Hilbert 90 Geometry: Remarks on Birationality

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1 Remarks on Birationality

Let L/k be a cyclic field extension of degree n, with $p := \operatorname{char} k$. Let $G := \operatorname{Gal}(L/k) = \langle \sigma \rangle$. Recall the maps

$$\begin{aligned} f : L^\times &\to L^x \\ x &\mapsto x/\sigma(x) \end{aligned}$$

and

$$\begin{aligned} h : L \times L &\to L \\ (y, z_0) &\mapsto \sigma^0(z_0) + y \cdot \sigma(y) \cdot \sigma^2(z_0) + \ldots + y \cdot \sigma(y) \cdot \ldots \cdot \sigma(y)^{n-2} \cdot \sigma^{n-1}(z_0) \end{aligned}$$

We know f maps surjectively onto the subset $X := \{y \in L \mid N(y) = 1\}$. From Miles' note, we know that $z_0 \in L$ can be fixed so that $\operatorname{Tr}(z_0) \neq 0$ in order for $h(1, z_0) \neq 0$, which guarantees that $h_{z_0} := h|_{L \times \{z_0\}}$, thought of as a polynomial endomorphism in the coefficients of $L \cong k^n$, is at least nonzero at $1 \in L$. If we instead regard $h : L \to \operatorname{Hom}_k(L, L)$ as a function of y, we can also recall from the note that

$$\begin{aligned} h \circ f : L^\times &\to \operatorname{Hom}_k(L, L) \\ x &\mapsto (z_0 \mapsto x \cdot \operatorname{Tr}(\frac{z_0}{x})). \end{aligned}$$

Let $1, x_1, \ldots, x_{n-1}$ be a k-basis for L, and let

$$x = a_0 + a_1 x_1 + \ldots + x_{n-1}$$

be an element of L for some coefficients $a_i \in K$. Let H be the hypersurface in $\mathbb{P}^n_k = \operatorname{Proj} k[a_0, \ldots, a_{n-1}, y]$ given by the equation $N(x) = y^n$. By an abuse of notation, let f denote the map $\mathbb{P}^{n-1}_k = \operatorname{Proj} k[a_0, \ldots, a_{n-1}] \to H$ induced by f as referenced above, and let g denote the map $H \to \mathbb{P}^{n-1}_k$ induced by h_{z_0}.

In this note, we will build up to the following proposition:

1.1. Proposition: — *If L/k as above is either an Artin-Schreier or Kummer extension, then f is a birational morphism with inverse g.*

We first establish some basic facts. Let $x_0, \ldots, x_{n-1}$ denote a k-basis for L such that we represent an arbitrary element $x \in L$ by the sum

$$x = \sum_{i=0}^{n-1} a_i x_i$$

for some indeterminates $a_0, \ldots, a_{n-1}$ representing a choice of coefficients in k. With this notation, the field norm $N(x) \coloneqq N_{L/k}(x)$ can be regarded as a polynomial in the indeterminates $a_0, \ldots a_{n-1}$. We recall the following theorem:

1.2. Theorem ([Fla53])**: —** *Let $K = k(\theta)$ be a primitive extension of k. Then the general norm $N(x)$ (as a polynomial function of the a_i) is irreducible in $k[x]$.*

1.3. Lemma: — *The polynomial $1 - N(x)$ is irreducible.*

Proof. By [Fla53, Theorem 1] above, we know that the polynomial $N(x)$ in the a_i is irreducible. We first show that $1 - N(x)$ is irreducible over k. To show that $1 - N(x)$ is irreducible, it suffices to show that its homogenization in $\mathbb{P}^n_k$, namely $y^n - N(x)$, is irreducible. By [Vak23, Exercise 4.5.F(c)], one can check the primeness of a homogeneous ideal or element by considering its divisibility by homogeneous elements.

