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Exploiting Imprecise Information Sources
in Sequential Decision Making Problems
under Uncertainty

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INTRODUCTION

CONTEXT

PROVIDING the autonomy to a robot consists in computing a function which returns the name of the actions to be triggered at a given moment, with respect to the data from its sensors. The features of interest of the robot and its surroundings form a *system*. In general, for a given sequence of actions performed by the robot, the evolution of this system is not fixed for sure, but its behavior may be known performing tests on the robot or using information from expert knowledge. As well, the raw or processed data from the robot sensors are not generally a deterministic outcome of the state of the system nor the taken actions: nevertheless, these data, called also *observations* of the system, depend on robot's actions and system states. Relations between observations, system states and actions may be known through tests of the sensors in various situations, or by taking into account the sensors description, the data processing, or any related expert information. For instance, in the case of a robot using Computer Vision (CV), the output of the picture processing algorithm employed is considered as an observation of the system since it is the result of processed sensor data: here data are pictures from camera. For a given camera, and a given vision algorithm, the behavior of the observation is related to the action and the system state during the picture taking process.

Thus, in order to make a robot autonomously fulfill a chosen *mission*, we are looking for a function returning actions conditional on the sequence of system observations, and taking into account uncertainty about the system evolution and the observation of it. Such functions may be called *strategies*. The research domain associated to this kind of problem, *i.e.* strategy computation, is not restricted to robotics and is called *sequential decision making under uncertainty*: in the general case, the entity which has to act is called the *agent*. In this thesis, the problem of strategy computation is studied in the context of autonomous robotics, and the agent is the decisional part of the robot. Computing a strategy for a given robotic mission needs a proper framework: the best known model describes the state and observation behaviors using Probability Theory.

A probabilistic model for strategy computation

Markov Decision Processes (MDPs) define a useful formalism to express sequential decision problems under probabilistic uncertainty [7]. It is a well suited framework if the actual system state is known by the agent at each point in time. In the robotic context, this assumption means that the considered mission allows to assume that the robot has full knowledge of the features of interest via its sensors. In this model, a system state is denoted by the letter s , and the finite set of all the possible states is \mathcal{S} . The finite set \mathcal{A} consists of all possible actions $a \in \mathcal{A}$ available to the agent. The time is discretized into integers $t \in \mathbb{N}$ which represent time steps of the action sequence.

The state dynamics is assumed to be *Markovian*: at each time step t , the next system state $s_{t+1} \in \mathcal{S}$, only depends on the current one $s_t \in \mathcal{S}$ and the chosen action $a_t \in \mathcal{A}$. This relation is described by a transition function $\mathbf{p}(s_{t+1} \mid s_t, a_t)$ which is defined as the probability

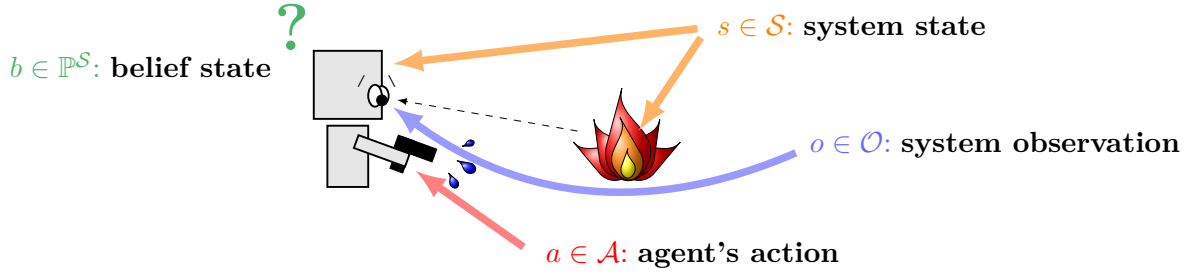


Figure 1 – Use of a POMDP for the firefighter robot mission modeling: in this toy example, the mission of the robot is the fire prevention. The **states of the system** $s \in \mathcal{S}$ encode for instance the robot location, the water jet orientation, the amount of water used, the fire location and its level on a scale between “minor fire” and “big fire”, etc. Using vision and heat sensors, the robot gets **observations** $o \in \mathcal{O}$ which are the raw or processed values from the sensors: the output of a classifier whose input is a picture of the scene (see Figure 3 and 5), and which returns the fire level or location may be encoded in an observation. Finally, the **actions of the robot** $a \in \mathcal{A}$ are for instance the rotor activations impacting the rotation of the robot’s wheels, the water pumping, the orientation of the water jet or sensors etc. The **reward function** $r(s, a)$ decreases with the fire level state, and is decreased by a cost proportional to the amount of water used: as an optimal strategy maximizes the mean of the sum of the rewards, the goal of the robot is thus to attack fires without wasting water. This mean can be computed knowing the probabilities describing the uncertainty dynamic of the system. The robot actions $a \in \mathcal{A}$ have a probabilistic effect on the system, as described by the **transition function** $\mathbf{p}(s' | s, a)$: for instance, the activation of wheel rotors modifies the location of the robot, and the probability of each possible next locations, given the current system state, takes part in the definition of the POMDP. An other example is the action modifying the water jet orientation, which redefines the probability of the next fire level given the current system state. The robot actions $a \in \mathcal{A}$ and next states $s' \in \mathcal{S}$ may also influence the observations from the sensors, as defined by the **observation function** $\mathbf{p}(o' | s', a)$: for instance, the orientation of the vision sensor may modify the probability of fire detection or fire level evaluation, which are parts of the observations $o' \in \mathcal{O}$. Finally, the **belief state** is the conditional probability distribution of the current system state conditional on all observations and actions up to the current time step: as observations and actions are the only data available to the robot, the belief state can be seen as the robot’s guess.

distribution on the next system states s_{t+1} conditional for each action: if the action $a_t \in \mathcal{A}$ is selected by the agent, and the current system state is $s_t \in \mathcal{S}$, the next state $s_{t+1} \in \mathcal{S}$ is reached with the probability denoted by $\mathbf{p}(s_{t+1} | s_t, a_t)$.

The mission of the agent is described in terms of rewards: a reward function $r(s, a) \in \mathbb{R}$ is defined for each action $a \in \mathcal{A}$ and system state $s \in \mathcal{S}$, and models the goal of the agent. The more rewards are gathered during an execution of the process, the better: a realization of a sequence of system states and actions is considered as well fulfilling the desired mission if encountered rewards $r(s_t, a_t)$ are high. Solving an infinite horizon MDP consists in computing an optimal strategy, *i.e.* a function prescribing actions $a \in \mathcal{A}$ to be taken over time, and maximizing the mean of the sum of rewards gathered during an execution: this mean is computed with respect to the probabilistic behavior of the system state encoded by transition functions $\mathbf{p}(s_{t+1} | s_t, a_t)$. For instance, a preferred strategy may be a function d defined on \mathcal{S} , as the current state is available to the agent, and with values in \mathcal{A} . A criterion measuring the accuracy of the strategy d may then be the expected discounted total reward:

$$\mathbb{E} \left[\sum_{t=0}^{+\infty} \gamma^t r(s_t, d_t) \right], \quad (1)$$

where $d_t = d(s_t) \in \mathcal{A}$ and $0 < \gamma < 1$ is a discount factor assuring the convergence of the sum.

The assumption that the agent has a perfect knowledge of the system state is quite strong: in particular, in the case of robots realizing tasks with conventional captors, the latter are

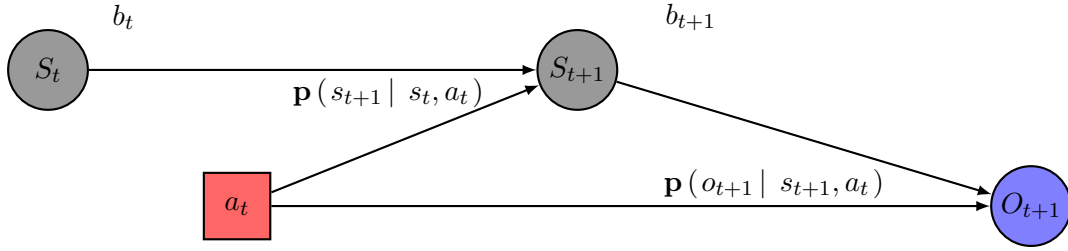


Figure 2 – Bayesian Network illustrating the belief update: the states are the gray circular nodes, the action is the red square node, and the observation is the blue circular node. The random variable S_t representing the next state s_{t+1} depends on the current one s_t and the current action a_t . The random variable O_{t+1} representing the next observation o_{t+1} depends on the next state s_{t+1} and the current action a_t too. The belief state b_t (resp. b_{t+1}) is the probabilistic estimation of the current (resp. next) system state s_t (resp. s_{t+1}).

usually unable to provide all the features of interest for the mission to the robot. Thus, a more flexible model has been built, allowing a *partial observability* of the system state by the agent.

Indeed a Partially Observable MDP (POMDP) [113] makes a step further in the modeling flexibility, handling situations in which the agent does not know directly the current state of the system: it finely models an agent acting under uncertainty in a partially hidden environment.

The set of system states \mathcal{S} , the set of actions \mathcal{A} , the transition function $\mathbf{p}(s_{t+1} | s_t, a_t)$ and the reward function $r(s, a)$ remain the same as for the MDP definition. In this model, since the current system state $s \in \mathcal{S}$ cannot be used as available information for the agent, the agent knowledge about the actual system state comes from observations $o \in \mathcal{O}$, where \mathcal{O} is a finite set. The observation function $\mathbf{p}(o_{t+1} | s_{t+1}, a_t)$ gives for each action $a_t \in \mathcal{A}$ and reached system state $s_{t+1} \in \mathcal{S}$, the probability over possible observations $o_{t+1} \in \mathcal{O}$. Finally, the *initial belief state* $b_0(s)$ defines the *prior* probability distribution over the system state space \mathcal{S} . An example of usage of a POMDP is presented in Figure 1.

Solving a POMDP consists in computing a strategy which returns a proper action at each process step, according to all received observations and selected actions *i.e.* all of the data available to the agent: a criterion for the strategy may be also the expected discounted sum of rewards (I.2).

Most of the POMDP algorithms reason about the *belief state*, defined as the probability of the actual system state knowing all the system observations and agent actions from the beginning. This belief is updated at each time step using the Bayes rule and the new observation. At a given time step $t \in \mathbb{N}$, the belief state $b_t(s)$ is defined as the probability that the t^{th} state is $s \in \mathcal{S}$ conditional on all the past actions and observations, and with the prior b_0 : it estimates the actual system state using the available data, as the latter is not directly observable.

It can be easily recursively computed using Bayes rule: at time step t , if the belief state is b_t , chosen action $a_t \in \mathcal{A}$ and new observation $o_{t+1} \in \mathcal{O}$, next belief is

$$b_{t+1}(s') \propto \mathbf{p}(o_{t+1} | s', a_t) \cdot \sum_{s \in \mathcal{S}} \mathbf{p}(s' | s, a_t) \cdot b_t(s). \quad (2)$$

as illustrated by the Bayesian Network in Figure 2.

As successive beliefs are computed with the observations perceived by the agent, they are considered as visible for the agent. Let us denote by $\mathbb{P}^{\mathcal{S}}$ the infinite set of probability distributions over \mathcal{S} . An optimal strategy can be looked for as a function d defined on $\mathbb{P}^{\mathcal{S}}$ such that successive $d_t = d(b_t) \in \mathcal{A}$ maximize the expected reward I.2: the agent decisions are then based on the belief state.

The POMDP framework is a flexible model for autonomous robotics, as illustrated by the firefighter example, see Figure 1: it allows to describe all the robotic and surrounding system,



Figure 3 – Example of an observation method in a robotic context: the robot, here a drone, is equipped with a camera and uses a classifier computed from a picture dataset (as NORB, see Figure 4): such a classifier is described by Figure 5. The classifier is computed before the mission (off-line) with a picture dataset (see the right part of the illustration), and the classifier output is used during the mission (online) as an observation, for the agent (see the left part). Here, observations are thus generated by computer vision.

as well as the robot mission, and it is commonly used in robotics [91, 83, 77, 25, 26]. It takes into account that the robot receives data from its sensors only, and thus has to figure out the actual system state using these data, called observations, in order to fulfill the mission. However the POMDP model raises some issues, in particular in the robotic context.

PRACTICAL ISSUES OF THE POMDP FRAMEWORK

Complexity

Solving a POMDP *i.e.* computing an optimal strategy, is PSPACE-hard in finite horizon [86] and even undecidable in infinite horizon [76]. Moreover a space exponential in the problem description may be required for an explicit specification of such a strategy (see [78] for a more detailed complexity analysis of POMDPs).

This high complexity is well-known by POMDP users: optimality can be reached for tiny problems, or highly structured ones. Classical approaches try to solve this problem using Dynamic Programming [21]. Otherwise, only approximate solutions can be computed, and thus the strategy has no optimality guaranty. For instance, popular approaches such as point-based methods [90, 69, 114], grid-based ones [58, 19, 11] or Monte Carlo approach [111], use approximate computations. The next POMDP's practical issues that will be highlighted, concern modeling flaws of this model, illustrated by robotic situations.

Parameter imprecision and computer vision

Consider now robots using visual perception, and whose observations come from computer vision algorithms based on statistical learning (see Figure 3). In this situation, the robot uses a *classifier* to recognize objects in pictures: the classifier is supposed to return the name of the object actually in the picture, and makes some mistakes with a low probability (see confusion matrix of Figure 6).

The classifier is computed using a *training dataset* of pictures (as NORB, see Figure 4, authors made it available at <http://www.cs.nyu.edu/~ylclab/data/norb-v1.0/>). A powerful gradient-based learning algorithm meant to compute classifiers using picture datasets is described in Figure 5: the associated framework is called Convolutional Nextwork [73]. The figures (4), (5) and (6) illustrate the example of a classifier computed for a drone mission where features of interests (system states of the problem) are related with the presence (or absence) of animals, cars, humans, planes or trucks: the statistical problem of computing a classifier recognizing such objects in pictures is called *multiclass classification*.

As the classifier is learned based on a picture dataset (see weights learned in Figure 5), its

NORB dataset: $(\text{picture}_i, \text{label}_i)_{i=1}^N$

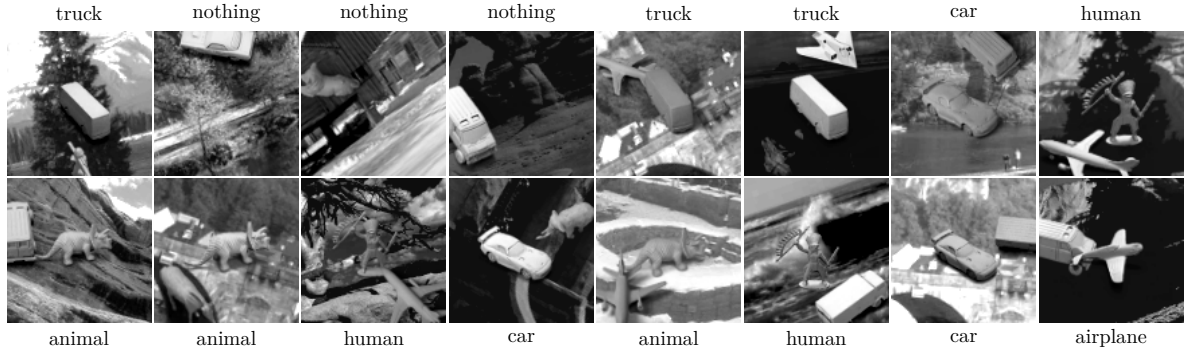


Figure 4 – Example of picture dataset for computer vision: the labeled picture dataset NORB [72]. The size of NORB is higher than 3.10^5 , and pictures from this dataset represents objects among the 5 classes: “animal”, “car”, “human”, “nothing”, “plane” and “truck”. Each element of a labeled picture dataset is composed of a picture (e.g. a picture showing a car) and a label corresponding to the class of the object represented by the picture (in the previous example, the label is “car”). This dataset can be used for supervised learning to compute a classifier (see Figure 5). In order to be able to discern locations of targets, a picture is labeled with the name of centered object (“nothing” if there is nothing in the picture center).

behavior, and thus its performances (*i.e.* how well it predicts objects in images) are inevitably dependent on the dataset. It is a problem if the picture variability in the dataset is too low: in this case, the probabilistic behavior of the classifier will be dependent on these particular pictures, and the robotic system will have poor observation capabilities when the considered mission involves pictures too different from the ones from the dataset.

Some large picture datasets with a high image variability exists (e.g. NORB, Figure 4, although variability could be ideally higher): note however that with such datasets, the vision performances are reduced, or good performances are, at least, harder to reach.

A confusion matrix can be computed (see Figure 6) using such a labeled picture dataset, not used for the training, and called *testing dataset*: observation frequencies can be deduced from this matrix, normalizing rows into probabilities. A row corresponds to an object in the scene, and probabilities on this row are observation probabilities, *i.e.* each probability is the frequency with which the classifier returns the name of the object of the corresponding column. These probabilities can be used to define the observation function $\mathbf{p}(o' | s', a)$ introduced above. This approach raises the issue of knowing if the testing dataset is quite representative of the mission reality. If not, the observation probabilities may be not reliable, and the POMDP badly defined: however, as shown by the equation (2) the belief update needs a perfect knowledge of the observation probability distributions.

More generally, observations of the agent may be outputs of image processing algorithms whose semantics (image correlation, object matching, class inference, preprocessing followed by classifiers such as the one computed in Figure 5 etc.) are so complex that probabilities of occurrence are hard to rigorously extract.

Finally, if the considered datasets are labeled more precisely (as NORB, which includes information such as the lighting condition, or the object scale), we can imagine that the computed observation probabilities (from the confusion matrix) were more reliable, or the vision performances upgraded (since separation when learning the classifier is easier). However, as more observation or states are involved in this case, the POMDP is harder to solve. Moreover, as the number of pictures per class is reduced (since there are more complex and numerous classes), a poorer confusion matrix is obtained when testing (in terms of confidence).

As a conclusion, the POMDP model supposes the knowledge of all the involved probability

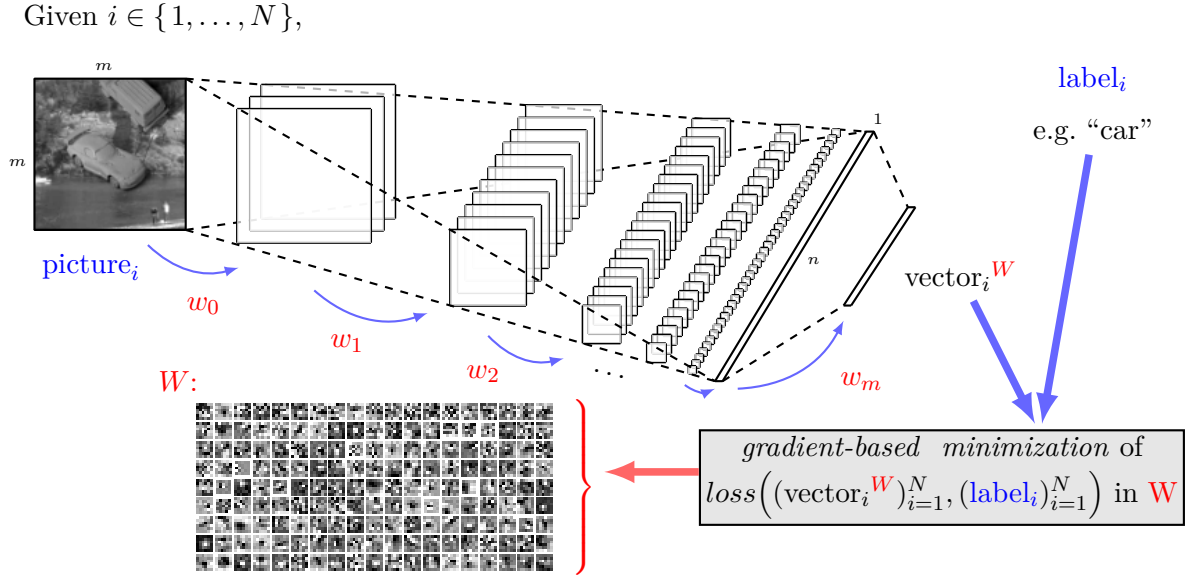


Figure 5 – Example of classifier training for computer vision: the labeled picture dataset NORB, see Figure 4, is used to train and test a classifier. The learning algorithm is based on Convolutional Network [73, 107] using gradient methods [14, 71, 15]. The weights $W = (w_0, \dots, w_m)$ are the parameters of a particular transformation (see the “bi-pyramid” of successive transformation stages) from the picture to a vector representing a label among { animal, car, human, nothing, plane, truck }. These weights are learned in order to minimize a given loss function i.e. a proper criterion representing the error of the classifier over the dataset. A classical loss function is the Mean Squared Error (MSE). The environment Torch7 (based on lua and C languages, [29]) has been used to compute the displayed weights.

distributions: unfortunately the actual frequencies used to compute them are not precisely known in practice. The imprecision about these probabilities, for instance the imprecision related to the vision algorithm behavior with real world pictures, has to be taken into account to make the robot autonomous under any circumstances. In general, the computation of the probability distributions of a POMDP needs enough tests for each possible system state and action, which is hard to perform in practice.

Works handling parameter imprecision

Some variations of the POMDP framework have been built in order to take into account the imprecision about the probability distributions of the model, also called *parameter impreci-*

animal	human	plane	truck	car	nothing		
3688	575	256	48	144	149	animal	75.885%
97	4180	81	20	225	257	human	86.008%
292	136	3906	237	202	87	plane	80.370%
95	1	44	4073	514	133	truck	83.807%
129	3	130	1283	3283	32	car	67.551%
154	283	36	63	61	4263	nothing	87.716%

Figure 6 – Example of confusion matrix for multiclass classification: this matrix is computed with a testing dataset of pictures, different from the training dataset. Each row only considers the pictures of a given object, and numbers represents the answers of the classifier: for instance, 3688 pictures of animals are well recognized, but 575 are confused with a human. Average row correct is here 80.223%. Torch7 environment [29] has also been used to compute this matrix from the classifier and the testing dataset.

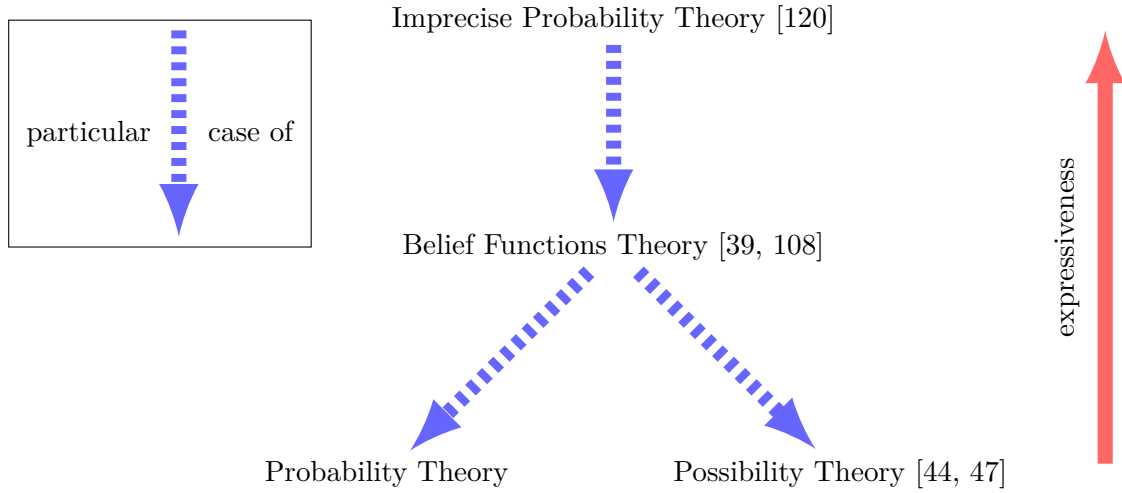


Figure 7 – Most known uncertainty theories and their relations. Imprecise Probability Theory (IPT) considers sets of probability measures to represent uncertainty of events and knowledge of it: this theory is the most general one. The Belief Functions Theory (BFT or Dempster-Shafer Theory, or yet Theory of Evidence), which is a particular case of IPT, considers a mass function m defined on all the subsets of the universe, and which sums to 1. For instance, if the function m returns 1 for the universe, and 0 for all other subsets, it expresses the total ignorance about the actual probability distribution: it corresponds to the set of all the possible probability distributions in IPT. If the mass function is equal to zero for each non singleton subsets, it is simply a probability distribution. If the mass function is positive only on a sequence of nested subsets, it is a possibility distribution. Probability and Possibility Theory are thus particular cases of BFT and a fortiori of IPT.

sion. POMDPs with Imprecise Probabilities (POMDPIP) [63] second order belief (imprecision and frequency are confused, no position taken): no criterion asserting how imprecision is taken into account, belief discretization using all the possible models, analysis??

Bounded Parameters POMDPs (BPPOMDP) [79]

robust + NOTES

robust [84] it answers to the robust question (approx), but at least the POMDP complexity, optim require linear program, and lack of knowledge is represented by (convex) credal sets, PO version of the robust model for non stationary uncertain MDP models, or of the robust approximation for uncertain MDPs [80] (lowerbound of the maximin criterion)

Toutefois, il est très difficile, en termes de calculs, d'extraire une stratégie de ces modèles. Cependant, ils sont plus complexes que les POMDPs, eux-même étant déjà au moins PSPACE-hard

autant travailler sur un already discretized model

While there are fascinating questions of algorithms no learning because we are interested in robotic applications which have to quite well behave in first tests, (but can learn in each use e.g. long term learning) for instance in order to begin Reinforcement Learning (RL, [40, 97]) problem of solving the problem offline:

Agent ignorance modeling

The initial belief, or *a priori* probability distribution over the system states, takes part in the definition of the POMDP problem. However in practice, the initial system state can be

unknown with absolutely no probabilistic information: for instance, in a robotic exploration context, the initial location of the agent, or the presence of an entity in the scene. Defining the process with a uniform probability distribution as initial belief (*e.g.* over all locations or over entity presence) is a subjectivist answer [34], *i.e.* all probabilities are the same because no event is more plausible than another: it corresponds to equal betting rates. However following belief updates will eventually mix frequentist probability distributions defining the POMDP problem with this initial belief which is a subjective probability, and it does not always make sense.

La connaissance de b_0 pose aussi un problème: il existe des missions où le robot ne connaît initialement rien de l'état du système, ce que ne modélise pas tout à fait une probabilité uniforme. Lorsqu'aucune expérience passée n'est disponible, il n'est pas évident d'attribuer une fréquence à un évènement initial...

and knowledge of the agent CARO

Moreover, consider missions where a part of the system state, describing something that the robot is supposed to infer by itself, is initially fully unknown: for instance the location or the nature of a target. Classical approaches initialize the belief state as a subjective uniform probability distribution. In this case the belief update mixes subjective probabilities and frequencies, which is questionable.

- **Initial belief** b_0 (*prior* information on the system state).

uniform = subjectif, mix up with frequencies!

The use of the POMDPs raises however some practical issues, such as the difficulty to encode robot ignorance, Caroline [23, 24]

De plus, dès le début du processus, l'agent n'observe que partiellement l'état du système: une croyance initiale est nécessaire pour définir entièrement le POMDP. Dans le cas d'une mission de repérage de cible, il faut en effet attribuer une distribution de probabilité à la position initiale de cette dernière. Si cette position est totalement méconnue, une distribution de probabilité uniforme sur toutes les positions possibles peut-être une solution, ce qui en fait une distribution de probabilité subjective (elle est choisie uniforme par prudence compte tenu de la méconnaissance). Cependant, les autres probabilités définissant le modèle étant des fréquences, cette approche mène à un mélange incongru de probabilités subjectives et de fréquences. Les π -POMDPs permettent une modélisation formelle de la méconnaissance de l'agent, bien que ce modèle n'ait pas été étudié de manière approfondie.

Consider situations where the agent totally ignores the system's initial state, for instance a robot that is for the first time in a room with an unknown exit location (initial belief) and has to find the exit and reach it. In practice, no experience can be repeated in order to extract a frequency of the exit's location. In this kind of situation, uncertainty is not due to a random fact, but to a lack of knowledge: no frequentist initial belief can be used to define the model. A uniform probability distribution is often chosen in order to assign the same mass to each state. This choice can be justified based on the subjective probability theory [?] (the probability distribution represents then an exchangeable bet) but subjective probabilities and observation frequencies are combined during the belief update.

In other cases, the agent may strongly believe that the exit is located in a wall as in the vast majority of rooms, but it still grants a very small probability p_ϵ to the fact that the exit may be a staircase in the middle of the room. Even if this is very unlikely to be the case, this second option must be taken into account in the belief, otherwise Bayes' rule cannot correctly update it if the exit is actually in the middle of the room. Eliciting p_ϵ without past experience

is not obvious at all and does not rely on any rational reasons, yet it dramatically impacts the agent's policy. On the contrary, possibilistic uncertainty models allow the agent to elicit beliefs with imprecise unbiased knowledge.

GENERAL PROBLEM

problem of solving the problem offline: fulfill the mission at the first execution, with a model built from imprecise data and using reasonable computation time. + MODELIZATION and interested by the actual knowledge of the agent (CARO) C'est ce qui justifie cette étude. Elle consiste à développer un algorithme de calcul de stratégie pour une mission robotique autonome, adapté à l'incertitude liée à la vision artificielle, et permettant la modélisation formelle de la méconnaissance initiale de l'agent.

The study of other uncertainty theories may bring useful properties to deal with this problem. DEMPSTER: no, belief plausibility evidence: a probability value for each $2^{\# \Omega}$ (to complex too) poss quant no, poss qual!!

plausibilité $Pl(A) = \sum_{\{B \mid B \cap A \neq \emptyset\}} m(B)$ et la croyance $Bel(A) = \sum_{\{B \subset A\}} m(B)$. Dempster-Shafer Theory [108] less complex than IP but more than proba alone

A qualitative possibilistic model

Qualitative Possibility Theory: \rightarrow simplification, ignorance and imprecision modeling. The π -POMDP model is a possibilistic and qualitative counterpart of the probabilistic POMDP model [?]: it allows a formal modeling of total ignorance using a possibility distribution equal to 1 on all the states. This distribution means that all states are equally possible independently of how likely they are to happen (no necessary state).

Finding qualitative estimates of their recognition performance is easier: the π -POMDP model only require qualitative data, thus it allows to construct the model without using more information than really available. Constat que les possibilitees qualitatives sont qualitatives, et peux modeliser la connaissance The Qualitative Possibility Theory allows to handle imprecise data and model the lack of knowledge (pas forcement le modele) tropical algebra. (SIMPLIFY or ENCODE AVAILABLE DATA)

The use of the Qualitative Possibility Theory [52] is studied here, as it appears capable to both simplify the POMDPs, and model imprecision and ignorance related to robotic missions.

In our context, distributions defined in the Possibility Theory framework are valued in a totally ordered scale $\mathcal{L} = \{1 = l_1, l_2, \dots, 0\}$ with $l_1 > l_2 > \dots > 0$. A possibility measure Π defined on \mathcal{S} is a fuzzy measure valued in \mathcal{L} , such that $\forall A, B \subset \mathcal{S}, \Pi(A \cup B) = \max \{\Pi(A), \Pi(B)\}$, $\Pi(\emptyset) = 0$ and $\Pi(\mathcal{S}) = 1$. It follows that this measure is entirely defined by the associated possibility distribution, *i.e.* the measure of the singletons: $\forall s \in \mathcal{S}, \pi(s) = \Pi(\{s\})$. Properties of this measure lead to the possibilistic normalization:

$$\max_{s \in \mathcal{S}} \pi(s) = \Pi(\mathcal{S}) = 1. \quad (3)$$

If $\bar{s}, \underline{s} \in \mathcal{S}$ are such that $\pi(\bar{s}) < \pi(\underline{s})$, it means that \bar{s} is less plausible than \underline{s} . States with possibility degree 0, *i.e.* states $s \in \mathcal{S}$ such that $\pi(s) = 0$, are impossible (same meaning as $\mathbf{p}(s) = 0$), and those such that $\pi(s) = 1$ are entirely possible (but not necessary the most probable one).

note the similarities between Possibility and Probability Theory, replacing max by + and min by \times . Moreover, Possibility Theory has its own counterpart of the Bayes rule [49]:

the indicator (characteristic) function of this set. Unlike classical sets, values of a fuzzy set indicator function π are not only in $\{0, 1\}$. Recall that the indicator function of a classical set $A \subseteq \mathcal{S}$ is $\mathbb{1}_A(s) = 1$ if $s \in A$ and 0 otherwise. Values of a fuzzy set indicator function are chosen in a totally ordered scale $\mathcal{L} = \{1 = l_1, l_2, \dots, 0\}$ with $l_1 > l_2 > \dots > 0$: $\pi : \mathcal{S} \rightarrow \mathcal{L}$. If $s \in \mathcal{S}$ is such that $\pi(s) = l_i$, s is in the fuzzy set described by π , with degree l_i . Possibilistic beliefs used in this work will represent fuzzy sets of possible states. If the current possibilistic belief coincide with the distribution $\pi(s) = 1 \forall s \in \mathcal{S}$, all system states are totally possible, and it models therefore a total ignorance about the current system state: qualitative possibilistic beliefs can model agent initial ignorance. The full knowledge of the current state, say $\tilde{s} \in \mathcal{S}$, is encoded by a possibility distribution equal to the classical indicator function of the singleton $\pi(s) = \mathbb{1}_{\{s=\tilde{s}\}}(s)$. Between these two extrema, current knowledge of the system is described by a set of entirely possible states, $\{s \in \mathcal{S} \text{ s.t. } \pi(s) = 1\}$, and successive sets of less plausible ones $\{s \in \mathcal{S} \text{ s.t. } \pi(s) = l_i\}$ down to the set of impossible states $\{s \in \mathcal{S} \text{ s.t. } \pi(s) = 0\}$.

Enfin, un homologue des POMDPs appelé π -POMDP a été développé dans le cadre de la Théorie des Possibilités [101], réduisant la complexité du problème. De plus, ce cadre possibiliste décrit de manière qualitative l'incertitude de l'agent, ce qui permet de rester prudent dans la définition des paramètres du problème. Cependant, ce modèle n'a été que très peu étudié, et les recherches le concernant peu abouties.

Qualitative possibilistic POMDPs, π -POMDPs [101], are alternative processes defined using a qualitative evaluation of events plausibility instead of probabilities: it allows to formally represent agent ignorance, and imprecision on observations hazard. As number of belief finite, and MDP P complexity [55], at most exponential the process description.

DESCRIPTION OF OUR STUDY

The goal of this thesis is to show what the Qualitative Possibility Theory can bring in Planning under Uncertainty and Sequential uncertainty management in practice, in particular through Graphical models, in terms of simplification and modeling. ...

This paper presents recent contributions in the use of the Qualitative Possibility Theory for planning under uncertainty, studied to answer to these concerns.

Our contributions consists in : ...

JUSTIFIER LES IDEES FORTES, ET VERIFIER QUIL Y A DEUX IDEES FORTES
par sous section de chaque chapitre

State of the art

The *first chapter* presents in parallel

piPOMDP detailed, as never detailed much to the best of our knowledge

The work developed in this paper remains in the classical MDP and POMDP frameworks, which are recalled in this chap: possibilistic material necessary to build the promised translation are then presented.

Ces modèles probabilistes sont conçus pour représenter de manière simple une situation où une entité (appelée *agent*) et le monde qui l'entoure peuvent être dans différents états $s \in \mathcal{S}$ au cours du temps (ce dernier, modélisé par l'ensemble des entiers naturels \mathbb{N}). L'agent choisit une action parmi celles dont il dispose à chaque étape de temps $t \in \mathbb{N}$, et le système comprenant l'agent et son environnement évolue de manière Markovienne dans \mathcal{S} ([?],[?]).

Le modèle POMDP, autour duquel ce travail est développé, est utilisé lorsque le système peut se modéliser à l'aide de probabilités. Il permet aussi bien de modéliser les incertitudes sur l'observation du système, que sur sa dynamique. Nous commencerons donc par présenter un modèle plus simple (MDP, où l'observation est parfaite) afin de construire plus facilement le modèle POMDP.

Dans le modèle POMDP, les distributions de probabilité sont supposées parfaitement connues a priori. En pratique, ce n'est pas toujours le cas, notamment lorsque les observations de l'environnement sont filtrées par des algorithmes de traitement du signal. De plus, l'agent est supposé avoir une croyance initiale de l'état du système: sa croyance initiale n'est pas précise en pratique (extraire des fréquences d'événements initiaux n'est pas forcément possible), et une probabilité uniforme ne représente pas réellement cette méconnaissance. Il existe aussi des situations dans lesquelles, donner de l'importance à l'information de la croyance (au sens entropique) améliore la politique en pratique [23].

Ces problèmes pratiques peuvent trouver des solutions grâce à d'autres modélisations de l'incertitude que la théorie des probabilités. C'est pourquoi ce rapport s'intéresse ensuite à des modèles utilisant la théorie des possibilités (π -MDP et π -POMDP). Ces deux modèles introduits par R.Sabbadin sont présentés, puis une preuve est donnée à l'homologue de l'algorithme d'itération sur les valeurs. Enfin, l'astuce de O.Buffet *et al.* est adaptée pour le modèle π -POMDP.

Natural update of the possibilistic model

The *second chapter* proposes

It begins with an update of the work of Sabbadin about a possibilistic counterpart of POMDPs called π -POMDP. a possibilistic version of Mixed-Observable MDPs [83], called π -MOMDP [41], is first presented to reduce dramatically the complexity of solving π -POMDPs, some state variables of which are fully observable. An algorithm for missions with unbounded durations is next proposed: returned strategies can outperform probabilistic POMDP ones for a target recognition problem where the agent's observations dynamics is not properly defined. [41]

La démarche a consisté dans un premier temps à adapter les processus décisionnels possibilistes qualitatifs aux missions robotiques car ils modélisent formellement la méconnaissance initiale. Pour cela, le problème π -POMDP est défini pour un horizon infini, afin que les missions puissent être définies sans décider a priori de la durée de la mission. Sachant l'état du système n'est pas entièrement caché en pratique lors des missions [83], les π -MOMDPs, pour (Mixed-Observable MDPs) sont introduits afin de profiter de cette observation dans la réduction du temps de calcul de la stratégie. Par exemple, le niveau de batterie d'un drone est accessible directement par le système robotique.

L'algorithme associé à ces contributions ainsi que sa preuve de convergence est alors nécessaire dans le but de calculer une stratégie [41]. Enfin, les performances de ce nouveau modèle sont illustrées à l'aide de simulations de problèmes particuliers pour différents critères. De plus, ces simulations illustrent aussi la dynamique de la croyance de l'agent: sa mise à jour possède la caractéristique intéressante d'accroître la connaissance associée. Cependant, la résolution proposée reste limitée à des problèmes de faible taille.

Factorization work on the models

The *third chapter* is

[42] Then strategies computation time is decreased with the use of Algebraic Decision Diagrams (ADDs) and benefiting from the problem structure. We compare performances of our solvers with those of its probabilistic counterparts, in terms of computation time, and with criteria measuring the mission achievement. While this possibilistic framework provides good results, some highlighted issues are finally discussed: Then factorized π -MOMDPs are defined making possible the processing of large structured planning problems. Building upon the probabilistic SPUDD algorithm [62], we conceived an algorithm named PPUDD [42] for solving factorized π -MOMDPs using *Algebraic Decision Diagrams* (ADD). Our experiments and the results of the International Probabilistic Planning Competition (IPPC 2014) show that this possibilistic approach can involve lower computation time and produce better policies than its probabilistic counterparts: it highlights also some issues of these qualitative models. Finally, points raised are added to the symbolic computations limits, and lead to the description of a future work. We participated in the competition in order to test the performance of our algorithm in the case of well known probabilistic models.

LETAPE SUIVANTE A ETE DADAPTER LALGO AFIN QUE... Une fois ce modèle adapté aux besoins robotiques pratiques, nous avons souhaité pouvoir résoudre des problèmes de décision séquentielle dans l'incertain de tailles plus générales. Profitant à la fois du caractère qualitatif de ces processus, ainsi que de la structure des problèmes en pratique, un algorithme symbolique de résolution [42] est proposé. Pour cela, nous introduisons les π -MOMDPs factorisés, et décrivons sur les hypothèses d'indépendance possibles pour de tels processus. AFIN DE..

ON a voulu que cet algorithme utilise les ADD pour pouvoir synthétiser les calculs.

L'algorithme résultant, utilisant des arbres de décision algébriques (ADDs) afin de synthétiser les calculs, est nommé PPUDD pour *Possibilistic Planning using Decision Diagrams* POUR CELA, on a simulé des problèmes de planification variés. et a été testé lors de la compétition internationale de planification probabiliste (IPPC14).

EN TERMES DE REWARD EXPECTED

Ces tests sur des problèmes de planification variés mènent à l'observation que ces méthodes à ADDs, dites *symboliques*, ne font pas le poids face aux approches procédant à une recherche heuristique dans l'espace d'état [65] (en termes de rewards... expliquer!). Nous avons pu aussi prendre note des inconvénients du formalisme possibiliste qualitatif pour la modélisation: le critère utilisé étant global, son choix restreint la généralité du modèle.

Qualitative possibilistic framework process for human-machine interaction modeling

The *fourth chapter* is

Les processus possibilistes qualitatifs ont toutefois leur place lorsque toutes les distributions de probabilité ne peuvent clairement pas être définies dans le contexte: dans les systèmes modélisant le comportement humain, seules des données expertes peuvent être utilisées pour la modélisation. Pour cela, nous appliquons ces processus... **Application des processus possibilistes en diagnostique dans l'interaction homme-machine**

Processus π -HMP hidden markov processes, outils de diagnostic pour l'Intéraction Homme-Machine (avec Sergio Pizziol)

- **occurrences:** états de la machine et actions humaines;
- **évaluation humaine** (de l'état de la machine);
- **effets:** transitions, classées par degrés de possibilité.
- **estimation** de l'état selon l'opérateur humain;
- **détection** des erreurs humaines d'évaluation de l'état;
- causes plausibles de ces erreurs (**diagnostique**).

Probabilistic-possibilistic approach: an hybrid perspective

Finally, the *fifth chapter* is

argues for a hybrid POMDP model with both probabilistic and possibilistic settings.

STATE OF THE ART

The main topic of this thesis is the Partially Observable Markov Decision Process (POMDP). The practical use of this model has been criticized in Introduction, however it sums up accurately the principal features of a robotic system. As it is possible to make this thesis mathematically self-contained the POMDP model is built in Section I.1 from low level objects of Probability Theory. The main ways to compute strategies from this model are then summarized in Section I.1.11. Next, Possibility Theory is presented in order to introduce the Qualitative Possibilistic Markov Decision Processes (π -MDPs) and the Partially Observable ones (π -POMDPs) which are the starting point of this work.

I.1 FROM MARKOV CHAINS TO PARTIALLY OBSERVABLE MARKOV DECISION PROCESSES

The first theoretical object behind the POMDP model, as its name suggests, is the *Markov Chain*. In order to present it, let us look back one century ago.

I.1.1 Markov Chains

In the early years of the twentieth century Andreï Markov, a doctoral student of Pafnouti Tchebychev, set up Markov Chains. Studying successive letters in the words of novels, he had the idea to define this kind of sequence of random variables: indeed, each letter depended primarily on the previous one. As usual, a random variable is a measurable function defined on a set Ω equipped with a σ -algebra \mathcal{F} and a probability measure \mathbb{P} .

Definition I.1.1 (*Markov Chain*)

Let \mathcal{S} be a countable set called *set of states* and $(S_t)_{t \in \mathbb{N}}$ a sequence of random variables whose values are in \mathcal{S} . The sequence $(S_t)_{t \in \mathbb{N}}$ is a *Markov Chain* if $\forall t \geq 1, \forall (s_0, s_1, \dots, s_{t+1}) \in \mathcal{S}^{t+2}$

$$\mathbb{P}(S_{t+1} = s_{t+1} \mid S_0 = s_0, S_1 = s_1, \dots, S_t = s_t) = \mathbb{P}(S_{t+1} = s_{t+1} \mid S_t = s_t) \quad (\text{I.1})$$

i.e. $\forall t \geq 1$, the random variable S_{t+1} is independent from all previous variables $\{S_i \mid i \leq t-1\}$ conditional on the random variable S_t : the value of S_0 , or its probability distribution is given, and the probability distribution of S_{t+1} only depends on the value of S_t and on the time step $t \geq 0$ (see Figure I.1).

Figure I.1 describes the Bayesian Network [88, 87] of a Markov Chain. In a Bayesian Network, or directed acyclic graphical model, the variables are represented by nodes. The absence of an arrow between two random variables (nodes) represents an assumption about the conditional independence of the variables. Let S' be a random variable: the set of variables from which an arrow starts and leads to S' is called the set of *parents* of S' , and denoted by $\text{parents}(S') =$

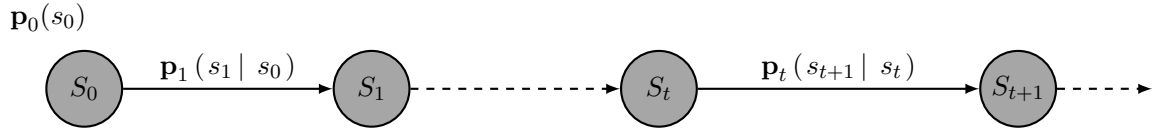


Figure I.1 – *Bayesian Network of a Markov Chain*: each node (black circle) represents a random variable. Each variable S_{t+1} has only one parent S_t : $\forall t \geq 1 \ S_{t+1} \perp\!\!\!\perp \{S_0, \dots, S_{t-1}\} | S_t$.

$\{S \mid S \rightarrow S'\}$. If $S \in \text{parents}(S')$, we say that S' is a *children* of S : the set of the children of S is denoted by $\text{children}(S)$. The set of the *descendants* of a random variable S is the smallest set of variables $\text{descend}(S)$ which contains all the children of S and such that $\forall S' \in \text{descend}(S)$, $\text{children}(S') \subset \text{descend}(S)$ i.e. all the children of a descendant is a descendant. The assumption taken through a Bayesian Network is that each variable S is independant from its *non-descendants* $\text{nondescend}(S) = \{\tilde{S} \notin \text{descend}(S) \cup \{S\} \cup \text{parents}(S)\}$ conditional on its parents $\text{parents}(S)$, denoted by:

$$S \perp\!\!\!\perp \text{nondescend}(S) \mid \text{parents}(S).$$

Figure I.1 is also called Dynamic Bayesian Networks (DBNs) [35] since the arrows model also successive time steps.

Here, the *Markov Property* i.e. Equation I.1 of Definition I.1.1, implies that $\forall t \in \mathbb{N}$, the variable S_{t+1} has only one arrow pointing to it from S_t i.e. the variable S_{t+1} has only one parent in the Network, which is S_t .

As \mathcal{S} is countable, let us number its elements, the *states*: $\mathcal{S} = \{s^{(1)}, s^{(2)}, \dots\}$. The probabilistic dynamics of a Markov Chain can be represented by a sequence of stochastic matrices¹ $(M_t)_{t \in \mathbb{N}}$ defined by $(M_t)_{i,j} = \mathbb{P}(S_{t+1} = s^{(j)} \mid S_t = s^{(i)})$. Note that the model is entirely defined given this sequence of matrices and the distribution of the first random variable S_0 : $\mathbf{p}_0(s) = \mathbb{P}(S_0 = s)$. For a better readability, $\mathbb{P}(S_{t+1} = s' \mid S_t = s)$ is denoted by $\mathbf{p}_t(s' \mid s)$, and called the *transition probability distribution*.

An important result about the Markov Chains is the following:

Property I.1.1

Let $(S_t)_{t \in \mathbb{N}}$ be a Markov Chain whose values are in the countable set of states \mathcal{S} .
 $\forall t \in \mathbb{N}, \forall f : \mathcal{S} \rightarrow \mathbb{R}$ bounded, $\forall (s_0, \dots, s_t) \in \mathcal{S}^{t+1}$,

$$\begin{aligned} \mathbb{E}[f(S_{t+1}) \mid S_0 = s_0, \dots, S_t = s_t] &= \sum_{s' \in \mathcal{S}} f(s') \cdot \mathbb{P}(S_{t+1} = s' \mid S_t = s_t) \\ &= \mathbb{E}[f(S_{t+1}) \mid S_t = s_t]. \end{aligned}$$

The proof is given in Annex A.2.

This result will help to rigorously set up Markov Decision Processes, presented in the next section.

I.1.2 The Markov Decision Process

The MDP [96] was built to model *systems* subject to a probabilistic uncertainty dependent on *actions* over *time*. In its classical formulation, the finite set \mathcal{S} defines the possible *states of the system* $s \in \mathcal{S}$. Here, for the needs of the POMDP model building, \mathcal{S} is defined as a countable set of states, as stated earlier. The set of the non-negative integers \mathbb{N} models the *time*, or *stages*

¹A stochastic matrix is a matrix whose values are non-negative real numbers and whose rows sum to one.

of the process. Possible actions, denoted by a , are chosen from a finite set \mathcal{A} . In order to get a clear overview of what information is used to decide each action, the *agent* is defined as the entity responsible for decision making, *i.e.* must choose the successive actions. In our case, a system state $s \in \mathcal{S}$ consists of the features of both a robot and its environment, needed to describe the robotic mission. The agent is here the decision making part of the robot.

If the sequence of chosen actions $a \in \mathcal{A}$ is known, an MDP is a Markov Chain: the system state of an MDP at stage $t + 1$, represented by the variable S_{t+1} , only depends, in the probabilistic meaning of the term, on the previous system state variable S_t and on the time step $t \geq 0$. However, the agent influences the probabilistic system dynamics by selecting the actions $a \in \mathcal{A}$ at each time step $t \in \mathbb{N}$.

In the MDP framework, it is assumed that the agent is informed of the current system state $s_t \in \mathcal{S}$ at each time step. Its decision can then be based on the current system state and all previous ones. A stochastic matrix M^d can be defined for each *decision rule* $\begin{cases} d : \mathcal{S} \rightarrow \mathcal{A} \\ s \mapsto d(s) \end{cases}$ describing transition probability distributions of the Markov Chain $(S_t)_{t \in \mathbb{N}}$ if the decision rule d is used: $(M^d)_{i,j}$ is the probability of reaching the state $s^{(j)}$ from $s^{(i)}$ when the action $d(s^{(i)})$ is selected by the agent. Consider a sequence of decision rules $(d_t)_{t \in \mathbb{N}}$: such a sequence is called *strategy* and defines a sequence of stochastic matrices $(M_t)_{t \in \mathbb{N}}$ with $M_t = M^{d_t}$. Each strategy thus defines the parameters of a Markov Chain entirely.

A reward $r_t(s, a) \in \mathbb{R}$ is associated to each triple $(s, a, t) \in \mathcal{S} \times \mathcal{A} \times \{0, \dots, H - 1\}$ to model the importance of going through the state s and selecting action a at time t . The function $r : \mathcal{S} \times \mathcal{A} \times \{0, \dots, H - 1\} \rightarrow \mathbb{R}$ is assumed to be bounded: this assumption is not necessary if \mathcal{S} is finite, as $r_t(s, a) \leq \max_{s' \in \mathcal{S}} \max_{a' \in \mathcal{A}} \max_{0 \leq t' \leq H-1} \{r_{t'}(s', a')\}$, however, \mathcal{S} is here countable and thus may be infinite. The value of a system finishing in state s is described by the terminal reward $R(s)$, defined for each state s . The function R is assumed to be bounded on \mathcal{S} .

For instance in the Navigation problem of the International Probabilistic Planning Competition, [106], a robot in a grid has to reach a goal: if the state s encodes a robot location which is not the goal, $r_t(s, a) = -1$, and $r_t(s, a) = 0$ otherwise. If the designers of the model want the robot to be in sleep mode at the end of the mission, terminal reward could have been defined as $R(s) = 1$ if the system state s encodes the sleep mode, and $R(s) = 0$ otherwise. If some of the moving actions of the robot were more energy demanding than others, $\forall s \in \mathcal{S}$ the functions $a \mapsto r_t(s, a)$ may be non-constant, etc.

Solving the optimal control problem for an MDP with horizon $H \in \mathbb{N}$, *i.e.* for a process whose number of stages is H , consists in finding a strategy maximizing the expectation of the sum of the rewards, or *expected total reward*:

$$V_H(s, (d_t)_{t=0}^{H-1}) := \mathbb{E} \left[\sum_{t=0}^{H-1} r_t(S_t, d_t(S_t)) + R(S_H) \mid S_0 = s \right]. \quad (\text{I.2})$$

The set of strategies of horizon $H \in \mathbb{N}$ *i.e.* the sequence of H decision rules d_t numbered from 0 to $H - 1$ is denoted by \mathcal{D}_H . The criterion V_H is a function of the initial state $s \in \mathcal{S}$, the horizon size $H \in \mathbb{N}$ and the strategy $(d_t)_{t=0}^{H-1} \in \mathcal{D}_H$: this function is called *value function*.

I.1.3 Dynamic Programming

In 1949, the Research ANd Development (RAND) corporation hired Richard Bellman to work on multi-stage decision processes. Richard Bellman found a really efficient method called *Dynamic Programming* [6] to solve a class of problems by decomposing them into several simpler subproblems. The optimal control of MDPs, *i.e.* the computation of a strategy maximizing

the expected total reward, is part of this class, and is classically performed using Dynamic Programming [96].

The notation (d) is used from now on to represent a strategy $(d_t)_{t=0}^{H-1} \in \mathcal{D}_H$. Theorem I.3 highlights the opportunity to use the Dynamic Programming in order to compute the optimal value function:

$$V_H^*(s) = \sup_{(d) \in \mathcal{D}_H} V_H(s, (d)) \quad (\text{I.3})$$

and a strategy $(d^*) = (d_t^*)_{t=0}^{H-1} \in \mathcal{D}_H$ such that $\forall s \in \mathcal{S}, V_H^*(s) = V_H(s, (d^*))$. As shown in the proof of the following Theorem 1, this strategy exists, and it is thus possible to rewrite the optimal value function I.3, as follows: $V_H^*(s) = \max_{(d) \in \mathcal{D}_H} V_H(s, (d))$.

At step $0 \leq t < H$, the probability that the system state becomes $s' \in \mathcal{S}$ from state $s \in \mathcal{S}$ when the agent selects action $a \in \mathcal{A}$ is denoted by $\mathbf{p}_t(s' | s, a) = \mathbb{P}(S_{t+1} = s' | S_t = s, a)$. However, we assume that $\forall t \in \{0, \dots, H-1\}, \forall s \in \mathcal{S}, \forall a \in \mathcal{A}$, the support of the transition probability distributions $\mathbf{p}_t(\cdot | s, a)$ is finite, *i.e.* $\exists \mathcal{S}_{s,a,t} \subset \mathcal{S}$, such that $\#(\mathcal{S}_{s,a,t}) < +\infty$, and $\forall s' \in \mathcal{S} - \mathcal{S}_{s,a,t}, \mathbf{p}_t(s' | s, a) = 0$. For $i \in \{0, \dots, H-1\}$, the partial value function V_i is defined as the value function for the process starting at time step $t = H - i$, and $V_i^*(s') = \max_{(d_t)_{t=H-i}^{H-1} \in \mathcal{D}_i} V_i(s')$.

Theorem 1

The following Dynamic Programming equations for an horizon $H \in \mathbb{N}$ computes successive functions V_i^* for each $\forall 0 \leq i \leq N$ and an optimal strategy (d^*) : $\forall s \in \mathcal{S}$,

$$\begin{aligned} V_0^*(s) &= R(s), \\ \text{and } \forall 1 \leq i \leq H, \quad V_i^*(s) &= \max_{a \in \mathcal{A}} \left\{ r_{H-i}(s, a) + \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{H-i}(s' | s, a) V_{i-1}^*(s') \right\}. \end{aligned}$$

As well, $\forall 1 \leq i \leq H, \forall s \in \mathcal{S}$,

$$d_{H-i}^*(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \left\{ r_{H-i}(s, a) + \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{H-i}(s' | s, a) V_{i-1}^*(s') \right\}.$$

The proof of this theorem is given in Annex A.3 and uses Property I.1.1.

If the state space \mathcal{S} is finite, Algorithm 1 solves the optimal control problem of a Markov Chain, or briefly the MDP problem: computations are performed recursively, using the Dynamic Programming equations.

I.1.4 Infinite Horizon MDP

The previous section was devoted to present the finite horizon version of the MDP framework. In some situations, designers of the MDP model do not know when the process will end, or they want the agent to manage the system forever: the MDP problem has to be expressed whatever the horizon H , or, more generally, for an infinite horizon. For instance, when the MDP models a robotic mission, it sounds better not to bound the time of the process in order to let the mission be completed, in case of delay.

An infinite horizon MDP is defined by the 6-uplet $\langle \mathcal{S}, \mathcal{A}, T, r, s_0, \gamma \rangle$:

- \mathcal{S} is the countable set of potential states of the system, *i.e.* states of the agent and its environment;

Algorithm 1: Dynamic Programming Algorithm for finite state space MDP

```

1  $V_0^* \leftarrow R;$ 
2 for  $i \in \{1, \dots, H\}$  do
3   for  $s \in \mathcal{S}$  do
4      $V_i^*(s) \leftarrow \max_{a \in \mathcal{A}} \left\{ r_{H-i}(s, a) + \sum_{s' \in \mathcal{S}} \mathbf{p}_{H-i}(s' | s, a) V_{i-1}^*(s') \right\};$ 
5      $d_{H-i}^*(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \left\{ r_{H-i}(s, a) + \sum_{s' \in \mathcal{S}} \mathbf{p}_{H-i}(s' | s, a) V_{i-1}^*(s') \right\};$ 
6 return  $V_H^*, (d^*)_{t=0}^{H-1};$ 

```

- \mathcal{A} is the finite set of actions which can be selected by the agent;
- T is the set of transition probability distributions: $\forall s \in \mathcal{S}, \forall a \in \mathcal{A}$, T contains the probability distribution over the reached state $s' \in \mathcal{S}$ from system state s and selecting action a , *i.e.* the function $s' \mapsto \mathbf{p}(s' | s, a) \in T$ defined on \mathcal{S} . Note that in this formulation, the probability distribution is stationary *i.e.* does not depend on the stage of the process: $\forall t \in \mathbb{N}$, $\mathbf{p}_t(s' | s, a) = \mathbf{p}(s' | s, a)$. As previously, we assume that $\forall s \in \mathcal{S}$, $\forall a \in \mathcal{A}$, the support of the transition probability distributions $\mathbf{p}(\cdot | s, a)$ is finite, *i.e.* $\exists \mathcal{S}_{s,a} \subset \mathcal{S}$, such that $\#(\mathcal{S}_{s,a}) < +\infty$, and $\forall s' \in \mathcal{S} - \mathcal{S}_{s,a}$, $\mathbf{p}(s' | s, a) = 0$;
- $r(s, a)$ the reward obtained when the agent selects action $a \in \mathcal{A}$ and the system is in state $s \in \mathcal{S}$. This function is assumed to be bounded on $\mathcal{S} \times \mathcal{A}$. Note that the reward is stationary too here: $\forall s \in \mathcal{S}, a \in \mathcal{A}, \forall t \in \mathbb{N}$, $r_t(s, a) = r(s, a)$;
- the initial state $s_0 \in \mathcal{S}$, is the state where the process begins;
- the discount factor γ , a real number such that $0 < \gamma < 1$.

Consider a *plan* *i.e.* a sequence of actions indexed by the stage of the process $t \in \mathbb{N}$: $(a_t)_{t \in \mathbb{N}}$. The system is initially in state s_0 . Next, at each step of the process ($t = 0, 1, \dots$), the system is in state $s_t \in \mathcal{S}$, the agent selects action $a_t \in \mathcal{A}$ and the system reaches a state $s_{t+1} \in \mathcal{S}$ according to the transition probability distribution $\mathbf{p}(\cdot | s_t, a_t) = \mathbb{P}(S_{t+1} = s' | S_t = s_t, a_t)$. Finally, the agent gets the reward $r(s_t, a_t) \in \mathbb{R}$ which is aggregated to the previous ones with a sum. Figure I.2 graphically sums up the stationary MDP: the presented figure is an Influence Diagram, *i.e.* it represents the relations between successive variables, just like a Dynamic Bayesian Network, but also including actions and rewards.

As the horizon is infinite, the set of infinite sequences of decision rules, or the set of strategies, is denoted by $\mathcal{D}_\infty = \{(d_t)_{t \in \mathbb{N}} \mid \forall t \in \mathbb{N}, d_t : \mathcal{S} \rightarrow \mathcal{A}\}$. The *discounted reward* at time step t is $\gamma^t \cdot r(s_t, a_t)$, with $0 < \gamma < 1$. The real number γ makes the total sum of discounted rewards converge: $\sum_{t=0}^{+\infty} \gamma^t r(s_t, d_t(s_t))$ whatever the trajectory $(s_t)_{t \in \mathbb{N}} \in \mathcal{S}^\mathbb{N}$ and the strategy $(d_t)_{t \in \mathbb{N}} \in \mathcal{D}_\infty$. Indeed, as the reward function r is bounded, $\exists c > 0$ such that $\forall s \in \mathcal{S}, \forall a \in \mathcal{A}$, $r(s, a) < c$. Thus $\sum_{t=0}^{+\infty} \gamma^t r(s_t, d_t(s_t)) \leq c \sum_{t=0}^{+\infty} \gamma^t \leq \frac{c}{1-\gamma} < +\infty$. The discount factor γ can be defined as the probability that the process goes on, *i.e.* does not terminate, after each stage $t \in \mathbb{N}$: at each time step t , whatever the current state $s \in \mathcal{S}$ and action $a \in \mathcal{A}$, the process has a probability $1 - \gamma$ to reach an artificial absorbing and reward-free state s_{end} modeling the end of the process, *i.e.* $\mathbb{P}(S_{t+1} = s_{\text{end}} \mid S_t = s, a) = 1 - \gamma$, with $\forall a \in \mathcal{A}$, $r(s_{\text{end}}, a) = 0$

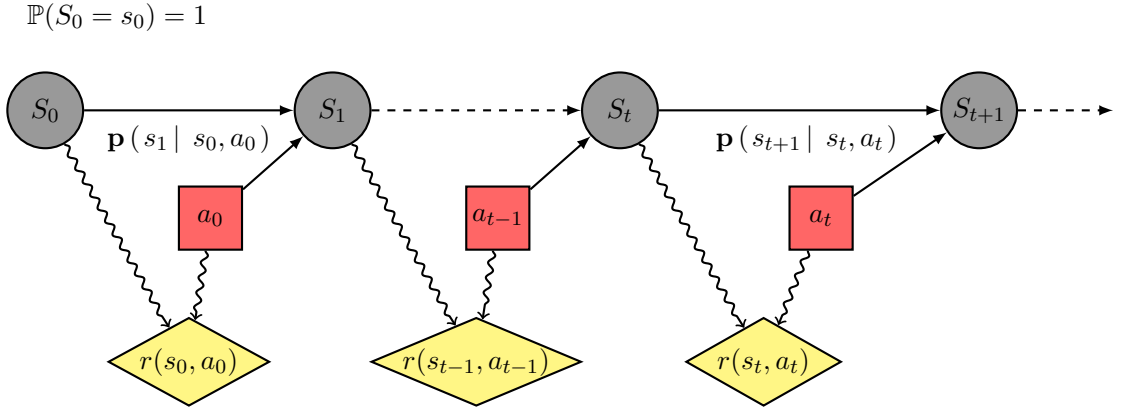


Figure I.2 – Influence Diagram of an MDP: black circles represent successive system states, red squares are selected actions, and yellow diamonds are the rewards. The Bayesian Network resulting from removing rewards and wavy arrows asserts that $\forall t \geq 1$, $S_{t+1} \perp\!\!\!\perp \{S_0, \dots, S_{t-1}\} \mid \{S_t, A_t\}$, where A_t represents the action at time step t seen as a random variable.

and $\mathbb{P}(S_{t+1} = s_{end} \mid S_t = s_{end}, a) = 1$. If such a terminal state s_{end} does not make sense in the particular context, the discount factor γ models only the fact that long term rewards are less important than short term ones.

Thus, once a problem has been defined in terms of an MDP, solving the problem consists in computing a strategy maximizing the expected sum of the discounted rewards. This quantity, called as previously the value function, is denoted by V and depends on the initial state and the strategy $(d) \in \mathcal{D}_\infty$:

$$V(s, (d)) := \mathbb{E} \left[\sum_{t=0}^{+\infty} \gamma^t r(S_t, d_t(S_t)) \mid S_0 = s \right]. \quad (\text{I.4})$$

Note that, for a given strategy, we denote by $V^d : \mathcal{S} \rightarrow \mathbb{R}$ the function such that $V^d(s) = V(s, (d))$.

I.1.5 The Value Iteration algorithm

The Value Iteration algorithm is a solver for Infinite Horizon MDPs: it is based on the Bellman equation, which is derived in Annex A.4. Let $(d)_{t \in \mathbb{N}} \in \mathcal{D}_\infty$:

Definition I.1.2 (Bellman Equation)

$$V^d(s) = r(s, d_0(s)) + \gamma \sum_{s' \in \mathcal{S}_{s, d_0(s)}} \mathbf{p}(s' \mid s, d_0(s)) V^{d^+}(s'). \quad (\text{I.5})$$

where $\forall t \in \mathbb{N}, \forall s \in \mathcal{S}, d_t^+(s) = d_{t+1}(s)$.

The set of bounded functions defined on \mathcal{S} and with real values, is denoted by $\mathcal{F}_B(\mathcal{S}, \mathbb{R})$. The Bellman operator defined as

$$\begin{array}{ccc} \mathcal{B}^d : \mathcal{F}_B(\mathcal{S}, \mathbb{R}) & \longrightarrow & \mathcal{F}_B(\mathcal{S}, \mathbb{R}) \\ V & \longmapsto & s \longmapsto r(s, d_0(s)) + \gamma \sum_{s' \in \mathcal{S}_{s, d_0(s)}} \mathbf{p}(s' \mid s, d_0(s)) V(s'). \end{array}$$

The Bellman equation I.5 can then be written

$$V^d = \mathcal{B}^d V^{d^+}. \quad (\text{I.6})$$

Consider the sup norm on $\mathcal{F}_B(\mathcal{S}, \mathbb{R})$: $\|V\|_\infty = \sup_{s \in \mathcal{S}} |V(s)|$.

Theorem 2

$(\mathcal{F}_B(\mathcal{S}, \mathbb{R}), \|\cdot\|_\infty)$ is a Banach space.

The proof is given in Annex A.5.

Property I.1.2

\mathcal{B}^d is a contracting operator of $(\mathcal{F}_B(\mathcal{S}, \mathbb{R}), \|\cdot\|_\infty)$ whose contraction is γ .

The proof is given in Annex A.6.

If the strategy (d) is stationary, i.e. $\forall t \in \mathbb{N}, \forall s \in \mathcal{S}, d_t(s) = d(s)$, then $V^d = V^{d^+}$, and the Bellman equation I.6 becomes $V^d = \mathcal{B}^d V^d$. The Fixed-Point Theorem assures that the equation $V = \mathcal{B}^d V$ has a unique solution, and it is thus the value function $V^d \in \mathcal{F}_B(\mathcal{S}, \mathbb{R})$. This equation is a characteristic equation of the value function V^d for a given stationary strategy (d) .

In order to simplify the next formulae, the operator \mathcal{B}^a , for $a \in \mathcal{A}$, is introduced:

$$\begin{array}{ccc} \mathcal{B}^a : \mathcal{F}_B(\mathcal{S}, \mathbb{R}) & \longrightarrow & \mathcal{F}_B(\mathcal{S}, \mathbb{R}) \\ V & \longmapsto & s \mapsto r(s, a) + \gamma \sum_{s' \in \mathcal{S}_{s,a}} \mathbf{p}(s' | s, a) V(s'). \end{array}$$

Let us also introduce now the Dynamic Programming operator \mathcal{B}^* : $\forall V \in \mathcal{F}_B(\mathcal{S}, \mathbb{R}), \forall s \in \mathcal{S}$,

$$(\mathcal{B}^* V)(s) = \max_{a \in \mathcal{A}} \mathcal{B}^a V(s)$$

Property I.1.3

The Dynamic programming operator \mathcal{B}^* is also a contracting operator of $\mathcal{F}_B(\mathcal{S}, \mathbb{R})$ whose contraction is γ .

The proof is given in Annex A.7.

The Fixed-Point Theorem assures that the Dynamic Programming equation

$$V = \mathcal{B}^* V \tag{I.7}$$

has a unique solution in $\mathcal{F}_B(\mathcal{S}, \mathbb{R})$, denoted by V^* .

Using this assertion, the following theorem assures that V^* is the optimal value function:

Theorem 3

The solution V^* of the dynamic programming equation (I.7) is equal to the optimal value function

$$V^*(s) = \sup_{d \in \mathcal{D}_\infty} V^d(s) = \sup_{d \in \mathcal{D}_\infty} \mathbb{E} \left[\sum_{t=0}^{+\infty} \gamma^t \cdot r(S_t, d_t(S_t)) \mid S_0 = s \right].$$

Let us define the decision rule $d^* : s \mapsto a^* \in \operatorname{argmax}_{a \in \mathcal{A}} (\mathcal{B}^a V^*)(s)$. The associated stationary strategy, i.e. the strategy $(d_t^*)_{t \in \mathbb{N}} \in \mathcal{D}_\infty$ such that $\forall t \in \mathbb{N}, \forall s \in \mathcal{S}, d_t^*(s) = d^*(s)$, is called greedy strategy: **the greedy strategy is optimal**.

The proof is given in Annex A.8.

If the state space \mathcal{S} is finite, the *Value Iteration Algorithm 2*, directly derived from the Fixed-Point Theorem, leads to the computation of the optimal value function. This theorem assures indeed that $\forall V^0 \in \mathcal{F}_B(\mathcal{S}, \mathbb{R}), (\mathcal{B}^*)^n V^0 \rightarrow V^*$, in the sense of the norm $\|\cdot\|_\infty$, when $n \rightarrow +\infty$.

Algorithm 2: Value Iteration Algorithm for MDP

```

1  $N \leftarrow$  number of iterations;
2  $V \leftarrow V^0$ ;
3  $i \leftarrow 1$ ;
4 while  $i \leq N$  do
5   for  $s \in \mathcal{S}$  do
6      $V'(s) \leftarrow \max_{a \in \mathcal{A}} \left\{ r(s, a) + \gamma \sum_{s' \in \mathcal{S}} \mathbf{p}(s' | s, a) V(s') \right\} \quad (= (\mathcal{B}^* V)(s));$ 
7    $V \leftarrow V'$ ;
8    $i \leftarrow i + 1$ ;
9 for  $s \in \mathcal{S}$  do
10   $d(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \left\{ r(s, a) + \gamma \sum_{s' \in \mathcal{S}} \mathbf{p}(s' | s, a) V(s') \right\};$ 
11 return  $V, d$ 

```

Error analysis

Let us denote by $(V^i)_{i \geq 0}$ the successive functions computed by the algorithm 2. The optimal value function is denoted by V^* . First, the following theorem informs us about the convergence of V^N :

Theorem 4

$$\|V^N - V^*\|_\infty \leq \frac{\gamma^N}{1 - \gamma} \cdot \|V^0 - V^1\|_\infty. \quad (\text{I.8})$$

The proof is given in Annex A.9.

However, we are more interested in an error bound of V^d , where (d) is the strategy returned by the algorithm. The control of the strategy error is given by the next theorem:

Theorem 5

$$\|V^d - V^*\|_\infty \leq \frac{2 \cdot \gamma^N}{1 - \gamma} \|V^1 - V^0\|_\infty.$$

The proof is given in Annex A.10.

The number of iteration N can be set up using this bound: if the desired maximal error is $\varepsilon > 0$, N has to be greater than $\log \left(\frac{\varepsilon(1-\gamma)}{2\|V^1 - V^0\|_\infty} \right) / (\log(\gamma))$.

Another classical way to solve an MDP is the use of the *Policy Iteration algorithm*, also called Howard's algorithm [96]: this algorithm converges in a finite number of iterations but implies to solve at each iteration the linear system (I.6) with $d^+ = d$.

I.1.6 Partially Observable Markov Decision Process

POMDPs generalize MDPs allowing the agent to misperceive, or partially observe, with a given probability, the current state of the system: the system state $s \in \mathcal{S}$ is not given as input to the agent. The latter has to figure it out using the observations $o \in \mathcal{O}$ of the system received at each time step.

Definition

A POMDP, in the infinite horizon settings, is defined by a 8-uple $\langle \mathcal{S}, \mathcal{A}, \mathcal{O}, T, O, r, b_0, \gamma \rangle$:

- \mathcal{S} the **finite** set of system states: the current state s_t is not given as input to the agent;
- \mathcal{A} the finite set of actions which can be selected by the agent;
- \mathcal{O} the finite set of observations that the agent may receive;
- T the set of transition probability distributions of the system state: $\forall s \in \mathcal{S}, a \in \mathcal{A}$, probability distributions $\mathbf{p}(\cdot | s, a) \in T$ are defined on \mathcal{S} ;
- O , the set of observation probability distributions, *i.e.* the set of probability distributions over \mathcal{O} , defining the probability that the agent observes o' , after selecting action a , if the system has reached the state s' : $\forall s' \in \mathcal{S}, a \in \mathcal{A}$, probability distributions $\mathbf{p}(\cdot | s', a) \in O$ are defined over the set of observations \mathcal{O} ;
- $r(s, a)$ the reward given to the agent in the situation where it selects action $a \in \mathcal{A}$ when the system is in state $s \in \mathcal{S}$.
- b_0 , a probability distribution defining the uncertainty on the initial state represented by the random variable S_0 : $S_0 \sim b_0$, *i.e.* $\forall s \in \mathcal{S}, b_0(s) = \mathbb{P}(S_0 = s)$. This probability distribution is called *initial belief* about the system state. It is an *epistemic* probability distribution, because it estimates the actual initial system state, since it is not available for the agent.

Note that the set of system states \mathcal{S} is assumed to be finite here: this is a sufficient assumption for the robotic mission features we would like to model, and a classical way to define POMDPs. Previously, in the Fully Observable case (MDP, see Section I.1.2), the space \mathcal{S} was assumed countable, which is a less restrictive assumption (if \mathcal{S} is countable, it may be infinite): the main results about the POMDP resolution comes from the results of Section I.1.2 about (Fully Observable) MDPs with countable state space \mathcal{S} . The finite set of the observations \mathcal{O} and the set of the observation probability distributions O , are both new in the Partially Observable model in comparison with the Fully Observable MDP model. However, the transition probability distributions T , the finite set of actions \mathcal{A} and the reward function $r : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$, were already introduced in the section about MDPs.

Consider a plan $(a_t)_{t \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$. Initially, the system is in state s_0 with probability b_0 . Next, just like the MDP model, at each time step ($t = 0, 1, \dots$) the system is in state $s_t \in \mathcal{S}$, the agent selects action $a_t \in \mathcal{A}$ and the system state changes, reaching $s_{t+1} \in \mathcal{S}$ according to the probability distribution $\mathbf{p}(\cdot | s_t, a_t)$. The agent gets the reward $r(s_t, a_t)$ and also finally gets the observation o_{t+1} with probability $\mathbf{p}(o_{t+1} | s_{t+1}, a_t) = \mathbb{P}(O_{t+1} = o_{t+1} | S_{t+1} = s_{t+1}, a_t)$: for each time step $t \geq 1$, the observation random variable O_t is such that $O_t \sim \mathbf{p}(\cdot | s_t, a_{t-1})$, where $\mathbf{p}(\cdot | s_t, a_{t-1}) \in O$ is the probability distribution over the observations. The bottom of Figure I.3 fully sums up the definition of this process: the *belief updating process* in green at the top of this figure is defined in the next section.

I.1.7 The belief updating process

The independence assumptions highlighted by the Bayesian Network (black straight-lined arrows and associated nodes) of Figure I.3, assert that the probability distributions defining the POMDP fully describe the uncertainty related to the system: at time step $t \geq 1$, the probability that the t first observations are (o_1, \dots, o_t) and the $t + 1$ first system states are (s_0, \dots, s_t)

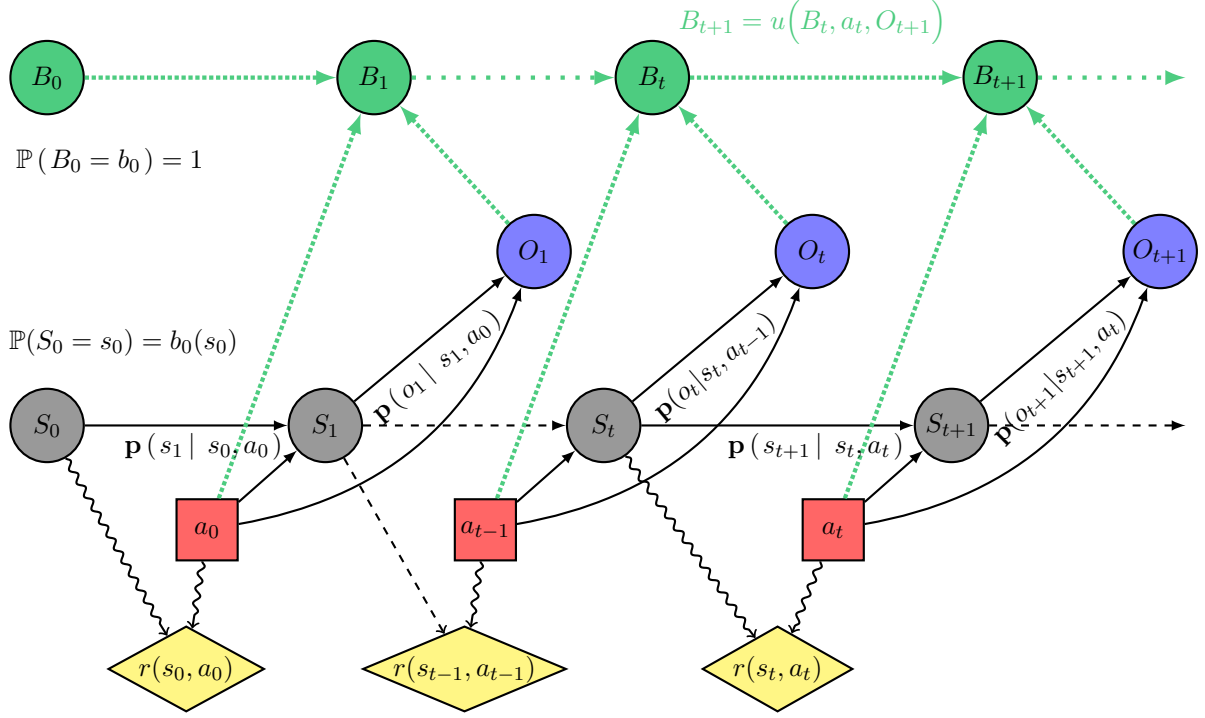


Figure I.3 – *Influence Diagram of a POMDP and its belief updating process*: black circles represent successive system states S_t , blue ones represent successive observations O_t , red squares are selected actions a_t , and yellow diamonds are the associated rewards. Green circles at the top of the figure are the successive belief states B_t constituting the belief updating process, computed using the update $B_{t+1} = u(B_t, a_t, O_{t+1})$: thus, just like the wavy lines leading to rewards, the green dotted lines represent a deterministic influence. The Bayesian Network resulting from removing belief states and rewards, asserts that $\forall t \geq 1$, $S_{t+1} \perp\!\!\!\perp \{S_0, S_1, O_1, \dots, S_{t-1}, O_{t-1}\} \mid \{S_t, A_t\}$, where A_t represents the action at time step t seen as a random variable. As well, $\forall t \geq 1$, O_t is independent from all other variables conditional on $\{S_t, A_t\}$.

conditional on the t first selected actions (a_0, \dots, a_{t-1}) is

$$\begin{aligned} & \mathbb{P}(S_0 = s_0, \dots, S_t = s_t, O_1 = o_1, \dots, O_t = o_t \mid a_0, \dots, a_{t-1}) \\ &= b_0(s_0) \cdot \prod_{i=1}^{t-1} \mathbf{p}(s_{i+1} \mid s_i, a_i) \cdot \prod_{i=1}^{t-1} \mathbf{p}(o_{i+1} \mid s_{i+1}, a_i). \end{aligned}$$

Let us define the belief updating process which is classically a basis for the strategy computation: at time step $t \in \mathbb{N}$, if the observation sequence from the beginning is $(o_1, \dots, o_t) \in \mathcal{O}^t$ given as input to the agent, and if the successive selected actions are $(a_0, \dots, a_{t-1}) \in \mathcal{A}^t$, the *belief state* at time step t is the probability distribution of the system state conditioned on the observation and action sequences.

Definition I.1.3 (Belief State and Information)

The **belief state** at time step t is the function $b_t : \mathcal{S} \rightarrow \mathbb{R}$ defined as

$$b_t(s) = \mathbb{P}(S_t = s \mid O_1 = o_1, \dots, O_t = o_t, a_0, \dots, a_{t-1}) = \mathbb{P}(S_t = s \mid I_t = i_t), \quad (\text{I.9})$$

where $i_t = \{o_1, \dots, o_t, a_0, \dots, a_{t-1}\}$, is the **information** gathered by the agent at time step t . The random variable version of i_t is denoted by $I_t = \{O_1, \dots, O_t, a_0, \dots, a_{t-1}\}$.

The belief state at time step t is thus the *a posteriori* probability distribution of the sys-

tem state, given the initial probability distribution b_0 and the probability distributions O and T defining the POMDP, and conditional on the information i_t . The belief process, which is the sequence of belief states, can be computed recursively using Bayes rule.

Theorem 6

If the belief state at time step t is b_t , the selected action is $a_t \in \mathcal{A}$, and the next observation is o_{t+1} , the next belief state b_{t+1} is computed as follows:

$$b_{t+1}(s') = \frac{\sum_{s \in \mathcal{S}} \mathbf{p}(o_{t+1} | s', a_t) \cdot \mathbf{p}(s' | s, a_t) \cdot b_t(s)}{\sum_{\tilde{s} \in \mathcal{S}} \sum_{\tilde{s}' \in \mathcal{S}} \mathbf{p}(o_{t+1} | \tilde{s}', a_t) \cdot \mathbf{p}(\tilde{s}' | \tilde{s}, a_t) \cdot b_t(\tilde{s})}. \quad (\text{I.10})$$

*This formula is called the **belief update**, and since the belief state b_{t+1} is shown to be a function of b_t , a_t and o_{t+1} , we denote it by*

$$b_{t+1} = u(b_t, a_t, o_{t+1}).$$

The proof is given in Annex A.11.

The belief update formula (I.10) is more simply written

$$b_{t+1}(s') \propto \sum_{s \in \mathcal{S}} \mathbf{p}(o_{t+1} | s', a_t) \cdot \mathbf{p}(s' | s, a_t) \cdot b_t(s)$$

as $\sum_{\tilde{s} \in \mathcal{S}} \sum_{\tilde{s}' \in \mathcal{S}} \mathbf{p}(o_{t+1} | \tilde{s}', a_t) \cdot \mathbf{p}(\tilde{s}' | \tilde{s}, a_t) \cdot b_t(\tilde{s})$ is nothing more than a normalization constant.

