Discrete Mathematics Problem Set 1

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1 Problem 1

Let $A = \{n \in \mathbb{Z} : n^2 < 10\}$ and $B = \{n \in \mathbb{Z} : 0 \leq n \leq 5\}$

(a) Rewrite the sets A and B using the roster form.

$$A = \{0, -1, 1, -2, 2, -3, 3\}, \quad B = \{0, 1, 2, 3, 4, 5\}.$$

(b) Determine $A \times B$

$$A \times B = \{(a, b) \mid a \in \{0, -1, 1, -2, 2, -3, 3\} \; and \; b \in \{0, 1, 2, 3, 4, 5\}\}$$

$$or \; A \times B = \{(0, 0), (0, 1), ..., (0, 5), (-1, 0), (-1, 1), ..., (3, 5)\}$$

$$|A \times B| = |A| * |B| = 42$$

2 Problem 2

Given the collection: $C = \{A_n : n \in \mathbb{N}\}$, where $A_n = \{x \in \mathbb{R} : x^2 < n\}$.

(a) Determine $\bigcup_{n=1}^{100} A_n$.

We know that:

$$A_n = \{x \in \mathbb{R} : x^2 < n\} = (-\sqrt{n}, \sqrt{n}).$$

Since the intervals A_n are nested, with each interval A_n being contained in A_{n+1}:

$$A_1 \subset A_2 \subset \cdots \subset A_{100}.$$

Thus, the union of all these intervals is equal to the largest interval:

$$\bigcup_{n=1}^{100} A_n = A_{100} = (-10, 10).$$

3 Problem 3

Let $A_n = \{n+1, n+2, n+3, n+4\}, n \in \mathbb{Z}$

(a) Find $\cup_{n=-3}^{3} A_n$ and $\cap_{n=-3}^{3} A_n$

$$\cup_{n=-3}^{3} A_n = \{x \in \mathbb{Z} \mid -2 \leq x \leq 7\}, \quad \cap_{n=-3}^{3} A_n = \emptyset$$

(b) Which values of $k \in \mathbb{N}$ make the subcollection $\{A_{kn} : n \in \mathbb{N}\}$ mutually disjoint.

$$A_{kn} = \{kn+1, kn+2, kn+3, kn+4\} \; for \; n \in \mathbb{N}$$

Sets A_{kn} and A_{km}, where $n \neq m$, will be mutually disjoint if:

$$A_{kn} \cap A_{km} = \emptyset$$

For this to be true, kn must be ≥ 5 from km:

$$|kn - km| \geq 5 \; for \; all \; n \neq m$$

For this to be true for all cases m and n:

$$k = 5$$

(c) Which values of $k \in \mathbb{N}$ which make $\cup_{n \in \mathbb{N}} A_{kn} = \mathbb{Z}$

We define A_{kn}: $A_{kn} = \{kn | n \in \mathbb{N}\}$

$$k = 1$$

4 Problem 4

For each condition below, give a nested collection C of distinct subsets of $\mathbb{R}$ which satisfies each of the following conditions:

1. $\cap C = \emptyset$
$$C_n = \{(-\infty, n) \mid n \in \mathbb{Z}\}$$
2. $\cap C = [-1, 1]$
$$C_n = \{[-1 - \frac{1}{n}, 1 + \frac{1}{n}] \mid n \in \mathbb{N}\}$$
3. $\cup C = [-1, 1]$
$$C_n = \{-1 + \frac{1}{n}, 1 - \frac{1}{n} \mid n \in \mathbb{N}\}$$
4. $\cup C = \mathbb{R}$
$$C_n = \{(-\infty, n) \mid n \in \mathbb{Z}\}$$

5 Problem 5

Let A, B, C be non-empty sets. Show that

$$A * (B \cap C) = (A \times B) \cap (A \times C)$$

Part A WTS $A \times B(B \cap C) \subseteq (A \times B) \cap (A \times C)$
Let $(a, x) \in A \times (B \cap C)$

This must mean that: $a \in A$ and $x \in B \cap C$
Since $x \in B \cap C$, $x \in B$ and $x \in C$.
Therefore, $(a, x) \in A \times B$ and $(a, x) \in A \times C$.
Therefore, $(a, x) \in (A \times B) \cap (A \times C)$.
Therefore, $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

Part B WTS $(A \times B) \cap (A \times C) \subseteq A * (B \cap C)$
Let $(a, x) \in (A \times B) \cap (A \times C)$
By definition of intersection, $(a, x) \in A \times B$ and $(a, x) \in A \times C$
$a \in A$, $x \in B$, $x \in C$
Hence, $x \in B \cap C$
Therefore, $(a, x) \in A \times (B \cap C)$
Therefore, $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

Conclusion Because

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$$

and

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$$

we can conclude that:

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

6 Problem 6

For each $n \in \mathbb{N}$, define

$$C_n = \{x \in \mathbb{R} \, : \, n - 1 < x^2 < n\}$$

Show that $\{C_n\}_{n \in \mathbb{N}}$ is a partition of $\mathbb{R}$.

