IDEA ABOUT INVERSE RATIONAL MAP

Let L/k be a cyclic extension, with Galois group $\langle\sigma\rangle = G = \mathrm{Gal}(L/k)$, and degree $n = [L : k]$. Hilbert's Theorem 90 tells us that the map

$$f : x \mapsto x/\sigma(x)$$

is a surjection from $L^\times$ onto the set

$$X = \{y \in L \mid N(y) = 1\}$$

Identifying L with k^n, f can be viewed as a rational map from $\mathbb{P}^{n-1}$ onto a hypersurface in $\mathbb{A}^n$.

Given $y \in L$ with $N(y) = 1$, the proof of Hilbert's Theorem 90 finds an x with $y = f(x)$ as follows. It suffices to find a nonzero x such that $\sigma(x) = y^{-1} \cdot x$. To this end, we consider the following L-linear combination of elements of G:

$$h^y = \sigma^0 + y \cdot \sigma^1 + y \cdot \sigma(y) \cdot \sigma^2 + \cdots + y \cdot \cdots \cdot \sigma^{n-2}(y) \cdot \sigma^{n-1}$$

Then we see that

$$\sigma \circ h^y = \sigma + \sigma(y) \cdot \sigma + \sigma(y) \cdot \sigma^2(y) \cdot \sigma^3 + \cdots + \sigma(y) \cdot \cdots \cdot \sigma^{n-2}(y) \cdot \sigma^{n-1} + y^{-1} \cdot N(y) \cdot \sigma^0$$

Using the fact that $N(y) = 1$, this is equal to $y^{-1} \cdot h$. Thus, it suffices to find a z such that $h^y(z) \neq 0$. Such a z exists by the linear independence of characters. Now we change perspective, and view $h^y(z)$ as a function of y with $z = z_0$ fixed. Identifying L with k^n, $y \mapsto h^y(z_0)$ is a polynomial map $k^n \to k^n$, which must be nonzero for y in a (possibly empty) Zariski open set.

This should give us a rational map g from the norm 1 hypersurface $X \subset \mathbb{A}^n$ to $\mathbb{P}^{n-1}$ which satisfies $f \circ g = \mathrm{id}$ on k-points where it is defined. If k-points are Zariski dense in X, this equality should hold on a (possibly empty) Zariski open set in X. It remains to choose an appropriate z_0, so that $h^y(z_0) \neq 0$ for some y on X, so that this works on a nonempty subset of X. For this, we set $y = 1$, and choose z_0 so that $h^1(z_0) \neq 0$.

$$h^1(z_0) = \sigma^0(z_0) + \sigma^1(z_0) + \cdots + \sigma^{n-1}(z_0) = \mathrm{tr}(z_0)$$

so it suffices to choose z_0 with $\mathrm{tr}(z_0) \neq 0$. $z_0 = 1$ works for n prime to the characteristic, giving a simple expression for g. For Artin-Schreier extensions, we need to choose some other z_0.

I now compute the other composition $g \circ f$, returning to the perspective of L as a field extension of k. Given $x \in L^\times$, we compute

$$\begin{aligned} h^{x/\sigma(x)}(z_0) &= z_0 + \frac{x}{\sigma(x)}\sigma(z_0) + \frac{x}{\sigma^2(x)} + \sigma^2(z_0) + \cdots + \frac{x}{\sigma^{n-1}(x)}\sigma^{n-1}(z_0) \\ &= x\left[\frac{z_0}{x} + \frac{\sigma(z_0)}{\sigma(x)} + \frac{\sigma^2(z_0)}{\sigma^2(x)} + \cdots + \frac{\sigma^{n-1}(z_0)}{\sigma^{n-1}(x)}\right] = x \cdot \mathrm{tr}\left(\frac{z_0}{x}\right) \end{aligned}$$

As $\mathrm{tr}\left(\frac{z_0}{x}\right)$ is an element of k, for $z_0, x \in L$, this represents the same point in $\mathbb{P}^n$ as does x, provided it is nonzero. Thus $g \circ f = \mathrm{id}$ on k-points of $\mathbb{P}^n$ where it is defined. If k-points are Zariski dense in $\mathbb{P}^n$, we should have $g \circ f = \mathrm{id}$ generically on $\mathbb{P}^n$.

Now some words about Zariski density. If k is an infinite field, k-points are Zariski dense in $\mathbb{P}^n$. This fails if k is finite. My idea for how to remedy this was to replace an extension of finite fields L/k by the extension of infinite fields $L(t)/k(t)$, and argue that the polynomials defining f and g for the extension L/k are the same as those for $L(t)/k(t)$, so that the required equations in the polynomial ring over k still hold. I don't know how one could show that k-points are Zariski dense in the norm hypersurface X (assuming that k is infinite). Of course, this will be true if X really is rational. If X is known to be irreducible, one might be able to argue this by arguing that the image of f contains a nonempty open subset of X, once we know that $g \circ f = \mathrm{id}$ generically on $\mathbb{P}^n$.

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