Note that b_0 may be seen as the actual value of a random variable B_0 , like the value s_0 of the random variable S_0 in the MDP section I.1.4: $\mathbb{P}(B_0 = b_0) = 1$. Now, if B_t is a random variable representing the belief state at time t , $B_{t+1} = u(B_t, a_t, O_t)$ is a random variable as a function of random variables. In the top of Figure I.3 the belief process is represented by the belief state variables $(B_t)_{t \geq 0}$: this figure highlights the links between the belief process and the POMDP process (green dotted arrows). More formally the belief state variable B_t is $B_t(s) = \mathbb{P}(S_t = s | I_t) = \mathbb{E}[\mathbb{1}_{\{S_t = s\}} | I_t]$, $\forall s \in \mathcal{S}$, i.e. the conditional expectation² of the characteristic function of the set $\{S_t = s\} \subseteq \Omega$.

I.1.8 A belief dependent value function

As the agent has only access to the information $i_t = \{o_1, \dots, o_t, a_0, \dots, a_{t-1}\}$ at time step $t \geq 1$, the sequence of decision rules $(d_t)_{t \in \mathbb{N}}$ is such that $\forall t \in \mathbb{N}$, $d_t : i_t \mapsto a \in \mathcal{A}$. Let us present the criterion, or value function, which has to be maximized by choosing the right strategy: for an initial belief state b_0 , which defines the probability distribution of S_0 , the value function for an infinite horizon and a strategy based on the current information $(d_t)_{t \in \mathbb{N}}$, is the expected discounted total reward:

$$V^d(b_0) = \mathbb{E} \left[\sum_{t \geq 0} \gamma^t \cdot r(S_t, d_t(I_t)) \right]. \quad (\text{I.11})$$

The following work leads to a formulation of the value function where the belief update process $(B_t)_{t \in \mathbb{N}}$ appears. It considers as well the action sequence as a sequence of random variables: $(A_t)_{t \in \mathbb{N}}$. It covers then the case $A_t = d_t(I_t)$ proposed in the value function definition

²The general definition of the probabilistic conditional expectation (as a random variable) is given in Annex, Definition A.1.

(I.11). The information random variable I_t becomes then $\{O_1, \dots, O_t, A_1, \dots, A_t\}$. Thanks to Fubini theorem and the linearity of the probabilistic expectation,

$$\begin{aligned}
 V^d(b_0) &= \sum_{t \geq 0} \gamma^t \cdot \mathbb{E}[r(S_t, A_t)] \\
 &= \sum_{t \geq 0} \gamma^t \cdot \mathbb{E}\left[\mathbb{E}[r(S_t, A_t) \mid I_t]\right] \text{ as a consequence of Definition A.1} \\
 &= \sum_{t \geq 0} \gamma^t \cdot \mathbb{E}\left[\mathbb{E}\left[\sum_{s \in \mathcal{S}} r(s, A_t) \cdot \mathbb{1}_{\{S_t=s\}} \mid I_t\right]\right] \\
 &= \sum_{t \geq 0} \gamma^t \cdot \mathbb{E}\left[\sum_{s \in \mathcal{S}} r(s, A_t) \cdot \mathbb{E}\left[\mathbb{1}_{\{S_t=s\}} \mid I_t\right]\right] \tag{I.12}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{t \geq 0} \gamma^t \cdot \mathbb{E}\left[\sum_{s \in \mathcal{S}} r(s, A_t) \cdot B_t(s)\right] \text{ by defition of } B_t \\
 &= \sum_{t \geq 0} \gamma^t \cdot \mathbb{E}[r(B_t, A_t)] \tag{I.13} \\
 &= \mathbb{E}\left[\sum_{t \geq 0} \gamma^t \cdot r(B_t, A_t)\right]
 \end{aligned}$$

where the belief reward function $r : (b, a) \mapsto r(b, a) := \sum_s r(s, a) \cdot b(s)$ is introduced at line (I.13). Line I.12 uses the linearity of the conditional expectation and Property A.1 (as a function of I_t , $A_t = d_t(I_t)$ is $\sigma(I_t)$ -measurable, see Property A.3).

Another way to see that the expectation of the discounted reward is equal to the expected discounted belief reward is to compute it conditional on the action sequence:

Theorem 7

$$\mathbb{E}[r(S_t, A_t) \mid \hat{A}_t = \hat{a}_t] = \mathbb{E}[r(B_t, A_t) \mid \hat{A}_t = \hat{a}_t]. \tag{I.14}$$

using the notations $\hat{A}_t = \{A_0, \dots, A_t\}$ and $\hat{a}_t = \{a_0, \dots, a_t\}$.

The proof is given in Annex A.12.

I.1.9 A POMDP as a belief-MDP

In this section a POMDP is expressed as a MDP, whose states are the belief states: the resulting MDP is denoted by $\langle \tilde{\mathcal{S}}, \mathcal{A}, \tilde{T}, \tilde{r}, \tilde{s}_0, \gamma \rangle$. The state space $\tilde{\mathcal{S}}$ is the set of all reachable belief states from b_0 , denoted by $\mathbb{P}_{b_0}^{\mathcal{S}}$. This set is countable: indeed, as \mathcal{A} and \mathcal{O} are finite, each reachable belief state has a finite number of possible successors. As there is only one initial belief state, each set of belief states generated for each time step is finite. Numbering them is thus as easy as numbering each belief states of each successive finite set, following the time index t . The set of all probability distributions is denoted by $\mathbb{P}^{\mathcal{S}}$: thus, $\mathbb{P}_{b_0}^{\mathcal{S}} \subset \mathbb{P}^{\mathcal{S}}$.

Let b_t a given belief state, *i.e.* a probability distribution in $\mathbb{P}_{b_0}^{\mathcal{S}}$. The sequence $(B_t)_{t \in \mathbb{N}}$ is a sequence of random variables: as highlighted by the belief update (I.10), if $B_t = b_t$, and the selected action is a_t , the value of the next variable B_{t+1} is a deterministic function of the observation O_{t+1} .

Before defining the belief-MDP, the belief process is shown to be a Markov process:

Theorem 8

$$\forall a \in \mathcal{A}, \forall b' \in \mathbb{P}_{b_0}^{\mathcal{S}},$$

$$\mathbb{P}(B_{t+1} = b' \mid I_t = i_t, a) = \mathbb{P}(B_{t+1} = b' \mid B_t = b_{b_0}^{i_t}, a), \quad (\text{I.15})$$

where $b_{b_0}^{i_t}$ is the current belief state if the initial belief is b_0 and the information gathered is i_t .

The proof is given in Annex A.13.

As highlighted by the equation (21) in the proof, if $B_t = b_t$, the probability that the next belief B_{t+1} is b_{t+1} , is the sum of all probabilities of the observations o' such that $u(b_t, a_t, o') = b_{t+1}$, *i.e.* of the observations leading to the belief state b_{t+1} : it defines the transition probability distributions of the belief process, *i.e.* elements of \tilde{T} , as follows: $\forall t \geq 0$,

$$\begin{aligned} \mathbf{p}(b_{t+1} \mid b_t, a_t) &= \mathbb{P}(B_{t+1} = b_{t+1} \mid B_t = b_t, a_t) \\ &= \sum_{\substack{o' \in \mathcal{O} \text{ s.t.} \\ u(b_t, a_t, o') = b_{t+1}}} \mathbb{P}(O_{t+1} = o' \mid B_t = b_t, a_t) \\ &= \sum_{\substack{o' \in \mathcal{O} \text{ s.t.} \\ u(b_t, a_t, o') = b_{t+1}}} \sum_{(s, s') \in \mathcal{S}^2} \mathbf{p}(o' \mid s', a_t) \cdot \mathbf{p}(s' \mid s, a_t) \cdot b_t(s) \\ &= \sum_{\substack{o' \in \mathcal{O} \text{ s.t.} \\ u(b_t, a_t, o') = b_{t+1}}} \mathbf{p}(o' \mid b_t, a_t). \end{aligned} \quad (\text{I.16})$$

where $\sum_{(s, s') \in \mathcal{S}^2} \mathbf{p}(o' \mid s', a_t) \cdot \mathbf{p}(s' \mid s, a_t) \cdot b_t(s)$ is denoted by $\mathbf{p}(o' \mid b_t, a_t)$.

Finally, the reward associated with the belief b_t is defined as previously

$$\begin{aligned} \tilde{r} : \mathbb{P}_{b_0}^{\mathcal{S}} \times \mathcal{A} &\rightarrow \mathbb{R} \\ (b, a) &\mapsto \sum_{s \in \mathcal{S}} r(s) \cdot b(s), \end{aligned} \quad (\text{I.17})$$

and the initial state denoted by \tilde{s}_0 is the belief state b_0 .

As $B_0 = b_0$, we can write that $\mathbb{P}(B_0 = b_0) = 1$, and then previous section demonstrates that

$$\mathbb{E} \left[\sum_{t \geq 0} \gamma^t \cdot r(S_t, d_t(I_t)) \right] = \mathbb{E} \left[\sum_{t \geq 0} \gamma^t \cdot r(B_t, d_t(I_t)) \mid B_0 = b_0 \right].$$

The right part of the equation is the value function of the MDP $\langle \tilde{\mathcal{S}}, \mathcal{A}, \tilde{T}, \tilde{r}, \tilde{s}_0, \gamma \rangle$. As highlighted by Dynamic Programming exposed in the sections I.1.3 and I.1.4, the criterion may varies for two belief states, but not if the information varies but leads to the same belief state: action sequence has to be chosen as a sequence of functions of the current belief, *i.e.* a maximizing action sequence $(A_t)_{t \geq 0}$ is given by a strategy $(d_t)_{t \geq 0}$: $A_t = d_t(B_t)$. As this criterion can be computed directly with the belief-MDP model $\langle \tilde{\mathcal{S}}, \mathcal{A}, \tilde{T}, \tilde{r}, \tilde{s}_0, \gamma \rangle$, no information is lost in focusing our efforts in solving the belief-MDP instead of the initial POMDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{O}, T, O, r, b_0, \gamma \rangle$ whose criterion is the left part of the equation.

The translation of a POMDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{O}, T, O, b_0, \gamma \rangle$ into the so-called belief-MDP $\langle \tilde{\mathcal{S}}, \mathcal{A}, \tilde{T}, \tilde{r}, \tilde{s}_0, \gamma \rangle$ is summed up here:

- $\tilde{\mathcal{S}} = \mathbb{P}_{b_0}^{\mathcal{S}}$, the set of all reachable belief states from the initial one b_0 ;

- \tilde{T} contains all transition probability distributions of the beliefs: $\forall a \in \mathcal{A}, \forall b \in \mathbb{P}_{b_0}^{\mathcal{S}}$, the belief transition probability distribution defined by the equation (I.16), $\mathbf{p}(\cdot | b, a)$ is in \tilde{T} ;
- the reward function \tilde{r} is defined by the equation (I.13),
- the initial state $\tilde{s}_0 = b_0$.

Note that the action set \mathcal{A} , as well as the discount factor γ remain the same. Note also that this belief-MDP fulfills the conditions defined in Section I.1.4. First, $\tilde{\mathcal{S}} = \mathbb{P}_{b_0}^{\mathcal{S}}$ is countable. Second, the successors of b_t for the action $a \in \mathcal{A}$ form the set $\left\{ u(b_t, a, o') \in \mathbb{P}_{b_0}^{\mathcal{S}} \mid o' \in \mathcal{O} \right\}$, which is finite as \mathcal{O} is finite. For each $b \in \tilde{\mathcal{S}}$ and $a \in \mathcal{A}$, a finite number of beliefs $b' \in \tilde{\mathcal{S}}$ are such that $\mathbf{p}(b' | b, a) > 0$, *i.e.* $\forall b \in \tilde{\mathcal{S}}, \forall a \in \mathcal{A}$, the support of the transition probability distribution $\mathbf{p}(\cdot | b, a)$ is finite: $\exists \tilde{\mathcal{S}}_{b,a} \subset \tilde{\mathcal{S}}$, such that $\#(\tilde{\mathcal{S}}_{b,a}) < +\infty$, and $\forall b' \in \tilde{\mathcal{S}} - \tilde{\mathcal{S}}_{b,a}, \mathbf{p}(b' | b, a) = 0$, making the transition function \tilde{T} of the MDP $\langle \tilde{\mathcal{S}}, \mathcal{A}, \tilde{T}, \tilde{r}, \tilde{s}_0, \gamma \rangle$ satisfying the condition stated in Section I.1.4.

Bellman Equation I.5 stated in page 26 can be rewritten in the context of POMDPs: given a strategy $(d_t)_{t \geq 0}, \forall b \in \tilde{\mathcal{S}}$,

$$\begin{aligned} V^d(b) &= \left(\mathcal{B}^d V^{d^+} \right)(b) \\ &= \tilde{r}(b, d_0(b)) + \gamma \cdot \sum_{b' \in \tilde{\mathcal{S}}_{b, d_0(b)}} \mathbf{p}(b' | b, d_0(b)) \cdot V^{d^+}(b') \\ &= \sum_{s \in \mathcal{S}} \tilde{r}(s, d_0(b)) \cdot b(s) + \gamma \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' | b, d_0(b)) \cdot V^{d^+}(u(b, d_0(b), o')) \end{aligned}$$

using transition formula (I.16), and where u is the update function defined in Theorem 6.

The Dynamic Programming operator is obtained adding $\max_{a \in \mathcal{A}}$ at the beginning, and replacing $d_0(b)$ by $a \in \mathcal{A}$ (see the Dynamic Programming equation I.7):

$$\mathcal{B}^* V(b) = \max_{a \in \mathcal{A}} \left\{ \sum_{s \in \mathcal{S}} r(s, a) \cdot b(s) + \gamma \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' | b, a) \cdot V(u(b, a, o')) \right\}. \quad (\text{I.18})$$

The Dynamic Programming Equation $V^* = \mathcal{B}^* V^*$ characterizes

$$\sup_{d \in \mathcal{D}_\infty} V^d(b) = \sup_{d \in \mathcal{D}_\infty} \mathbb{E} \left[\sum_{t=0}^{+\infty} \gamma^t \cdot \tilde{r}(B_t, d_t(B_t)) \mid B_0 = b \right].$$

I.1.10 Solving a POMDP

Given an action $a \in \mathcal{A}$, the reward \tilde{r} of a belief $b \in \mathbb{P}^{\mathcal{S}}$ is a linear function of the belief b : $\tilde{r}(b, a) = \sum_{s \in \mathcal{S}} r(s, a) \cdot b(s) = \langle \tilde{r}_a, b \rangle_{\mathbb{R}^{\mathcal{S}}}$, where \tilde{r}_a is the vector from $\mathbb{R}^{\mathcal{S}}$ such that for each index $s \in \mathcal{S}$, the value is $r(a, s) \in \mathbb{R}$. The function $(x, y) \mapsto \langle x, y \rangle_{\mathbb{R}^{\mathcal{S}}}$ is the classical scalar product of x and y in the vector space $\mathbb{R}^{\mathcal{S}}$.

For each belief $b \in \mathbb{P}^{\mathcal{S}}$, the value $V^0(b) = \max_{a \in \mathcal{A}} \tilde{r}(b, a)$, is the optimal expected reward for an horizon 0 (only one decision step), starting with the belief b . This function is PieceWise Linear and Convex (PWLC), *i.e.* there exists a finite set of vectors $\Gamma \subset \mathbb{R}^{\mathcal{S}}$ such that $V^0(b) = \max_{\alpha \in \Gamma} \langle \alpha, b \rangle_{\mathbb{R}^{\mathcal{S}}}$. As shown by R D. Smallwood and E J. Sondik [113] successive

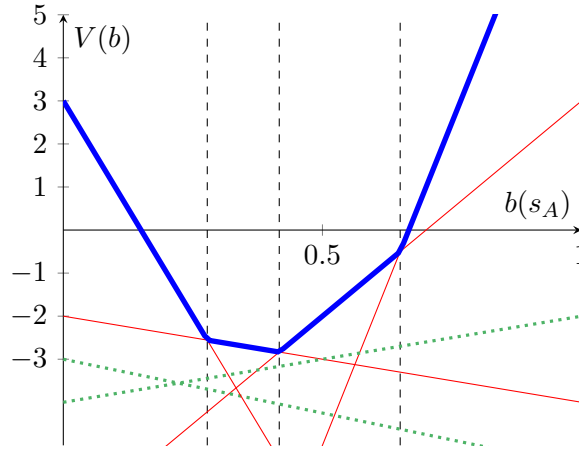


Figure I.4 – Example of useful alpha vectors (red), value function PWLC (thick blue), and useless (bad) alpha vectors (dotted green) of a POMDP at a given iteration: the state space is $\mathcal{S} = \{s_A, s_B\}$: x -axis represents $b(s_A)$ and y -axis represents $V(b)$. As $\#\mathcal{S} = 2$, $b(s_B) = 1 - b(s_A)$.

$V^i = \mathcal{B}^* V^{i-1}$ are PWLC, i.e. $V^i(b) = \max_{\alpha \in \Gamma} \langle \alpha, b \rangle_{\mathbb{R}^{\mathcal{S}}}$, where $\alpha \in \mathbb{R}^{\mathcal{S}}$ is called “ α -vector”:

Theorem 9

PWLC functions becomes PWLC functions after the application of the operator \mathcal{B}^* . More specifically, if a function $V : \mathbb{P}^{\mathcal{S}} \rightarrow \mathbb{R}$ is PWLC, i.e. such that

$$V(b) = \max_{\alpha \in \Gamma} \langle \alpha, b \rangle_{\mathbb{R}^{\mathcal{S}}} = \max_{\alpha \in \Gamma} \sum_{s \in \mathcal{S}} \alpha(s) \cdot b(s) \text{ with } \Gamma \subset \mathbb{R}^{\mathcal{S}}, \#\Gamma < +\infty,$$

then, $\forall b \in \mathbb{P}^{\mathcal{S}}$,

$$(B^*V)(b) = \max_{\alpha' \in \Gamma'} \langle \alpha', b \rangle_{\mathbb{R}^{\mathcal{S}}} = \max_{a \in \mathcal{A}} \max_{(\alpha_o) \in \Gamma^{\mathcal{O}}} \langle \alpha_{a,(\alpha_o)}, b \rangle_{\mathbb{R}^{\mathcal{S}}}, \quad (\text{I.19})$$

with the new α -vectors $\alpha_{a,(\alpha_o)} \in \Gamma'$ defined as

$$\alpha_{a,(\alpha_o)}(s) = r(s, a) + \gamma \cdot \sum_{o' \in \mathcal{O}, s' \in \mathcal{S}} \mathbf{p}(o' \mid s', a) \cdot \mathbf{p}(s' \mid s, a) \cdot \alpha_{o'}(s), \quad (\text{I.20})$$

where $(\alpha_o) \in \Gamma^{\mathcal{O}}$ is the notation for a vector such that the coordinate of index $o \in \mathcal{O}$ is an α -vector $\alpha_o \in \Gamma$.

The proof is given in Annex A.14.

This result has inspired many POMDP solvers. Indeed, it makes possible to compute the optimal value function and the associated strategy in finite horizon settings, and to approach them for an infinite horizon POMDP, through a few modifications of the Value Iteration algorithm, Algorithm 2: while the state space is infinite, the value function is summed up in a set of α -vectors. Let us start with a PWLC function $V^0(b) = \max_{\alpha \in \Gamma^0} \sum_{s \in \mathcal{S}} \alpha(s) \cdot b(s)$, where Γ^0 is the initial set of α -vectors. In order to remain close to the finite horizon case, it is possible to start with the *reward vectors* as initial α -vectors: $\Gamma^0 = \{\alpha \in \mathbb{R}^{\mathcal{S}} \mid \forall s \in \mathcal{S}, \alpha(s) = r(s, a) \text{ with } a \in \mathcal{A}\}$. The function encoded by Γ^0 is in this case the optimal initial reward for an agent whose belief is b : $V^0(b) = \max_{\alpha \in \Gamma^0} \sum_{s \in \mathcal{S}} \alpha(s) \cdot b(s) = \max_{a \in \mathcal{A}} \sum_{s \in \mathcal{S}} r(s, a) \cdot b(s) = \max_{a \in \mathcal{A}} \mathbb{E}_{S \sim b} [r(S, a)]$. Iterations of the operator \mathcal{B}^* starting from V^0 create a sequence of functions $(V^i)_{i \in \mathbb{N}}$ whose theoretical limit is $\sup_{(d) \in \mathcal{D}_{\infty}} V^d$ as shown in Section I.1.4: here, each function of the sequence can be summed up with a finite set of alpha vectors, which makes computations possible in practice.

Given the number of iterations N for a specified error bound $\varepsilon > 0$ (see the error analysis

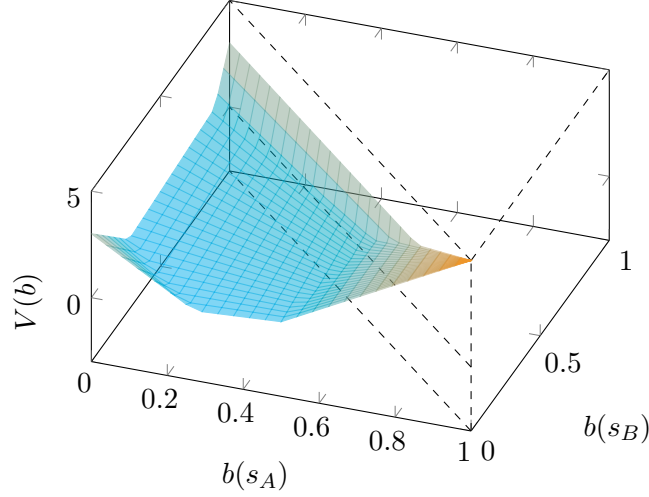


Figure I.5 – Value function PWLC at a given iteration when the state space is $\mathcal{S} = \{s_A, s_B, s_C\}$: x -axis represents $b(s_A)$, y -axis represents $b(s_B)$ and z -axis represents $V(b)$. As $\#\mathcal{S} = 3$, $b(s_C) = 1 - b(s_A) - b(s_B)$. $V(b)$ is the maximum of a set of hyperplans.

of Section I.1.5), Algorithm 3 leads to a value function in the form of a set of α -vectors Γ_ε : $V^\varepsilon(b) = \max_{\alpha \in \Gamma_\varepsilon} \sum_{s \in \mathcal{S}} \alpha(s) \cdot b(s)$. The associated strategy, approximately optimal, is

$$\begin{aligned} d^\varepsilon(b) &\in \operatorname{argmax}_{a \in \mathcal{A}} \max_{(\alpha_o) \in (\Gamma_\varepsilon)^\mathcal{O}} \langle \alpha_{a,(\alpha_o)}, b \rangle_{\mathbb{R}^S} \\ &\in \operatorname{argmax}_{a \in \mathcal{A}} \max_{(\alpha_o) \in \Gamma_\varepsilon^\mathcal{O}} \sum_{s \in \mathcal{S}} r(s, a) + \gamma \cdot \sum_{o' \in \mathcal{O}} \sum_{s' \in \mathcal{S}} \mathbf{p}(o' | s', a) \cdot \mathbf{p}(s' | s, a) \cdot \alpha_{o'}(s'). \end{aligned}$$

At execution, if the agent has a belief b , and the α -vector $\alpha_{a,(\alpha_o)}$ is such that $\forall \alpha \in \Gamma_\varepsilon$, $\langle \alpha_{a,(\alpha_o)}, b \rangle_{\mathbb{R}^S} \leq \langle \alpha, b \rangle_{\mathbb{R}^S}$, then action a is approximately optimal (with error ε).

Algorithm 3: Value Iteration Algorithm for POMDP

```

1  $\Gamma \leftarrow \Gamma^0$ ;
2  $i \leftarrow 1$ ;
3 while  $i \leq N$  do
4   for  $a \in \mathcal{A}$ ,  $(\alpha_{o'}) \in \Gamma^\mathcal{O}$  do
5     for  $s \in \mathcal{S}$  do
6        $\alpha(s) \leftarrow r(s, a) + \gamma \cdot \sum_{o' \in \mathcal{O}} \sum_{s' \in \mathcal{S}} \mathbf{p}(o' | s', a) \cdot \mathbf{p}(s' | s, a) \cdot \alpha_{o'}(s')$ ;
7      $\Gamma' \leftarrow \{\Gamma', \alpha\}$ ;
8    $\Gamma \leftarrow \Gamma'$ ;
9    $i++$ ;
10 return  $\Gamma$ ;

```

This algorithm is really naive since the number of α -vectors increases as a double exponential with iterations: if $\forall n \in \mathbb{N}$, Γ^n is the set of α -vectors at the end of iteration n , and $g_n = \#\Gamma^n$, then $g_{n+1} = \#\mathcal{A} \cdot g_n^{\#\mathcal{O}}$. Thus, as $\#\mathcal{A}$ is the initial number of α -vectors, $g_n = (\#\mathcal{A}) \sum_{i=0}^{n-1} (\#\mathcal{O})^i \cdot (\#\mathcal{A}) (\#\mathcal{O})^n$.

A first improvement consists in removing, at each iteration $i = 1, \dots, N$, dominated α -vectors $\alpha_{bad} \in \Gamma^i$, *i.e.* α -vectors such that $\max_{\alpha} \sum_{s \in \mathcal{S}} \alpha(s) \cdot b(s) > \sum_{s \in \mathcal{S}} \alpha_{bad}(s) \cdot b(s)$, $\forall b \in \mathbb{P}^{\mathcal{S}}$: Cassandra's algorithms use linear programs to prune these dominated α -vectors [22, 21].

Whereas the resolution of finite state MDPs (MDPs with $\#\mathcal{S} < \infty$) is a P-complete problem [86], solving a finite horizon POMDP is PSPACE-hard [86], and solving an infinite horizon POMDP is undecidable [76]. These theoretical complexities are a faithful representation of the difficulty of solving POMDPs in practice. Algorithm 3 or Cassandra's improvements [22, 21] solve only really small POMDPs, *i.e.* POMDPs with a few system states and observations. For instance in the case of robotic mission problems, the number of system states may be quite big, as well as the number of observations, and classical computations are intractable: other computation methods are necessary to compute efficiently a satisfactory strategy. The next section is devoted to the presentation of the most notorious algorithms producing approximate strategies within reasonable time.

I.1.11 Computation of Strategies in Practice

This section is meant to sum up the main ways to approximate POMDP optimal strategies, as strategy computation is a difficult task in practice and theoretically intractable [86, 76].

First consider \hat{V} and \tilde{V} two functions mapping the set of all probability distributions $\mathbb{P}^{\mathcal{S}}$ to \mathbb{R} . Suppose that $\forall b \in \mathbb{P}^{\mathcal{S}}, \hat{V}(b) \leq \tilde{V}(b)$. Then, $\forall b \in \mathbb{P}^{\mathcal{S}}$,

$$(\mathcal{B}^* \hat{V})(b) = \max_{a \in \mathcal{A}} \left\{ \sum_{s \in \mathcal{S}} r(s, a) \cdot b(s) + \gamma \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' | b, a) \cdot \hat{V}(u(b, a, o')) \right\} \leq (\mathcal{B}^* \tilde{V})(b).$$

Thus, if $\forall b \in \mathbb{P}^{\mathcal{S}}, V(b) \leq V^*(b)$, *i.e.* if V is a lower bound of the optimal value function V^* , $\forall b \in \mathbb{P}^{\mathcal{S}}, (\mathcal{B}^* V)(b) \leq (\mathcal{B}^* V^*)(b) = V^*(b)$. As well, if V is an upper bound of the optimal value function, $\forall b \in \mathbb{P}^{\mathcal{S}}, (\mathcal{B}^* V)(b) \geq (\mathcal{B}^* V^*)(b) = V^*(b)$. This means that the iterations of the Dynamic Programming operator \mathcal{B}^* on a lower (resp. upper) bound of V^* return a lower (resp. upper) bound of V^* .

Defining $r_{\min} = \min_{s \in \mathcal{S}, a \in \mathcal{A}} r(s, a)$, the constant function $\underline{V}_0(b) = \sum_{t \geq 0} \gamma^t \cdot r_{\min} = \frac{r_{\min}}{1-\gamma}$, $\forall b \in \mathbb{P}^{\mathcal{S}}$, is an example of initial lower bound of the optimal value function: the worst reward is gathered at each time step. The only α -vector representing this function is $\alpha_0(s) = \frac{r_{\min}}{1-\gamma}$, $\forall s \in \mathcal{S}$.

Let us start from an initial set of α -vectors denoted by $\underline{\Gamma}_0$, defining a lower bound of the optimal value function, *i.e.* such that $\underline{V}_0(b) = \max_{\alpha \in \underline{\Gamma}_0} \sum_{s \in \mathcal{S}} \alpha(s) \cdot b(s) \leq V^*(b)$: for instance $\underline{\Gamma}_0 = \{\alpha_0\}$. The α -vectors $\alpha \in \underline{\Gamma}_1$ computed using the current α -vector set $\underline{\Gamma}_0$ and the equation (I.20) of Theorem 9, take part in the definition of $\underline{V}_1(b) = (\mathcal{B}^* \underline{V}_0)(b)$. As noted above, \underline{V}_1 is also a lower bound: $\underline{V}_1(b) = \max_{\alpha \in \underline{\Gamma}_1} \sum_{s \in \mathcal{S}} \alpha(s) \cdot b(s) = (\mathcal{B}^* \underline{V}_0)(b) \leq V^*(b)$, $\forall b \in \mathbb{P}^{\mathcal{S}}$. Thus, as \underline{V}_0 and \underline{V}_1 are lower bounds of V^* , $\max\{\underline{V}_0, \underline{V}_1\}$ too, *i.e.* $\max_{\alpha \in \underline{\Gamma}_0 \cup \underline{\Gamma}_1} \sum_{s \in \mathcal{S}} \alpha(s) \cdot b(s)$ is a lower bound of $V^*(b)$, and the best available at the moment. Hence, it is sufficient to maintain a set of α -vectors $\underline{\Gamma}$: the α -vectors computed from $\underline{\Gamma}$ may be added into $\underline{\Gamma}$, and the dominated ones may be removed. Thanks to the convergence of $\left((\mathcal{B}^*)^n \underline{V}_0\right)_{n \in \mathbb{N}}$ towards V^* , the computation of new α -vectors tends to improve the lower bound.

The presented mechanism of incremental improvement of the lower bound of V^* using new α -vectors does not allow to compute an upper bound: starting from an upper bound \overline{V}_0 , if the computed α -vectors represent a function \overline{V}_1 (also an upper bound of V^*) which tends to be closer to V^* than \overline{V}_0 (thanks to the convergence), then $\max\{\overline{V}_0, \overline{V}_1\}$ is not a better upper bound: it is actually the worst.

Another method can be used to maintain and improve an upper bound of the value function: an upper bound \overline{V}_0 of the optimal value function is only computed over a set of $n > 0$ beliefs, leading to the belief-value mappings $\{b_i, \bar{v}_i\}_{i=1}^n$, where $\overline{V}_0(b_i) = \bar{v}_i$ and $(b_i, v_i) \in \mathbb{P}^{\mathcal{S}} \times \mathbb{R}$. As

the limit of a sequence of convex functions is convex, the optimal value function V^* is known to be convex. As the mappings are such that $v_i \geq V^*(b_i)$, and as V^* is convex, any convex combination of the beliefs $(b_i)_{i=1}^n$ has an optimal value lower or equal to the same convex combination of the upper bound values $(\bar{v}_i)_{i=1}^n$. Indeed, if $(w_i)_{i=1}^n$ are convex coefficients, *i.e.* $\sum_{i=1}^n w_i = 1$ and $\forall i \in \{1, \dots, n\}$, $w_i \geq 0$, then the optimal value function at $\sum_{i=1}^n w_i \cdot b_i$ is bounded by the convex combination of $(\bar{v}_i)_{i=1}^n$ with respect to $(w_i)_{i=1}^n$: $V^*(\sum_{i=1}^n w_i \cdot b_i) \leq \sum_{i=1}^n w_i \cdot V^*(b_i) \leq \sum_{i=1}^n w_i \cdot \bar{v}_i$. In order to get an upper bound of V^* defined on a larger set of beliefs, the belief-value mappings $\{b_i, \bar{v}_i\}_{i=1}^n$ may be completed with the couples (b, v) such that $b \in \mathbb{P}^{\mathcal{S}}$ is in the convex hull of $(b_i)_{i=1}^n$, and v is the lowest value of $\sum_{i=1}^n w_i \cdot \bar{v}_i$, where $(w_i)_{i=1}^n$ convex coefficients such that $b = \sum_{i=1}^n w_i \cdot b_i$. These coefficients defining the interpolation of the belief-value mappings can be computed using linear programming or with approximate computations [60, 94]. Finally, consider $b_j \in (b_i)_{i=1}^n$ such that $\forall a \in \mathcal{A}$, $\forall o' \in \mathcal{O}$, $u(b_j, a, o')$ is in $(b_i)_{i=1}^n$. The upper bound value \bar{v}_j associated to b_j can be replaced by $(\mathcal{B}^* \bar{V}_0)(b_j)$, computed using the equation (I.18) defining the Dynamic Programming operator \mathcal{B}^* :

$$(\mathcal{B}^* \bar{V}_0)(b_j) = \max_{a \in \mathcal{A}} \left\{ \sum_{s \in \mathcal{S}} r(s, a) \cdot b_j(s) + \gamma \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' | b_j, a) \bar{V}_0(u(b_j, a, o')) \right\},$$

where $\bar{V}_0(u(b_j, a, o')) = \bar{v}_k$ with $u(b_j, a, o') = b_k \in (b_i)_{i=1}^n$. This value update replaces then the value \bar{v}_j and leads to an improved upper bound of the optimal value function in b_j . A famous and simple method to compute an initial upper bound of V^* is called the Q_{MDP} method [74]: it consists in computing an optimal value function for the underlying MDP *i.e.* for the MDP built ignoring the observations and the observation probabilities, and considering that the state is fully observable. A MDP strategy is looked for among a more general set of functions of the data available at execution, than a POMDP strategy. First, the reward is defined on the system states and the actions, which are directly available during execution in fully observable settings. Second, as the uncertainty over the observations is conditional on the system state, the observation random variables O_t can be written as measurable functions of the state and action variables S_t and A_{t-1} : the functions from the actions and the observations consist thus of a subset of the functions from the system states and the actions. We can conclude that, the optimal value function of the POMDP starting from the deterministic belief $b_0^A(s) = \mathbb{1}_{\{s=s_A\}}$ with $s_A \in \mathcal{S}$, namely

$$V^*(b_0^A) = \max_{\substack{(d_t)_{t \geq 0} \text{ s.t.} \\ d_t: \mathcal{I}_t \rightarrow \mathcal{A}}} \mathbb{E}_{S_0 \sim b_0^A} \left[\sum_{t \geq 0} \gamma^t \cdot r(S_t, d(I_t)) \right],$$

is lower or equal to the optimal value function for the associated MDP starting from s_A :

$$V_{MDP}^*(s_A) = \max_{\substack{(d_t)_{t \geq 0} \text{ s.t.} \\ d_t: \mathcal{S} \rightarrow \mathcal{A}}} \mathbb{E} \left[\sum_{t \geq 0} \gamma^t \cdot r(S_t, d(S_t)) \middle| S_0 = s_A \right].$$

Indeed, the maximum operator of the latter is performed on a set of functions including the one used to maximize the POMDP value function at b_0^A (first equation). Hence, an initial upper bound of the POMDP optimal value function V^* can be computed for the deterministic beliefs *i.e.* the beliefs $b \in \mathbb{P}^{\mathcal{S}}$ such that $b(s) = \mathbb{1}_{\{s=s_A\}}$ with $s_A \in \mathcal{S}$: $\bar{V}_0(b) \geq V_{MDP}^*(s_A)$. The convex hull of these deterministic beliefs is the full set of probability distributions $\mathbb{P}^{\mathcal{S}}$. Thus, using the previous interpolation trick, an upper bound of V^* is available in the full belief space $\mathbb{P}^{\mathcal{S}}$. Other methods to compute bounds for V^* are available for instance in [59].

Some recent POMDP solvers are said to be *point-based* as they maintain a set of beliefs $(b_i)_{i=1}^n$ (the *belief points*) and the associated α -vectors $(\alpha_i)_{i=1}^n$. The current approximation V of the optimal value function is described by $(\alpha_i)_{i=1}^n$: $\forall b \in \mathbb{P}^{\mathcal{S}}$,

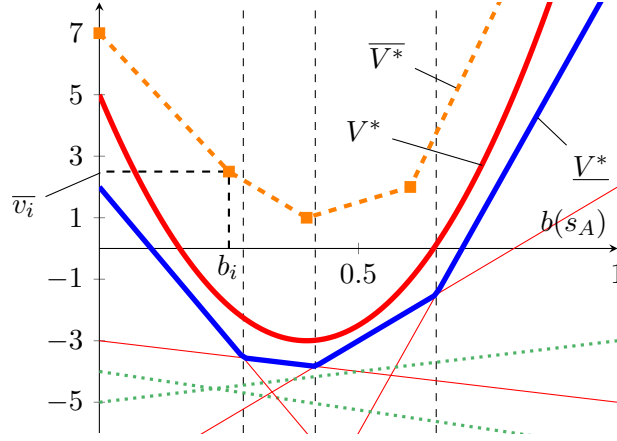


Figure I.6 – Bounds of the optimal value function V^* . The latter is represented by the regular thick red line. The lower bound \underline{V}^* is the piecewise linear function represented by the thick blue line. Useful α -vectors are represented by the thin red lines, and dominated ones are represented by dotted green lines. The upper bound \overline{V}^* is represented by the piecewise linear dashed orange line: the squares represent the belief-value mappings.

$V(b) = \max_{i=1}^n \sum_{s \in \mathcal{S}} \alpha_i(s) \cdot b(s)$. Each belief of $(b_i)_{i=1}^n$ is such that $V(b_i) = \sum_{s \in \mathcal{S}} b_i(s) \cdot \alpha_i(s)$. The point-based Bellman backup of a belief b_i is the replacement of its α -vector α_i by one of the new α -vectors (Equation I.20) which maximize the equation (I.19) with the belief b_i , i.e. replacing it by $\alpha' \in \arg\max_{\alpha' \in \Gamma'} \langle \alpha', b_i \rangle_{\mathbb{R}^S}$.

The first point-based algorithm was PBVI (*Point-Based Value Iteration*) [90] which starts with a single belief point b_0 . At a given iteration, the point-based Bellman backup of each belief of the current set $(b_i)_{i=1}^n$ are computed and lead to a new set of α -vectors $(v'_i)_{i=1}^n$. This operation is repeated until the convergence of the α -vectors. Next, for each belief in $(b_i)_{i=1}^n$, one successor is selected: the most distant one from $(b_i)_{i=1}^n$, with respect to a given distance metric. These successors are added to the set, which becomes $(b_i)_{i=1}^{2n}$, and the next iteration begins.

Another point-based POMDP solver is the *Perseus* solver [115]: this solver starts running trials of random exploration of the belief space, sampling $a \in \mathcal{A}$ and observation $o' \in \mathcal{O}$ at each time step of a trial to compute the next belief $b' = u(b, a, o')$ from the current one $b \in \mathbb{P}^S$. The set of all reached beliefs is $(b_i)_{i=1}^n$ and does not evolve anymore. A lower bound PWLC \underline{V} is used to approach the value function: the associated set of α -vectors is denoted by $\underline{\Gamma}$. At each iteration, \mathbb{B} is initialized as a copy of $(b_i)_{i=1}^n$, and $\Gamma = \emptyset$. While $\mathbb{B} \neq \emptyset$, an arbitrary belief b is selected in \mathbb{B} , and the associated new α -vector α' is computed with the point-based Bellman backup on b and using $\underline{\Gamma}$. If $\langle \alpha', b \rangle_{\mathbb{R}^S} \geq \underline{V}(b)$, all the beliefs whose value is improved by α' , i.e. the beliefs $\tilde{b} \in \mathbb{B}$ such that $\langle \alpha', \tilde{b} \rangle_{\mathbb{R}^S} \geq \underline{V}(\tilde{b})$, are removed from \mathbb{B} . The new α' is added to Γ . Otherwise, if $\langle \alpha', b \rangle_{\mathbb{R}^S} < \underline{V}(b)$, b is removed from \mathbb{B} and an α -vector from $\underline{\Gamma}$ $\tilde{\alpha} \in \arg\max_{\alpha \in \underline{\Gamma}} \langle \alpha, b \rangle_{\mathbb{R}^S}$ is added into Γ . When $\mathbb{B} = \emptyset$, $\underline{\Gamma}$ is set to Γ , and a new iteration begins.

The HSVI solver (*Heuristic Search Value Iteration*) [114] is also a point-based algorithm. This solver maintains both an upper and a lower bound of V^* : \overline{V} and \underline{V} . It takes into account that the error of the approximation of V^* is less important for much later successors of $b_0 \in \mathbb{P}^S$, due to the discount factor γ : given an error $\varepsilon > 0$, a sequence of beliefs $(b_t)_{t \geq 0}$ is generated from b_0 until $\overline{V}(b_t) - \underline{V}(b_t) < \frac{\varepsilon}{\gamma^t}$. The generation of the belief sequence is done selecting actions according to the upper bound: if the current belief is b , the chosen action is in $\arg\max_{a \in \mathcal{A}} r(s, a) + \gamma \sum_{o' \in \mathcal{O}} \overline{V}(u(b, a, o'))$. This trick tends to force the improvement of the upper bound: if the selected action is not optimal, as successors for this action are selected, the computations will focussed on these beliefs, and will decrease (improve) the upper bound for them. As well, the observation selected $o' \in \mathcal{O}$ is such that the value

$\bar{V}(u(b, a, o')) - \underline{V}(u(b, a, o')) \cdot \mathbf{p}(o' | b, a)$ is the greatest: the probable beliefs for which \bar{V} and \underline{V} are poor bounds are preferred, in order to focus the computational efforts on the belief space subsets where the bounds are the worst. Then, both bounds are updated starting with the last reached belief back in time step to b_0 : this order makes these updates more efficient. The scheme starting with a belief sequence generation, and updating the bounds on it, is repeated until $\bar{V}(b_0) - \underline{V}(b_0) < \varepsilon$.

The solver SARSOP [69], inspired by HSVI and FSVI [109], refines the generation method of the belief sequence. First of all, the belief space is clustered using a simple learning technique: the features are for instance the initial upper bound \underline{V}_0 and the entropy of beliefs. This discretization is used to maintain an estimation of the optimal value function denoted by \hat{V} : this estimation is constant over each cluster, equal to the average of the estimated optimal value of the beliefs in this cluster. Let b be a belief reached during a generation of a belief sequence: L_1 is real number such that $L_1 \leq V^*(b)$. Let $L_2 = (\mathcal{B}^* \underline{V})(b)$. Thus, the lower bound of the optimal value function \underline{V} on b is likely to be improved by selecting the next belief $b' = u(b, a, o')$ (where $a \in \mathcal{A}$ and $o' \in \mathcal{O}$ are selected as in HSVI) if the optimal value of b' is likely to be big enough for it: that is, if $r(b, a) + \gamma \cdot \left(\mathbf{p}(o' | b, a) \cdot \hat{V}(b') + \sum_{\tilde{o} \neq o'} \mathbf{p}(\tilde{o} | b, a) \cdot \underline{V}(u(b, a, \tilde{o})) \right)$ is greater than $L = \max \{L_1, L_2\}$. In this case, if L'_1 such that $L = r(b, a) + \gamma \cdot \left(\mathbf{p}(o' | b, a) \cdot L'_1 + \sum_{\tilde{o} \neq o'} \mathbf{p}(\tilde{o} | b, a) \cdot \underline{V}(u(b, a, \tilde{o})) \right)$, the condition $\hat{V}(u(b, a, \tilde{o})) \geq L'_1$ is a good indicator that the selection of the belief b' is likely to improve the current lower bound at b : if $\hat{V}(b') \geq L'_1$ (and if, as in HSVI, $\bar{V}(b') - \underline{V}(b') \leq \frac{\varepsilon}{\gamma_t}$), the belief b' is selected and the same test is performed on its successor knowing that $V^*(b')$ is likely to be greater than L'_1 . In this way, the belief sequences selected may be longer than in HSVI, as sequence generation continues if the next beliefs are likely to improve optimal value function estimation. Next, when the generation of the sequence ends, the bound updates start from the last belief to b_0 , following the generated belief sequence in the reverse order (as with HSVI). Finally, SARSOP proposes also another major computation simplification: all the previous belief sequence generations can be summed up in a tree representing the different transitions $b \rightarrow u(b, a, o')$ and which is recorded in the memory. Let us define $\underline{Q}(b, a) = r(b, a) + \gamma \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' | b, a) \cdot \underline{V}(u(b, a, o'))$, and $\bar{Q}(b, a)$ with the same formula replacing \underline{V} by \bar{V} . If b is a belief, a an action, and $\exists a' \in \mathcal{A}$, $\exists b' \in \mathbb{P}_{b_0}^S$ such that $\bar{Q}(b, a) \leq \underline{Q}(b', a')$, then, all the successors of b by selecting action a are removed from the tree, and the associated α -vectors deleted. Indeed, less stored α -vectors and beliefs speed up the POMDP resolution, as lots of useless computations are avoided: the computations are focused on the beliefs which seems to be in $\mathbb{P}_{b_0, *}^S$, the notation for the subset of $\mathbb{P}_{b_0}^S$ containing the beliefs reached with an optimal strategy.

While the α -vector (or Sondic's) representation is used by a large part of the POMDP solvers, the *grid-based* POMDP solvers, which does not use it, are also popular algorithms [10, 75, 19, 11]. These solvers are based on a discretization of the belief space \mathbb{P}^S which leads to an MDP over the finite set of discretized beliefs: computed strategies map any cluster of beliefs to an action $a \in \mathcal{A}$.

Another POMDP solver based on a discretization of the belief space is RTDP-bel [58]. The discretization is only used to store a finite number of values during the computations. The approximation maintains the approximate optimal value function as a piecewise constant function: two beliefs in the same discretization group have the same value. The algorithm operates in the same way as RTDP (*Real Time Dynamic Programming*) [5], a *Goal-MDP* solver which converges to the optimal value function without considering all the system states.

A Goal-MDP [9] is an MDP whose all rewards are negative and with a subset of the system state $\mathcal{G} \subset \mathcal{S}$ called *set of goals*. The system states in \mathcal{G} are absorbing and cost-free: $\forall (s, a) \in \mathcal{G} \times \mathcal{A}$, $r(s, a) = 0$ and $\mathbf{p}(s \mid s, a) = 1$. The criterion of a Goal-MDP is the expected (undiscounted) total reward, *i.e.* the expectation of the sum of the rewards over time steps (without factors γ^t). As well, a *Goal-POMDP* is a POMDP with only negative rewards and a set of goals \mathcal{G} which are absorbing, cost-free and fully observable system states *i.e.* \mathcal{O} contains \mathcal{G} too, and $\forall s' \in \mathcal{G}$, $\forall a \in \mathcal{A}$, $\forall t \geq 0$, $\mathbb{P}(O_{t+1} = s' \mid S_{t+1} = s', a) = 1$. In fact RTDP-bel is a Goal-POMDP solver. It initializes the value function V with a (piecewise constant) upper bound called *admissible heuristique*. Then, the repetition of trials starting from b_0 improves the approximation of the optimal value function and the associated strategy. At a given stage of a trial, the current belief is denoted by b . The Q -value is $Q(b, a) = \left\{ r(b, a) + \sum_{o' \in \mathcal{O}} \mathbf{p}(o' \mid b, a) \cdot V(u(b, a, o')) \right\}$. An action $a \in \arg\max_{\tilde{a} \in \mathcal{A}} Q(b, \tilde{a})$ is selected, and $V(b)$ is updated to $\max_{a \in \mathcal{A}} Q(b, a)$. Then s is sampled from b , as well as s' according to $\mathbf{p}(s' \mid s, a)$, and o' using $\mathbf{p}(o' \mid s', a)$. If $b' = u(b, a, o')$ is such that $\forall s \in \mathcal{S} \setminus \mathcal{G}$, $b'(s) = 0$, then another trial begins. Otherwise, the next stage of the trial consider b' . Any classical (discounted) POMDP (Section I.1.6) can be translated into a Goal-POMDP [13], then RTDP-bel can solve any POMDP.

Finally, POMCP [111] is the partially observable counterpart of the MDP solver UCT [66]. The latter is based on the UCB (Upper Confidence Bound) strategy for stochastic bandits [3] and is an instance of MCTS (Monte-Carlo Tree Search) [27]. A decision tree whose nodes are the reached states and arrows are the actions, is built during simulations to maintain at each node the counts $N(s, a)$ of the visit of the couple (s, a) , $\forall a \in \mathcal{A}$. The estimation V of the optimal value function is computed by Monte-Carlo simulations. The Q -value is $Q(s, a) = \left\{ r(s, a) + \gamma \cdot \sum_{s' \in \mathcal{S}} \mathbf{p}(s' \mid s, a) \cdot V(s') \right\}$. The UCB-inspired exploration-exploitation strategy is to select $a \in \arg\max_{a \in \mathcal{A}} \left\{ Q(s, a) + c \cdot \sqrt{\frac{\log N(s)}{N(s, a)}} \right\}$, where $N(s) = \sum_{a \in \mathcal{A}} N(s, a)$ and $c > 0$ is the relative ratio between exploration to exploitation: the more c is small, the more actions with high values are selected (exploitation), the more $c > 0$ is big, the more the actions are selected with about the same rate, without paying attention to the estimated values. The term $c \cdot \sqrt{\frac{\log N(s)}{N(s, a)}}$ is meant to force the actions rarely tried before to be selected (exploration). In the case of POMCP, during computations, the belief is approximated by an unweighted particle filter $B_t(s) \approx \frac{1}{K} \sum_{i=1}^K \mathbb{1}_{S_i=s}$ where $\forall i = 1, \dots, K$, $S_i \sim B_t$, and the nodes of the tree represent possible successive information $i_t = \{a_0, o_1, \dots, a_{t-1}, o_t\}$, instead of the system states like in UCT.

The algorithms presented in this section are part of the state of the art POMDP solvers, and some of them will be used in the next chapters in order to propose comparisons with our work. Now, the second main subject of this thesis is presented: Possibility Theory, and the qualitative possibilistic counterpart of the Partially Observable Markov Decision Processes.

I.2 QUALITATIVE POSSIBILISTIC MDPs

This section presents the uncertainty framework studied in this thesis: namely, the model π -POMDP. First of all, the Possibility Theory is presented, with a particular emphasis on the qualitative part of the theory. Qualitative conditioning is then presented, as well as notions of independence. Finally, the qualitative possibilistic counterpart of the MDPs called *Qualitative Possibilistic MDPs*, or π -MDPs are defined, followed by the *Qualitative Possibilistic POMDPs*, or π -POMDPs.

I.2.1 Possibility Theory

The “fuzzy sets” introduced by Lotfi Zadeh [126], have been studied by Didier Dubois [32] and Henri Prade and their contributions have led to the foundation of Possibility Theory [48].

As in Probability Theory, this theory is based on the definition of an uncertainty measure, called *possibility measure*. Unlike the probability measure \mathbb{P} which is a classical measure (Definition I.2.1), the *possibility measure*, denoted by Π , is a *fuzzy measure*, or *capacity*. For simplicity, a fuzzy measure is not supposed to be additive, but just *monotone*, as highlighted by Definition I.2.2. In this thesis, Possibility Theory will only concern finite sets such as \mathcal{S} and \mathcal{O} , that is why definitions only concern finite sets.

Definition I.2.1 (*Measure*)

A classical measure \mathbb{M} on the finite set Ω is a function from 2^Ω to \mathbb{R}^+ such that

- $\mathbb{M}(\emptyset) = 0$ (null empty set);
- $\forall A, B \subseteq \Omega$ such that $A \cap B = \emptyset$, $\mathbb{M}(A \cup B) = \mathbb{M}(A) + \mathbb{M}(B)$ (additivity).

Definition I.2.2 (*Fuzzy Measure*)

A fuzzy measure \mathfrak{M} on the finite set Ω is a function from 2^Ω to \mathbb{R}^+ such that

- $\mathfrak{M}(\emptyset) = 0$ (null empty set);
- $\forall A, B \subseteq \Omega$ such that $A \subseteq B$, $\mathfrak{M}(A) \leq \mathfrak{M}(B)$ (monotonicity).

Note that a classical measure is a particular case of fuzzy measure since classical measures are monotone: if $A \subseteq B$,

$$\mathbb{M}(B) = \mathbb{M}\left((A \cap B) \cup (\bar{A} \cap B)\right) = \mathbb{M}(A) + \mathbb{M}(\bar{A} \cap B) \geq \mathbb{M}(A),$$

where \bar{A} is the complementary set of A in Ω *i.e.* $\bar{A} \cap A = \emptyset$, and $\bar{A} \cup A = \Omega$.

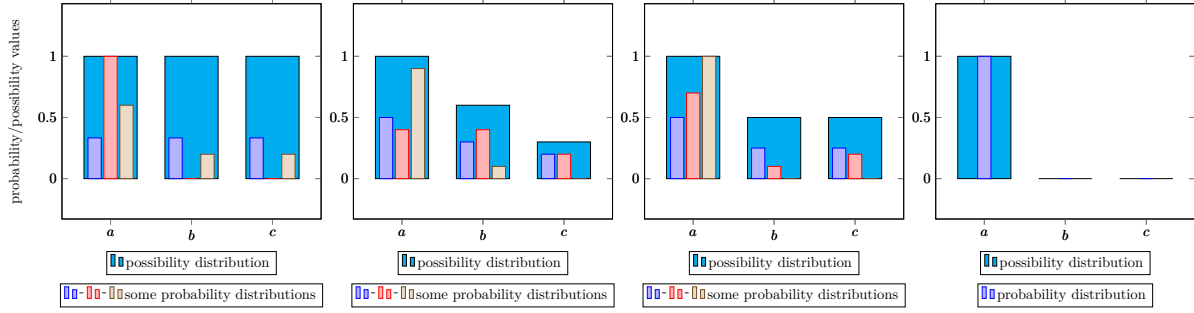
A possibility measure is also a particular case of fuzzy measure:

Definition I.2.3 (*Possibility Measure*)

A possibility measure on the finite set Ω is a fuzzy measure such that

- $\Pi(\Omega) = 1$ (normalization);
- $\forall A, B \subseteq \Omega$, $\Pi\{A \cup B\} = \max\{\Pi(A), \Pi(B)\}$ (maxitivity).

Probability Theory models the uncertainty due to the variability of events: in practice, used probabilities are estimated frequencies of events stated as the actual variability model of events. Another view of this theory is De Finetti’s one [34]: the probability value of an event is an exchangeable bet, *i.e.* the value in $[0, 1]$ that a given person is willing to give for the bet winning 1 if the event is true. However this person takes into account in her/his choice of value that the bet can be reversed just before verifying if the event is true. Indeed, she/he may be asked to get the chosen value, and to give 1 if the event is true. As the probability values depend on a natural person, who wants to guess the actual probability distribution as well as possible with respect to all her/his information about the event, they are called *subjective probabilities*: that is why the theory based on this definition is called Subjective Probability Theory.



(a) ignorant distribution (b) three different degrees (c) two different degrees (d) determinism

Figure I.7 – Example of quantitative possibility distributions (thick blue), and some of the associated probability distributions (thin) i.e. some of the probability measures encoded by the possibility distribution. Distributions are defined on $\Omega = \{a, b, c\}$.

Possibility Theory is devoted to uncertainty due to a lack of knowledge or imprecision about an event. Quantitative Possibility Theory can be seen as a special case of imprecise probabilities i.e. a possibility measure Π represents a set of probability measures defined on Ω , denoted by \mathcal{P}^Π , and each probability measure $\mathbb{P} \in \mathcal{P}^\Pi$ is a guess about the actual probabilistic model. The set \mathcal{P}^Π is the set of each probability measure \mathbb{P} such that $\forall A \subset \Omega, \mathbb{P}(A) \leq \Pi(A)$. In this case, possibility measures are thus “inflated” probability measures, in order to model that frequencies are not well known, as illustrated by Figure I.7. The possibility measure is then $\forall A \subset \Omega, \Pi(A) = \max_{\mathbb{P} \in \mathcal{P}^\Pi} \mathbb{P}(A)$.

If the set of probability distributions consists of all the probability distributions defined on Ω , i.e. the probabilistic model is completely unknown, and then $\forall A \subset \Omega, \Pi(A) = \max_{\mathbb{P} \in \mathcal{P}^\Pi} \mathbb{P}(A) = 1$: the ignorant possibility measure is equal to 1 for each set $A \subset \Omega$, as illustrated Figure I.7a. On the contrary, if the actual elementary event is known to be $\omega_A, \forall \omega \in \Omega$ such that $\omega \neq \omega_A, \Pi(\{\omega\}) = \mathbb{P}(\{\omega\}) = 0$ and $\Pi(\{\omega_A\}) = \mathbb{P}(\{\omega_A\}) = 1$, as illustrated Figure I.7d. It is worth noting that there exists some sets of probability distributions which are not represented by any quantitative possibility distribution: for instance, the set of probability distributions on $\Omega = \{\omega_A, \omega_B\}, \{\mathbb{P} \mid \mathbb{P}(\omega_A) \geq 0.1, \mathbb{P}(\omega_B) \geq 0.1\}$.

Unlike quantitative one, Qualitative Possibility Theory uses possibility measures whose values are defined in any ordered scale. This theory allows to reason under a lack of quantitative information: the only information given by a qualitative possibility measure is the rank between events i.e. $\forall A, B \subseteq \Omega$, the information “event A is less plausible than event B ”, which is written $\Pi(A) \leq \Pi(B)$. Hence qualitative possibility measures Π are often defined as functions $2^\Omega \rightarrow \mathcal{L}$, where \mathcal{L} is a finite set called *possibility scale* and equipped with a total order. In this work, and more specifically for the following three chapters of this thesis, the possibility scale is defined as $\mathcal{L} = \{0, \frac{1}{k}, \dots, 1\}$ to simplify notations.

The structure of Possibility Theory is easily understood using the terminology of *fuzzy sets*. A classical set A of elements of Ω can be defined through a characteristic (or membership) function

$$\mathbb{1}_A : \begin{cases} \Omega & \rightarrow & \{0, 1\} \\ \omega & \mapsto & \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

In practice, some problems may need to be more flexible about the membership of elements using *membership degrees*: a *fuzzy set* \mathfrak{A} is defined by a characteristic function $\mathbb{1}_{\mathfrak{A}} : \Omega \rightarrow \mathcal{L}$ whose possible values are not only in $\{0, 1\}$ but may be in the possibility scale \mathcal{L} : $\mathbb{1}_{\mathfrak{A}}(\omega) \in \mathcal{L}$ is the *membership degree* of $\omega \in \Omega$. If $\mathbb{1}_{\mathfrak{A}}(\omega) = 0$, then $\omega \notin \mathfrak{A}$. If $\mathbb{1}_{\mathfrak{A}}(\omega) = 1$, then $\omega \in \mathfrak{A}$. And, finally, if $\mathbb{1}_{\mathfrak{A}}(\omega) = \lambda \in \mathcal{L} \setminus \{0, 1\}$, then $\omega \in \mathfrak{A}$ with membership degree λ .

Let us define the *possibility distribution* as $\pi(\omega) = \Pi(\{\omega\})$: according to Definition I.2.3,

the possibility measure is entirely defined with the distribution π . Consider that the set Ω is the set of states \mathcal{S} . Let S be a variable representing a state of the problem, and whose values are in \mathcal{S} . Consider an expert description of the actual value of S , given for instance in natural language: “the state is near state $s_A \in \mathcal{S}$ (in some sense) and is not s_B ”. This description given by the expert knowledge leads to a fuzzy set \mathfrak{T} : $\mathbb{1}_{\mathfrak{T}}(s)$ is the degree of membership of $s \in \mathcal{S}$, *i.e.* the degree of how well s respects the description. For instance, in the previous example, $\mathbb{1}_{\mathfrak{T}}(s_B) = 0$, and $\mathbb{1}_{\mathfrak{T}}(s) > \mathbb{1}_{\mathfrak{T}}(s')$ if s is “closer” (in some sense) to s_A than s' . In summary, the function $\mathbb{1}_{\mathfrak{T}} : \mathcal{S} \rightarrow \mathcal{S}$ gives the degree of similarity of a prototype, which corresponds to the expert description. It is assumed that at least one state $s \in \mathcal{S}$ is fully consistent with the prototype: $\mathbb{1}_{\mathfrak{T}}(s) = 1$. Consider again the variable S : its actual value is unknown, but the fuzzy set \mathfrak{T} associated to the expert description leads to the possibility distribution of this variable: the possibility distribution π is defined as the membership function of the fuzzy set \mathfrak{T} : $\forall s \in \mathcal{S}, \pi(s) = \mathbb{1}_{\mathfrak{T}}(s)$. A possibility distribution of the variable $S \in \mathcal{S}$ is a characteristic function of a fuzzy set based on \mathcal{S} where the elementary events $\{S = s\}$ are mutually exclusive: $\forall s_A \neq s_B \in \mathcal{S}, \Pi(S = s_A, S = s_B) = 0$.

Now the possibility degree of the event $S \in \{s_A, s_B\}$, *i.e.* of the event “the state is s_A or s_B ” is $\Pi(\{s_A, s_B\}) = \max\{\mathbb{1}_{\mathfrak{T}}(s_A), \mathbb{1}_{\mathfrak{T}}(s_B)\} = \max_{s \in \{s_A, s_B\}} \pi(s)$: this is a maximum, as an extension of the logical “or” (\vee), usually defined on $\{0, 1\}$ or $\{\top, \perp\}$, and here defined on \mathcal{L} . This is easily generalized for more than two elementary events: $\forall A \subseteq \mathcal{S}, \Pi(A) = \max_{s \in A} \pi(s)$, which is also a consequence of Definition I.2.3. The possibility measure evaluates an event $A \subseteq \mathcal{S}$ by the most plausible elementary event in the event A . Hence, the normalization condition of Definition I.2.3, becomes $\max_{s \in \mathcal{S}} \pi(s) = 1$, which looks more like the probabilistic normalization $\sum_{s \in \mathcal{S}} \mathbf{p}(s) = 1$. As a conclusion, a possibility degree $\pi(s)$ can be seen as a “non-surprise” degree, since $1 - \pi(s)$ is considered as the “surprise” degree of the event $\{S = s\}$.

For each possibility measure, a dual measure called *necessity* can be defined: the necessity degree of an event increases if the possibility degree of the opposite event decreases.

Definition I.2.4 (*Necessity Measure associated to Π*)

The necessity measure associated to Π is the fuzzy measure $\mathcal{N} : 2^\Omega \rightarrow [0, 1]$ such that $\forall A \subset \Omega$,

$$\mathcal{N}(A) = 1 - \Pi(\bar{A}),$$

where \bar{A} is the complementary event of A in Ω .

Note that, as $\mathcal{N}(\emptyset) = 1 - \Pi(\Omega) = 0$, and as $\forall A, B$ subsets of Ω such that $A \subseteq B$, $\mathcal{N}(A) = 1 - \Pi(\bar{A}) \leq 1 - \Pi(\bar{B}) = \mathcal{N}(B)$, the necessity is indeed a fuzzy measure.

Note also that if an event $A \subseteq \Omega$ is not entirely possible ($\Pi(A) < 1$), then this event is not necessary at all ($\mathcal{N}(A) = 0$). As well, if the necessity degree of the event $A \subseteq \Omega$ is positive ($\mathcal{N}(A) > 0$), then it is entirely possible $\Pi(A) = 1$. Indeed, $\Pi(A \cap \bar{A}) = \Pi(\Omega) = 1$ and $\Pi(A \cap \bar{A}) = \max\{\Pi(A), \Pi(\bar{A})\}$. Thus $\max\{\Pi(A), \Pi(\bar{A})\} = 1$. Then, if $\Pi(A) < 1$, $\Pi(\bar{A}) = 1$ and $\mathcal{N}(A) = 0$. As well, if $\mathcal{N}(A) > 0$, $\Pi(\bar{A}) < 1$ and then $\Pi(A) = 1$.

Consider now $A, B \subseteq \Omega$: whereas $\Pi\{A \cup B\} = \max\{\Pi(A), \Pi(B)\}$,

$$\mathcal{N}(A \cap B) = \min\{\mathcal{N}(A), \mathcal{N}(B)\}.$$

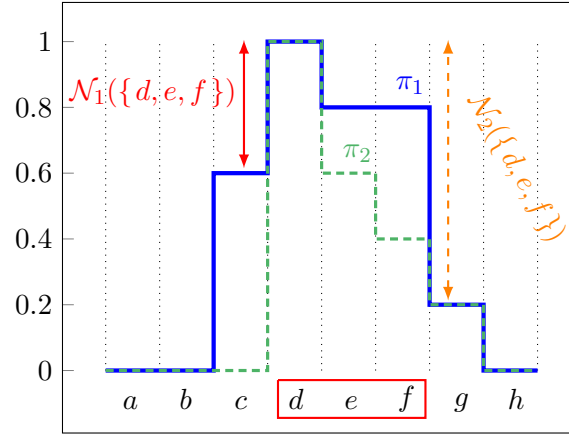


Figure I.8 – Example of two possibility distributions over $\Omega = \{a, b, c, d, e, f, g, h\}$: π_1 (solid blue line) and π_2 (dashed green one), with π_2 more specific than π_1 . The necessity measure \mathcal{N}_1 associated with π_1 is assessed on the event $\{d, e, f\} \subset \Omega$: the necessity degree is equal to $0.4 = 1 - 0.6$, as illustrated with solid red arrows. The necessity measure is assessed on the same event: the necessity degree is equal to $0.8 = 1 - 0.2$, as illustrated with dashed orange arrows.

Indeed,

$$\begin{aligned}
 \mathcal{N}(A \cap B) &= 1 - \Pi(\overline{A \cap B}) \\
 &= 1 - \Pi(\overline{A} \cup \overline{B}) \\
 &= 1 - \max \{ \Pi(\overline{A}), \Pi(\overline{B}) \} \\
 &= \min \{ 1 - \Pi(\overline{A}), 1 - \Pi(\overline{B}) \} \\
 &= \min \{ \mathcal{N}(A), \mathcal{N}(B) \}.
 \end{aligned}$$

The total ignorance is modeled by a possibility distribution π such that $\forall \omega \in \Omega, \pi(\omega) = 1$ i.e. any elementary event is possible. As well $\forall A \subseteq \Omega, A \neq \Omega, \mathcal{N}(A) = 1 - \Pi(\overline{A}) = 1 - \max_{\omega \in \overline{A}} \Pi(\omega) = 0$: apart from the universe Ω , no event is necessary.

On the contrary, the perfect knowledge that the actual elementary event is $\omega_A \in \Omega$ is modeled by a possibility distribution π such that $\pi(\omega_A) = 1$ and $\pi(\omega) = 0, \forall \omega \neq \omega_A$. The necessity of the singleton $\{\omega_A\}$ is also equal to one: $\mathcal{N}(\{\omega_A\}) = 1 - \Pi(\overline{\{\omega_A\}}) = 1$. The elementary event ω_A is necessary, and all the other have a null necessity degree: if $\omega_B \neq \omega_A, \mathcal{N}(\{\omega_B\}) = 1 - \Pi(\overline{\{\omega_B\}}) = 1 - \pi(\omega_A) = 0$.

These two particular cases give the intuition to formalize the knowledge, or the information, provided by a possibility distribution. This idea is led by the word *specificity*:

Definition I.2.5 (Specificity)

A possibility distribution π_2 is more specific (i.e. more informative) than another possibility distribution π_1 , if $\forall \omega \in \Omega$,

$$\pi_2(\omega) \leq \pi_1(\omega).$$

Both notions of specificity and necessity are illustrated by Figure I.8.

The main concepts of Possibility Theory have been presented. Possibilistic planning models studied in this thesis are based on *conditional possibility distributions* and some independence assumptions, that is why the next section focuses on these notions.

I.2.2 Qualitative Conditioning and Possibilistic Independence

In practice, two kinds of independence between variables can be distinguished. The first one is the *decompositional independence*: two variables $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ are said independent

in this sense, if the joint distribution of these two variables can be decomposed into two marginal distributions (one for each variable) without losing any information provided by the joint distribution. In the probabilistic framework, variables X and Y are independent in the decompositional sense, if $\forall E \subset \mathcal{X}, F \subset \mathcal{Y}$,

$$\mathbb{P}(X \in E, Y \in F) = \mathbb{P}(X \in E) \cdot \mathbb{P}(Y \in F). \quad (\text{I.21})$$

The second kind of independence is called the *causal independence*: a variable X is independent of a variable Y in the causal sense, if the distribution of X is not modified when something about Y is learned *i.e.* there is no causality from Y towards X , or yet, Y does not influence X . Note that this independence relation is not necessarily symetric: “ Y does not influence X ” does not imply that “ X does not influence Y ”. In terms of probability measures, the causal independence of X from Y can be written $\forall E \subset \mathcal{X}, F \subset \mathcal{Y}$,

$$\mathbb{P}(X \in E | Y \in F) = \mathbb{P}(X \in E). \quad (\text{I.22})$$

In Probability Theory, both equations (I.21) and (I.22) are equivalent, if probabilities are positive, and then the causal independence is symetric: in this theory, decompositional and causal independence are equal and called simply independence.

In Possibility Theory, Zadeh [125] introduced the *non-interactivity independence*, or NI-independence, a decompositional independence inspired from Fuzzy Logic: as the fuzzy generalization of the “and” (\wedge) operator is the minimum (\min), if events A and B do not interact together, the degree of truth of $A \cap B$ (*i.e.* of the event “ A and B occur”), or its possibility degree, is $\Pi(A \cap B) = \min \{ \Pi(A), \Pi(B) \}$. The analogy is also possible with the framework of fuzzy sets, as the fuzzy “intersection” (\cap) is represented by the minimum between the membership functions. This leads to the definition of the NI-independence for variables $(X, Y) \in \mathcal{X} \times \mathcal{Y}$, replacing the event A (resp. B) by $\{X \in E\} \subseteq \Omega$ with $E \subseteq \mathcal{X}$ (resp. $\{Y \in F\} \subseteq \Omega$ with $F \subseteq \mathcal{Y}$):

Definition I.2.6 (Non Interactivity Independence)

Two events $A, B \subset \Omega$ are NI-independent if

$$\Pi(A \cap B) = \min \{ \Pi(A), \Pi(B) \}.$$

Then, two variables $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ are NI-independent if $\forall E \subset \mathcal{X}, F \subset \mathcal{Y}$,

$$\Pi(X \in E, Y \in F) = \min \{ \Pi(X \in E), \Pi(Y \in F) \}.$$

Finally, in terms of possibility distributions, it simply asserts that the joint one is equal to the minimum between the marginal ones: $\forall x \in \mathcal{X}, y \in \mathcal{Y}$,

$$\pi(x, y) = \min \{ \pi(x), \pi(y) \},$$

where $\pi(x) = \Pi(\{X = x\})$, $\pi(y) = \Pi(\{Y = y\})$, and the joint possibility distribution $\pi(x, y)$ is $\Pi(\{X = x\} \cap \{Y = y\})$.

Note that, as Π is a fuzzy measure, Π is monotone, and then $\forall A, B \subset \Omega$, $\Pi(A \cap B) \leq \Pi(A)$ and $\Pi(A \cap B) \leq \Pi(B)$: thus, the inequality $\Pi(A \cap B) \leq \min \{ \Pi(A), \Pi(B) \}$ is always true, with an equality when events are NI-independent. Figure I.9 illustrates a joint possibility distribution $\pi(x, y)$ over $\mathcal{X} \times \mathcal{Y}$ whose corresponding variables are not NI-independent, whereas Figure I.10 represents a similar distribution whose variables are NI-independent. Note that, if a joint distribution $\pi(x, y)$ is given, $\pi(x)$ can be computed from it by marginalization using the max

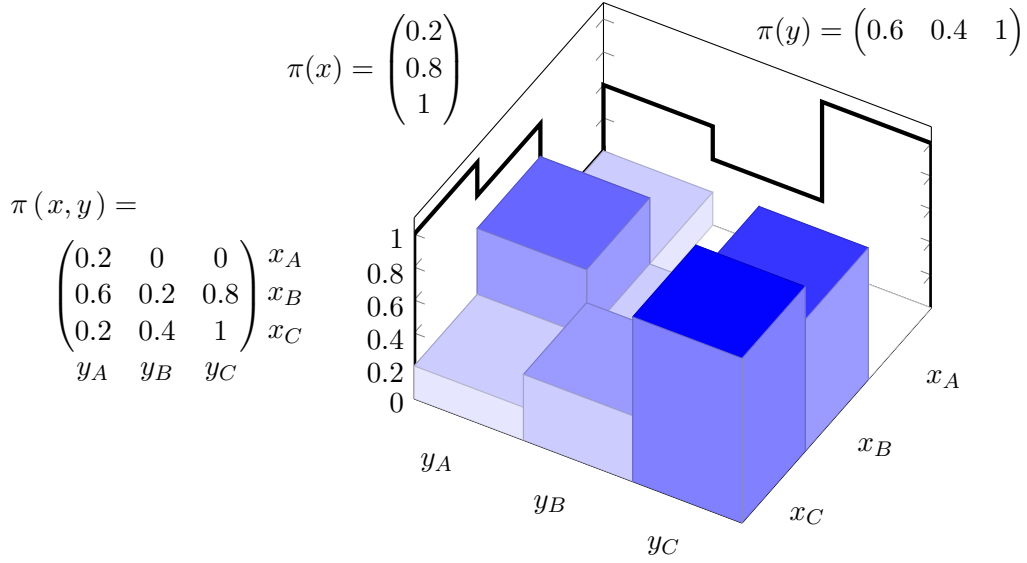


Figure I.9 – Example of a joint distribution over $\mathcal{X} \times \mathcal{Y} = \{x_A, x_B, x_C\} \times \{y_A, y_B, y_C\}$ without any independence.

operator over \mathcal{Y} :

$$\begin{aligned}
 \pi(x) &= \Pi(\{X = x\}) \\
 &= \Pi(\cup_{y \in \mathcal{Y}} \{X = x\} \cap \{Y = y\}) \\
 &= \max_{y \in \mathcal{Y}} \Pi(\{X = x\} \cap \{Y = y\}) \\
 &= \max_{y \in \mathcal{Y}} \pi(x, y).
 \end{aligned}$$

The NI-independence leads to a first qualitative possibilistic conditioning, introduced by Hisdal [61]. Indeed, the conditional possibility degree of an event $A \subset \Omega$ given an event $B \subset \Omega$, namely $\Pi(A|B)$, can be obtained from $\Pi(B)$ and $\Pi(A \cap B)$, as a solution of the following equation:

$$\Pi(A \cap B) = \min \{ \Pi(A|B), \Pi(B) \}. \quad (\text{I.23})$$

Intuitively, once conditioned on the event B , the event A (or rather “ $A|B$ ”) does not interact with the event B anymore. Moreover Equation I.23 is close to the probabilistic equation $\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \cdot \mathbb{P}(B)$ which comes from the definition of the probabilistic conditioning. Possibility Theory looks very similar to Probability Theory observing that the addition (+) in Probability Theory becomes a maximum (max) in Possibility Theory, and the multiplication (\times) becomes a minimum (min). However, Quantitative Possibility Theory keeps the multiplication operator (\times) for conditioning and computing joint possibility measures: in this theory, the classical conditioning is then equivalent to the Dempster rule of conditioning [39], an evidentialist extension of the well-known Bayes rule.

The classical qualitative possibilistic conditioning, counterpart of Bayes rule, is the least specific solution of Equation I.23:

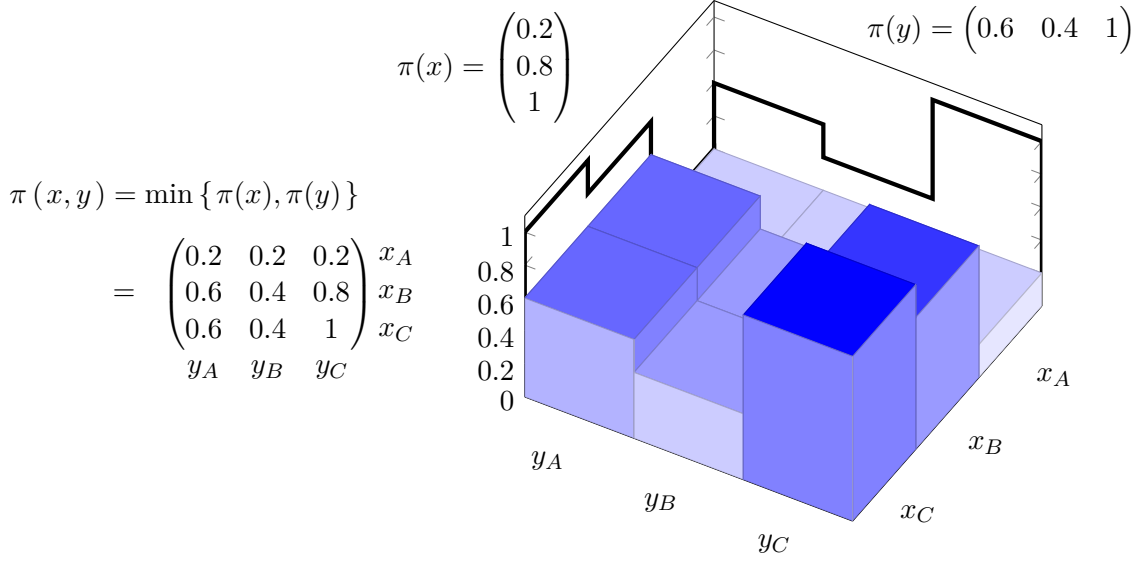


Figure I.10 – Example of a joint distribution over $\mathcal{X} \times \mathcal{Y} = \{x_A, x_B, x_C\} \times \{y_A, y_B, y_C\}$ when X and Y are NI-independent (no M-independence).

Definition I.2.7 (Qualitative Possibilistic Conditioning)

$\forall A, B \subset \Omega$, such that $\Pi(B) > 0$,

$$\Pi(A|B) = \begin{cases} 1 & \text{if } \Pi(B) = \Pi(A \cap B), \\ \Pi(A \cap B) & \text{otherwise.} \end{cases} \quad (\text{I.24})$$

The conditional possibility distributions of the variable $X \in \mathcal{X}$ knowing variable $Y \in \mathcal{Y}$ is thus, $\forall x \in \mathcal{X}$, $\forall y \in \mathcal{Y}$ such that $\pi(y) > 0$,

$$\pi(x|y) = \begin{cases} 1 & \text{if } \pi(y) = \pi(x, y), \\ \pi(x, y) & \text{otherwise.} \end{cases} \quad (\text{I.25})$$

where $\pi(x|y) = \Pi(X = x | Y = y)$.

The meaning of the conditioning (I.24) can be explained as follows: when the event A contains the elementary event which has the highest possibility degree in B , i.e. if $\Pi(B) = \Pi(A \cap B)$, then $\Pi(A|B) = 1$. Indeed, as conditioning on B assumes that the new set of possible elementary events (universe) is B , A has the maximal possibility degree among the new universe B : the possibility measure conditional on B is normalized, setting to 1 the possibility degree of most plausible events in B . As qualitative possibility distributions only define a ranking between the events, the possibility degree of events A such that $\Pi(A \cap B) < \Pi(B)$ (second case of Equation I.24) are simply set to $\Pi(A \cap B)$. As $\min\{\Pi(A|B), \Pi(B)\} = \Pi(A|B)$, Equation I.24 is true thanks to this choice.

Defining the qualitative possibilistic conditioning as previously (I.2.7), the non-interactivity independence (Definition I.2.6) between variables X and Y corresponds to the fact that $\forall x, y \in \mathcal{X} \times \mathcal{Y}$ either the knowledge that the variable X is equal to x does not decrease the possibility degree of $Y = y$, or the knowledge that the variable Y is equal to y does not decrease the possibility degree of $X = x$.

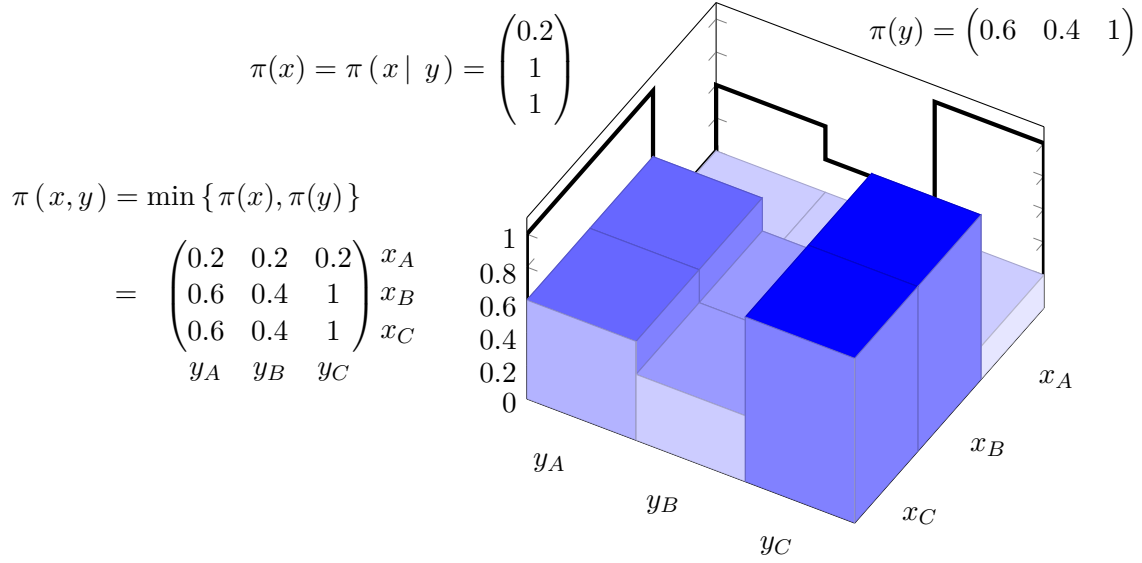


Figure I.11 – Example of a joint distribution over $\mathcal{X} \times \mathcal{Y} = \{x_A, x_B, x_C\} \times \{y_A, y_B, y_C\}$ when X is M -independent from Y .

Theorem 10

$$\forall (x, y) \in \mathcal{X} \times \mathcal{Y}, \quad \pi(x, y) = \min \{ \pi(x), \pi(y) \}$$

(non interactivity independence)



$$\forall (x, y) \in \mathcal{X} \times \mathcal{Y}, \quad \pi(x) \leq \pi(x | y) \text{ or } \pi(y) \leq \pi(y | x)$$

(either the knowledge of Y does not bring any knowledge about X , or the knowledge of X doesn't bring any knowledge about Y .)

The proof is given in Annex A.15.

Another independence called the *Min-based independence*, or M -independence, comes from the conditioning (I.2.7):

Definition I.2.8 (*Min-based “Causal” Independence*)

The event $A \subset \Omega$ is M -independent from the event $B \subset \Omega$ if

$$\Pi(A | B) = \Pi(A).$$

As well, the variable X is M -independent from the variable Y if their distributions are such that

$$\pi(x | y) = \pi(x).$$

Note that, using Theorem 10, it follows that the M -independence implies the NI -independence: it can be also observed replacing $\Pi(A | B)$ by $\Pi(A)$ in the equation I.23, as both possibility degrees are equal.

