rightarrow F(\mathcal{B}_{p_{\tilde{\mathcal{B}}}})$. Since $F = (\Psi, \Phi_*)$ is a positive enumerable functor, $F(p_{\tilde{\mathcal{B}}}^{-1}) = \Phi_*^{\tilde{\mathcal{B}} \oplus p_{\tilde{\mathcal{B}}}^{-1} \oplus \tilde{\mathcal{B}}_{p_{\tilde{\mathcal{B}}}}}$. The oracles are uniformly computable from $\tilde{\mathcal{B}}$, therefore $F(p_{\tilde{\mathcal{B}}}^{-1})$ is also uniformly computable from $\tilde{\mathcal{B}}$.

In the proof of Lemma 4.6 we constructed isomorphisms $\tilde{m}_{\tilde{\mathcal{B}}} : \mathcal{I}_{\mathcal{A}}^{\tilde{\mathcal{B}}} \rightarrow \mathcal{I}^F(\tilde{\mathcal{B}})$ and showed that these are uniformly computable in the copies of \mathcal{B} . Therefore, we

can define $\Lambda^{\tilde{\mathcal{B}}} = \tilde{m}_{\tilde{\mathcal{B}}} \circ f^{\tilde{\mathcal{B}}} \circ F(p_{\tilde{\mathcal{B}}}^{-1})$. We may assume that $\mathcal{I}_{\mathcal{A}}^{\mathcal{B}}$ and $\mathcal{I}_{\mathcal{A}}^{\tilde{\mathcal{B}}}$ have as universe the least elements in the $\sim^{\mathcal{B}}$, respectively, $\sim^{\tilde{\mathcal{B}}}$ -equivalence classes. Then given $h : \mathcal{B} \cong \tilde{\mathcal{B}}$, \bar{h} is then an isomorphism from $\mathcal{I}_{\mathcal{A}}^{\mathcal{B}}$ to $\mathcal{I}_{\mathcal{A}}^{\tilde{\mathcal{B}}}$. It remains to show that the diagram below commutes.

$$\begin{array}{ccccc}
 I^F(\mathcal{B}) & \xleftarrow{\tilde{m}_{\mathcal{B}}} & \mathcal{I}_{\mathcal{A}}^{\mathcal{B}} & \xleftarrow{f^{\mathcal{B}} \circ F(p_{\mathcal{B}}^{-1})} & F(\mathcal{B}) \\
 \downarrow I^F(h) & & \downarrow \bar{h} & & \downarrow F(h) \\
 I^F(\tilde{\mathcal{B}}) & \xleftarrow{\tilde{m}_{\tilde{\mathcal{B}}}} & \mathcal{I}_{\mathcal{A}}^{\tilde{\mathcal{B}}} & \xleftarrow{f^{\tilde{\mathcal{B}}} \circ F(p_{\tilde{\mathcal{B}}}^{-1})} & F(\tilde{\mathcal{B}})
 \end{array}$$

As $I^F(h)$ is defined as $I^F(h) = \tilde{m}_{\tilde{\mathcal{B}}} \circ \bar{h} \circ \tilde{m}_{\mathcal{B}}^{-1}$ the square on the left clearly commutes. To show that the square on the right commutes, it is sufficient to show

$$(f^{\tilde{\mathcal{B}}})^{-1} \circ \bar{h} \circ f^{\mathcal{B}} = F(p_{\tilde{\mathcal{B}}}^{-1} \circ h \circ p_{\mathcal{B}}).$$