So, suppose that $y^n - N(x) = pq$ for two homogeneous polynomials $p, q \in k[a_0, \ldots, a_{n-1}, y]$. We can regard each of p, q as polynomials in y with coefficients in $k[a_0, \ldots, a_{n-1}]$. With this view, it is clear that the product of the terms of p and q which do not involve y is just $N(x)$. But by the irreducibility of $N(x)$, this means that one of p or q must be 1. Then our homogeneity assumption forces one of p or q to be an element of k and therefore a unit, and this finishes.

To see that $y^n - N(x)$ is geometrically irreducible, we suppose again that there is some $p, q \in \overline{k}[a_0, \ldots, a_{n-1}, y]$ for which $y^n - N(x) = pq$. Let l/k denote the finite field extension generated by the coefficients of p and q over k. Then $y^n - N(x)$ is reducible over l. But $N(x)$ describes the field norm of the degree n extension of l, so this reducibility contradicts the theorem above. ■

1.4. Corollary: — *The hypersurface X is integral.*

1.5. Lemma: — *The maps f and g are rational. If k is infinite, then g is also dominant.*

Proof. Both f and g are clearly rational – they are regular over the complement of the simultaneous vanishing loci of their respective coordinates. If k is infinite, then the subset of k-points of $\mathbb{P}^{n-1}_k$ is dense. Let $P \subset \mathbb{P}^{n-1}_k$ denote the subset of k-points. The computations from Miles' note show that $(g \circ f)|_P = \mathrm{id}_P$. Hence, g surjects onto the k-points of $\mathbb{P}^{n-1}_k$, so the image of g is dense in $\mathbb{P}^{n-1}_k$, and this finishes. ■

Since g is dominant, we can realize the function field $K(X)$ as an extension of $K(\mathbb{P}^{n-1}_k) = k(t_1, \ldots, t_{n-1})$. By the principal ideal theorem, we also know that $\dim X = n - 1$, so $K(X)$ is also a field of transcendence degree $n - 1$. It is then a straightforward algebraic exercise to show that the extension $K(X)/K(\mathbb{P}^{n-1}_k)$ must be algebraic.

1.6. Claim: — *There is a dense open subset of $\mathbb{P}_k^{n-1}$ over which the following identity holds:*

$$g \circ f = \mathrm{id}_{\mathbb{P}_k^{n-1}}.$$

Sketch of idea. It suffices to show this identity after base-change to the algebraic closure, so we may suppose that $k = \overline{k}$. Since k can be taken to be infinite in this way, the subset of k-points, which are precisely the closed points, of $\mathbb{P}_k^{n-1}$ is dense. It would be nice to write some formula for the maps which would give a bijection on these $\overline{k}$ points. If one can show that X is normal, then it might be possible to use Zariski's Main Theorem (e.g.[Sta24, Lemma 03GW] and [(s]) to argue that f and g must be birational inverses.

Another possibility is working within $k \neq \overline{k}$. Without modifying the existing coordinates, it might be possible to show that the $g \circ f$ and $\mathrm{id}_{\mathbb{P}_k^{n-1}}$ induce the same endomorphism of the $\overline{k}$-points $\mathbb{P}_k^{n-1}(\overline{k})$ to apply the fact referenced here. One could then immediately apply this to the field extension discussed above to see that $K(X) \cong K(\mathbb{P}_k^{n-1})$, and this would finish.

Applying these ideas to Artin-Schreier extensions of finite fields would then be a matter of applying descent: since $\mathscr{H}\!om$ is a sheaf in any Grothendieck topology, we can change base by the faithfully flat map Spec $k(t) \to$ Spec k to apply the ideas above which require an infinite base field. ■

References

[Fla53] Harley Flanders, *The norm function of an algebraic field extension*, Pacific J. Math. **3** (1953), 103–113. MR55376

[(s] Jason Starr (https://mathoverflow.net/users/13265/jason starr), *Bijection implies isomorphism for algebraic varieties*. URL:https://mathoverflow.net/q/264216 (version: 2017-03-09).

[Sta24] The Stacks Project Authors, *Stacks Project*, 2024.

[Vak23] Ravi Vakil, *The rising sea: Foundations of algebraic geometry*, Math 216: Foundations of Algebraic Geometry, 2023.