This independence is causal, and not symmetric. Figure I.11 displays an example of joint distribution $\pi(x, y)$ such that $\pi(x | y) = \pi(x)$. However $\pi(y | x) \neq \pi(y)$: indeed, if $X = x_A$,

Y is fully unknown, *i.e.* $\pi(y | x_A) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, if $X = x_B$, $\pi(y | x_B) = \begin{pmatrix} 0.6 & 0.4 & 1 \end{pmatrix}$, and the same for $X = x_C$. This illustrates the fact that the M-independence is not symmetric.

The symetrized version of the M-independence is the *Symetric Min-Based Independence* or MS-independence:

Definition I.2.9 (*Symetric Min-Based Independence*)

Variables X and Y are said to be MS-independent if X is M-independent from Y and Y is M-independent from X .

However, this symetric independence is too restrictive: one of the variable has to be entirely unknown, as highlighted by the following theorem.

Theorem 11

If X and Y are MS-independent, then $\forall x \in \mathcal{X}, \Pi(X = x) = 1$ or $\forall y \in \mathcal{Y}, \Pi(Y = y) = 1$.

The proof is given in Annex A.16.

Finally, we present a second qualitative possibilistic conditioning, proposed in [33], based on the previous one (Definition I.2.7): this one ensure that the posterior distribution is not less specific than the prior.

Definition I.2.10 (*Alternative Qualitative Possibilistic Conditioning*)

This alternative conditioning is denoted by $\pi(x||y)$ and is a modified version of the classical one (Definition I.2.7): $\forall (x, y) \in \mathcal{X} \times \mathcal{Y}$,

$$\pi(x||y) = \begin{cases} \pi(x) & \text{if } \pi(x' | y) \geq \pi(x'), \forall x' \in \mathcal{X}, \\ \pi(x | y) & \text{otherwise.} \end{cases}$$

A more general presentation of the various possibilistic conditioning and independence and their consequences on possibilistic graphical models is available in the thesis of N.Ben Amor [8].

This section just introduced that, as the studied models will be qualitative, the operators min and max will be present in most equations below. Some properties about these operators, which are used in the following sections, end this one.

Property I.2.1

Consider the functions $f : \Omega \rightarrow \mathcal{L}$ and $\lambda \in \mathcal{L}$:

$$\max_{\omega \in \Omega} \{1 - f(\omega)\} = 1 - \min_{\omega \in \Omega} f(\omega), \quad (\text{I.26})$$

$$\min_{\omega \in \Omega} \{1 - f(\omega)\} = 1 - \max_{\omega \in \Omega} f(\omega), \quad (\text{I.27})$$

$$\min_{\omega \in \Omega} \min \{f(\omega), \lambda\} = \min \left\{ \min_{\omega \in \Omega} f(\omega), \lambda \right\}, \quad (\text{I.28})$$

$$\min_{\omega \in \Omega} \max \{f(\omega), \lambda\} = \max \left\{ \min_{\omega \in \Omega} f(\omega), \lambda \right\}, \quad (\text{I.29})$$

$$\max_{\omega \in \Omega} \min \{f(\omega), \lambda\} = \min \left\{ \max_{\omega \in \Omega} f(\omega), \lambda \right\} \quad (\text{I.30})$$

$$\text{and } \operatorname{argmax}_{\omega \in \Omega} f(\omega) \subseteq \operatorname{argmax}_{\omega \in \Omega} \min \{f(\omega), \lambda\}, \quad (\text{I.31})$$

$$\max_{\omega \in \Omega} \max \{f(\omega), \lambda\} = \max \left\{ \max_{\omega \in \Omega} f(\omega), \lambda \right\}, \quad (\text{I.32})$$

$$\text{and } \operatorname{argmax}_{\omega \in \Omega} f(\omega) \subseteq \operatorname{argmax}_{\omega \in \Omega} \max \{f(\omega), \lambda\}. \quad (\text{I.33})$$

Let us introduce now $g : \Omega \rightarrow \mathcal{L}$ and suppose that $\exists \omega^* \in \Omega$ such that $g(\omega^*) = 0$:

$$\min_{\omega \in \Omega} \max \left\{ \min \{ \lambda, f(\omega) \}, g(\omega) \right\} = \min_{\omega \in \Omega} \min \left\{ \lambda, \max \{ f(\omega), g(\omega) \} \right\}. \quad (\text{I.34})$$

As well, if we introduce $h : \Omega \rightarrow \mathcal{L}$ and suppose that $\exists \omega^* \in \Omega$ such that $h(\omega^*) = 1$,

$$\max_{\omega \in \Omega} \min \left\{ \max \{ \lambda, f(\omega) \}, h(\omega) \right\} = \max_{\omega \in \Omega} \max \left\{ \lambda, \min \{ f(\omega), h(\omega) \} \right\}. \quad (\text{I.35})$$

The proof is given in Annex A.17.

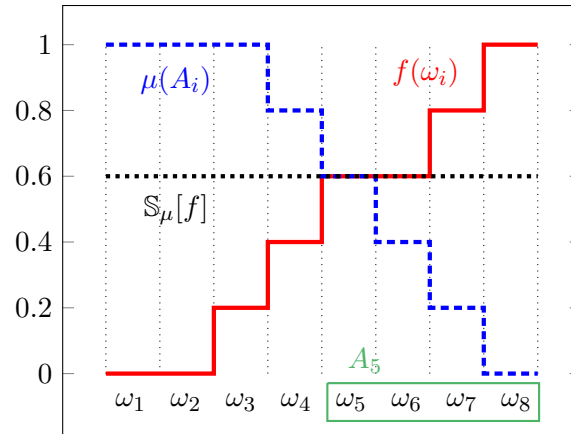


Figure I.12 – Illustration of the result of the Sugeno Integral: the x-axis represents the set $\Omega = \{\omega_1, \dots, \omega_{\#\Omega}\}$, where $\forall i \in \{1, \dots, \#\Omega - 1\}$, $f(\omega_i) \leq f(\omega_{i+1})$. The y-axis is \mathcal{L} . The red line represents the degrees $f(\omega_i)$, the dashed blue one represents the degrees $\mu(A_i)$ with $A_i = \{\omega_i, \dots, \omega_{\#\Omega}\}$, and the black dotted one is the result of the Sugeno Integral.

I.2.3 Qualitative Criteria

As detailed in the first section of this chapter, the criterion used to quantify the quality of a strategy for a given (PO)MDP is the expectation of the reward, *i.e.* the integral of the reward function with respect to the probability measure. The concept of Integral has been extended to the framework of fuzzy measures. When the measure is quantitative, the extension is called the *Choquet Integral* [28]. In the case of a qualitative measure, the resulting object is the *Sugeno Integral* [117]. As the planning framework that will be studied is qualitative, we define here the Sugeno Integral:

Definition I.2.11 (*Sugeno Integral*)

The Sugeno Integral of a function $f : \Omega \rightarrow \mathcal{L}$ with respect to the capacity (fuzzy measure) $\mu : 2^\Omega \rightarrow \mathcal{L}$ is

$$\mathbb{S}_\mu[f] = \max_{i=1}^{\#\Omega} \min \{ f(\omega_i), \mu(A_i) \} \quad (\text{I.36})$$

$$= \min_{i=1}^{\#\Omega} \max \{ f(\omega_i), \mu(A_{i+1}) \} \quad (\text{I.37})$$

where $f(\omega_1) \leq \dots \leq f(\omega_{\#\Omega})$, $A_i = \{\omega_i, \omega_{i+1}, \dots, \omega_{\#\Omega}\}$ and $A_{\#\Omega+1} = \emptyset$.

The proof of the equality is given in Annex A.18.

As illustrated by Figure I.12, the Sugeno integral of $f : \Omega \rightarrow \mathcal{L}$ with respect to the fuzzy measure μ is the highest degree $\lambda \in \mathcal{L}$ such that the measure μ of $\{\omega \mid f(\omega) \geq \lambda\}$ is higher or equal to λ . As an example, the *h-index* (or *Hirsh index*) is the Sugeno integral of the function $\text{paper} \mapsto \# \text{citations}$ with respect to the counting measure.

The Sugeno integral with respect to a possibility measure, and the one with respect to a necessity measure, lead to two criteria: an optimistic and a cautious one. The following theorem rewrites both integral in a more simple way:

Theorem 12 (*Sugeno Integrals with respect to Possibility and Necessity measures*)

$$\mathbb{S}_\Pi[f] = \max_{i=1}^{\#\Omega} \min \{ f(\omega_i), \pi(\omega_i) \} \quad (\text{optimistic}), \quad (\text{I.38})$$

$$\mathbb{S}_\mathcal{N}[f] = \min_{i=1}^{\#\Omega} \max \{ f(\omega_i), 1 - \pi(\omega_i) \} \quad (\text{cautious}). \quad (\text{I.39})$$

are rewritings of the Sugeno integrals with respect to possibility and necessity measures.

The proof is given in Annex A.19.

These integrals can be seen as the possibilistic expectations of the variable $f : \Omega \rightarrow \mathcal{L}$. Let us introduce the variable $S : \Omega \rightarrow \mathcal{S}$ whose possibility distribution is $\pi(s) = \Pi(\{S = s\}) = \Pi(\{\omega \in \Omega \mid S(\omega) = s\}) = \max_{\{\omega \in \Omega \mid S(\omega) = s\}} \pi(\omega)$. We can note for instance that the Sugeno Integral of the (classical) characteristic function of the event $\{S \in A\}$ with $A \subseteq \mathcal{S}$, namely

$\mathbb{1}_{\{S \in A\}}(\omega) = \begin{cases} 1 & \text{if } S(\omega) \in A \\ 0 & \text{otherwise} \end{cases}$, is equal to the possibility degree of this event:

$$\begin{aligned} \mathbb{S}_\Pi [\mathbb{1}_{\{S \in A\}}] &= \max_{\omega \in \Omega} \min \{ \mathbb{1}_{\{S \in A\}}(\omega), \pi(\omega) \} \\ &= \max_{s \in \mathcal{S}} \max_{\{ \omega \in \Omega \mid S(\omega) = s \}} \min \{ \mathbb{1}_A(s), \pi(\omega) \} \\ &= \max_{s \in \mathcal{S}} \min \{ \mathbb{1}_A(s), \pi(s) \} \end{aligned} \quad (\text{I.40})$$

$$\begin{aligned} &= \max_{s \in \mathcal{S}} \begin{cases} \pi(s) & \text{if } s \in A \\ 0 & \text{otherwise} \end{cases} \\ &= \max_{s \in A} \pi(s) = \Pi(S \in A). \end{aligned} \quad (\text{I.41})$$

where line (I.40) comes from equation (I.30) of Property (I.2.1). In the same way,

$$\begin{aligned} \mathbb{S}_\mathcal{N} [\mathbb{1}_{\{S \in A\}}] &= \min_{s \in \mathcal{S}} \max \{ \mathbb{1}_A(s), 1 - \pi(s) \} \\ &= \min_{s \in \mathcal{S}} \{ 1 - \min \{ 1 - \mathbb{1}_A(s), \pi(s) \} \} \end{aligned} \quad (\text{I.42})$$

$$\begin{aligned} &= 1 - \max_{s \in \mathcal{S}} \min \{ \mathbb{1}_{\bar{A}}(s), \pi(s) \} \\ &= 1 - \max_{s \in \bar{A}} \pi(s) = 1 - \Pi(S \in \bar{A}) = \mathcal{N}(\{S \in A\}). \end{aligned} \quad (\text{I.43})$$

where lines (I.42) and (I.43) come from equation (I.26) and (I.27) of Property (I.2.1). These remarks are the counterparts of the probabilistic equality $\mathbb{E} [\mathbb{1}_{\{S \in A\}}] = \mathbb{P}(\{S \in A\})$.

Qualitative Possibilistic Decision Criteria, *i.e.* functions $\mathcal{A} \rightarrow \mathcal{L}$ measuring the accuracy of actions given a possibilistic and a preference model, have been proposed in [104, 52, 50], based on Sugeno integrals (I.38) and (I.39). Let us recall that the set \mathcal{S} (resp. \mathcal{A}) is as previously the finite set of system states s (resp. of actions a). The variable representing the system state is $S \in \mathcal{S}$. Let $(\pi_a)_{a \in \mathcal{A}}$ be a family of possibility distributions over \mathcal{S} , *i.e.* $\forall a \in \mathcal{A}$, $\pi_a(s) = \Pi_a(\{S = s\})$ is the possibility degree of the situation $\{S = s\} \subset \Omega$ after selecting action $a \in \mathcal{A}$. Let function $\rho : \mathcal{S} \rightarrow \mathcal{L}$ be the preference function, defining the preference degree of each system state $s \in \mathcal{S}$.

Definition I.2.12 (Qualitative Decision Criteria)

Let π_a be the possibility distribution describing the uncertainty about the system state given that action $a \in \mathcal{A}$ has been selected, and $\rho(s)$ the preference of the system state $s \in \mathcal{S}$. Using the formula (I.38) with $f = \rho(S)$, the Sugeno integral of the preference with respect to the possibility measure Π_a leads to an optimistic criteria for $a \in \mathcal{A}$:

$$\mathbb{S}_{\Pi_a}[\rho(S)] = \max_{s \in \mathcal{S}} \min \{ \rho(s), \pi_a(s) \}. \quad (\text{I.44})$$

As well, using the formula (I.39) with $f = \rho(S)$, the Sugeno integral of the preference with respect to the necessity measure associated to Π_a , \mathcal{N}_a , leads to a cautious criteria for $a \in \mathcal{A}$:

$$\mathbb{S}_{\mathcal{N}_a}[\rho(S)] = \min_{s \in \mathcal{S}} \max \{ \rho(s), 1 - \pi_a(s) \}. \quad (\text{I.45})$$

These criteria can be understood best with the fuzzy sets vision: a possibility distribution $\pi_a : \mathcal{S} \rightarrow \mathcal{L}$ is the characteristic (or membership) function of the fuzzy set of the possible system states after selecting $a \in \mathcal{A}$, denoted by \mathfrak{T}^a *i.e.* $\pi(s) = \mathbb{1}_{\mathfrak{T}^a}(s)$. The preference function

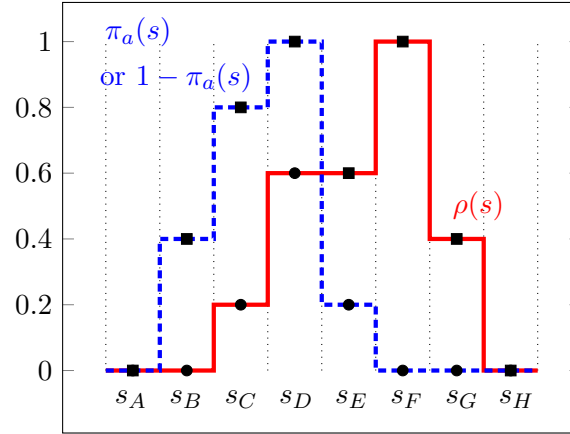


Figure I.13 – Illustration of the qualitative criteria. The solid red line is $\rho(s) = \mathbb{1}_{\mathbf{R}}(s)$. The dashed blue line may represent $\pi_a(s) = \mathbb{1}_{\mathbf{T}^a}(s)$, and the black circles represent the characteristic function of the intersection $\mathbb{1}_{\mathbf{T}^a \cap \mathbf{R}}(s) = \min\{\rho(s), \pi_a(s)\}$. Actions maximizing the optimistic criterion (I.44), maximizes the highest membership degree $\max_{s \in \mathcal{S}} \mathbb{1}_{\mathbf{T}^a \cap \mathbf{R}}(s)$, which is here equal to 0.6, the degree of s_D . The dashed blue line may also represent $1 - \pi_a(s) = \mathbb{1}_{\overline{\mathbf{T}^a}}(s)$, and the black squares represent the characteristic function of the union $\mathbb{1}_{\overline{\mathbf{T}^a} \cup \mathbf{R}}(s) = \max\{\rho(s), 1 - \pi_a(s)\}$. Actions maximizing the pessimistic criterion (I.45), maximizes the lowest membership degree $\min_{s \in \mathcal{S}} \mathbb{1}_{\overline{\mathbf{T}^a} \cup \mathbf{R}}(s)$, which is here equal to 0, as s_A and s_H are totally possible and unpleasant.

$\rho : \mathcal{S} \rightarrow \mathcal{L}$ can also be viewed as the characteristic function of the fuzzy set denoted by \mathbf{R} representing the preferred system states: $\rho(s) = \mathbb{1}_{\mathbf{R}}(s)$. Finally, the characteristic function of the fuzzy set of plausible and preferred states after selecting action $a \in \mathcal{A}$, i.e. $\mathbf{T}^a \cap \mathbf{R}$, is $\mathbb{1}_{\mathbf{T}^a \cap \mathbf{R}} = \min\{\mathbb{1}_{\mathbf{T}^a}, \mathbb{1}_{\mathbf{R}}\} = \min\{\pi_a, \rho\}$.

An action $a \in \mathcal{A}$ maximizing the optimistic criterion (I.44), is thus an action that maximizes the highest membership degree of $\mathbf{T}^a \cap \mathbf{R}$, i.e. of the fuzzy set of possible and preferred states. This criterion is optimistic because it maximizes the degree of the best situation, but does not ensure that unwanted states are avoided by the system.

The characteristic function of the complementary set of \mathbf{T}^a , denoted by $\overline{\mathbf{T}^a}$ is $\mathbb{1}_{\overline{\mathbf{T}^a}} = 1 - \mathbb{1}_{\mathbf{T}^a} = 1 - \pi_a$: the fuzzy set of implausible system state. An action $a \in \mathcal{A}$ maximizing the pessimistic criterion (I.45), is thus an action that maximizes the lowest membership degree of $\overline{\mathbf{T}^a} \cup \mathbf{R}$, i.e. of the fuzzy set of the system states which are implausible or preferred. An action that maximizes the lowest degree of this fuzzy set, tries to make all the system states either implausible or preferred, i.e. to ensure that if any system state is plausible it is preferred: it maximizes the “degree” of the inclusion $\mathbf{T} \subseteq \mathbf{R}$.

Note that, for a function $f : \mathcal{A} \rightarrow \mathcal{L}$, $\operatorname{argmax}_{a \in \mathcal{A}} f(a) = 1 - \operatorname{argmin}_{a \in \mathcal{A}} f(a)$, which can be shown as equation (I.27) of Property I.2.1. Thus,

$$\operatorname{argmax}_{a \in \mathcal{A}} \left\{ \min_{s \in \mathcal{S}} \max\{\rho(s), 1 - \pi_a(s)\} \right\} = \operatorname{argmin}_{a \in \mathcal{A}} \left\{ 1 - \min_{s \in \mathcal{S}} \max\{\rho(s), 1 - \pi_a(s)\} \right\}.$$

As $1 - \min_{s \in \mathcal{S}} \max\{\rho(s), 1 - \pi_a(s)\} = \max_{s \in \mathcal{S}} \{1 - \max\{\rho(s), 1 - \pi_a(s)\}\} = \max_{s \in \mathcal{S}} \min\{1 - \rho(s), \pi_a(s)\}$ (see equations (I.26) and (I.27) of Property I.2.1), the action $a \in \mathcal{A}$ minimizes the highest membership degree of the fuzzy set $\overline{\mathbf{T}^a} \cap \mathbf{R}$, i.e. the fuzzy set of plausible and unwanted system states: the pessimistic criterion tries to keep down all membership degrees of this set.

The pessimistic criterion (I.45) tends to avoid unwanted system states, whereas the optimistic criterion (I.44) wants to make possible that the system reaches preferred ones. Figure (I.13) illustrates the result of the criteria for a given action $a \in \mathcal{A}$.

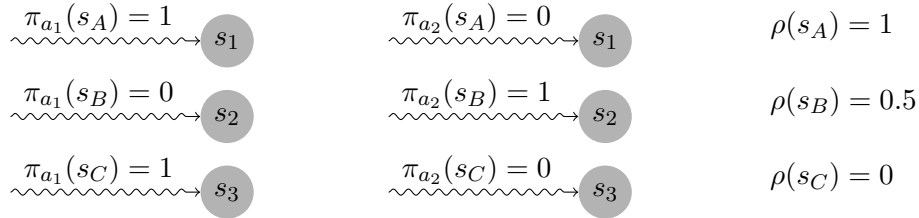


Figure I.14 – Illustration of the situation in the example of Section I.2.3 about qualitative criteria. Action a_1 maximizes the optimistic criterion (I.44), which can lead to the best state (s_A), but also the worst (s_C). On the contrary, action a_2 maximizes the pessimistic criterion (I.45) since the worst state is not reachable with this action.

The following toy example, illustrated in Figure I.14 is meant to present the two criteria in practice: let the set of system states \mathcal{S} be $\{s_A, s_B, s_C\}$ and the set of actions $\mathcal{A} = \{a_1, a_2\}$. The preference model and the uncertainty model are described respectively by ρ and $(\pi_a)_{a \in \mathcal{A}}$:

- $1 = \rho(s_A) > \rho(s_B) > \rho(s_C) = 0$;
- selecting action a_1 , $\pi_{a_1}(s_A) = \pi_{a_1}(s_C) = 1$, and $\pi(s_B) = 0$;
- selecting action a_2 , $\pi_{a_2}(s_A) = \pi_{a_2}(s_C) = 0$, and $\pi(s_B) = 1$, *i.e.* the system is in state s_B deterministically.

As $\min\{\rho(s), \pi_{a_1}(s)\} = \begin{cases} 1 & \text{if } s = s_A, \\ 0 & \text{otherwise.} \end{cases}$, the optimistic criterion is equal to $S_{\Pi_{a_1}}[r(S)] = 1$

for a_1 . Now, as $\min\{\rho(s), \pi_{a_2}(s)\} = \begin{cases} \rho(s_B) & \text{if } s = s_B, \\ 0 & \text{otherwise.} \end{cases}$, the optimistic criterion is equal to $S_{\Pi_{a_2}}[r(S)] = \rho(s_B)$ for a_2 . Thus, as $\rho(s_B) < 1$, a_1 maximizes the optimistic criterion (I.44).

The optimistic criterion is maximized by the action a_1 , because with this action, the best system state, s_A , is entirely possible. However, this action makes also the worst system state, s_C entirely possible: state s_A is not necessary at all: $\mathcal{N}(\{s_A\}) = 1 - \pi(\{s_B, s_C\}) = 1 - \max(\pi(s_B), \pi(s_C)) = 0$. A more cautious action is a_2 , whose the preference of the reached state (s_B) is lower than 1, but certain.

As expected, the action a_2 maximizes the cautious criterion (I.45): $\max\{\rho(s), 1 - \pi_{a_1}(s)\} = \begin{cases} 1 & \text{if } s = s_A \text{ or } s_B, \\ 0 & \text{otherwise.} \end{cases}$ and then the cautious criterion is equal to 0 for a_1 . It is greater with

a_2 : $\max\{\rho(s), \pi_{a_2}(s)\} = \begin{cases} 1 & \text{if } s = s_A \text{ or } s_C, \\ \rho(s_B) & \text{otherwise.} \end{cases}$ and thus the criterion is equal to $\rho(s_B) > 0$ for a_2 . This choice is more cautious since $\mathcal{N}_{a_2}(\{s_B\}) = 1 - \Pi(\{s_A, s_C\}) = 1$, *i.e.* the preference of the state will be $\rho(s_B)$ with certainty.

This section ends with a remark about this qualitative framework: possibility degrees are compared to preference degrees in the presented criteria. This requests a *commensurability* assumption, *i.e.* these comparisons must mean something. When values of ρ are in $\{0, 1\}$, this assumption is not necessary as system states with a preference of 1 are the goals, and other states are not. How to model problems in practice with these settings will be detailed in experimental parts.

I.2.4 π -MDPs

This model, presented in [103, 102, 101], is a qualitative possibilistic version of the probabilistic MDPs detailed in Section I.1.2, based on the criteria (optimistic and pessimistic) presented

in Section I.2.3: this version is called *Qualitative Possibilistic Markov Decision Process*, or π -MDP.

The finite set of system states, describing the agent and its environment, remains denoted by \mathcal{S} , as seen in Section I.1.2 where probabilistic MDPs are presented. The finite set of action is always \mathcal{A} and \mathcal{L} is the possibility scale $\left\{0, \frac{1}{k}, \dots, 1\right\}$, with $k \geq 2$.

As in the probabilistic case, this model considers that successive system states, represented by the sequence of variables $(S_t)_{t \in \mathbb{N}}$ with $S_t \in \mathcal{S} \forall t \geq 0$, is Markovian. In this qualitative possibilistic framework, it means that the sequence $(S_t)_{t \in \mathbb{N}}$ is such that $\forall t \geq 0, \forall (s_0, s_1, \dots, s_{t+1}) \in \mathcal{S}^{t+2}$ and for each action sequence of action $(a_t)_{t \geq 0} \in \mathcal{A}^{\mathbb{N}}$, S_{t+1} is M -independent (see Definition I.2.8) from variables $\{S_0, \dots, S_{t-1}\}$, conditioned on $\{S_t = s\}$ and a_t :

$$\Pi\left(S_{t+1} = s_{t+1} \mid S_t = s_t, a_t\right) = \Pi\left(S_{t+1} = s_{t+1} \mid S_t = s_t, S_{t-1} = s_{t-1}, \dots, S_0 = s_0, (a_t)_{t \geq 0}\right). \quad (\text{I.46})$$

Using this property, dynamics of the system are fully described with possibilistic transitions $\pi_t(s' \mid s, a) = \Pi\left(S_{t+1} = s' \mid S_t = s, a\right) \in \mathcal{L} : \forall t \geq 0, (s, s') \in \mathcal{S}^2$ and $a \in \mathcal{A}$, $\pi_t(s' \mid s, a)$ is the possibility degree, that at time step t , the system reaches the state s' when the agent selects action a , conditioned to the fact that the current state is s . Finally a π -MDP is entirely defined with the sequence of preference functions $(\rho_t)_{t=0}^{H-1}$, where $\forall s \in \mathcal{S}, \forall a \in \mathcal{A}, \rho_t(s, a)$ is the preference degree when the system state is s and the agent selects action a at time step t . The final preference function, Ψ , gives for each system state $s \in \mathcal{S}$, the preference degree if $S_H = s$: $\Psi(s)$. With the previous notations for the preference functions, $\Psi(s) = \rho_H(s, a)$, $\forall s \in \mathcal{S}, a \in \mathcal{A}$. Figure I.2 in the MDP section (Section I.1.2) is a good representation of a π -MDP, replacing rewards by preferences, and probability distributions by possibility distributions.

In order to easily derive the MDP criteria from the qualitative possibilistic criteria (I.44) and (I.45), let us introduce, for an horizon $H \geq 0$, a H -length trajectory $\mathcal{T} = (s_1, \dots, s_H)$, and $\mathcal{T}_H = \mathcal{S}^H$ the set of such trajectories. A decision rule is denoted by $\delta : \mathcal{S} \rightarrow \mathcal{L}$, and a H -length strategy is a sequence of decision rules $\delta_t : (\delta_t)_{t=0}^{H-1}$. The set of all the H -length strategies is denoted by Δ_H . In [100], for a given H -length strategy $(\delta) \in \Delta_H$, a given sequence of system states $\mathcal{T} = (s_1, \dots, s_H) \in \mathcal{T}_H$, and a given initial state $s_0 \in \mathcal{S}$, the *preference of an H -length trajectory from s_0* is defined as the lowest preference degree along s_0 and the trajectory:

$$\rho(\mathcal{T}, (\delta)) = \min \left\{ \min_{t=0}^{H-1} \rho(s_t, \delta_t(s_t)), \Psi(s_H) \right\}.$$

This is the possibilistic counterpart of the sum $\sum_{t=0}^{H-1} r_t(S_t, d_t(S_t)) + R(S_H)$, reward aggregation of the probabilistic framework. Note that any preference aggregation of the qualitative possibilistic framework has to result in a degree $\lambda \in \mathcal{L}$.

Using the Markov property of this system state process, for a given initial system state $s_0 \in \mathcal{S}$, a horizon $H \in \mathbb{N}$, and a strategy $(\delta_t)_{t=0}^{H-1}$, the possibility degree of the trajectory $\mathcal{T} = (s_1, \dots, s_H)$ is

$$\Pi\left(S_H = s_H, S_{H-1} = s_{H-1}, \dots, S_1 = s_1 \mid S_0 = s_0, (\delta_t)_{t=0}^{H-1}\right) = \min_{t=0}^{H-1} \pi_{t+1}(s_{t+1} \mid s_t, \delta_t(s_t)) \quad (\text{I.47})$$

denoted by $\pi(\mathcal{T} \mid s_0, (\delta))$.

The Sugeno integral of the preference of the trajectory with respect to this distribution is denoted by

$$\mathbb{S}_{\Pi} \left[\rho(\mathcal{T}, (\delta)) \mid S_0 = s_0, (\delta) \right] = \mathbb{S}_{\Pi} \left[\min \left\{ \min_{t=0}^{H-1} \rho(s_t, \delta_t(s_t)), \Psi(s_H) \right\} \mid S_0 = s_0, (\delta) \right]$$

and defines the optimistic criterion defining optimal strategies, *i.e.* an optimistic value function:

$$\overline{U}_H(s_0, (\delta_t)_{t=0}^{H-1}) = \max_{\mathcal{T} \in \mathcal{T}_H} \min \left\{ \rho(\mathcal{T}, (\delta)), \pi(\mathcal{T} | s_0, (\delta)) \right\}. \quad (\text{I.48})$$

It is equivalent to the optimistic qualitative possibilistic criterion (I.44), however, the expectation is over trajectories \mathcal{T}_H , and the preference depends on the strategy. The *optimal optimistic strategy* $\bar{\delta}^*$ is the strategy maximizing the optimistic value function (I.48), and the *optimal optimistic value function* is the maximal optimistic value function among strategies $(\delta) \in \Delta_H$:

Definition I.2.13 (Optimal Optimistic Value Function and Strategy)

$\forall s \in \mathcal{S}$,

$$\overline{U}_H^*(s) = \max_{(\delta) \in \Delta_H} \left\{ \overline{U}_H(s, (\delta)) \right\} \quad (\text{optimal optimistic value function}), \quad (\text{I.49})$$

$$\bar{\delta}^*(s) \in \operatorname{argmax}_{(\delta) \in \Delta_H} \left\{ \overline{U}_H(s, (\delta)) \right\} \quad (\text{optimal optimistic strategy}). \quad (\text{I.50})$$

As well, the pessimistic qualitative possibilistic criterion (I.45) leads to a cautious criterion for strategies: the pessimistic value function is the Sugeno integral of the preference trajectory with respect to the necessity measure which comes from the possibility distribution over trajectories \mathcal{T}_H (I.47) with the strategy $(\delta) \in \Delta_H$:

$$\underline{U}_H(s_0, (\delta_t)_{t=0}^{H-1}) = \min_{\mathcal{T} \in \mathcal{T}_H} \max \left\{ \rho(\mathcal{T}, (\delta)), 1 - \pi(\mathcal{T} | s_0, (\delta)) \right\}. \quad (\text{I.51})$$

denoted by $\mathbb{S}_N \left[\rho(\mathcal{T}, (\delta)) \mid S_0 = s, (\delta) \right]$. As previously for the optimistic case, the *optimal cautious strategy* $\underline{\delta}^*$ is the strategy maximizing the pessimistic value function (I.51), and the *optimal pessimistic value function* is the maximal pessimistic value function among strategies $(\delta) \in \Delta_H$:

Definition I.2.14 (Optimal Pessimistic Value Function and Strategy)

$\forall s \in \mathcal{S}$,

$$\underline{U}_H^*(s) = \max_{(\delta) \in \Delta_H} \left\{ \underline{U}_H(s, (\delta)) \right\} \quad (\text{optimal pessimistic value function}), \quad (\text{I.52})$$

$$\underline{\delta}^*(s) \in \operatorname{argmax}_{(\delta) \in \Delta_H} \left\{ \underline{U}_H(s, (\delta)) \right\} \quad (\text{optimal pessimistic strategy}). \quad (\text{I.53})$$

As for the probabilistic MDPs (see Section I.1.3 Theorem 1), optimal value functions and strategies can be computed with Dynamic Programming:

Theorem 13 (Dynamic Programming for π -MDPs)

The optimal optimistic criterion and an associated optimal strategy can be computed as follows: $\forall s \in \mathcal{S}$,

$$\begin{aligned} \overline{U}_0^*(s) &= \Psi(s), \quad \text{and, } \forall 1 \leq i \leq H, \\ \overline{U}_i^*(s) &= \max_{a \in \mathcal{A}} \min \left\{ \rho_{H-i}(s, a), \max_{s' \in \mathcal{S}} \min \left\{ \pi_{H-i}(s' \mid s, a), \overline{U}_{i-1}^*(s') \right\} \right\}. \end{aligned} \quad (\text{I.54})$$

$$\overline{\delta_{H-i}^*}(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \min \left\{ \rho_{H-i}(s, a), \max_{s' \in \mathcal{S}} \min \left\{ \pi_{H-i}(s' | s, a), \overline{U_{i-1}^*}(s') \right\} \right\}. \quad (\text{I.55})$$

As well, the optimal pessimistic criterion and an associated optimal strategy can be computed as follows: $\forall s \in \mathcal{S}$,

$$\begin{aligned} \underline{U}_0^*(s) &= \Psi(s), \quad \text{and, } \forall 1 \leq i \leq H, \\ \underline{U}_i^*(s) &= \max_{a \in \mathcal{A}} \min \left\{ \rho_{H-i}(s, a), \min_{s' \in \mathcal{S}} \max \left\{ 1 - \pi_{H-i}(s' | s, a), \underline{U_{i-1}^*}(s') \right\} \right\}. \end{aligned} \quad (\text{I.56})$$

$$\underline{\delta_{H-i}^*}(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \min \left\{ \rho_{H-i}(s, a), \min_{s' \in \mathcal{S}} \max \left\{ 1 - \pi_{H-i}(s' | s, a), \underline{U_{i-1}^*}(s') \right\} \right\}. \quad (\text{I.57})$$

The proof is given in Annex A.20.

In this theorem, the horizon i is the opposite modulo H of the stage of the process t : during execution, $\delta_t = \delta_{H-i}$ is used at time step t , *i.e.* when it remains i steps. These Dynamic Programming formulae lead to the optimistic algorithm, Algorithm 4 and the pessimistic one Algorithm 5, qualitative possibilistic counterpart of Algorithm 1 for finite horizon probabilistic MDPs.

Algorithm 4: Dynamic Programming Algorithm for Optimistic π -MDP

```

1  $\overline{U}_0^* \leftarrow \Psi$ ;
2 for  $i \in \{1, \dots, H\}$  do
3   for  $s \in \mathcal{S}$  do
4      $\overline{U}_i^*(s) \leftarrow \max_{a \in \mathcal{A}} \min \left\{ \rho_{H-i}(s, a), \max_{s' \in \mathcal{S}} \min \left\{ \pi_{H-i}(s' | s, a), \overline{U_{i-1}^*}(s') \right\} \right\}$ ;
5      $\overline{\delta_{H-i}^*}(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \min \left\{ \rho_{H-i}(s, a), \max_{s' \in \mathcal{S}} \min \left\{ \pi_{H-i}(s' | s, a), \overline{U_{i-1}^*}(s') \right\} \right\}$ ;
6 return  $\overline{U}_H^*, \overline{\delta^*}$ ;
```

Algorithm 5: Dynamic Programming Algorithm for Pessimistic π -MDP

```

1  $\underline{U}_0^* \leftarrow \Psi$ ;
2 for  $i \in \{1, \dots, H\}$  do
3   for  $s \in \mathcal{S}$  do
4      $\underline{U}_i^*(s) \leftarrow \max_{a \in \mathcal{A}} \min \left\{ \rho_{H-i}(s, a), \min_{s' \in \mathcal{S}} \max \left\{ 1 - \pi_{H-i}(s' | s, a), \underline{U_{i-1}^*}(s') \right\} \right\}$ ;
5      $\underline{\delta_{H-i}^*}(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \min \left\{ \rho_{H-i}(s, a), \min_{s' \in \mathcal{S}} \max \left\{ 1 - \pi_{H-i}(s' | s, a), \underline{U_{i-1}^*}(s') \right\} \right\}$ ;
6 return  $\underline{U}_H^*, \underline{\delta^*}$ ;
```

Note that a broader class of MDP models, including both probabilistic and qualitative possibilistic MDPs presented above, is called *Algebraic MDPs* [89].

The next section is devoted to the presentation of the qualitative possibilistic counterpart of the POMDPs denoted by π -POMDPs: the π -POMDP model is the partially observable version of the π -MDP one. This model has been presented first in [101] in pessimistic settings. The algorithm to solve it has been also presented in case no intermediate preference degree

is involved *i.e.* in case where preference functions ρ_t are not used: in these settings only the terminal preference function Ψ models the goal of the mission: an optimistic strategy maximizes the plausibility of strategies which end with a good preference, and a cautious strategy minimizes the plausibility of strategies ending in unwanted states. As the preference of a system state trajectory $\mathcal{T} = (s_1, \dots, s_H)$ is simply $\rho(\mathcal{T}) = \Psi(s_H)$, while the preference of a π -MDP trajectory is $\left\{ \min_{t=0}^{H-1} \rho(s_t, a_t), \Psi(s_H) \right\}$, it is sufficient to consider a classical π -MDP such that $\rho_t(s, a) = 1, \forall s \in \mathcal{S}, \forall a \in \mathcal{A}$ and $\forall t \in \{0, \dots, H-1\}$. The π -MDP criteria are simplified as follows:

Definition I.2.15 (Criteria for π -MDP with Terminal Preference Only)

The optimistic (resp. pessimistic) criterion is the Sugeno integral of the last state preference $\Psi(s_H)$ with respect to the possibility measure (resp. necessity measure) associated to the possibility distribution

$$\Pi \left(S_H = s_H \mid S_0 = s_0, (\delta_t)_{t=0}^{H-1} \right) = \max_{(s_1, \dots, s_{H-1}) \in \mathcal{S}^{H-1}} \min_{t=0}^{H-1} \pi \left(s_{t+1} \mid s_t, \delta_t(s_t) \right)$$

denoted by $\pi(s_H \mid s_0, (\delta))$: these integrals may be denoted by $\mathbb{S}_\Pi \left[\Psi(s_H) \mid S_0 = s_0, (\delta) \right]$ (resp. $\mathbb{S}_\mathcal{N} \left[\Psi(s_H) \mid S_0 = s_0, (\delta) \right]$).

Optimistic criterion:

$$\overline{U}_H \left(s_0, (\delta_t)_{t=0}^{H-1} \right) = \max_{s_H \in \mathcal{S}} \min \left\{ \Psi(s_H), \pi(s_H \mid s_0, (\delta)) \right\}. \quad (\text{I.58})$$

Pessimistic criterion:

$$\underline{U}_H \left(s_0, (\delta_t)_{t=0}^{H-1} \right) = \min_{s_H \in \mathcal{S}} \max \left\{ \Psi(s_H), 1 - \pi(s_H \mid s_0, (\delta)) \right\}. \quad (\text{I.59})$$

In this case the π -MDP focuses on the preference over terminal states, whatever the intermediate ones.

I.2.5 π -POMDPs

The qualitative possibilistic POMDP (π -POMDP) model has been first presented in [101]. As explained in Section I.1.6 which presents the classical probabilistic POMDP model, in partially observable settings the system state is not given anymore as input to the agent: the agent has to infer it using the observations $o \in \mathcal{O}$ received at each time step, represented by the observation process $(O_t)_{t \in \mathbb{N}}$. The uncertainty about successive observation variables O_t only depends on the current action and the reached state: if the agent selected action $a \in \mathcal{A}$ at time step t , and the system has reached state $s' \in \mathcal{S}$ at time step $t+1$, the observation $o' \in \mathcal{O}$ is received with possibility degree $\pi_t(o' \mid s', a) = \Pi(O_{t+1} = o' \mid S_{t+1} = s', a)$: conditional to the next system state s' and the current action a , the next observation variable is M-independent (see Definition I.2.9) from all other variables. Figure I.3 of Section I.1.6 illustrates just as well the dynamic and the structure of a π -POMDP: however, the rewards r and R have to be replaced by preferences ρ and Ψ , and transition (resp. observation) probability distributions \mathbf{p} have to be replaced by the possibility distribution $\pi_t(s' \mid s, a)$ (resp. $\pi_t(o' \mid s', a)$).

As with the probabilistic model, the computation of strategies is performed by translating of the π -POMDP into a fully observable π -MDP. The state space of the later is the set of possible *qualitative possibilistic belief states* $\beta : \mathcal{S} \rightarrow \mathcal{L}$ describing the knowledge about the actual system state, *i.e.* the set of all the possibility distributions over \mathcal{S} . This set is denoted

by $\Pi_{\mathcal{L}}^{\mathcal{S}} = \{\pi : \mathcal{S} \rightarrow \mathcal{L} \mid \max_{s \in \mathcal{S}} \pi(s) = 1\}$. Note first that the number of possible possibilistic beliefs about the actual system state is

$$(\#\mathcal{L})^{\#\mathcal{S}} - (\#\mathcal{L} - 1)^{\#\mathcal{S}}. \quad (\text{I.60})$$

Indeed, there are $\#\mathcal{L}^{\#\mathcal{S}}$ different functions from \mathcal{S} to \mathcal{L} , and $(\#\mathcal{L} - 1)^{\#\mathcal{S}}$ non-normalized ones *i.e.* functions $f : \mathcal{S} \rightarrow \mathcal{L}$ such that $\max_{s \in \mathcal{S}} f(s) < 1$. The number of possibility distributions over \mathcal{S} is the number of normalized functions from \mathcal{S} to \mathcal{L} , that is the total number of functions minus the number of non-normalized ones.

First of all, let us formally defined a π -POMDP as the 7-uple $\langle \mathcal{S}, \mathcal{A}, \mathcal{O}, T^{\pi}, O^{\pi}, \Psi, \beta_0 \rangle$:

- \mathcal{S} , a finite set of hidden system states;
- \mathcal{A} a finite set of actions;
- \mathcal{O} a finite set of observations;
- T^{π} the set of transition possibility distributions containing for each time step $t \in \mathbb{N}$, each current system state $s \in \mathcal{S}$ and each current action $a \in \mathcal{A}$, the possibility distributions over the next system state $s' \in \mathcal{S}$, $\pi_t(s' \mid s, a)$;
- O^{π} , the set of observation possibility distributions: for each time step $t \in \mathbb{N}$, each current action $a \in \mathcal{A}$, each next state s' , the possibility distribution over the next observation $o' \in \mathcal{O}$, $\pi_t(o' \mid s', a)$ is part of O^{π} .
- Ψ the preference function, defining for each state $s \in \mathcal{S}$, the preference assigned to the situations where the system terminates in state s .
- β_0 , the possibilistic *initial belief state*, is the possibility distribution defining the uncertainty about the initial state: $\forall s \in \mathcal{S}, \beta_0(s) = \Pi(S_0 = s)$.

At each time step the current qualitative possibilistic belief state is computed from these objects: the possibilistic counterpart of the probabilistic belief defined in Definition I.1.3 of Section I.1.6. The initial belief state $\beta_0 \in \Pi_{\mathcal{L}}^{\mathcal{S}}$ is part of the definition of a π -POMDP. At a time step $t \geq 1$, the belief state is the possibility distribution over the current system state, conditional to all the data available to the agent.

Definition I.2.16 (*Qualitative Possibilistic Belief state*)

$$\beta_t(s) = \Pi(S_t = s \mid O_1 = o_1, \dots, O_t = o_t, a_0, \dots, a_{t-1}) = \Pi(S_t = s \mid I_t = i_t) \quad (\text{I.61})$$

where $i_t = \{o_1, \dots, o_t, a_0, \dots, a_{t-1}\}$ is the information available to the agent at time t ($i_0 = \{\} = \emptyset$), and I_t the variable version (as in the probabilistic POMDP presentation).

The possibilistic belief updating process consists in the sequence of belief states, which can be computed recursively:

Theorem 14 (*Qualitative Possibilistic Belief Update*)

If the belief state at time step t is β_t , the selected action is $a_t \in \mathcal{A}$, and the next observation is o_{t+1} , the next belief state β_{t+1} is computed as follows:

$$\beta_{t+1}(s') = \begin{cases} 1 & \text{if } \pi_t(s', o_{t+1} \mid \beta_t, a_t) = \pi_t(o_{t+1} \mid \beta_t, a_t), \\ \pi_t(s', o_{t+1} \mid \beta_t, a_t) & \text{otherwise.} \end{cases} \quad (\text{I.62})$$

where the joint distribution over system state variable S_{t+1} and observation variable O_{t+1} conditional on the current information, is denoted by $\pi_t(s', o' \mid \beta_t, a_t) =$

$\min \left\{ \pi_t(o' \mid s', a_t), \max_{s \in \mathcal{S}} \min \left\{ \pi_t(s' \mid s, a_t), \beta_t(s) \right\} \right\}$. The notation $\pi(o' \mid \beta_t, a_t)$ is also used for $\max_{s' \in \mathcal{S}} \pi_t(s', o' \mid \beta_t, a_t)$.

This formula is called the **possibilistic belief update**, and since the belief state β_{t+1} is shown to be a function of β_t , a_t and o_{t+1} , we denote it by

$$\beta_{t+1} = \nu(\beta_t, a_t, o_{t+1}),$$

with ν called the belief update function.

The proof is given in Annex A.21.

The possibilistic belief update (I.62) is denoted by

$$\beta_{t+1}(s') \propto^\pi \pi_t(s', o_{t+1} \mid \beta_t, a_t)$$

as it only consists in normalizing the function $s' \mapsto \pi(s', o_{t+1} \mid \beta_t, a_t)$ in a possibilistic sense ($\max_s \pi(s) = 1$).

We denote by B_t^π the belief state when considered as a variable, i.e. B_0^π is deterministic equal to β_0 (but S_0 is uncertain with possibility distribution β_0) and $B_{t+1}^\pi = \nu(B_t^\pi, a_t, O_{t+1})$ where O_{t+1} is the observation variable at time step $t + 1$.

To make things clear, the π -POMDP model is defined here only with a terminal preference function Ψ , and no intermediate ones ρ_t . The next chapter will address formally the model with intermediate preference degree: the pessimistic one presented in [101] and an optimistic one. The criteria, or optimistic and pessimistic value functions of the π -POMDP model with terminal preference only, are thus similar to criteria (I.58) and (I.59). Note that the optimistic criterion has not been presented yet to the best of our knowledge, and is proposed now in parallel with the pessimistic one [101].

Definition I.2.17 (π -POMDP Criteria with Terminal Preference Only)

This is the same criteria as in the fully observable case (terminal preference case, Definition I.2.15): however, these criteria depends here on the initial belief state.

The optimistic π -POMDP criterion, or **optimistic value function**, is the Sugeno integral of the terminal preference with respect to the possibility measure of the system process for a given strategy $(\delta_t)_{t=0}^{H-1}$: the strategy which is looked for is a sequence of function of the available information i_t , i.e. $(\delta) = (\delta_t)_{t=0}^{H-1}$ with $\delta_t : i_t \mapsto \delta(i_t) \in \mathcal{A}$.

$$\overline{U}_H(\beta_0, (\delta)_{t=0}^{H-1}) = \max_{s_H \in \mathcal{S}} \min \left\{ \Psi(s_H), \pi(s_H \mid \beta_0, (\delta)) \right\}. \quad (\text{I.63})$$

As well, the π -POMDP **pessimistic value function** is the Sugeno integral of the terminal preference with respect to the necessity measure of the system process given such a strategy:

$$\underline{U}_H(\beta_0, (\delta)_{t=0}^{H-1}) = \min_{s_H \in \mathcal{S}} \max \left\{ \Psi(s_H), 1 - \pi(s_H \mid \beta_0, (\delta)) \right\}. \quad (\text{I.64})$$

where

$$\begin{aligned} \pi(s_H \mid \beta_0, (\delta)) &= \Pi(S_H = s_H \mid (\delta)) \\ &= \max_{(s_0, \dots, s_{H-1}) \in \mathcal{S}^H} \min \left\{ \min_{t=0}^{H-1} \pi_t(s_{t+1} \mid s_t, \delta(i_t)), \beta_0(s_0) \right\} \end{aligned}$$

is the possibility distribution over the last system state given the strategy. Thus the optimistic criterion may be denoted by $\mathbb{S}_\Pi[\Psi(S_H) \mid \beta_0, (\delta)]$ and the pessimistic one

$\mathbb{S}_{\mathcal{N}}[\Psi(S_H)|\beta_0, (\delta)]$, as they are optimistic and pessimistic Sugeno integrals based on the distribution $\pi(s_H|\beta_0, (\delta))$.

As in the probabilistic framework, Section I.1.8, these criteria can be rewritten based on a belief-dependent preference. Consider as previously, a strategy $(\delta_t)_{t=0}^{H-1}$ based on the current information $i_0 = \emptyset$, $i_1 = \{a_0, o_1\}$, $i_2 = \{a_0, o_1, a_1, o_2\}$, etc: for each time step $t \geq 0$, $\delta_t : i_t \mapsto \delta_t(i_t) \in \mathcal{A}$. Recall that ν is the belief update function defined in Theorem 14. We denote by $\hat{O}_t = (O_i)_{i=1}^t$ the successive observations until time step t seen as variables, and I_t the associated information. The belief state at time step $t+1$, seen as a variable, can be written $B_{t+1}^\pi = \nu^{\delta_t, \hat{O}_{t+1}}(B_t^\pi)$, $\forall t \geq 0$, with $(\delta_t)_{t=0}^{H-1}$ such a strategy, and the notation $\nu^{\delta_t, \hat{O}_{t+1}} : \beta \mapsto \nu(\beta, \delta_t(I_t), O_{t+1})$. Thus,

$$B_H^\pi = \left(\circ_{t=0}^{H-1} \nu^{\delta_t, \hat{O}_{t+1}} \right) (B_0^\pi),$$

where \circ is the function composition operator. Then, knowing that $\hat{O}_H = \hat{o}_H$ with $\hat{o}_H = \{o_1, \dots, o_H\} \in \mathcal{O}^H$ a sequence of observations, B_H^π is known: it is denoted by $\beta_{\beta_0}^{\delta, \hat{o}_H}$, and is called the belief state generated by the observation sequence \hat{o}_H and the strategy $(\delta_t)_{t=0}^{H-1}$.

Theorem 15 (π -POMDP Criteria Rewritings – Terminal Preference Case)

Let $\hat{o}_H = \{o_1, \dots, o_H\}$ a sequence of observations, and $(\delta) = (\delta_t)_{t=0}^{H-1}$ be a strategy such that δ_{t+1} is a function of the information $i_{t+1} = \{\delta_t(i_t), o_{t+1}\}$. The possibility distribution over the possible sequences of observations is denoted by

$$\begin{aligned} \pi(\hat{o}_H | (\delta), \beta_0) &= \Pi(\hat{O}_H = \hat{o}_H | (\delta), \beta_0) \\ &= \max_{(s_0, \dots, s_H) \in \mathcal{S}^{H+1}} \min \left\{ \pi_t(o_{t+1} | s_{t+1}, \delta_t(i_t)), \pi_t(s_{t+1} | s_t, \delta_t(i_t)), \beta_0(s_0) \right\}. \end{aligned}$$

The optimistic π -POMDP criterion is equal to the Sugeno integral of the belief-based optimistic preference $\bar{\Psi}(B_H^\pi) = \max_{s \in \mathcal{S}} \min \{ \Psi(s), B_H^\pi(s) \}$ with respect to the possibility measure over the observation sequences. That is, denoting by $\beta_{\beta_0}^{\delta, \hat{o}_H}$ the belief state generated by the observation sequence \hat{o}_H and the strategy $(\delta_t)_{t=0}^{H-1}$, the **optimistic criterion can be rewritten**

$$\begin{aligned} \bar{U}_H(\beta_0, (\delta)_{t=0}^{H-1}) &= \max_{\hat{o}_H} \min \left\{ \bar{\Psi}(\beta_{\beta_0}^{\delta, \hat{o}_H}), \pi(\hat{o}_H | (\delta), \beta_0) \right\} \\ &= \max_{\hat{o}_H} \min \left\{ \max_{s \in \mathcal{S}} \min \left\{ \Psi(s), \beta_{\beta_0}^{\delta, \hat{o}_H}(s) \right\}, \pi(\hat{o}_H | (\delta), \beta_0) \right\}. \end{aligned} \quad (\text{I.65})$$

As well, the pessimistic criterion is equal to the Sugeno integral of the belief-based pessimistic preference $\underline{\Psi}(B_H^\pi) = \min_{s \in \mathcal{S}} \max \{ \Psi(s), 1 - B_H^\pi(s) \}$, with respect to the necessity measure over the observation sequences. The π -POMDP **pessimistic criterion can be rewritten**

$$\begin{aligned} \underline{U}_H(\beta_0, (\delta)_{t=0}^{H-1}) &= \min_{\hat{o}_H} \max \left\{ \underline{\Psi}(\beta_{\beta_0}^{\delta, \hat{o}_H}), 1 - \pi(\hat{o}_H | (\delta), \beta_0) \right\} \\ &= \min_{\hat{o}_H} \max \left\{ \min_{s \in \mathcal{S}} \max \left\{ \Psi(s), 1 - \beta_{\beta_0}^{\delta, \hat{o}_H}(s) \right\}, 1 - \pi(\hat{o}_H | (\delta), \beta_0) \right\}. \end{aligned} \quad (\text{I.66})$$

The proof is given in Annex A.22.

Written as a Sugeno integral, the optimistic criterion $\mathbb{S}_\Pi[\Psi(S_H)|\beta_0, (\delta)]$, which is based, as in Definition I.2.17, on $\pi(s_H|\beta_0, (\delta))$ (the possibility distribution over the last system state s_H), becomes in the previous theorem

$$\mathbb{S}_\Pi\left[\max_{s \in \mathcal{S}} \min\{\Psi(s), B_H^\pi(s)\} \mid \beta_0, (\delta)\right] = \mathbb{S}_\Pi[\overline{\Psi}(B_H^\pi) \mid \beta_0, (\delta)].$$

These two equal Sugeno integrals are based on the possibility distribution over the observation sequence $\pi(\hat{o}_H|(\delta), \beta_0)$. As well, in Theorem 15, the pessimistic criterion $\mathbb{S}_\mathcal{N}[\Psi(S_H)|\beta_0, (\delta)]$ is rewritten

$$\mathbb{S}_\mathcal{N}\left[\min_{s \in \mathcal{S}} \max\{\Psi(s), 1 - B_H^\pi(s)\} \mid \beta_0, (\delta)\right] = \mathbb{S}_\mathcal{N}[\underline{\Psi}(B_H^\pi) \mid \beta_0, (\delta)].$$

Note that the belief-based preferences $\overline{\Psi}$ and $\underline{\Psi}$, are the π -POMDP counterparts of the POMDP belief-based reward $r(b, a) = \sum_{s \in \mathcal{S}} r(s, a) \cdot b_t(s)$. As well, these rewritings are the possibilistic counterpart of the rewriting $\mathbb{E}\left[r(S_t, d_t(i_t))\right] = \mathbb{E}\left[\sum_{s \in \mathcal{S}} B_t(s) \cdot r(s, d_t(i_t))\right]$.

This theorem assures us that the criteria can be expressed as Sugeno integrals of a function of the possibilistic belief state B_H^π : this result leads to the definition of π -MDPs whose states are the qualitative possibilistic belief states: these π -MDPs are denoted by $\langle \tilde{S}^\pi, \mathcal{A}, \tilde{T}^\pi, \tilde{\Psi} \rangle$. The state space \tilde{S}^π is the finite set of possibilistic belief states $\Pi_\mathcal{L}^\mathcal{S} = \{\beta \mid \beta : \mathcal{S} \rightarrow \mathcal{L}, \max_{s \in \mathcal{S}} \beta(s) = 1\}$.

Let β_t a given qualitative possibilistic belief, *i.e.* a possibility distribution in $\Pi_\mathcal{L}^\mathcal{S}$. The sequence of variables $(B_t^\pi)_{t \in \mathbb{N}}$ is the sequence of the belief functions seen as random variables. As highlighted by the possibilistic belief update (I.62), if $B_t^\pi = \beta_t$, and the selected action is a_t , the value of the next variable B_{t+1}^π is a deterministic function of the observation O_{t+1} .

A belief π -MDP is defined since the qualitative possibilistic belief updating process is shown to be a possibilistic Markov process *i.e.* $\forall a \in \mathcal{A}, \forall \beta' \in \Pi_\mathcal{L}^\mathcal{S}$, B_{t+1}^π is M-independent from all previous variables conditional to the current belief B_t^π and the selected action $a_t \in \mathcal{A}$:

Theorem 16

The qualitative possibilistic belief updating process is a Markov process, i.e.

$$\Pi(B_{t+1}^\pi = \beta' \mid I_t = i_t, a_t) = \Pi(B_{t+1}^\pi = \beta' \mid B_t^\pi = \beta_{b_0}^{i_t}, a_t), \quad (\text{I.67})$$

where $\beta_{b_0}^{i_t}$ is the qualitative belief state reached starting with β_0 and with the information $i_t = \{a_0, o_1, a_1, o_2, \dots, a_{t-1}, o_t\}$.

The proof is given in Annex A.23.

As highlighted by the equation (44) in the proof, if $B_t^\pi = \beta$ and the selected action is $a \in \mathcal{A}$, the possibility degree that the next belief B_{t+1}^π is $\beta' \in \Pi_\mathcal{L}^\mathcal{S}$, is the maximum of all the possibility degrees of observations o' such that $\nu(\beta, a, o') = \beta'$: it defines the transition probability distributions of the belief process, *i.e.* elements of \tilde{T} , as follows: $\forall t \geq 0$,

$$\pi_t(\beta' \mid \beta, a) = \max_{\substack{o' \in \mathcal{O} \text{ s.t.} \\ \nu(\beta, a, o') = \beta'}} \pi_t(o' \mid \beta, a), \quad (\text{I.68})$$

where $\pi_t(o' \mid \beta, a) = \max_{(s, s') \in \mathcal{S}^2} \min\{\pi_t(o' \mid s', a_t), \pi_t(s' \mid s, a_t), \beta(s)\}$, is the possibility degree of observing o' conditional on all the previous information.

Finally, the preference functions associated with the possibilistic belief β_H are defined as highlighted by Theorem I.65: for an optimistic π -POMDP, the preference function is, $\forall \beta \in \Pi_{\mathcal{L}}^S$,

$$\overline{\Psi}(\beta) = \max_{s \in \mathcal{S}} \min \{ \Psi(s), \beta(s) \} \quad (\text{I.69})$$

and for a pessimistic one, $\forall \beta \in \Pi_{\mathcal{L}}^S$,

$$\underline{\Psi}(\beta) = \min_{s \in \mathcal{S}} \max \{ \Psi(s), 1 - \beta(s) \}. \quad (\text{I.70})$$

As for each belief $\beta \in \Pi_{\mathcal{L}}^S$, the possibility and necessity measure conditional on the information i don't vary if $i \in \{i \mid \beta_{\beta_0}^i = \beta\}$, *i.e.* if the information i leads to β (see for instance the proof of Theorem 16), it is sufficient to look for a belief-based strategy $(\delta_t)_{t=0}^{H-1}$, such that $\forall t \geq 0$, $\delta_t : \beta_t \mapsto \delta_t(\beta_t) \in \mathcal{A}$.

The π -MDP $\langle \tilde{\mathcal{S}}^\pi, \mathcal{A}, \tilde{T}^\pi, \overline{\Psi}$ or $\underline{\Psi} \rangle$ built from of a π -POMDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{O}, T^\pi, O^\pi, \beta_0 \rangle$ is finally:

- $\tilde{\mathcal{S}}^\pi = \Pi_{\mathcal{L}}^S$, the set of all qualitative possibilistic beliefs;
- \tilde{T}^π contains all transition possibility distributions of the possibilistic beliefs: $\forall a \in \mathcal{A}$, $\forall \beta \in \Pi_{\mathcal{L}}^S$, the belief transition possibility distribution defined by the equation (I.68), $\pi_t(\cdot \mid \beta, a)$ is in \tilde{T}^π ;
- preference functions $\overline{\Psi}$ if the computed criterion is optimistic, see equation (I.69), or $\underline{\Psi}$ if it is pessimistic, equation (I.70).

Note now that, using the belief state transition definition (I.68), and the equation (I.30) of Property I.2.1, for each function from the belief space to \mathcal{L} , $U : \Pi_{\mathcal{L}}^S \rightarrow \mathcal{L}$,

$$\begin{aligned} \max_{\beta' \in \Pi_{\mathcal{L}}^S} \min \{ \pi_t(\beta' \mid \beta, a), U(\beta') \} &= \max_{\beta' \in \Pi_{\mathcal{L}}^S} \min \left\{ \max_{\substack{o' \in \mathcal{O} \text{ s.t.} \\ \nu(\beta, a, o') = \beta'}} \pi_t(o' \mid \beta, a), U(\beta') \right\} \\ &= \max_{\beta' \in \Pi_{\mathcal{L}}^S} \max_{\substack{o' \in \mathcal{O} \text{ s.t.} \\ \nu(\beta, a, o') = \beta'}} \min \{ \pi_t(o' \mid \beta, a), U(\beta') \} \\ &= \max_{o' \in \mathcal{O}} \min \{ \pi_t(o' \mid \beta, a), U(\nu(\beta, a, o')) \}, \end{aligned}$$

This observation leads to Algorithm 6 which is the π -MDP algorithm (4) with terminal preference criteria (I.58), applied to the π -MDP $\langle \tilde{\mathcal{S}}^\pi, \mathcal{A}, \tilde{T}^\pi, \overline{\Psi} \rangle$.

Algorithm 6: Dynamic Programming Algorithm for Optimistic π -POMDP with Terminal Preference Only

```

1  $\overline{U}_0^* \leftarrow \overline{\Psi}$ ;
2 for  $i \in \{1, \dots, H\}$  do
3   for  $\beta \in \Pi_{\mathcal{L}}^S$  do
4      $\overline{U}_i^*(\beta) \leftarrow \max_{a \in \mathcal{A}} \max_{o' \in \mathcal{O}} \min \{ \pi_t(o' \mid \beta, a), \overline{U}_{i-1}^*(\nu(\beta, a, o')) \}$ ;
5      $\overline{\delta}_{H-i}(\beta) \in \arg\max_{a \in \mathcal{A}} \max_{o' \in \mathcal{O}} \min \{ \pi_t(o' \mid \beta, a), \overline{U}_{i-1}^*(\nu(\beta, a, o')) \}$ ;
6 return  $\overline{U}_H^*, (\overline{\delta}^*)$ ;

```

As well, the equations (I.26) and (I.27) of Property I.2.1 leads to

$$\min_{\beta' \in \Pi_{\mathcal{L}}^S} \max \{ 1 - \pi_t(\beta' \mid \beta, a), U(\beta') \} = 1 - \max_{\beta' \in \Pi_{\mathcal{L}}^S} \min \{ \pi_t(\beta' \mid \beta, a), 1 - U(\beta') \},$$

for each function $U : \Pi_{\mathcal{L}}^{\mathcal{S}} \rightarrow \mathcal{L}$. Thus, using the belief state transition definition (I.68),

$$\min_{\beta' \in \Pi_{\mathcal{L}}^{\mathcal{S}}} \max \{ 1 - \pi_t(\beta' \mid \beta, a), U(\beta') \} = \min_{o' \in \mathcal{O}} \max \{ 1 - \pi_t(o' \mid \beta, a), U(\nu(\beta, a, o')) \}.$$

It leads to Algorithm 7, which is the π -MDP algorithm (5), with terminal preference criteria (I.59), applied to the π -MDP $\langle \tilde{\mathcal{S}}^{\pi}, \mathcal{A}, \tilde{T}^{\pi}, \underline{\Psi} \rangle$.

Algorithm 7: Dynamic Programming Algorithm for Pessimistic π -POMDP with Terminal Preference Only

```

1  $\underline{U}_0^* \leftarrow \underline{\Psi}$ ;
2 for  $i \in \{1, \dots, H\}$  do
3   for  $\beta \in \Pi_{\mathcal{L}}^{\mathcal{S}}$  do
4      $\underline{U}_i^*(\beta) \leftarrow \max_{a \in \mathcal{A}} \min_{o' \in \mathcal{O}} \max \{ 1 - \pi_t(o' \mid \beta, a), \underline{U}_{i-1}^*(\nu(\beta, a, o')) \};$ 
5      $\underline{\delta}_{H-i}(\beta) \in \operatorname{argmax}_{a \in \mathcal{A}} \min_{o' \in \mathcal{O}} \max \{ 1 - \pi_t(o' \mid \beta, a), \underline{U}_{i-1}^*(\nu(\beta, a, o')) \};$ 
6 return  $\underline{U}_H^*, (\underline{\delta}^*)$ ;
```

In this first chapter, probabilistic and qualitative possibilistic POMDPs have been built one after the other putting some light on the similarities between both models: with probabilistic POMDPs, system state dynamics and observation uncertainty are described with probabilities $\mathbf{p}(s' \mid s, a) \in \mathbb{R}$ and $\mathbf{p}(o' \mid s', a) \in \mathbb{R}$ while they are defined by possibility distributions $\pi(s' \mid s, a) \in \mathcal{L} = \{0, \frac{1}{k}, \dots, 1\}$ (with $k \geq 1$) and $\pi(o' \mid s', a) \in \mathcal{L}$ in the π -POMDP framework. Moreover, the probabilistic framework measures the benefit from passing through a system state $s \in \mathcal{S}$ and using action $a \in \mathcal{A}$ with the additive reward functions $r(s, a) \in \mathbb{R}$ (and $R(s)$ for the last state in case of finite-horizon problem); the possibilistic framework uses qualitative preferences $\rho(s, a) \in \mathcal{L}$ and $\Psi(s) \in \mathcal{L}$. Thus, the probabilistic criterion (the value function) for a given strategy is the expectation of the rewards written $\mathbb{E}[\text{rewards}((S_t)_{t \geq 0})] \in \mathbb{R}$, and the possibilistic framework has two criteria (value functions) which are Sugeno integrals of preferences written $\mathbb{S}_{\Pi}[\text{preferences}((S_t)_{t \geq 0})] \in \mathcal{L}$ for the optimistic one, and $\mathbb{S}_{\mathcal{N}}[\text{preferences}((S_t)_{t \geq 0})] \in \mathcal{L}$ for the pessimistic one. A POMDP (resp. π -POMDP), is redefined in terms of fully observable MDP (resp. π -MDP) where the system states are the belief states $b_t \in \mathbb{P}_{b_0}^{\mathcal{S}}$ (resp. $\beta_t \in \Pi_{\mathcal{L}}^{\mathcal{S}}$), *i.e.* probability (resp. possibility) distributions over the system states of the initial POMDP: the belief-based reward has to be defined $r(b, a) = \mathbb{E}_{S \sim b}[r(S, a)] = \sum_{s \in \mathcal{S}} r(s, a) \cdot b(s)$ in the probabilistic case. In the possibilistic case, the belief-based preference can be written $\bar{\rho}(b, a) = \mathbb{S}_{\Pi, S \sim \beta}[\rho(S, a)] = \max_{s \in \mathcal{S}} \min \{ \rho(s, a), \beta(s) \}$ for the optimistic criterion, and $\underline{\rho}(b, a) = \mathbb{S}_{\mathcal{N}, S \sim \beta}[\rho(S, a)] = \min_{s \in \mathcal{S}} \max \{ \rho(s, a), 1 - \beta(s) \}$ for the pessimistic one.

Next chapter proposes some improvements of the qualitative possibilistic model: first, criteria are discussed, concerning the preference aggregation, and the impact of the choice of the (optimistic or pessimistic) criterion. Next, the Mixed-Observability property is defined: as for the probabilistic model, the complexity of solving π -POMDPs having this property is reduced. Finally the infinite horizon problem is formally defined and the proposed solving algorithm is shown to return an optimal strategy for a given criterion.

UPDATES AND PRACTICAL STUDY OF THE QUALITATIVE POSSIBILISTIC PARTIALLY OBSERVABLE MARKOV DECISION PROCESSES

The end of the previous chapter presented the π -POMDPs, a qualitative possibilistic counterpart of the classical probabilistic POMDPs. Recall that, in the qualitative possibilistic framework, the set of belief states is finite, $\#\Pi_{\mathcal{L}}^{\mathcal{S}} < +\infty$ (see Equation I.60) while the set of belief states is infinite in the probabilistic framework $\mathbb{P}_{b_0}^{\mathcal{S}}$: for this reason, π -POMDPs can be seen as a simpler model for sequential decision making under uncertainty, than the probabilistic one. This is a good point as solving probabilistic POMDPs is at least a PSPACE problem [86, 76]. Moreover, the qualitative possibility theory allows to model the total ignorance as noted in Introduction: it is a motivation for the study of this model. The distribution $\forall s \in \mathcal{S}, \beta(s) = 1$ means that all states are plausible according to the belief state β , *i.e.* the agent considers that all system states are possible. Finally, possibility distributions only sort events and does not assign qualitative values to them: if the probability distributions defining the POMDP are not known in practice, as in the robotic vision example given in Introduction, a qualitative description of the problem is more suitable.

The π -POMDP model has been previously defined in the pessimistic settings only [101]. This section is then devoted to the update of this promising model: first, optimistic and pessimistic models with intermediate preferences are built and discussed. This discussion leads to other criteria. Next, the *Mixed-Observability* property [83, 2] is defined, describing the systems some state variables of which are fully observable: it is shown that this property, if correctly taken into account, reduces dramatically the complexity of solving π -POMDPs. Finally, a value iteration algorithm for infinite horizon π -MDPs (Fully and Partially Observable) is next proposed and the optimality of the returned strategy (for a specified criterion) is shown assuming the existence of a “stay” action in some goal states. Experimental work finally illustrates performances of the strategies computed from different criteria. It is also shown that strategies computed from π -POMDPs can outperform probabilistic POMDP strategies for a target recognition problem where the agent’s observations are imprecise.

II.1 INTERMEDIATE PREFERENCES IN π -POMDPs

In the previous chapter, π -MDPs have been defined with intermediate preference degrees, and after that, the particular case of terminal preferences only has been presented. The global preference degree of a trajectory $\mathcal{T} = (s_0, \dots, s_H) \in \mathcal{S}^{H+1}$ has been defined there as the

minimum of all the preference degrees of the encountered states:

$$\rho(\mathcal{T}, (a_t)_{t=0}^{H-1}) = \min \left\{ \min_{t=0}^{H-1} \rho_t(s_t, a_t), \Psi(s_H) \right\}. \quad (\text{II.1})$$

Another global preference based on the max operator can be also proposed. Let us explicitly define the global preferences for system states: they are the possibilistic counterparts of the sum $\sum_{t=0}^{H-1} r_t(s_t, a_t) + R(s_H)$ in the probabilistic model, given in the equation (I.2).

Definition II.1.1 (Global Preferences on System State Trajectory)

Let $(A_t)_{t=0}^{H-1}$ be a sequence of action variables modeling the successive agent decisions. Global preferences over system state trajectories $(S_t)_{t=0}^H$ are denoted as follows:

- a new one, **maximum-based**,

$$\overline{\mathcal{G}}((S_t)_{t=0}^H, (A_t)_{t=0}^{H-1}) = \max \left\{ \max_{t=0}^{H-1} \left\{ \rho_t(S_t, A_t) \right\}, \Psi(S_H) \right\}, \quad (\text{II.2})$$

- the classical **minimum-based** one, which has been used until now,

$$\underline{\mathcal{G}}((S_t)_{t=0}^H, (A_t)_{t=0}^{H-1}) = \min \left\{ \min_{t=0}^{H-1} \left\{ \rho_t(S_t, A_t) \right\}, \Psi(S_H) \right\}, \quad (\text{II.3})$$

i.e. with previous notations, if the system state trajectory is denoted by $\mathcal{T} = (s_0, \dots, s_H) \in \mathcal{S}^{H+1}$ and the strategy $(\delta) = (\delta_t)_{t=0}^{H-1}$ with $\forall t \in \{1, \dots, H-1\}$, $\delta : \mathcal{S} \rightarrow \mathcal{A}$,

- $\overline{\mathcal{G}}(\mathcal{T}, (\delta)) = \overline{\mathcal{G}}((s_t)_{t=0}^H, (\delta_t(s_t))_{t=0}^{H-1})$, and
- $\underline{\mathcal{G}}(\mathcal{T}, (\delta)) = \underline{\mathcal{G}}((s_t)_{t=0}^H, (\delta_t(s_t))_{t=0}^{H-1}) = \rho(\mathcal{T}, (a)_{t=0}^{H-1})$, see the equation (II.1).

Note that a π -MDP with an optimistic criterion (or optimistic value function, see the equation (I.48) of Section I.2.4) and a maximum-based global preference degree $\overline{\mathcal{G}}$ has not been defined yet (all π -MDPs seen previously had a minimum-based global preference $\underline{\mathcal{G}}$), and we propose it now:

Definition II.1.2 (Optimistic π -MDP Criterion – Maximum-based Global Preference)

Let us denote as previously the initial state $s_0 \in \mathcal{S}$, an H -length trajectory $\mathcal{T} = (s_t)_{t=0}^H$, and $\mathcal{T}_H = \mathcal{S}^H$ the set of such trajectories. The **value function** of the Maximum-based Optimistic π -MDP is

$$\overline{\overline{U}}_H(s_0, (\delta_t)_{t=0}^{H-1}) = \max_{\mathcal{T} \in \mathcal{T}_H} \min \left\{ \overline{\mathcal{G}}(\mathcal{T}, (\delta)), \pi(\mathcal{T} | s_0, (\delta)) \right\},$$

where, $\pi(\mathcal{T} | s_0, (\delta))$ is the possibility degree of the trajectory \mathcal{T} given the strategy $(\delta) = (\delta_t)_{t=0}^{H-1}$, defined as in Section I.2.4, see the equation (I.47). As the Sugeno integral of the global reward with respect to the possibility measure, it can then be denoted by

$$\overline{\overline{U}}_H(s, (\delta)) = \mathbb{S}_{\Pi} \left[\overline{\mathcal{G}}((S_t)_{t=0}^H, (\delta_t(S_t))_{t=0}^{H-1}) \mid S_0 = s, (\delta) \right].$$

This optimistic π -MDP also can be solved using Dynamic Programming (DP), just like the π -MDPs presented in the previous chapter (see Section I.2.4), and as described by the next theorem:

Theorem 17 (DP for optimistic π -MDPs with Maximum-based Global Preference)

The optimal optimistic criterion with the maximum-based global preference $\bar{\mathcal{G}}$, denoted by \bar{U}_H^* , and an associated optimal strategy $(\bar{\delta}^*)_{t=0}^{H-1}$, can be computed as follows:
 $\forall s \in \mathcal{S}$,

$$\begin{aligned} \bar{U}_0^*(s) &= \Psi(s), \quad \text{and, } \forall 1 \leq i \leq H, \\ \bar{U}_i^*(s) &= \max_{a \in \mathcal{A}} \max \left\{ \rho_{H-i}(s, a), \max_{s' \in \mathcal{S}} \min \left\{ \pi_{H-i}(s' | s, a), \bar{U}_{i-1}^*(s') \right\} \right\}. \end{aligned} \quad (\text{II.4})$$

$$\bar{\delta}_{H-i}^*(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \max \left\{ \rho_{H-i}(s, a), \max_{s' \in \mathcal{S}} \min \left\{ \pi_{H-i}(s' | s, a), \bar{U}_{i-1}^*(s') \right\} \right\}. \quad (\text{II.5})$$

This theorem can be shown in exactly the same way as the proof of Theorem 13 (see Annex A.20), however the equation (I.35) of Property I.2.1 has to be used.

The optimistic and pessimistic π -MDPs with a minimum-based global preference have been presented in the previous chapter. The optimistic π -MDP with a maximum-based global preference has been defined just above. Note that a last π -MDP may be pessimistic with a maximum-based global preference but is not defined here.

Let us recall that, in the partially observable case, the strategy $(\delta_t)_{t=0}^{H-1}$ is *a priori* an *information-based* one *i.e.* it is such that, for the time step $t \in \{0, \dots, H-1\}$, δ_t maps the current information $i_t = \{a_0, \dots, a_{t-1}, o_1, \dots, o_t\} \in \mathcal{A}^t \times \mathcal{O}^t$ to an action $a_t \in \mathcal{A}$. As shown below, two π -POMDP criteria with global preferences, *i.e.* with intermediate preferences, allow to translate the partially observable processes into fully observable ones called belief π -MDPs, as in the case of a terminal preference only (see Theorem 15):

- **the optimistic criterion with maximum-based global preference**, partially observable version of the Definition II.1.2,

$$\bar{U}_H(\beta_0, (\delta)) = \max_{\mathcal{T} \in \mathcal{T}_H} \min \left\{ \bar{\mathcal{G}}(\mathcal{T}, (\delta)), \pi(\mathcal{T} | \beta_0, (\delta)) \right\}. \quad (\text{II.6})$$

In this formula, $\bar{\mathcal{G}}(\mathcal{T}, (\delta))$ is the maximum-based global preference, defined by the equation (II.2), where the strategy consists in a function of the current information i_t . The possibility degree $\pi(\mathcal{T} | \beta_0, (\delta)) = \min \left\{ \min_{t=0}^{H-1} \pi_t(s_{t+1} | s_t, \delta_t(s_t)), \beta(s_0) \right\}$ is the possibility degree of the trajectory \mathcal{T} given the strategy (δ) and the initial belief state $\beta_0 \in \Pi_{\mathcal{L}}^{\mathcal{S}}$. This criterion can thus be denoted by $\mathbb{S}_{\Pi} \left[\bar{\mathcal{G}} \left((S_t)_{t=0}^H, (\delta_t(I_t))_{t=0}^{H-1} \right) \middle| \beta_0, (\delta) \right]$, where $I_t = \{O_t, \delta_{t-1}(I_{t-1}), I_{t-1}\}$ is the variable representing the current information.

- **the pessimistic criterion with minimum-based global preference**, partially observable version of the equation I.45 of Section I.2.4,

$$U_H(\beta_0, (\delta)) = \min_{\mathcal{T} \in \mathcal{T}_H} \max \left\{ \underline{\mathcal{G}}(\mathcal{T}, (\delta)), 1 - \pi(\mathcal{T} | \beta_0, (\delta)) \right\}. \quad (\text{II.7})$$

Here, $\underline{\mathcal{G}}(\mathcal{T}, (\delta))$ is the minimum-based global preference, see the equation (II.3), with an information-based strategy (δ) . The possibility degree of the trajectory is denoted by $\pi(\mathcal{T} | \beta_0, (\delta))$, and criterion may be denoted by $\mathbb{S}_{\mathcal{N}} \left[\underline{\mathcal{G}} \left((S_t)_{t=0}^H, (\delta_t(I_t))_{t=0}^{H-1} \right) \middle| \beta_0, (\delta) \right]$.

Note that, in Section I.1.8, the probabilistic criterion is rewritten as a sum of rewards defined on the belief states: this is possible because of the linearity of the probabilistic expectation. In order to propose a global preference degree such as preferences (II.2) and (II.3) for the π -POMDPs, some properties of the Sugeno Integral are needed. These properties are the counterparts of the linearity of the probabilistic expectation:

Property II.1.1 (*Maxitivity and Minitivity of the possibilistic Sugeno integrals*)

Let f and g two functions from Ω to \mathcal{L} . Then,

$$\mathbb{S}_{\Pi} [\max \{f, g\}] = \max \{ \mathbb{S}_{\Pi} [f], \mathbb{S}_{\Pi} [g] \}, \quad (\text{II.8})$$

$$\mathbb{S}_{\mathcal{N}} [\min \{f, g\}] = \min \{ \mathbb{S}_{\mathcal{N}} [f], \mathbb{S}_{\mathcal{N}} [g] \}. \quad (\text{II.9})$$

where the Sugeno integrals \mathbb{S}_{Π} and $\mathbb{S}_{\mathcal{N}}$ are defined in Section I.2.3 (see Theorem 12).

The proof is given in Annex B.2.

These properties offer rewritings of the Sugeno integrals (with respect to the possibility and necessity measures) of the global system state preferences $\overline{\mathcal{G}}$ and $\underline{\mathcal{G}}$, as Sugeno integrals of the global belief state preference. As the presented π -POMDP value functions, *i.e.* criteria (II.6) and (II.7), are Sugeno integrals of the global system state preference, they can be rewritten in the form of belief-dependent value functions similar to the one of Section I.1.8 for probabilistic POMDPs:

Theorem 18 (*Rewritings of the π -POMDP Value Functions*)

Recall that the sequence of variables representing the successive belief states is denoted by $(B_t^{\pi})_{t=0}^{H-1}$, and the sequence of action variables by A_t (it includes the case $A_t = \delta_t(I_t)$). Following equalities are true:

$$\mathbb{S}_{\Pi} [\overline{\mathcal{G}}((S_t)_{t=0}^H, (A_t)_{t=0}^{H-1})] = \mathbb{S}_{\Pi} [\overline{\mathcal{G}}((B_t^{\pi})_{t=0}^H, (A_t)_{t=0}^{H-1})], \quad (\text{II.10})$$

$$\mathbb{S}_{\mathcal{N}} [\underline{\mathcal{G}}((S_t)_{t=0}^H, (A_t)_{t=0}^{H-1})] = \mathbb{S}_{\mathcal{N}} [\underline{\mathcal{G}}((B_t^{\pi})_{t=0}^H, (A_t)_{t=0}^{H-1})]. \quad (\text{II.11})$$

where the global preference degrees of a belief state trajectory $(B_t^{\pi})_{t=0}^H$ are:

$$\overline{\mathcal{G}}((B_t^{\pi})_{t=0}^H, (A_t)_{t=0}^{H-1}) = \max \left\{ \max_{t=0}^{H-1} \{ \overline{\rho}_t(B_t^{\pi}, A_t) \}, \overline{\Psi}(B_H^{\pi}) \right\}$$

$$\underline{\mathcal{G}}((B_t^{\pi})_{t=0}^H, (A_t)_{t=0}^{H-1}) = \min \left\{ \min_{t=0}^{H-1} \{ \underline{\rho}_t(B_t^{\pi}, A_t) \}, \underline{\Psi}(B_H^{\pi}) \right\}.$$

The global preference degrees of a belief state trajectory $(B_t^{\pi})_{t=0}^H$ are defined as functions of the intermediate preference degrees, denoted by $\overline{\rho}_t$ and $\underline{\rho}_t$:

$$\overline{\rho}_t(B_t^{\pi}, A_t) = \max_{s \in \mathcal{S}} \min \{ \rho_t(s, A_t), B_t^{\pi}(s) \}$$

$$\underline{\rho}_t(B_t^{\pi}, A_t) = \min_{s \in \mathcal{S}} \max \{ \rho_t(s, A_t), 1 - B_t^{\pi}(s) \}.$$

Finally, the terminal preference degrees are defined as in Theorem 15 in the previous chapter: $\overline{\Psi}(B_H^{\pi}) = \max_{s \in \mathcal{S}} \min \{ \Psi(s), B_H^{\pi}(s) \}$, $\underline{\Psi}(B_H^{\pi}) = \min_{s \in \mathcal{S}} \max \{ \Psi(s), 1 - B_H^{\pi}(s) \}$.

The proof is given in Annex B.3 and uses Theorem I.65 and Property II.1.1.