Suppose $F(p_{\tilde{\mathcal{B}}}^{-1} \circ h \circ p_{\mathcal{B}})(i) = j$, then $\Phi_*^{\mathcal{B}_{p_{\mathcal{B}}} \oplus (p_{\tilde{\mathcal{B}}}^{-1} \circ h \circ p_{\mathcal{B}}) \oplus \tilde{\mathcal{B}}_{p_{\tilde{\mathcal{B}}}}}(i) = j$. Looking for the use, we can find $s \in \omega$, \bar{b} and \bar{c} that are initial segments of B , respectively, \tilde{B} , such that $(\bar{b}, \langle s, i \rangle) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$, $(\bar{c}, \langle s, j \rangle) \in \text{Dom}_{\mathcal{A}}^{\tilde{\mathcal{B}}}$ and the computation $\Phi_*^{\mathcal{B}_{p_{\mathcal{B}}} \oplus (p_{\tilde{\mathcal{B}}}^{-1} \circ h \circ p_{\mathcal{B}}) \oplus \tilde{\mathcal{B}}_{p_{\tilde{\mathcal{B}}}}}(i) \downarrow = j$ has its use bounded by the oracle $P(\bar{b})_s \oplus (p_{\tilde{\mathcal{B}}}^{-1} \circ h \circ p_{\mathcal{B}}) \upharpoonright |\bar{b}| \oplus P(\bar{c})_s$. Since isomorphisms preserve pull-backs, $P(\bar{b})_s = P(h(\bar{b}))_s$. We further assume that \bar{c} is large enough such that all elements of $h(\bar{b})$ appear in \bar{c} . Then $\bar{b}' = h(\bar{b}) - \bar{c}$ is empty. Let $\bar{c}' = \bar{c} - h(\bar{b})$. It can be checked that $h(\bar{b})(i) = \bar{c}((p_{\tilde{\mathcal{B}}}^{-1} \circ h \circ p_{\mathcal{B}})(i))$ for $0 \leq i < |\bar{b}|$. Therefore $(p_{\tilde{\mathcal{B}}}^{-1} \circ h \circ p_{\mathcal{B}}) \upharpoonright |\bar{b}|$ can be extended to a finite permutation σ taking $h(\bar{b})\bar{c}'$ to \bar{c} . Therefore

$$\Phi_*^{P(h(\bar{b})\bar{c}')_s \oplus \sigma \oplus P(\bar{c})_s}(i) = \Phi_*^{P(\bar{b})_s \oplus (p_{\tilde{\mathcal{B}}}^{-1} \circ h \circ p_{\mathcal{B}}) \upharpoonright |\bar{b}| \oplus P(\bar{c})_s}(i) = j$$

by the use principle. This witnesses $(h(\bar{b}), \langle s, i \rangle) \sim^{\tilde{\mathcal{B}}} (\bar{c}, \langle s, j \rangle)$. Giving us $(f^{\tilde{\mathcal{B}}})^{-1} \circ \bar{h} \circ f^{\mathcal{B}}(i) = (f^{\tilde{\mathcal{B}}})^{-1} \circ \bar{h}([h(\bar{b}), \langle s, i \rangle]) = (f^{\tilde{\mathcal{B}}})^{-1}([h(\bar{b}), \langle s, i \rangle]) = (f^{\tilde{\mathcal{B}}})^{-1}([\bar{c}, \langle s, j \rangle]) = j$, as desired. \square

Positive enumerable bi-transformations and u.p.e.t. reductions also have syntactic analogues.

Definition 4.8. Two structures \mathcal{A} and \mathcal{B} are Σ_1^c *bi-interpretable* if and only if there exists Σ_1^c interpretations $f_{\mathcal{A}}^{\mathcal{B}}, f_{\mathcal{B}}^{\mathcal{A}}$ witnessing that \mathcal{A} and \mathcal{B} are Σ_1^c interpretable in each other, and the compositions

$$f_{\mathcal{B}}^{\mathcal{A}} \circ \overline{f_{\mathcal{A}}^{\mathcal{B}}} : \text{Dom}_{\mathcal{B}}^{(\text{Dom}_{\mathcal{A}}^{\mathcal{B}})} \rightarrow \text{Dom}_{\mathcal{B}}^{\mathcal{A}} \rightarrow B$$

and

$$f_{\mathcal{A}}^{\mathcal{B}} \circ \overline{f_{\mathcal{B}}^{\mathcal{A}}} : \text{Dom}_{\mathcal{A}}^{(\text{Dom}_{\mathcal{B}}^{\mathcal{A}})} \rightarrow \text{Dom}_{\mathcal{A}}^{\mathcal{B}} \rightarrow A$$

are u.r.i. computable in $\mathcal{A} \oplus \mathcal{B}$ (Recall that $\overline{f_{\mathcal{A}}^{\mathcal{B}}}$ and $\overline{f_{\mathcal{B}}^{\mathcal{A}}}$ refer to the natural extension of $f_{\mathcal{A}}^{\mathcal{B}}$ and $f_{\mathcal{B}}^{\mathcal{A}}$ to tuples).