Part A WTS $C_n = \emptyset$ for all $n \in \mathbb{N}$
For any $n \in \mathbb{N}$, there must be $x \in \mathbb{R}$ that satisfies $n - 1 < x^2 < n$
Because $n - 1 \geq 0$ for all $n \geq 1$ and $n - 1 < n$, the interval $(n - 1, n)$ is non-empty and contains positive values.
Thus, there exists $x \in \mathbb{R}$ such that $x^2 \in (n - 1, n)$.

Therefore, for each $n \in \mathbb{N}$, C_n is non-empty, $C_n \neq \emptyset$.

Part B WTS $C_n \cap C_m = \emptyset$ for $n \neq m$
Suppose $x \in C_n$ and $x \in C_m$ for some $n \neq m$.
By definition of C_n, this means:

$$n - 1 < x^2 < n \quad and \quad m - 1 < x^2 < m$$

If $n \neq m$, then the intervals $(n-1, n)$ and $(m-1, m)$ are non-overlapping because:

$$n - 1 < n < m - 1 for n < m$$

Therefore, it is impossible for x^2 to lie in both intervals, meaning $C_n \cap C_m = \emptyset$ for $n \neq m$.

Part C WTS

$$\bigcup_{n \in \mathbb{N}} C_n = \mathbb{R}$$

For any $x \in \mathbb{R}$ there must exist $x^2 \geq 0$.
For any $x \in \mathbb{R}$, there exists some $n \in \mathbb{N}$ such that $n - 1 < x^2 < n$.
Specifically, for $x^2 \in (n-1, n)$, we have $x \in C_n$.
Therefore, $\bigcup_{n \in \mathbb{N}} C_n = \mathbb{R}$

7 Problem 7

Let A, B, C be sets such that $A \subset B$. Show that

$$A \cap B = A$$

Part A WTS $A \cap \subset A$
Let $x \in A \cap B$.
By the definition of intersection:

$$x \in A \quad and \quad x \in B$$

Since $A \subset B$, if $x \in A$, then $x \in B$
Therefore, $x \in A$ shows that every element $A \cap B$ is also in A.
Hence, $A \cap B \subset A$

Part B WTS $A \subset A \cap B$
Let $x \in A$. Since $A \subset B$, it follows that $x \in B$
Therefore, $x \in A$ and $x \in B$, which implies that $x \in A \cap B$.
Thus, for all $x \in A$, $x \in A \cap B$.
Hence, $A \subset A \cap B$.

Conclusion

Because:

$$A \cap B \subset A$$

and:

$$A \subset A \cap B$$

therefore:

$$A \cap B = A$$

8 Problem 8

For each $\lambda \in \mathbb{R}$, define $A_\lambda = \{(x, y) \in \mathbb{R} \ : \ y = x^2 + \lambda\}$. Prove that the collection $\{A_\lambda \ : \ \lambda \in \mathbb{R}\}$ forms a partition of $\mathbb{R}^2$.

Part A WTS A_λ are disjoint: $A_{\lambda_1} \cap A_{\lambda_2}$ for $\lambda_1 \neq \lambda_2$

Let $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Consider the sets:

$$A_{\lambda 1} = \{(x, y) \in \mathbb{R}^2 \ : \ y = x^2 + \lambda_1\} \quad A_{\lambda_2} = \{(x, y) \in \mathbb{R}^2 \ : \ y = x^2 + \lambda_2$$

Suppose that $(x, y) \in A_{\lambda 1} \cap A_{\lambda 2}$. Then, by definition, we must have:

$$y = x^2 + \lambda_1 \quad y = x^2 + \lambda_2$$

Which can be rewritten as:

$$x^2 + \lambda_1 = x^2 + \lambda_2$$

Which implies:

$$\lambda_1 = \lambda_2$$

Which can not because $\lambda_1 \neq \lambda_2$. Therefore $A_{\lambda^{\cdot}} \cap A_{\lambda 2} = \emptyset$ when $\lambda_1 \neq \lambda_2$.
Thus, A_λ are disjointed.
Part B WTS $\bigcup_{\lambda \in \mathbb{R}} A_\lambda = \mathbb{R}^2$.
Let $(x, y) \in \mathbb{R}^2$.
By definition A_λ, $(x, y) \in A_\lambda \iff y = x^2 + \lambda$
Rearranging the equation gives: $\lambda = y - x^2$
Since $\lambda = y - x^2 \in \mathbb{R}$ for any $(x, y) \in \mathbb{R}$, there exists some $\lambda \in \mathbb{R}$ such that $(x, y) \in A_\lambda$
Thus:

$$\bigcup_{\lambda \in \mathbb{R}} A_\lambda = \mathbb{R}^2$$

Conclusion

1. A_λ are mutually disjoint, $A_{\lambda 1} \cap A_{\lambda 2} = \emptyset$ for $\lambda_1 \neq \lambda_2$
2. $\bigcup_{\lambda \in \mathbb{R}} A_\lambda = \mathbb{R}^2$