Two equivalent belief π -MDPs can be then defined from the π -POMDP criteria (II.6) and (II.7): as explained in Section I.2.5, their state space is $\tilde{\mathcal{S}}^{\pi} = \Pi_{\mathcal{L}}^{\mathcal{S}}$, *i.e.* the set of all belief states $\{\beta \mid \max_{s \in \mathcal{S}} \beta(s) = 1\}$. The set of the transition possibility distribu-

tion, denoted by \tilde{T}^π , contains $\forall \beta \in \Pi_{\mathcal{L}}^S, \forall t \in \{0, \dots, H-1\}$, the possibility distribution $\forall \beta' \in \Pi_{\mathcal{L}}^S, \pi_t(\beta' | \beta, a) = \max_{\substack{o' \in \mathcal{O} \text{ s.t.} \\ \nu(\beta, a, o') = \beta'}} \pi_t(o' | \beta, a)$, where $\pi_t(o' | \beta, a)$ is a notation for $\max_{(s, s') \in \mathcal{S}^2} \min \{ \pi_t(o' | s', a_t), \pi_t(s' | s, a_t), \beta(s) \}$, and $\nu : \Pi_{\mathcal{L}}^S \times \mathcal{A} \times \mathcal{O} \rightarrow \Pi_{\mathcal{L}}^S$ is the belief update function (see Theorem 14). Finally,

- for the optimistic π -POMDP with maximum-based global preference $\bar{\mathcal{G}}$, the preference function of the resulting π -MDP is $\forall t \in \{1, \dots, H-1\}, \forall \beta \in \Pi_{\mathcal{L}}^S, \forall a \in \mathcal{A}$,

$$\bar{\rho}_t(\beta, a) = \max_{s \in \mathcal{S}} \min \{ \rho_t(s, a), \beta(s) \}$$

and terminal preference function is $\bar{\Psi}(\beta) = \max_{s \in \mathcal{S}} \min \{ \Psi(s), \beta(s) \}$. The resulting π -MDP criterion is the one with the maximum-based global preference $\bar{\mathcal{G}}$ (see Definition II.1.2).

- for the pessimistic π -POMDP with minimum-based global preference $\underline{\mathcal{G}}$, the preference function of the resulting π -MDP is $\forall t \in \{1, \dots, H-1\}, \forall \beta \in \Pi_{\mathcal{L}}^S, \forall a \in \mathcal{A}$,

$$\underline{\rho}_t(\beta, a) = \min_{s \in \mathcal{S}} \max \{ \rho_t(s, a), 1 - \beta(s) \}$$

and terminal preference function is $\underline{\Psi}(\beta) = \min_{s \in \mathcal{S}} \max \{ \Psi(s), 1 - \beta(s) \}$. Finally, the resulting π -MDP criterion is the one with the minimum-based global preference $\underline{\mathcal{G}}$ (see the equation (I.51) of Section I.2.4).

Note that line II.8 of Property II.1.1 offers the rewriting of the optimistic π -POMDP (π -POMDP with criterion II.6) if the global preference degree is maximum-based. However, such a rewriting is impossible with the minimum-based global preference. Indeed, in order to keep a criterion based on the minimum (II.1), the following equality should be true: $\mathbb{S}_{\Pi}[\min \{f, g\}] = \min \{ \mathbb{S}_{\Pi}[f], \mathbb{S}_{\Pi}[g] \}$. However, the following counterexample confirms that this equality is not true in general: consider $\Omega = \{\omega_1, \omega_2\}$, $f : \Omega \rightarrow \mathcal{L}$ and $g : \Omega \rightarrow \mathcal{L}$ such that $f(\omega_1) = 1, f(\omega_2) = 0$, and $g = 1 - f$. Consider the total ignorance possibility distribution: $\pi(\omega_1) = \pi(\omega_2) = 1$. As $\min \{f(\omega), g(\omega)\} = 0, \forall \omega \in \Omega$,

$$\mathbb{S}_{\Pi}[\min \{f, g\}] = \max_{\omega \in \Omega} \min \{f(\omega), g(\omega), \pi(\omega)\} = 0,$$

whereas $\mathbb{S}_{\Pi}[f] = \max_{\omega \in \Omega} \min \{f(\omega), \pi(\omega)\} = \max \{1, 0\} = 1$ and $\mathbb{S}_{\Pi}[g] = \max \{0, 1\} = 1$ as well, thus

$$\min \{ \mathbb{S}_{\Pi}[f], \mathbb{S}_{\Pi}[g] \} = 1.$$

It can be shown that $\mathbb{S}_{\mathcal{N}}[\max \{f, g\}] = \max \{ \mathbb{S}_{\mathcal{N}}[f], \mathbb{S}_{\mathcal{N}}[g] \}$ is not true in general, with the same counterexample.

The rewritings of Theorem 18 lead to the Dynamic Programming (DP) algorithms (8) and (9): Algorithm (8) corresponds to the DP scheme of Theorem 17, and Algorithm 9 is the π -MDP algorithm (5):

- Algorithm 8 computes an optimal strategy for the π -MDP $\langle \tilde{\mathcal{S}}^\pi, \mathcal{A}, \tilde{T}^\pi, (\bar{\rho}_t)_{t=0}^{H-1}, \bar{\Psi} \rangle$ with the optimistic criterion (using the Sugeno integral \mathbb{S}_{Π}) and a maximum-based global preference ($\bar{\mathcal{G}}$, see Definition II.1.1), also optimal for the π -POMDP $\langle \mathcal{S}, \mathcal{A}, T^\pi, (\rho_t)_{t=0}^{H-1}, \Psi \rangle$ with the optimistic criterion (using \mathbb{S}_{Π}), and maximum-based global preference ($\bar{\mathcal{G}}$).
- Algorithm 9 computes an optimal strategy for the π -MDP $\langle \tilde{\mathcal{S}}^\pi, \mathcal{A}, \tilde{T}^\pi, (\underline{\rho}_t)_{t=0}^{H-1}, \underline{\Psi} \rangle$, with the pessimistic criterion (using the Sugeno integral $\mathbb{S}_{\mathcal{N}}$) and the classical global minimum-based preference ($\underline{\mathcal{G}}$, see the equation (II.1) or Definition II.1.1), also optimal for the π -POMDP $\langle \mathcal{S}, \mathcal{A}, T^\pi, (\rho_t)_{t=0}^{H-1}, \Psi \rangle$, with the pessimistic criterion ($\mathbb{S}_{\mathcal{N}}$) and the classical global minimum-based preference ($\underline{\mathcal{G}}$).

Algorithm 8: DP Algorithm for Optimistic π -POMDP with intermediate preferences

```

1  $\overline{U}_0^* \leftarrow \overline{\Psi}$ ;
2 for  $i \in \{1, \dots, H\}$  do
3   for  $\beta \in \Pi_{\mathcal{L}}^{\mathcal{S}}$  do
4      $\overline{U}_i^*(\beta) \leftarrow \max_{a \in \mathcal{A}} \max \left\{ \overline{\rho}_t(\beta, a), \max_{o' \in \mathcal{O}} \min \left\{ \pi_t(o' \mid b_t, a), \overline{U}_{i-1}^*(\nu(\beta, a, o')) \right\} \right\}$ ;
5      $\overline{\delta}_{H-i}(\beta) \in \operatorname{argmax}_{a \in \mathcal{A}} \max \left\{ \overline{\rho}_t(\beta, a), \max_{o' \in \mathcal{O}} \min \left\{ \pi_t(o' \mid b_t, a), \overline{U}_{i-1}^*(\nu(\beta, a, o')) \right\} \right\}$ ;
6 return  $\overline{U}_H^*, (\overline{\delta}^*)$ ;
```

Algorithm 9: DP Algorithm for Pessimistic π -POMDP with intermediate preferences

```

1  $\underline{U}_0^* \leftarrow \underline{\Psi}$ ;
2 for  $i \in \{1, \dots, H\}$  do
3   for  $\beta \in \Pi_{\mathcal{L}}^{\mathcal{S}}$  do
4      $\underline{U}_i^*(\beta) \leftarrow \max_{a \in \mathcal{A}} \min \left\{ \underline{\rho}_t(\beta, a), \min_{o' \in \mathcal{O}} \max \left\{ 1 - \pi_t(o' \mid b_t, a), \underline{U}_{i-1}^*(\nu(\beta, a, o')) \right\} \right\}$ ;
5      $\underline{\delta}_{H-i}(\beta) \in \operatorname{argmax}_{a \in \mathcal{A}} \min \left\{ \underline{\rho}_t(\beta, a), \min_{o' \in \mathcal{O}} \max \left\{ 1 - \pi_t(o' \mid b_t, a), \underline{U}_{i-1}^*(\nu(\beta, a, o')) \right\} \right\}$ ;
6 return  $\underline{U}_H^*, (\underline{\delta}^*)$ ;
```

II.1.1 Discussion

The maximum-based global preference $\overline{\mathcal{G}}$ (II.2) cares about the fact that at least one encountered state has a high preference. As explained just above, this global preference has been introduced in order to make possible both the definition of belief-based preferences ($\overline{\rho}$ and $\overline{\Psi}$ as in the Terminal Preference case Section I.2.5), and a global preference for belief trajectories (the maximum of the belief-based preference degrees too), when the optimistic criterion is used (*i.e.* using \mathbb{S}_{Π}). Moreover, defining $\forall t \in \{0, \dots, H-1\}, \forall s \in \mathcal{S}, \forall a \in \mathcal{A}, \rho_t(s, a) = 0$, the use of the maximum-based global preference goes back to the case of the π -MDP with terminal preference only, *i.e.* to use the criterion (I.58) or (I.59).

The minimum-based global preference $\underline{\mathcal{G}}$ (II.3) has been used until this chapter and is the one proposed in [103, 102, 101]: using this global preference, a trajectory has a high preference degree if all the states of this trajectory have a high preference degree. It allows also, when the pessimistic criterion is used (*i.e.* using $\mathbb{S}_{\mathcal{N}}$), the definition of belief-based preferences ($\underline{\rho}$, $\underline{\Psi}$), and a global preference for belief state trajectories (also the minimum of the belief-based preference degrees). As noted before, if all intermediate preferences are set to 1, *i.e.* $\forall t \in \{0, \dots, H-1\}, \forall s \in \mathcal{S}, \forall a \in \mathcal{A}, \rho_t(s, a) = 1$, the use of the minimum-based global preference goes back to use a π -MDP with terminal preference only.

These global preferences suffer from the *drowning effect* [45]. Indeed, using the minimum-based global preference, if one of the encountered state has a low preference degree, the trajectory has a low preference degree no matter the other preference states. There is the same issue with the maximum-based global preference: no distinction will be made between a trajectory with only high preferences, and a trajectory with only one state with a high preference. As it will be presented and tested in this thesis, the *lexi* approaches, or other criteria [121] can bypass some qualitative issues.

Finally, it can be noted that, in the optimistic case, a π -POMDP is satisfied by a total

ignorant belief state: for instance, if $\forall s \in \mathcal{S}, \beta(s) = 1$, $\bar{\Psi}(\beta) = \max_{s \in \mathcal{S}} \min \{ \Psi(s), \beta(s) \} = \max_{s \in \mathcal{S}} \Psi(s)$. Thus, an uninformative belief leads to the best preference. A criterion mixing a pessimistic belief-based preference $\bar{\Psi}$ and an optimistic one over the belief preferences can be introduced, even if it means that this criterion is not related anymore on any preference over system states:

Definition II.1.3 (*Mixed Optimistic-Pessimistic Criterion, Terminal Preference Case*)

Given a strategy $(\delta) = (\delta_t)_{t=0}^{H-1}$ and an initial belief state $\beta_0 \in \Pi_{\mathcal{L}}^S$, the mixed π -POMDP value function can be defined by one of the three following equivalent formulae:

$$\begin{aligned} U(\beta_0, (\delta)) &= \max_{\beta_H \in \Pi_{\mathcal{L}}^S} \min \left\{ \min_{s \in \mathcal{S}} \max \{ \Psi(s), 1 - \beta_H(s) \}, \pi(\beta_H | \beta_0, (\delta)) \right\} \\ &= \max_{\beta_H \in \Pi_{\mathcal{L}}^S} \min \left\{ \underline{\Psi}(S_H), \pi(\beta_H | \beta_0, (\delta)) \right\} \\ &= \mathbb{S}_{\Pi} [\underline{\Psi}(S_H) | \beta_0, (\delta)]. \end{aligned} \quad (\text{II.12})$$

In a nutshell, the general form of the Dynamic Programming equation of a π -POMDP is,

$$\forall \beta \in \Pi_{\mathcal{L}}^S, \quad \hat{U}_0^*(\beta) = \tilde{\Psi}(\beta),$$

and, $\forall i \in \{1, \dots, H\}, \forall \beta \in \Pi_{\mathcal{L}}^S$,

$$\hat{U}_i^*(\beta) = \max_{a \in \mathcal{A}} \widehat{M} \left\{ \tilde{\rho}_t(\beta, a), \widehat{\mathbb{S}} \left(\pi_t(o' | \beta, a), \widehat{U}_{i-1}^*(\nu(\beta, a, o')) \right) \right\}, \quad (\text{II.13})$$

$$\hat{\delta}_{H-i}^*(\beta) \in \operatorname{argmax}_{a \in \mathcal{A}} \widehat{M} \left\{ \tilde{\rho}_t(\beta, a), \widehat{\mathbb{S}} \left(\pi_t(o' | \beta, a), \widehat{U}_{i-1}^*(\nu(\beta, a, o')) \right) \right\} \quad (\text{II.14})$$

For instance, in the case of the optimistic π -POMDP value function (*i.e.* based on \mathbb{S}_{Π}) with a maximum-based global preference ($\bar{\mathcal{G}}$),

- \hat{U}_i^* is denoted by \overline{U}_i^* ,
- $\tilde{\Psi}(\beta) = \bar{\Psi}(\beta) = \max_{s \in \mathcal{S}} \min \{ \Psi(s), \beta(s) \}$,
- $\tilde{\rho}_t(\beta, a) = \bar{\rho}_t(\beta) = \max_{s \in \mathcal{S}} \min \{ \rho_t(s, a), \beta(s) \}$,
- \widehat{M} is the maximum operator,
- $\widehat{\mathbb{S}} \left(\pi_t(o' | \beta, a), \widehat{U}_{i-1}^*(\nu(\beta, a, o')) \right) = \max_{o' \in \mathcal{O}} \min \left\{ \pi_t(o' | \beta, a), \overline{U}_{i-1}^*(\nu(\beta, a, o')) \right\}$,

see Algorithm 8.

Another example is the case of the pessimistic π -POMDP value function (*i.e.* based on $\mathbb{S}_{\mathcal{N}}$) with a minimum-based global preference ($\underline{\mathcal{G}}$),

- \hat{U}_i^* is denoted by \underline{U}_i^* ,
- $\tilde{\Psi}(\beta) = \underline{\Psi}(\beta) = \min_{s \in \mathcal{S}} \max \{ \Psi(s), 1 - \beta(s) \}$,
- $\tilde{\rho}_t(\beta, a) = \underline{\rho}_t(\beta) = \min_{s \in \mathcal{S}} \max \{ \rho_t(s, a), 1 - \beta(s) \}$,
- \widehat{M} is the minimum operator,

$$\bullet \widehat{\mathbb{S}}\left(\pi_t(o' | \beta, a), \widehat{U_{i-1}^*}(\nu(\beta, a, o'))\right) = \min_{o' \in \mathcal{O}} \max \left\{ 1 - \pi_t(o' | \beta, a), \underline{U_{i-1}^*}(\nu(\beta, a, o')) \right\},$$

see Algorithm 9.

Finally, in the case of the mixed optimistic-pessimistic π -POMDP value function,

- $\widehat{U_i^*}$ is denoted by U_i^* ,
- $\tilde{\Psi}(\beta) = \underline{\Psi}(\beta) = \min_{s \in \mathcal{S}} \max \{ \Psi(s), 1 - \beta(s) \}$,
- $\tilde{\rho}_t(\beta, a) = \underline{\rho}_t(\beta) = \min_{s \in \mathcal{S}} \max \{ \rho_t(s, a), 1 - \beta(s) \}$,
- \widehat{M} has not been defined, as the case of intermediate preferences was not considered for this criterion: it may be defined as the minimum or the maximum operator,
- $\widehat{\mathbb{S}}\left(\pi_t(o' | \beta, a), \widehat{U_{i-1}^*}(\nu(\beta, a, o'))\right) = \max_{o' \in \mathcal{O}} \min \left\{ \pi_t(o' | \beta, a), U_{i-1}^*(\nu(\beta, a, o')) \right\},$

see just above Definition II.1.3 for the case of terminal preference only.

Let us recall now that $\tilde{\mathcal{S}}^\pi = \Pi_{\mathcal{L}}^{\mathcal{S}}$ is a finite set of cardinal $\#\tilde{\mathcal{S}}^\pi = \#\mathcal{L}^{\#\mathcal{S}} - (\#\mathcal{L} - 1)^{\#\mathcal{S}}$ (the total number of $\#\mathcal{S}$ -size vectors valued in \mathcal{L} , minus $(\#\mathcal{L} - 1)^{\#\mathcal{S}}$ non-normalized distributions). For concrete problems, the state space can be dramatically large: $\#\tilde{\mathcal{S}}^\pi$ explodes and computations become intractable like in standard probabilistic POMDPs. The next section presents a way to exploits a specific structure of the problem that is very common in practice.

II.2 MIXED-OBSERVABILITY AND π -MOMDPs

The complexity issue of π -POMDP solving is due to the fact that the size of the belief state space $\Pi_{\mathcal{L}}^{\mathcal{S}}$ exponentially grows with the size of the state space \mathcal{S} , see the equation (I.60) of Section I.2.5. However, in practice, states are rarely completely hidden. Using mixed-observability can be a solution: inspired by a similar recent work in probabilistic POMDPs [83, 2], we present in this section a structured modeling that takes into account situations where the agent directly observes some part of the state. a π -POMDP which models such a situation respects the

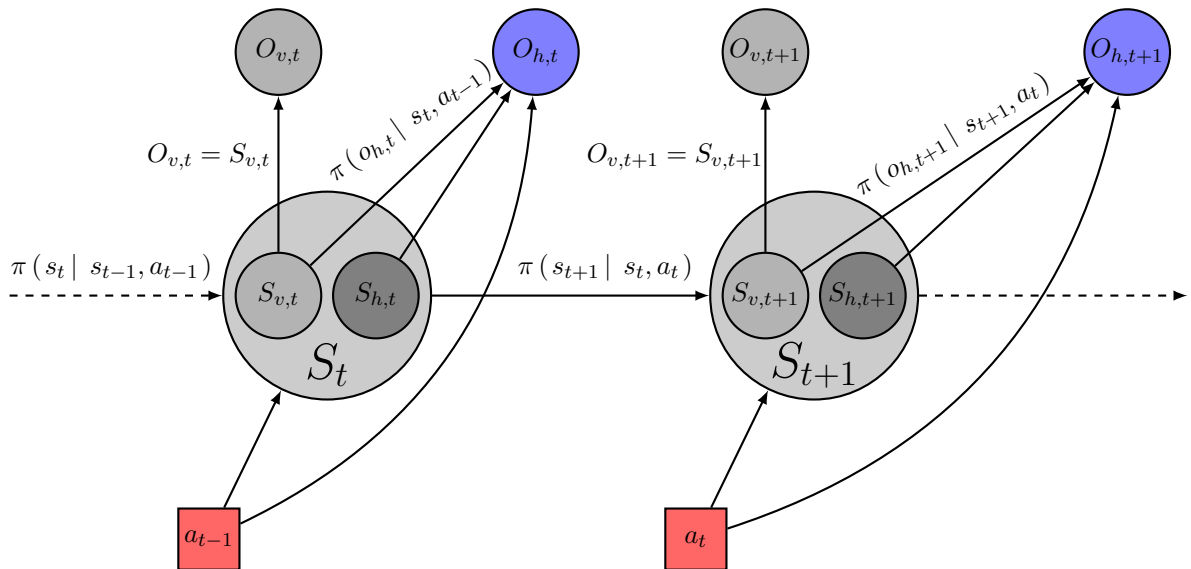


Figure II.1 – *Dynamic Bayesian Network of a π -MOMDP*: at time step t , the system state is described by variable $S_t = (S_{v,t}, S_{h,t})$. The received observation is $O_t = (O_{v,t}, O_{h,t})$ with $O_{v,t} = S_{v,t}$, and $O_{h,t}$ depending on S_t and action a_t .

mixed-observable property. Belief states are then used only for the partially observed components and the size of the belief state space is substantially reduced. Thus, this model generalizes both π -MDPs and π -POMDPs.

Like in [2], we assume that the state space \mathcal{S} of a Qualitative Possibilistic Mixed-Observable MDP (π -MOMDP) can be written as a Cartesian product of a visible state space \mathcal{S}_v and a hidden one \mathcal{S}_h : $\mathcal{S} = \mathcal{S}_v \times \mathcal{S}_h$. Let $s = (s_v, s_h)$ be a state of the system. The component s_v is directly observed by the agent and s_h is only partially observed through the observations of the set \mathcal{O}_h : we denote by $\pi_t(o'_h | s', a)$, the possibility distribution over the future observation $o'_h \in \mathcal{O}_h$ at time step t , knowing the future state $s' \in \mathcal{S}$ and the current action $a \in \mathcal{A}$. Figure II.1 illustrates the structure of this Mixed-Observable model.

The visible state space is integrated to the observation space: $\mathcal{O}_v = \mathcal{S}_v$ and $\mathcal{O} = \mathcal{O}_v \times \mathcal{O}_h$. Then, knowing that the current visible component of the state is s_v , the agent *necessarily* observes $o_v = s_v$ (if $o'_v \neq s_v$, $\pi_t(o'_v | s_v) = 0$). Formally, seen as a π -POMDP, its observation possibility distribution can be written as:

$$\begin{aligned} \pi_t(o' | s', a) &= \pi_t(o'_v, o'_h | s'_v, s'_h, a) \\ &= \min \{ \pi_t(o'_h | s'_v, s'_h, a), \pi_t(o'_v | s'_v) \} \\ &= \begin{cases} \pi_t(o'_h | s', a) & \text{if } o'_v = s'_v \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (\text{II.15})$$

since $\pi_t(o'_v | s'_v) = 1$ if $s'_v = o'_v$ and 0 otherwise. The following theorem, based on this equality enables the belief over hidden states to be defined.

Theorem 19 (Nature of Reachable Belief States)

Each reachable belief state of a π -MOMDP can be written as an element of $\mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$ where $\Pi_{\mathcal{L}}^{\mathcal{S}_h}$ is the set of possibility distributions over \mathcal{S}_h : any $\beta \in \Pi_{\mathcal{L}}^{\mathcal{S}_h}$ can be written as (s_v, β_h) with $\beta_h(s_h) = \max_{\bar{s}_v \in \mathcal{S}_v} \beta(\bar{s}_v, s_h)$ and $s_v = \operatorname{argmax}_{\bar{s}_v \in \mathcal{S}_v} \beta(\bar{s}_v, s_h)$.

The proof is given in Annex B.4

As all reachable belief states are in $\mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$ when the mixed-observability property holds, the next theorem rewrites the belief update function for the beliefs $\beta_h \in \Pi_{\mathcal{L}}^{\mathcal{S}_h}$ over the hidden system states $s_h \in \mathcal{S}_h$.

Theorem 20 (Belief Update for a π -MOMDP)

If a problem can be modeled by a π -MOMDP

$$\langle \mathcal{S}_v \times \mathcal{S}_h, \mathcal{A}, \mathcal{O}_h, T^\pi, O^\pi, (\rho_t)_{t=0}^{H-1}, \Psi, \beta_0 = (s_{v,0}, \beta_{h,0}) \rangle,$$

a new belief update function ν_h can be defined: if, at time step t , the current visible state is $s_{v,t} \in \mathcal{S}_v$, the current belief state about the hidden system state is $\beta_{h,t} \in \Pi_{\mathcal{L}}^{\mathcal{S}_h}$, the selected action is $a_t \in \mathcal{A}$, the next visible state is $s_{v,t+1} \in \mathcal{S}_v$ and the next observation is $o_{h,t+1} \in \mathcal{O}_h$, then the next belief state about the hidden system state is

$$\beta_{h,t+1}(s'_h) = \begin{cases} 1 & \text{if } \begin{aligned} &\pi_t(s'_h, s_{v,t+1}, o_{h,t+1} | s_{v,t}, \beta_{h,t}, a_t) \\ &= \pi_t(s_{v,t+1}, o_{h,t+1} | s_{v,t}, \beta_{h,t}, a_t) \end{aligned} \\ \pi_t(s'_h, s_{v,t+1}, o_{h,t+1} | s_{v,t}, \beta_{h,t}, a_t) & \text{otherwise} \end{cases}, \quad (\text{II.16})$$

where

$$\pi_t(s'_v, s'_h, o'_h | s_v, \beta_h, a) = \min \left\{ \pi_t(o'_h | s', a), \max_{s_h \in \mathcal{S}_h} \min \{ \pi_t(s' | s_v, s_h, a), \beta_h(s_h) \} \right\}$$

is the joint possibility distribution over hidden system states $s'_h \in \mathcal{S}_h$ and visible objects (visible system state and observation) $s'_v \in \mathcal{S}_v$ and $o'_h \in \mathcal{O}_h$. The notation $\pi_t(s'_v, o'_h | s_v, \beta_h, a)$ is for the possibility degree of the visible objects $\max_{s'_h \in \mathcal{S}_h} \pi_t(s'_h, s'_v, o'_h | s_v, \beta_h, a)$ (using the notation $s' = (s'_v, s'_h) \in \mathcal{S} = \mathcal{S}_v \times \mathcal{S}_h$). This belief update is denoted by

$$\beta'_h = \nu_h(s_v, \beta_h, a, s'_v, o'_h).$$

The proof is given in Annex B.5.

The state space of the belief π -MDP resulting from a π -MOMDP can be restricted to the product space $\mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$, *i.e.* a finer belief π -MDP than those presented previously, benefiting from the Mixed-Observability, can be defined: $\langle \tilde{S}^\pi, \tilde{T}^\pi, \mathcal{A}, (\tilde{\rho}_t)_{t=0}^{H-1}, \tilde{\Psi} \rangle$, where

- the state space of the belief π -MDP is defined as $\tilde{S}^\pi = \mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$,
- a transition possibility distribution in \tilde{T}^π is such that $\forall \{0, \dots, H-1\}, \forall a \in \mathcal{A}, \forall [(s_v, \beta_h), (s'_v, \beta'_h)] \in (\tilde{S}^\pi)^2$,

$$\pi_t((s'_v, \beta'_h) | (s_v, \beta_h), a) = \max_{\substack{o'_h \in \mathcal{O}_h \text{ s.t.} \\ \nu_h(s_v, \beta_h, a, s'_v, o'_h) = \beta'_h}} \pi_t(s'_v, o'_h | s_v, \beta_h, a),$$

where $\pi_t(s'_v, o'_h | s_v, \beta_h, a)$ is defined just above,

- If the belief-based preferences are optimistic *i.e.* $\tilde{\rho}_t = \overline{\rho}_t$ and $\tilde{\Psi} = \overline{\Psi}$, then $\forall t \in \{0, \dots, H-1\}, \forall s_v \in \mathcal{S}_v, \forall \beta_h \in \Pi_{\mathcal{L}}^{\mathcal{S}_h}, \forall a \in \mathcal{A}$, the preference functions can be rewritten

$$\overline{\rho}_t(s_v, \beta_h) = \max_{s_h \in \mathcal{S}_h} \min \{ \rho_t(s_v, s_h), \beta_h(s_h) \},$$

and

$$\overline{\Psi}(s_v, \beta_h) = \max_{s_h \in \mathcal{S}_h} \min \{ \Psi(s_v, s_h), \beta_h(s_h) \}.$$

Indeed, $\beta(\overline{s}_v, s_h) = 0$ if \overline{s}_v is not the actual visible state s_v , thus, for instance

$$\begin{aligned} \overline{\Psi}(\beta) &= \max_{s \in \mathcal{S}} \min \{ \Psi(s), \beta(s) \} \\ &= \max_{s_h \in \mathcal{S}_h} \min \{ \Psi(s_v, s_h), \beta(s_h, s_v) \}. \end{aligned}$$

- If the belief-based preferences are pessimistic *i.e.* $\tilde{\rho}_t = \underline{\rho}_t$ and $\tilde{\Psi} = \underline{\Psi}$, then $\forall t \in \{0, \dots, H-1\}, \forall s_v \in \mathcal{S}_v, \forall \beta_h \in \Pi_{\mathcal{L}}^{\mathcal{S}_h}, \forall a \in \mathcal{A}$, the preference functions can be rewritten

$$\underline{\rho}_t(s_v, \beta_h) = \min_{s_h \in \mathcal{S}_h} \max \{ \rho_t(s_v, s_h), 1 - \beta_h(s_h) \},$$

and

$$\underline{\Psi}(s_v, \beta_h) = \min_{s_h \in \mathcal{S}_h} \max \{ \Psi(s_v, s_h), 1 - \beta_h(s_h) \}.$$

Indeed, $1 - \beta(\overline{s}_v, s_h) = 1$ if \overline{s}_v is not the actual visible state s_v , thus, for instance

$$\begin{aligned} \underline{\Psi}(\beta) &= \min_{s \in \mathcal{S}} \max \{ \Psi(s), 1 - \beta(s) \} \\ &= \min_{s_h \in \mathcal{S}_h} \max \{ \Psi(s_v, s_h), 1 - \beta(s_h, s_v) \}. \end{aligned}$$

Theorem 21 (Dynamic Programming Equation of a π -MOMDP)

The dynamic programming equation becomes:

$$\forall (s_v, \beta_h) \in \mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}, \quad \widehat{U}_0^*(s_v, \beta_h) = \tilde{\Psi}(s_v, \beta_h),$$

and, $\forall i \in \{1, \dots, H\}, \forall (s_v, \beta_h) \in \mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h},$

$$\widehat{U}_i^*(s_v, \beta_h) = \max_{a \in \mathcal{A}} \widehat{M} \left\{ \tilde{\rho}_t(s_v, \beta_h, a), \widehat{\mathbb{S}} \left(\pi_t(s'_v, o'_h \mid s_v, \beta_h, a), \widehat{U}_{i-1}^*(\nu_h(s_v, \beta_h, a, s'_v, o'_h)) \right) \right\},$$

$$\widehat{\delta}_i^*(s_v, \beta_h) \in \operatorname{argmax}_{a \in \mathcal{A}} \widehat{M} \left\{ \tilde{\rho}_t(s_v, \beta_h, a), \widehat{\mathbb{S}} \left(\pi_t(s'_v, o'_h \mid s_v, \beta_h, a), \widehat{U}_{i-1}^*(\nu_h(s_v, \beta_h, a, s'_v, o'_h)) \right) \right\}, \quad (\text{II.17})$$

where ν_h is the new belief update function (Theorem 20), and the notations come from the general Dynamic Programming Equation II.13.

The proof is given in Annex B.6.

A standard algorithm would have computed $\widehat{U}_i^*(\beta)$ for each $\beta \in \Pi_{\mathcal{L}}^{\mathcal{S}}$ while this new dynamic programming equation leads to an algorithm which computes it only for all $(s_v, \beta_h) \in \mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$, since only this kind of belief states can be encountered. The size of the new belief space is

$$\#(\mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}) = \#\mathcal{S}_v \times (\#\mathcal{L}^{\#\mathcal{S}_h} - (\#\mathcal{L} - 1)^{\#\mathcal{S}_h}),$$

which is exponentially smaller than the size of standard π -POMDPs' belief space:

$$\#\Pi_{\mathcal{L}}^{\mathcal{S}} = \#\mathcal{L}^{\#\mathcal{S}_v \times \#\mathcal{S}_h} - (\#\mathcal{L} - 1)^{\#\mathcal{S}_v \times \#\mathcal{S}_h}.$$

An even finer belief π -MDP would be defined on the set of reachable belief states starting from the initial belief state $\beta_0: \Pi_{\mathcal{L}, \beta_0}^{\mathcal{S}}$ which is a subset of $\Pi_{\mathcal{L}}^{\mathcal{S}_h}$.

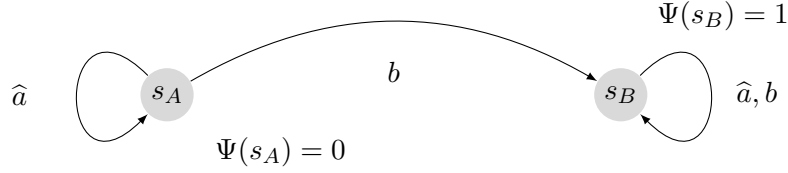
II.3 INFINITE HORIZON SETTINGS

A finite strategy for possibilistic MOMDPs can now be computed for larger problems using the dynamic programming equation of Theorem 21 and selecting maximizing actions for each state $(s_v, \beta_h) \in \mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$ (see the equation II.17), as done in the equation (II.14) for each $\beta \in \Pi_{\mathcal{L}}^{\mathcal{S}}$. However, for many problems in practice, it is difficult to determine a horizon size H . The goal of this section is to present an algorithm to solve optimistic π -MOMDPs with terminal preference only, under infinite horizon: it is the first proved algorithm to solve such π -(MO)MDPs.

II.3.1 The π -MDP case

Previous work, [101, 103], on solving π -MDPs proposed a Value Iteration algorithm that was proved to compute optimal value functions, but not necessarily optimal strategies for some problems with cycles. There is a similar issue in *undiscounted* probabilistic MDPs where the greedy strategy at convergence of Value Iteration does not need to be optimal [96]. It is not surprising that we are facing the same issue in π -MDPs since the possibilistic dynamic programming operator does not rely on algebraic products so that it cannot be contracted by some *discount factor* $0 < \gamma < 1$.

Figure II.2 – Deterministic example showing the limits of previous algorithms



For infinite horizon problems, the optimisitic π -MDP model has to undergo a little change. The dynamic is defined as stationary, as in the probabilistic case, see Section I.1.4: $\forall t \geq 0$, $\forall (s, s') \in \mathcal{S}^2$, $\forall a \in \mathcal{A}$,

$$\pi_t(s' | s, a) = \pi(s' | s, a).$$

The case of terminal preference only, is considered: starting from the general optimistic π -MDP (with intermediate preferences and minimum-based global preference, see Section I.2.4) the preference functions can be set equivalently to $\forall t \geq 0$, $\forall s \in \mathcal{S}$, $\forall a \in \mathcal{A}$,

$$\rho_t(s, a) = 1,$$

and thus, only the terminal preference function Ψ has an effect on the criterion, and has to be defined for the instantiation of the π -MDP.

Algorithm 10: Optimistic π -MDP VI Algorithm – Terminal Preference Only

```

1 for  $s \in \mathcal{S}$  do
2    $\overline{U}^*(s) \leftarrow 0$  ;
3    $\overline{U}^c(s) \leftarrow \Psi(s)$  ;
4    $\overline{\delta}^*(s) \leftarrow \hat{a}$  ;
5 while  $\overline{U}^* \neq \overline{U}^c$  do
6    $\overline{U}^* = \overline{U}^c$  ;
7   for  $s \in \mathcal{S}$  do
8      $\overline{U}^c(s) \leftarrow \max_{a \in \mathcal{A}} \max_{s' \in \mathcal{S}} \min \{ \pi(s' | s, a), \overline{U}^*(s') \}$  ;
9     if  $\overline{U}^c(s) > \overline{U}^*(s)$  then
10       $\overline{\delta}^*(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \max_{s' \in \mathcal{S}} \min \{ \pi(s' | s, a), \overline{U}^*(s') \}$  ;
11 return  $\overline{U}^*, \overline{\delta}^*$  ;

```

To the best of our knowledge, we propose here the first Value Iteration algorithm for π -MDPs, that provably returns an optimal strategy, and that is different from the one of [103]. Indeed, in the deterministic example of Figure II.2, action \hat{a} , which is clearly suboptimal in state s_A , was found to be optimal in s_A with this algorithm: however it is clear that since $\pi(s_B | s_A, b) = 1$ and $\Psi(s_B) = 1$, $\overline{U}_1^*(s_A) = 1$. Obviously, $\overline{U}_1^*(s_B) = 1$ and since $\pi(s_A | s_A, \hat{a}) = 1$, $\max_{s' \in \mathcal{S}} \min \{ \pi(s' | s_A, a), \overline{U}_1^*(s') \} = 1 \forall a \in \{\hat{a}, b\} = \mathcal{A}$, *i.e.* all actions are optimal in s_A . The “if” condition of Algorithm 10 permits to select the optimal action b during the first step. This condition and the initialization, which were not present in previous algorithms of the literature, are needed to prove the optimality of the strategy. The proof, which is quite lengthy and intricate, is presented in Annex C. This sound algorithm for π -MDPs will then be extended to π -MOMDPs in the next section.

As mentioned in [101], we assume the existence of an action “stay”, denoted by \hat{a} , which lets the system in the same state with necessity 1. This action is the possibilistic counterpart of

the discount parameter γ in the probabilistic model, as it guarantees convergence of the Value Iteration algorithm. However, we will see that action \hat{a} is finally used only on some particular satisfactory states. Note that a similar assumption is used to compute optimal strategies in the framework of deterministic processes (classical planning) whose horizon is not specified [70].

We denote by $\hat{\delta}$ the decision rule such that $\forall s \in \mathcal{S}$, $\hat{\delta}(s) = \hat{a}$. The set of all the finite strategies is $\Delta = \cup_{i \geq 1} \Delta_i$, and $\# \delta$ is the size of a strategy (δ) in terms of decision epochs. We can now define the optimistic criterion for an infinite horizon: if $(\delta) \in \Delta$,

$$\bar{U}(s_0, (\delta)) = \max_{\mathcal{T} \in \mathcal{T}_{\# \delta}} \min \left\{ \pi(\mathcal{T} | s_0, (\delta)), \Psi(s_{\# \delta}) \right\}, \quad (\text{II.18})$$

where $\mathcal{T} = (s_1, \dots, s_{\# \delta})$ is a trajectory of system states, $\mathcal{T}_{\# \delta}$ the set of such trajectories, and

$$\pi(\mathcal{T} | s_0, (\delta)) = \min_{i=0}^{\# \delta - 1} \pi(s_{i+1} | s_i, \delta_i(s_i)).$$

Theorem 22 (*Optimality of the VI Algorithm for Optimistic π -MDPs*)

If there exists an action \hat{a} such that, for each $s \in \mathcal{S}$, $\pi(s' | s, \hat{a}) = 1$ if $s' = s$ and 0 otherwise, then Algorithm 10 computes the maximum optimistic criterion and an optimal strategy, i.e. maximizing the criterion (II.18), which is stationary (i.e. which does not depend on the stage of the process t).

The proof is given in Annex C. Note that, as in the probabilistic case (see Section I.1.5), the computed optimal strategy is stationary i.e. does not depend on the time step of the process.

Let s be a state such that $\bar{\delta}^*(s) = \hat{a}$, where $\bar{\delta}^*$ is the returned strategy. By looking at Algorithm 10, it can be noted that $\bar{U}^*(s)$ always remains equal to $\Psi(s)$ during the iterations of the algorithm after the first entry in the while loop. Thus, $\forall s' \in \mathcal{S}$, either $\forall a \in \mathcal{A}$, $\Psi(s) \geq \pi(s' | s, a)$, or $\Psi(s) \geq \bar{U}^*(s')$. If the problem is non trivial, it means that s is a goal ($\Psi(s) > 0$) and that degrees of possibility of transitions to better goals are lower than the degree of preference for s .

II.3.2 Value Iteration for π -MOMDPs

We are now ready to propose the Value Iteration algorithm for π -MOMDPs which reduces to a belief π -MDP which is optimistic and with terminal preference only (whose Value Iteration algorithm has been presented in the previous section). This Value Iteration algorithm is, for instance, devoted to π -MOMDPs with the π -POMDP mixed optimistic-pessimistic criterion, see Definition II.1.3: the terminal preference function is then $\Psi(s_v, \beta_h) = \min_{s_h \in \mathcal{S}_h} \max \{ \Psi(s_v, s_h), 1 - \beta_h(s_h) \}$. An other example of appropriate π -MOMDP with the optimistic π -POMDP criterion (see Algorithm 8, removing intermediate preferences): the terminal preference function is in this case $\bar{\Psi}(s_v, \beta_h) = \max_{s_h \in \mathcal{S}_h} \min \{ \Psi(s_v, s_h), \beta_h(s_h) \}$.

Note that Algorithm 11 has the same structure as Algorithm 10. Note as well that a π -MOMDP is a π -MDP over $\mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$. Recall that the transition possibility distribution is

$$\pi_t((s'_v, \beta'_h) | (s_v, \beta_h), a) = \max_{\substack{o'_h \in \mathcal{O}_h \text{ s.t.} \\ \nu_h(s_v, \beta_h, a, s'_v, o'_h) = \beta'_h}} \pi_t(s'_v, o'_h | s_v, \beta_h, a).$$

To satisfy the assumption of Theorem 22, it suffices to ensure that $\pi_t((s'_v, \beta'_h) | (s_v, \beta_h), a) = 1$ if $s'_v = s_v$ and $\beta'_h = \beta_h$, and 0 otherwise. This property is verified if the two following conditions hold: $\pi(s'_v, s'_h | s_v, s_h, \hat{a}) = 1$ if $(s'_v, s'_h) = (s_v, s_h)$, and 0 otherwise, and there exists

Algorithm 11: π -MOMDP Value Iteration Algorithm

```

1 for  $s_v \in \mathcal{S}_v$  and  $\beta_h \in \Pi_{\mathcal{L}}^{\mathcal{S}_h}$  do
2    $\bar{U}^*(s_v, \beta_h) \leftarrow 0$  ;
3    $\bar{U}^c(s_v, \beta_h) \leftarrow \tilde{\Psi}(s_v, \beta_h)$  ;
4    $\bar{\delta}^*(s_v, \beta_h) \leftarrow \hat{a}$  ;
5 while  $\bar{U}^* \neq \bar{U}^c$  do
6    $\bar{U}^* \leftarrow \bar{U}^c$  ;
7   for  $s_v \in \mathcal{S}_v$  and  $\beta_h \in \Pi_{\mathcal{L}}^{\mathcal{S}_h}$  do
8      $\bar{U}^c(s_v, \beta_h) \leftarrow \max_{a \in \mathcal{A}} \max_{s'_v \in \mathcal{S}} \max_{o'_h \in \mathcal{O}_h} \min \left\{ \pi(s'_v, o'_h | s_v, \beta_h, a), \bar{U}^*(s'_v, \nu_h(s_v, \beta_h, s'_v, o'_h)) \right\}$  ;
9     if  $\bar{U}^c(s_v, \beta_h) > \bar{U}^*(s_v, \beta_h)$  then
10       $\bar{\delta}^*(s_v, \beta_h) \in \operatorname{argmax}_{a \in \mathcal{A}} \max_{s'_v \in \mathcal{S}} \max_{o'_h \in \mathcal{O}_h} \min \left\{ \pi(s'_v, o'_h | s_v, \beta_h, a), \bar{U}^*(s'_v, \nu_h(s_v, \beta_h, s'_v, o'_h)) \right\}$  ;
11 return  $u^*, \delta^*$  ;

```

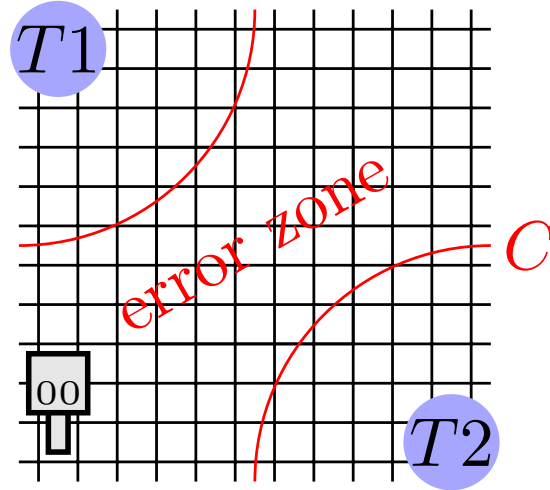
an observation “nothing” \hat{o}_h that is required for each state when \hat{a} is chosen *i.e.* $\forall (s'_v, s'_h) \in \mathcal{S}$, $\pi(o'_h | s'_v, s'_h, \hat{a}) = 1$ if $o'_h = \hat{o}_h$ and 0 otherwise. Indeed, it means that \hat{o}_h is received for sure when \hat{a} is selected: no information is provided by this observation, and the belief does not evolve.

II.4 RESULTS ON A ROBOTIC MISSION AND POSSIBILISTIC BELIEF STATE BEHAVIOUR

This section is devoted to the use of strategies computed by Algorithm 11 in the context of a concrete robotic problem. Consider a robot over a grid of size $g \times g$, with $g > 1$. It always perfectly knows its location on the grid $(x, y) \in \{1, \dots, g\}^2$, which forms the visible state space \mathcal{S}_v . It starts at location $s_{v,0} = (1, 1)$. Two targets are located at $(x_1, y_1) = (1, g)$ (“target 1”) and $(x_2, y_2) = (g, 1)$ (“target 2”) on the grid, and the robot perfectly knows their positions. One of the targets is A , the other B and the robot’s mission is to identify and reach target A as soon as possible. The robot does not know which target is A : the two situations, “target 1 is A ” ($A1$) and “target 2 is A ” ($A2$), constitute the hidden state space \mathcal{S}_h . The moves of the robot are deterministic and its actions \mathcal{A} consist in moving in the four directions plus the action “stay”. At each stage of the process, the robot analyzes pictures of each target and gets then an observation of the targets’ natures: the two targets (oAA) can be observed as A , or target 1 (oAB), or target 2 (oBA) or no target (oBB).

In the probabilistic framework, the probability of having a good observation of target $i \in \{1, 2\}$, is not really known but approximated by $\mathbf{p}(\text{good}_i | x, y) = \frac{1}{2} \left[1 + \exp \left(-\frac{\sqrt{(x-x_i)^2 + (y-y_i)^2}}{D} \right) \right]$ where $(x, y) = s_v \in \{1, \dots, g\}^2$ is the location of the robot, (x_i, y_i) the position of target i , and D a normalization constant. We suppose that the observations of both targets are independant: then, for instance, the probability $\mathbf{p}(oAB | (x, y), A1)$ is equal to $\mathbf{p}(\text{good}_1 | (x, y)) \cdot \mathbf{p}(\text{good}_2 | (x, y))$, $\mathbf{p}(oAA | (x, y), A1)$ to $\mathbf{p}(\text{good}_1 | (x, y)) \cdot [1 - \mathbf{p}(\text{good}_2 | (x, y))]$, and so on. Each step of the process before reaching a target costs 1, reaching target A is rewarded by 100, and -100 for B . The probabilistic strategy was computed in mixed-observability settings with APPL¹ based on SARSOP [83, 69] (see

¹The used software is available at <http://bigbird.comp.nus.edu.sg/pmwiki/farm/appl/>

Figure II.3 – Illustration of a robotic mission, first experiment on π -MOMDPs

Section I.1.11), using a precision of 0.046 (the memory limit is reached for higher precisions) and a discount factor $\gamma = 0.99$. This problem cannot be solved with the exact algorithm for MOMDPs [2] because it consumes the entire RAM after 15 iterations.

In the framework of Qualitative Possibility Theory, it is considered always possible to observe the good target: $\pi(\text{good} \mid x, y) = 1$. Secondly, further the robot is from target i , the more likely it is to badly observe it (e.g. observe A instead of B), which is a reasonable assumption if the actual probabilistic observation model is imprecisely known: $\pi(\text{bad}_i \mid x, y) = \frac{(x-x_i)^2 + (y-y_i)^2}{2(g-1)^2}$, and thus, \mathcal{L} can be defined by $\{0, \frac{1}{2(g-1)}, \dots, 1\}$, or any other scale preserving the fact that the possibility degree of misperceiving increases with the distance from the considered target. The observation of a target is also considered as NI-independent from the the observation of the other target (see Definition I.2.6 of Section I.2.2). Thus for instance, $\pi(oAB \mid (x, y), A1) = 1$, $\pi(oAA \mid (x, y), A1) = \pi(\text{bad}_2 \mid x, y)$, $\pi(oBA \mid (x, y), A1) = \min\{\pi(\text{bad}_1 \mid x, y), \pi(\text{bad}_2 \mid x, y)\}$, etc. Note that the construction of this model with a probability-possibility transformation [53] would have been equivalent. The terminal preference function Ψ is equal to 0 for all the system's states and to 1 for states $[(x_1, y_1), A1]$ and $[(x_2, y_2), A2]$ where (x_i, y_i) is the position of target i . As mentioned in [101], the computed strategy guarantees a shortest path to a goal state. The strategy then aims at reducing the mission duration. The mixed optimistic-pessimistic criterion, see Definition II.1.3, is used here to compute the strategy.

Standard π -POMDPs, which do not exploit mixed-observability contrary to our π -MOMDP model, could not solve even very small 3×3 grids. Indeed, for this problem, $\#\mathcal{L} \geq 5$, $\#\mathcal{S}_v = 9$, and $\#\mathcal{S}_h = 2$. Thus, $\#\mathcal{S} = \#\mathcal{S}_v \times \#\mathcal{S}_h = 18$ and the number of belief states is then $\#\Pi_{\mathcal{L}}^{\mathcal{S}} = \mathcal{L}^{\#\mathcal{S}} - (\mathcal{L} - 1)^{\#\mathcal{S}} \geq 5^{18} - 4^{18} \geq 3.7 \cdot 10^{12}$ instead of 81 states with a π -MOMDP. Therefore, the following experimental results could **not** be conducted with standard π -POMDPs, which indeed justifies our present work on π -MOMDPs.

In order to compare performances of the probabilistic and possibilistic models, we compare the average of their total (undiscounted) rewards at execution, *i.e.* a reward-based criterion really close to the probabilistic criterion (the same with $\gamma = 1$): since the situation (the nature of the targets) is fully known by the agent when the robot is at a target's location, it can not end up choosing target B . If k is the number of time steps to identify and reach the correct target, then the total reward is $100 - k$.

We consider now that, in reality (thus here for the simulations), and contrary to what is described by the model, used image processing algorithms badly perform when the robot is far away from targets, *i.e.*, if $\forall i \in \{1, 2\}$, $\sqrt{(x - x_i)^2 + (y - y_i)^2} > C$, with C a positive constant,

then $\mathbf{p}(good_i | x, y) = 1 - P_{bad} < \frac{1}{2}$. In all other cases, we assume that the probabilistic model is the good one. Figure II.3 illustrates this problem, and indicates the zone where the robot misperceives calling it “error zone”. For the following numerical experiments, we used 10^4 simulations to compute the statistical mean of the total reward at execution. The grid was 10×10 , $D = 10$ and $C = 4$.

Figure II.4.a shows that the probabilistic model is more affected by the introduced error than the possibilistic one: it shows the total reward at execution of each model as a function of P_{bad} , the probability of badly observing targets when the robot’s location is such that $\sqrt{(x - x_i)^2 + (y - x_i)^2} > C$, *i.e.* when the robot is in the “error zone”. This is due to the fact that the possibilistic update of the belief state does not take into account new observations when the robot has already obtained a more reliable one, whereas the probabilistic model modifies the current belief at each step. Indeed, as there are only two hidden states ($A1$ and $A2$) that we now denote by s_h^1 and s_h^2 , if $\beta_h(s_h^1) < 1$, then $\beta_h(s_h^2) = 1$ (possibilistic normalization). As the hidden state does not change during the mission, the joint possibility distribution over the hidden state and the observation is the minimum of the possibility distribution over the system state (described by the current belief state) and the observation possibility degree: *e.g.* for s_h^1 , the joint distribution is $\min\{\pi(o_h | s_v, s_h^1, a), \beta_h(s_h^1)\}$, with $o_h \in \{oAA, oAB, oBA, oBB\}$. It implies that the joint possibility of s_h^1 and the observation o_h , is smaller than $\beta_h(s_h^1)$. The possibilistic counterpart of the belief update equation, see the equation (I.62) or the equation (II.16) for Mixed-Observability settings, ensures that the next belief is either more skeptic about s_h^1 if the observation is more reliable and confirms the prior belief ($\pi(o_h | s_v, s_h^1, a)$ is smaller than $\beta_h(s_h^1)$); or changes to the opposite belief if the observation is more reliable and contradicts the prior belief ($\pi(o_h | s_v, s_h^2, a)$ is smaller than both $\beta_h(s_h^1)$ and $\pi(o_h | s_v, s_h^1, a)$); or yet simply remains unchanged if the observation is not more informative than the current belief.

The following theorem gives sufficient conditions leading to an informative possibilistic belief update *i.e.* which make the resulting belief state more specific (see Definition I.2.5 of Section I.2.1) than the previous one: a belief state $\beta_1 \in \Pi_{\mathcal{L}}^S$ is said more specific than a belief state $\beta_2 \in \Pi_{\mathcal{L}}^S$ if $\forall s \in \mathcal{S}, \beta_1(s) \leq \beta_2(s)$. In order to get a total order on $\Pi_{\mathcal{L}}^S$, the ranking relation \preceq is defined to sort belief states with respect to their specificity:

$$\beta_1 \preceq \beta_2 \Leftrightarrow \sum_{s \in \mathcal{S}} \beta_1(s) \leq \sum_{s \in \mathcal{S}} \beta_2(s).$$

Note that if β_1 is more specific than β_2 , then $\beta_1 \preceq \beta_2$.

Theorem 23 (Conditions for an increasing specificity of the belief states)

Let $\beta_0 \in \Pi_{\mathcal{L}}^S$ be the initial belief state modeling the total ignorance *i.e.* $\forall s \in \mathcal{S}, \beta_0(s) = 1$. If the transition function $\pi(s' | s, a)$ is deterministic, and if the observations are not informative, *i.e.* $\forall s' \in \mathcal{S}, \forall a \in \mathcal{A}, \forall o' \in \mathcal{O}, \pi(o' | s', a) = 1$, then $\beta_{t+1} \preceq \beta_t$, where $\beta_{t+1} \in \Pi_{\mathcal{L}}^S$ is the result of an update (I.62) of the belief state $\beta_t \in \Pi_{\mathcal{L}}^S$,

The result $\beta_{t+1} \preceq \beta_t$ remains true if, for each action $a \in \mathcal{A}$, the transition possibility distributions $\pi(s' | s, a) = \mathbb{1}_{\{s=s'\}}$ (*i.e.* is equal to 1 if $s' = s$ and 0 otherwise), and $\forall o' \in \mathcal{O}, \forall a \in \mathcal{A}, \forall s', \tilde{s} \in \mathcal{S}, \pi(o' | s', a) \neq \pi(o' | \tilde{s}, a)$.

The proof is given in Annex B.7. Note that this theorem was devoted to π -POMDP. However, the same result holds for π -MOMDPs, replacing $s \in \mathcal{S}$ by $s_h \in \mathcal{S}_h$ and $o \in \mathcal{O}$ by $o_h \in \mathcal{O}_h$. Note also that the conditioning presented in Definition I.2.10 of Section I.2.2 leads to another belief update: with this one, if β_{t+1} is the update of the belief state β_t , β_{t+1} is not less specific than β_t . However, it does not ensure that $\beta_{t+1} \preceq \beta_t$.

The probabilistic belief update does not have these capabilities to directly change to the opposite belief and to disregard less reliable observations: the robot then proceed towards the

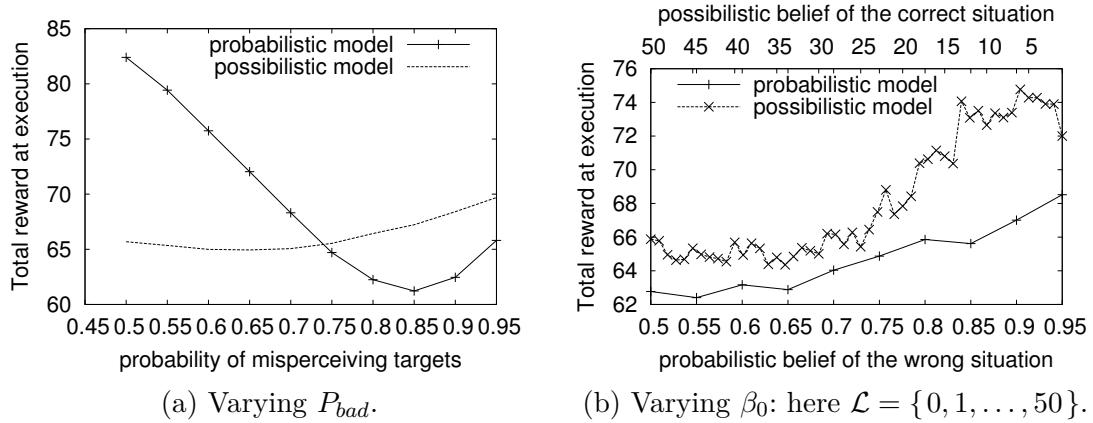


Figure II.4 – Comparison of the total reward gathered at execution for possibilistic and probabilistic models.

wrong target because it is initially far away and thus badly observes targets (without knowing it). When it is close to this target, it gets good observations and gradually modifies its belief which becomes true enough to convince it to go towards the right target. However it has to cross a remote area away from targets: this yet gradually modifies its belief, which becomes wrong, and the robot finds itself in the same initial situation: it loses thus a lot of time to get out of this loop. We can observe that the total reward increases for high probabilities of misperceiving P_{bad} : this is because this high error leads the robot to reach the wrong target faster, thus to entirely know that the true target is the other one.

Now if we set $P_{bad} = 0.8$ and evaluate the total reward at execution for different wrong initial beliefs, we get Figure II.4.b with the same parameters: we compare here the possibilistic model and the probabilistic one when the initial belief is strongly oriented towards the wrong hidden states (i.e. the agent strongly believes that target 1 is B whereas it is A in reality). Note that the possibilistic belief of the good target decreases when the necessity of the bad one increases. This figure shows that the possibilistic model yields higher rewards at execution if the initial belief is wrong and the observation function is imprecise ².

II.5 CONCLUSION

We have proposed a Value Iteration algorithm for possibilistic MDPs, which can produce optimal stationary strategies in infinite horizon contrary to previous methods. We have provided a complete proof of convergence that relies on the existence of intermediate “stay” actions that vanish for non goal states in the final optimal strategy. Finally, we have extended this algorithm to a new Mixed-Observable possibilistic MDP model, whose complexity is exponentially smaller than possibilistic POMDPs, so that we could compare π -MOMDPs with their probabilistic counterparts on realistic robotic problems. Our experimental results show that possibilistic strategies can outperform probabilistic ones when the observation function is not precisely known.

A value iteration algorithm for the pessimistic π -MDPs can be easily written on the basis of the optimistic value iteration algorithm, Algorithm 10. However, the optimality of the returned strategy seems hard to prove, essentially because it is not enough to construct a maximizing

²The implementation of the solver, as well as a generator of descriptions of such recognition problems (expressed in the RDDL language [105]) which is the input of the solver, are available on the repository <https://github.com/drougui/ppudd> : executions can be simulated using the possibilistic optimal strategy which is the output of the solver.

trajectory, as the proof in Annex C does. The works [122, 95] may be useful materials to help us to get results about pessimistic π -MDP in infinite horizon settings.

Note that, if some probabilistic information is really known by the designers of the model, the π -POMDP is a qualitative approximation of the probabilistic POMDP: in such cases the quantitative information which is available is not taken into account, in order to simplify the problem resolution. Indeed, this model only implies maximum and minimum operators. Note also that if the model has been built from expert qualitative information about the plausible behaviour of the system, an arbitrary probabilistic POMDP has no more guarantees than the possibilistic one which only uses really available description of the problem.

Finally, as highlighted by the experiment, while the π -POMDPs are based on a computationally simpler uncertainty model than the probabilistic POMDP, the possibilistic belief updating process may have an interesting behaviour. Under some sufficient assumptions given by Theorem 23, the belief state is not modified by less reliable information than the previously gathered information, but is able to change to a quite opposite belief state if an information which suggests it and which is more reliable is received. More complex problems have to be studied to have a better overview of this behaviour in a wider set of situations. However, π -POMDPs with a large system state (or π -MOMDPs with large \mathcal{S}_h) cannot be solved with reasonable computation times by algorithms developed until now. The next section presents and uses another problem structure, possibilistic counterpart of *factored POMDPs*, which leads to easier computations of optimal possibilistic strategies and making wider experiments possible: the developed solvers use Algebraic Decision Diagrams avoiding some useless computations and making handled data more compact.

DEVELOPMENT OF SYMBOLIC ALGORITHMS TO SOLVE π -POMDPs

III

In this chapter, we propose the study of factored π -MOMDP models in order to solve large structured planning problems under qualitative uncertainty, or considered as qualitative approximations of probabilistic problems. Building upon the SPUDD algorithm for solving factored (probabilistic) MDPs, we conceived a symbolic algorithm named PPUDD for solving factored π -MOMDPs. Whereas SPUDD's decision diagrams' leaves may be as large as the state space since their values are real numbers aggregated through additions and multiplications, PPUDD's ones always remain in the finite scale \mathcal{L} via min and max operations only. Finally, we present a sound transformation from factored mixed-observable possibilistic problems with both hidden and visible state variables to fully observable ones, on which PPUDD is run. Our experiments show that PPUDD's computation time is much lower than SPUDD, Symbolic-HSVI and APPL for possibilistic and probabilistic versions of the same benchmarks under either total or mixed-observability, while still providing high-quality strategies.

III.1 INTRODUCTION

As explained at the end of the previous chapter, starting from a probabilistic MOMDP [83, 2], the use of Possibility Theory instead of Probability Theory leads to an approximation of the initial probabilistic model [102]: probabilities and rewards are replaced by qualitative statements that lie in a finite scale (as opposed to continuous ranges in the probabilistic framework), which results in simpler computations. Possibilities and probabilities have similar behaviors for problems with low entropy probability distributions [46]. However, the decision resulting from both models can be completely different in practice. Consider for example a situation in which three actions a_A , a_B and a_C lead to three different sets of system states:

- action a_A leads to $\mathcal{S}_A = \{s_A^1, s_A^2\}$, with $\mathbf{p}(s_A^1 | a_A) = \mathbf{p}(s_A^2 | a_A) = 0.5$, $r(s_A^1, a_A) = 1$ and $r(s_A^2, a_A) = 0$;
- action a_B leads to $\mathcal{S}_B = \{s_B^1\}$ ($\mathbf{p}(s_B^1 | a_B) = 1$) with $r(s_B^1, a_B) = 0.5$;
- action a_C leads to $\mathcal{S}_C = \{s_C^1, \dots, s_C^7\}$ with $\mathbf{p}(s_C^1 | a_C) = 0.4$, $\mathbf{p}(s_C^2 | a_C) = \dots = \mathbf{p}(s_C^7 | a_C) = 0.1$, $r(s_C^1, a_C) = 0$ and $r(s_C^2, a_C) = \dots = r(s_C^7, a_C) = 1$.

Actions a_A and a_B lead to the same average gain: 0.5. However, action a_C leads to a better one: 0.6. Now, let us define a possibilistic model respecting the ranking of event plausibilities:

- $\pi(s_A^1 | a_A) = \pi(s_A^2 | a_A) = 1$, $\rho(s_A^1, a_A) = 1$ and $\rho(s_A^2, a_A) = 0$;
- $\pi(s_B^1 | a_B) = 1$ and $\rho(s_B^1, a_B) = \frac{3}{4}$;
- $\pi(s_C^1 | a_C) = \frac{2}{4}$, $\pi(s_C^2 | a_C) = \dots = \pi(s_C^7 | a_C) = \frac{1}{4}$, $\rho(s_C^1, a_C) = 0$ and $\rho(s_C^2, a_C) = \dots = \rho(s_C^7, a_C) = 1$.

The action a_A is chosen by the optimistic approach. Indeed, it is entirely possible to reach s_A^1 ($\pi(s_A^1 | a_A) = 1$) whose preference is 1. The action a_B is chosen by the pessimistic approach since the preference $\frac{3}{4}$ is reached with certainty with this action: action a_A leads potentially to s_A^1 with the preference $\rho(s_A^2, a_A) = 0$, and a_C leads potentially to s_C^1 with $\rho(s_C^1, a_C) = 0$. Thus, in some cases, the three approaches, (the optimistic, pessimistic possibilistic approaches and the probabilistic approach) may select three different actions.

Recall that the possibilistic approach benefits from computations on *finite* belief state spaces, whereas probabilistic MOMDPs tackle *infinite* ones. It means that the same algorithmic techniques can be used to solve π -MDPs, π -POMDPs or π -MOMDPs. What is lost in precision of the uncertainty model is saved in computational complexity. Problems where the uncertainty model is imprecisely known are also naturally well-modeled by π -MOMDPs [41]: e.g., occurrence frequencies of observations resulting from a complex image processing algorithm depend on many environmental factors like lightening conditions so that they are generally not precisely known.

Previously presented works on π -(MO)MDPs do not totally take advantage of the problem structure, i.e. visible or hidden parts of the state can be themselves factored into many state variables, which are flattened by current possibilistic approaches. In probabilistic settings, factored MDPs and Symbolic Dynamic Programming (SDP) frameworks [17, 62] have been extensively studied in order to reason directly at the level of state variables rather than state space in extension. However, factored probabilistic MOMDPs have not yet been proposed to the best of our knowledge, probably because of the intricacy of reasoning with a mixture of a finite state subspace and an infinite belief state subspace due to the probabilistic model – contrary to the possibilistic case where both subspaces are finite. The famous algorithm SPUDD [62] solves factored probabilistic MDPs by using symbolic functional representations of value functions and strategies in the form of Algebraic Decision Diagrams (ADDs) [4], which compactly encode real-valued functions of Boolean variables: ADDs are directed acyclic graphs whose nodes represent state variables and leaves are the function's values. Instead of updating state values individually at each iteration of the algorithm, they are aggregated within ADDs and operations are symbolically and directly performed on ADDs over many states at once. However, SPUDD suffers from manipulation of potentially huge ADDs in the worst case: for instance, expectation involves additions and multiplications of real values (probabilities and rewards), creating other values in-between, in such a way that the number of ADD leaves may equal the size of the state space, which is exponential in the number of state variables. Therefore, the work presented here is motivated by the simple observation that **symbolic operations with possibilistic MDPs would necessarily limit the size of ADDs**: indeed, this formalism operates over a *finite* possibilistic scale \mathcal{L} with only max and min operations involved, which implies that all manipulated values remain in the finite scale \mathcal{L} , which is generally far smaller than the number of states.

Figure III.1 shows that ADDs used in the possibilistic settings have a limited number of nodes since the number of their leaves are at most equal to the cardinal of the possibilistic finite scale \mathcal{L} : the maximal size (maximal number of nodes) of an ADD whose leaves are in \mathcal{L} , is represented as a function of $\#\mathcal{L}$, in the case of 8 and 10 variables. The biggest ADD in quantitative settings with 8 (resp. 10) variables have $2^9 - 1$ (resp. $2^{11} - 1$) nodes, as represented also in Figure III.1. Operating ADDs in the possibilistic framework would behave much like manipulating Binary Decision Diagrams (BDDs) [20], which are generally more compact and thus more efficient than ADDs.

In this chapter we present a Symbolic Dynamic Programming algorithm for solving factored π -MOMDPs named Possibilistic Planning Using Decision Diagram (PPUDD). This contribution alone is insufficient, since it relies on a belief state variable whose number of values is exponential in the size of the state space. Therefore, our second contribution is a theorem to

Maximal number of nodes of an ADD: leaves in \mathcal{L} versus in \mathbb{R}

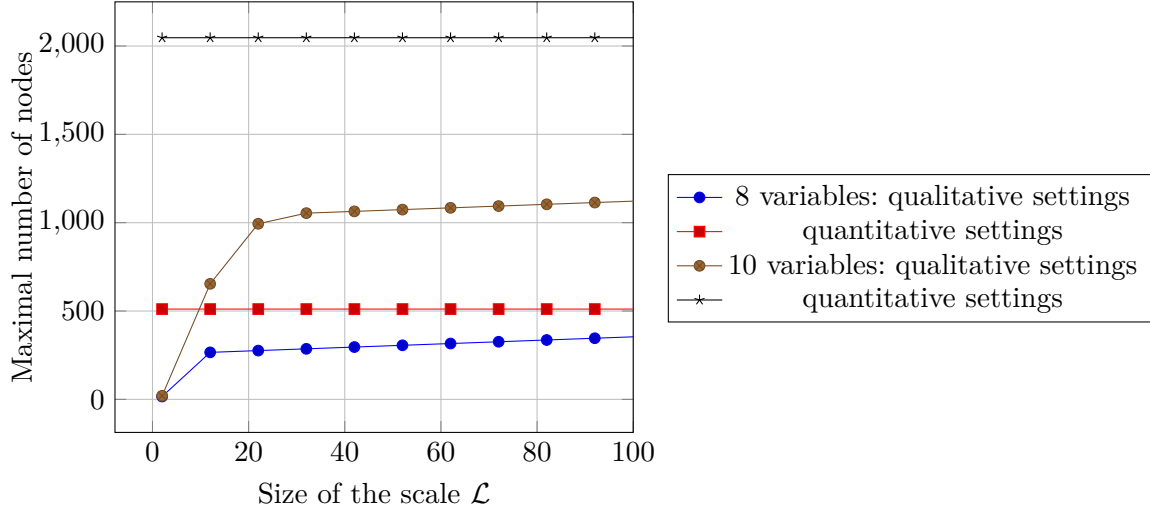


Figure III.1 – The maximal size (total number of nodes) of an ADD whose values are in \mathcal{L} , i.e. in qualitative settings, is limited: the upper bound is represented in blue and brown lines with circles, as a function of the size of \mathcal{L} . When the leaves of the ADD are in \mathbb{R} , the number of its nodes is potentially exponential in the number of variables: the upper bound is represented with red squares and black stars (as a constant function of the size of \mathcal{L}).

factorize the belief state itself in many variables under some assumptions about dependence relationships between state and observation variables of a π -MOMDP, which makes our algorithm more tractable while still exact and optimal. We note that our idea of factorizing the belief state under mixed-observability is sufficiently general to be reused in probabilistic models. Then, we experimentally assess our approach on possibilistic and probabilistic versions of the same benchmarks: PPUDD against SPUDD and APRICODD [116] under total observability to demonstrate that generality of our approximate approach does not penalize performances on restrictive submodels; PPUDD against symbolic HSVI [112] (a symbolic version of HSVI, see Section I.1.11) and APPL [69, 83] (already used in previous chapter, and based on SARSOP, see Section I.1.11) under mixed-observability. These promising results were a motivation to take part in the International Probabilistic Planning Competition 2014 (IPPC): results of PPUDD on the fully observable track of IPPC 2014 are then provided and discussed. A general practical solver for solving π -MOMDP using ADDs and available on the repository <https://github.com/drougui/ppudd> is finally detailed: performances of this solver on the problems of the Partially Observable track of IPPC are also presented.

III.2 SOLVING FACTORED π -MOMDPs USING SYMBOLIC DYNAMIC PROGRAMMING

Factored MDPs [62] have been used to efficiently solve structured sequential decision problems under probabilistic uncertainty, by symbolically reasoning on functions of states via decision diagrams rather than on individual states. Inspired by this work this section sets up a symbolic resolution of factored π -MOMDPs, which assumes that the visible state space \mathcal{S}_v , the hidden one \mathcal{S}_h and the set of observations \mathcal{O}_h are each cartesian products of finite sets described by variables. It boils down to solving a finite-space belief π -MDP whose state space is in the form of $\mathcal{S}_v^1 \times \dots \times \mathcal{S}_v^m \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$, where each of those spaces is finite. We will see in the next section how $\Pi_{\mathcal{L}}^{\mathcal{S}_h}$ can be further factorized thanks to the factorization of \mathcal{S}_h and \mathcal{O}_h . While probabilistic belief factorization in [18, 110] is approximate, the one presented here relies on

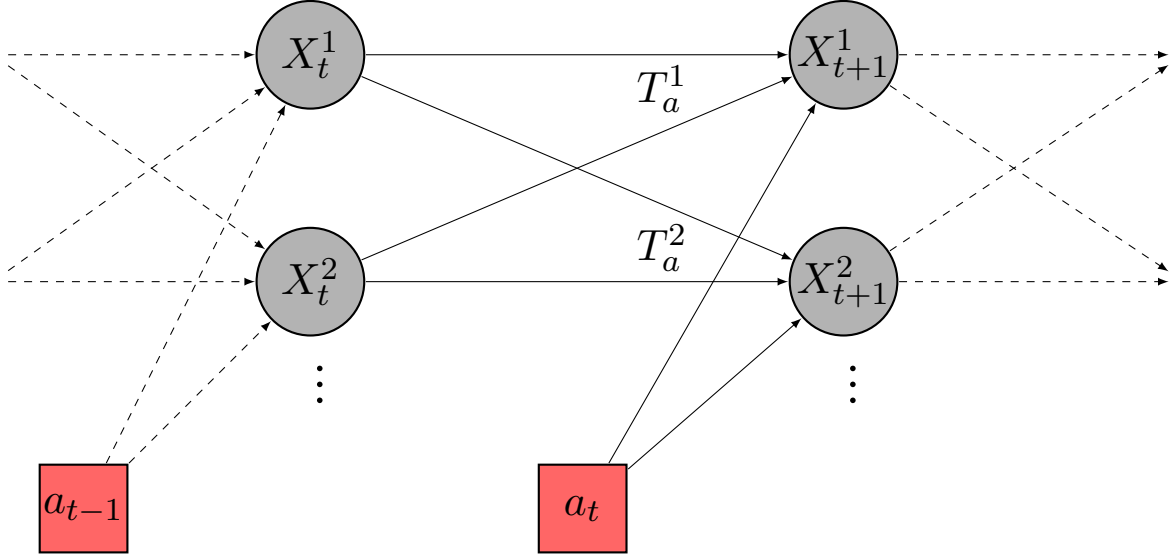


Figure III.2 – *Dynamic Bayesian Network of a factored (π -)MDP: in the possibilistic (resp. probabilistic) framework T_a^i is the transition possibility (resp. probability) distribution of next variable X_{t+1}^i conditional to the selected action $a \in \mathcal{A}$ and its parents $\text{parents}(X_{t+1}^i) \subseteq \{X_t^1, \dots, X_t^n\}$ (i.e. $\text{parents}(X_{t+1}^i)$ is a subset of the current state variables) where $n \geq 1$ is the number of variables describing the state space.*

some assumptions but is exact. For now, as finite spaces of size K can be themselves factored into $\lceil \log_2 K \rceil$ binary-variable spaces (see [62]), we can assume that we are reasoning about a factored belief state π -MDP whose state space is denoted by \mathcal{X} and fully described by variables (X^1, \dots, X^n) , with $n \in \mathbb{N}^*$ and $\forall i, X^i \in \{\top, \perp\}$: $\mathcal{X} = \{\top, \perp\}^n$.

Recall that Dynamic Bayesian Networks (DBNs) [35] already used in Section I.1 (for instance taking part of the influence diagrams in Figure I.2 and Figure I.3) and in the previous chapter (Figure II.1 illustrating the Mixed-Observable structure) are a useful graphical representation of studied processes. A DBN representing the structure of a factored π -MDP is depicted in Figure III.2: the state variables at a given time step $t \geq 0$ are denoted by $X_t = (X_t^i)_{i=1}^n$ (current variables), and $(X_{t+1}^i)_{i=1}^n$ are the state variables at step $t+1$ (next variables). In DBN semantics $\text{parents}(X_{t+1}^i)$ is the set of state variables on which the next state variable X_{t+1}^i “depends”, i.e. a variable Y , represented by a node in the DBN, is in $\text{parents}(X_{t+1}^i)$ if and only if there is an arrow from Y to X_{t+1}^i . We assume that $\text{parents}(X_{t+1}^i) \subseteq \{X_t^1, \dots, X_t^n\}$, i.e. parents of the next state variable X_{t+1}^i are a part of the current state variables $\{X_t^1, \dots, X_t^n\}$: there cannot be any arrow between state variables of the same time step. Methods are discussed in the literature to circumvent this restrictive assumption [16].

As recalled in Section I.1.1, in probabilistic settings, the absence of an arrow in a Bayesian Network represents an independence assumption. Let us consider a set of variables $\{Y_1, \dots, Y_p\}$, with $p > 1$, and such that $\forall i \in \{1, \dots, p\}, Y_i \in \mathcal{Y}_i$ where \mathcal{Y}_i is a finite set. For each $i \in \{1, \dots, p\}$, the set of the parents of variables Y_i can be denoted by $\text{parents}(Y_i) = \{Y_{pa_i(1)}, \dots, Y_{pa_i(p_i)}\} \subseteq \{Y_1, \dots, Y_p\}$, where $pa_i : \{1, \dots, p_i\} \rightarrow \{1, \dots, p\}$ is an increasing function. (note that $p_i \leq p$). Let us recall that $\text{children}(Y_i)$ is the set of all the variables $Y_j \in \{Y_1, \dots, Y_p\}$ such that there is an arrow from Y_i to Y_j . The set $\text{descend}(Y_i)$ is the set of the descendants of variable Y_i : $\text{descend}(Y_i)$ is defined as the smallest set containing $\text{children}(Y_i)$, and such that $\forall Y_j \in \text{descend}(Y_i), \text{children}(Y_j) \subset \text{descend}(Y_i)$. The set of non-descendants $\text{nondescend}(Y_i) = \{Y_j \in \{Y_1, \dots, Y_p\} \mid Y_j \notin \{Y_i\} \cup \text{descend}(Y_i) \cup \text{parents}(Y_i)\}$ can be also denoted by $\{Y_{nd_i(1)}, \dots, Y_{nd_i(d_i)}\} \subset \{Y_1, \dots, Y_p\}$ where $nd_i : \{1, \dots, d_i\} \rightarrow \{1, \dots, n\}$ is an increasing function (and $d_i \leq p$). Hence, a DBN makes the assumption that Y_i is independent

from its non-descendants conditional on its parents: $Y_i \perp\!\!\!\perp \text{nondescend}(Y_i) \mid \text{parents}(Y_i)$. In the probabilistic case, it can be written $\forall(y_1, \dots, y_p) \in \mathcal{Y}_1 \times \dots \times \mathcal{Y}_p, \forall i \in \{1, \dots, p\}$,

$$\begin{aligned} \mathbb{P}\left(Y_i = y_i \mid Y_{pa_i(1)} = y_{pa_i(1)}, \dots, Y_{pa_i(p_i)} = y_{pa_i(p_i)}, Y_{nd_i(1)} = y_{nd_i(1)}, \dots, Y_{nd_i(d_i)} = y_{nd_i(d_i)}\right) \\ = \mathbb{P}\left(Y_i = y_i \mid Y_{pa_i(1)} = y_{pa_i(1)}, \dots, Y_{pa_i(p_i)} = y_{pa_i(p_i)}\right). \end{aligned} \quad (\text{III.1})$$

This equation holds with a subset of the non-descendants $N \subset \text{nondescend}(Y_i)$: it suffices to consider the probability distribution over the set $\text{nondescend}(Y_i) \setminus N$, and compute the mean of each parts of the equation with respect to it. The upper part is the probability distribution conditional on the parents of Y_i and on C , and the lower part remains the same.

Using this formula, if we consider the state variables $(X_t^i)_{t \in \{0, \dots, H\}}^{i \in \{1, \dots, n\}}$ and actions $(a_t)_{t=0}^{H-1}$ of a factored MDP, denoting by X_t the variables $(X_t^i)_{i=1}^n$, we get from the DBN of Figure III.2 that $\forall t \in \{0, \dots, H-1\}, \forall(x_0, \dots, x_{t+1}) \in \{\top, \perp\}^{t+2}, \forall(a_0, \dots, a_t) \in \mathcal{A}^{t+1}$,

$$\mathbb{P}(X_{t+1} = x_{t+1} \mid X_0 = x_0, \dots, X_t = x_t, a_0, \dots, a_{t-1}) = \mathbb{P}(X_{t+1} = x_{t+1} \mid X_t = x_t, a_t),$$

which is nothing more than the Markov property, justifying the definition of the transition probability distributions $\mathbf{p}(x'|x, a), \forall(x, x') \in \{\top, \perp\}^{2n}$ and $\forall a \in \mathcal{A}$.

Section I.2.2 details that Qualitative Possibility Theory admits more than one independence definition: for instance, the non-interactivity independence (NI-independence, see Definition I.2.6) and the causal or min-based independence (M-independence, see Definition I.2.9) are two useful independence definitions for this chapter. If a DBN models M-dependences (causal dependences) between variables, then Equation III.1 holds replacing the probability measure \mathbb{P} by the possibility measure Π :

$$\begin{aligned} \Pi\left(Y_i = y_i \mid Y_{pa_i(1)} = y_{pa_i(1)}, \dots, Y_{pa_i(p_i)} = y_{pa_i(p_i)}, Y_{nd_i(1)} = y_{nd_i(1)}, \dots, Y_{nd_i(d_i)} = y_{nd_i(d_i)}\right) \\ = \Pi\left(Y_i = y_i \mid Y_{pa_i(1)} = y_{pa_i(1)}, \dots, Y_{pa_i(p_i)} = y_{pa_i(p_i)}\right), \end{aligned} \quad (\text{III.2})$$

which also holds considering only a subset of the descendants. The possibilistic Markov property is also deduced from the DBN of Figure III.2 the same result for π -MDPs: $\forall t \in \{0, \dots, H-1\}, \forall(x_0, \dots, x_{t+1}) \in \{\top, \perp\}^{t+2}, \forall(a_0, \dots, a_t) \in \mathcal{A}^{t+1}$,

$$\Pi(X_{t+1} = x_{t+1} \mid X_0 = x_0, \dots, X_t = x_t, a_0, \dots, a_{t-1}) = \Pi(X_{t+1} = x_{t+1} \mid X_t = x_t, a_t),$$

which has been already defined in Section I.2.4, see Equation I.46.

Consider again variables $(Y)_{i=1}^n$. Let A and B be two disjoint subsets of $\{1, \dots, n\}$. In order to simplify notations, if the probability $\mathbb{P}((Y_i = y_i)_{i \in A} \mid (Y_j = y_j)_{j \in B})$ is considered $\forall(y_i)_{i \in A} \in \times_{i \in A} \mathcal{Y}_i$ and $\forall(y_j)_{j \in B} \in \times_{j \in B} \mathcal{Y}_j$, the probability can be denoted by $\mathbb{P}((Y_i)_{i \in A} \mid (Y_j)_{j \in B})$. For instance, Equation III.1 can be rewritten

$$\mathbb{P}\left(Y_i \mid \text{parents}(Y_i), \text{nondescend}(Y_i)\right) = \mathbb{P}\left(Y_i \mid \text{parents}(Y_i)\right).$$

Let $(i_k)_{k=1}^s$ be an increasing sequence of indices with $s \leq p$, defining the set of variables $\{Y_{i_1}, \dots, Y_{i_s}\}$. The parents of variables $\{Y_{i_1}, \dots, Y_{i_s}\}$ are

$$\text{parents}\left((Y_{i_k})_{k=1}^s\right) := \bigcup_{k=1}^s \text{parents}(Y_{i_k}).$$

Suppose that the set of variables $\{Y_{i_1}, \dots, Y_{i_s}\}$ is such that $\text{parents}\left((Y_{i_k})_{k=1}^s\right) \cap \{Y_{i_1}, \dots, Y_{i_s}\} = \emptyset$, i.e. the parents of variables $\{Y_{i_1}, \dots, Y_{i_s}\}$ are not in $\{Y_{i_1}, \dots, Y_{i_s}\}$.