Definition 4.9. A class \mathfrak{C} is *reducible via Σ_1^c bi-interpretations* to \mathfrak{D} if for every $\mathcal{C} \in \mathfrak{C}$ there is $\mathcal{D} \in \mathfrak{D}$ such that \mathcal{C} and \mathcal{D} are Σ_1^c bi-interpretable and the formulas defining the bi-interpretations and isomorphisms are independent of \mathcal{C} and \mathcal{D} .

In [Har+17] it was shown that two structures are effectively bi-interpretable if and only if they are computably bi-transformable. Following their proof but using Theorem 4.4 and Theorem 4.5 instead of the analogues for computable functors and effective interpretability we get the following theorem and its corollary.

Theorem 4.10. *Two structures \mathcal{A} and \mathcal{B} are positively enumerably bi-transformable if and only if they are positively enumerably bi-interpretable.*

Corollary 4.11. *A class \mathfrak{C} is u.p.e.t. reducible to \mathfrak{D} if and only if \mathfrak{C} is reducible via Σ_1^c bi-interpretations to \mathfrak{D} .*

REFERENCES

- [Ash+89] Chris Ash, Julia Knight, Mark Manasse, and Theodore Slaman. “Generic Copies of Countable Structures”. In: *Annals of Pure and Applied Logic* 42.3 (1989), pp. 195–205. DOI: 10.1016/0168-0072(89)90015-8.
- [Chi90] John Chisholm. “Effective Model Theory vs. Recursive Model Theory”. In: *The Journal of Symbolic Logic* 55.03 (1990), pp. 1168–1191. DOI: 10.2307/2274481.
- [Gon80] S. S. Goncharov. “Problem of the Number of Non-Self-Equivalent Constructivizations”. In: *Algebra and Logic* 19.6 (1980), pp. 401–414.
- [Har+17] Matthew Harrison-Trainor, Alexander Melnikov, Russell Miller, and Antonio Montalbán. “Computable Functors and Effective Interpretability”. In: *The Journal of Symbolic Logic* 82.1 (2017), pp. 77–97.
- [HMM18] Matthew Harrison-Trainor, Russell Miller, and Antonio Montalbán. “Borel Functors and Infinitary Interpretations”. In: *The Journal of Symbolic Logic* 83.4 (2018), pp. 1434–1456.
- [Mil+18] Russell Miller, Bjorn Poonen, Hans Schoutens, and Alexandra Shlapentokh. “A Computable Functor from Graphs to Fields”. In: *The Journal of Symbolic Logic* 83.1 (2018), pp. 326–348.
- [Mon12] Antonio Montalbán. “Rice Sequences of Relations”. In: *Philosophical Transactions of the Royal Society of London A* 370.1971 (2012), pp. 3464–3487.
- [Mon18] Antonio Montalbán. *Computable Structure Theory*. draft. Jan. 20, 2018.
- [Ros17] Dino Rossegger. “On Functors Enumerating Structures”. In: *Siberian Electronic Mathematical Reports* 14 (2017), pp. 690–702.
- [Ros19] Dino Rossegger. “Computable structure theory with respect to equivalence relations”. Technische Universität Wien, 2019.
- [Sos04] Ivan N. Soskov. “Degree Spectra and Co-Spectra of Structures”. In: *Ann. Univ. Sofia* 96 (2004), pp. 45–68.
- [SS17] Alexandra Soskova and Mariya Soskova. “Enumeration Reducibility and Computable Structure Theory”. In: *Computability and Complexity*. Springer, 2017, pp. 271–301.

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