Then, using the equation (III.1), we can write

$$\begin{aligned} \mathbb{P} \left((Y_{i_k})_{k=1}^s \mid \text{parents}((Y_{i_k})_{k=1}^s) \right) \\ = \mathbb{P} \left(Y_{i_1} \mid \text{parents}((Y_{i_k})_{k=1}^s) \right) \cdot \mathbb{P} \left(Y_{i_2} \mid Y_{i_1}, \text{parents}((Y_{i_k})_{k=1}^s) \right) \cdot \\ \dots \cdot \mathbb{P} \left(Y_{i_s} \mid Y_{i_1}, \dots, Y_{i_{s-1}}, \text{parents}((Y_{i_k})_{k=1}^s) \right) \\ = \prod_{k=1}^s \mathbb{P}(Y_{i_k} \mid \text{parents}(Y_{i_k})). \end{aligned}$$

As already noted, the variables of the factored MDP model depicted by Figure III.2, are such that $\forall t \geq 0, \forall i \in \{1, \dots, n\}, \text{parents}(X_{t+1}^i) \subseteq \{X_t^1, \dots, X_t^n\}$: thus, using the previous general equation (with variables $(Y_i)_{i=1}^p$),

$$\begin{aligned} \mathbb{P}(X_{t+1} \mid X_t, a_t) &= \mathbb{P}(X_{t+1}^1, \dots, X_{t+1}^n \mid X_t, a_t) \\ &= \prod_{i=1}^n \mathbb{P}(X_{t+1}^i \mid \text{parents}(X_{t+1}^i), a_t). \end{aligned}$$

It shows that the transition probability distributions $\mathbf{p}(x' \mid x, a)$ can be computed from the simpler distributions $\mathbb{P}(X_{t+1}^i \mid \text{parents}(X_{t+1}^i), a_t), \forall i \in \{1, \dots, n\}$. These transition distributions are denoted by $\mathbf{p}(X'_i \mid \text{parents}(X_i), a)$, and by T_a^i in Figure III.2.

If we consider that the DBN involving variables $(Y_i)_{i=1}^k$ represents the M-dependences, Equation III.2 holds, and using the definition of the qualitative possibilistic conditioning (Definition I.2.7), we can also write

$$\begin{aligned} \Pi \left((Y_{i_k})_{k=1}^s \mid \text{parents}((Y_{i_k})_{k=1}^s) \right) \\ = \min \left\{ \Pi \left(Y_{i_1} \mid \text{parents}((Y_{i_k})_{k=1}^s) \right), \Pi \left(Y_{i_2} \mid Y_{i_1}, \text{parents}((Y_{i_k})_{k=1}^s) \right), \right. \\ \left. \dots, \Pi \left(Y_{i_s} \mid Y_{i_1}, \dots, Y_{i_{s-1}}, \text{parents}((Y_{i_k})_{k=1}^s) \right) \right\} \\ = \min_{k=1}^s \Pi(Y_{i_k} \mid \text{parents}(Y_{i_k})), \end{aligned}$$

under the same assumption: $\text{parents}((Y_{i_k})_{k=1}^s) \cap \{Y_{i_1}, \dots, Y_{i_k}\} = \emptyset$. Thus, as we consider that the DBN in Figure III.2 represents the M-dependences,

$$\begin{aligned} \Pi(X_{t+1} \mid X_t, a_t) &= \Pi(X_{t+1}^1, \dots, X_{t+1}^n \mid X_t, a_t) \\ &= \min_{i=1}^n \Pi(X_{t+1}^i \mid \text{parents}(X_{t+1}^i), a_t). \end{aligned}$$

The transition possibility distributions $\pi(x' \mid x, a)$ can be computed from the simpler distributions $\Pi(X_{t+1}^i \mid \text{parents}(X_{t+1}^i), a_t), \forall i \in \{1, \dots, n\}$, denoted by $\pi(X_{t+1}^i \mid \text{parents}(X_{t+1}^i), a)$, and by T_a^i in Figure III.2.

With the π -MOMDP notations, assumptions of the Bayesian Network in Figure III.2 allows us to compute the joint possibility transition as $\pi(s'_v, \beta'_h \mid s_v, \beta_h, a) = \pi(X' \mid X, a) = \min_{i=1}^n \pi(X'_i \mid \text{parents}(X_i), a)$, where, given a time step t , primed variables are variables concerning the time step $t+1$ (next variables), and non-primed variables are current variables

(at time step t): for instance, X'_i is the notation for X_{t+1}^i , and X_i the one for X_t^i . Thus, a factored π -MOMDP can be defined with transition functions $T_a^i = \pi(X'_i | \text{parents}(X'_i), a)$ for each action a and variable X'_i (if transitions are assumed stationary).

Each transition function can be compactly encoded in an Algebraic Decision Diagram (ADD) [4]. An ADD, as illustrated in Figure III.3a, is a directed acyclic graph which compactly represents a real-valued function of binary variables, whose identical sub-graphs are merged and zero-valued leaves are not memorized. The following notations are used to make it explicit that we are working with symbolic functions encoded as ADDs:

- $\boxed{\min} \{f, g\}$ where f and g are 2 ADDs;
- $\boxed{\max}_{X_i} f = \boxed{\max} \{f^{X_i=0}, f^{X_i=1}\}$,

which can be easily computed because ADDs are constructed on the basis of the Shannon expansion: $f = \overline{X_i} \cdot f^{X_i=0} + X_i \cdot f^{X_i=1}$ where $f^{X_i=1}$ and $f^{X_i=0}$ are sub-ADDs representing the positive and negative Shannon cofactors (see Fig. III.3a).

The optimistic possibilistic update of dynamic programming, *i.e.* line 8 of the Value Iteration Algorithm 10 in the previous chapter, (or line 8 of the VI Algorithm 11 for π -MOMDPs) can be rewritten in a symbolic form, so that states are now globally updated at once instead of individually: denoting by $X = (X_1, \dots, X_n)$ the current variable and $X' = (X'_1, \dots, X'_n)$ the next one, the possibilistic Q-value of an action $a \in \mathcal{A}$ is $\overline{q^a} = \overline{q^a}(X) = \boxed{\max}_{X'} \boxed{\min} \{ \pi(X' | X, a), \overline{U^*}(X') \}$. The computation of this ADD ($\overline{q^a}$) can be decomposed into independent computations thanks to the following proposition:

Property III.2.1 (Possibilistic regression of the Value Function)

Consider the current value function $\overline{U^*} : \{\top, \perp\}^n \rightarrow \mathcal{L}$. For a given action $a \in \mathcal{A}$, let us define:

- $\overline{q_0^a} = \overline{U^*}(X'_1, \dots, X'_n)$,
- $\overline{q_i^a} = \max_{X'_i \in \{\top, \perp\}} \min \{ \pi(X'_i | \text{parents}(X'_i), a), \overline{q_{i-1}^a} \}$.

Then, the possibilistic Q-value of action a is: $\overline{q^a} = \overline{q_n^a}$, which depends on variables X_1, \dots, X_n , and the next value function is $\overline{U^*}(X_1, \dots, X_n) = \max_{a \in \mathcal{A}} \overline{q_n^a}(X_1, \dots, X_n)$.

The proof is given in Annex D.1. Note that the same trick can be used to compute pessimistic value functions, using the equation (I.29) of Property I.2.1:

- $\underline{q_0^a} = \underline{U^*}(X'_1, \dots, X'_n)$,
- $\underline{q_i^a} = \min_{X'_i \in \{\top, \perp\}} \max \{ 1 - \pi(X'_i | \text{parents}(X'_i), a), \underline{q_{i-1}^a} \}$,

and next value function is $\underline{U^*}(X_1, \dots, X_n) = \max_{a \in \mathcal{A}} \underline{q_n^a}(X_1, \dots, X_n)$.

The Q-value of action a , represented as an ADD, can be then iteratively regressed over successive post-action state variables $X'_i, 1 \leq i \leq n$. Figure III.3b illustrates the possibilistic regression of the Q-value of an action for the first state variable X_1 and leads to the intuition that ADDs should be far smaller in practice under possibilistic settings, since their leaves lie in \mathcal{L} instead of \mathbb{R} , thus yielding more sub-graph simplifications.

Algorithm 12 is a symbolic version of the π -MOMDP Value Iteration Algorithm (Algorithm 11 in the previous chapter), which relies on the regression scheme defined in Proposition III.2.1. Inspired by SPUDD [62], PPUDD means *Possibilistic Planning Using Decision Diagrams*. As for SPUDD, it needs to swap unprimed state variables to primed ones in the ADD encoding the current value function before computing the Q-value of an action a (see Line 5 of Algorithm

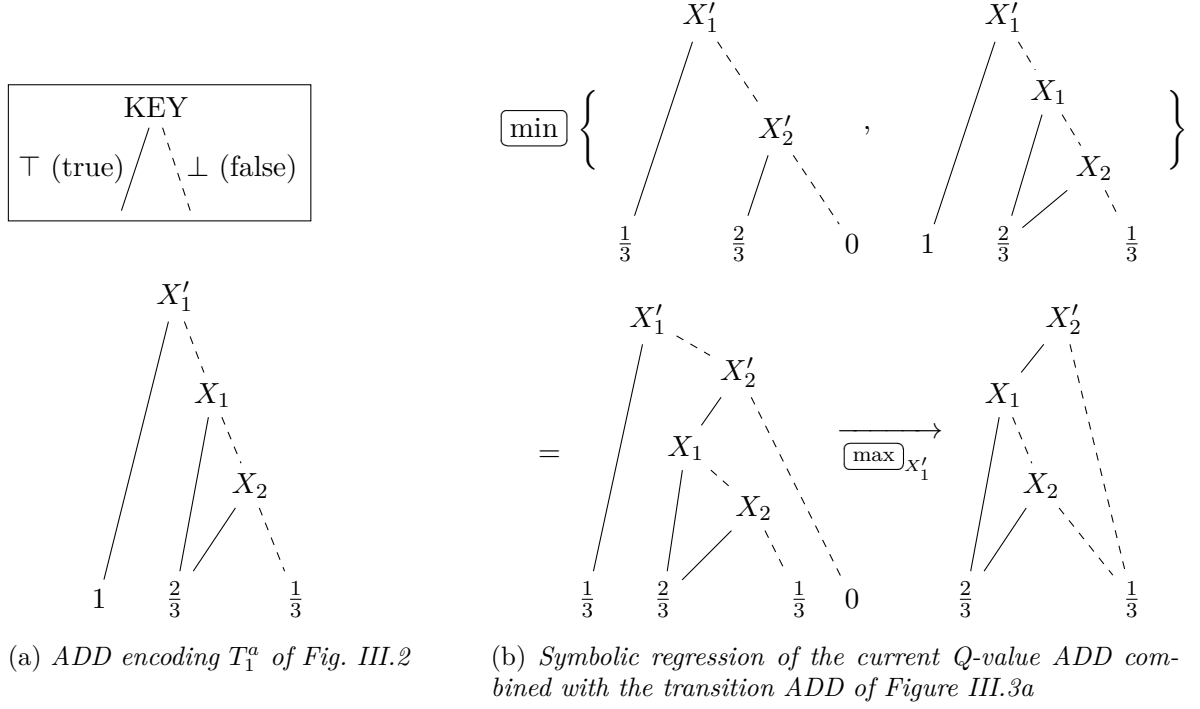


Figure III.3 – Algebraic Decision Diagrams for PPUDD

Algorithm 12: PPUDD (infinite horizon resolution)

```

1  $\overline{U}^* \leftarrow 0$  ;  $\overline{U}^c \leftarrow \Psi$  ;  $\bar{\delta} \leftarrow \hat{a}$  ;
2 while  $\overline{U}^* \neq \overline{U}^c$  do
3    $\overline{U}^* \leftarrow \overline{U}^c$  ;
4   for  $a \in \mathcal{A}$  do
5      $\overline{q}^a \leftarrow$  swap each  $X_i$  variable in  $\overline{U}^*$  with  $X'_i$  ;
6     for  $1 \leq i \leq n$  do
7        $\overline{q}^a \leftarrow \min \left\{ \overline{q}^a, \pi(X'_i \mid \text{parents}(X'_i), a) \right\}$  ;
8        $\overline{q}^a \leftarrow \max_{X'_i} \overline{q}^a$  ;
9      $\overline{U}^c \leftarrow \max \left\{ \overline{q}^a, \overline{U}^c \right\}$  ;
10    update  $\bar{\delta}$  to  $a$  where  $\overline{q}^a = \overline{U}^c$  and  $\overline{U}^c > \overline{U}^*$  ;
11 return  $\overline{U}^*, \bar{\delta}^*$  ;

```

12 and Figure III.3b). This operation is required to differentiate the next state represented by primed variables from the current one when operating on ADDs. Lines 4-9 apply Proposition III.2.1 and correspond to Line 8 of Algorithm 10.

We mentioned at the beginning of this section that belief state space $\Pi_{\mathcal{L}}^{\mathcal{S}_h}$ could be described by $\lceil \log_2 K \rceil$ binary variables where $K = \#\mathcal{L}^{\#\mathcal{S}_h} - (\#\mathcal{L} - 1)^{\#\mathcal{S}_h}$. However, this K can be very large so we propose in the next section a method to exploit the factorization of \mathcal{S}_h and \mathcal{O}_h in order to factorize $\Pi_{\mathcal{L}}^{\mathcal{S}_h}$ itself into small belief subvariables, which will decompose the possibilistic transition ADD into an aggregation of smaller ADDs. Note that PPUDD can solve π -MOMDPs even if this belief factorization is not feasible, but it will manipulate bigger ADDs.

Previous chapter highlight that a pre-treatment is required to translate a π -MOMDP into a π -MDP whose state space is \mathcal{X} . We can then reason on the state space accessible to the agent $\mathcal{X} = S_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$ and solve the π -MOMDP as a π -MDP. Next section links the structured properties of a π -MOMDP, concerning dependencies of original variables (visible, hidden and observation ones), to the factorization of the treated problem *i.e.* of the resulting π -MDP on $S_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$ (concerning dependencies of visible and belief variables).

Finally, we note that we could have used *complete action diagrams* (CADs) introduced in [116], which directly encode the transition matrix of each action as a single ADD. On one hand, CADs are simpler to manipulate than a set of transition ADDs for each state variable, and enable to deal with correlated action effects. On the other hand, they require to operate bigger ADDs while preventing intermediate simplifications that are yet offered by reasoning about separate state variables as we do (Lines 4-9 of Algorithm 12) or SPUDD does [62].

III.3 π -MOMDP BELIEF FACTORIZATION

Factorizing the belief variable requires three structural assumptions on the π -MOMDP's DBN, which are illustrated by the Rocksample benchmark [114].

III.3.1 Motivating example.

A rover navigating in a $g \times g$ grid has to collect scientific samples from interesting ("good") rocks among R ones and then to reach the exit. It is fitted with a noisy long-range sensor that can be used to determine if a rock is "good" or not. It knows the locations of the R rocks $(x_i, y_i)_{i=1}^R$ but not which ones are actually of interest (called "good" rocks). However, sampling a rock is expensive: the rover is fitted with a noisy long-range sensor that can be used to determine if a rock is "good" or not ("bad"). When a rock is sampled, it becomes (or stays) "bad" (no more interesting). At the end of the mission, the rover has to reach the exit location at the right side of the grid:

- \mathcal{S}_v consists of all the possible locations of the rover in addition to the exit ($\#\mathcal{S}_v = g^2 + 1$);
- \mathcal{S}_h consists of all the possible natures of the rocks: $\mathcal{S}_h = \mathcal{S}_h^1 \times \dots \times \mathcal{S}_h^R$, with $\forall 1 \leq i \leq R$, $\mathcal{S}_h^i = \{good, bad\}$;
- \mathcal{A} contains the (deterministic) moves in the 4 directions ($a_{north}, a_{east}, a_{south}, a_{west}$), checking rock i , (a_{check_i}) $\forall 1 \leq i \leq R$, and sampling the current rock, (a_{sample});
- $\mathcal{O} = \{o_{good}, o_{bad}\}$ are the possible sensor's answers for the current rock.

The rationale behind observation dynamics is the following: the more the rover is close to the checked rock, the better it observes its nature. In the original probabilistic model, the probability of a correct observation equals $\frac{1}{2} \left(1 + e^{-c\sqrt{(x_r - x_i)^2 + (y_r - y_i)^2}} \right)$ with $c > 0$. a constant

(the smaller is c , the more effective is the sensor). The rover gets the reward $+10$ (resp. -10) for each good (resp. bad) sampled rock, and $+10$ when it reaches the exit.

In the possibilistic model, the observation function is approximated using a critical distance $d > 0$ beyond which checking a rock is uninformative: $\pi(o'_i | s'_i, a, s_v) = 1 \ \forall o'_i \in \mathcal{O}_i$. The possibility degree of erroneous observation becomes zero if it stands at the checked rock, and lowest non zero possibility degree otherwise. Finally, as possibilistic semantics does not allow sums of rewards, an additional visible state variable $s_v^2 \in \{1, \dots, R\}$ which counts the number of checked rocks is introduced. The qualitative dislike of sampling is modeled as $\Psi(s) = \frac{R+2-s_v^2}{R+2}$ if the location is terminal and zero otherwise. The location of the rover is finally denoted by $s_v^1 \in \mathcal{S}_v^1$ and the visible state is then $s_v = (s_v^1, s_v^2) \in \mathcal{S}_v^1 \times \mathcal{S}_v^2 = \mathcal{S}_v$.

Observations $\{o_{good}, o_{bad}\}$ for the current rock can be equivalently modeled as a Cartesian product of observations $\{o_{good_1}, o_{bad_1}\} \times \dots \times \{o_{good_R}, o_{bad_R}\}$ for each rock. By using this equivalent modeling, state and observation spaces are both respectively factored as $\mathcal{S}_v^1 \times \dots \times \mathcal{S}_v^m \times \mathcal{S}_h^1 \times \dots \times \mathcal{S}_h^l$ and $\mathcal{O} = \mathcal{O}^1 \times \dots \times \mathcal{O}^l$, and we can now map each observation variable $O^j \in \mathcal{O}^j$ to its hidden state variable $S_h^j \in \mathcal{S}_h^j$. It allows us to reason about the DBN of Figure III.4, which expresses three important assumptions that will help us factorize the belief state itself:

1. all state variables $S_v^1, S_v^2, \dots, S_v^m, S_h^1, S_h^2, \dots, S_h^l$ are post-action independent variables, and next visible variables does not depend on current hidden ones. Thus, there is no arrow between two state variables at the same time step, as $S_{v,t}^2$ and $S_{h,t}^1$, nor arrow from a current hidden variable to a next visible one, as $S_{h,t}^1$ and $S_{v,t+1}^1$;
2. a hidden variable does not depend on previous other hidden variables: the nature of a rock is independent from the previous nature of other rocks. For instance, there is no arrow from $S_{h,t}^1$ to $S_{h,t+1}^2$;
3. an observation variable is available for each hidden state variable, and depends on it. It does not depend on other hidden state variables nor current visible ones, but on previous visible state variables and action: for instance, there is no arrow between $S_{h,t+1}^1$ and O_{t+1}^2 , nor between $S_{v,t+1}^1$ and O_{t+1}^1 .

Each observation variable is indeed only related to the nature of the corresponding rock. The observation quality yet depends on the rover's location *i.e.* a current visible state variable, not allowed by the DBN: fortunately, as moves are deterministic, we avoid this issue considering observations depend on previous location and action.

III.3.2 Consequences of the factorization assumptions

In this section, we formally demonstrate how the three previous independence assumptions can be used to factorize $\Pi_{\mathcal{L}}^{S_h}$ as the Cartesian product $\bigtimes_{j=1}^l \Pi_{\mathcal{L}}^{S_h^j}$, *i.e.* represent the belief state

β_h about the hidden states $s_h \in \mathcal{S}_h$ with marginal belief states $\beta_h^j \in \Pi_{\mathcal{L}}^{S_h^j}$ about hidden states $s^j \in \mathcal{S}_h^j, \forall j \in \{1, \dots, l\}$.

To this end, we will use *d-Separation criterion* [118] in order to show some independence between variables from the DBN. As explained in Section III.2, a DBN can be drawn from independence relations. Let us denote by $X \perp\!\!\!\perp Y \mid Z$ the assertion “ X is independent from Y conditional on Z ”: recall that for a given definition of the used independence relation, *e.g.* probabilistic, non interactivity (NI), or minimum based (M, causal) independence, the DBN is drawn such that for each node (variable) X , $X \perp\!\!\!\perp \text{nondescend}(X) \mid \text{parents}(X)$. If the used

independence relation obey the *semi-graphoids* axioms [88, 119], the graphical criterion called d-Separation can be used to identify some independences between variables of the DBN.

This criterion is for instance used in probabilistic settings in [123]. Recall that the M-independence is not symetric (see Section I.2.2), and thus does not obey the axioms of semi-graphoids. However, the NI-independence leads to a semi-graphoids, as proved in [57].

Let us recall that M-independence implies NI-independence (see Theorem 10 of Section I.2.2). The DBN of Figure III.4 represents M-independences between variables: thus the DBN representing NI-independences (which is not drawn in this work) has potentially less arrows, *i.e.* assumes potentially more independences than the DBN representing the M-independences. Assuming that the DBN of Figure III.4 represents NI-independences is a relaxation *i.e.* we potentially forget some NI-independence assumptions by doing this assumption. However, all NI-independences proved using d-Separation criterion on the DBN are true.

First of all, the DBN of Figure III.4 representing the M-independence assumptions, some probability distributions can be defined from the fact that each node is M-independent from its non-descendants conditional on its parents: given a time step $t \geq 0$, an action $a_t \in \mathcal{A}$, and a current state $s = (s_{v,t}, s_{h,t}) = (s_{v,t}^1, \dots, s_{v,t}^m, s_{h,t}^1, \dots, s_{h,t}^l) \in \mathcal{S}$,

- $\forall i \in \{1, \dots, m\}$, the transition possibility distribution over the i^{th} visible state variable

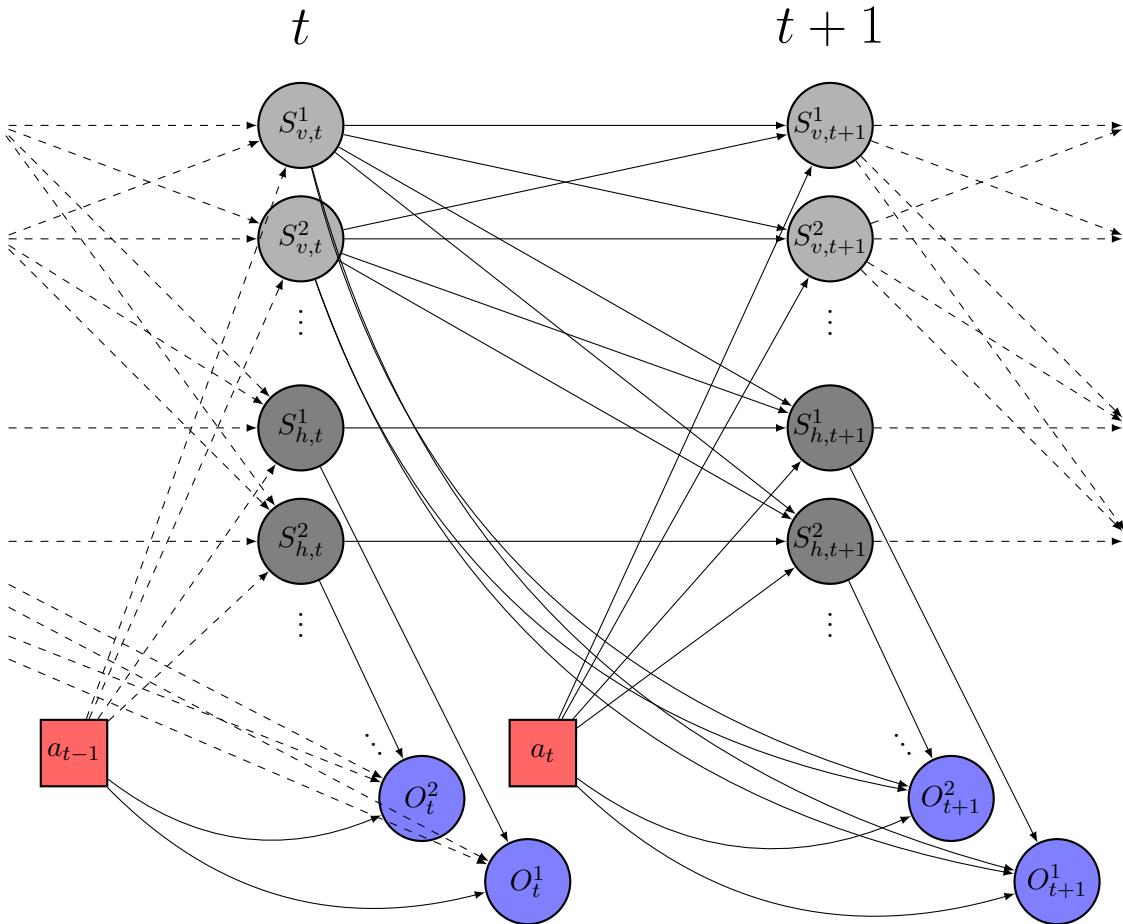


Figure III.4 – DBN summing up independence assumptions of a π -MOMDP leading to marginal beliefs and a π -MDP with a factored transition function *i.e.* a factored belief π -MDP. Parents of a visible state variable are the previous visible state variables. Parents of a hidden state variable are the previous visible state variables and the corresponding previous hidden state variable. Finally, parents of an observation variable are the previous visible state variables, and the corresponding current hidden state variable.

$$s_{v,t+1}^i \in \mathcal{S}_v^i:$$

$$\pi \left(s_{v,t+1}^i \mid s_{v,t}, a_t \right) = \Pi \left(S_{v,t+1}^i = s_{v,t+1}^i \mid S_{v,t} = s_{v,t}, a_t \right); \quad (\text{III.3})$$

- $\forall j \in \{1, \dots, l\}$, the transition possibility distribution over the j^{th} hidden state variable $s_{h,t+1}^j \in \mathcal{S}_h^j$:

$$\pi \left(s_{h,t+1}^j \mid s_{v,t}, s_{h,t}, a_t \right) = \Pi \left(S_{h,t+1}^j = s_{h,t+1}^j \mid S_{h,t} = s_{h,t}, a_t \right); \quad (\text{III.4})$$

- $\forall j \in \{1, \dots, l\}$, the observation possibility distribution over the j^{th} observation variable $o^j \in \mathcal{O}^j$:

$$\pi \left(o_{t+1}^j \mid s_{v,t}, s_{h,t+1}, a_t \right) = \Pi \left(O_{t+1}^j = o_{t+1}^j \mid S_{v,t} = s_{v,t}, S_{h,t+1} = s_{h,t+1}, a_t \right). \quad (\text{III.5})$$

With these distributions, the dynamics of the process of a π -MOMDP respecting the assumptions of Figure III.4 is entirely defined.

Let us define the information i_t known by the agent at time step $t \geq 1$ when the model is a (π -)MOMDP: $i_0 = \{s_{v,0}\}$, and for each time step $t \geq 1$, $i_t = \{o_t, s_{v,t}, a_{t-1}, i_{t-1}\}$: the corresponding variable is denoted by I_t . The next theorem shows that the current belief can be decomposed into marginal belief states dependent on the current information i_t .

Theorem 24 (Independence of hidden state variables and marginal belief states)

Consider a π -MOMDP described by the DBN of Figure III.4. If initial hidden variables $S_{h,0}^1, \dots, S_{h,0}^l$ are NI-independent, then at each time step $t > 0$ the belief over hidden states can be written as

$$\beta_{h,t} = \min_{j=1}^l \beta_{h,t}^j$$

with $\forall s \in \mathcal{S}_h^j$, $\beta_{h,t}^j(s) = \Pi \left(S_{h,t}^j = s \mid I_t = i_t \right)$ the belief state concerning hidden states of the set \mathcal{S}_h^j .

The proof is given in Annex D.2.

Thanks to the previous theorem, the state space accessible to the agent can now be rewritten as $\mathcal{S}_v^1 \times \dots \times \mathcal{S}_v^m \times \Pi_{\mathcal{L}}^{\mathcal{S}_h^1} \times \dots \times \Pi_{\mathcal{L}}^{\mathcal{S}_h^l}$ with $\Pi_{\mathcal{L}}^{\mathcal{S}_h^j} \subsetneq \mathcal{L}^{\mathcal{S}_h^j}$. The size of $\Pi_{\mathcal{L}}^{\mathcal{S}_h^j}$ is $\#\mathcal{L}^{\#\mathcal{S}_h^j} - (\#\mathcal{L} - 1)\#\mathcal{S}_h^j$ (see Equation I.60). If all state variables are binary, $\#\Pi_{\mathcal{L}}^{\mathcal{S}_h^j} = 2\#\mathcal{L} - 1$ for all $1 \leq j \leq l$, so that $\#\mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h} = 2^m(2\#\mathcal{L} - 1)^l$: contrary to probabilistic settings, **hidden state variables and visible ones have a similar impact on the solving complexity**, i.e. both singly-exponential in the number of state variables. In the general case, by noting $\kappa = \max\{\max_{1 \leq i \leq m} \#\mathcal{S}_{v,i}, \max_{1 \leq j \leq l} \#\mathcal{S}_{h,j}\}$, there are $\mathcal{O}(\kappa^m(\#\mathcal{L})^{(\kappa-1)l})$ flattened belief states, which is indeed exponential in the arity of state variables too.

In the Section I.2.5 about π -POMDP, the belief state variable at time step $t \geq 0$ is denoted by B_t^π , and its possible values are $\beta \in \Pi_{\mathcal{L}}^{\mathcal{S}}$. Now that $\Pi_{\mathcal{L}}^{\mathcal{S}_h}$ has been factorized, we can consider the *marginal belief state variables* $B_{h,t}^{\pi,j}$, $\forall j \in \{1, \dots, l\}$, whose possible values are $\beta_h^j \in \Pi_{\mathcal{L}}^{\mathcal{S}_h^j}$, i.e. belief states concerning hidden states $s^j \in \mathcal{S}_h^j$. We want now to show that successive variables $S_{v,t}^1, \dots, S_{v,t}^m, B_{h,t}^{\pi,1}, \dots, B_{h,t}^{\pi,l}$ respect the assumptions of the DBN of Figure III.2, i.e. are independent post-action variables, as successive variables X_t^1, \dots, X_t^n . This result is based on Lemma III.3.1, which shows how marginal belief state are actually updated.

Lemma III.3.1 (Update of the marginal belief states)

¹ At time $t \geq 0$, if the system is in the visible state $s_{v,t} = (s_{v,t}^1, \dots, s_{v,t}^m) \in \mathcal{S}_v$, in the belief

state over the j^{th} hidden state $\beta_{h,t}^j \in \Pi_{\mathcal{L}}^{S_h^j}$, and if the agent selects action $a_t \in \mathcal{A}$ and then gets observation $o_{t+1}^j \in \mathcal{O}^j$, the update of the belief state about hidden system states $s^j \in \mathcal{S}_h^j$ is:

$$\beta_{h,t+1}^j(s_{t+1}^j) = \begin{cases} 1 & \text{if } \pi(o_{t+1}^j, s_{h,t+1}^j \mid s_{v,t}, \beta_{h,t}^j, a_t) = \pi(o_{t+1}^j \mid s_{v,t}, \beta_{h,t}^j, a_t) \\ \pi(o_{t+1}^j, s_{h,t+1}^j \mid s_{v,t}, \beta_{h,t}^j, a_t) & \text{otherwise.} \end{cases} \quad (\text{III.6})$$

where $\pi(o_{t+1}^j, s_{h,t+1}^j \mid s_{v,t}, \beta_{h,t}^j, a)$ is the notation for

$$\max_{s^j \in \mathcal{S}_h^j} \min \left\{ \pi(o_{t+1}^j \mid s_{v,t}, s_{h,t+1}^j, a), \pi(s_{h,t+1}^j \mid s_{v,t}, s^j, a), \beta_h^j(s^j) \right\},$$

using distributions (III.4) and (III.5), and

$$\pi(o_{t+1}^j \mid s_{v,t}, \beta_{h,t}^j, a) = \max_{s_{h,t+1}^j \in \mathcal{S}_h^j} \pi(o_{t+1}^j, s_{h,t+1}^j \mid s_{v,t}, \beta_{h,t}^j, a).$$

The proof is given in Annex D.3. The associated **belief update function** is ν^j :

$$\beta_{h,t+1}^j = \nu^j(s_{v,t}, \beta_{h,t}^j, a_t, o_{t+1}^j),$$

which can be denoted by

$$\beta_{h,t+1}^j(s_{h,t+1}^j) \propto^\pi \pi(o_{t+1}^j, s_{h,t+1}^j \mid s_{v,t}, \beta_{h,t}^j, a_t)$$

as it consists of the possibilistic normalization of the joint possibility distribution over the j^{th} hidden state variable and the j^{th} observation.

Hence, the possibility degree that the marginal belief state variables $B_{h,t+1}^{\pi,j}$ is $\beta_{h,t+1}^j \in \Pi_{\mathcal{L}}^{S_h^j}$ conditional on $B_{h,t}^{\pi,j} = \beta_{h,t}^j$ and the action $a_t \in \mathcal{A}$, can be computed:

$$\Pi(B_{h,t+1}^{\pi,j} = \beta_{h,t+1}^j \mid S_{v,t} = s_{v,t}, B_{h,t}^{\pi,j} = \beta_{h,t}^j, a_t) = \max_{\substack{o^j \in \mathcal{O}^j \text{ s.t.} \\ \nu^j(s_{v,t}, \beta_{h,t}^j, a_t, o^j) = \beta_{h,t+1}^j}} \pi(o^j \mid s_{v,t}, \beta_{h,t}^j, a_t) \quad (\text{III.7})$$

defining the transition possibility distribution of marginal belief states $\pi(\beta_{h,t+1}^j \mid s_{v,t}, \beta_{h,t}^j, a_t)$.

Finally, Theorem 25 relies on Lemma III.3.1 to ensure independence of all post-action variables of the belief π -MDP resulting from the factorization, conditional on the current state: this allows us to write the possibilistic transition function of the belief-state π -MDP in a factored form:

Theorem 25 (Factored expression of the transition possibility distribution)

If independence assumptions of a π -MOMDP are described by the DBN of Figure III.4, then $\forall \beta_{h,t} = (\beta_{h,t}^1, \dots, \beta_{h,t}^l) \in \Pi_{\mathcal{L}}^{S_h}, \beta_{h,t+1} = (\beta_{h,t+1}^1, \dots, \beta_{h,t+1}^l) \in \Pi_{\mathcal{L}}^{S_h}, \forall (s_{v,t}, s_{v,t+1}) \in (\mathcal{S}_v)^2, \forall a_t \in \mathcal{A}$,

$$\pi(s_{v,t+1}, \beta_{h,t+1} \mid s_{v,t}, \beta_{h,t}, a)$$

$$= \min \left\{ \min_{i=1}^m \pi(s_{v,t+1}^i \mid s_{v,t}, a_t), \min_{j=1}^l \pi(\beta_{h,t+1}^j \mid s_{v,t}, \beta_{h,t}^j, a_t) \right\},$$

where the transition possibility distributions of visible state variables is given in the equation (III.3) and the one of marginal belief state variables in the equation (III.7).

The proof is given in Annex D.4. Using this result, such a factorized expression of the transition possibility distribution allows to compute the value function with $n = m + l$ stages, as described in the previous section: the π -MOMDP is indeed a factored π -MDP since variables $(S_v^1, \dots, S_v^m, B_h^{\pi,1}, \dots, B_h^{\pi,l})$, can play the role of variables X_1, \dots, X_n in Algorithm 12.

Note that Probability Theory does not make any difference between causal independence (M-independence in Possibility Theory, see Definition I.2.8) and decompositional independence (NI-independence in Possibility Theory, see Definition I.2.6). Moreover, probabilistic independence relation obey the axioms of semi-graphoids: so previous independence results due to d-Separation are also true in probabilistic settings. If independence assumptions between variables of a probabilistic MOMDP [83, 2] are described by the DBN of Figure III.4, then a similar factorization result can be deduced:

Theorem 26 (Factored expression of the transition probability distribution)

Consider a probabilistic MOMDP with independence assumptions described by the DBN of Figure III.4. The following probability distributions define entirely dynamics of variables:

- the transition probability distributions of visible state variables: $\forall i \in \{1, \dots, m\}$,

$$\mathbf{p}(s_{v,t+1}^i \mid s_{v,t}, a_t) = \mathbb{P}(S_{v,t+1}^i = s_{v,t+1}^i \mid S_{v,t} = s_{v,t}, a_t);$$

- the transition probability distributions of hidden state variables: $\forall j \in \{1, \dots, l\}$,

$$\mathbf{p}(s_{h,t+1}^j \mid s_{v,t}, s_{h,t}^j, a_t) = \mathbb{P}(S_{h,t+1}^j = s_{h,t+1}^j \mid S_{v,t} = s_{v,t}, S_{h,t}^j = s_{h,t}^j, a_t);$$

- the observation probability distributions: $\forall j \in \{1, \dots, l\}$,

$$\mathbf{p}(o_{t+1}^j \mid s_{v,t}, s_{h,t+1}^j, a_t) = \mathbb{P}(O_{t+1}^j = o_{t+1}^j \mid S_{v,t} = s_{v,t}, S_{h,t+1}^j = s_{h,t+1}^j, a_t).$$

Let us define probabilistic marginal belief states: $\forall j \in \{1, \dots, l\}$, $\forall s \in \mathcal{S}_h^j$,

$$b_{h,t}^j(s) = \mathbb{P}(S_{h,t}^j = s \mid I_t = i_t).$$

Given such a marginal belief state $b_{h,t}^j$ and a visible state $s_{v,t} \in \mathcal{S}_v$, if the action $a_t \in \mathcal{A}$ is selected and the observation $o_{t+1}^j \in \mathcal{O}^j$ is received, then the next belief state about the j^{th} hidden state variable is $\forall s_{h,t+1}^j \in \mathcal{S}_h^j$,

$$\beta_{h,t+1}^j(s_{h,t+1}^j) \propto \mathbf{p}(o_{t+1}^j \mid s_{v,t}, s_{h,t+1}^j, a_t) \cdot \sum_{s_{h,t}^j \in \mathcal{S}_h^j} \mathbf{p}(s_{h,t+1}^j \mid s_{v,t}, s_{h,t}^j, a_t) \cdot b(s_{h,t}^j),$$

denoted by $\beta_{h,t+1}^j(s_{h,t+1}^j) = u^j(s_{v,t}, \beta_{h,t}^j, a_t, o_{t+1}^j)$.

The transition probability distribution can be written as follows: $\forall b_{h,t} = (b_{h,t}^1, \dots, b_{h,t}^l) \in \mathbb{P}^{\mathcal{S}_h}$, $b_{h,t+1} = (b_{h,t+1}^1, \dots, b_{h,t+1}^l) \in \mathbb{P}^{\mathcal{S}_h}$, $\forall (s_{v,t}, s_{v,t+1}) \in (\mathcal{S}_v)^2$, $\forall a_t \in \mathcal{A}$,

$$\mathbf{p}(s_{v,t+1}, b_{h,t+1} \mid s_{v,t}, b_{h,t}, a) = \prod_{i=1}^m \mathbf{p}(s_{v,t+1}^i \mid s_{v,t}, a_t) \cdot \prod_{j=1}^l \mathbf{p}(b_{h,t+1}^j \mid s_{v,t}, b_{h,t}^j, a_t),$$

where

$$\mathbf{p}(b_{h,t+1}^j \mid s_{v,t}, b_{h,t}^j, a_t) = \sum_{\substack{o_{t+1}^j \in \mathcal{O}^j \text{ s.t.} \\ u^j(s_{v,t}, \beta_{h,t}^j, a_t, o_{t+1}^j) = \beta_{h,t+1}^j}} \mathbf{p}(o_{t+1}^j \mid b_{h,t}^j, a_t),$$

and

$$\mathbf{p}(o_{t+1}^j \mid b_{h,t}^j, a_t) = \sum_{s_{h,t+1}^j \in \mathcal{S}_h^j} \mathbf{p}(o_{t+1}^j \mid s_{v,t}, s_{h,t+1}^j, a_t) \cdot \sum_{s_{h,t}^j \in \mathcal{S}_h^j} \mathbf{p}(s_{h,t+1}^j \mid s_{v,t}, s_{h,t}^j, a_t) \cdot b(s_{h,t}^j).$$

The MDP built from such a probabilistic MOMDP is thus a factored MDP.

Previous theorems allow to express the transition distribution of the (π) -MDP resulting from a (π) -MOMDP with distributions which concern less variables. The value function update is then divided into $n = m + l$ stages in the possibilistic case, as depicted by the *for* loop of Algorithm 12. Qualitative possibilistic MOMDPs can however also be solved using ADDs even if the independence assumptions do not hold: in this case, one global transition distribution, encoded as a big ADD concerning all variables $(S_v^1, \dots, S_v^m, B_h^\pi)$ is used, and the number of potential values $\beta_h \in \Pi_{\mathcal{L}}^{S_h}$ of the global belief state variable B_h^π increases exponentially with the number of hidden states: $\#\Pi_{\mathcal{L}}^{S_h} = (\#\mathcal{L})^{\#S_h} - (\#\mathcal{L} - 1)^{\#S_h}$ (see Equation I.60). Nevertheless, if the factorization of the transition distribution is possible, handled ADDs have less nodes and computations should be faster. These results are used in the next section to compute more efficiently optimal strategies of π -MOMDPs.

III.4 EXPERIMENTAL RESULTS

The main expected advantages of using factored π -(MO)MDPs over their probabilistic counterparts are:

1. values of ADDs are in the finite scale \mathcal{L} rather than \mathbb{R} , so that the number of their leaves is at most $\#\mathcal{L} \ll 2^n$ (probabilistic models' ADDs can have up to 2^n leaves, where n is the number of variables involved in the ADD);
2. π -MOMDPs boil down to factored *finite-state* belief π -MDPs that can be solved by PPUDD assuming some independence assumptions on the underlying DBNs;
3. π -MOMDPs are in the same complexity class as π -MDPs *if all hidden state variables are binary* (in probabilistic models, partially-observable problems are always in a higher complexity class).

Of course, we have to pay a price: namely, possibilistic models can be seen as approximations of probabilistic ones (except if probabilities in the model are not precisely known and uncertainty of the problem is better described in a qualitative form). Yet, many state-of-the-art probabilistic algorithms are approximate, e.g. MDP solver PROST [65] (based on UCT algorithm [66]) and POMDP solvers described in Section I.1.11. Our PPUDD algorithm, however, is exact.

In the case of the infinite horizon probabilistic MOMDPs, it is sufficient to look for a stationary strategy *i.e.* an optimal strategy can be defined as a function $d^* : \mathcal{S}_v \times \mathbb{P}^{\mathcal{S}_h} \rightarrow \mathcal{A}$ maximizing the value function given an initial visible state $s_{v,0} \in \mathcal{S}_v$ and an initial belief state $b_{h,0} \in \mathbb{P}^{\mathcal{S}_h}$:

$$V(s_{v,0}, b_{h,0}, d^*) = \sup_{d: \mathcal{S}_v \times \mathbb{P}^{\mathcal{S}_h} \rightarrow \mathcal{A}} V(s_{v,0}, b_{h,0}, d),$$

where $V(s_{v,0}, b_{h,0}, d) = \mathbb{E} \left[\sum_{t \geq 0} \gamma^t \cdot r(S_t, d(S_{v,t}, B_{h,t})) \mid S_{v,0} = s_{v,0}, B_{h,0} = b_{h,0} \right]$, and the action defining probabilistic dynamics at time step t is $d(S_{v,t}, B_{h,t})$ if the strategy is d (see Section I.1.6 and [83, 2]).

Consider now qualitative possibilistic MOMDPs [41]: first, the optimistic criterion of an infinite horizon π -MDP can be written

$$\max_{t \geq 0} \max_{\mathcal{T} \in \mathcal{T}_t} \min \{ \pi(\mathcal{T} \mid s_0, (\delta)), \Psi(s_t) \},$$

using Equation II.18 in the previous chapter and Lemma C.1 of Annex C. In this formula, the set of t -length trajectories $(s_1, \dots, s_t) \in \mathcal{S}^t$ is denoted by \mathcal{T}_t and recall that $\Psi : \mathcal{S} \rightarrow \mathcal{L}$ is the

terminal preference function. Let us now denote by x_t the couple of visible and belief states $(s_{v,t}, \beta_{h,t}) \in \mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$, available to the agent at time step $t \geq 0$: $\mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}$ is denoted by \mathcal{X} . Theorem 22 assures that at least one stationary strategy is optimal if a “stay” action, which maintains the system in the same state and denoted by $\hat{a} \in \mathcal{A}$, is available: recall that this action is only used in some goal states in the optimal strategy. The qualitative infinite horizon criterion for a mixed optimistic-pessimistic π -MOMDP (see the sections II.1.1 and II.2) can thus be written

$$\bar{U}(x_0, (\delta)) = \max_{t \geq 0} \max_{\mathcal{T} \in \mathcal{T}_t} \min \left\{ \pi(\mathcal{T} \mid x_0, (\delta)), \underline{\Psi}(x_t) \right\}, \quad (\text{III.8})$$

where $\mathcal{T}_t = (x_1, \dots, x_t)$ is a trajectory of couples $x_i = (s_{v,i}, \beta_{h,i})$, \mathcal{T}_t the set of such trajectories, and

$$\pi(\mathcal{T} \mid x_0, (\delta)) = \min_{i=0}^{t-1} \pi(x_{i+1} \mid x_i, \delta(x_i))$$

the possibility degree of such trajectories, with the transition possibility distribution $\pi(x_{i+1} \mid x_i, \delta(x_i))$ defined in Section II.2. Recall also that the pessimistic terminal preference is denoted by $\underline{\Psi}(x) = \min_{s_h \in \mathcal{S}_h} \max \{ \Psi(s_v, s_h), 1 - \beta_h(s_h) \}$ with $x = (s_v, \beta_h)$. An optimal strategy δ^* is thus such that

$$\bar{U}(s_{v,0}, \beta_0, \delta^*) = \sup_{\delta: \mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h} \rightarrow \mathcal{A}} \bar{U}(s_{v,0}, \beta_0, \delta),$$

see Section II.3.2 of the previous chapter.

In the probabilistic case, as the belief state $B_{h,t}$ is a deterministic function of the current available information I_t , a strategy of an infinite probabilistic MOMDP can be defined as a sequence of functions based on the current information (and thus non stationary): $(d_t)_{t \geq 0}$, with d_t a function of the visible state s_v and the information $i_t = \{s_{v,0}, a_0, o_1, \dots, s_{v,t-1}, a_{t-1}, o_t\}$. In this way, the strategy only depends on the variables of the initial problem (and not on the current belief state). As in the probabilistic case, the qualitative possibilistic belief state $B_{h,t}^\pi$ is fully specified by the current available information I_t . The strategy of an infinite horizon π -MOMDP can thus be defined as a sequence of functions based on the current information: $(\delta_t)_{t \geq 0}$, with δ_t a function of the visible state $s_{v,t}$ and the information i_t . This being so, the strategy does not depend on the current possibilistic belief state, but on the variables of the π -MOMDP.

At execution, the agent knows the successive visible system states and the available information: taking into account the previous paragraph, he/she may thus use the optimal strategy provided by either the probabilistic model using d^* , or the possibilistic one using δ^* . Obviously, if the performance of the strategy is measured using the probabilistic criterion V , *i.e.* evaluating the average of rewards obtained using the given strategy during many trials, the optimal possibilistic strategy δ^* is less efficient than d^* :

$$V(s_v, b_h, d^*) \geq V(s_v, b_h, \delta^*).$$

However, as dimensions of considered problems may be high enough to make the probabilistic computations intractable or to make probabilistic solvers require too many computation time resources, strategies returned by probabilistic solvers are approximations of d^* which may be less efficient than δ^* even in the probabilistic sense. Note also that

$$\bar{U}(s_v, \beta_h, \delta^*) \geq \bar{U}(s_v, \beta_h, d^*),$$

but this criterion will not be used during the experiments as it is not a standard performance measure when planning under uncertainty. Qualitative criteria can however be good performance measure of strategies in practice if the probabilities of the model are in fact given arbitrarily from a qualitative evaluation of variables dynamics.

In this section, we compare our possibilistic approach against probabilistic solvers in order to answer the following question: what is the efficacy/quality tradeoff achieved by reasoning about an approximate model (π -MOMDP) but with an exact efficient algorithm (PPUDD)? Despite radically different methods, possibilistic strategies and probabilistic ones are both able to return an action for each possible visible state variable and current information: they are thus directly comparable and statistically evaluated under identical settings *i.e.* using transition and reward functions defined by the probabilistic model (criterion V).

It has been shown in [102] that the optimistic π -MDP criterion (see Equation I.48 of Section I.2.4) leads to better strategies than the pessimistic one (see Equation I.51) when the goal is to approximate an optimal strategy of a probabilistic fully observable MDP. Moreover, we proposed an algorithm for infinite horizon π -MDPs which has been proved to produce an optimal strategy with the optimistic criterion, see Section II.3 of previous chapter: Algorithm 10. This algorithm is also devoted to problems without intermediate preferences. The optimistic criterion, as well as the case of terminal preference only (see Definition I.2.15), are thus preferred in the following experimentations. In the case of mixed-observability, the mixed optimistic-pessimistic criterion (see Definition II.1.3) is a good choice as the π -MOMDP boils down to an optimistic π -MDP. Moreover, with this criterion, the more a belief state is specific, the higher is its preference, unlike the purely optimistic ones.

III.4.1 Robotic missions

We first assessed PPUDD performances on totally observable factored problems since PPUDD is also the first algorithm to solve factored π -MDPs (by inclusion in π -MOMDPs). To this end, we compared PPUDD against SPUDD on the *Navigation problem* used in the International Probabilistic Planning Competition 2011 [106]. In this domain, a robot navigates in a grid where it must reach some goal location most reliably. It can apply actions going north, east, south, west and stay: all these actions cost 1 except on the goal. When moving, it can suddenly disappear with some probability defined as a Bernoulli distribution, so that a good policy tries to reach the goal by avoiding situations where it may disappear. This probabilistic model is approximated by two possibilistic ones where: the preference of reaching the goal is 1; in the first model (M1) the highest probability of each Bernoulli distribution is replaced by 1 (for possibility normalization reasons) and the same value for the lowest probability is kept; for the second model (M2), the probability of disappearing is replaced by 1 and the other one is kept. Figure III.5a shows that SPUDD runs out of memory from the 6th problem, and PPUDD computation's time outperforms SPUDD's one by many orders of magnitude for the two models. Intuitively, this result comes from the fact that PPUDD's ADDs should be smaller because their leaves' values are in the finite scale \mathcal{L} rather than \mathbb{R} , which is indeed demonstrated in Figure III.5b. Performances were evaluated with two relevant criteria: frequency of runs where the policy reaches the goal (see Figure III.5c), and average length of execution runs that reach the goal (see Figure III.5d), that are both functions of the problem's instance. As expected, model (M2) is more cautious than model (M1) and gets a better reached goal frequency (similar to SPUDD's one for the instances it can solve). The later is more optimistic and gets a better average length of execution runs than model (M2) due to its dangerous behavior. For fairness reasons, we also compared ourselves against APRICODD [116], which is an approximate algorithm for factored MDPs: however parameters impacting the approximation are hard to tune (either huge computation times, or zero qualities) and it is largely outperformed by PPUDD in both time and quality whatever the parameters (curves are not shown since uninformative).

Finally, we compared PPUDD on the *Rocksampling problem* (RS), described in Section

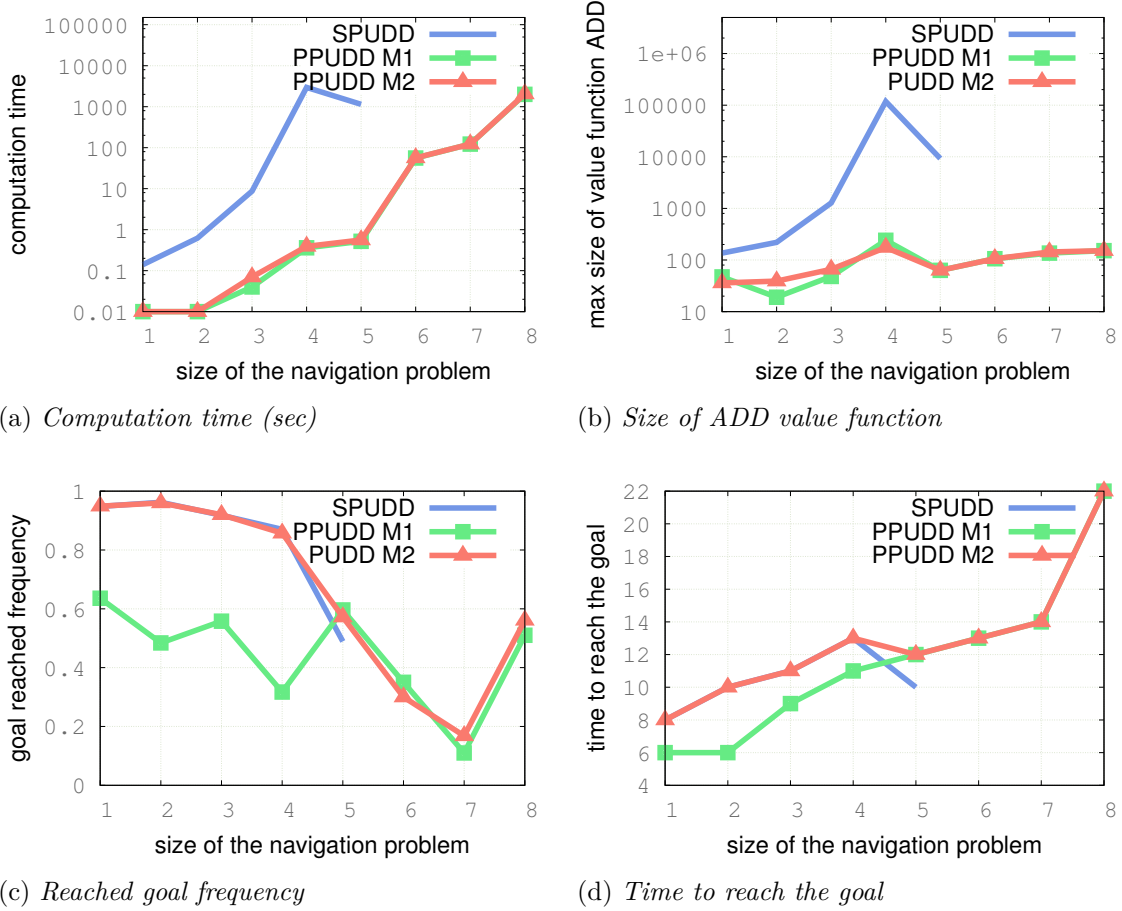


Figure III.5 – PPUDD vs. SPUDD on the Navigation problem: the x -axis represents indexes of problem instances, increasing with the problem sizes.

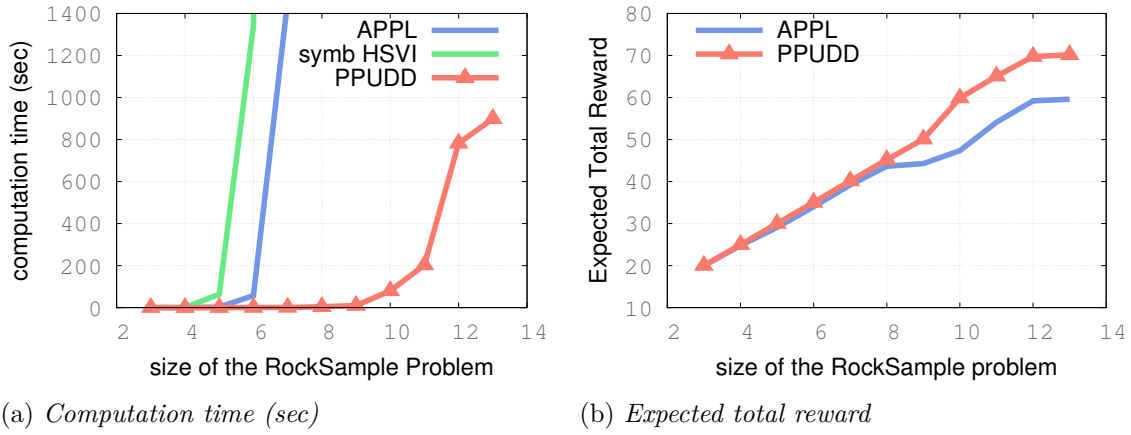


Figure III.6 – PPUDD vs. APPL and symb HSVI on the RockSample problem: the x-axis represents indexes of problem instances, increasing with the problem sizes.

III.3.1, against a recent probabilistic MOMDP planner, APPL [83], and a POMDP planner using ADDs, symbolic HSVI [112]. Both algorithms are approximate and anytime, so we decided to stop them when they reached a precision of 1. Figure III.6a, where problem instances increase with grid size and number of rocks, shows that APPL runs out of memory at the 8th problem instance, symbolic HSVI at the 7th one, while PPUDD outperforms them by many orders of magnitude. Instead of precision, computation time of APPL can be fixed at PPUDD’s computation time in order to compare their expected total rewards (using probabilistic model’s rewards) after they consumed the same CPU time. Surprisingly, Figure III.6b shows that rewards gathered are higher with PPUDD than with APPL. The reason is that APPL is in fact an approximate probabilistic planner, which shows that our approach consisting in exactly solving an approximate model can outperform algorithms that approximately solve an exact model. Moreover, exact POMDP planners are unable to scale to problems of the size of the RockSample ones. Finally, it is worth noting that probabilities of the observation model, which represent uncertainties of sensor outputs, may be difficult to precisely know in practice, in which case possibilistic models may be more physically accurate. In fact for this example the policy produced by PPUDD is the best to get all possible rewards: this is essentially because the rover can be sure of a rock’s nature checking when it is on it.

These results assured us that it was not unreasonable to present PPUDD in the International Probabilistic Planning Competition 2014, even though the computation of strategies for probabilistic problems is not the initial vocation of this solver. The next section describes the competition context as well as the presented versions of PPUDD, and discusses the results of the different competitor solvers.

III.4.2 International Probabilistic Planning Competition 2014

The fully observable track of the International Probabilistic Planning Competition (IPPC) allows to fairly compare performances of MDP solvers. The competitors’ solvers have to compute strategies for some problems which are not known in advance. Given one of these problems, solvers have a limited amount of time to send actions to the competition server which simulates the evolution of the system state: successive states are sampled by the competition server using the transition probability distributions of the MDP defining the problem, and sent to a given competitor’s solver. For each system state received, the solver has to send back the action it has computed. These data exchanges are conducted during few trials of finite horizon and the

score of the solver for the considered problem is the average (over trials) of the undiscounted and finite sum of rewards along the trajectory generated by the trial.

Materials about this competition are available at the official web page of the competition https://cs.uwaterloo.ca/~mgrzes/IPPC_2014/. Problems are grouped in *domains*, which are MDPs whose a finite number of parameters are undefined: the problem, or MDP, used in practice during the competition is an *instance* of a domain, *i.e.* a domain whose parameters have been set. In this competition, 8 domains have been proposed, called respectively *Academic advising*, *Crossing traffic*, *Elevators*, *Skill teaching*, *Tamarisk*, *Traffic*, *Triangle tireworld* and *Wildfire*. Three possible encodings of the instances of these domains are proposed, *i.e.* three different languages can be used to describe the instances: the first is the *Planning Domain Definition Language* (PPDDL, [124]); the second is a *LISP*-like language introduced with symbolic algorithms such as SPUDD which defines explicitly transition probability distributions and reward function as ADDs (see http://users.cecs.anu.edu.au/~ssanner/IPPC_2011/); finally, the third is the *Relational Dynamic Influence Diagram Language* (RDDDL, [105]) which is simpler and more expressive than the previous ones. The competition consists in evaluating the solver over 10 instances per domain with 30 runs per instance and 18 minutes per instance: it takes 24 hours in total.

In order to ensure that everyone has the same computational power, each competitor solver is set up in a remote server whose RAM is 7.5Gb with 2 cores. The client and server for the competition are available in the open source *RDDLSim* software, which is available online at <http://code.google.com/p/rddlsim/>. Four solvers have been proposed for this competition:

- *PROST* [65], based on *Upper Confidence bound applied to Trees* (UCT, [66]) and using directly RDDDL encoding;
- *GOURMAND* [68, 67], based on *Labeled Real Time Dynamic Programming* (LRTDP, [12]) using PPDDL encoding;
- *symbolic LRTDP*, using ADDs and LISP-like encoding [38];
- our algorithm PPUDD, using LISP-like encoding too.

As the score given to solvers only depends on the 40 first stages of the process, the presented version of PPUDD consists of the Algorithm 12 with the “while condition” $\bar{U}^* \neq \bar{U}^c$ at line 2 replaced by the condition “iteration ≤ 40 ”. It also incrementally augments the planning horizon while maintaining a mask stored in form of a Binary Decision Diagram (BDD, *i.e.* an ADD with leaves in $\{0, 1\}$) representing the states reachable from the initial state: the computation of the current value function is then restricted to the reachable states only. While PPUDD is an offline algorithm, we proposed also *AnyTime PPUDD* (ATPPUDD) which is an anytime version which learns computation times of Bellman backups while dispatching the computational effort accordingly over the remaining planning horizon much like GOURMAND does in the probabilistic world (see [68]).

When encoded with the LISP-like format, problems of the competition, *i.e.* instances of each domains, are described as factored MDPs with boolean system state variables: for each action $a \in \mathcal{A}$ and for each next boolean system state variable X'_i , one ADD representing the corresponding transition probability distribution $\mathbf{p}(X'_i \mid \text{parents}(X'_i), a)$ is given. In order to define the π -MDP which will be solved by PPUDD, we simply normalize these distributions in the possibilistic sense: we set to 1 the possibility degree of an assignment of X'_i when its probability value is maximal, and to the probability value otherwise. For instance, for a given assignment of the previous variables $\text{parents}(X'_i)$, if the probability value of the assignment (or event) $X'_i = 1$ is 0.7 (and thus probability 0.3 that $X'_i = 0$), then the possibility degree of $X'_i = 1$ is set to 1, and the one of $X'_i = 0$ is set to 0.3.

In terms of ADDs, it can be computed as follows: let us first recall the notation $\mathbf{p}(X'_i \mid \text{parents}(X'_i), a)^{X'_i=0}$, used to represent the subtree of the ADD $\mathbf{p}(X'_i \mid \text{parents}(X'_i), a)$ setting X'_i to false (*i.e.* to 0). As well, $\mathbf{p}(X'_i \mid \text{parents}(X'_i), a)^{X'_i=1}$ is the subtree of the same ADD, setting X'_i to true (*i.e.* to 1). Let us denote by $\mathbb{1}_{\mathbf{p}_\top > \mathbf{p}_\perp}$ the BDD equal to 1 for each variable assignment such that

$$\mathbf{p}(X'_i \mid \text{parents}(X'_i), a)^{X'_i=0} < \mathbf{p}(X'_i \mid \text{parents}(X'_i), a)^{X'_i=1},$$

and equal to 0 for other assignments. The BDD always equal to 1 is denoted by $\mathbb{1}$. The BDD $\mathbb{1}_{\mathbf{p}=0.5}$ is equal to 1 for variable assignments such that the probability of the event $X'_i = 1$ (or $X'_i = 0$) is equal to 0.5, and this BDD is equal to 0 otherwise. We can also denote by $\mathbb{1}_{\mathbf{p}_\top < \mathbf{p}_\perp}$ the BDD which is equal to 1 for assignments of variables in $\text{parents}(X'_i)$ such that the probability of event $X'_i = 1$ is lower than the probability of event $X'_i = 0$: this BDD can be computed from previous BDDs, $\mathbb{1} \ominus \mathbb{1}_{\mathbf{p}_\top > \mathbf{p}_\perp} \ominus \mathbb{1}_{\mathbf{p}=0.5}$, where \ominus is the minus operator, applied to trees. The possibility transition distribution for the i^{th} variable is

$$\pi(X'_i \mid \text{parents}(X'_i), a) = \boxed{\max} \left\{ \begin{array}{l} \mathbb{1}_{\mathbf{p}=0.5}, \\ \boxed{\min} \left\{ \mathbb{1}_{\mathbf{p}_\top > \mathbf{p}_\perp}, \mathbf{p}(X'_i \mid \text{parents}(X'_i), a) \right\}, \\ \boxed{\min} \left\{ \mathbb{1}_{\mathbf{p}_\top < \mathbf{p}_\perp}, \mathbf{p}(X'_i \mid \text{parents}(X'_i), a) \right\} \end{array} \right\}$$

As well, for each action $a \in \mathcal{A}$, an ADD representing the reward function for this action is provided and denoted by $r(X_1, \dots, X_n, a)$. Let us define then for each $s \in \mathcal{S}$ and $a \in \mathcal{A}$,

$$\Psi(s, a) = \frac{r(s, a) - \min_{s,a} r(s, a)}{\max_{s,a} r(s, a) - \min_{s,a} r(s, a)} \in [0, 1].$$

The terminal preference function is set to $\Psi(s) = \max_{a \in \mathcal{A}} \Psi(s, a)$, and the strategy is initialized by $\delta^*(s) \in \arg\max_{a \in \mathcal{A}} \Psi(s, a)$ at the beginning of the algorithm.

Note that possibility and preference degrees are not in a scale \mathcal{L} as previously defined (*i.e.* $\{0, \frac{1}{k}, \frac{2}{k}, \dots, 1\}$ for some $k \geq 1$). Indeed, possibility degrees comes from probability values, and preferences are normalized rewards. However, only max and min operators are used, so it has no impact on the qualitative results of the computations.

The library used to perform computations with ADDs is the *CU Decision Diagram Package* (CUDD, <http://vlsi.colorado.edu/~fabio/CUDD/>), and the described versions of PPUDD are available at the adress <https://github.com/drougui/ppudd>.

Following figures illustrates the results of IPPC 2014: the score is given in function of the instance index, which generally increases with the difficulty (or the size) of the associated problem.

Figure III.7 presents the scores obtained by each solver for each of the 10 instances of the *Academic advising* domains, *i.e.* the average over 30 trials of the sum of the encountered rewards. Performances of our algorithms are close to the best ones. However, an unexplained and unwanted bug occurred with ATPPUDD for the 2nd instance, as only 3 runs have been performed by this solver. For the other instances, PPUDD and ATPPUDD produce strategies with performances like PROST and GOURMAND, and better than Symbolic LRTDP. This has been less true for the *Crossing traffic* problem, whose results are also described by Figure



III.7. This problem models a robot which has to get to a goal which is across an highway with a lot of cars. These cars arrive randomly and move left. As the possibility degree of the fact that no car arrive is set to 1 by our naive MDP to π -MDP translation, the optimistic criterion leads to decide to cross the street, even if an unseen car may arrive (with a probability < 0.5 but bit enough to be cautious). This explain the poor quality of the produced strategies for this domain. Note however that, for the 6 last instances (most difficult problems) our approach leads to better strategies than the probabilistic solver Symbolic LRTDP.

In the *Elevators* problem, people arrive randomly and have to be transported to the correct building stage: as the frequentist information is lost using the possibilistic approach and seems important in this problem (people do not want to wait once arrived), scores of our algorithms are poorer than the ones of PROST and GOURMAND. The toy example at the beginning of the introduction of this chapter illustrates that the possibilistic approaches can select actions probabilistically clearly suboptimal when the probability values are at the heart of the problem. PPUDD and ATPPUDD are however better than Symbolic LRTDP, as shown by Figure III.8, and than doing nothing (“noop strategy”) or choosing actions randomly (“random strategy”). PPUDD and ATPPUDD have quite good behaviours with the *Skill teaching* problem as illustrated by the same figure. Moreover, ATPPUDD leads to better results for the last three instances: as these instances are the *Skill teaching* problems with the largest system space, the anytime version, which manages the computation time, produces strategies with better performances than PPUDD, which classically solve the associated π -MDP, but cannot complete computations and lead to a poorer strategy.

With respect to other solvers, possibilistic solvers have good results with the *Tamarisk* domain, as shown in Figure III.9.. However, some instances (e.g. the 6th, the 8th and the 10th) are not even run as the ADD instantiation takes too long. Symbolic LRTP faces the same issue as it uses also the LISP-like encoding of the problem. We think that this is an issue specific to the competition, as each problem has to be equivalently translated into three different languages (RDDL, PPDDL and LISP-like), which produces sometimes artificially complex encodings of the problems. The *Traffic* domain is really hard to solve by PPUDD and ATPPUDD (see Figure III.9). Actually the least scores are obtained with this domain, and even the random and the noop strategies are better strategies. Note that we did not implement any “watchdog” returning random actions when the computed strategy is less effective than the random one. However, this kind of gadget is essential to improve results for such large and risky problem. As mentioned above for the *Crossing traffic* problem, the optimistic criterion may lead to dangerous actions, as it does here. Moreover, as this problem involves frequentist information (car arrivals) an high suboptimality of the strategy produced by the possibilistic approach is confirmed for this kind of problems (see the *Elevator* problems). Finally, the *Traffic* problem is known to be one of the hardest domain, so ADD instantiation takes long, as well as computations, which are then not proceeded enough to produce satisfying results.

Finally, the two last domains, whose results are described in Figure III.10, are called *Triangle Tireworld* and *Wildfire*. Firt, ATPPUDD faces an unexplained bug for each instance of the *Triangle Tireworld* domain: no trial is performed from the 5th instance, and maximum 2 trials are performed for other instances (which explains the poor score for each instance). As already mentioned for the *Tamarisk* domain, ADD instantiation takes too long for the last instances, and no trial is performed for the last 4 instances with PPUDD too: Symbolic LRTDP faces the same issue. The last domain, called *Wildfire*, leads to highly frequentist problems: it involves random fire starts. That is why PPUDD and ATPPUDD strategies are not really efficient, but not too distant from Symbolic LRTDP solver’s results.

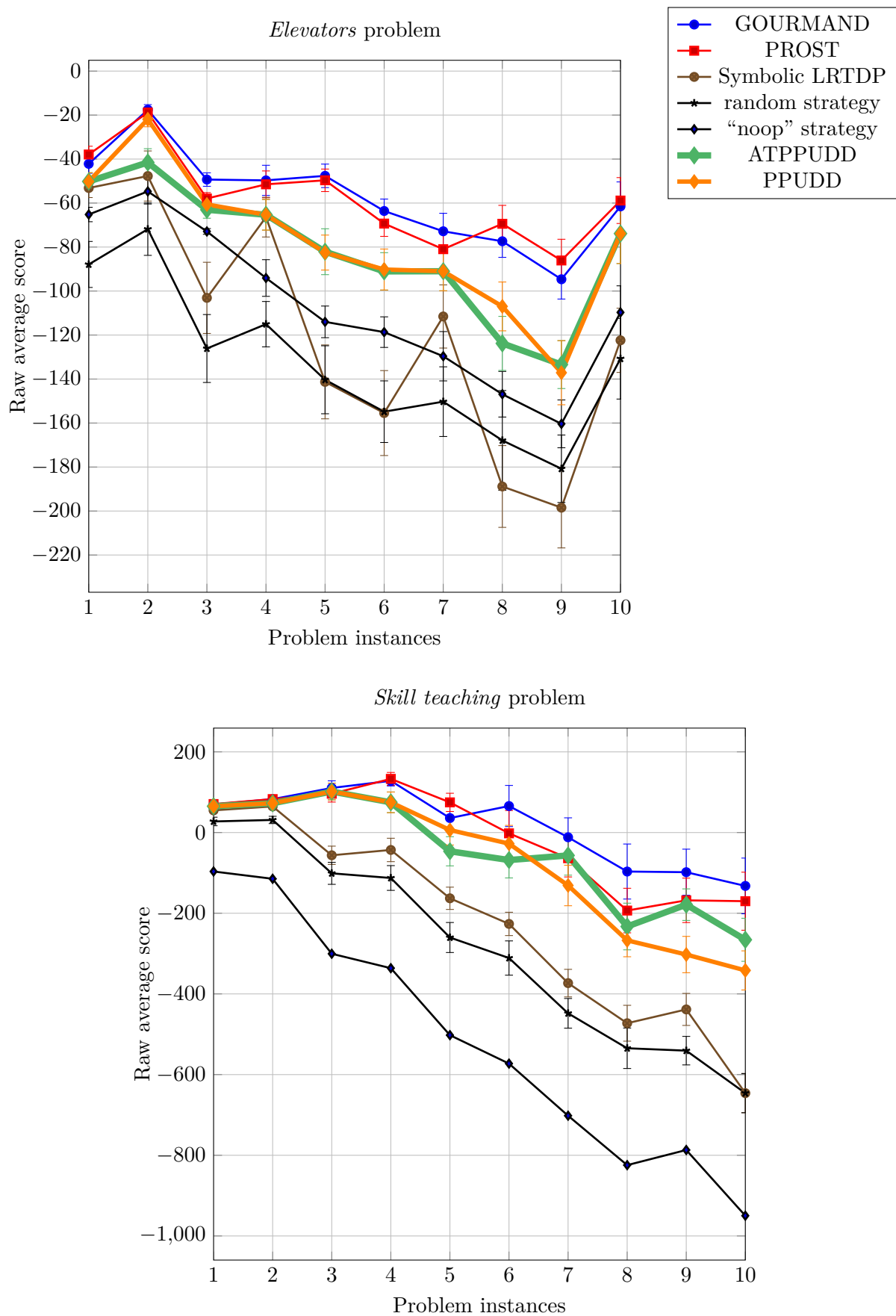


Figure III.8 – Results of the International Probabilistic Planning Competition – Fully Observable track



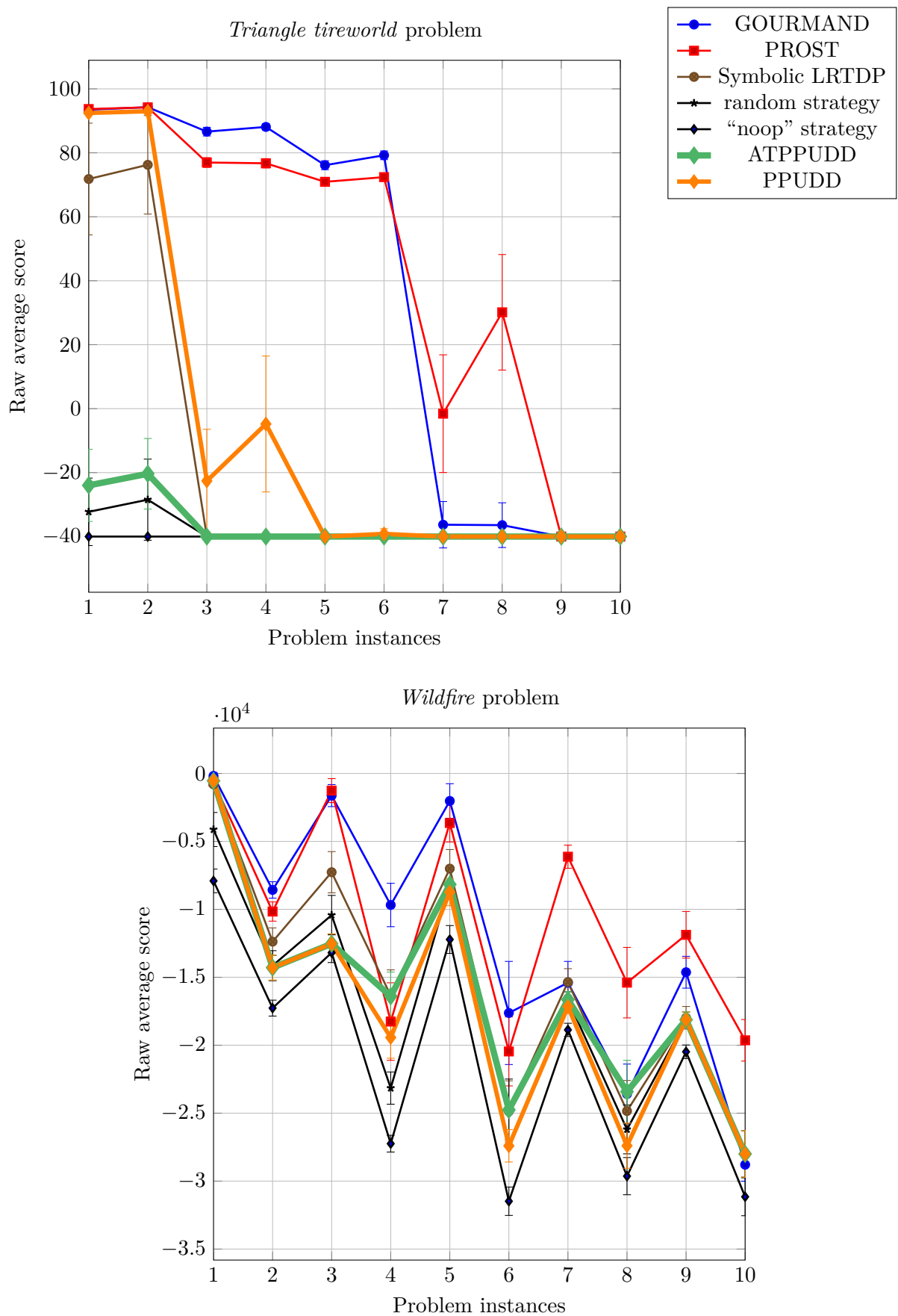


Figure III.10 – Results of the International Probabilistic Planning Competition – Fully Observable track

III.5 CONCLUSION

We presented PPUDD, the first algorithm to the best of our knowledge that solves factored possibilistic qualitative (MO)MDPs with symbolic calculations. In our opinion, possibilistic models are a good tradeoff between non-deterministic ones, whose uncertainties are not at all quantified yielding a very approximate model, and probabilistic ones, where uncertainties are fully specified, sometimes arbitrarily in practice. Moreover, π -MOMDPs reason about finite values in a qualitative scale \mathcal{L} whereas probabilistic MOMDPs deal with values in \mathbb{R} , which implies larger ADDs for symbolic algorithms. Also, the former reduce to finite-state belief π -MDPs contrary to the latter that yield *continuous*-state belief MDPs of significantly higher complexity. Our experimental results highlight the point that using an exact algorithm (PPUDD) for an approximate model (π -MDPs) can bring significantly faster computations than reasoning about complex exact models, while providing better strategies than approximate algorithms (APPL) for exact models. In the future, we would like to develop a probabilistic algorithm using the generalization of our possibilistic belief factorization theory to probabilistic settings (see Theorem 26): related but slightly different results have been proposed for probabilistic POMDPs [110]. These results also does not concern the case of mixed observability.

This chapter finally presents the results of our possibilistic approach during IPPC 2014: the highlighted bottleneck of our possibilistic algorithms resides on the translation from probabilities to possibilities: the naive automated translation presented before the description of the results leads to poor policies in benchmarks with complex dynamics and reward structures. Another issue is the size of the input LISP-like encoded domains whose ADD instantiation before optimization takes a very long time or does not even fit into memory for many difficult benchmarks: this difficulty is shared with the Symbolic LRTDP solver. However, there is almost no discretization of the initial probability values defining the MDP in order to produce the possibility degrees during the instantiation of the ADD defining the π -MDP: the maximal difference between two possibility degrees is set to 10^{-3} . Stronger discretizations have not been tested yet, and could improve scores of our solvers for problems with such memory issues. Modeling issues have been also highlighted, namely the fact that some problems request a cautious behaviour, not provided by the use of the optimistic criterion (see Definition III.8) used during the competition. Moreover, as illustrated in introduction, these experiments show that problems with high entropy events are outperformed by probabilistic approaches since the possibilistic approach does not take into account the frequentist information about the problem. The use of *lexi*-approaches, as used in the following chapter, may be a possibilistic stratagem to get around this issue. Note finally that the partially observable version of PPUDD (with the generation of a mask of reachable belief states, avoiding useless computations on unreachable beliefs) is also available on the repository <https://github.com/drougui/ppudd>.

The next chapter, Chapter IV, deals with *Human-Machine Interaction* (HMI) problems: the uncertainty dynamics of the system are in this context typically not known in terms of probability values, and the qualitative possibilistic approach is shown to be a natural approach to produce efficient diagnosis of human errors.

Finally, the last chapter, Chapter V, takes into account the remarks made using the results of IPPC14: an approach using Probability and Possibility Theory in order to benefit from both approaches in the resolution of factored POMDPs is presented: quantitative information of the problem is kept to avoid the highlighted modeling issues, and the belief state is handled in a possibilistic way, in order to get a smart discretization of it and to benefit from a finite and factorized belief state spaces. This approach leads to a factored probabilistic MDP which can be solved for instance by GOURMAND or PROST (which does not use the memory constraining LISP-like encoding).

APPLICATION OF QUALITATIVE POSSIBILISTIC HIDDEN MARKOV PROCESSES FOR DIAGNOSIS IN HUMAN-MACHINE INTERACTION

The work developed in this chapter is quite independent from the main theme of this thesis, namely the problem of *Decision Making under Uncertainty*. This work, performed in collaboration with Sergio Pizziol and extending work of his PhD thesis [92], contributes to modelling human-machine interactions. It is part of this thesis since it is a great example of the need of qualitative models (such as those presented in previous chapters) in some practical situations.

We formalize here a framework providing an estimate of the human assessment of the machine state, an automated detection of human operator attentional errors, and finally an estimate of the most plausible causes of these errors. A qualitative possibilistic approach is used to deal with uncertainty about the human operator assessment.

The human-machine context is first introduced to point out the need for a new modelling method for human attentional error. Then a human error model is derived from the machine logic, using expert assumptions on human errors and their plausibility. A human-machine interaction model results from the combination of the machine logic and the error model. Using the Possibility Theory, an analysis model estimating the human assessment is built on the interaction model, summed up in a *Possibilistic Hidden Markov Processes* (π -HMPs). The possibilistic analysis is first performed on a toy example. Finally the soundness of the approach is shown through tests with pilots performing a flight simulator mission.

IV.1 INTRODUCTION

In human-machine interaction studies, the problem of the correct human assessment of the machine state has been widely discussed. The main issue is that a human operator with a wrong assessment of the machine state is likely to perform *erroneous actions*, *i.e.* actions whose outcome is different from what is intended [64].

Many different approaches have been proposed to deal with this issue: among them, mental models and situation awareness [54, 99], formal inference rules [82], or *error models* for the human misinterpretation of the machine feedbacks [98]. Those approaches are based on a deterministic model for the human error, suited for error dynamic analysis but not for error detection. Moreover, they do not benefit from the flexibility provided by uncertainty representations.

The human assessment of the machine state is not observable during the interaction with the machine: nevertheless it may be estimated via uncertainty modelling for example thanks

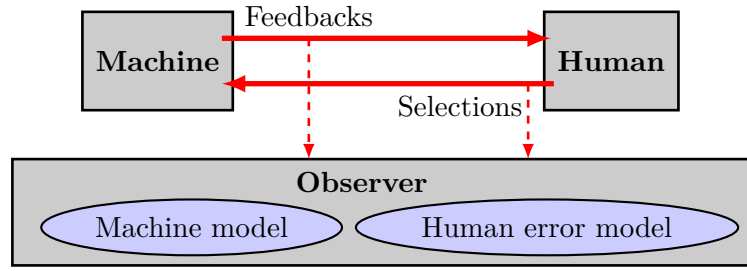


Figure IV.1 – The three actors involved in the study. The red arrows represent information flows.

to Probability Theory [81]. However probability values can be difficult to define in practice because of a lack of quantitative information related to the human operator's behaviour. A method to build an interaction model relying on less informative data is needed. This chapter proposes a new method building such a model with qualitative expert data and using the machine logic.

The human assessment of the machine state (shortly *assessment*) is mainly based on the *feedbacks* provided by the machine. *Feedbacks* are pieces of information the machine sends (via visual or aural alerts/signs) in order to inform the human operator about its current state. Some formal approaches have been proposed in order to estimate whether the human receives enough feedbacks [85, 30]. Nevertheless even if enough feedbacks are provided, the problem of their correct reception remains. If the human does not perceive some machine feedbacks during the interaction, the human assessment may be incorrect, leading to erroneous actions based on such a wrong assessment: the main objective of this work is to formalise an analysis model detecting assessment errors through human assessment estimation.

Three actors are involved in this framework: a machine, a human operator acting on the machine, and an observer analysing the human-machine interaction. Human operator actions on the machine are called *selections*.

The observer knows the machine logic and contemplates possible human assessment errors. Moreover, during the human machine interaction, they observe a sequence of data generated by the machine and the human operator: these data are called *observable occurrences*, and consist in each successive feedback (from the machine) and selection (from the human operator), as illustrated Figure IV.1. Machine state changes corresponding to those feedbacks and selections can also be considered as part of the observable data: indeed, the observer perfectly knows the initial machine state as well as the deterministic model of the machine, so the current machine state is easily determined.

The analysis model, which is the objective of this study, is meant to help the observer in the analysis of the human-machine interaction: using successive observable occurrences, as well as a machine model and a human error model, it provides an estimate of the human operator assessment of the machine state, a detection of human assessment errors, and an explanation for those errors.

The system designer could later modify the machine logic to make it take into account the assessment errors detection and diagnosis performed by the observer using the analysis model. For instance they could provide new specific feedbacks meant to correct the human operator assessment [37, 36]. Note that those applications are out of the scope of this work. This work focuses on a method to set up a human-machine interaction model from the machine model and on the definition of the analysis model; experiments on a flight simulator are also provided, showing that this approach is repponsive in practice.

Some key concepts are detailed in the next section, as the definition of the machine model

describing the machine logic. The human assessment error model (shortly *error model*) is then defined starting from the machine model. Next the human-machine interaction model is presented, resulting from the combination of the machine logic and the error model. It exhaustively defines all the human assessment transitions considered as possible by the observer. Later some working assumptions are given: assumed by the observer and expressed in natural language, they involve the plausibility of these human assessment transitions. An analysis model can then be set up based on these assumptions.

The analysis model presented in this work uses the Qualitative Possibility Theory as it is well suited to encode qualitative expert knowledge: the possibilistic analysis model is formally described, in the form of a Hidden Markov Process. It provides a human assessment estimation, assessment error detection and explanation of the error.

A detailed example of this approach is provided Section IV.4 analysing interactions for a simple three-state machine. In the last section, the method is tried through tests with pilots performing a flight simulator mission.

IV.2 FRAMEWORK FOR HUMAN-MACHINE INTERACTIONS MODELLING INCLUDING ASSESSMENT ERRORS

Hereafter we call *state* the machine state represented by the notation $s \in \mathcal{S}$. The actual human assessment of the state is represented by $h \in \mathcal{S}$: the equality $h = s^* \in \mathcal{S}$ means that the human operator thinks that the state of the machine is s^* . Note that this work is based on the simplifying assumption that the human operator is certain about the state of the machine: the representation of the human knowledge is limited to a unique machine state $h = s^*$. This unique state can also be seen as the most plausible one from the human operator's point of view, *i.e.* the one on which she/he bases her/his selections. Remember that a selection is a human operator action on the human-machine interface.

If no assessment error arises, assessment h coincides with actual state s . However the sending of feedbacks does not guarantee the correct receipt of the information, in particular for the *automated state changes* [56], *i.e.* state changes that are not fired by a selection. So assessment errors may occur.

The actual assessment h is not observable since the observer has no access to the human assessment of the situation: the main contribution of this chapter is then to provide a possibilistic estimation for it. Successive *observable occurrences* (shortly *occurrences*) *i.e.* each successive feedback (from the machine) and selection (from the human operator), are represented by the variable v , and are used to update this estimation. Occurrences can be divided into three categories:

- human selections on the machine interface;
- automated machine state changes with relevant feedback sending;
- the initialization, representing the beginning of the interaction process.

The observer knowing the initial state and machine model is able to deduce the actual state at each occurrence (so the machine state is considered as observable as well). Next section details how this machine logic is described, starting point for a human error model derivation.

IV.2.1 Machine model

The machine logic is summarized through a *logic table* representation [56]. As this representation has been developed to describe the deterministic behaviour of the machine, the logic

table takes into account the machine state s , but not the human assessment h . Machine state transitions are represented as triplets (previous state s , current occurrence v' , current state s'). These machine state transitions are summarized in pairs of (*situation*, *behaviour*):

Definition IV.2.1 (*Situation*)

A situation is defined as the conjunction between a proposition about current occurrence $\mathcal{P}(v')$ and a proposition concerning the previous state $\mathcal{P}(s)$: $\mathcal{P}(v') \wedge \mathcal{P}(s)$. In practice, the proposition about the current occurrence is a disjunction of occurrences.

Moreover the machine state is described by a tuple of state variables: $s = (s^1, s^2, \dots, s^n)$. The proposition about the previous state is a Conjunctive Normal Form (CNF) of these state variables, i.e. a logic conjunction between disjunctions of assignments of the same state variable.

For instance consider a set of possible occurrences $\{v_A, v_B, v_C\}$, and a set of states described by variables $(s^1, s^2) \in \{s_A^1, s_B^1\} \times \{s_A^2, s_B^2, s_C^2\}$. An example of proposition about the current occurrence might be $\mathcal{P}(v') = (v' = [v_A \vee v_C])$. Current occurrences and previous state variables are either defined explicitly or take the parametric value “no matter which occurrence/assignment” denoted by “[*]”. Proposition $\mathcal{P}(s) = (s^1 = [*]) \wedge (s^2 = [s_A^2 \vee s_B^2])$ is an example of CNF, or proposition about the current state. The situation is finally fully described with $\mathcal{P}(v') \wedge \mathcal{P}(s)$. In this example, the situation is expressed in natural language as: “occurrence is either v_A or v_C , variable s^1 takes any value, and variable s^2 is either s_A^2 or s_B^2 ”.

Definition IV.2.2 (*Behaviour*)

A behaviour, which is the result of a situation, is a proposition $\mathcal{P}(s')$ defined as a logic conjunction between assignments of the different state variables.

These assignments are either defined explicitly, or take the parametric value “same assignment as the corresponding previous state variable assignment” denoted by “[*]”.

An example of proposition describing a behaviour for the previous situation example might be $\mathcal{P}(s') = (s'^1 = [*]) \wedge (s'^2 = [s_C^2])$. In this example, the behaviour is expressed as: “variable s'^1 assignment is the same as s^1 , and variable s'^2 assignment is s_C^2 ”.

A complete set of (situation, behaviour) pairs can be summed up in a logic table.

Definition IV.2.3 (*Logic table*)

The set of pairs (situation, behaviour) is represented in an explicit way with a table called logic table. The first column of the table contains the occurrence variable notation v and state variables names, the second column contains possible occurrences, and possible state variables assignments. Pairs (situation, behaviour) are represented in the next columns (1, 2, 3, etc). In those columns, boxes containing 1 mean that the current occurrence variable or state variable is equal to the current line value (assignment). If for some situation all the boxes corresponding to the occurrence variable or a state variable are empty, the occurrence variable or state variables take value [*] (no matter which occurrence/assessment). Note that this is equivalent to fill those boxes with many 1: the only purpose of this convention is the table readability. If for some behaviours all the boxes for a state variable are empty, the variable takes value [*] (same as the previous state).

As a toy example let us consider the case of a machine whose state can be represented with one boolean variable $s \in \{s_A, s_B\}$. The set of possible occurrences is $\{v_A, v_B, v_C\}$. Table IV.1 gives the logic table of the following (situation, behaviour) pairs:

1. • situation: $(v' = [v_A \vee v_C]) \wedge (s = [s_A])$;
 • behaviour: $(s' = [*])$.

		1	2	3
SITUATION				
v'	v_A	1		
	v_B		1	
	v_C	1		
s	s_A	1	1	
	s_B			1
BEHAVIOUR				
s'	s_A			
	s_B		1	

Table IV.1 – *Example of logic table for a machine with one boolean machine state variables and three possible occurrences: each pair (situation, behaviour) is described by a column.*

2.
 - situation: $(v' = [v_B]) \wedge (s = [s_A])$;
 - behaviour: $(s' = [s_B])$.
3.
 - situation: $(v' = [**]) \wedge (s = [s_B])$;
 - behaviour: $(s' = [*])$.

The total number of columns is equal to 3, as the number of pairs (situation, behaviour) describing the state machine. Column 1 of table IV.1 has to be read: “if $v' = v_A \vee v_B$ and $s = s_A$, then machine state remains the same”.

As the machine logic depiction has been presented, a human error model can be now deduced, leading to a full human-machine interaction model.

IV.2.2 Derivation of an error model

Classically, human-machine interaction models are based on the machine logic, and, for each occurrence, only on the expected (or feared) consequence on the human assessment of the machine state [98, 93]. The presented interaction model provides a more expressive representation of the assessment dynamics: many consequences on the human assessment of the machine state, called *effects* and denoted by variable $e \in E$, are considered possible for each occurrence. Nevertheless, they may be defined as more or less plausible by experts. In other words the term *effect* means “the (non observable) effect (of an observable occurrence) on the human assessment of the machine state”. For instance one possible effect is the correct human perception and interpretation of a feedback. Other possible effects, for the same observable occurrence (*i.e.* the feedback sending) could be that the feedback goes unperceived or misinterpreted.

An effect can be formally defined as the result of a partial function $f_e : S \times V \times S \rightarrow E$ of previous assessment h , current occurrence v' and current assessment h' : $e = f_e(h, v', h')$. Indeed that function defines the effect of the occurrence v' on the assessment dynamics, *i.e.* on the transition from h to h' . Partial function f_e is undefined for all triplets $(h, v', h') \in S \times V \times S$. Indeed, in the context of occurrence v' some assessment transitions are not assumed to be possible by experts: if h cannot become h' , $f_e(h, v', h')$ is not defined, and no effect is associated with this transition. Effects of v' are thus each $f_e(h, v', h')$, $\forall (h, h') \in \mathcal{S}^2$ when defined.

For a given occurrence, the effect when no assessment error arises *i.e.* when the human assessment transition is equal to the actual machine state transition, is the *nominal effect*. Nominal effects are then already defined given the logic table, replacing machine state variables s and s' with human assessment variables h and h' .

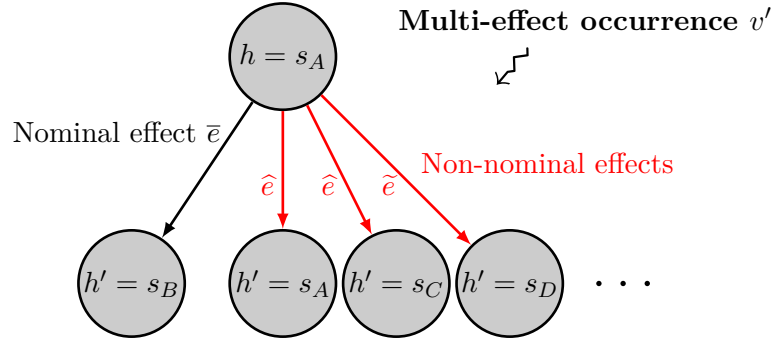


Figure IV.2 – Nominal effect and non-nominal effects of the occurrence v' on the assessment $h \in \mathcal{S}$ which becomes $h' \in \mathcal{S}$.

Non-nominal effects could also take place and correspond to assessment errors. Nevertheless for some occurrences only nominal effects are taken into account (*i.e.* the observer does not foresees any human assessment error for those occurrences). Occurrences with more than one possible effect for at least one possible assessment h are called *multi-effect* occurrences, and are illustrated by Figure IV.2. Formally if v' and h are such that $\exists(h'_A, h'_B) \in \mathcal{S}^2$ and $h'_A \neq h'_B$ for which $f_e(h, v', h'_A)$ and $f_e(h, v', h'_B)$ are defined, and $f_e(h, v', h'_A) \neq f_e(h, v', h'_B)$, v' is a *multi-effect* occurrence.

The error model is completed once all non-nominal effects have been defined: the logic table can be enhanced adding non-nominal effects for some occurrences. The way those non-nominal effects are described is the same as for the nominal effects: by pairs (situation, behaviour) represented by columns of the logic table. Referring to the example of the logic table given above (see Table IV.1), the expert knowledge could for instance assert that occurrence v_C could possibly make the human believe that if the machine state is initially s_A it finally changes to s_B . This potential effect is described by column 4 in Table IV.2. This new column does not replace the nominal case that is unaltered and still described by column 1. For readability columns corresponding to non-nominal effects are written in red. Occurrence v_C is thus a multi-effect occurrence. A logic table that includes assessment errors as Table IV.2 is called an *enhanced logic table*. The expert knowledge may again enhance the error model, stating that occurrence v_A could also lead to the same kind of human assessment error (see column 5).

Remember that effects e concern the human assessment dynamics h , which is of course not observable: actual effects are thus not observable, as a result of $f_e(h, v', h')$. Effects can however be sorted according to their plausibility, as presented right now.

IV.2.3 Effect plausibility

In this study nominal effects are considered as generally more plausible than the corresponding non-nominal ones *i.e.* than the corresponding human assessment errors starting from the same previous assessment h , and under the same occurrence v' : the human operator is thus assumed to know the machine logic and to have a quite good perception of the feedbacks.

Experts, after the enumeration of the potential non-nominal effects, have also to sort all effects according to their plausibility, dividing them into categories: for instance, effects whose plausibility is normal \bar{e} (shortly *normal effects*), effects whose plausibility is less than normal but not unusual \hat{e} (shortly *less than normal effects*), effects whose plausibility is unusual \underline{e} (shortly *unusual effects*), or even effects whose plausibility is very rare \tilde{e} (shortly *very rare effects*). The line “effect” is thus added to Table IV.2 specifying effects plausibility. For instance, nominal effect described by column 1 is normal according to the expert (\bar{e}), but nominal effect described by column 2 is very rare (\tilde{e}).

We have made the assumption that for each assessment h , there always exists at least one

columns		1	2	3	4	5
SITUATION						
v'	v_A	1				1
	v_B		1			
	v_C	1			1	
h	s_A	1	1		1	
	s_B			1		1
BEHAVIOUR						
h'	s_A					1
	s_B		1		1	
EFFECT		\bar{e}	\tilde{e}	\bar{e}	\hat{e}	\underline{e}
POSSIBILITY		1	ε	1	λ	δ

Table IV.2 – Enhanced logic table of the logic table IV.1: occurrences v_A and v_C have both a non-nominal effect, described respectively by columns 5 and 4. Each column represent a pair (situation, behaviour), and effect row represents the effect plausibility label. Last row assigns a possibility degree to each effect (see section IV.3.1).

occurrence v' with a normal effect. Thus, for each current human assessment h , it exists one possible occurrence and next human assessment considered as normal *i.e.*

$$\exists v', h' \text{ such that } f_e(h, v', h') = \bar{e}. \quad (\text{IV.1})$$

This remark is essential and forms a rule imposed in practice when filling the effect row of the enhanced enhanced logic table. It means that for each human assessment, at least one next step of the human-machine interaction is normal. Moreover, this property suits the possibilistic analysis model construction, as recalled section IV.3.

In the next section, the analysis model is defined from a given effect plausibility ranking, using a plausibility measure on the human-machine system dynamics: more details about the chosen measure are given in section IV.3. The last part of this section (IV.2.5) will derive the effect ranking from a set of expert rules, leading to a fully defined human-machine interaction model.

Note that an effect is *normal* if its plausibility is considered as normal (by the expert, or the system designer). Nominal effects and normal effects must not be confused (see Figure IV.3): the words *normal*, *unusual* are used to define the plausibility of effects *i.e.* to sort them, in order to fully define the interaction model, as just explained in this section IV.2.3 and performed from general assumptions in section IV.2.5. On the other hand, effects are said *Nominal* if they represent a human assessment transition without error, *i.e.* a human assessment transition corresponding to the machine state transition (ideal human understanding). Thus, effects are said *non-nominal* if they represent assessment errors, and are added to the logic table using expert knowledge.

In the following section IV.2.4 the human-machine interaction system dynamic is detailed.

IV.2.4 System dynamics: trajectories and exceptions

After the manifestation of $m \geq 0$ occurrences, the sequence of machine states is called the $(m+1)$ -length *state trajectory* and is denoted by $\mathcal{S}_m = (s_0, s_1, \dots, s_m)$. This trajectory is considered as observable, as well as the $(m+1)$ -length *occurrence trajectory* $\mathcal{V}_m = (v_0, v_1, \dots, v_m)$. In other words the observer defined in introduction section IV.1 and Figure IV.1, thanks to their knowledge of the machine logic, is able to provide $\forall 0 \leq t \leq m$,

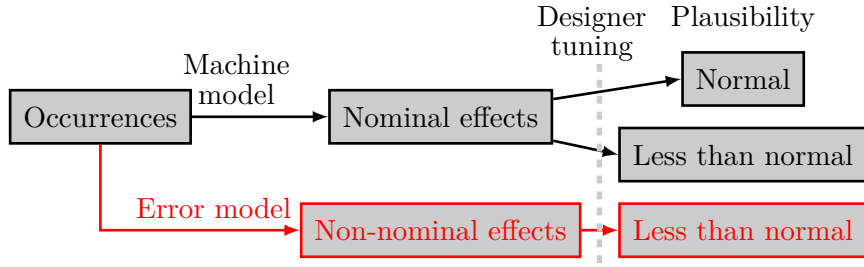


Figure IV.3 – Nominal effects, non-nominal ones defining the error model, and plausibility evaluation.

- the occurrence at stage t , v_t (e.g. selection, or automated state change),
- and actual state of the machine s_t , deduced from the machine logic.

However the actual *effect trajectory* (e_0, e_1, \dots, e_m) and *assessment trajectory* (h_0, h_1, \dots, h_m) are not observable. They may be estimated using the possibilistic analysis model described in section IV.3. Remember that each occurrence may have many effects. So a $(m+1)$ -length occurrence trajectory corresponds to many possible $(m+1)$ -length effect trajectories: $\forall 0 \leq t \leq m$, e_t is an effect¹ of the occurrence v_t . Each time a multi-effect occurrence is fired, several *assessment trajectories* are possible, one or more for each possible effect (see Figure IV.2). The set of possible effects trajectories is denoted by \mathcal{E}_m and the set of possible assessments trajectories \mathcal{H}_m .

In practice, possible effects and assessments trajectories are stored together in the form of *non-observable trajectories*: $(e_0, h_0, e_1, h_1, \dots, e_m, h_m)$ with $\forall 0 \leq t \leq m$, $e_t = f_e(h_{t-1}, v_t, h_t)$ if f_e is defined for this triplet, and removing h_{-1} for $t = 0$. The firing of a new occurrence v_{m+1} updates the set of non-observable trajectories adding possible effects and assessments. Each non-observable trajectory ending with $h \in \mathcal{S}$ at stage m , is completed with each pair $(f_e(h, v_{m+1}, h'), h')$ such that $f_e(h, v_{m+1}, h')$ is defined, *i.e.* stored in the enhanced logic table. Because several multi-effect occurrences may be fired the number of possible non-observable trajectories may increase significantly.

Initially the most plausible non-observable trajectory is the one that includes only nominal effects (*i.e.* no assessment errors). The corresponding assessment trajectory (removing effects from the non-observable trajectory) is called the *objective assessment trajectory* and is equal to the machine state trajectory. After the firing of an occurrence whose effects are all considered at the most as unusual in the actual situation, the objective assessment trajectory is no longer considered as normal. This situation is called an *exception* and the occurrence that led to the exception a *triggering occurrence*. Typical triggering occurrences are selections considered as erroneous in the particular context.

Let us describe the mechanism and the behaviour of the wanted analysis model which will be set up in section IV.3.1. If an exception is detected by the analysis model, the model itself verifies if there is a non-nominal effect (*i.e.* an assessment error) in the past history that, if considered as the actual effect, could lead to a situation in which the firing of the triggering occurrence is not unusual, but instead normal. For instance the analysis models may verify if there is a human feedback misinterpretation that could explain a human selection otherwise considered as erroneous. This non-nominal effect is called *exception explanation* and the assessment trajectory embedding this exception explanation is considered as the new most plausible one. The formerly most plausible assessment trajectory is no longer coherent with the actual observations of the system. Therefore, its plausibility is decreased. On the other hand, if no exception explanation is found, the plausibility of the assessment trajectories remains

¹Remember that effects are not observables.

unchanged. The *possibilistic Bayes rule*, also presented in the section IV.3, formally defines how to implement these concepts.

The following section IV.2.5 presents the expert rules chosen to complete the human-machine interaction model: these assumptions about effects are used in applications presented in last sections IV.4 and IV.5.

IV.2.5 Working assumptions

In order to define in practice non-nominal effects modelling human operator assessment errors, as well as their plausibility, two methods are possible: designing them one by one *by hand* with the help of experts of the domain, or deriving them *mechanically* from a limited number of general assumptions formulated by the experts. Nevertheless a mixed approach is also possible: for instance designers could start with the *mechanical* assumption-based method and successively suppress or add *by hand* some non-nominal effects, or even rank *by hand* the plausibility of some effects.

Hereafter some assumptions for the mechanical approach about possible effects and their plausibilities are defined. Defining a generic set of assumptions is out of the scope of this work. Note that these rules have been defined by experts for the experiment presented in section IV.5, and may be unsuitable for other applications. They are defined here to set an example of interaction model (including the ranking of effects), before the building of the possibilistic analysis model.

Here is the list of the chosen working assumptions:

- the human knowledge of the machine behaviour is correct;
- the human should perceive feedbacks, but she/he can possibly miss them;
- the human knowledge of the initial state is uncertain, but is likely to coincide with the real one;
- selections that do not change the machine state are considered as *slips*, *i.e.* unmeant selections;
- the missing of a feedback is more likely to happen than a slip or a mistake: a *mistake* is a selection or a lack of selection defined as erroneous by the system designer.
- the missing of $n < n_{max}$ feedbacks is more likely to happen than missing $n + 1$ feedbacks, or a slip.

The first assumption implies that the human is sufficiently trained to use the machine and knows its behaviour: nominal effects are then generally considered as more plausible than the corresponding non-nominal ones. as previously announced.

According to the next two assumptions, the interaction model takes into account two multi-effect occurrences. The first one is the feedback sending: feedbacks should be perceived (nominal effect) but could go unseen (non-nominal effect). The second one is the initial human appreciation of the machine state: the initial assessment should be correct (nominal effect), but could be wrong (non-nominal effect).

The fourth assumption means that selections which do not change the machine state are not voluntary. The last assumption states that the more feedbacks are lost, the less the situation is plausible.

For this interaction model, the following occurrences and corresponding effects are thus used:

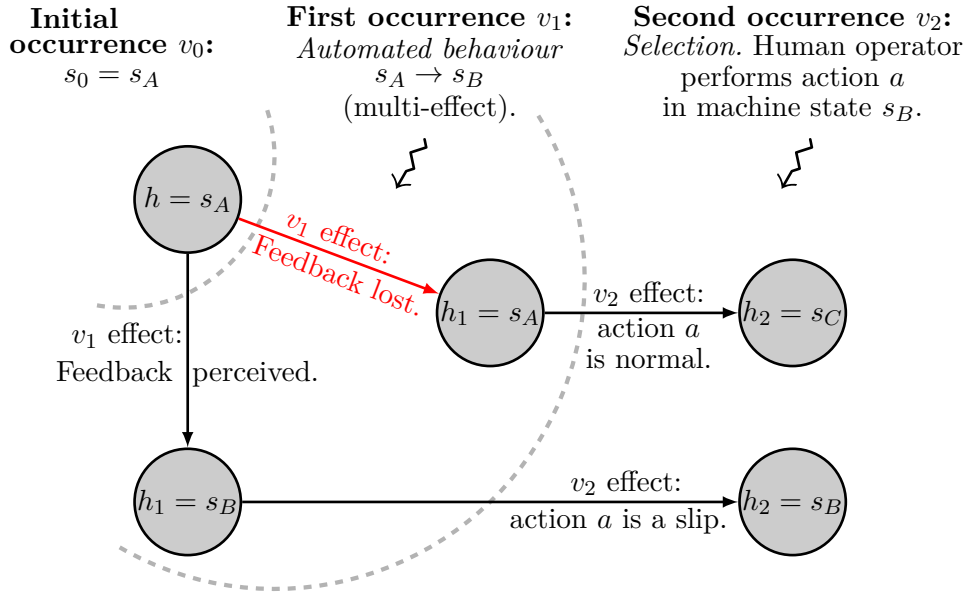


Figure IV.4 – Loss of feedback as an exception explanation.

- the execution of selections which have one nominal effect; depending on the current assessment they are classified as normal selections, or as slips/mistakes which are unusual;
- an *automated state change* with two possible effects: the reception and well interpretation of the relevant feedback (nominal), which is normal, and the loss of the feedback (non-nominal), which is not normal but more likely to happen than an unusual effect;
- the state initialization with two possible effects: correct initialization (nominal), which is normal, and wrong initialization (non-nominal), which is unusual.

The interaction model is so fully defined. The analysis model detailed in the next section is based on the Qualitative Possibility Theory (see Section I.2.1) providing a formal way to sort effects according to their plausibility. Starting from a human-machine interaction model, built as presented in the current section, the analysis model leads to a possibilistic estimation of the human assessment at each step of the process.

Before detailing the analysis model let us show with an example the desired comportment of the model for an automated state change followed by an action that is a slip in the actual state of the machine (see Figure IV.4).

Let us start with the automated state change with machine initial state $s_A \in \mathcal{S}$ and final state $s_B \in \mathcal{S}$. The correct reception of the relevant feedback has to be considered as the most plausible effect. The objective assessment trajectory $(h_0 = s_A, h_1 = s_B)$ has to be normal and so the most plausible one. The second trajectory $(h_0 = s_A, h_1 = s_A)$ corresponds to the loss of feedback and it has to be less than normal but more than unusual.

Later on the human performs a selection (action a) that does not modify the machine state. This selection is a slip in the actual state of the machine: s_B . Therefore it is a slip also for a human operator for which $h_1 = s_B$. The objective assessment trajectory becomes $(h_0 = s_A, h_1 = s_B, h_2 = s_B)$, and its plausibility should be reduced to that of a slip (unusual).

If the human assessment has not been updated in step 1 due to a loss of feedback (second trajectory) the assessment is still $h_1 = s_A$. Suppose that in this machine state s_A action a is totally normal. The second trajectory becomes $(h_0 = s_A, h_1 = s_A, h_2 = s_C)$ and its plausibility should remain unchanged (less than normal but more than unusual).

An exception should be detected and the analysis model should identify the exception explanation in the loss of the relevant feedback.

IV.3 HUMAN ASSESSMENT ESTIMATION, ERROR DETECTION AND DIAGNOSTIC

Possibility Theory is well suited to encode qualitative expert knowledge such as the working assumptions presented in the previous section. This section begins with a short presentation of this theory. Later on the first step of the possibilistic analysis model is presented: starting from the interaction model, natural language knowledge is expressed in terms of possibility degrees: basically, what was “plausible” becomes “possible”, what was “less normal” is defined as “less possible”, what was “unusual” becomes “far less possible”, and what was “very rare” becomes “almost impossible”. This section ends with formal computations leading to successive human assessment estimations, the error assessment detection and the exception explanation.

Here the problem deals with uncertainty related to the effects of each occurrence, *i.e.* the human assessment transitions: in situations where no wide enough experiments dataset are available, the corresponding uncertainty cannot be modelled precisely with frequencies leading to transition probabilities. Possibility Theory allows the definition of a model using only the available information about the system, which is however enough rich to build a useful model.

IV.3.1 Possibilistic analysis model

In order to perform the possibilistic analysis the interaction model has to be fully defined, *i.e.* all the effects have to be defined and sorted according to their plausibility. The effect ranking can be encoded defining an appropriate qualitative scale \mathcal{L} and assigning possibility degrees from this scale to the effects. The example of table IV.2 defines the qualitative scale \mathcal{L} as $\{0, \varepsilon, \delta, \lambda, 1\}$ with $0 < \varepsilon < \delta < \lambda < 1$, and assigns possibility degree 1 to normal effects, $\pi(\bar{e}) = 1$, degree λ to less normal effects, $\pi(\hat{e}) = \lambda$, degree δ to unusual effects $\pi(\underline{e}) = \delta$, and degree ε to very rare effects, $\pi(\tilde{e}) = \varepsilon$. Of course, if a transition is impossible, *i.e.* if function f_e is not defined, corresponding possibility degree is 0. This last modelling step leads to the full definition of a possibilistic hidden Markov process whose states are the successive human assessments, sound framework for human assessment estimation.

The interaction model used in this work has been set up in section IV.2.5, providing definition of occurrence effects and the ranking of those effects. The possibility degrees have still to be assigned to them. Occurrence effects possibility degrees are in the qualitative scale $\mathcal{L} = \{0, \varepsilon, \lambda, 1\}$ with $0 < \varepsilon < \lambda < 1$. The following notations defining classes of effects are useful to assign possibility degrees:

- $e_{0c} \doteq$ the correct initialization – nominal effect of an “initialization” occurrence,
 $\pi(e_{0c}) = 1$;
- $e_{0w} \doteq$ a wrong initialization – non-nominal effect of an “initialization” occurrence,
 $\pi(e_{0w}) = \varepsilon$;
- $e_f \doteq$ the correct reception of a feedback – nominal effect of an “automated behaviour” occurrence: $\pi(e_f) = 1$;
- $e_l \doteq$ a missed feedback – non-nominal effect of an “automated behaviour” occurrence:
 $\pi(e_l) = \lambda$;

- $e_n \doteq$ a generic occurrence effect considered as normal, other than e_{0c} and e_f (for instance a normal selection): $\pi(e_n) = 1$;
- $e_s \doteq$ the arising of a slip – nominal effect of a selection occurrence, $\pi(e_s) = \varepsilon$;
- $e_m \doteq$ a mistake – nominal effect of a selection or “absence of selection” occurrence, $\pi(e_m) = \varepsilon$.

The human assessment of the initial state is uncertain, but is likely to coincide with the real one, *i.e.* a correct initialization is more plausible than a wrong one: $1 = \pi(e_{0c}) > \pi(e_{0w}) = \varepsilon$. Moreover, the good reception of a feedback is more plausible than a loss of one of them, which is more likely to happen than a slip or a mistake: $\pi(e_f) = 1 > \pi(e_l) = \lambda > \pi(e_s) = \pi(e_m) = \varepsilon$.

Before starting the human assessment estimation, it is important to understand the link between possibility degrees of effects, and possibilistic system dynamics. An effect encodes the manifestation of an occurrence v' and the transition from current human assessment $h \in \mathcal{S}$ to the next one $h' \in \mathcal{S}$: an effect is then plausible when occurrence v' and assessment h' are plausible knowing previous one h . That leads to the following equation defining the joint possibility distribution over next assessment and occurrence:

$$\pi(v', h' | h) = \pi(f_e(h, v', h')) \quad (\text{IV.2})$$

i.e. the possibility degree of the effect of an occurrence v' is the joint possibility degree of the this occurrence (v') and the next assessment (h') associated with the effect, knowing current assessment (h). For the initialization occurrence (beginning of the human-machine interaction), as no previous assessment is available, equation IV.2 becomes simply: $\pi(v_0, h_0) = \pi(f_e(v_0, h_0))$. Note that initialization occurrence is artificially added to define initial uncertainty: it is thus considered as a totally normal occurrence: $\pi(v_0) = 1$. Then $\pi(v_0, h_0) = \min\{\pi(v_0), \pi(h_0 | v_0)\} = \pi(h_0 | v_0)$, *i.e.*

$$\pi(h_0 | v_0) = \pi(f_e(v_0, h_0)). \quad (\text{IV.3})$$

Recall that the possibility degree 0 is of course assigned to each triplets (h, v', h') for which f_e is not defined: no such effects have been declared possible by experts.

As explained around equation IV.1 for each assessment h , there exists an occurrence v' and a human assessment h' entirely possible: there is always a couple (v', h') such as $\pi(v', h' | h) = 1$. Thus $\pi(v', h' | h)$ defines actually a joint possibility distribution, as possibilistic normalization is naturally ensured.

IV.3.2 Human assessment estimation

Initial possibility distribution over human assessment h is denoted by $\pi_0(h) = \pi(h_0 = h | v_0)$ which depends on initial occurrence v_0 defining initial machine state s_0 . It encodes the initial estimation of the human assessment about the machine state, using assumptions of the interaction model: knowing the initial machine state, as part of data available for the observer, a positive possibility degree is assigned to each potential human operator initial assessment of the machine state, using equation IV.3.

Given occurrence v_{t+1} , the possibilistic dynamics of human belief (assessment), *i.e.* the possibility degree of each assessment transition, is summed up in the transition function $(h, h') \mapsto T_{t+1}(h, h') = \pi(h_{t+1} = h' | h_t = h, v_{t+1})$.

Definition IV.3.1 (*Transition function*)

As $\pi(v_{t+1}, h' | h)$ is given by the interaction model, using equation IV.2, and as v_{t+1} is

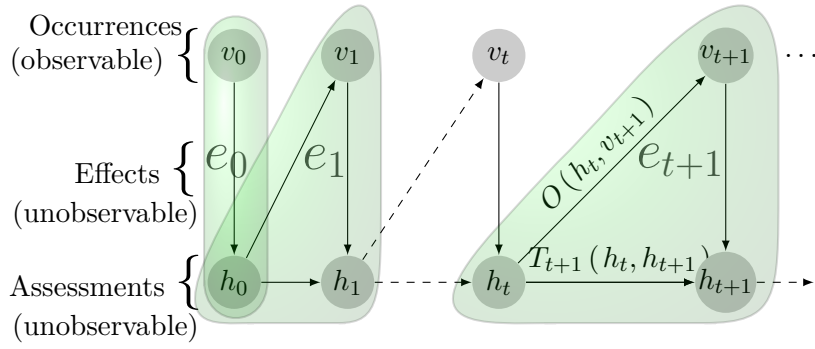


Figure IV.5 – *Dynamic Bayesian Network of the problem: relations between occurrences (v_t), and corresponding effects (e_t) on assessments evolution (h_t).*

$$\left\{ \begin{array}{l} \text{known, } T_{t+1} \text{ is computed with normalization} \\ T_{t+1}(h, h') = \begin{cases} 1 & \text{if } h' \in \underset{h' \in \mathcal{S}}{\operatorname{argmax}} \pi(v_{t+1}, h' | h) \\ \pi(v_{t+1}, h' | h) & \text{otherwise,} \end{cases} \end{array} \right. \quad \text{which comes directly from possibilistic conditioning I.2.7.}$$

The occurrences sequence $(v_t)_{t \in \mathbb{N}}$ only concerns actual facts (*e.g.* human operator selection, automated behaviour, ...) that are available (fully observable): T_t is then in fact a transition possibility distribution of the non-stationary possibilistic Markov Chain $(h_t)_{t \in \mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ with initial possibility distribution π_0 . This formalism is the same as the one used in the previous chapter for planning under uncertainty [41]. Here, unlike in planning problems, no action has to be chosen in this framework: however hidden state (here human assessment) is inferred in the same way. As successive human assessments constitute states of the Markov process, and are not directly observable, the analysis model is in fact summed up in a possibilistic hidden Markov process illustrated by Figure IV.5.

At occurrence step t , the *possibilistic estimation of the human assessment* is defined by the following possibility distribution:

Definition IV.3.2 (Possibilistic estimation of the human assessment)

$$\pi_t(h) = \pi(h = h_t | v_0, \dots, v_t) \quad (\text{IV.4})$$

As illustrated in Figure IV.5, some occurrences (as selections), depends on the previous belief (assessment) of the human operator *i.e.* possibility degree of occurrence v_{t+1} depends on h_t , and this dependence is defined by the observation function $(h, v') \mapsto O(h, v') = \pi(v_{t+1} = v' | h_t = h)$: this information is used to update the current estimation π_t .

Definition IV.3.3 (Observation function)

As $\pi(v_{t+1}, h' | h)$ is given by equation IV.2, observation function is given by marginalization $O(h, v_{t+1}) = \max_{h' \in \mathcal{S}} \pi(v_{t+1}, h' | h)$

The set of assessments h such that $\pi_t(h) = 1$ is denoted by H_t^* (human assessments of the machine state that are totally possible). At step t the next occurrence v_{t+1} can contradict estimation of the assessment: an exception arises when the possibility degree of this occurrence v_{t+1} knowing one of the most plausible $h \in H_t^*$ is less than the same possibility degree knowing another human assessment $\tilde{h} \notin H_t^*$, $O(h, v_{t+1}) \leq O(\tilde{h}, v_{t+1})$, and less than current estimation of the latter $O(h, v_{t+1}) \leq \pi_t(\tilde{h})$. More generally, information given by next occurrence v_{t+1} is used to update estimation, using next theorem.

Theorem 27

Human assessment estimation update, $\pi'_t(h) = \pi(h_t = h \mid v_0, v_1, \dots, v_{t+1})$, can be computed as follow:

$$\pi'_t(h) = \begin{cases} 1 & \text{if } h \in \underset{S}{\operatorname{argmax}} \min \{O(h, v_{t+1}), \pi_t(h)\}, \\ \min \{O(h, v_{t+1}), \pi_t(h)\} & \text{otherwise.} \end{cases} \quad (\text{IV.5})$$

This equation does not modify assessment estimation ($\pi_t \equiv \pi'_t$) if the following sufficient condition holds: $\forall h \in H_t^*, O(h, v_{t+1}) = 1$ and $\forall h \notin H_t^*, \pi_t(h) \leq O(h, v_{t+1})$.

Proof: Qualitative possibilistic conditioning I.2.7 provides equation IV.5, observation function playing the role of $\pi(y_{obs} \mid x)$, occurrence v_{t+1} the observation role, and assessment h the role of the state. Now, if $\forall h \in H_t^*, O(h, v_{t+1}) = 1$, as $\pi_t(h) = 1$, the first case occurs and $\pi'_t(h) = \pi_t(h) = 1$. The same equation holds $\forall h \notin H_t^*$: as $\pi_t(h) < 1$, the second case ("otherwise") occurs, and as $\pi_t(h) \leq O(h, v_{t+1})$, $\pi'_t(h) = \min \{\pi_t(h), O(h, v_{t+1})\} = \pi_t(h)$. ■

It is now possible to formally define an exception: **an exception is detected at step $t+1$ when update IV.5 makes the possibility degree of an assessment h different from the actual state s_t and such that $\pi_t(h) < 1$ become $\pi'_t(h) = 1$, i.e.** when the occurrence underlying the estimation update leading to π'_t , contradicts the previous estimation π_t and make an assessment different from the actual machine state becomes totally possible.

Once π'_t is computed using observation function $O(h, v_{t+1})$, estimation π_{t+1} is easily deduced using π' and transition function T_{t+1} :

Theorem 28

Assume $\pi'_t(h) = \pi(h_t = h \mid v_0, \dots, v_{t+1})$ is available: next possibilistic estimation of the human assessment $\pi_{t+1}(h) = \pi(h_{t+1} = h \mid v_0, \dots, v_{t+1})$ is computed as follow, propagating assessment estimation over one step of the possibilistic Markov process:

$$\pi_{t+1}(h') = \max_{h \in S} \min \{T_{t+1}(h, h'), \pi'_t(h)\} \quad (\text{IV.6})$$

Proof: As $T_{t+1}(h, h') = \pi(h_{t+1} = h' \mid h_t = h, v_{t+1})$, then $\pi_{t+1}(h') = \max_{h \in S} \pi(h_{t+1} = h', h_t = h \mid v_0, \dots, v_{t+1})$

$$\begin{aligned} &= \max_{h \in S} \min \{T_{t+1}(h, h'), \pi(h_t = h \mid v_0, \dots, v_{t+1})\} \\ &= \max_{h \in S} \min \{T_{t+1}(h, h'), \pi'_t(h)\}. \end{aligned} \quad \blacksquare$$

IV.3.3 Exception explanation

As all possible non-observable trajectories are recorded as described in section IV.2.4, set of possible effects (respectively assessments) trajectories at step m , \mathcal{E}_m , (respectively \mathcal{H}_m) is available removing assessments (respectively effects): they are used in case of exception to provide, if existing, an exception explanation. The explanation search uses operator leximin [43]. Operator leximin is a function similar to the minimum that may discriminate trajectories whose minimum possibility degree is the same. The idea is to compare effects trajectories at first via the simple minimum of effect possibility degrees, i.e. for a possible effects trajectory $(e_0, \dots, e_m) \in \mathcal{E}_m$, via $\min_{t=0}^m \pi(e_t)$.

Using equality IV.2, it appears that the minimum of effects possibility degrees corresponds to the joint possibility degree of the observed occurrences trajectory $(v_0, \dots, v_m) \in \mathcal{V}_m$ and

the assessments trajectory $(h_0, \dots, h_m) \in \mathcal{H}_m$ such that, $\forall 0 \leq t \leq m$, $e_t = f_e(h_t, v_{t+1}, h_{t+1})$ (removing " h_{-1} " for $t = 0$): $\min_{t=0}^m \pi(e_t) = \min_{t=0}^m \pi(v_{t+1}, h_{t+1} \mid h_t)$ (removing " $|h_{-1}$ " for $t = 0$) which is equal to $\pi(h_0, \dots, h_m, v_0, \dots, v_m)$.

When an exception occurs, effects trajectories which maximize this quantity are the most plausible explanations, and associated assessment trajectories can inform the observer about what the human operator thought. If more than one effects trajectory maximize this quantity, leximin operator can help to discriminate them. It counts in each trajectory the multiplicity of effects which have the minimum possibility degree and chose as lexi-minimal the effects trajectory with largest multiplicity (and then the most plausible ones are the effects trajectories with lowest multiplicity). If some trajectories have the same number of effects having the minimal possibility degree, leximin operator remove these effects, and counts multiplicity of the new minimal possibility degree, etc.

Definition IV.3.4 (*Leximin*)

Consider finite sequences of elements from a totally ordered space Π . A sequence $(\pi_1, \dots, \pi_m) \in \Pi^m$ is lower than a sequence $(\tilde{\pi}_1, \dots, \tilde{\pi}_m) \in \Pi^m$ in leximin sense, i.e. $(\pi_1, \dots, \pi_m) >^{\text{leximin}} (\tilde{\pi}_1, \dots, \tilde{\pi}_m)$ if and only if, assuming that elements $\pi \in \Pi$ are classified in increasing order in both sequences, $\exists 1 \leq i \leq m$ such that $(\pi_1, \dots, \pi_i) = (\tilde{\pi}_1, \dots, \tilde{\pi}_i)$ and $\pi_{i+1} > \tilde{\pi}_{i+1}$.

For a deeper investigation of the leximin operator, see [43] which explains how to find most credible trajectories in max leximin sence, i.e. $\arg\max_{\mathcal{E}_m} \min_{t=0}^m \pi(e_t)$, using dynamic programming.

Finally, as stated when defining our particular interaction model in section IV.2.5, not perceiving $n < n_{max}$ feedbacks is more likely to happen than not perceiving $n + 1$ feedbacks. Moreover not perceiving $n < n_{max}$ feedbacks (denoted by e_l^n) is more likely to happen than slips, i.e. $\forall n < n_{max}$

$$\pi(e_l^n) > \pi(e_s) \text{ and } \pi(e_l^n) > \pi(e_l^{n+1}).$$

This condition is taken into account for exception explanation as leximin naturally encodes it. However, assessment estimation as presented here, suffers from *drowning effect* due to the min operator [43]. It is possible to get around this issue redefining \mathcal{L} and min operator: $\mathcal{L} = \{0, \varepsilon, \lambda_{n_{max}}, \dots, \lambda_2, \lambda_1, 1\}$, and $\min\{\lambda_t, \lambda_j\} = \lambda_{t+j}$, with $\lambda_{t+j} = \lambda_{n_{max}}$ if $i + j \geq n_{max}$ (min operator remains the classical one for other values). The use of the leximin operator for h may solve this issue as well (but more discriminating).

In the following sections a mock-up example and a real case example are detailed, illustrating and validating the possibilistic analysis model based on the particular interaction model presented in section IV.2.5.

IV.4 INTERACTING WITH A THREE-STATE MACHINE

This example is meant to detail the estimation of assessment h as well as the error detection and explanation over few occurrences: human selections and automated state changes. Let us consider a machine with only one state variable with three values L , M and H (for respectively "low", "medium" and "high"), two possible selections $selU$ and $selD$ (for respectively "up" and "down"), and three automated state changes: acL (machine state becomes low), acM (machine state becomes medium) and acH (machine state becomes high). The interaction model is described in the enhanced logic table IV.3: an additional row "DETAILS" appears, containing for each column representing a non-nominal effect, a reference to the column number of the corresponding nominal effect. A description of each effect is given as well in the lower part of

columns		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
SITUATION																
v'																
Selection	$selU$	1	1	1												
	$selD$				1	1	1									
Autom. change	acL							1			1	1				
	acM								1				1	1		
	acH									1					1	1
h	L	1			1								1		1	
	M		1			1					1					1
	H			1			1					1		1		
BEHAVIOUR																
h'	L				1	1		1					1		1	
	M	1					1		1		1					1
	H		1	1						1		1		1		
EFFECT		e_n	e_n	e_s	e_s	e_n	e_n	e_f	e_f	e_f	e_l	e_l	e_l	e_l	e_l	e_l
POSSIBILITY		1	1	ε	ε	1	1	1	1	1	λ	λ	λ	λ	λ	λ
DETAILS	from										7	7	8	8	9	9
			Selection Up, from Low to Medium	Selection Up, from Medium to High	Selection Up, no effect because already High	Selection Down, no effect because already Low	Selection Down, from Medium to Low	Selection Down, from High to Medium	Automated behaviour to Low	Automated behaviour to Medium	Automated behaviour from Medium to Low, but unseen	Automated behaviour from High to Low, but unseen	Automated behaviour from Low to Medium, but unseen	Automated behaviour from High to Medium, but unseen	Automated behaviour from Low to High, but unseen	Automated behaviour from Medium to High, but unseen

Table IV.3 – Enhanced logic table for the three-state machine: the last row provides a natural language description for each effect.

the table. Note that columns 7, 8 and 9 represents each three nominal effects: for instance, column 7 represents transitions from $h = L$ (Low), from $h = M$ (Medium) and from $h = H$ (High), to $h' = L$ (Low). Note that the three effects of column 7 lead to the same assessment L (Low). The same is true for columns 8 and 9.

In this section the human assessment estimation evolution is shown for two scenarios consisting of three successive occurrences.

IV.4.1 Two successive selections

Initialization

The initial human assessment of the machine state is denoted by $h_0 \in \mathcal{S} = \{L, M, H\}$. The initial machine state, denoted by $s_0 \in \mathcal{S}$, is equal to M .

First occurrence v_0 encodes initialization $\{s_0 = M\}$ with two possible effects on human assessment (stated by experts in section IV.2.5): a correct initialization $e_{0c} = f_e(v_0, M)$ and a wrong one $e_{0w} = f_e(v_0, L) = f_e(v_0, H)$, which is less plausible. Then, using equation IV.3,

possibility distribution over h_0 is

$$\begin{aligned}\pi_0(h) &= \pi(h_0 = h \mid v_0) \\ &= \pi(f_e(v_0, h)) \\ &= \begin{cases} \pi(e_{0c}) = 1 & \text{if } h = M, \\ \pi(e_{0w}) = \varepsilon & \text{otherwise.} \end{cases}\end{aligned}$$

As three assessments are possible ($\forall h \in \mathcal{S}, \pi_0(h) > 0$), it leads to three possible non-observable trajectories at this initialization step: a good initialization (e_{0c}, M) and two wrong initializations, (e_{0w}, L) and (e_{0w}, H) .

Execution of a first *up* selection

After the execution of an *up* selection, machine state becomes $s_1 = H$: the occurrence is then $v_1 = selU$.

If $h_0 = M$ this occurrence v_1 has only one possible effect considered as normal e_n and making the assessment become $h_1 = H$ (see column 2 of table IV.3). Also if $h_0 = L$, occurrence v_1 has the same effect e_n and the new assessment is $h_1 = M$ (see column 1).

On the other hand if $h_0 = H$, occurrence v_1 has only one possible effect considered as unusual: in fact, as the new assessment is unchanged by the *up* selection ($h_1 = H$), this effect is a slip e_s (see column 3).

These effects, encoded by function $(h, h') \mapsto f_e(h, v_1, h')$, can be represented by a matrix whose rows are indexed by current assessments $h = L, M$ and then H , and columns by next assessment $h' = L, M$ and then H : $\begin{pmatrix} \emptyset & e_n & \emptyset \\ \emptyset & \emptyset & e_n \\ \emptyset & \emptyset & e_s \end{pmatrix}$ where \emptyset is put where $f_e(h, v', h')$ is not defined. Possibility distribution can then be represented as well using equation IV.2: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \varepsilon \end{pmatrix}$.

Using the definition of the observation function IV.3.3, by maximizing the previous matrix over column index h' (marginalization), $O(h, v_1) = \pi(v_1 \mid h_0 = h) = \begin{pmatrix} 1 & 1 & \varepsilon \end{pmatrix}$ represented by a vector indexed by variable h_0 with assignment L, M and then H . Indeed, this selection effect is considered totally possible except when human operator thinks machine state is H : in this case it has no reason to select *up* as machine state is already the highest (H), and this occurrence is considered as a slip (e_s with $\pi(e_s) = \varepsilon$). As $\pi_0(h) = \begin{pmatrix} \varepsilon & 1 & \varepsilon \end{pmatrix}$ (indexed by h_0), update of π_0 is not necessary, as sufficient condition stated by theorem 27 is satisfied.

Possibilistic transition function, computed from previous matrix using definition IV.3.1, can be expressed by the following matrix (h indexes rows with L, M and then H ; h' indexes columns in the same way), and increases deterministically the assessment: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Next human assessment is represented by variable h_1 . Its estimation (definition IV.4) is given by propagation equation IV.6: using a representation with matrices and the "max-min" matrix product \otimes which replaces sum and product of the classical matrix product \times by respectively max and min, this equation becomes

$$\begin{aligned}\pi_1(h_1) &= \max_{h \in \mathcal{S}} \min \{T_1(h, h_1), \pi_0(h)\} \\ &= \begin{pmatrix} \varepsilon & 1 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \varepsilon & 1 \end{pmatrix}\end{aligned}$$

where $h' = h_1$ indexes last vector with assignments L, M and then H . Finally, as T_1 encodes a deterministic transition, only three non-observable trajectories are possible at this step of the process: (e_{0w}, L, e_n, M) , (e_{0c}, M, e_n, H) and (e_{0w}, H, e_s, H) .

Execution of a second *up* selection

After the execution of a second *up* selection, machine state remains unchanged and then $s_2 = H$. The second occurrence is thus $v_2 = selU$. In this paragraph, vectors are indexed with h from L to H . Estimation update has now to be computed.

As $O(h, v_2) = \begin{pmatrix} 1 & 1 & \varepsilon \end{pmatrix}$, $\min\{\pi_1(h), O(h, v_2)\} = \begin{pmatrix} 0 & \varepsilon & \varepsilon \end{pmatrix}$, and finally, update IV.5 asserts that $\pi'_1(h) = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$. As assessment $h = M$ becomes entirely possible, this update **leads to an exception** because actual machine state is $s_2 = H$. This selection contradicts previous estimation π_1 since the human operator has no reason to select *up* if their assessment of the machine state is H .

As $v_2 = v_1$, transition function $T_2 = T_1$. Propagation equation IV.6 leads to the estimation of the next assessment h_2 from $\pi'_1(h)$, represented by the vector $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$, where h indexes this vector with L , M and then H . Using matrices representation of T_2 and "max-min" matrix product \otimes ,

$$\pi_2(h') = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

where h indexes rows of the matrix, $h' = h_2$ indexes columns and last vector. After the two occurrences v_1 and v_2 the possibilistic model returns a certitude for the human assessment: the human operator assessment for the machine state is H . Deterministic transition function conserves three possible non-observable trajectories: $(e_{0w}, L, e_n, M, e_n, H)$, $(e_{0c}, M, e_n, H, e_s, H)$ and $(e_{0w}, H, e_s, H, e_s, H)$. Associated effects trajectories have the same possibility degree: $\min\{\pi(e_0), \pi(e_1), \pi(e_2)\} = \varepsilon$. However the most credible trajectories can be found using leximin operator: $(e_{0w}, e_n, e_n), (e_{0c}, e_n, e_s) \in \operatorname{argmax}_{\mathcal{E}_3} \operatorname{leximin}\{\pi(e_0), \pi(e_1), \pi(e_2)\}$, i.e. the trajectory with a correct initialization and the trajectory beginning with $h_0 = L$ are the most plausible ones. The exception explanations are then a slip at the end, or a wrong initialization.

IV.4.2 Automated state change followed by a selection

This scenario shows the effects of the automated state changes on the human assessments.

Initialization and automated state change

Starting from the same machine state as in the previous scenario, the same initial occurrence $v_0 = \{s_0 = M\}$ occurs, and $\pi_0(h) = \begin{pmatrix} \varepsilon & 1 & \varepsilon \end{pmatrix}$. The same non-observable trajectories are recorded: (e_{0w}, L) , (e_{0c}, M) and (e_{0w}, H) .

Then machine state automatically goes to H : $v_1 = acH$. Occurrence v_1 can produce two effects: e_f if feedback of this automated state change is well received by human operator, which is totally possible $\pi(e_f) = 1$; and e_l if it is missed, with possibility degree $\pi(e_l) = \lambda$ (see columns 9, 14 and 15 of table IV.3). Variable h' (next assessment) indexing columns, and h (current one) indexing rows, with values L , M and then H , $f_e(h, v_1, h')$ can be written $\begin{pmatrix} e_l & 0 & e_f \\ 0 & e_l & e_f \\ 0 & 0 & e_f \end{pmatrix}$, and $\pi(f_e(h, v_1, h')) = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Using marginalization IV.3.3, yields the observation function $O(h, v_1) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ i.e. this automated behaviour is entirely possible whatever the human operator assessment. Then estimation update is not necessary as sufficient condition of theorem 27 is satisfied.

Using normalization IV.3.1, transition function is then $T_1(h, h') = \pi(h_1 = h' | h_0 = h, v_1) = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Finally, using equation IV.6 and "max-min" matrix product \otimes , $\pi_1(h') = \begin{pmatrix} \varepsilon & 1 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \varepsilon & \lambda & 1 \end{pmatrix}$. For each initial trajectory, except for the one beginning

with $h_0 = H$, next effect can be either e_f or e_l : (e_{0w}, L, e_f, H) , (e_{0w}, L, e_l, L) , (e_{0c}, M, e_f, H) , (e_{0c}, M, e_l, M) and (e_{0c}, H, e_f, H) .

Execution of an up selection

Human operator has no reason to execute this selection if their assessment of the machine state is H (estimated as the most plausible assessment). This occurrence $v_2 = selU$ corrects estimation using theorem 27: as previously, $O(h, v_2) = \begin{pmatrix} 1 & 1 & \varepsilon \end{pmatrix}$, indexed by h . Then $\min\{\pi_1(h), O(h, v_2)\} = \begin{pmatrix} \varepsilon & \lambda & \varepsilon \end{pmatrix}$. Finally update 27 concludes that $\pi'_1(h) = \begin{pmatrix} \varepsilon & 1 & \varepsilon \end{pmatrix}$. As $h_1 = H$ is no more the most plausible assessment, while machine state $s_1 = H$, it leads to an exception. Estimation of h_2 is computed thanks to deterministic transition produced by $v_2 = selU$, already used in the previous scenario: $\pi_2(h) = \begin{pmatrix} \varepsilon & 1 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon & 1 \end{pmatrix}$, where \otimes is yet the previously defined "max-min" matrix product.

As transition function T_2 is deterministic, number of trajectories remains 5:

- $(e_{0w}, L, e_f, H, e_s, H)$,
- $(e_{0w}, L, e_l, L, e_n, M)$,
- $(e_{0c}, M, e_f, H, e_s, H)$,
- $(e_{0c}, M, e_l, M, e_n, H)$,
- $(e_{0c}, H, e_f, H, e_s, H)$.

Computing $\arg\max_{\mathcal{E}_3} \text{leximin}\{\pi(e_0), \pi(e_1), \pi(e_2)\}$, the situation with a correct initialization, followed by a missed feedback and by a normal selection is the best guess of the analysis model: $(e_{0c}, M, e_l, M, e_n, H)$ explains thus the exception.

The current section was meant to get an intuition of the mechanism of the analysis model in estimating successive human assessments, and in detecting and explaining assessment errors. In practice, machines logics are much more complex: next section presents the results of our analysis model when facing a realistic human-machine system.

IV.5 INTERACTING WITH FLIGHT CONTROL AND GUIDANCE

In this section a real case application is presented: the AutoPilot (AP) of a flight simulator. An experience has been conducted with ten general aviation pilots in this flight simulator. The experience was originally meant to test the soundness of a method to detect dangerous situations called human-machine conflicts, in which the pilot actions are not coherent with the actual state of the machine. Those conflicts are the consequence of human attentional errors. For that reason the data collected in that experience is a valuable source of attentional errors in a realistic setting. This dataset is used in this section to test the possibilistic analysis model for the detection of human attentional errors. For more details about the used dataset, see [93].

As for the three-state machine example presented in section IV.4, the first step is the definition of the logic of the automation and the definition of some human assessment errors.

IV.5.1 System description

The definition of human assessment errors is easily automatized starting from the logic of the automation and using rules stated section IV.2.5. Hereafter are detailed state variables of the machine, possible occurrences and effects.

State variables: s_i

s_1 = AP state (On/Off);

s_2 = AutoTHRust (ATHR) state (On/Off);

s_3 = Airspeed (Underspeed, Normal, Near overspeed, Overspeed). “Underspeed/Overspeed” means that the airspeed is smaller/greater than minimum/maximum speed. Minimum and maximum speed are calculated by the autopilot and they depend on the flight envelope. “Near overspeed” means that the speed is between the maximum speed and the maximum speed minus five knots. “Normal” means that the speed is between the minimum speed and maximum speed minus five knots;

s_4 = Control stick (Actioning, Not actioning);

s_5 = Throttle lever (Actioning, Not actioning).

Occurrences: v

- initialisation:

v_A = *Initialization* \doteq Start of the experiment and creation of all the initial assessment trajectories;

- selections:

v_B = *AP button* \doteq Autopilot engagement/disengagement button pressed. The Autopilot is the part of the automation that if switched on is in charge for the control of the pitch, the roll and the yaw of the aircraft, *i.e.* of its attitude;

v_C = *ATHR button* \doteq Autothrust engagement/disengagement button pressed. The Autothrust is the part of the automation that if switched on is in charge of the control of the thrust;

v_D = *Control Stick On* \doteq Control stick activation;

v_E = *Control Stick Off* \doteq Control stick deactivation;

v_F = *Throttle lever On* \doteq Throttle lever activation;

v_G = *Throttle lever Off* \doteq Throttle lever deactivation;

- automated state changes:

v_H = *Speed Low* \doteq Airspeed takes value Underspeed and consequent AP disconnection;

v_I = *Speed Normal* \doteq Airspeed takes value Normal;

v_J = *Near overspeed* \doteq Airspeed takes value Near overspeed and consequent vertical speed constraint;

v_K = *Overspeed* \doteq Airspeed takes value Overspeed and consequent AP disconnection;

v_L = *Trajectory divergence* \doteq divergence between pilot selected trajectory and autopilot executed trajectory that is greater than 250 feet.

Effects: e

• Initialization effects:

The correct initialization (nominal effect): e_{0c} , with $\pi(e_{0c}) = 1$;

The wrong initializations (non-nominal effect): e_{0wi} , with $\pi(e_{0wi}) = \varepsilon$. Note that there are many possible initializations: as many as the cardinality of the cartesian product of the state variables. One of those is the correct initialization, and all the others are wrong. For computational issues, in this work, a limited number of possible wrong initializations is taken into account: the initialization may be wrong for only one variable at a time. The cardinality of the initial set of possible assessments is reduced to the number of variables. This reduced initialization set of possible assessments has shown to be rich enough to provide proper detections and explanations for the analysed dataset.

• Automated state changes effects:

nominal effects, e_f with $\pi(e_f) = 1$

e_{f1} = Pilot perception of Airspeed change to Underspeed;

e_{f2} = Pilot perception of Airspeed change to Normal;

e_{f3} = Pilot perception of Airspeed change to Near overspeed;

e_{f4} = Pilot perception of Airspeed change to Overspeed;

e_{f5} = Trajectory divergence greater than 250 feet when the AP is off (the pilot is in charge of the flight level).

non-nominal effects (lost feedbacks), e_l with $\pi(e_l) = \lambda$

e_{l1} = Airspeed takes value Underspeed (missed feedback);

e_{l2} = Airspeed takes value Normal (missed feedback);

e_{l3} = Airspeed takes value Near overspeed (missed feedback);

e_{l4} = Airspeed takes value Overspeed (missed feedback);

e_{l5} = Airspeed takes value Overspeed but just AP disconnection perceived;

e_{l6} = Airspeed takes value Underspeed but just AP disconnection perceived;

e_{l7} = Trajectory divergence greater than 250 feet when AP is on (missed feedback).

• Selection effects:

normal effects, e_n , with $\pi(e_n) = 1$

e_{n1} = AP/ATHR connection/disconnection in nominal condition;

e_{n2} = Control stick/Throttle lever activation/deactivation in nominal condition.

slips and mistakes: e_s , with $\pi(e_s) = \epsilon$

e_{s1} = AP connection during overspeed/underspeed (slip);

e_{s2} = Control stick/Throttle lever activation when AP/ATHR On (slip);

e_{s3} = Trajectory divergence consciously greater than 250 feet and increasing because of AP on (mistake). This effect is defined as a mistake by the designer by hand, so its possibility degree is not automatically generated starting from the logic table and the general assumptions. By that we mean that *the pilot may not voluntarily be aware of the increasing trajectory divergence and that the AP is on (so it is the cause of the divergence) without taking actions, passively accepting that their requests are not executed.*

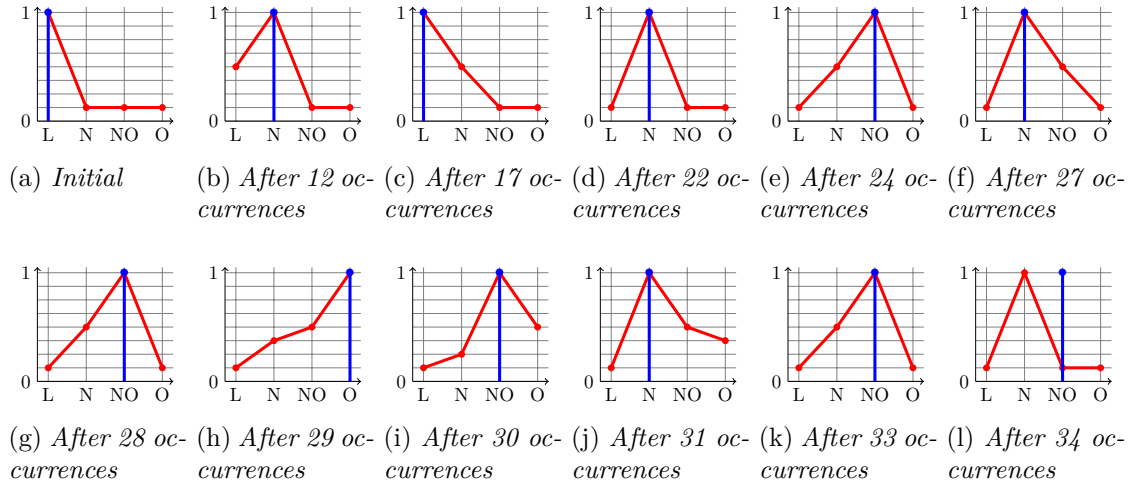


Figure IV.6 – *Experiment 1: Possibilistic estimation on human assessment of Airspeed (solid red curve), and actual machine state (blue bar). “L” is “Low speed”, “N” is “Normal speed”, “NO” is “Near Overspeed” and “O” is “Overspeed”.*

IV.5.2 Experiments

Data generated during the experience have been pre-processed to generate sequences of occurrences (among which are selections). Those occurrence sequences (one sequence for each pilot running the experiment in the flight simulator) have been then processed by our analysis model to automatically detect exceptions, *i.e.* when the objective assessment trajectory is no longer considered as normal. In those cases the exception is analysed: the exception explanation (if any is found) is specified by the model using the description of the relevant human assessment error, *i.e.* the less possible effect of the trajectory explanation, and the triggering occurrence is specified as well. That in textual form²:

- if an exception explanation is found it is labelled as an explained exception: **Exception description:** ‘Triggering occurrence’ because ‘exception explanation’;
- if no exception explanation is found it is labelled as a simple exception: **Exception description:** ‘Triggering occurrence’.

Hereafter the analysis performed for two participants of the experience is presented.

Example 1

The sequence of occurrences generated from the data recorded during the experience with the first participant is shown hereafter. A total of 57 occurrences (among which are some selections) have been generated:

‘Initialization’, ‘Control Stick Off’, ‘Throttle lever On’, ‘Throttle lever Off’, ‘Throttle lever On’, ‘Throttle lever Off’, ‘Throttle lever On’, ‘Throttle lever Off’, ‘Throttle lever On’, ‘Throttle lever Off’, ‘Control Stick On’, ‘Speed Normal’, ‘Control Stick Off’, ‘Control Stick On’, ‘ATHR button’, ‘Control Stick Off’, ‘SpeedLow’, ‘Speed Normal’, ‘Control Stick On’, ‘Control Stick Off’, ‘Control Stick On’, ‘AP button’, ‘Control Stick Off’, ‘Near overspeed’, ‘Control Stick On’, ‘Control Stick Off’, ‘Speed Normal’, ‘Near overspeed’, ‘Overspeed’, ‘Near

²The message that is automatically generated by our model could later be used to compose specific feedbacks meant to correct the human state assessment. By the way the definition of those feedbacks is out of the scope of this work. Note that the real time version of the algorithm is totally feasible: the computing time for a 15-min mission is less than one minute.

overspeed', 'Speed Normal', 'AP button', 'Near overspeed', 'Trajectory divergence', 'AP button', 'Control Stick On', 'Control Stick Off', 'Control Stick On', 'Control Stick Off', 'Control Stick On', 'Control Stick Off', 'Control Stick On', 'Control Stick Off', 'Control Stick On', 'Control Stick Off', 'Control Stick On', 'Control Stick Off', 'AP button', 'Trajectory divergence', 'AP button', 'Control Stick On', 'Speed Normal', 'Near overspeed', 'Control Stick Off'

The initial machine state is:

s_1 = AP state: Off;

s_2 = ATHR state: On;

s_3 = Airspeed: Underspeed;

s_4 = Control stick: Actioning;

s_5 = Throttle lever: Not actioning.

After the firing of the 25th occurrence, 119.5 seconds from the beginning of the experiment, the analysis model detects an exception:

Exception description: 'Control stick On when AP on!'.

After the firing of the 34th occurrence, 772.6 seconds from the beginning of the experiment, the analysis model detects an exception and explains it:

Exception description: 'Vertical speed divergence > 250 ft, unnoticed'
because 'Speed becomes >Vmax-5, but unseen!'.

After the firing of the 51st occurrence, 886.1 seconds from the beginning of the experiment, an exception is explained again:

Exception description: 'Vertical speed divergence > 250 ft, unnoticed'
because 'Speed becomes >Vmax-5, but unseen!'.

It is worth noting that the experimenters [93] reported two human automation conflicts corresponding to the second and third exceptions, and that their findings on those conflicts causes (based on the observation of the data, the video recording and the interview of the pilot) is in agree with the exception explanation provided by the analysis model here.

After the execution of the 57 occurrences, 196 assessment trajectories are considered as possible (with different possibility degrees). The computation time is 30 seconds. Some possibilistic estimations of the human assessment of the machine state variable "Airspeed" are represented in Figure IV.6: the possibility distribution over the human assessment h of the airspeed, $\pi_i(h)$, is indicated by the red curve. This possibilistic evaluation is qualitative, nevertheless in the graphic representation, quantitative values are arbitrarily assigned to the possibility degrees (respecting qualitative ordering) to plot them. The actual machine state s is stated by the blue bar with value 1 on the y-axis: if no exception arises, most possible assessment h should be the actual state (blue bar).

Remember that after the firing of 34 occurrences an exception is detected by the analysis model, which is graphically highlighted in Figure IV.6l: most possible human assessment is no more the real machine state.

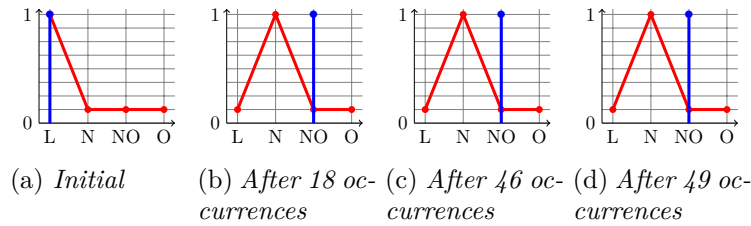


Figure IV.7 – *Experiment 2: Possibilistic estimation on human assessment of speed (solid red curve), actual machine state (blue bar).*

Example 2

A sequence of occurrences has been generated from the data recorded during experience with a second participant. A total of 85 occurrences (among which are some human selections) have been generated. Hereafter the beginning of the sequence:

‘Initialization’, ‘Throttle lever On’, ‘Throttle lever Off’, ‘ATHR button’, ‘Throttle lever On’ ...

Initial machine state variables are the same as previously, except of control stick, which is here initially actioning.

This second example was chosen because of the second occurrence, 30.2 seconds from the beginning of the experiment. The analysis model detects the following exception:

Exception description: ‘Throttle lever On when ATHR on!’ because ‘Wrong state initialization’

Initially the ATHR is on: operating the throttle lever has no result (this action is considered as a slip). The model explains this slip as the result of a wrong initial assessment: if the pilot initial assessment of the ATHR state was Off, that could explain the execution of this action as nominal. It is worth noting that after giving up this useless action (‘Throttle lever Off’) the participant deactivated the ATHR (‘ATHR button’) and she/he started again operating the throttle lever (‘Throttle lever On’), probably because she/he had a wrong initial situation assessment (as the found exception explanation) and she/he understood their assessment error.

The analysis model detected also three times the same exception as for the previous participant:

Exception description: ‘Vertical speed divergence > 250 ft, unnoticed’ because ‘Speed becomes >Vmax-5, but unseen!’

Figure IV.7 shows the estimation of the human assessment of the speed, initially, and when these exceptions occur.

IV.6 CONCLUSION AND PERSPECTIVES

This chapter proposes a model for the human-machine interaction based on a machine model and expert knowledge on an human assessment error model. The human-machine interaction is modelled as a possibilistic hidden Markov process. Qualitative Possibility Theory has been chosen because it is well suited to handle uncertainty defined by expert knowledge. The proposed possibilistic analysis model provides an estimation of the human assessment of the machine state and detects assessment errors. The analysis model is able to provide also an explanation (diagnosis) when an assessment error is detected.

This process of detection/identification could be used in real time applications in order to inform the human operator of their assessment errors. It can help to make them correct their situation awareness and prevent the execution of other errors.

This work is based on the simplifying assumption that the human operator is certain about the state of the machine: a possible extension of this model may be to drop this assumption using a more refined representation of the human state assessment, as a set of machine states, or an uncertainty measure over the machine states.

This article proposes a model for the human-machine interaction based on a machine model and expert knowledge on an human assessment error model. The human-machine interaction is modelled as a possibilistic hidden Markov process. Qualitative Possibility Theory has been chosen because it is well suited to handle uncertainty defined by expert knowledge. The proposed possibilistic analysis model provides an estimation of the human assessment of the machine state and detects assessment errors. The analysis model is able to provide also an explanation (diagnosis) when an assessment error is detected.

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A HYBRID MODEL: PLANNING IN PARTIALLY OBSERVABLE DOMAINS WITH FUZZY EPISTEMIC STATES AND PROBABILISTIC DYNAMICS

While the previous chapters dealt with purely qualitative possibilistic models, this one proposes to use the strength of both probabilistic and possibilistic approaches, in order to solve fully defined factored POMDPs, or partially defined ones. This idea comes from the analysis of the results of the experiments of Chapter III: qualitative modeling may lead to poor strategies for risky problems, or when the frequentist information defining the POMDP is at the heart of planning the problem. Here, a new translation from Partially Observable MDP into Fully Observable MDP is described. Unlike the classical translation (see Section I.1.9), the resulting problem state space is finite, making MDP solvers able to solve this simplified version of the initial partially observable problem: this approach encodes agent beliefs with fuzzy measures over states, leading to an MDP whose system state space is a finite set of epistemic states. The translation is described in a formal manner with semantic arguments. Then actual computations of this transformation are detailed, in order to highly benefits from the factorized structure of the initial POMDP in the the final MDP problem size reduction and structure. Finally size reduction and tractability of the resulting MDP is illustrated on a simple POMDP problem.

V.1 INTRODUCTION

The approach proposed here simplifies the belief space of a POMDP problem before solving it. The transformation described leads to a fully observable MDP on a finite number of epistemic states, *i.e.* a problem modeling an agent acting under uncertainty in a fully observable environment [96]. As such a finite state space MDP problem is P-complete [86] this transformation qualifies as a simplification, and any MDP solver can return a policy for this translated POMDP.

More than only a simplification of the initial POMDP problem, the theoretical framework used here for belief states representation formally models an agent's knowledge about the system state. Indeed the proposed translation defines the belief states as possibility distributions over system states $s \in \mathcal{S}$ which represents the fuzzy set of possible system states, as done with π -POMDP models.

The major originality of this work comes from the finiteness of the scale \mathcal{L} : indeed it follows that the number of possible belief states over the system state is, as well, finite (smaller than $\#(\mathcal{L}^{\mathcal{S}}) = (\#\mathcal{L})^{\#\mathcal{S}}$, see Equation I.60). What very clearly distinguishes this approach from the classical one is that the classical translation leads to an infinite set of belief states (the

continuous set of all probability distributions over \mathcal{S} , or the sequence of reachable belief states from an initial one, see Section I.1.9). The translation described here leads to an MDP whose system state space is the set of possible possibilistic belief states, or *epistemic states*, that is why its state space is finite.

In addition to POMDP simplification and knowledge modelling, this qualitative possibilistic framework offers some interesting properties: the possibilistic counterpart of the Bayes rule leads to a special belief state behaviour. Indeed the agent can possibly change their mind radically and rapidly, as described in Section II.4, experimental section of Chapter II. Moreover, under some conditions, the increased specificity of the belief state distribution is enforced, *i.e.* the knowledge about the current state is non decreasing with time steps (see Section II.4). Finally, in order to fully define the resulting MDP, the translation has to attach a reward function to its states: as a possibilistic belief state distributions constitute the new (epistemic) states of the problem, the definition of the rewards uses the Choquet integral adapted to fuzzy measures. This integral is used with the dual measure of the possibility measure defined by the belief state. The dual measure of a possibility measure, called the *necessity measure* (see Definition I.2.4), and the use of this integral makes the rewards values pessimistic about the potential lack of knowledge described by the associated belief.

However the number of possibilistic belief distributions, or *fuzzy epistemic states*, grows exponentially with the number of initial POMDP system states. The so called simplification of the problem does not transform the PSPACE POMDP problem into a polynomial one: as the new state space size is exponential in the previous one, the resulting problem is EXPTIME. The proposed translation tries to generate as few epistemic states as possible taking carefully into account potential factorized structures of the initial POMDP.

This chapter begins with the description of the first contribution of this work, which is the translation itself, presented in a formal way. As the resulting state space of the built MDP is too big to make this problem tractable without factorization tricks in practice, the next section details the proper way to preprocess its attributes. Finally, the last section illustrates the power of this approach, describing the translation in practice, and applying it on a simple factored POMDP problem.

V.2 A HYBRID POMDP

As claimed by Zadeh, “most information/intelligent systems will be of hybrid type” [127]: the idea developped here is to use a granulated representation of the agent knowledge using possibilistic beliefs instead of probabilistic beliefs in the POMDP framework. The first advantage of this granulation is that strategy computations is performed reasoning on a finite set of possibilistic beliefs called then epistemic states: the set of all possibility distributions defined over \mathcal{S} , denoted by $\Pi_{\mathcal{L}}^{\mathcal{S}}$ is $\#\Pi_{\mathcal{L}}^{\mathcal{S}} = \#\mathcal{L}^{\#\mathcal{S}} - (\#\mathcal{L} - 1)^{\#\mathcal{S}}$, due to the possibilistic normalization (see Equation I.60), while the set of probability distributions over \mathcal{S} is infinite. The π -MDPs studied in the first chapters of this thesis are quite different from the model exposed in this paper. For instance, Qualitative Possibilistic MDPs do not use quantitative data as probabilities or rewards. Dynamics is described in a purely qualitative possibilistic way. Frequentist information about the problem cannot be encoded: these frameworks are indeed dedicated to situations when the probabilistic dynamic of the studied system is lacking. Moreover, possible values of the reward function are chosen among the degrees of the qualitative possibilistic scale. A commensurability assumption between reward and possibility degrees, *i.e.* a meaning of why they share the same scale, is needed to use the criteria proposed in these frameworks. Our model bypass these demands: a real number is assigned to each possibilistic belief (epistemic state), instead of a qualitative utility degree: it represents the reward got by the agent when reaching this belief (in a MDP fashion) as detailed in Section V.2.2. Moreover, the dynamics

of our process is described with probability distributions: approximate probabilistic transition functions between current and next beliefs, or epistemic states, are given section V.2.1. Finally, our model can be solved by any MDP solver in practice: it becomes eventually a classical probabilistic fully observable MDP whose state space is the finite set $\Pi_{\mathcal{L}}^{\mathcal{S}}$.

Here, the term hybrid is used because the beliefs only are defined as possibility distributions, and all variables keep a probabilistic dynamic: the agent reasons based on a possibilistic analysis of the system state (the possibilistic belief, or epistemic state), and transition probability distributions are defined for its epistemic states. Such beliefs are formally defined in Section I.2.5. As the set of the possibilistic beliefs is finite, they define the finite state space of an MDP, whose the probabilistic transitions are defined in the next section. At this step, a Markov process based on epistemic states is thus defined. Finally rewards are defined on epistemic states using the discrete Choquet integral and leading to the definition of the resulting MDP.

V.2.1 Set transitions

First, we use here some notations: the transition probability distribution is denoted by $T(s, a, s') = \mathbf{p}(s' | s, a)$, and the observation probability distribution by $O(s', a, o') = \mathbf{p}(o' | s', a)$. If the agent selects action $a \in \mathcal{A}$ in the epistemic state $\beta \in \Pi_{\mathcal{L}}^{\mathcal{S}}$, the next epistemic state depends only on the next observation, as highlighted by possibilistic belief update (see Theorem 14). The probability distribution over observations conditionned on the reached state is part of the POMDP definition via the observation function O . The probability distribution over observations conditionned on previous state is obtained using transition function T :

$$\mathbf{p}(o' | s, a) = \sum_{s' \in \mathcal{S}} O(s', a, o') \cdot T(s, a, s').$$

This distribution and the possibilistic belief β about the system state, can lead to an approximate probability distribution over next observations. Indeed, a probability distribution over the system state, $\bar{\beta} \in \mathbb{P}_{\mathcal{S}}$, can be derived from β using extension of Laplace principle. Then approximate distribution over $o' \in \mathcal{O}$ is defined as

$$\mathbf{p}(o' | \beta, a) = \sum_{s \in \mathcal{S}} \mathbf{p}(o' | s, a) \cdot \bar{\beta}(s). \quad (\text{V.1})$$

Finally, summing over concerned observations, the transition probability distribution over epistemic states is defined as

$$\mathbf{p}(\tilde{\beta} | \beta, a) = \sum_{o' | u(\beta, a, o') = \tilde{\beta}} \mathbf{p}(o' | \beta, a). \quad (\text{V.2})$$

A way to construct a probability distribution $\bar{\beta}$ from a possibility one β is the use of the pignistic transformation [44] minimizing the arbitrariness in the translation into probability distribution: numbering system states with the order induced by distribution β , $1 = \beta(s_1) \geq \beta(s_2) \geq \dots \geq \beta(s_{\#\mathcal{S}+1}) = 0$, with $s_{\#\mathcal{S}+1}$ an artificial state such that $\pi(s_{\#\mathcal{S}+1}) = 0$ introduced to simplify the formula,

$$\bar{\beta}(s_i) = \sum_{j=i}^{\#\mathcal{S}} \frac{\beta(s_j) - \beta(s_{j+1})}{j} \quad (\text{V.3})$$

Note that this probability distribution corresponds to the center of gravity of the probability distributions family induced by the possibility measure defined by distribution β [53], and respects the Laplace principle of Insufficient Reason (ignorance leads to uniform probability).

Although possibilistic beliefs were so far defined in a qualitative way, degrees of \mathcal{L} are considered as numerical in this section and the following: the section about factorization will make it clear that possibility distributions can be computed from T and O if the sole purpose is to simplify the POMDP. Numerical values are then used to compute the observation probability distribution here, and in order to aggregate rewards according to the current epistemic state in the next section.

V.2.2 Reward aggregation

After the transition function, it remains to assign a reward to each epistemic state: in the classical probabilistic translation, the reward assigned to a belief b is the reward expectation according to the probability distribution b : $\sum_{s \in \mathcal{S}} r(s, a) \cdot b(s)$. Here, the agent knowledge is represented with a possibility distribution β : it sums up the frequentist uncertainty of the problem, and imprecision due to the possibilistic discretization and/or due to partial ignorance about actual probability distributions defining the situation. A way to define a reward being pessimistic about these imprecisions is to aggregate the reward using the dual measure of the possibility distribution, and the *Choquet integral*.

The dual measure of a possibility measure $\Pi : 2^{\mathcal{S}} \rightarrow \mathcal{L}$ is called *necessity measure* and is denoted by N . This measure is defined by $\forall A \subseteq \mathcal{S}, N(A) = 1 - \Pi(\bar{A})$ where \bar{A} is the complementary set of A : $\bar{A} = \mathcal{S} \setminus A$. Recall notation $\mathcal{L} = \{l_1 = 1, l_2, l_3, \dots, 0\}$. For a given action $a \in \mathcal{A}$, reward values, $\{r(s, a) \mid s \in \mathcal{S}\}$ are denoted by $\{r_1, r_2, \dots, r_k\}$ with $r_1 > r_2 > \dots > r_k$, with $k \leq \#\mathcal{S}$. An artificial value $r_{k+1} = 0$ is also introduced to simplify the formulae.

Discrete Choquet integral of the reward function against necessity measure N [1] is defined as follows:

$$\begin{aligned}
 Ch(r, N) &= \sum_{i=1}^k (r_i - r_{i+1}) \cdot N(\{r(s, a) \geq r_i\}) & (V.4) \\
 &= \sum_{i=1}^k (r_i - r_{i+1}) \cdot [1 - \Pi(\{r(s, a) < r_i\})] \\
 &= \sum_{i=1}^k (r_i - r_{i+1}) \cdot \left(1 - \max_{s \mid r(s, a) < r_i} \pi(s)\right) \\
 &= r_1 - r_{k+1} - \sum_{i=1}^k (r_i - r_{i+1}) \cdot \max_{s \mid r(s, a) < r_i} \pi(s) \\
 &= r_1 - (r_1 - r_2) \cdot \max_{s \mid r(s, a) < r_1} \pi(s) - \dots \\
 &\quad - (r_{k-1} - r_k) \cdot \max_{s \mid r(s, a) < r_{k-1}} \pi(s) - r_k \cdot \max_{s \mid r(s, a) < r_k} \pi(s) \\
 &= \sum_{i=1}^{\#\mathcal{L}-1} (l_i - l_{i+1}) \cdot \min_{s \mid \pi(s) \geq l_i} r(s) & (V.5)
 \end{aligned}$$

More on possibilistic Choquet integrals can be found in [31, 51].

This reward aggregation using the necessity measure leads to a pessimistic estimation of the reward: as an example, the reward $\min_{s \in \mathcal{S}} r(s, a)$ is assigned to the total ignorance.

Note that, if the necessity measure N is replaced by a probability measure \mathbb{P} , *e.g.* as the one induced by probability distribution β using V.3, Choquet integral coincides with the expected reward based on $\bar{\beta}$. This could be a good aggregation choice as well, but more optimistic than the one described above. The most optimistic way to aggregate the reward is to compute the Choquet integral with the possibilistic measure Π induced by distribution β , rather than with necessity one N , but this is not detailed here.

V.2.3 MDP with epistemic states

This section summarizes the complete translation using final equations of the previous sections. This translation takes for input a POMDP: $\langle \mathcal{S}, \mathcal{A}, T, \mathcal{O}, O, r \rangle$ and returns an epistemic states based MDP: $\langle \tilde{\mathcal{S}}, \mathcal{A}, \tilde{T}, \tilde{r} \rangle$ with

- $\tilde{\mathcal{S}} = \Pi_{\mathcal{L}}^{\mathcal{S}}$;
- \tilde{T} , such that $\forall (\beta, \tilde{\beta}) \in (\Pi_{\mathcal{L}}^{\mathcal{S}})^2, \forall a \in \mathcal{A}$
 $\tilde{T}(\beta, a, \tilde{\beta}) = \mathbf{p}(\tilde{\beta} \mid \beta, a)$ using V.1 and V.2;
- $\tilde{r}(a, \beta) = Ch(r(a, \cdot), N_{\beta})$, using Equation V.5 and where N_{β} is the necessity measure computed from β .

Finally, as in the probabilistic framework (see Section I.1.4), the criterion of this MDP is the expected total reward:

$$\mathbb{E}_{(\beta_t) \sim \tilde{T}} \left[\sum_{t=0}^{+\infty} \gamma^t \tilde{r}(\beta_t, d_t) \right].$$

While the resulting state space is finite, only really small POMDP problems can be solved with this translation without computation tricks. Indeed, $\Pi_{\mathcal{L}}^{\mathcal{S}}$ grows exponentially with the number of system states (see Equation I.60), which makes the problem intractable even for state of the art MDP solvers.

V.3 BENEFIT FROM FACTORIZATION

This section carefully derives a tractable MDP problem from a factored POMDP: the resulting MDP is equivalent to the former translation, but some factorization and computational tricks are described here to reduce its size and to fit to the factorized structure. First, the definition of a factored POMDP is quickly exposed, followed by some dependency notations helpful for describing how distributions are dealt with. Next, a classification of the state variables is made to strongly adapt computations according to the nature of the state. Then follows the definition of possibility distributions, and the description of the use of the possibilistic Bayes rule in practice ends this section.

V.3.1 Factored POMDP

Partially Observable Markov Decision Processes can be defined in a factorized way.

- state space $\mathcal{S} = s_1 \times \dots \times s_m$ with $\forall j \in \{1, \dots, m\}, s_j$ boolean variable. The set of boolean state variables is denoted by $\mathbb{S} = \{s_1, \dots, s_m\}$;
- observation space $\mathcal{O} = o_1 \times \dots \times o_n$ with $\forall i \in \{1, \dots, n\}, o_i$ boolean variable. In the same way as to state variables, the set of observation variables is denoted by $\mathcal{O} = \{o_1, \dots, o_n\}$;
- action space \mathcal{A} , a finite set of actions $a \in \mathcal{A}$.

Note that a problem with non boolean variables can be easily reduced to such a problem with the boolean variables assumption. For simplicity, and as state $s_j \in \mathcal{S}$ and observation $o_i \in \mathcal{O}$ notations are no longer reused in this paper, only variables are denoted with these letters from now: $s \in \mathbb{S}$ and $o \in \mathcal{O}$.

Non-primed variables correspond to the current time step, and primed variables to the next time-step. This notation is also used for sets of variables: \mathbb{S}' is the set of next state variables and \mathcal{O}' the set of next observable ones. The factorized description continues with following probability distributions:

- $\forall j \in \{1, \dots, m\}, \forall a \in \mathcal{A}$, a transition function is defined:

$$T_j^a(s_1, \dots, s_m, s'_j) = \mathbf{p}(s'_j \mid s_1, \dots, s_m, a);$$

- One observation function is also given for each observation variable: $\forall i \in \{1, \dots, n\}, \forall a \in \mathcal{A}$,

$$O_i^a(s'_1, \dots, s'_m, o'_i) = \mathbf{p}(o'_i \mid s'_1, \dots, s'_m, a);$$

- and reward function $r : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$.

These definitions lead to the following observations: $\{s'_j\}_{j \in \{1, \dots, m\}}$ are post-action independent, and $\{o'_i\}_{i \in \{1, \dots, n\}}$ post-transition independent.

V.3.2 Notations and Observation Functions

Transitions of the final MDP make it more handy if each variable depends on only few previous variables: the procedure to avoid blocking such simplifications brought by the structure of the initial POMDP during the translation, needs the following notations. In practice, for each $i \in \{1, \dots, n\}$ not all state variables influence observation variable o'_i ; similarly, for each $j \in \{1, \dots, m\}$, not all current state variables influence next state variable s'_j :

- for each action $a \in \mathcal{A}$, observation variable o'_i depends on some state variables which are denoted by

$$\mathcal{P}_a(o'_i) = \{s'_j \in \mathbb{S}' \text{ s.t. } o'_i \text{ depends on } s'_j \text{ when } a \text{ applied}\}$$

They are called *parents* as they appears as “parents nodes” in a dynamic Bayesian network [35] illustrating dependencies of the process.

- as well, for each action $a \in \mathcal{A}$, probability distribution of next state variable s'_j depends on some current ones, denoted by

$$\mathcal{P}_a(s'_j) = \{s_k \in \mathbb{S} \text{ s.t. } s'_j \text{ depends on } s_k \text{ when } a \text{ applied}\}$$

and illustrated in Figure V.1.

Now, let us define parents whatever the chosen action: $\forall i = 1, \dots, n$,

$$\mathcal{P}(o'_i) = \cup_{a \in \mathcal{A}} \mathcal{P}_a(o'_i) \subseteq \mathbb{S}'$$

and $\forall j = 1, \dots, m$,

$$\mathcal{P}(s'_j) = \cup_{a \in \mathcal{A}} \mathcal{P}_a(s'_j) \subseteq \mathbb{S}$$

It leads to the following rewriting of probability distributions:

$$T_j^a(\mathcal{P}(s'_j), s'_j) = \mathbf{p}(s'_j \mid \mathcal{P}(s'_j), a)$$

and

$$O_i^a(\mathcal{P}(o'_j), o'_i) = \mathbf{p}(o'_i \mid \mathcal{P}(o'_j), a).$$

Following subset of \mathbb{S} is useful to specify observation dynamic:

$$\mathcal{Q}(o'_i) = \{s_k \in \mathbb{S} \text{ s.t. } \exists s'_j \in \mathcal{P}(o'_i) \text{ s.t. } s_k \in \mathcal{P}(s'_j)\} = \cup_{s'_j \in \mathcal{P}(o'_i)} \mathcal{P}(s'_j) \subseteq \mathbb{S}$$

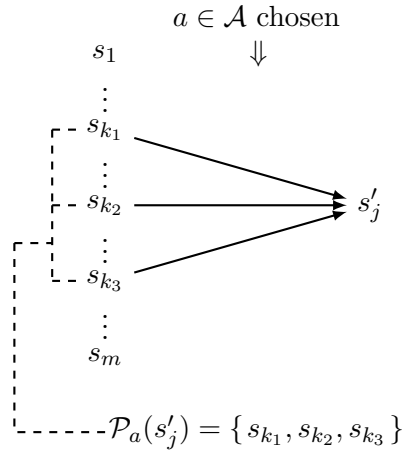


Figure V.1 – For action $a \in \mathcal{A}$, only tree variables influence variable s'_j in this Bayesian network: s_{k_1} , s_{k_2} and s_{k_3} , which constitute $\mathcal{P}_a(s'_j)$.

and is illustrated in Figure V.2.

In order to simplify notations, and as it causes no confusion, \mathbb{S} , $\mathcal{P}(o'_i)$, $\mathcal{P}(s'_i)$ and $\mathcal{Q}(o'_i)$ designate as well a set of variables, or a vector comprised of these variables (with an arbitrary order). Distribution over $\mathcal{P}(o'_i)$ assignments benefits from previous rewritings:

$$\mathbf{p}(\mathcal{P}(o'_i) \mid \mathbb{S}, a) = \prod_{s'_j \in \mathcal{P}(o'_i)} T_j^a(\mathcal{P}(s'_j), s'_j) = \prod_{s'_j \in \mathcal{P}(o'_i)} \mathbf{p}(s'_j \mid \mathcal{P}(s'_j), a) = \mathbf{p}(\mathcal{P}(o'_i) \mid \mathcal{Q}(o'_i), a) \quad (\text{V.6})$$

Observation probability distributions, knowing previous state variables, are then defined $\forall i = 1, \dots, n$

$$\mathbf{p}(o'_i \mid \mathcal{Q}(o'_i), a) = \sum_{v \in 2^{\mathcal{P}(o'_i)}} \mathbf{p}(o'_i \mid v, a) \cdot \mathbf{p}(v \mid \mathcal{Q}(o'_i), a) \quad (\text{V.7})$$

Therefore a possibilistic belief defined on $2^{\mathcal{Q}(o'_i)}$ is enough to get the approximate probability distribution of an observation variable, Equation V.1: such an epistemic state leads via transformation V.3 to a probability distribution $\bar{\beta}$ over $2^{\mathcal{Q}(o'_i)}$. Finally, the approximate probability distribution of the observation variable i , factored counterpart of former equation V.1, is:

$$\mathbf{p}(o'_i \mid \beta, a) = \sum_{v \in 2^{\mathcal{Q}(o'_i)}} \mathbf{p}(o'_i \mid v, a) \cdot \bar{\beta}(v). \quad (\text{V.8})$$

V.3.3 State variables classification

State variables $s \in \mathbb{S}$ do not play the same role in the process: as already studied in the literature [83], some variables can be visible for the agent, and namely this *mixed-observability* leads to important computational simplifications. Moreover, some variables do not affect observation variables, and factorization of the POMDP is then easily transmitted to the epistemic state based MDP. Finally, using rewritings of previous sections, useless computations are highlighted.

- A state variable s_j is said to be **visible**, if $\exists o'_i \in \mathbb{O}$, observation variable, such that $\mathcal{P}(o'_i) = \{s'_j\}$ and $\forall a \in \mathcal{A}$, $\mathbf{p}(o'_i \mid s'_j, a) = \mathbb{1}_{\{o'_i = s'_j\}}$ i.e. if $o'_i = s'_j$ almost surely. The set of visible state variables is denoted by $\mathbb{S}_v = \{s_{v,1}, s_{v,2}, \dots, s_{v,m_v}\}$;

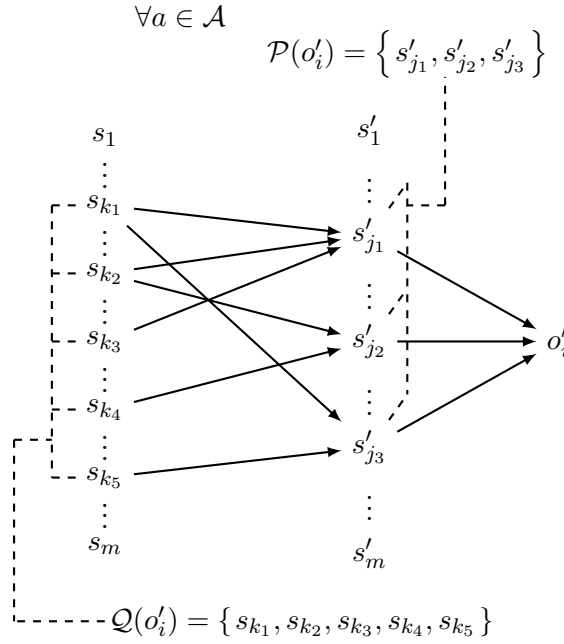


Figure V.2 – Whatever the action $a \in \mathcal{A}$, only five state variables influence variable o'_i in this Bayesian network: $s_{k_1}, s_{k_2}, s_{k_3}, s_{k_4}, s_{k_5}$ which constitute $Q(o'_i)$.

Observation variables corresponding to visible state variables can be removed from the set of observation variables: the number of observation variables becomes \tilde{n} , and remaining observation variables are denoted by $o_1, \dots, o_{\tilde{n}}$.

- **Inferred hidden variables** are simply $\cup_{i=1}^{\tilde{n}} \mathcal{P}(o'_i)$, *i.e.* all hidden variables influencing (remaining) observation variables. The set of inferred hidden variables is $\mathbb{S}_h = \{s_{h,1}, s_{h,2}, \dots, s_{h,m_h}\}$ and contains possibly visible variables.
- **Non-inferred hidden variables** or **fully hidden variables**, denoted by \mathbb{S}_f , consists of hidden state variables which do not influence any observation, *i.e.* all remaining state variables. The fully hidden variables are denoted by $s_{f,1}, s_{f,2}, \dots, s_{f,m_f}$, and the corresponding set is \mathbb{S}_f .

Of course, this classification leads to a partition of the initial set of state variables if potential visible variables are removed from inferred hidden variables: denoting purely inferred hidden variables by $\bar{\mathbb{S}}_h = \mathbb{S}_h \setminus \mathbb{S}_v$, and $\bar{m}_h = \#\bar{\mathbb{S}}_h$ the state variables partition is $\mathbb{S} = \mathbb{S}_v \sqcup \bar{\mathbb{S}}_h \sqcup \mathbb{S}_f$ and $m = m_v + \bar{m}_h + m_f$.

The classification defined here is used to avoid some computations for visible variables: if $s_v \in \mathbb{S}_v$ is visible, and $o_v \in \mathbb{O}$ is the associated observation ($s_v = o_v$ almost surely), computations of the distribution over $\mathcal{P}(o'_v)$, Equation V.6, and of the distribution over o'_v , Equation V.7, are unnecessary: the distribution over $s'_v (= o'_v)$ needed is simply given by $\mathbf{p}(s'_v | \mathcal{P}(s'_v), a)$, data of the original problem. The counterpart of Equation V.8 is then

$$\mathbf{p}(s'_v | \beta, a) = \sum_{2^{\mathcal{P}(s'_v)}} \mathbf{p}(s'_v | \mathcal{P}(s'_v), a) \cdot \bar{\beta}(\mathcal{P}(s'_v)), \quad (\text{V.9})$$

where $\bar{\beta}$ is the probability distribution over $2^{\mathcal{P}(s'_v)}$ extracted from the possibilistic belief over the same space, using transformation (V.3).

V.3.4 Joint possibility distributions

Possibility distributions of the problem are here nothing more than probability distributions previously defined, normalized in a possibilistic sense: if $v \in \{\top, \perp\}$ maximizes $\mathbf{p}(v)$, $\pi(v) = 1$, otherwise $\pi(v) = \mathbf{p}(v)$.

Joint possibility distributions over $\mathcal{O}'_i \times \mathcal{P}(\mathcal{O}'_i)$, needed for belief process definition (possibilistic belief update, Theorem 14), are computed in the following way:

$$\begin{aligned} \pi(\mathcal{O}'_i, \mathcal{P}(\mathcal{O}'_i) \mid \mathcal{Q}(\mathcal{O}'_i), a) &= \min \{ \pi(\mathcal{O}'_i \mid \mathcal{P}(\mathcal{O}'_i), a), \pi(\mathcal{P}(\mathcal{O}'_i) \mid \mathcal{Q}(\mathcal{O}'_i), a) \} \\ &= \min \left\{ \pi(\mathcal{O}'_i \mid \mathcal{P}(\mathcal{O}'_i), a), \min_{s'_j \in \mathcal{P}(\mathcal{O}'_i)} \pi(s'_j \mid \mathcal{P}(s'_j), a) \right\}. \end{aligned}$$

V.3.5 Beliefs process definition and handling

This section is meant to define marginal belief distributions instead of a global one, in order to benefit from the factorized structure of the initial POMDP. Indeed, possibilistic belief distributions have different definitions according to which class of state variables they concern:

- as visible state variables are directly observed, there is no uncertainty over these variables. Two epistemic states (possibilistic belief distribution) are possible for visible state variable $s'_{v,j}$: $b'_{v,T}(s'_{v,j}) = \mathbb{1}_{\{s'_{v,j}=\top\}}$ and $b'_{v,F}(s'_{v,j}) = \mathbb{1}_{\{s'_{v,j}=\perp\}}$. As a consequence, one boolean variable $\beta'_{v,j} \in \{\top, \perp\}$ per visible state variables is enough to represent this belief distribution in practice: if $s'_{v,j} = \top$, then next belief is $b' = b'_{v,T}$ represented by belief variable assignment $\beta'_{v,j} = \top$, otherwise, next belief is $b' = b'_{v,F}$, and $\beta'_{v,j} = \perp$. A belief variable of a visible state variable is denoted by β_v .
- for each $i \in 1, \dots, \tilde{n}$, each inferred hidden variable constituting $\mathcal{P}(\mathcal{O}'_i)$ is an input of the same possibilistic belief distribution: non-normalized belief is, $\forall i = 1, \dots, \tilde{n}$

$$\tilde{b}'(\mathcal{P}(\mathcal{O}'_i)) = \max_{v \in 2^{\mathcal{Q}(\mathcal{O}'_i)}} \min \{ \pi(\mathcal{O}'_i, \mathcal{P}(\mathcal{O}'_i) \mid v, a), b(v) \}. \quad (\text{V.10})$$

A possibilistic normalization finalizes the belief update: for $w \in 2^{\mathcal{P}(\mathcal{O}'_i)}$,

$$b'(w') = \begin{cases} 1 & \text{if } w' \in \operatorname{argmax}_{v' \in 2^{\mathcal{P}(\mathcal{O}'_i)}} \tilde{b}'(v'); \\ \tilde{b}'(w') & \text{otherwise.} \end{cases} \quad (\text{V.11})$$

In practice, if $l = \#\mathcal{L}$ is the size of the possibility scale, and $p_i = \#\mathcal{P}(\mathcal{O}'_i)$, the number of belief states is $l^{2^{p_i}} - (l-1)^{2^{p_i}}$, and then the number of belief variables is $n_{h,i} = \lceil \log_2(l^{2^{p_i}} - (l-1)^{2^{p_i}}) \rceil$. A belief variable of an inferred hidden state variable is denoted by β_h .

- for each $j \in 1, \dots, m_f$, non-normalized belief defined on fully hidden variable $s'_{f,j}$ is

$$\tilde{b}'(s'_{f,j}) = \max_{v \in 2^{\mathcal{P}(s'_{f,j})}} \min \{ \pi(s'_{f,j} \mid v, a), b(v) \}, \quad (\text{V.12})$$

which leads to the actual new belief b' after normalization (V.11). In practice, as each fully hidden variable is considered independently from the others, following the previous reasoning for vector of inferred hidden s.v., the number of belief variables is $n_f = \lceil \log_2(l^2 - (l-1)^2) \rceil = \lceil \log_2(2l-1) \rceil$. A belief variable of a fully hidden state variable is denoted by β_f .

Finally the actual global epistemic state $b'(\mathbb{S})$ is upper bounded by

$$b'(\mathbb{S}) = \min \left\{ \min_{j=1}^{m_v} b'(s'_{v,j}), \min_{i=1}^{\tilde{n}} b'(\mathcal{P}(\mathcal{O}'_i)), \min_{k=1}^{m_f} b'(s'_{f,k}) \right\}, \quad (\text{V.13})$$

where \mathbb{S} has to be seen as a vector composed of all state variables.

The latter is considered as the agent belief to make the final MDP factorized.

V.3.6 Select and use belief variables

Starting with initial beliefs defined for each visible state variable $s_{v,j}$, for each vector of inferred hidden variables $\mathcal{P}(o'_i)$ and for each fully hidden variable $s_{f,j}$, these beliefs are updated at each time step according to the transformations described above.

However previous formulae V.10 and distribution over observation variable $o'_i \in \mathbb{O}$, Equation V.8, depend on belief distribution over $\mathcal{Q}(o'_i) \subseteq \mathbb{S}$. They can be computed from the available beliefs as follow: $\forall i = 1, \dots, \tilde{n}$,

$$b(\mathcal{Q}(o'_i)) = \max_{v \in 2^{\mathcal{K}_i}} \min \left\{ \min_{s_v \in \mathcal{Q}(o'_i) \cap \mathbb{S}_v} b(s_v), \min_{j \in \mathcal{J}_i} b(\mathcal{P}(o_j)), \min_{s_f \in \mathcal{Q}(o'_i) \cap \mathbb{S}_f} b(s_f) \right\}, \quad (\text{V.14})$$

where

- $\mathcal{J}_i = \{j \in \{1, \dots, \tilde{n}\} \text{ s.t. } \mathcal{P}(o_j) \cap \mathcal{Q}(o'_i) \neq \emptyset\}$, *i.e.* \mathcal{J}_i is the set of indices j for which $\mathcal{P}(o_j)$ shares (inferred hidden) state variables with $\mathcal{Q}(o'_i)$, and
- $\mathcal{K}_i = \{\cup_{j \in \mathcal{J}_i} \mathcal{P}(o_j)\} \setminus \mathcal{Q}(o'_i) \subseteq \mathbb{S}_h$, *i.e.* \mathcal{K}_i is the set of (inferred hidden) state variables which are not present in $\mathcal{Q}(o'_i)$, but are present in a set $\mathcal{P}(o_j)$ sharing state variables with $\mathcal{Q}(o'_i)$.

As well, belief update for fully hidden state variables, Equation V.12, needs a belief distribution over variables $\mathcal{P}(s'_{f,j})$: $\forall j = 1, \dots, m_f$,

$$b(\mathcal{P}(s'_{f,j})) = \max_{v \in 2^{\mathcal{N}_j}} \min \left\{ \min_{s_v \in \mathcal{P}(s'_{f,j}) \cap \mathbb{S}_v} b(s_v), \min_{k \in \mathcal{M}_j} b(\mathcal{P}(o_k)), \min_{s_f \in \mathcal{P}(s'_{f,j}) \cap \mathbb{S}_f} b(s_f) \right\},$$

where

- $\mathcal{M}_j = \{k \in \{1, \dots, m_f\} \text{ s.t. } \mathcal{P}(o_k) \cap \mathcal{P}(s'_{f,j}) \neq \emptyset\}$, *i.e.* \mathcal{M}_j is the set of indices k for which $\mathcal{P}(o_k)$ shares (inferred hidden) state variables with $\mathcal{P}(s'_{f,j})$, and
- $\mathcal{N}_j = \{\cup_{k \in \mathcal{M}_j} \mathcal{P}(o_k)\} \setminus \mathcal{P}(s'_{f,j}) \subseteq \mathbb{S}_h$, *i.e.* \mathcal{N}_j is the set of (inferred hidden) state variables which are not present in $\mathcal{P}(s'_{f,j})$, but are present in a set $\mathcal{P}(o_k)$ sharing state variables with $\mathcal{P}(s'_{f,j})$.

Finally, a belief distribution over $\mathcal{P}(s'_{v,i})$ needed to define an approximate probability distribution over visible state variables (Equation V.9), can be defined in the same way, marginalizing (max) over unused variables.

V.4 SOLVING A POMDP WITH A DISCRETE MDP SOLVER

The previous section leads to a factored MDP, whose the version used in practice is defined here. A concrete POMDP problem and its resulting MDP are then described in order to highlight the power in state space size reduction of the possibilistic structured translation.

V.4.1 Resulting factored MDP:

Section V.3.3 classifies state variables in order to define epistemic states b over sets of state variables (respectively $\forall j = 1, \dots, m_v$, $\{s_j^v\}$, $\forall i = 1, \dots, \tilde{n}$, $\mathcal{P}(o'_i)$, and $\forall k = 1, \dots, m_f$, $\{s_j^f\}$) and set of variables encoding them (respectively β_v , β_h and β_f) independently to each other. As belief updates are deterministic knowing the observation, a simple trick is used to keep this determinism in the final MDP: a *flipflop* boolean variable is introduced, changing its state

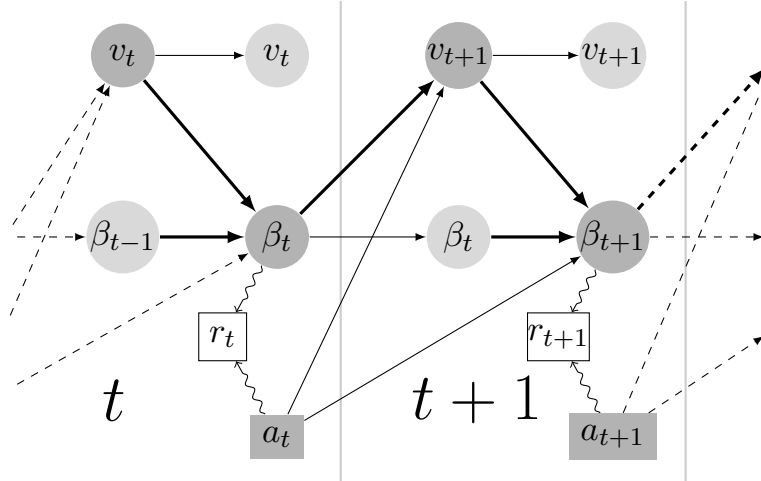


Figure V.3 – Practical DBN of the resulting MDP: thickest arrows illustrate transitions which are not identity transitions.

at each step, denoted by f . It artificially divides a classical time step of the POMDP into two phases. During the first phase, called *the observation generation phase*, non-identity transition functions (*i.e.* which do not let the variable remain the same) are the probability distributions over observation variables V.8 and visible state variables V.9.

During the second phase, called *the belief update phase*, non-identity transition functions are the deterministic transitions of the belief variables:

- variable β_v is updated knowing value of the corresponding visible variable s_v ;
- variables $\beta_h^1, \dots, \beta_h^{n_{h,i}}$ are updated knowing value of observation variable o_i , and using update V.10, V.11;
- variables $\beta_f^1, \dots, \beta_f^{n_f}$ using update V.12.

The state space is then defined as:

$\mathcal{S} = f \times s_v^1 \times \dots \times s_v^{m_v} \times o^1 \times \dots \times o^{\tilde{n}} \times \beta_v^1 \times \dots \times \beta_v^{m_v} \times \beta_h^1 \times \dots \times \beta_h^{\tilde{n}} \times \beta_f^1 \times \dots \times \beta_f^{m_f}$, where $\forall i = 1, \dots, \tilde{n}$, β_h^i represents boolean variables $\beta_h^{1,i}, \dots, \beta_h^{n_{h,i},i}$, and $\forall k = 1, \dots, m_f$, β_f^k represents boolean variables $\beta_f^{1,k}, \dots, \beta_f^{n_{f,k},k}$.

The resulting MDP is illustrated in Figure V.3 where β_t represents all belief variables, and v_t the visible variables: flipflop variable f , observations o_i and visible state variables s_v .

This trick makes the belief update phase deterministic. Each belief variable transition can be then deterministically defined, and independently from each other: as visible state and observation variables are already post-action independent, the resulting MDP is a factored MDP.

V.4.2 Results for a concrete POMDP problem

A problem inspired by the RockSample problem [114] is described in this section to illustrate the factorized possibilistic discretization of the agent belief, from a factored POMDP: a rover is navigating in a place described by a finite number of locations l_1, \dots, l_n , and where stand m rocks. Some of these m rocks have an interest in the scientific mission of the rover, and it has to sample them. However, sampling a rock is an expensive operation. The rover is thus fitted with a long range sensor making him able to estimate if the rock has to be sampled. Finally operating time of the rover is limited, but its battery level is available.

Variables of this problem can now be set, and classified as in Section V.3.3: as the battery level is directly observable by the agent (the rover), the set of visible state variables consists of the boolean variables encoding it: $\mathbb{S}_v = \{B_1, B_2, \dots, B_k\}$. The agent knows the different locations of the rocks, however the nature of a rock is estimated. The set of inferred hidden state variables consists of m boolean variables R_i encoding the nature of the i^{th} rock, \top for “scientifically good” and \perp otherwise: $\mathbb{S}_h = \{R_1, R_2, \dots, R_m\}$. When the i^{th} rock is observed using the sensor, it returns a noisy observation of the rock in $\{\top, \perp\}$, modeled by the boolean variable O_i : the set of observation variables is then $\mathbb{O} = \{O_1, O_2, \dots, O_m\}$. Finally, no localization equipment is provided: the agent estimates its location from its initial information, and its dynamics. Each location of the rover is formally described by a variable L_j , which equals \top if the rover is at the j^{th} location, and \perp otherwise. The set of fully hidden variables consists thus of these n variables: $\mathbb{S}_f = \{L_1, L_2, \dots, L_n\}$.

Initial location is known, described by variable L_1 , and leading to a deterministic initial belief: $\beta_0(\mathcal{S}_h) = 1$ if $L_1 = \top$ and $L_j = \perp \forall j \neq 1$, 0 otherwise. However initial nature of each rock is not known. Instead of a uniform probability distribution over the rocks nature (“rock has to be sampled”, or “rock is not interesting”), the Possibility Theory allows to represent this initial ignorance with the marginal belief $\beta_0(\mathcal{S}_h) = 1$, for each assignment of the hidden inferred state variables modelling nature of the rocks.

Finally, the factorization trick leads to a reduction of the domain size: with a flat translation of this POMDP, the size of the resulting state space is described with $\lceil \log_2(\#\mathcal{L}^{2^{n+m+k}} - (\#\mathcal{L} - 1)^{2^{n+m+k}}) \rceil$ boolean variables. Taking advantage of the POMDP structure, the resulting state space is encoded with $1 + 2 \cdot \dots \cdot k + m + (m + n) \cdot \lceil \log_2(2\#\mathcal{L} - 1) \rceil$ Boolean variables: the flipflop variable, the visible variables and associated beliefs variables, the observation variables, and the belief variables associated to the fully hidden and inferred hidden variables.

Moreover, the dynamic of the resulting MDP is factorized: all variables are independent post-action, and lots of them are deterministic, thank to the flipflop variable trick. These structures are beneficial to the MDP solvers, leading to faster computations.

V.5 CONCLUSION

This chapter described a hybrid translation of a POMDP problem into a finite state space MDP one: the Qualitative Possibility Theory, is used here to maintain an epistemic state during the process. The MDP problem, result of this translation, is entirely built defining transition and reward functions over these epistemic states. Definitions of these functions use respectively the pignistic transformation, used to recover a probability distribution from an epistemic state, and the Choquet integral with respect to the necessity, making the agent pessimistic about the potential ignorance described by its epistemic state. A practical way to implement this translation is then described: with these computations, a factored POMDP leads to a factored and tractable MDP problem. The essential particularity of this translation is the granular modeling of the agent belief using a qualitative fuzzy knowledge representation. Finally this promising approach will be tested on RDDDL files of the IPPC competition [106] using a state of the art MDP planner like PROST [65]. Indeed these files describe factored POMDP problems as introduced in Section V.3.

ANNEX

A PROOFS OF CHAPTER I

A.1 Preliminaries

Firstly recall the more general definition of the *conditional expectation* with respect to a random variable: a random variable is a measurable function defined on the set Ω equipped with the σ -algebra \mathcal{F} and the probability measure \mathbb{P} .

Definition A.1 (*Expectation of X Conditional on Y : $\mathbb{E}[X | Y]$)*

Let X et Y be two random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$: values of X are in \mathbb{R} equipped with the Borel σ -algebra $\mathcal{B}(\mathbb{R})$ and X is integrable. Values of Y are in a set \mathcal{Y} equipped with a σ -algebra \mathcal{V} . Let us denote by $\sigma(Y)$ the σ -algebra generated by Y i.e. $\sigma(Y) = \{Y^{-1}(V) \mid V \in \mathcal{V}\} \subset \mathcal{F}$.

The **expectation of X conditional on Y** , denoted by $\mathbb{E}[X | Y]$, is **the unique random variable in $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ which is**

- $\sigma(Y)$ -measurable,
- and such that $\forall A \in \sigma(Y), \int_A \mathbb{E}[X | Y](\omega) d\mathbb{P}(\omega) = \int_A X(\omega) d\mathbb{P}(\omega)$.

As the classical expectation, the conditional expectation is linear: if X_1 and X_2 are two random variables, $\forall c \in \mathbb{R}, \mathbb{E}[c \cdot X_1 + X_2 | Y] = c \cdot \mathbb{E}[X_1 | Y] + \mathbb{E}[X_2 | Y]$.

First, note that if X is $\sigma(Y)$ -measurable, $X = \mathbb{E}[X | Y]$ \mathbb{P} -almost surely. Indeed, the function X meets both conditions to be $\mathbb{E}[X | Y]$. Note also that the second point of Definition A.1 implies that

$$\mathbb{E}[\mathbb{E}[X | Y]] = \int_{\Omega} \mathbb{E}[X | Y](\omega) d\mathbb{P}(\omega) = \int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \mathbb{E}[X].$$

This second point can be replaced by an other characterization, given by the following property:

Property A.1

The random variable $\mathbb{E}[X | Y]$ can be defined as the unique function $\sigma(Y)$ -measurable such that $\forall Z : \mathcal{Y} \rightarrow \mathbb{R}$ $\sigma(Y)$ -measurable,

$$\mathbb{E}[Z \cdot \mathbb{E}[X | Y]] = \mathbb{E}[Z \cdot X].$$

Proof: Indeed if $Z = \mathbb{1}_A$ with $A \in \sigma(Y)$, we fall back into the second point of Definition A.1.

Thanks to the linearity of the expectation, it remains true if Z is a linear combination of characteristic functions of elements of the σ -algebra. Finally, consider a non decreasing sequence $(Z_n)_{n \in \mathbb{N}}$, with $\forall n \in \mathbb{N}, Z_n$ a combination of characteristic functions, and whose limit is a measurable function Z . Equality holds for each Z_n , and thanks to Beppo-Levi Theorem, the result is true for the measurable function Z . ■

This result makes the following property easier to show:

Property A.2

Let X be a real and integrable random variable, and Y_1, Y_2 two random variables:

$$\mathbb{E}[\mathbb{E}[X | Y_1, Y_2] | Y_2] = \mathbb{E}[X | Y_2] \text{ } \mathbb{P}\text{-almost surely.}$$

As well,

$$\mathbb{E}[\mathbb{E}[X | Y_2] | Y_1, Y_2] = \mathbb{E}[X | Y_2] \quad \mathbb{P}\text{-almost surely.}$$

Proof: The second equality is obvious because if X is $\sigma(Y)$ -measurable, $\mathbb{E}[X | Y] = X$ \mathbb{P} -almost surely. Indeed, X meets the first condition to be $\mathbb{E}[X | Y]$ in Definition A.1, and of course, the second condition is met. Here, the random variable $\mathbb{E}[X | Y_2]$ is $\sigma(Y_2)$ -measurable by definition, and thus it is $\sigma(Y_1, Y_2)$ -measurable: thus the second equality holds.

For the first equality, since both conditional expectations are $\sigma(Y_2)$ -measurable by definition, it is sufficient to show that $\forall Z$ $\sigma(Y_2)$ -measurable, $\mathbb{E}\left[Z \cdot \mathbb{E}[\mathbb{E}[X | Y_1, Y_2] | Y_2]\right] = \mathbb{E}[Z \cdot X]$, as $\mathbb{E}[X | Y_2]$ is the only random variable in $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that for each $\sigma(Y_2)$ -measurable random variable Z , $\mathbb{E}[Z \cdot \mathbb{E}[X | Y_2]] = \mathbb{E}[Z \cdot X]$. Let Z be $\sigma(Y_2)$ -measurable: as Z is $\sigma(Y_2)$ -measurable, it is *a fortiori* $\sigma(Y_1, Y_2)$ -measurable: $\sigma(Y_2) \subseteq \sigma(Y_1, Y_2)$. Thus

$$\mathbb{E}\left[Z \cdot \mathbb{E}[\mathbb{E}[X | Y_1, Y_2] | Y_2]\right] = \mathbb{E}[Z \cdot \mathbb{E}[X | Y_1, Y_2]] = \mathbb{E}[Z \cdot X]. \quad \blacksquare$$

The conditional expectation has been defined as a random variable $\mathbb{E}[X | Y] : \Omega \rightarrow \mathbb{R}$. However, the following property implies that the $\sigma(Y)$ -measurability condition of Definition A.1 may be replaced by “ $\exists \varphi : (\mathcal{Y}, \mathcal{V}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ measurable such that $\mathbb{E}[X | Y] = \varphi(Y)$ ”. The function φ is called *Expectation of X Conditional on the Values of Y* and may be denoted by $\varphi(y) = \mathbb{E}[X | Y = y]$, $\forall y \in \mathcal{Y}$.

Property A.3

The function $Z : \Omega \rightarrow \mathbb{R}$ is $\sigma(Y)$ -measurable, where $Y : (\Omega, \mathcal{F}) \rightarrow (\mathcal{Y}, \mathcal{V})$
 $\Leftrightarrow \exists \varphi : (\mathcal{Y}, \mathcal{V}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ measurable such that $Z = \varphi(Y)$.

Proof: The set $\{\varphi(Y) | \varphi : (\mathcal{Y}, \mathcal{V}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ measurable}\}$ is denoted by Φ . If $Z \in \Phi$, then $Z = \varphi(Y)$ with φ measurable, and as Y is $\sigma(Y)$ -measurable, Z is $\sigma(Y)$ -measurable: in short, if $Z \in \Phi$, then Z is $\sigma(Y)$ -measurable.

Now let us show that any $\sigma(Y)$ -measurable function can be written $\varphi(Y)$ with φ measurable. By definition $\forall A \in \sigma(Y)$, $\exists B \in \mathcal{V}$ such that $A = \{Y \in B\} = Y^{-1}(B)$. Thus $\mathbb{1}_A = \mathbb{1}_{\{Y \in B\}} = \mathbb{1}_B(Y)$ is in Φ as a function of Y . Now, as linear combinations of such characteristic functions are in Φ , and as non-decreasing limits of sequences of functions in Φ are in Φ , it can be concluded that Φ contains all $\sigma(Y)$ -measurable functions. \blacksquare

Definition A.2 (Expectation of X Conditional on the Values of Y)

Let $X \in \mathbb{R}$ and $Y \in \mathcal{Y}$ two random variables: as $\mathbb{E}[X | Y]$ can be written $\varphi(Y)$ with $\varphi : \mathcal{Y} \rightarrow \mathbb{R}$ measurable,

$$\mathbb{E}[X | Y = y] = \varphi(y)$$

is called the expectation of X conditional on the event $\{Y = y\}$. For each $y \in \mathcal{Y}$ such that $\mathbb{P}(Y = y) > 0$, it is the expectation of X if we know that $Y = y$.

If \mathcal{Y} is countable, and $\mathbb{P}(Y = y) > 0$,

$$\mathbb{E}[X | Y = y] = \int_{\{Y=y\}} \frac{\mathbb{E}[X | Y](\omega)}{\mathbb{P}(Y = y)} d\mathbb{P}(\omega).$$

Consider that the values of variables X and Y are in \mathcal{S} , and let us introduce $f : \mathcal{S} \rightarrow \mathbb{R}$ measurable: $f(X)$ is a random variable whose values are in \mathbb{R} . The expectation of $f(X)$ conditional on the values of $Y \in \mathcal{S}$ is denoted by

$\mathbb{E}[f(X) | Y = y]$, $\forall y \in \mathcal{S}$. If \mathcal{S} is a countable set equipped with the σ -algebra $\mathcal{P}(\mathcal{S})$ (the set of sets included in \mathcal{S}), $\varphi(y) = \mathbb{E}[f(X) | Y = y]$ can be computed explicitly.

Property A.4

Let X and Y two random variables whose values are in a countable set \mathcal{S} , and $f : \mathcal{S} \rightarrow \mathbb{R}$ a measurable function: the expectation of $f(X)$ conditional on the values of Y is

$$\forall y \in \mathcal{S} \text{ such that } \mathbb{P}(Y = y) > 0, \mathbb{E}[f(X) | Y = y] = \sum_{x \in \mathcal{S}} f(x) \cdot \mathbb{P}(X = x | Y = y).$$

where $\mathbb{P}(X = x | Y = y) = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)}$.

Proof: For clarity, $\mathbb{E}[f(X) | Y = y]$ is denoted by $\varphi(y)$ in this proof. By only considering singletons $\{Y = y\} \subseteq \Omega$ in the second condition of Definition A.1, it leads to $\forall y \in \mathcal{S}$,

$$\int_{\{Y=y\}} \varphi(Y) d\mathbb{P} = \int_{\{Y=y\}} f(X) d\mathbb{P} \Leftrightarrow \varphi(y) \cdot \mathbb{P}(Y = y) = \int_{\{Y=y\}} f(X) d\mathbb{P}$$

and if $\mathbb{P}(Y = y) > 0$,

$$\varphi(y) = \int_{\Omega} f(X) \cdot \frac{\mathbb{1}_{\{Y=y\}} d\mathbb{P}}{\mathbb{P}(Y=y)} = \int_{\Omega} \sum_{x \in \mathcal{S}} f(x) \cdot \mathbb{1}_{\{X=x\}} \cdot \frac{\mathbb{1}_{\{Y=y\}} d\mathbb{P}}{\mathbb{P}(Y=y)} = \sum_{x \in \mathcal{S}} f(x) \cdot \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)}$$

thanks to the Fubini theorem. Note that if $f(X) = \mathbb{1}_{\{X=x\}}$, we get $\mathbb{E}[\mathbb{1}_{\{X=x\}} | Y = y] = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)}$, which is the discrete conditional probability $\mathbb{P}(X = x | Y = y)$. ■

A.2 Proof of Property I.1.1

Proof: Let $t \in \mathbb{N}$, $(s_0, s_1, \dots, s_t) \in \mathcal{S}^t$ and $s' \in \mathcal{S}$: on the one hand,

$$\int_{\{S_0=s_0, \dots, S_t=s_t\}} \mathbb{1}_{\{S_{t+1}=s'\}} d\mathbb{P} = \mathbb{P}(S_0 = s_0, \dots, S_t = s_t, S_{t+1} = s'),$$

and on the other hand, if $\mathbb{E}[\mathbb{1}_{\{S_{t+1}=s'\}} | S_t]$ is denoted by $\mathbb{P}(S_{t+1} = s' | S_t)$,

$$\int_{\{S_0=s_0, \dots, S_t=s_t\}} \mathbb{P}(S_{t+1} = s' | S_t) d\mathbb{P} = \mathbb{P}(S_{t+1} = s' | S_t = s_t) \cdot \mathbb{P}(S_0 = s_0, S_1 = s_1, \dots, S_t = s_t).$$

Both integrals are equal as $(S_t)_{t \in \mathbb{N}}$ is a Markov Chain. Since events of $\sigma(S_0, \dots, S_t)$ are unions of events which can be written $\{S_0 = s_0, S_1 = s_1, \dots, S_t = s_t\}$, for all $B \in \sigma(S_0, \dots, S_t)$

$$\int_B \mathbb{1}_{\{S_{t+1}=s'\}} d\mathbb{P} = \int_B \mathbb{P}(S_{t+1} = s' | S_t) d\mathbb{P}$$

and then, using Definition A.1 and as $\mathbb{P}(S_{t+1} = s' | S_t)$ is $\sigma(S_0, \dots, S_t)$ -measurable

$$\mathbb{P}(S_{t+1} = s' | S_t) = \mathbb{E}[\mathbb{1}_{S_{t+1}=s'} | S_0, \dots, S_t] \quad \mathbb{P}\text{-almost surely.}$$

Now,

$$\begin{aligned} \mathbb{E}[f(S_{t+1}) | S_0, \dots, S_t] &= \mathbb{E}\left[\sum_{s' \in \mathcal{S}} f(s') \cdot \mathbb{1}_{\{S_{t+1}=s'\}} \middle| S_0, \dots, S_t\right] \\ &= \sum_{s' \in \mathcal{S}} f(s') \cdot \mathbb{E}[\mathbb{1}_{\{S_{t+1}=s'\}} | S_0, \dots, S_t] \\ &= \sum_{s' \in \mathcal{S}} f(s') \cdot \mathbb{P}(S_{t+1} = s' | S_t) \quad \mathbb{P}\text{-almost surely.} \end{aligned}$$

The random variable $\sum_{s' \in \mathcal{S}} f(s') \cdot \mathbb{P}(S_{t+1} = s' | S_t)$ is $\sigma(S_t)$ -measurable, thus $\mathbb{E}[f(S_{t+1}) | S_0, \dots, S_t]$ is $\sigma(S_t)$ -measurable too, and then,

$$\mathbb{E}[f(S_{t+1}) | S_0, \dots, S_t] = \mathbb{E}\left[\mathbb{E}[f(S_{t+1}) | S_0, \dots, S_t] \middle| S_t\right] = \mathbb{E}[f(S_{t+1}) | S_t],$$

because of Property A.2. Finally, the equalities of Property I.1.1 are achieved by integrating both parts of the equations over $\{S_0 = s_0, \dots, S_t = s_t\}$ and then by dividing them by $\mathbb{P}(S_0 = s_0, \dots, S_t = s_t)$. ■

A.3 Proof of Theorem 1

Proof: First of all, consider that the process is at the stage $\tilde{t} \in \mathbb{N}$. All states from the beginning of the process are given as input to the agent: it has to choose the best action knowing these $\tilde{t} + 1$ first system states $\{s_0, s_1, \dots, s_{\tilde{t}}\} \in \mathcal{S}^{\tilde{t}+1}$. Regardless the previously gathered rewards, its goal is to maximize the expectation of the sum of the next rewards. We show by induction on \tilde{t} from $H - 1$ to 0, that the highest expected total reward can be reached with a strategy $(d_t)_{t=0}^{H-1}$ *i.e.* a sequence of decision rules $d_t : \mathcal{S} \rightarrow \mathcal{A}$. A sequence of functions from all the states that the system has gone through, *i.e.* a sequence $(d_t)_{t=0}^{H-1}$ of functions $d_t : \mathcal{S}^t \rightarrow \mathcal{A}$, is not necessary.

Let $\tilde{t} = H - 1$: the agent has to find the action a maximizing

$$\mathbb{E} \left[r_{H-1}(S_{H-1}, a) + R(S_H) \mid S_0 = s_0, \dots, S_{H-1} = s_{H-1} \right]$$

which is equal to $\mathbb{E} \left[r_{H-1}(S_{H-1}, a) + \mathbb{E}[R(S_H) \mid S_{H-1}] \mid S_0 = s_0, \dots, S_{H-1} = s_{H-1} \right]$

$$\begin{aligned} &= \mathbb{E}[f_a(S_{H-1}) \mid S_0 = s_0, \dots, S_{H-1} = s_{H-1}] \text{ with } f_a : \mathcal{S} \rightarrow \mathbb{R} \text{ measurable,} \\ &= \mathbb{E}[f_a(S_{H-1}) \mid S_{H-1} = s_{H-1}] \end{aligned}$$

because of Property (I.1.1). Then, the value to be maximized depends on the state $s_{H-1} \in \mathcal{S}$ only: a decision rule $d_{H-1}^* : \mathcal{S} \rightarrow \mathcal{A}$ such that $\forall s \in \mathcal{S}$

$$d_{H-1}^*(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \mathbb{E}[f_a(S_{H-1}) \mid S_{H-1} = s]$$

is sufficient. Now, assume that this result is true for the time step $\tilde{t} + 1 \leq H - 1$, and consider that a strategy $(d_t)_{t=\tilde{t}+1}^{H-1}$ has been computed. The agent has to find the action a maximizing

$$\mathbb{E} \left[r_{\tilde{t}}(S_{\tilde{t}}, a) + \sum_{t=\tilde{t}+1}^{H-1} r_t(S_t, d_t(S_t)) + R(S_H) \mid S_0 = s_0, \dots, S_{\tilde{t}} = s_{\tilde{t}} \right]$$

which is equal to $\mathbb{E} \left[r_{\tilde{t}}(S_{\tilde{t}}, a) + \mathbb{E} \left[\sum_{t=\tilde{t}+1}^{H-1} r_t(S_t, d_t(S_t)) + R(S_H) \mid S_{\tilde{t}} \right] \mid S_0 = s_0, \dots, S_{\tilde{t}} = s_{\tilde{t}} \right]$

$$\begin{aligned} &= \mathbb{E}[f_a(S_{\tilde{t}}) \mid S_0 = s_0, \dots, S_{\tilde{t}} = s_{\tilde{t}}] \text{ with } f_a : \mathcal{S} \rightarrow \mathbb{R} \text{ measurable,} \\ &= \mathbb{E}[f_a(S_{\tilde{t}}) \mid S_{\tilde{t}} = s_{\tilde{t}}] \end{aligned}$$

and then, the same conclusion holds: it is sufficient to compute a decision rule $d_{\tilde{t}}^* : \mathcal{S} \rightarrow \mathcal{A}$ such that $\forall s \in \mathcal{S}$

$$d_{\tilde{t}}^*(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \mathbb{E}[f_a(S_{\tilde{t}}) \mid S_{\tilde{t}} = s].$$

We just proved by induction that it suffices to look for strategies of \mathcal{D}_H as defined above, *i.e.* to look for a sequence of decision rules $(d_t)_{t=0}^{H-1}$, maximizing the criterion I.2.

In the following, we set up the Dynamic Programming equations used to compute the optimal value function and the optimal strategy. The size of the horizon i is the index used for the incremental computation of the optimal value function V^* . It is also the opposite modulo H of the stage of the process t , index used for the strategy. The initialization $V_0^*(s) = R(s)$ is obvious: when the horizon is zero, no action has to be selected by the agent, and it receives only the terminal reward. Next, let $s \in \mathcal{S}$, $i \in \{1, \dots, H\}$, and set $t_0 = H - i$:

$$V_i^*(s) = \sup_{(d_t)_{t=t_0}^{H-1} \in \mathcal{D}_{H-t_0}} \mathbb{E} \left[\sum_{t=t_0}^{H-1} r_t(S_t, d_t(S_t)) + R(S_H) \mid S_{t_0} = s \right].$$

$$\begin{aligned}
& \text{Yet } \mathbb{E} \left[\sum_{t=t_0}^{H-1} r_t(S_i, d_i(S_i)) + R(S_H) \mid S_{t_0} \right] \\
&= \mathbb{E} \left[r_{t_0}(S_{t_0}, d_{t_0}(S_{t_0})) + \mathbb{E} \left[\sum_{t=t_0+1}^{H-1} r_t(S_i, d_i(S_i)) + R(S_H) \mid S_{t_0}, S_{t_0+1} \right] \mid S_{t_0} \right] \\
&= \mathbb{E} \left[r_{t_0}(S_{t_0}, d_{t_0}(S_{t_0})) + \mathbb{E} \left[V_{i-1}(S_{t_0+1}, (d_t)_{t=t_0+1}^{H-1}) \mid S_{t_0}, S_{t_0+1} \right] \mid S_{t_0} \right] \\
&= \mathbb{E} \left[r_{t_0}(S_{t_0}, d_{t_0}(S_{t_0})) + \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid S_{t_0}, d_{t_0}(S_{t_0})) \cdot V_{i-1}(s', (d_t)_{t=t_0+1}^{H-1}) \mid S_{t_0} \right]
\end{aligned}$$

using properties (A.2) and (I.1.1). By integrating both part of this equality over $\{S_{t_0} = s\}$ and dividing them by $\mathbb{P}(S_{t_0} = s) > 0$, it becomes

$$\begin{aligned}
V_i^*(s) &= \sup_{(d)_{t=t_0}^{H-1} \in \mathcal{D}_{H-t_0}} \left\{ r_t(s, d_{t_0}(s)) + \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, d_{t_0}(s)) \cdot V_{i-1}(s', (d_t)_{t=t_0+1}^{H-1}) \right\} \\
&= \sup_{(a, (d')_{t=t_0+1}^{H-1}) \in \mathcal{D}_{H-t_0}} \left\{ r_t(s, a) + \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \cdot V_{i-1}(s', (d'_t)) \right\} \\
&= \max_{a \in \mathcal{A}} \left\{ r_t(s, a) + \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \cdot \sup_{(d'_t)_{t=t_0+1}^{H-1} \in \mathcal{D}_{H-t_0-1}} V_{i-1}(s', (d'_t)) \right\} \quad (15) \\
&= \max_{a \in \mathcal{A}} \left\{ r_t(s, a) + \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \cdot V_{i-1}^*(s') \right\}
\end{aligned}$$

where 15 is justified by:

- as $\sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \cdot V_{i-1}(s', (d_t)) \leq \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \sup_{(d'_t) \in \mathcal{D}_{H-t_0-1}} V_{i-1}(s', (d'_t))$
for each strategy $(d_t)_{t=t_0+1}^{H-1} \in \mathcal{D}_{H-t_0-1}$,
 $\sup_{(d'_t) \in \mathcal{D}_{H-t_0-1}} \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) V_{i-1}(s', (d'_t)) \leq \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \sup_{(d'_t) \in \mathcal{D}_{H-t_0-1}} V_{i-1}^*(s').$
- let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers, such that $\varepsilon_n \rightarrow 0$ when $n \rightarrow \infty$. For each $n \in \mathbb{N}$, let $(d_t^{\varepsilon_n})_{t=t_0+1}^{H-1} \in \mathcal{D}_{H-t_0-1}$ be a strategy such that

$$V_{i-1}^*(s') - \varepsilon \leq V_{i-1}(s', (d_t^{\varepsilon_n})) \leq V_{i-1}^*(s'), \quad \forall s' \in \mathcal{S}.$$

Computing the mean with respect to $\mathbf{p}_{t_0}(\cdot \mid s, a)$ whose support is finite, the inequality on the left becomes

$$\sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \cdot V_{i-1}^*(s') - \varepsilon_n \leq \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \cdot V_{i-1}(s', (d_t^{\varepsilon_n}))$$

where the right part is obviously lower than $\sup_{(d) \in \mathcal{D}_{H-t_0-1}} \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \cdot V_{i-1}(s', (d))$.

Now, as this couple of inequalities are true for each $n \in \mathbb{N}$, making $n \rightarrow \infty$, the result is

$$\sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \cdot V_{i-1}^*(s') \leq \sup_{(d) \in \mathcal{D}_{H-t_0-1}} \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \cdot V_{i-1}(s', (d)).$$

Therefore, if at each iteration $i \in \{1, \dots, H\}$, the decision rule d_{H-i}^* is defined as $\forall s \in \mathcal{S}$, $d_{H-i}^*(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \left\{ r_t(s, a) + \sum_{s' \in \mathcal{S}_{s,a,t}} \mathbf{p}_{t_0}(s' \mid s, a) \cdot V_{i-1}^*(s') \right\}$, $V_i^*(s, (d_t^*)_{t=H-i}^{H-1}) = V_i^*(s)$: with $i = H$, it shows that $(d_t^*)_{t=0}^{H-1} \in \mathcal{D}_H$ is optimal. ■

A.4 Proof of the Bellman Equation (I.5)

Proof: The following calculus lines lead to the Bellman Equation. Let $(d)_{t \in \mathbb{N}} \in \mathcal{D}_\infty$.

$$\begin{aligned}
V^d(s) &:= \mathbb{E} \left[\sum_{t=0}^{+\infty} \gamma^t \cdot r(S_t, d_t(S_t)) \mid S_0 = s \right] \\
&= \mathbb{E} \left[r(S_0, d_0(S_0)) + \sum_{t=1}^{+\infty} \gamma^t \cdot r(S_t, d_t(S_t)) \mid S_0 = s \right] \\
&= r(s, d_0(s)) + \mathbb{E} \left[\mathbb{E} \left[\sum_{t=1}^{+\infty} \gamma^t \cdot r(S_t, d_t(S_t)) \mid S_1 \right] \mid S_0 = s \right] \\
&= r(s, d_0(s)) + \gamma \cdot \sum_{s' \in \mathcal{S}_{s, d_0(s)}} \mathbf{p}(s' \mid s, d_0(s)) \cdot \mathbb{E} \left[\sum_{t=1}^{+\infty} \gamma^{t-1} \cdot r(S_t, d_t(S_t)) \mid S_1 = s' \right] \\
&= r(s, d_0(s)) + \gamma \cdot \sum_{s' \in \mathcal{S}_{s, d_0(s)}} \mathbf{p}(s' \mid s, d_0(s)) \cdot \mathbb{E} \left[\sum_{t'=0}^{+\infty} \gamma^{t'} \cdot r(S_{t'}^+, d_{t'}^+(S_{t'}^+)) \mid S_0^+ = s' \right] \\
&= r(s, d_0(s)) + \gamma \cdot \sum_{s' \in \mathcal{S}_{s, d_0(s)}} \mathbf{p}(s' \mid s, d_0(s)) \cdot V^{d^+}(s').
\end{aligned}$$

The third and fourth lines come from the properties A.2 and I.1.1. In the fifth line, $(S_t^+)_{t \in \mathbb{N}}$ is defined as $S_t^+ = S_{t+1}$. As well, $\forall t \in \mathbb{N}, \forall s \in \mathcal{S}, d_t^+(s) = d_{t+1}(s)$. ■

A.5 Proof of Theorem 2

Proof: Let $(V_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $(\mathcal{F}_B(\mathcal{S}, \mathbb{R}), \|\cdot\|_\infty)$. For each state $s \in \mathcal{S}$, $(V_n(s))_{n \in \mathbb{N}}$ is a Cauchy sequence of $(\mathbb{R}, |\cdot|)$ because $|V_n(s)| \leq \|V_n\|_\infty$. Since $(\mathbb{R}, |\cdot|)$ is complete, $\forall s \in \mathcal{S}$, $(V_n(s))_{n \in \mathbb{N}}$ has a limit in \mathbb{R} that we denote by $V(s)$. A Cauchy sequence is bounded, therefore $\exists M > 0$ such that $\forall n \in \mathbb{N}, \|V_n\|_\infty \leq M$. Thus, $\forall s \in \mathcal{S}, |V_n(s)| \rightarrow |V(s)| \leq M$, and then $V \in \mathcal{F}_B(\mathcal{S}, \mathbb{R})$. Finally, as (V_n) is a Cauchy sequence,

$$\begin{aligned}
\forall \varepsilon > 0, \exists N \geq 0, \text{ such that } & \forall n \geq N, p \geq 0, \forall s \in \mathcal{S}, |V_n(s) - V_{n+p}(s)| < \varepsilon \\
\Rightarrow & \forall n \geq N, \forall s \in \mathcal{S}, |V_n(s) - V(s)| < \varepsilon \\
\Rightarrow & \forall n \geq N, \|V_n - V\|_\infty < \varepsilon
\end{aligned}$$

Thus $V_n \rightarrow V$ in $\mathcal{F}_B(\mathcal{S}, \mathbb{R})$ when $n \rightarrow +\infty$, and then $(\mathcal{F}_B(\mathcal{S}, \mathbb{R}), \|\cdot\|_\infty)$ is a Banach space. ■

A.6 Proof of Property I.1.2

Proof: Let $(V, V') \in \mathcal{F}_B(\mathcal{S}, \mathbb{R})^2$,

$$\begin{aligned}
\|\mathcal{B}^d V - \mathcal{B}^d V'\|_\infty &\leq \gamma \cdot \sup_{s \in \mathcal{S}} \sum_{s' \in \mathcal{S}_{s, d_0(s)}} \mathbf{p}(s' \mid s, d_0(s)) \cdot |V(s') - V'(s')| \\
&\leq \gamma \cdot \sup_{s \in \mathcal{S}} \sum_{s' \in \mathcal{S}_{s, d_0(s)}} \mathbf{p}(s' \mid s, d_0(s)) \cdot \|V - V'\|_\infty \\
&\leq \gamma \cdot \|V - V'\|_\infty
\end{aligned}$$

since $\forall s \in \mathcal{S}, \mathbf{p}(\cdot \mid s, d_0(s))$ is a probability distribution. ■

A.7 Proof of Property I.1.3

First, let us present the following result:

Property A.5

Let f and g be two functions defined on the finite set \mathcal{A} and with values in \mathbb{R} :

$$\left| \max_{a \in \mathcal{A}} f(a) - \max_{a \in \mathcal{A}} g(a) \right| \leq \max_{a \in \mathcal{A}} |f(a) - g(a)| \quad (16)$$

Proof:

$$\begin{aligned} \forall a \in \mathcal{A}, \quad f(a) - g(a) &\leq \max_{a' \in \mathcal{A}} |f(a') - g(a')| \\ \Rightarrow \forall a \in \mathcal{A}, \quad f(a) &\leq \max_{a' \in \mathcal{A}} g(a') + \max_{a' \in \mathcal{A}} |f(a') - g(a')| \\ \Rightarrow \max_{a' \in \mathcal{A}} f(a') - \max_{a' \in \mathcal{A}} g(a') &\leq \max_{a' \in \mathcal{A}} |f(a') - g(a')|. \end{aligned}$$

Finally, the same inequalities hold starting with $g(a) - f(a)$, thus we get the result 16. \blacksquare

Here is the proof of Property I.1.3:

Proof: The contraction inequality of the operator \mathcal{B}^d for $(V, V') \in \mathcal{F}_B(\mathcal{S}, \mathbb{R})^2$ is true for each strategy $(d) \in \mathcal{D}_\infty$, as stated by Property I.1.2:

$$\begin{aligned} \forall (d) \in \mathcal{D}_\infty, \quad &\|\mathcal{B}^d V - \mathcal{B}^d V'\|_\infty \leq \gamma \cdot \|V - V'\|_\infty \\ \Rightarrow \forall (d) \in \mathcal{D}_\infty, \forall s \in \mathcal{S}, \quad &|(\mathcal{B}^d V)(s) - (\mathcal{B}^d V')(s)| \leq \gamma \cdot \|V - V'\|_\infty \\ \Rightarrow \forall a \in \mathcal{A}, \forall s \in \mathcal{S}, \quad &|(\mathcal{B}^a V)(s) - (\mathcal{B}^a V')(s)| \leq \gamma \cdot \|V - V'\|_\infty \\ \Rightarrow \forall s \in \mathcal{S}, \quad &\max_{a \in \mathcal{A}} |(\mathcal{B}^a V)(s) - (\mathcal{B}^a V')(s)| \leq \gamma \cdot \|V - V'\|_\infty \end{aligned}$$

$$\begin{aligned} \text{and Property A.5 leads then to } \forall s \in \mathcal{S} \quad &\left| \max_{a \in \mathcal{A}} (\mathcal{B}^a V)(s) - \max_{a \in \mathcal{A}} (\mathcal{B}^a V')(s) \right| \leq \gamma \cdot \|V - V'\|_\infty \\ \Rightarrow \|\mathcal{B}^* V - \mathcal{B}^* V'\|_\infty &\leq \gamma \cdot \|V - V'\|_\infty \end{aligned} \quad \blacksquare$$

A.8 Proof of Theorem 3

Proof: Here is how we show that the function V^* is the optimal value function, *i.e.*

$$V^*(s) = \sup_{d \in \mathcal{D}_\infty} V^d(s) = \sup_{d \in \mathcal{D}_\infty} \mathbb{E} \left[\sum_{t=0}^{+\infty} \gamma^t \cdot r(S_t, d_t(S_t)) \mid S_0 = s \right].$$

Let $s \in \mathcal{S}$ and $d = (a, d^+)$ the strategy which consists in selecting action a at time step $t = 0$, and then actions $a_1 = d_0^+(s_1) := d_1(s_1)$, $a_2 = d_1^+(s_2) := d_2(s_2)$, etc. Note that $d^+ \in \mathcal{D}_\infty$ as well,

and numbered from 0.

$$\begin{aligned}
\sup_{(d) \in \mathcal{D}_\infty} V^d(s) &:= \sup_{(d) \in \mathcal{D}_\infty} \mathbb{E} \left[\sum_{t=0}^{+\infty} \gamma^t \cdot r(S_t, d_t(S_t)) \mid S_0 = s \right] \\
&= \sup_{(a, d^+) \in \mathcal{D}_\infty} \left\{ r(s, a) + \gamma \cdot \sum_{s' \in \mathcal{S}_{s,a}} \mathbf{p}(s' \mid s, a) \cdot V^{d^+}(s') \right\} \\
&= \max_{a \in \mathcal{A}} \left\{ r(s, a) + \gamma \cdot \sum_{s' \in \mathcal{S}_{s,a}} \mathbf{p}(s' \mid s, a) \cdot \sup_{(d^+) \in \mathcal{D}_\infty} V^{d^+}(s') \right\} \quad (17) \\
&= \max_{a \in \mathcal{A}} \left\{ r(s, a) + \gamma \cdot \sum_{s' \in \mathcal{S}_{s,a}} \mathbf{p}(s' \mid s, a) \cdot \sup_{(d^+) \in \mathcal{D}_\infty} V^{d^+}(s') \right\} \\
&= (\mathcal{B}^* \sup_{(d^+) \in \mathcal{D}_\infty} V^{d^+})(s) = (\mathcal{B}^* \sup_{(d) \in \mathcal{D}_\infty} V^d)(s).
\end{aligned}$$

The equality 17 is justified just like at the line 15 of the proof of Theorem 1.

Thus $V^* = \sup_{d \in \mathcal{D}_\infty} V^d$ as stated by the Fixed-Point Theorem. Indeed, as \mathbb{B}^* is a contracting operator (see Property I.1.3), the solution V^* of the Dynamic Programming equation I.7 is unique.

First, as $d^* : s \mapsto a^* \in \operatorname{argmax}_{a \in \mathcal{A}} (\mathcal{B}^a V^*)(s)$,

$$\mathcal{B}^{d^*} V^* = \max_{a \in \mathcal{A}} (\mathcal{B}^a V^*)(s) = \mathcal{B}^* V^*$$

The function V^* is then a fixed-point of the contracting operator \mathcal{B}^{d^*} (see Property I.1.2). As noted earlier, V^{d^*} is a fixed-point of \mathcal{B}^{d^*} too, and thus $V^{d^*} = V^* = \max_{(d) \in \mathcal{D}_\infty} V^d$. It means that d^* is an optimal strategy. It is thus shown that it is sufficient to look for stationary strategies because at least one of them, d^* , is optimal. ■

A.9 Proof of Theorem 4

Proof: First,

$$\|V^N - V^*\|_\infty = \|\mathcal{B}^* V^{N-1} - \mathcal{B}^* V^*\|_\infty \leq \gamma \cdot \|V^{N-1} - V^*\|_\infty \leq \gamma \cdot (\|V^{N-1} - V^N\|_\infty + \|V^N - V^*\|_\infty)$$

and then $\|V^N - V^*\|_\infty \leq \frac{\gamma}{1-\gamma} \cdot \|V^{N-1} - V^N\|_\infty$. Moreover,

$$\|V^{N-1} - V^N\|_\infty = \|(\mathcal{B}^*)^{N-1} V^0 - (\mathcal{B}^*)^{N-1} V^1\|_\infty \leq \gamma^{N-1} \cdot \|V^0 - V^1\|_\infty.$$

Finally,

$$\|V^N - V^*\|_\infty \leq \frac{\gamma^N}{1-\gamma} \cdot \|V^0 - V^1\|_\infty. \quad \blacksquare$$

A.10 Proof of Theorem 5

Proof: The Bellman equation for the strategy (d) , is $V^d = \mathcal{B}^d V^d$, and the last iteration of the algorithm is $V^{N+1} = \mathcal{B}^* V^N = \mathcal{B}^d V^N$ (as (d^*) is greedy with respect to V_N , we consider V_{N+1} even if it is not actually computed). Thanks to these two equalities, it is possible to write

$$\begin{aligned}
\|V^d - V^{N+1}\|_\infty &= \|\mathcal{B}^d V^d - \mathcal{B}^d V^N\|_\infty \leq \gamma \cdot \|V^d - V^N\|_\infty \\
&\leq \gamma \cdot (\|V^d - V^{N+1}\|_\infty + \|V^{N+1} - V^N\|_\infty) \\
\Rightarrow \|V^d - V^{N+1}\|_\infty &\leq \frac{\gamma}{1-\gamma} \cdot \|V^{N+1} - V^N\|_\infty \leq \frac{\gamma^{N+1}}{1-\gamma} \cdot \|V^1 - V^0\|_\infty. \quad (18)
\end{aligned}$$

Finally, thanks to results (I.8) and (18), we get the control of the strategy error:

$$\|V^d - V^*\|_\infty \leq \|V^d - V^N\|_\infty + \|V^N - V^*\|_\infty \leq \frac{2 \cdot \gamma^N}{1 - \gamma} \|V^1 - V^0\|_\infty. \quad \blacksquare$$

A.11 Proof of theorem 6

Proof: If i_{t+1} is the current information, $\forall s' \in \mathcal{S}$,

$$\begin{aligned} & b_{t+1}(s') \\ &:= \mathbb{P}(S_{t+1} = s' \mid I_{t+1} = i_{t+1}) \\ &= \frac{\mathbb{P}(S_{t+1} = s', O_{t+1} = o_{t+1} \mid I_t = i_t, a_t)}{\mathbb{P}(O_{t+1} = o_{t+1} \mid I_t = i_t, a_t)} \\ &= \frac{\sum_{s \in \mathcal{S}} \mathbb{P}(S_{t+1} = s', O_{t+1} = o_{t+1} \mid S_t = s, I_t = i_t, a_t) \cdot \mathbb{P}(S_t = s \mid I_t = i_t, a_t)}{\sum_{\tilde{s} \in \mathcal{S}} \mathbb{P}(O_{t+1} = o_{t+1} \mid S_t = \tilde{s}, I_t = i_t, a_t) \cdot \mathbb{P}(S_t = \tilde{s} \mid I_t = i_t, a_t)} \\ &= \frac{\sum_{s \in \mathcal{S}} \mathbb{P}(O_{t+1} = o_{t+1} \mid S_{t+1} = s', S_t = s, I_t = i_t, a_t) \cdot \mathbb{P}(S_{t+1} = s' \mid S_t = s, I_t = i_t, a_t) \cdot b_t(s)}{\sum_{\tilde{s} \in \mathcal{S}} \sum_{\tilde{s}' \in \mathcal{S}} \mathbb{P}(O_{t+1} = o_{t+1}, S_{t+1} = \tilde{s}' \mid S_t = \tilde{s}, I_t = i_t, a_t) \cdot b_t(\tilde{s})} \\ &= \frac{\sum_{s \in \mathcal{S}} \mathbf{p}(o_{t+1} \mid s', a_t) \cdot \mathbf{p}(s' \mid s, a_t) \cdot b_t(s)}{\sum_{\tilde{s} \in \mathcal{S}} \sum_{\tilde{s}' \in \mathcal{S}} \mathbf{p}(o_{t+1} \mid \tilde{s}', a_t) \cdot \mathbf{p}(\tilde{s}' \mid \tilde{s}, a_t) \cdot b_t(\tilde{s})} := u(b_t, a_t, o_{t+1})(s'). \end{aligned} \quad (19)$$

where line 19 is simply given by the fact that, for A, B, C subsets of Ω , $\mathbb{P}(A \mid B \cap C) = \frac{\mathbb{P}(A \cap B \mid C)}{\mathbb{P}(B \mid C)}$. \blacksquare

A.12 Proof of Theorem 7

Proof: We use here the following notations: $\hat{A}_t = \{A_0, \dots, A_t\}$, $\hat{a}_t = \{a_0, \dots, a_t\}$, $\hat{O}_t = \{O_1, \dots, O_t\}$, and $\hat{o}_t = \{o_1, \dots, o_t\}$. The belief at time step t can be written as a function of \hat{o}_t , \hat{a}_{t-1} and b_0 : the belief b_t is then denoted by $b_{b_0}^{i_t} = b_{b_0}^{\hat{o}_t, \hat{a}_{t-1}}$, and

$$b_{b_0}^{i_t}(s) = b_{b_0}^{\hat{o}_t, \hat{a}_{t-1}}(s) = \frac{\mathbb{P}(S_t = s, \hat{O}_t = \hat{o}_t \mid \hat{A}_{t-1} = \hat{a}_{t-1})}{\mathbb{P}(\hat{O}_t = \hat{o}_t \mid \hat{A}_{t-1} = \hat{a}_{t-1})} = u\left(u\left(\dots u(b_0, a_0, o_1), \dots\right), a_{H-1}, o_H\right)(s). \quad (20)$$

The following notation will be useful for the follow-up:

$$\mathbf{p}(\hat{o}_t, s_t \mid \hat{a}_{t-1}, s_0) = \mathbf{p}(o_t \mid s_t, a_{t-1}) \cdot \sum_{s_1, \dots, s_{t-1}} \mathbf{p}(s_t \mid s_{t-1}, a_{t-1}) \cdot \prod_{i=1}^{t-1} \mathbf{p}(o_i \mid s_i, a_{i-1}) \cdot \mathbf{p}(s_i \mid s_{i-1}, a_{i-1}).$$

The numerator of the fraction (20) may be written

$$\mathbb{P}(S_t = s, \hat{O}_t = \hat{o}_t \mid \hat{A}_{t-1} = \hat{a}_{t-1}) = \sum_{s_0 \in \mathcal{S}} \mathbf{p}(\hat{o}_t, s_t \mid \hat{a}_{t-1}, s_0) \cdot b_0(s_0).$$

As well, the denominator of the fraction (20) is

$$\mathbb{P}(\hat{O}_t = \hat{o}_t \mid \hat{A}_{t-1} = \hat{a}_{t-1}) = \sum_{s_t \in \mathcal{S}} \sum_{s_0 \in \mathcal{S}} \mathbf{p}(\hat{o}_t, s_t \mid \hat{a}_{t-1}, s_0) \cdot b_0(s_0).$$

Finally, the probability of the system state at time step t conditioned on the random sequence of actions is

$$\begin{aligned}\mathbb{P}\left(S_t = s \mid \hat{A}_{t-1} = \hat{a}_{t-1}\right) &= \sum_{\substack{(s_0, \dots, s_{t-1}) \\ \in \mathcal{S}^t}} \prod_{i=1}^t \mathbf{p}(s_i \mid s_{i-1}, a_{i-1}) \cdot b_0(s_0) \\ &= \sum_{\hat{o}_t} \sum_{s_0 \in \mathcal{S}} \mathbf{p}(\hat{o}_t, s_t \mid \hat{a}_{t-1}, s_0) \cdot b_0(s_0).\end{aligned}$$

Then, the expectation of the reward conditioned on the random sequence of actions is

$$\begin{aligned}\mathbb{E}\left[r(S_t, A_t) \mid \hat{A}_t = \hat{a}_t\right] &= \sum_{s \in \mathcal{S}} r(s, a_t) \cdot \mathbb{P}\left(S_t = s \mid \hat{A}_{t-1} = \hat{a}_{t-1}\right) \\ &= \sum_{s \in \mathcal{S}} r(s, a_t) \cdot \sum_{s_0, s_1, \dots, s_{t-1}} \prod_{i=1}^t \mathbf{p}(s_i \mid s_{i-1}, a_{i-1}) \cdot b_0(s_0) \\ &= \sum_{s \in \mathcal{S}} r(s, a_t) \cdot \sum_{\hat{o}_t} \sum_{s_0 \in \mathcal{S}} \mathbf{p}(\hat{o}_t, s \mid \hat{a}_{t-1}, s_0) \cdot b_0(s_0) \\ &= \sum_{\hat{o}_t} \sum_{(s_0, s) \in \mathcal{S}^2} \mathbf{p}(\hat{o}_t, s \mid \hat{a}_{t-1}, s_0) \cdot b_0(s_0) \cdot r(s, a_t).\end{aligned}$$

Next, multiplying each term of the sum over observations \hat{o}_t by

$$1 = \frac{\sum_{(s^{(1)}, s'^{(1)}) \in \mathcal{S}^2} \mathbf{p}(\hat{o}_t, s'^{(1)} \mid \hat{a}_{t-1}, s^{(1)}) \cdot b_0(s^{(1)})}{\sum_{(s^{(2)}, s'^{(2)}) \in \mathcal{S}^2} \mathbf{p}(\hat{o}_t, s'^{(2)} \mid \hat{a}_{t-1}, s^{(2)}) \cdot b_0(s^{(2)})},$$

we get

$$\begin{aligned}&\sum_{\hat{o}_t} \frac{\sum_{(s^{(1)}, s'^{(1)}) \in \mathcal{S}^2} \mathbf{p}(\hat{o}_t, s'^{(1)} \mid \hat{a}_{t-1}, s^{(1)}) \cdot b_0(s^{(1)})}{\sum_{(s^{(2)}, s'^{(2)}) \in \mathcal{S}^2} \mathbf{p}(\hat{o}_t, s'^{(2)} \mid \hat{a}_{t-1}, s^{(2)}) \cdot b_0(s^{(2)})} \cdot \sum_{(s_0, s) \in \mathcal{S}^2} \mathbf{p}(\hat{o}_t, s \mid \hat{a}_{t-1}, s_0) \cdot b_0(s_0) \cdot r(s, a_t) \\ &= \sum_{\hat{o}_t} \sum_{(s^{(1)}, s'^{(1)})} \mathbf{p}(\hat{o}_t, s'^{(1)} \mid \hat{a}_{t-1}, s^{(1)}) \cdot b_0(s^{(1)}) \cdot \sum_{s \in \mathcal{S}} \frac{\sum_{s_0 \in \mathcal{S}} \mathbf{p}(\hat{o}_t, s \mid \hat{a}_{t-1}, s_0) \cdot b_0(s_0)}{\sum_{(s^{(2)}, s'^{(2)})} \mathbf{p}(\hat{o}_t, s'^{(2)} \mid \hat{a}_{t-1}, s^{(2)}) \cdot b_0(s^{(2)})} \cdot r(s, a_t) \\ &= \sum_{\hat{o}_t} \sum_{(s^{(1)}, s'^{(1)})} \mathbf{p}(\hat{o}_t, s'^{(1)} \mid \hat{a}_{t-1}, s^{(1)}) \cdot b_0(s^{(1)}) \cdot \sum_{s \in \mathcal{S}} \frac{\sum_{s_0 \in \mathcal{S}} \mathbf{p}(\hat{o}_t, s \mid \hat{a}_{t-1}, s_0) \cdot b_0(s_0)}{\sum_{(s^{(2)}, s'^{(2)})} \mathbf{p}(\hat{o}_t, s'^{(2)} \mid \hat{a}_{t-1}, s^{(2)}) \cdot b_0(s^{(2)})} \cdot r(s, a_t) \\ &= \sum_{\hat{o}_t} \sum_{(s^{(1)}, s'^{(1)})} \mathbf{p}(\hat{o}_t, s'^{(1)} \mid \hat{a}_{t-1}, s^{(1)}) \cdot b_0(s^{(1)}) \cdot \sum_{s \in \mathcal{S}} b_{b_0}^{\hat{o}_t, \hat{a}_{t-1}}(s) \cdot r(s, a_t) \\ &= \sum_{\hat{o}_t} \mathbb{P}\left(\hat{O}_t = \hat{o}_t \mid \hat{A}_{t-1} = \hat{a}_{t-1}\right) \cdot \sum_{s \in \mathcal{S}} b_{b_0}^{\hat{o}_t, \hat{a}_{t-1}}(s) \cdot r(s, a_t) \\ &= \mathbb{E}\left[\sum_{s \in \mathcal{S}} B_t(s) \cdot r(s, A_t) \mid \hat{A}_t = \hat{a}_t\right] \\ &= \mathbb{E}\left[r(B_t, A_t) \mid \hat{A}_t = \hat{a}_t\right]\end{aligned}$$

where $r(b, a) = \sum_{s \in \mathcal{S}} r(s, a) \cdot b(s)$. ■

A.13 Proof of Theorem 8

Proof: First, $\forall a_t \in \mathcal{A}, \forall o' \in \mathcal{O}$,

$$\begin{aligned} \mathbb{P}(O_{t+1} = o' \mid I_t = i_t, a_t) &= \sum_{s' \in \mathcal{S}} \mathbb{P}(O_{t+1} = o', S_{t+1} = s' \mid I_t = i_t, a_t) \\ &= \sum_{(s, s') \in \mathcal{S}^2} \mathbf{p}(o' \mid s', a_t) \cdot \mathbf{p}(s' \mid s, a_t) \cdot \mathbb{P}(S_t = s \mid I_t = i_t) \\ &= \sum_{(s, s') \in \mathcal{S}^2} \mathbf{p}(o' \mid s', a_t) \cdot \mathbf{p}(s' \mid s, a_t) \cdot b_t(s). \end{aligned}$$

where $b_t = b_{b_0}^{i_t}$ i.e. b_t is the belief obtained starting from b_0 and computed with information i_t . For the sake of readability, the result $\sum_{(s, s') \in \mathcal{S}^2} \mathbf{p}(o' \mid s', a_t) \cdot \mathbf{p}(s' \mid s, a_t) \cdot b_t(s)$ is denoted by $\mathbf{p}(o' \mid b_t, a_t)$. Then,

$$\begin{aligned} \mathbb{P}(B_{t+1} = b' \mid I_t = i_t, a_t) &= \mathbb{E}[\mathbb{1}_{\{B_{t+1}=b'\}} \mid I_t = i_t, a_t] \\ &= \sum_{o' \in \mathcal{O}} \mathbb{P}(O_{t+1} = o' \mid I_t = i_t, a_t) \cdot \mathbb{1}_{\{u(b_{b_0}^{i_t}, a_t, o')=b'\}} \\ &= \sum_{o' \in \mathcal{O}} \sum_{(s, s') \in \mathcal{S}^2} \mathbf{p}(o' \mid s', a_t) \cdot \mathbf{p}(s' \mid s, a_t) \cdot b_t(s) \cdot \mathbb{1}_{\{u(b_t, a_t, o')=b'\}} \\ &= \sum_{o' \in \mathcal{O}} \mathbf{p}(o' \mid b_t, a_t) \cdot \mathbb{1}_{\{u(b_t, a_t, o')=b'\}}, \end{aligned} \quad (21)$$

and thus $\mathbb{P}(B_{t+1} = b' \mid I_t, a_t)$ is $b_{b_0}^{i_t}$ -measurable, i.e. B_t -measurable. Indeed, it is shown that $\mathbb{P}(B_{t+1} = b' \mid I_t = i_t, a_t)$ is a measurable function of $b_t = b_{b_0}^{i_t}$ when $a_t \in \mathcal{A}$ is fixed.

In order to show the equality (I.15) asserting that the belief process is a Markov process, it is sufficient to show the following equation (see Definition A.1): $\forall (b, b') \in (\mathbb{P}_{b_0}^{\mathcal{S}})^2$:

$$\int_{\{B_t=b\}} \mathbb{E}[\mathbb{1}_{\{B_{t+1}=b'\}} \mid I_t, a_t] d\mathbb{P} = \int_{\{B_t=b\}} \mathbb{1}_{\{B_{t+1}=b'\}} d\mathbb{P}. \quad (22)$$

Indeed, as $\mathbb{P}(B_{t+1} = b' \mid I_t, a_t) = \mathbb{E}[\mathbb{1}_{\{B_{t+1}=b'\}} \mid I_t, a_t]$ is B_t -measurable, it remains to show the equality (22) to prove that $\mathbb{E}[\mathbb{1}_{\{B_{t+1}=b'\}} \mid I_t, a_t] = \mathbb{E}[\mathbb{1}_{\{B_{t+1}=b'\}} \mid B_t, a_t]$ \mathbb{P} -almost surely, i.e. to show equality (I.15).

On the one hand, the left part of the equation (22) is

$$\begin{aligned} &\int_{\{B_t=b\}} \sum_{o' \in \mathcal{O}} \mathbf{p}(o' \mid b_t, a_t) \cdot \mathbb{1}_{\{u(b_t, a_t, o')=b'\}} d\mathbb{P} \\ &= \int_{\Omega} \mathbb{1}_{\{B_t=b\}} d\mathbb{P} \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' \mid b, a_t) \cdot \mathbb{1}_{\{u(b, a_t, o')=b'\}}. \end{aligned} \quad (23)$$

thanks to the equality (21).

On the other hand, the right part of equation 22 is

$$\begin{aligned} \int_{\{B_t=b\}} \mathbb{1}_{\{B_{t+1}=b'\}} d\mathbb{P} &= \int_{\Omega} \mathbb{1}_{\{B_t=b\}} \cdot \mathbb{1}_{\{u(b, a_t, O_{t+1})=b'\}} d\mathbb{P} \\ &= \mathbb{E}[\mathbb{1}_{\{B_t=b\}} \cdot \mathbb{1}_{\{u(b, a_t, O_{t+1})=b'\}}] \\ &= \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\{B_t=b\}} \cdot \mathbb{1}_{\{u(b, a_t, O_{t+1})=b'\}} \mid I_t, a_t]\right] \end{aligned} \quad (24)$$

$$= \mathbb{E}\left[\mathbb{1}_{\{B_t=b\}} \cdot \mathbb{E}[\mathbb{1}_{\{u(b, a_t, O_{t+1})=b'\}} \mid I_t, a_t]\right] \quad (25)$$

$$= \mathbb{E}\left[\mathbb{1}_{\{B_t=b\}} \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' \mid b, a_t) \cdot \mathbb{1}_{\{u(b, a_t, o')=b'\}}\right], \quad (26)$$

which is also equal to result (23). Line (24) comes from Definition A.1. Line (25) comes from Property A.1 and the fact that B_t (and thus $\mathbb{1}_{\{B_t=b\}}$) is $\sigma(I_t)$ -measurable. The last line (26) is given by the result (21).

The belief process $(B_t)_{t \geq 0}$ is thus a Markov process. ■

A.14 Proof of Theorem 9

Proof: Let $V : \mathbb{P}_{b_0}^S \rightarrow \mathbb{R}$ be a PWLC function. Then, $\exists \Gamma \subset \mathbb{R}^S$, $\#\Gamma < +\infty$ such that

$$V(b) = \max_{\alpha \in \Gamma} \left\{ \sum_{s \in S} b(s) \cdot \alpha(s) \right\} = \max_{\alpha \in \Gamma} \langle \alpha, b \rangle_{\mathbb{R}^S}.$$

For $b \in \mathbb{P}_{b_0}^S$, the Dynamic Programming Equation is

$$\begin{aligned} (\mathcal{B}^*V)(b) &= \max_{a \in \mathcal{A}} \left\{ \sum_{s \in S} r(s, a) \cdot b(s) + \gamma \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' | b, a) \cdot V(u(b, a, o')) \right\} \\ &= \max_{a \in \mathcal{A}} \left\{ \sum_{s \in S} r(s, a) \cdot b(s) + \gamma \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' | b, a) \cdot \max_{\alpha \in \Gamma} \left\{ \sum_{s' \in S} u(b, a, o')(s') \cdot \alpha(s') \right\} \right\} \\ &= \max_{a \in \mathcal{A}} \left\{ \sum_{s \in S} r(s, a) \cdot b(s) + \max_{(\alpha_o)_{o \in \mathcal{O}} \in \Gamma^{\mathcal{O}}} \left\{ \gamma \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' | b, a) \cdot \sum_{s' \in S} u(b, a, o')(s') \cdot \alpha_{o'}(s') \right\} \right\} \\ &= \max_{a \in \mathcal{A}} \max_{(\alpha_o)_{o \in \mathcal{O}} \in \Gamma^{\mathcal{O}}} \left\{ \sum_{s \in S} r(s, a) \cdot b(s) + \left\{ \gamma \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' | b, a) \cdot \sum_{s' \in S} u(b, a, o')(s') \cdot \alpha_{o'}(s') \right\} \right\} \end{aligned}$$

where $(\alpha_o)_{o \in \mathcal{O}}$ is an element of $\Gamma^{\mathcal{O}}$ i.e. is a set of vectors from $\Gamma \subset \mathbb{R}^S$: each vector $\alpha_o \in \Gamma$ is indexed by an observation $o \in \mathcal{O}$.

Thereafter, given a belief $b \in \tilde{\mathcal{S}}$ and an action $a \in \mathcal{A}$, we use the notation $\mathbf{p}(s' | b, a) = \sum_{s \in S} \mathbf{p}(s' | s, a) \cdot b(s)$. Then, the belief update (I.10) becomes $\forall s' \in S$,

$$u(b, a, o')(s') = \frac{\mathbf{p}(o' | s', a) \cdot \mathbf{p}(s' | b, a)}{\mathbf{p}(o' | b, a)}.$$

The result applying \mathcal{B}^* to V is then

$$\begin{aligned} (\mathcal{B}^*V)(b) &= \max_{a \in \mathcal{A}} \max_{(\alpha_{o'})_{o' \in \mathcal{O}} \in \Gamma^{\mathcal{O}}} \sum_{s \in S} r(s, a) \cdot b(s) + \gamma \cdot \sum_{o' \in \mathcal{O}} \mathbf{p}(o' | b, a) \cdot \sum_{s' \in S} \frac{\mathbf{p}(o' | s', a) \cdot \mathbf{p}(s' | b, a)}{\mathbf{p}(o' | b, a)} \cdot \alpha_{o'}(s') \\ &= \max_{a \in \mathcal{A}, (\alpha_{o'})_{o' \in \mathcal{O}} \in \Gamma^{\mathcal{O}}} \sum_{s \in S} r(s, a) \cdot b(s) + \gamma \cdot \sum_{o' \in \mathcal{O}} \sum_{s' \in S} \mathbf{p}(o' | s', a) \cdot \sum_{s \in S} \mathbf{p}(s' | s, a) \cdot b(s) \cdot \alpha_{o'}(s') \\ &= \max_{a \in \mathcal{A}, (\alpha_{o'})_{o' \in \mathcal{O}} \in \Gamma^{\mathcal{O}}} \sum_{s \in S} \left(r(s, a) + \gamma \cdot \sum_{o' \in \mathcal{O}} \sum_{s' \in S} \mathbf{p}(o' | s', a) \cdot \mathbf{p}(s' | s, a) \cdot \alpha_{o'}(s') \right) \cdot b(s) \\ &= \max_{\alpha' \in \Gamma'} \sum_{s \in S} \alpha'(s) \cdot b(s) = \max_{\alpha' \in \Gamma'} \langle \alpha', b \rangle_{\mathbb{R}^S}. \end{aligned}$$

where

$$\Gamma' = \left\{ \alpha'(s) = r(s, a) + \gamma \cdot \sum_{o' \in \mathcal{O}} \sum_{s' \in S} \mathbf{p}(o' | s', a) \cdot \mathbf{p}(s' | s, a) \cdot \alpha_{o'}(s') \mid a \in \mathcal{A}, \text{ and } \forall o' \in \mathcal{O}, \alpha_{o'} \in \Gamma \right\}$$

which is finite, and which size is $\#\mathcal{A} \cdot (\#\Gamma)^{\#\mathcal{O}}$.

The function $b \mapsto \mathcal{B}^*V(b)$ is thus PWLC and elements of Γ' are called α -vectors. ■

A.15 Proof of Theorem 10

Proof: First, note that

$$\begin{aligned}
& \forall (x, y) \in \mathcal{X} \times \mathcal{Y}, \quad \pi(x, y) < \min \{ \pi(x), \pi(y) \} \\
& \quad \Updownarrow \\
& \forall (x, y) \in \mathcal{X} \times \mathcal{Y}, \quad \pi(x, y) < \max_{x'} \pi(x', y) = \pi(y) \text{ and } \pi(x, y) < \max_{y'} \pi(x, y') = \pi(x) \\
& \quad \Updownarrow \\
& \forall (x, y) \in \mathcal{X} \times \mathcal{Y}, \quad \pi(x | y) = \pi(x, y) < \pi(x) \text{ and } \pi(y | x) = \pi(x, y) < \pi(y)
\end{aligned}$$

Thus, as $\pi(x, y) \leq \min \{ \pi(x), \pi(y) \}$ is always true, $\forall (x, y) \in \mathcal{X} \times \mathcal{Y}$, $\pi(x, y) = \min \{ \pi(x), \pi(y) \}$
 $\Leftrightarrow \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$, $\pi(x | y) \geq \pi(x)$ or $\pi(y | x) \geq \pi(y)$. ■

A.16 Proof of Theorem 11

Proof: Let us suppose that X and Y are MS-independent: in terms of possibility distributions, it can be written $\forall x \in \mathcal{X}, \forall y \in \mathcal{Y}, \pi(x) = \pi(x | y)$ and $\pi(y) = \pi(y | x)$. Suppose also that $\exists x_0 \in \mathcal{X}$ such that $\pi(x_0) < 1$ i.e. variable X is not fully unknown. Thus, using the qualitative possibilistic conditioning (Definition I.2.7), we know that, since $\pi(x_0 | y) = \pi(x_0) < 1, \forall y \in \mathcal{Y}$, $\pi(x_0 | y) = \pi(x_0, y)$. Thus, $\pi(x_0, y) < 1$. Using also the conditioning of Definition I.2.7, we get that $\forall y \in \mathcal{Y}, \pi(y | x_0) = 1$, since the equality $\pi(x_0) = \pi(x_0, y)$ is the condition which leads to the possibility degree 1. As $\forall x \in \mathcal{X}, \forall y \in \mathcal{Y}, \pi(y) = \pi(y | x), \forall y \in \mathcal{Y}, \pi(y) = 1$. As a conclusion, if X is not fully unknown, Y is fully unknown: thus, if Y is not fully unknown, X is fully unknown. ■

A.17 Proof of Property I.2.1

Proof: First, if $f^* = \min_{\omega \in \Omega} f(\omega)$, then $\forall \omega \in \Omega, f^* \leq f(\omega)$. Thus $\forall \omega \in \Omega, 1 - f^* \geq 1 - f(\omega)$, i.e. $1 - f^* = 1 - \min_{\omega \in \Omega} f(\omega) = \max_{\omega \in \Omega} \{ 1 - f(\omega) \}$: the equation (I.26) is proved. The equation (I.27) can be shown using the previous one: setting $f = 1 - g$, we get $\forall g : \Omega \rightarrow \mathcal{L}$ $\max_{\omega \in \Omega} g(\omega) = 1 - \min_{\omega \in \Omega} \{ 1 - g(\omega) \}$ i.e. it equality is shown.

Now, the equation (I.30) is shown considering the case where $\lambda \geq \max_{\omega \in \Omega} f(\omega)$: on the one hand, in this case, both parts of the equality are equal to $\max_{\omega \in \Omega} f(\omega)$ since $\forall \omega \in \Omega, \lambda \geq f(\omega)$. On the other hand, when $\lambda \leq \max_{\omega \in \Omega} f(\omega)$, right part of equation (I.30) is equal to λ . As $\forall \omega \in \Omega, \min \{ \lambda, f(\omega) \} \leq \lambda$, and for each ω such that $f(\omega) \geq \lambda$, $\min \{ \lambda, f(\omega) \} = \lambda$, the maximum is equal to λ : the left part of the equation (I.30) is equal to λ too. The equation (I.29) can be proved using the previous equation: setting f to $1 - g$, and λ to $1 - \mu$, we get $\max_{\omega \in \Omega} \min \{ 1 - g(\omega), 1 - \mu \} = \min \{ \max_{\omega \in \Omega} \{ 1 - g(\omega) \}, 1 - \mu \}$. And then, using equations (I.26) and (I.27), $\forall \mu \in \mathcal{L}, \forall g : \Omega \rightarrow \mathcal{L}, 1 - \min_{\omega \in \Omega} \max \{ g(\omega), \mu \} = 1 - \max \{ \min_{\omega \in \Omega} g(\omega), \mu \}$, i.e. the equation is shown.

The equations (I.28) and (I.32) are trivial: with the operator \min (resp. \max), no matter what is the order of elements and the repetitions, as they are associative¹.

The inclusions (I.31) and (I.33) are shown as follows: let $\omega^* \in \operatorname{argmax}_{\omega \in \Omega} f(\omega)$. If $f(\omega^*) \leq \lambda$, $\forall \omega \in \Omega, \min \{ f(\omega), \lambda \} = f(\omega)$ and thus $\operatorname{argmax}_{\omega \in \Omega} \min \{ f(\omega), \lambda \} = \operatorname{argmax}_{\omega \in \Omega} f(\omega)$. Now, if $f(\omega^*) > \lambda$, $\max_{\omega \in \Omega} \min \{ f(\omega), \lambda \} = \lambda$, and then $\operatorname{argmax}_{\omega \in \Omega} \min \{ f(\omega), \lambda \} = \{ \omega | f(\omega) \geq \lambda \}$: thus $\operatorname{argmax}_{\omega \in \Omega} f(\omega) \subseteq \operatorname{argmax}_{\omega \in \Omega} \min \{ f(\omega), \lambda \}$, i.e. the inclusion (I.31) is shown. For the inclusion (I.33), if $f(\omega^*) > \lambda$, then the maximizing elements are in $\omega \{ \omega | f(\omega) > \lambda \}$, where $f(\omega) = \max \{ f(\omega), \lambda \}$. Thus maximizing elements are the same $\operatorname{argmax}_{\omega \in \Omega} f(\omega) = \operatorname{argmax}_{\omega \in \Omega} \max \{ f(\omega), \lambda \}$. Otherwise, if $f(\omega^*) \leq \lambda$, $\forall \omega \in \Omega, \max \{ f(\omega), \lambda \} = \lambda$, then $\operatorname{argmax}_{\omega \in \Omega} \max \{ f(\omega), \lambda \} = \Omega$, and thus obviously $\operatorname{argmax}_{\omega \in \Omega} f(\omega) \subseteq \operatorname{argmax}_{\omega \in \Omega} \max \{ f(\omega), \lambda \}$.

¹An operator $*$ over \mathcal{L} is associative if $\forall (\lambda_1, \lambda_2, \lambda_3) \in \mathcal{L}^3, (\lambda_1 * \lambda_2) * \lambda_3 = \lambda_1 * (\lambda_2 * \lambda_3)$.

Finally, let us show the equality (I.34). Note first that functions $A : \omega \mapsto \max \{ \min \{ \lambda, f(\omega) \}, g(\omega) \}$ and $B : \omega \mapsto \min \{ \lambda, \max \{ f(\omega), g(\omega) \} \}$ are equal to $\omega \mapsto \max \{ f(\omega), g(\omega) \}$ on the set $\Omega_1 = \{ \omega \mid \lambda \geq \max \{ f(\omega), g(\omega) \} \}$. It is trivial for B , and as $\lambda \geq f(\omega)$ on this set, the result comes for A . On the set $\Omega_2 = \overline{\Omega_1} = \{ \omega \mid \lambda \leq \max \{ f(\omega), g(\omega) \} \}$, B is of course equal to λ , and if $f(\omega) \geq \lambda$, $A(\omega) = \max \{ \lambda, g(\omega) \} \geq \lambda$, otherwise if $f(\omega) \leq \lambda$, $A(\omega) = \max \{ f(\omega), g(\omega) \} \geq \lambda$ (on the set Ω_2). A and B are thus smaller on Ω_1 . Indeed, they are both equal to $\omega \mapsto \max \{ f(\omega), g(\omega) \} \leq \lambda$ on Ω_1 , whereas on Ω_2 , $N = \lambda$ and $A \geq \lambda$. Thus, if $\Omega_1 \neq \emptyset$, then the result is shown: the minimum is on Ω_1 , where functions are equal. Otherwise, if $\Omega_1 = \emptyset$, $\forall \omega \in \Omega$, $\lambda \leq \max \{ f(\omega), g(\omega) \}$. As $g(\omega^*) = 0$, then $f(\omega^*) \geq \lambda$, and $A(\omega^*) = \max \{ \lambda, g(\omega^*) \} = \lambda = B(\omega^*)$. For the other $\omega \in \Omega$, $A(\omega) = \max \{ \lambda, g(\omega) \} \geq \lambda$ or $A(\omega) = \max \{ f(\omega), g(\omega) \} \geq \lambda$, thus the minimum of both functions, λ , is reached with ω^* , where A and B are equal. The equality (I.35) can be proved using the equality (I.34): if we set $g = 1 - h$, g fulfills the condition $\exists \omega^* \in \Omega$ such that $g(\omega^*) = 0$. Setting $\lambda = 1 - \mu$ and $f' = 1 - f$, we get

$$\min_{\omega \in \Omega} \max \left\{ \min \{ 1 - \mu, 1 - f'(\omega) \}, 1 - h(\omega) \right\} = \min_{\omega \in \Omega} \min \left\{ 1 - \mu, \max \{ 1 - f'(\omega), 1 - h(\omega) \} \right\}.$$

Using equations (I.26) and (I.27) of Property I.2.1, we get

$$1 - \max_{\omega \in \Omega} \min \left\{ \max \{ \mu, f'(\omega) \}, h(\omega) \right\} = 1 - \max_{\omega \in \Omega} \max \left\{ \mu, \min \{ f'(\omega), h(\omega) \} \right\},$$

i.e. one minus equation (I.35). ■

A.18 Proof of the equality of Definition I.2.11

Proof: Let us show that (I.36) is equal to (I.37). Note first that, by definition, $f(\omega_i)$ is non-decreasing with $i \in \{1, \dots, \#\Omega\}$. Note as well that μ is monotone, and then $\mu(A_i) = \mu(\{\omega_i, \dots, \omega_{\#\Omega}\}) \geq \mu(A_{i+1})$ since $A_{i+1} \subset A_i$. Let i^* be the highest $i \in \{1, \dots, \#\Omega\}$ such that $\mu(A_{i^*}) \geq f(\omega_{i^*})$. As $\mu(A_1) = \mu(\Omega) = 1 \geq f(\omega_1)$, i^* exists. For each $i \leq i^*$, $\min \{ f(\omega_i), \mu(A_i) \} = f(\omega_i)$, and for each $i > i^*$, $\min \{ f(\omega_i), \mu(A_i) \} = \mu(A_i)$, thanks to the definition of i^* . As $f(\omega_i)$ is non-decreasing and $\mu(A_i)$ is non-increasing with i , highest values of $\left(\min \{ f(\omega_i), \mu(A_i) \} \right)_{i=1}^{\#\Omega-1}$ are $f(\omega_{i^*})$ and $\mu(A_{i^*+1})$.

If $f(\omega_{i^*}) \leq \mu(A_{i^*+1})$, then (I.36) is equal to $\mu(A_{i^*+1})$. As well, $\max \{ f(\omega_{i^*}), \mu(A_{i^*+1}) \} = \mu(A_{i^*+1})$. Using the definition of i^* , and as μ is monotone, $f(\omega_{i^*+1}) > \mu(A_{i^*+1}) \geq \mu(A_{i^*+2})$. This implies that $\max \{ f(\omega_{i^*+1}), \mu(A_{i^*+2}) \} = f(\omega_{i^*+1})$. As $\mu(A_{i+1})$ is non-increasing with i , and $f(\omega_i)$ non-decreasing, $\mu(A_{i^*+1})$ and $f(\omega_{i^*+1})$ are the lowest values of $\left(\max \{ f(\omega_i), \mu(A_{i+1}) \} \right)_{i=1}^{\#\Omega-1}$. By definition of i^* , $f(\omega_{i^*+1}) > \mu(A_{i^*+1})$, and thus formula (I.37) is also equal to $\mu(A_{i^*+1})$: (I.36) and (I.37) are equal.

If $f(\omega_{i^*}) > \mu(A_{i^*+1})$, then formula (I.36) is equal to $f(\omega_{i^*})$, and $\max \{ f(\omega_{i^*}), \mu(A_{i^*+1}) \} = f(\omega_{i^*})$. As $f(\omega_i)$ is non-decreasing with i , and thanks to the definition of i^* , $f(\omega_{i^*-1}) \leq f(\omega_{i^*}) \leq \mu(A_{i^*})$ and thus $\max \{ f(\omega_{i^*-1}), \mu(A_{i^*}) \} = \mu(A_{i^*})$. As previously ($\mu(A_{i+1})$ non-increasing and $f(\omega_i)$ non-decreasing), $\mu(A_{i^*})$ and $f(\omega_{i^*})$ are the lowest values of $\left(\max \{ f(\omega_i), \mu(A_{i+1}) \} \right)_{i=1}^{\#\Omega-1}$. By definition of i^* , $f(\omega_{i^*}) \leq \mu(A_{i^*})$, and thus formula (I.37) is also equal to $f(\omega_{i^*})$.

Finally, formula (I.36) and (I.37) are equal. ■

A.19 Proof of Theorem 12

Proof: First, let us rewrite the Sugeno integral of a function $f : \Omega \rightarrow \mathcal{L}$ with respect to a possibility measure Π , using formula (I.36):

$$\begin{aligned} \mathbb{S}_{\Pi}[f] &= \max_{i=1}^{\#\Omega} \min \{ f(\omega_i), \Pi(A_i) \} \\ &= \max_{i=1}^{\#\Omega} \min \left\{ f(\omega_i), \max_{j=i}^{\#\Omega} \pi(\omega_j) \right\} \\ &= \max_{\substack{(i,j) \in \{1, \dots, \#\Omega\}^2 \\ \text{s.t. } i \leq j}} \min \{ f(\omega_i), \pi(\omega_j) \}. \end{aligned} \quad (27)$$

Now, note that $\forall (i, j) \in \{1, \dots, \#\Omega\}^2$ such that $i < j$, $\min \{ f(\omega_i), \pi(\omega_j) \} \leq \min \{ f(\omega_j), \pi(\omega_j) \}$ since $f(\omega_i)$ is non-decreasing with i . As just shown, such pairs have a minimum lower than the minimum of an other pair: such pairs can thus be removed from the maximum operator in the equation (27): we get then the formula (I.38).

As well, using formula (I.37), the Sugeno integral of f with respect to a necessity measure \mathcal{N} is

$$\begin{aligned} \mathbb{S}_{\mathcal{N}}[f] &= \min_{i=1}^{\#\Omega} \max \{ f(\omega_i), \mathcal{N}(A_{i+1}) \} \\ &= \min_{i=1}^{\#\Omega} \max \left\{ f(\omega_i), 1 - \Pi(\{\omega_1, \dots, \omega_i\}) \right\} \\ &= \min_{i=1}^{\#\Omega} \max \left\{ f(\omega_i), \min_{j=1}^i \{ 1 - \pi(\omega_j) \} \right\} \\ &= \min_{\substack{(i,j) \in \{1, \dots, \#\Omega\}^2 \\ \text{s.t. } i \geq j}} \min \{ f(\omega_i), 1 - \pi(\omega_j) \}. \end{aligned} \quad (28)$$

Note that the terms of $\min_{\substack{(i,j) \in \{1, \dots, \#\Omega\}^2 \\ \text{s.t. } i \geq j}} \min \{ f(\omega_i), 1 - \pi(\omega_j) \}$, such that $i > j$, *i.e.* $\max \{ f(\omega_i), 1 - \pi(\omega_j) \}$, are greater or equal to $\max \{ f(\omega_j), 1 - \pi(\omega_j) \}$ (as $f(\omega_i)$ is non-decreasing with i) and can be removed from min: the equation (I.39) is then deduced. ■

A.20 Proof of Theorem 13

Proof: For the case $i = 0$, $\overline{U}_0^*(s) = \Psi(s)$, is obvious since no action has to be chosen. The case $i = 1$ consists in applying the formula (I.30) of Property I.2.1. Following sequence of equalities come from properties (I.2.1). Let i be in $\{2, \dots, H-1\}$ and $j = H - i$, and \mathcal{T}_i the set of i -length system state trajectories $\mathcal{T} = (s_{j+1}, \dots, s_H): \forall s_j \in \mathcal{S}$,

$$\overline{U}_i^*(s_j)$$

$$\begin{aligned}
&= \max_{(\delta) \in \Delta_i} \max_{\mathcal{T} \in \mathcal{T}_i} \min \left\{ \min_{t=j}^H \rho_t(s_t, \delta_t(s_t)), \min_{t=j}^{H-1} \pi_t(s_{t+1} | s_t, \delta_t(s_t)) \right\} \\
&= \max_{(\delta) \in \Delta_i} \max_{s_{j+1} \in \mathcal{S}} \max_{\mathcal{T} \in \mathcal{T}_{i-1}} \min \left\{ \min \left\{ \rho_j(s_j, \delta_j(s_j)), \min_{t=j+1}^H \rho_t(s_t, \delta_t(s_t)) \right\}, \min_{t=j}^{H-1} \pi_t(s_{t+1} | s_t, \delta_t(s_t)) \right\} \\
&= \max_{(\delta) \in \Delta_i} \max_{s_{j+1} \in \mathcal{S}} \min \left\{ \rho_j(s_j, \delta_j(s_j)), \max_{\mathcal{T} \in \mathcal{T}_{i-1}} \min \left\{ \min_{t=j+1}^H \rho_t(s_t, \delta_t(s_t)), \min_{t=j}^{H-1} \pi_t(s_{t+1} | s_t, \delta_t(s_t)) \right\} \right\} \\
&= \max_{(\delta) \in \Delta_i} \max_{s_{j+1} \in \mathcal{S}} \min \left\{ \rho_j(s_j, \delta_j(s_j)), \min \left\{ \pi_j(s_{j+1} | s_j, \delta_j(s_j)), \right. \right. \\
&\quad \left. \left. \max_{\mathcal{T} \in \mathcal{T}_{i-1}} \min \left\{ \min_{t=j+1}^H \rho_t(s_t, \delta_t(s_t)), \min_{t=j+1}^{H-1} \pi_t(s_{t+1} | s_t, \delta_t(s_t)) \right\} \right\} \right\} \\
&= \max_{\delta_j(s_j) \in \mathcal{A}} \max_{s_{j+1} \in \mathcal{S}} \min \left\{ \rho_j(s_j, \delta_j(s_j)), \min \left\{ \pi_j(s_{j+1} | s_j, \delta_j(s_j)), \right. \right. \\
&\quad \left. \left. \max_{(\delta) \in \Delta_{i-1}} \max_{\mathcal{T} \in \mathcal{T}_{i-1}} \min \left\{ \min_{t=j+1}^H \rho_t(s_t, \delta_t(s_t)), \min_{t=j+1}^{H-1} \pi_t(s_{t+1} | s_t, \delta_t(s_t)) \right\} \right\} \right\} \\
&= \max_{a \in \mathcal{A}} \max_{s_{j+1} \in \mathcal{S}} \min \left\{ \rho_j(s_j, a), \min \left\{ \pi_j(s_{j+1} | s_j, a), \overline{U_{i-1}^*}(s_1) \right\} \right\} \\
&= \max_{a \in \mathcal{A}} \min \left\{ \rho_j(s_j, a), \max_{s_{j+1} \in \mathcal{S}} \min \left\{ \pi_j(s_{j+1} | s_j, a), \overline{U_{i-1}^*}(s_1) \right\} \right\}.
\end{aligned}$$

where the final preference function $\Psi(s)$ is denoted by $\rho_H(s, a)$ to simplify equations. This shows that the optimistic value function can be computed using the recursive formula (I.54). The strategy computed with formula (I.55) is indeed optimal, thanks to the inclusion (I.30) of Property I.2.1.

As well, for the pessimistic criterion, the case $i = 0$ is obvious too, and the case $i = 1$ consists in applying the formulae (I.26) and (I.29) of Property I.2.1. For $i \in \{2, \dots, H-1\}$, and $j = H-i$, using Property I.2.1,

$$\underline{U}_H^*(s_j)$$

$$\begin{aligned}
&= \max_{(\delta) \in \Delta_i} \min_{\mathcal{T} \in \mathcal{T}_i} \max \left\{ \min_{t=j}^H \rho_t(s_t, \delta_t(s_t)), 1 - \min_{t=j}^{H-1} \pi_t(s_{t+1} | s_t, \delta_t(s_t)) \right\} \\
&= \max_{(\delta) \in \Delta_i} \min_{s_{j+1} \in \mathcal{S}} \min_{\mathcal{T} \in \mathcal{T}_{i-1}} \max \left\{ \min \left\{ \rho_j(s_j, \delta_j(s_j)), \min_{t=j+1}^H \rho_t(s_t, \delta_t(s_t)) \right\}, 1 - \min_{t=j}^{H-1} \pi_t(s_{t+1} | s_t, \delta_t(s_t)) \right\} \\
&= \max_{(\delta) \in \Delta_i} \min_{s_{j+1} \in \mathcal{S}} \min \left\{ \rho_j(s_j, \delta_j(s_j)), \min_{\mathcal{T} \in \mathcal{T}_{i-1}} \max \left\{ \min_{t=j+1}^H \rho_t(s_t, \delta_t(s_t)), 1 - \min_{t=j}^{H-1} \pi_t(s_{t+1} | s_t, \delta_t(s_t)) \right\} \right\} \quad (29) \\
&= \max_{(\delta) \in \Delta_i} \min_{s_{j+1} \in \mathcal{S}} \min \left\{ \rho_j(s_j, \delta_j(s_j)), \min_{\mathcal{T} \in \mathcal{T}_{i-1}} \max \left\{ \min_{t=j+1}^H \rho_t(s_t, \delta_t(s_t)), \max_{t=j}^{H-1} \left\{ 1 - \pi_t(s_{t+1} | s_t, \delta_t(s_t)) \right\} \right\} \right\} \\
&= \max_{(\delta) \in \Delta_i} \min_{s_{j+1} \in \mathcal{S}} \min \left\{ \rho_j(s_j, \delta_j(s_j)), \max \left\{ 1 - \pi_j(s_{j+1} | s_j, \delta_j(s_j)), \right. \right. \\
&\quad \left. \left. \min_{\mathcal{T} \in \mathcal{T}_{i-1}} \max \left\{ \min_{t=j+1}^H \rho_t(s_t, \delta_t(s_t)), 1 - \min_{t=j+1}^{H-1} \pi_t(s_{t+1} | s_t, \delta_t(s_t)) \right\} \right\} \right\} \\
&= \max_{\delta_j(s_j) \in \mathcal{A}} \min_{s_{j+1} \in \mathcal{S}} \min \left\{ \rho_j(s_j, \delta_j(s_j)), \max \left\{ 1 - \pi_j(s_{j+1} | s_j, \delta_j(s_j)), \right. \right. \quad (30) \\
&\quad \left. \left. \max_{(\delta) \in \Delta_{i-1}} \min_{\mathcal{T} \in \mathcal{T}_{i-1}} \max \left\{ \min_{t=j+1}^H \rho_t(s_t, \delta_t(s_t)), 1 - \min_{t=j+1}^{H-1} \pi_t(s_{t+1} | s_t, \delta_t(s_t)) \right\} \right\} \right\} \\
&= \max_{a \in \mathcal{A}} \min_{s_{j+1} \in \mathcal{S}} \min \left\{ \rho_j(s_j, a), \max \left\{ 1 - \pi_j(s_{j+1} | s_j, a), \underline{U}_{H-1}^*(s_{j+1}) \right\} \right\} \\
&= \max_{a \in \mathcal{A}} \min \left\{ \rho_j(s_j, a), \min_{s_{j+1} \in \mathcal{S}} \max \left\{ 1 - \pi_j(s_{j+1} | s_j, a), \underline{U}_{H-1}^*(s_{j+1}) \right\} \right\}.
\end{aligned}$$

where the equation (I.34) of Property I.2.1 is used for the row (29). The row (30) is explained by the following result: consider the function $F : \mathcal{S} \times \mathcal{A} \rightarrow \mathcal{L}$. Then,

$$\max_{\delta: \mathcal{S} \rightarrow \mathcal{L}} \min_{s \in \mathcal{S}} F(s, \delta(s)) = \min_{s \in \mathcal{S}} \max_{\delta: \mathcal{S} \rightarrow \mathcal{A}} F(s, \delta(s)).$$

Indeed, $\forall \delta : \mathcal{S} \rightarrow \mathcal{A}, \forall s \in \mathcal{S}, \min_{s' \in \mathcal{S}} F(s', \delta(s')) \leq \max_{\delta': \mathcal{S} \rightarrow \mathcal{A}} F(s, \delta'(s))$, and thus, $\max_{\delta: \mathcal{S} \rightarrow \mathcal{A}} \min_{s' \in \mathcal{S}} F(s', \delta(s')) \leq \min_{s \in \mathcal{S}} \max_{\delta': \mathcal{S} \rightarrow \mathcal{A}} F(s, \delta'(s))$. Now, consider $\delta^* : \mathcal{S} \rightarrow \mathcal{A}$ such that $\forall s \in \mathcal{S}, \delta^*(s) \in \operatorname{argmax}_{a \in \mathcal{A}} F(s, a)$. Then, $\forall \delta : \mathcal{S} \rightarrow \mathcal{A}, \min_{s \in \mathcal{S}} F(s, \delta^*(s)) \geq \min_{s \in \mathcal{S}} F(s, \delta(s))$, which implies that $\min_{s \in \mathcal{S}} F(s, \delta^*(s)) \geq \max_{\delta: \mathcal{S} \rightarrow \mathcal{A}} \min_{s \in \mathcal{S}} F(s, \delta(s))$, i.e. $\min_{s \in \mathcal{S}} \max_{\delta: \mathcal{S} \rightarrow \mathcal{A}} F(s, \delta(s)) \geq \max_{\delta: \mathcal{S} \rightarrow \mathcal{A}} \min_{s \in \mathcal{S}} F(s, \delta(s))$: the equality is shown.

Finally, the strategy computation (I.57) is explained by inclusions (I.31) and (I.33) of properties (I.2.1). ■

A.21 Proof of Theorem 14

Proof: Suppose that the belief state at time step $t \in \mathbb{N}$ is β_t . The joint distribution of the system state variable S_{t+1} and the observation variable O_{t+1} is

$$\begin{aligned}
& \Pi(S_{t+1} = s', O_{t+1} = o' \mid I_t = i_t, a) \\
&= \max_{s \in \mathcal{S}} \Pi(S_{t+1} = s', O_{t+1} = o', S_t = s \mid I_t = i_t, a_t) \quad (31) \\
&= \max_{s \in \mathcal{S}} \min \left\{ \Pi(S_{t+1} = s', O_{t+1} = o' \mid S_t = s, I_t = i_t, a_t), \Pi(S_t = s \mid I_t = i_t, a_t) \right\} \quad (32) \\
&= \max_{s \in \mathcal{S}} \min \left\{ \Pi(O_{t+1} = o' \mid S_{t+1} = s', a_t), \Pi(S_{t+1} = s' \mid S_t = s, a_t), \beta_t(s) \right\} \quad (33) \\
&= \min \left\{ \Pi(O_{t+1} = o' \mid S_{t+1} = s', a_t), \max_{s \in \mathcal{S}} \min \left\{ \Pi(S_{t+1} = s' \mid S_t = s, a_t), \beta_t(s) \right\} \right\} \quad (34) \\
&= \min \left\{ \pi_t(o' \mid s', a_t), \max_{s \in \mathcal{S}} \min \left\{ \pi_t(s' \mid s, a_t), \beta_t(s) \right\} \right\},
\end{aligned}$$

denoted by $\pi_t(s', o' \mid \beta_t, a_t)$ to simplify notations: $\max_{s' \in \mathcal{S}} \pi_t(s', o' \mid \beta_t, a_t)$ is also denoted by $\pi(o' \mid \beta_t, a_t)$. Line (31) is the possibilistic marginalization over variable S_t . Line (32) is due to the definition of the conditioning, Definition I.2.7. Line (33) uses the Definition of the belief state, Definition I.2.16, and that S_t does not depend on the action a_t . Finally, line (34) is comes from equation (I.30) of Property I.2.1.

Suppose now that the observation received at time step $t + 1$ is o_{t+1} : as, by definition, $\beta_{t+1}(s') = \Pi(S_{t+1} = s' \mid I_t = i_t, O_{t+1} = o_{t+1}, a_t)$, using the qualitative possibilistic conditioning (Definition I.2.7), we conclude that, $\forall s \in \mathcal{S}$,

$$\beta_{t+1}(s') = \begin{cases} 1 & \text{if } \pi_t(s', o_{t+1} \mid \beta_t, a_t) = \pi_t(o_{t+1} \mid \beta_t, a_t), \\ \pi_t(s', o_{t+1} \mid \beta_t, a_t) & \text{otherwise.} \end{cases} \quad \blacksquare$$

A.22 Proof of Theorem 15

Proof: Let us denote by $\pi(s_H, \widehat{o}_H \mid \beta_0, (\delta))$ the joint possibility degree of the last system state $s_H \in \mathcal{S}$ and the observation sequence \widehat{o}_H when the strategy is $(\delta) = (\delta)_{t=0}^{H-1}$:

$$\begin{aligned}
& \Pi(S_H = s_H, \widehat{O}_H = \widehat{o}_H \mid (\delta)) \\
&= \max_{(s_0, \dots, s_{H-1}) \in \mathcal{S}^H} \min_{t=0}^{H-1} \min \left\{ \pi_t(o_{t+1} \mid s_{t+1}, \delta_t(i_t)), \pi_t(s_{t+1} \mid s_t, \delta_t(i_t)), \beta_0(s_0) \right\}.
\end{aligned}$$

The possibility distribution over the observation sequence is also denoted by

$$\pi(\widehat{o}_H \mid \beta_0, (\delta)) = \max_{s_H \in \mathcal{S}} \pi(s_H, \widehat{o}_H \mid \beta_0, (\delta)),$$

and the one over the last state $s_H \in \mathcal{S}$ is denoted by

$$\pi(s_H \mid \beta_0, (\delta)) = \max_{\widehat{o}_H} \pi(s_H, \widehat{o}_H \mid \beta_0, (\delta)).$$

By definition, the qualitative belief state at the end of the execution, *i.e.* at time step $t = H$, is equal to $\beta_{\beta_0}^{\delta, \widehat{o}_H}(s) = \Pi(S_H = s \mid \widehat{O}_H = \widehat{o}_H, (\delta_t(i_t))_{t=0}^{H-1})$, see Definition I.2.16. It can be written as a function of $\widehat{o}_H, (\delta)$ and β_0 , and this belief state is then denoted by $\beta_{\beta_0}^{\delta, \widehat{o}_H}$:

$$\beta_{\beta_0}^{\delta, \widehat{o}_H}(s) = \begin{cases} 1 & \text{if } \pi(s, \widehat{o}_H \mid \beta_0, (\delta)) = \pi(\widehat{o}_H \mid \beta_0, (\delta)), \\ \pi(s, \widehat{o}_H \mid \beta_0, (\delta)) & \text{otherwise,} \end{cases}$$

using the possibilistic conditioning, Definition I.2.7.

Following lines show the desired equality for the optimistic criterion: its rewriting, denoted by $\mathbb{S}_\Pi \left[\max_{s \in \mathcal{S}} \min \{ \Psi(s), B_H^\pi(s) \} \mid \beta_0, (\delta) \right]$, is equal to

$$\begin{aligned} & \max_{\widehat{o}_H} \min \left\{ \max_{s \in \mathcal{S}} \min \left\{ \Psi(s), \beta_{b_0}^{\delta, \widehat{o}_H}(s) \right\}, \pi(\widehat{o}_H \mid \beta_0, (\delta)) \right\} \\ &= \max_{\widehat{o}_H, s \in \mathcal{S}} \min \left\{ \Psi(s), \beta_{b_0}^{\delta, \widehat{o}_H}(s), \pi(\widehat{o}_H \mid \beta_0, (\delta)) \right\} \end{aligned} \quad (35)$$

$$= \max_{s \in \mathcal{S}} \min \left\{ \Psi(s), \max_{\widehat{o}_H} \min \left\{ \Pi \left(S_H = s \mid \widehat{O}_H = \widehat{o}_H, \left(\delta_t(i_t) \right)_{t=0}^{H-1} \right), \pi(\widehat{o}_H \mid \beta_0, (\delta)) \right\} \right\} \quad (36)$$

$$= \max_{s \in \mathcal{S}} \min \left\{ \Psi(s), \pi(s \mid \beta_0, (\delta)) \right\} \quad (37)$$

$$= \mathbb{S}_\Pi [\Psi(S_H) \mid \beta_0, (\delta)]. \quad (38)$$

Line (35) comes using equality (I.30) of Property I.2.1. Line (36) uses the definition of the possibilistic belief state. Line (37) uses the definition of $\pi(\widehat{o}_H \mid \beta_0, (\delta))$, and the possibilistic conditioning. Finally, line (38) is a notation: it is used as the Sugeno integral is based on the distribution $\pi(s_H \mid \beta_0, (\delta))$.

As regards the pessimistic criterion (I.66), the rewriting is shown in the same way:

$$\mathbb{S}_\mathcal{N} \left[\min_{s \in \mathcal{S}} \max \{ \Psi(s), 1 - B_H^\pi(s) \} \mid \beta_0, (\delta) \right]$$

$$\begin{aligned} &= \min_{\widehat{o}_H} \max \left\{ \min_{s \in \mathcal{S}} \max \left\{ \Psi(s), 1 - \beta_{b_0}^{\delta, \widehat{o}_H}(s) \right\}, 1 - \pi(\widehat{o}_H \mid \beta_0, (\delta)) \right\} \\ &= \min_{\widehat{o}_H, s \in \mathcal{S}} \max \left\{ \Psi(s), 1 - \beta_{b_0}^{\delta, \widehat{o}_H}(s), 1 - \pi(\widehat{o}_H \mid \beta_0, (\delta)) \right\} \end{aligned} \quad (39)$$

$$= \min_{s \in \mathcal{S}} \max \left\{ \Psi(s), \min_{\widehat{o}_H} \max \left\{ 1 - \Pi \left(S_H = s \mid \widehat{O}_H = \widehat{o}_H, \left(\delta_t(i_t) \right)_{t=0}^{H-1} \right), 1 - \pi(\widehat{o}_H \mid \beta_0, (\delta)) \right\} \right\} \quad (40)$$

$$= \min_{s \in \mathcal{S}} \max \left\{ \Psi(s), 1 - \max_{\widehat{o}_H} \min \left\{ \Pi \left(S_H = s \mid \widehat{O}_H = \widehat{o}_H, \left(\delta_t(i_t) \right)_{t=0}^{H-1} \right), \pi(\widehat{o}_H \mid \beta_0, (\delta)) \right\} \right\} \quad (41)$$

$$= \min_{s \in \mathcal{S}} \max \left\{ P(s), 1 - \pi(s \mid \beta_0, (\delta)) \right\} \quad (42)$$

$$= \mathbb{S}_\mathcal{N} [P(S_H) \mid \beta_0, (\delta)]. \quad (43)$$

Line (39) comes using the equality (I.29) of Property I.2.1 and line (40) uses the definition of the possibilistic belief state. Line (41) uses the equality (I.26) and the equality (I.27) of Property I.2.1. Line (42) uses the definition of $\pi(\widehat{o}_H \mid \beta_0, (\delta))$, and the possibilistic conditioning. Finally, as the Sugeno integral is based on the distribution $\pi(s_H \mid \beta_0, (\delta))$, the result can be denoted as line (43). ■

A.23 Proof of Theorem 16

Proof: First, $\forall a_t \in \mathcal{A}, \forall o' \in \mathcal{O}$,

$$\begin{aligned} \Pi(O_{t+1} = o' \mid I_t = i_t, a_t) &= \max_{s' \in \mathcal{S}} \Pi(O_{t+1} = o', S_{t+1} = s' \mid I_t = i_t, a_t) \\ &= \max_{(s, s') \in \mathcal{S}^2} \min \left\{ \pi_t(o' \mid s', a_t), \pi_t(s' \mid s, a_t), \Pi(S_t = s \mid I_t = i_t) \right\} \\ &= \max_{(s, s') \in \mathcal{S}^2} \min \left\{ \pi_t(o' \mid s', a_t), \pi_t(s' \mid s, a_t), \beta_{\beta_0}^{i_t}(s) \right\}. \end{aligned}$$

where $\beta_{\beta_0}^{i_t}$ is obtained starting from β_0 and with information i_t . To make the next equations clear, the next formula $\max_{(s, s') \in \mathcal{S}^2} \min \left\{ \pi_t(o' \mid s', a_t), \pi_t(s' \mid s, a_t), \beta_{\beta_0}^{i_t}(s) \right\}$ is denoted by $\pi_t(o' \mid \beta_{\beta_0}^{i_t}, a_t)$. Then, as the belief state is a deterministic function of the current observation, the previous action, and the previous belief state,

$$\begin{aligned} \Pi(B_{t+1}^\pi = \beta' \mid I_t = i_t, a_t) &= \Pi \left(\bigcup_{\substack{o' \in \mathcal{O} \text{ s.t.} \\ \nu(\beta, a_t, o') = \beta'}} \{O_{t+1} = o'\} \mid I_t = i_t, a_t \right) \\ &= \max_{\substack{o' \in \mathcal{O} \text{ s.t.} \\ \nu(\beta, a_t, o') = \beta'}} \Pi(O_{t+1} = o' \mid I_t = i_t, a_t) \\ &= \max_{\substack{o' \in \mathcal{O} \text{ s.t.} \\ \nu(\beta, a_t, o') = \beta'}} \pi_t(o' \mid \beta, a_t). \end{aligned} \tag{44}$$

where $\beta_{\beta_0}^{i_t}(s)$ is denoted by β . The set of all the possible information sequences at a time step $t \geq 1$ is denoted by $\mathcal{I}_t = \mathcal{A}^t \times \mathcal{O}^t$. Thanks to the equation (44) for each belief $\beta \in \Pi_{\mathcal{L}}^{\mathcal{S}}$, $\Pi(B_{t+1}^\pi = \beta' \mid I_t = i_t, a_t)$ does not depend on the information $i_t \in \mathcal{I}_t$, provided i_t is in $\{i \in \mathcal{I}_t \mid \beta_{\beta_0}^i = \beta\}$. Thus, as $\Pi(I_t = i \mid B_t^\pi = \beta, a_t) = 0$ if $\beta_{\beta_0}^i \neq \beta$, we can write

$$\begin{aligned} \Pi(B_{t+1}^\pi = \beta' \mid B_t^\pi = \beta, a_t) &= \max_{i \in \mathcal{I}_t} \min \left\{ \Pi(B_{t+1}^\pi = \beta' \mid I_t = i, a_t), \Pi(I_t = i \mid B_t^\pi = \beta, a_t) \right\} \\ &= \max_{i \in \mathcal{I}_t} \min \left\{ \Pi(B_{t+1}^\pi = \beta' \mid I_t = i^*, a_t), \Pi(I_t = i \mid B_t^\pi = \beta, a_t) \right\} \\ &= \min \left\{ \Pi(B_{t+1}^\pi = \beta' \mid I_t = i^*, a_t), \max_{i \in \mathcal{I}_t} \Pi(I_t = i \mid B_t^\pi = \beta, a_t) \right\} \\ &= \Pi(B_{t+1}^\pi = \beta' \mid I_t = i^*, a_t) \end{aligned}$$

where i^* is such that $\beta_{\beta_0}^{i^*} = \beta$, using the equation (I.30) of Property I.2.1, and the fact that $\max_{i \in \mathcal{I}_t} \Pi(I_t = i \mid B_t^\pi = \beta, a_t) = 1$. ■

B PROOFS OF CHAPTER II

B.1 Property linking \mathbb{S}_Π and $\mathbb{S}_\mathcal{N}$

Property B.1

Let f be a function from Ω to \mathcal{L} :

$$\mathbb{S}_\Pi [1 - f] = 1 - \mathbb{S}_\mathcal{N} [f]$$

Proof: By definition (see Definition I.2.11 and Theorem 12),

$$\begin{aligned} \mathbb{S}_\Pi [1 - f] &= \max_{\omega \in \Omega} \min \{1 - f(\omega), \pi(\omega)\} \\ &= \max_{\omega \in \Omega} \left\{ 1 - \max \{f(\omega), 1 - \pi(\omega)\} \right\} \\ &= 1 - \min_{\omega \in \Omega} \max \{f(\omega), 1 - \pi(\omega)\} \\ &= 1 - \mathbb{S}_\mathcal{N} [f], \end{aligned}$$

using equations (I.26) and (I.27) of Property I.2.1. ■

B.2 Proof of Property II.1.1

Proof: First, $\forall \omega \in \Omega$, $\max \{f(\omega), g(\omega)\} \geq f(\omega)$, and $\max \{f(\omega), g(\omega)\} \geq g(\omega)$. Thus, $\forall \omega \in \Omega$, $\max \{f(\omega), g(\omega)\} \geq \min \{f(\omega), \pi(\omega)\}$, and $\max \{f(\omega), g(\omega)\} \geq \min \{g(\omega), \pi(\omega)\}$. As well, note that $\forall \omega \in \Omega$, $\pi(\omega) \geq \min \{f(\omega), \pi(\omega)\}$, and $\forall \omega \in \Omega$, $\pi(\omega) \geq \min \{g(\omega), \pi(\omega)\}$. Thus, $\forall \omega \in \Omega$,

$$\min \left\{ \max \{f(\omega), g(\omega)\}, \pi(\omega) \right\} \geq \min \{f(\omega), \pi(\omega)\}$$

and

$$\min \left\{ \max \{f(\omega), g(\omega)\}, \pi(\omega) \right\} \geq \min \{g(\omega), \pi(\omega)\}.$$

Recall that the Sugeno integral $\mathbb{S}_\Pi [\max \{f, g\}]$ is equal to

$$\max_{\omega' \in \Omega} \min \left\{ \max \{f(\omega'), g(\omega')\}, \pi(\omega') \right\}.$$

Using previous inequalities, $\forall \omega \in \Omega$,

$$\mathbb{S}_\Pi [\max \{f, g\}] \geq \min \{f(\omega), \pi(\omega)\}$$

and

$$\mathbb{S}_\Pi [\max \{f, g\}] \geq \min \{g(\omega), \pi(\omega)\},$$

and then

$$\mathbb{S}_\Pi [\max \{f, g\}] \geq \max_{\omega \in \Omega} \min \{f(\omega), \pi(\omega)\} = \mathbb{S}_\Pi [f],$$

and

$$\mathbb{S}_\Pi [\max \{f, g\}] \geq \max_{\omega \in \Omega} \min \{g(\omega), \pi(\omega)\} = \mathbb{S}_\Pi [g].$$

We can conclude that $\mathbb{S}_\Pi [\max \{f, g\}] \geq \max \{\mathbb{S}_\Pi [f], \mathbb{S}_\Pi [g]\}$.

Now, let us show that $\mathbb{S}_\Pi [\max \{f, g\}] \leq \max \{\mathbb{S}_\Pi [f], \mathbb{S}_\Pi [g]\}$: let us denote by ω^* an element from Ω such that

$$\omega^* \in \operatorname{argmax} \min \left\{ \max \{f(\omega), g(\omega)\}, \pi(\omega) \right\}.$$

Thus, $\mathbb{S}_\Pi[\max\{f, g\}] = \min\{\max\{f(\omega^*), g(\omega^*)\}, \pi(\omega^*)\}$. On the one hand, if $f(\omega^*) \geq g(\omega^*)$,

$$\begin{aligned}\mathbb{S}_\Pi[\max\{f, g\}] &= \min\{f(\omega^*), \pi(\omega^*)\} \\ &\leq \max_{\omega \in \Omega} \min\{f(\omega), \pi(\omega)\} = \mathbb{S}_\Pi[f] \leq \max\{\mathbb{S}_\Pi[f], \mathbb{S}_\Pi[g]\}.\end{aligned}$$

On the other hand, if $f(\omega^*) \leq g(\omega^*)$,

$$\begin{aligned}\mathbb{S}_\Pi[\max\{f, g\}] &= \min\{g(\omega^*), \pi(\omega^*)\} \\ &\leq \max_{\omega \in \Omega} \min\{g(\omega), \pi(\omega)\} = \mathbb{S}_\Pi[g] \leq \max\{\mathbb{S}_\Pi[f], \mathbb{S}_\Pi[g]\}.\end{aligned}$$

Finally, $\mathbb{S}_\Pi[\max\{f, g\}] = \max\{\mathbb{S}_\Pi[f], \mathbb{S}_\Pi[g]\}$.

Using Property B.1, equation (I.27) of Property I.2.1, and the previous result,

$$\begin{aligned}\mathbb{S}_\mathcal{N}[\min\{f, g\}] &= \mathbb{S}_\mathcal{N}[1 - \max\{1 - f, 1 - g\}] \\ &= 1 - \mathbb{S}_\Pi[\max\{1 - f, 1 - g\}] \\ &= 1 - \max\{\mathbb{S}_\Pi[1 - f], \mathbb{S}_\Pi[1 - g]\} \\ &= \min\{1 - \mathbb{S}_\Pi[1 - f], 1 - \mathbb{S}_\Pi[1 - g]\} \\ &= \min\{\mathbb{S}_\mathcal{N}[f], \mathbb{S}_\mathcal{N}[g]\}.\end{aligned}$$

■

B.3 Proof of Theorem 18

Proof: In order to make the following calculus lines easier to read, $\Psi(S_H)$ is denoted by $\rho_H(S_H, A_H)$, $\bar{\Psi}(B_t^\pi)$ is denoted by $\bar{\rho}_H(B_H^\pi, A_H)$, and $\underline{\Psi}(B_t^\pi)$ is denoted by $\underline{\rho}_H(B_H^\pi, A_H)$:

$$\mathbb{S}_\Pi\left[\bar{\mathcal{G}}\left((S_t)_{t=0}^H, (A_t)_{t=0}^{H-1}\right)\right] = \mathbb{S}_\Pi\left[\max_{t=0}^H \rho_t(S_t, A_t)\right] \quad (45)$$

$$= \max_{t=0}^H \mathbb{S}_\Pi[\rho_t(S_t, A_t)] \quad (46)$$

$$= \max_{t=0}^H \mathbb{S}_\Pi[\bar{\rho}_t(B_t^\pi, A_t)] \quad (47)$$

$$= \mathbb{S}_\Pi\left[\max_{t=0}^H \bar{\rho}_t(B_t^\pi, A_t)\right] \quad (48)$$

$$= \mathbb{S}_\Pi\left[\bar{\mathcal{G}}\left((B_t^\pi)_{t=0}^H, (A_t)_{t=0}^{H-1}\right)\right], \quad (49)$$

where $\bar{\rho}_t(B_t^\pi, A_t) = \max_{s \in \mathcal{S}} \min\{\rho_t(s, A_t), B_t^\pi(s)\}$. The definition of $\bar{\mathcal{G}}$ (see Definition II.1.1) explains the first line (45). Line (46) comes from the maxitivity of \mathbb{S}_Π , described in Property II.1.1. The Theorem 15 is applied to each terms of the maximum operator in line (47). Finally, line (48) uses Property II.1.1 and line (49) uses the definition of $\bar{\mathcal{G}}$ for belief states.

$$\mathbb{S}_\mathcal{N}\left[\underline{\mathcal{G}}\left((S_t)_{t=0}^H, (A_t)_{t=0}^{H-1}\right)\right] = \mathbb{S}_\mathcal{N}\left[\min_{t=0}^H \rho_t(S_t, A_t)\right] \quad (50)$$

$$= \min_{t=0}^H \mathbb{S}_\mathcal{N}[\rho_t(S_t, A_t)] \quad (51)$$

$$= \min_{t=0}^H \mathbb{S}_\mathcal{N}[\underline{\rho}_t(B_t^\pi, A_t)] \quad (52)$$

$$= \mathbb{S}_\mathcal{N}\left[\min_{t=0}^H \underline{\rho}_t(B_t^\pi, A_t)\right] \quad (53)$$

$$= \mathbb{S}_\mathcal{N}\left[\underline{\mathcal{G}}\left((B_t^\pi)_{t=0}^H, (A_t)_{t=0}^{H-1}\right)\right], \quad (54)$$

where $\underline{\rho}_t(B_t^\pi, A_t) = \min_{s \in \mathcal{S}} \max\{\rho_t(s, A_t), 1 - B_t^\pi(s)\}$. The definition of $\underline{\mathcal{G}}$ (see Definition II.1.1) explains line (50). The minitivity of $\mathbb{S}_\mathcal{N}$, described in Property II.1.1, allows to write the line (51). For line (52), the Theorem 15 is applied to each terms of the minimum operator. Finally, Property II.1.1 is used for line (53), and line (54) comes from the definition of $\underline{\mathcal{G}}$ for belief states. ■

B.4 Proof of Theorem 19

Proof: We proceed by induction on $t \in \mathbb{N}$: as the initial visible state $s_{v,0}$ is known by the agent, only states $s = (s_v, s_h)$ for which $s_v = s_{v,0}$ are such that $\beta_0(s) > 0$. A belief over hidden states can be thus defined as $\beta_{h,0}(s_h) = \max_{s_v \in \mathcal{S}_v} \beta_0(s_v, s_h) = \beta_0(s_{v,0}, s_h)$.

At time t , if $\beta_t(s) = 0$ for each $s = (s_v, s_h) \in \mathcal{S}$ such that $s_v \neq s_{v,t}$, the same notation can be adopted: $\beta_{h,t}(s_h) = \beta_t(s_{v,t}, s_h)$. Thus, if the agent reaches state $s_{t+1} = (s_{v,t+1}, s_{h,t+1})$ using action $a_t \in \mathcal{A}$, and if $s' = (s'_v, s'_h)$ with $s'_v \neq s_{v,t+1}$, then $s'_v \neq o_{v,t+1}$ and:

$$\begin{aligned} \pi_t(o_{t+1}, s' \mid \beta_t, a_t) &= \min \left\{ \pi_t(o_{t+1} \mid s', a_t), \max_{s \in \mathcal{S}} \min \{ \pi_t(s' \mid s, a_t), \beta_t(s') \} \right\} \\ &= 0. \end{aligned}$$

thanks to Equation (II.15). Finally, belief update formula (I.62) ensures that $\beta_{t+1}(s') = 0$, since $0 = \pi_t(o_{t+1}, s' \mid \beta_t, a_t) < \pi_t(o_{t+1} \mid \beta_t, a_t)$ (as o_{t+1} is received, o_{t+1} is not impossible, otherwise the model is wrong). Then, β_{t+1} is entirely encoded by $(s_{v,t+1}, \beta_{h,t+1})$ with $s_{v,t+1} = o_{v,t+1}$ and $\beta_{h,t+1}(s_h) = \max_{s_v} \beta_{t+1}(s_v, s_h)$, $\forall s_h \in \mathcal{S}_h$. ■

B.5 Proof of Theorem 20

Proof: Starting from the standard belief update equation (I.62) of Theorem 14 with $o_{t+1} = (o_{v,t+1}, o_{h,t+1})$, and using equation (II.15) in the case of $o_{v,t+1} = s_{v,t+1}$ (i.e. $o'_v = s'_v$), we get that $\beta_{t+1}(s'_v, s'_h)$ is equal to the right part of the equation (II.16). As defined in Theorem 19, $\beta_{t+1,h}(s'_h) = \max_{\bar{s}_v \in \mathcal{S}_v} \beta_{t+1}(\bar{s}_v, s'_h)$; as shown in its proof, $\forall s'_v \in \mathcal{S}_v$ such that $s'_v \neq o_v$, $\beta_{t+1}(s_v, s_h) = 0$. Thus, $\beta_{h,t+1}(s'_h)$ is equal to $\beta_{t+1}(s'_v, s'_h)$ with $o'_v = s'_v$, i.e. to the right part of the equation. ■

B.6 Proof of Theorem 21

Proof: Using the classical dynamic programming equation, Theorem 19, and the facts $\mathcal{S}_v = \mathcal{O}_v$ and that $s'_v \neq o'_v$ is impossible,

$$\begin{aligned} \widehat{U}_i^*(s_v, \beta_h) &= \widehat{U}_i^*(\beta) \\ &= \max_{a \in \mathcal{A}} \widehat{M} \left\{ \widehat{\rho}_t(\beta, a), \widehat{\mathbb{S}} \left(\pi_t(o' \mid \beta, a), \widehat{U}_{i-1}^*(\nu(\beta, a, o')) \right) \right\} \\ &= \max_{a \in \mathcal{A}} \widehat{M} \left\{ \widehat{\rho}_t(\beta, a), \widehat{\mathbb{S}} \left(\pi_t(o'_v, o'_h \mid \beta, a), \widehat{U}_{i-1}^*(\nu(\beta, a, o'_v, o'_h)) \right) \right\} \\ &= \max_{a \in \mathcal{A}} \widehat{M} \left\{ \widehat{\rho}_t(s_v, \beta_h, a), \widehat{\mathbb{S}} \left(\pi_t(s'_v, o'_h \mid \beta, a), \widehat{U}_{i-1}^*(\nu(\beta, a, s'_v, o'_h)) \right) \right\} \\ &= \max_{a \in \mathcal{A}} \widehat{M} \left\{ \widehat{\rho}_t(s_v, \beta_h, a), \widehat{\mathbb{S}} \left(\pi_t(s'_v, o'_h \mid \beta, a), \widehat{U}_{i-1}^*(s'_v, \nu_h(s_v, \beta_h, a, s'_v, o'_h)) \right) \right\} \end{aligned}$$

where $\forall s_h \in \mathcal{S}_h$,

$$\begin{aligned} \beta'_h(s'_h) &= \nu_h(s_v, \beta_h, a, s'_v, o'_h)(s'_h) = \max_{\bar{s}_v \in \mathcal{S}_v} \nu(\beta, a, s'_v, o'_h)(\bar{s}_v, s'_h) = \nu(\beta, a, s'_v, o'_h)(s'_v, s'_h) \\ &= \beta_{t+1}(s'_v, s'_h). \end{aligned}$$

■

B.7 Proof of Theorem 23

Proof: Let β_t be a belief state in $\Pi_{\mathcal{L}}^{\mathcal{S}}$, a be an action in \mathcal{A} , and let us denote $\max_{s \in \mathcal{S}} \min \{ \pi(s' \mid s, a), \beta_t(s) \}$ by $\beta_t^a(s')$, $\forall s' \in \mathcal{S}$.

For each $s' \in \mathcal{S}$, we denote by $\mathcal{P}_a(s')$ the set of system states which lead to s' selecting action $a \in \mathcal{A}$: $\{s \in \mathcal{S} \mid \pi(s' \mid s, a) = 1\}$. As the transition possibility distribution $\pi(s' \mid s, a)$ is deterministic, $\forall s \in \mathcal{S}$, $\exists! s' \in \mathcal{S}$ such that $\pi(s' \mid s, a) = 1$. It implies that $\forall (s', \tilde{s}) \in \mathcal{S}^2$, $\mathcal{P}_a(s') \cap \mathcal{P}_a(\tilde{s}) = \emptyset$ (otherwise, $\exists s \in \mathcal{S}$ leading to two different successors selecting action a , namely s' and \tilde{s}). Moreover, $\cup_{s' \in \mathcal{S}} \mathcal{P}_a(s') = \mathcal{S}$ (otherwise $\exists s \in \mathcal{S}$ without any successor $s' \in \mathcal{S}$). Thus, $\forall s' \in \mathcal{S}$, two cases are possible: either there is no system state $s \in \mathcal{S}$ leading to s' selecting action a , *i.e.* $\mathcal{P}_a(s') = \emptyset$, and then $\beta_t^a(s') = 0$ (since $\forall s \in \mathcal{S}$, $\pi(s' \mid s, a) = 0$). Otherwise, $\mathcal{P}_a(s') \neq \emptyset$ and $\beta_t^a(s') = \max_{s \in \mathcal{P}_a(s')} \beta_t(s)$. As, $\forall s' \in \mathcal{S}$, $\max_{s \in \mathcal{P}_a(s')} \beta_t(s) \leq \sum_{s \in \mathcal{P}_a(s')} \beta_t(s)$,

$$\sum_{s' \in \mathcal{S}} \beta_t^a(s') \leq \sum_{s' \in \mathcal{S}} \sum_{s \in \mathcal{P}_a(s')} \beta_t(s) = \sum_{s \in \mathcal{S}} \beta_t(s),$$

i.e. $\beta_t^a \preceq \beta_t$. Finally, since $\pi(o' \mid s', a) = 1$, $\min\{\pi(o' \mid s', a), \beta_t^a(s')\} = \beta_t^a(s')$. Thus, using the belief update equation (I.62), $\beta_t^a = \beta_{t+1}$, and then $\beta_{t+1} \preceq \beta_t$. We can conclude that the first conditions lead to $\beta_{t+1} \preceq \beta_t$.

We prove now that the second conditions lead to the same conclusion. First note that if $\pi(s' \mid s, a) = \mathbf{1}_{\{s'=s\}}$, then $\beta_t^a = \beta_t$. Two cases are now distinguished. First, suppose that it exists a system state $s' \in \mathcal{S}$ such that $\min\{\pi(o' \mid s', a), \beta_t^a(s')\} = 1$: using the belief update (I.62), we get that $\forall s' \in \mathcal{S}$, $\beta_{t+1}(s') = \min\{\pi(o' \mid s', a), \beta_t^a(s')\}$. Yet, $\min\{\pi(o' \mid s', a), \beta_t^a(s')\} \leq \beta_t^a(s') = \beta_t(s')$, thus β_{t+1} is more specific than β_t (and then $\beta_{t+1} \preceq \beta_t$). Second, suppose that $\forall s' \in \mathcal{S}$, $\min\{\pi(o' \mid s', a), \beta_t^a(s')\} < 1$. Then, only one $s' \in \mathcal{S}$ can maximize $\min\{\pi(o' \mid s', a), \beta_t^a(s')\}$. Indeed, otherwise, it would be two system states s' and \tilde{s} such that one of the following equalities hold: $\pi(o' \mid s', a) = \pi(o' \mid \tilde{s}, a)$; or $\pi(o' \mid s', a) = \beta_t^a(\tilde{s})$; or $\beta_t^a(s') = \beta_t^a(\tilde{s})$. We know that the belief update (I.62) defines $\beta_{t+1}(s')$ as the possibility degree $\beta_t^a(s')$, or $\pi(o' \mid s', a)$, or yet 1. Since $\forall s \in \mathcal{S}$, $\beta_0(s) = 1$ and since $\forall o' \in \mathcal{O}$, $\forall a \in \mathcal{A}$, $\forall (s', \tilde{s}) \in \mathcal{S}^2$, $\pi(o' \mid s', a) \neq \pi(o' \mid \tilde{s}, a)$, belief states obtained from β_0 , using successive belief updates (I.62), never assign the same possibility degree to two different states, except if it is the possibility degree 1. It contradicts thus the fact that more than one system state $s' \in \mathcal{S}$ maximize $\min\{\pi(o' \mid s', a), \beta_t^a(s')\}$. Let us note the maximizing system state s^* : using belief update (I.62), this state is the only system state such that $\beta_{t+1}(s^*) = 1$. System states $s' \in \mathcal{S}$ which do not maximize the joint possibility distribution ($s' \neq s^*$) are such that $\beta_{t+1}(s') = \min\{\pi(o' \mid s', a), \beta_t^a(s')\} \leq \beta_t^a(s') = \beta_t(s')$. Thus if $\beta_t(s^*) = 1$, β_{t+1} is more specific than β_t (and then $\beta_{t+1} \preceq \beta_t$). In order to complete the proof, we now show that the inequality $\beta_{t+1} \preceq \beta_t$ remains true even if $\beta_t(s^*) < 1$. Let us denote by $\tilde{s} \in \mathcal{S}$ a system state such that $\beta_t(\tilde{s}) = 1$. We can affirm that $\beta_{t+1}(\tilde{s}) < \beta_t(s^*)$. Indeed, if this was not the case, then $\beta_{t+1}(\tilde{s}) \geq \beta_t(s^*)$, and

$$\beta_{t+1}(\tilde{s}) \geq \min\{\pi(o' \mid s^*, a), \beta_t(s^*)\} = \min\{\pi(o' \mid s^*, a), \beta_t^a(s^*)\},$$

where the last equality is due to the fact that $\beta_t^a = \beta_t$. Since $\beta_{t+1}(\tilde{s}) = \min\{\pi(o' \mid \tilde{s}, a), \beta_t^a(\tilde{s})\}$, s^* is not maximizing: it is a contradiction. Thus, $\beta_{t+1}(\tilde{s}) < \beta_t(s^*)$. Finally, $\sum_{s \in \mathcal{S} \setminus \{s^*, \tilde{s}\}} \beta_{t+1}(s) \leq \sum_{s \in \mathcal{S} \setminus \{s^*, \tilde{s}\}} \beta_t(s)$, $\beta_{t+1}(s^*) = \beta_t(\tilde{s}) = 1$ and $\beta_{t+1}(s^*) < \beta_t(\tilde{s})$, thus $\beta_{t+1} \preceq \beta_t$ (and even $\beta_{t+1} \prec \beta_t$). ■

C PROOF OF THEOREM 22: OPTIMALITY OF THE STRATEGY COMPUTED BY ALGORITHM 10

This appendix demonstrates that Algorithm 10 returns the maximum value of Equation (II.18) and an optimal strategy. Note that the computed optimal strategy is stationary, and is optimal regardless of the initial state. We recall that $\exists \hat{a} \in \mathcal{A}$ such that $\forall s \in \mathcal{S}, \pi(s' | s, \hat{a}) = 1$ if $s' = s$, and 0 otherwise. The existence of this action \hat{a} makes the maximum value of the criterion non-decreasing with respect to the horizon size. Let us denote by $(u_i^*)_{i \geq 0}$ the sequence of functions \overline{U}^* computed by the algorithm: $\forall i \geq 1, u_{i-1}^*$ is the function \overline{U}^* at the beginning of the i^{th} iteration (line 5). That is, $u_0^* = \Psi$, and $\forall i \geq 1, \forall s \in \mathcal{S}, u_i^*(s) = \max_{a \in \mathcal{A}} \min \{ \pi(s' | s, a), u_{i-1}^*(s) \}$. As well, $\forall s \in \mathcal{S}$, we denote by δ_i^* the strategy $\delta^* : \mathcal{S} \rightarrow \mathcal{A}$ computed after i iterations (according to the while loop). Finally, \mathcal{T}_i is the set of i -length system state trajectories (s_1, \dots, s_i) , and Δ_i the set of i -length strategies: $(\delta_t)_{t=0}^{i-1}$ such that $\delta : \mathcal{S} \rightarrow \mathcal{A}$.

Lemma C.1

$$\boxed{\forall s \in \mathcal{S}, \forall i \geq 0, u_i^*(s) \leq u_{i+1}^*(s).}$$

Proof: Let $s_0 \in \mathcal{S}$. Looking at Algorithm 4 for optimistic finite-horizon π -MDP, and looking at Algorithm 10, we see that $\forall i \geq 0, u_i^* = \overline{U}_i^*$ (in the case of terminal preference only, *i.e.* $\forall t \geq 0, \forall a \in \mathcal{A}, \forall s \in \mathcal{S}, \rho_t(s, a) = 1$ since the global preference (II.3) is based on the minimum operator, see Section I.2.4), and thus, $\forall p \geq 1$

$$u_{p+1}^*(s_0) = \max_{(\delta) \in \Delta_{p+1}} \max_{\tau \in \mathcal{T}_{p+1}} \min \left\{ \min_{i=0}^p \pi(s_{i+1} | s_i, \delta_i(s_i)), \Psi(s_{p+1}) \right\}.$$

Consider the particular trajectories $\tau' \in \mathcal{T}'_{p+1} \subset \mathcal{T}_{p+1}$ such that $\tau' = (s_1, \dots, s_p, s_{p+1})$ and $s_p = s_{p+1}$. Consider also the particular policies $(\delta') \in \Delta'_{p+1} \subset \Delta_{p+1}$ such that $(\delta') = (\delta_0, \dots, \delta_{p-1}, \hat{\delta})$, where $\hat{\delta}$ is the decision rule such that $\hat{\delta}(s) = \hat{a}$ (action “stay”). It is obvious that

$$u_{p+1}^*(s_0) \geq \max_{(\delta') \in \Delta'_{p+1}} \max_{\tau' \in \mathcal{T}'_{p+1}} \min \left\{ \min_{i=0}^p \pi(s_{i+1} | s_i, \delta'_i(s_i)), \Psi(s_{p+1}) \right\}.$$

The right part of this inequality can be rewritten as

$$\max_{(\delta) \in \Delta_p} \max_{\tau \in \mathcal{T}_p} \min \left\{ \min_{i=0}^{p-1} \pi(s_{i+1} | s_i, \delta_i(s_i)), \pi(s_p | s_p, \hat{a}), \Psi(s_p) \right\} = u_p^*(s_0)$$

since $\pi(s_p | s_p, \hat{a}) = 1$. Hence, $\forall p \geq 0, u_{p+1}^*(s_0) \geq u_p^*(s_0)$. ■

The meaning of this lemma is: it is always more possible to reach a state s from s_0 in at most $p+1$ steps than in at most p steps. As for each $s \in \mathcal{S}$, $(u_p^*(s))_{p \in \mathbb{N}} \leq 1$, Lemma C.1 insures that the sequence $(u_p^*(s))_{p \in \mathbb{N}}$ converges. The next lemma shows that the convergence of this sequence occurs in finite time.

Lemma C.2

For all $\forall s \in \mathcal{S}$, the number of iterations of the sequence $(u_p^(s))_{p \in \mathbb{N}}$ up to convergence is bounded by $\#\mathcal{S} \times \#\mathcal{L}$.*

Proof: Recall first that values of the possibility and preference distributions are in \mathcal{L} which is finite and totally ordered: we can write $\mathcal{L} = \{0, l_1, l_2, \dots, 1\}$ with $0 < l_1 < l_2 < \dots < 1$. If two successive functions u_k^* and u_{k+1}^* are equal, then $\forall s \in \mathcal{S}$ sequences $(u_p^*(s))_{p \geq k}$ are constant. Indeed this sequence can be defined by the recursive formula

$$u_p^*(s) = \max_{a \in \mathcal{A}} \max_{s' \in \mathcal{S}} \min \{ \pi(s' | s, a), u_{p-1}^*(s') \}.$$

Thus if $\forall s \in \mathcal{S}$, $u_p^*(s) = u_{p-1}^*(s)$ then the next iteration $(p+1)$ faces the same situation ($u_{p+1}^*(s) = u_p^*(s)$, $\forall s \in \mathcal{S}$). The slowest convergence can then be described as follows: for each $p \in \mathbb{N}$ only one $s \in \mathcal{S}$ is such that $u_{p+1}^*(s) > u_p^*(s)$. Moreover, for this s , if $u_p^*(s) = l_i$, then $u_{p+1}^*(s) = l_{i+1}$. We can conclude that for $p > \#\mathcal{L} \times \#\mathcal{S}$, the sequence is constant. ■

We conclude that the variable \overline{U}^* of the algorithm converges to the maximal value of the criterion for an $(\#\mathcal{L} \times \#\mathcal{S})$ -size horizon and can not be greater: the function \overline{U}^* returned is thus optimal with respect to Equation (II.18) and is computed in a finite number of steps.

In the following, we prove the optimality of the strategy, based on the decision rule $\overline{\delta}^* : \mathcal{S} \rightarrow \mathcal{A}$, returned by Algorithm 10. For this purpose, we will construct a trajectory $\tau = (s_1, \dots, s_p) \in \mathcal{T}_p$ of size p smaller than $\#\mathcal{S}$ which maximizes $\min \{ \pi(\tau \mid s_0, (\delta)), \Psi(s_p) \}$ with strategy $(\overline{\delta}^*) = (\overline{\delta}_t^*)_{t \geq 0}$ such that $\forall s \in \mathcal{S}$, $\forall t \geq 0$, $\overline{\delta}_t^*(s) = \overline{\delta}^*(s)$. The next two lemmas are needed for this construction and require some notations.

Let $s_0 \in \mathcal{S}$ and p be the smallest integer such that $\forall p' \geq p$, $u_{p'}^*(s_0) = \overline{U}^*(s_0)$, where \overline{U}^* is here the optimal value of the infinite horizon criterion of Equation (II.18) returned by Algorithm 10 (variable $\overline{U}^*(s)$ of Algorithm 10 does not increase after p iterations). Algorithm 4 for optimistic finite-horizon π -MDP (in the case of terminal preference only) can be used to return an optimal strategy in the finite-horizon sense, see criterion (I.58), denoted by $(\delta^{(s_0)}) \in \Delta_p$ (this strategy is not stationary in general). With this notation, $\forall s \in \mathcal{S}$, $\overline{\delta}^*(s) = \delta_0^{(s)}(s)$, since $\delta_0^{(s)}(s)$ is the last selected action before convergence of $(\overline{U}_i^*(s))_{i \geq 0}$ (see Algorithm 4).

Consider now a trajectory $\tau = (s_1, s_2, \dots, s_p)$ which maximizes

$$\min \left\{ \min_{i=0}^{p-1} \pi(s_{i+1} \mid s_i, \delta_i^{(s_0)}(s_i)), \Psi(s_p) \right\}.$$

This trajectory is called *optimal trajectory of minimal size from s_0* .

Lemma C.3

Let $\tau = (s_1, \dots, s_p)$ be an optimal trajectory of minimal size from s_0 . Then, $\forall k \in \{1, \dots, p-1\}$,

$$\overline{U}^*(s_0) \leq u_{p-k}^*(s_k) \leq \overline{U}^*(s_k),$$

where \overline{U}^* is the optimal value function returned by Algorithm 10, and $u_{p-k}^*(s_k)$ is equal to $\overline{U}_{p-k}^*(s_k)$ in Algorithm 4 (or defined above).

Proof: Let $k \in \{1, \dots, p-1\}$.

$$\begin{aligned} \overline{U}^*(s_0) &= \min \left\{ \min_{i=0}^{p-1} \pi(s_{i+1} \mid s_i, \delta_i^{(s_0)}(s_i)), \Psi(s_p) \right\} \\ &\leq \min \left\{ \min_{i=k}^{p-1} \pi(s_{i+1} \mid s_i, \delta_i^{(s_0)}(s_i)), \Psi(s_p) \right\} \leq \overline{U}_{p-k}^*(s_k) = u_{p-k}^*(s_k) \leq \overline{U}^*(s_k) \end{aligned}$$

since $(u_p^*)_{p \in \mathbb{N}}$ is non-decreasing (see Lemma C.1), and $\forall i \geq 0$, $\overline{U}_i^* = u_i^*$ (see Algorithm 4) whose limit is \overline{U}^* . ■

Lemma C.4

Let $\tau = (s_1, \dots, s_p)$ be an optimal trajectory of minimal size from s_0 and $k \in \{1, \dots, p-1\}$. If $\overline{U}^*(s_0) = \overline{U}^*(s_k)$, then $\overline{\delta}^*(s_k) = \delta_k^{(s_0)}(s_k)$.

Proof: Suppose that $\overline{U}^*(s_0) = \overline{U}^*(s_k)$. Since $\overline{U}^*(s_0) \leq u_{p-k}^*(s_k) \leq \overline{U}^*(s_k)$ (Lemma C.3), we obtain that $u_{p-k}^*(s_k) = \overline{U}^*(s_k)$. The criterion in s_k is thus optimized within a $(p-k)$ -horizon. Moreover a shorter horizon is not optimal: $\forall m \in \{1, \dots, p-k\}$, $u_{p-k-m}^*(s_k) < \overline{U}^*(s_k)$ i.e. with

a $(p - k - m)$ -size horizon the criterion in s_k is not maximized. Indeed if the contrary was true, the criterion in s_0 would be maximized within a $(p - m)$ -size horizon: the strategy

$$\delta' = (\delta_0^{(s_0)}, \delta_1^{(s_0)}, \dots, \delta_{k-1}^{(s_0)}, \delta_0^{(s_k)}, \dots, \delta_{p-k-m-1}^{(s_k)}) \in \Delta_{p-m}$$

would be optimal. Indeed,

$$\begin{aligned} \overline{U}^*(s_0) &= \min \left\{ \min_{i=0}^{k-1} \pi \left(s_{i+1} \mid s_i, \delta_i^{(s_0)}(s_i) \right), u_{p-k}^*(s_k) \right\} \\ &= \min \left\{ \min_{i=0}^{k-1} \pi \left(s_{i+1} \mid s_i, \delta_i^{(s_0)}(s_i) \right), \overline{U}^*(s_k) \right\}. \end{aligned}$$

Then let $\bar{\tau} = (\bar{s}_1, \dots, \bar{s}_{p-k-m}) \in \mathcal{T}_{p-k-m}$ be an optimal trajectory of minimal size from s_k . Setting $\bar{s}_0 = s_k$, $\bar{\tau}$ thus maximizes $\min \left\{ \min_{i=0}^{p-k-m-1} \pi \left(\bar{s}_{i+1} \mid \bar{s}_i, \delta_i^{(s_k)}(\bar{s}_i) \right), \Psi(\bar{s}_{p-k-m}) \right\}$, which is then equal to $\overline{U}^*(s_k)$. If $(s'_1, \dots, s'_{p-m}) = (s_1, \dots, s_{k-1}, \bar{s}_0, \dots, \bar{s}_{p-k-m})$,

$$\overline{U}^*(s_0) = \min \left\{ \min_{i=0}^{p-m-1} \pi \left(s'_{i+1} \mid s'_i, \delta'_i(s'_i) \right), \Psi(s'_{p-m}) \right\},$$

i.e. $\exists p' < p$ such that $\overline{U}^*(s_0) = u_{p'}^*(s_0)$: it contradicts the assumption that (s_1, \dots, s_p) is an optimal trajectory of minimum size. Thus $p - k$ is the smallest integer such that $u_{p-k}^*(s_k) = \overline{U}^*(s_k)$: we finally conclude that $\bar{\delta}^*(s_k) \left(:= \delta_0^{(s_k)}(s_k) \right) = \delta_k^{(s_0)}(s_k)$, as the action $\delta_k^{(s_0)}(s_k)$ is also selected during the iteration $p - k$, maximizing the same value function. ■

Theorem 29

Let $(\bar{\delta}^*)$ be the strategy returned by Algorithm 10; $\forall s_0 \in \mathcal{S}$, there exists $p^* \leq \#\mathcal{S}$ and a trajectory (s_1, \dots, s_{p^*}) such that

$$\overline{U}^*(s_0) = \min \left\{ \min_{i=0}^{p^*-1} \pi \left(s_{i+1} \mid s_i, \bar{\delta}^*(s_i) \right), \Psi(s_{p^*}) \right\},$$

i.e. $(\bar{\delta}^*)$ is an optimal strategy.

Proof: Let s_0 be in \mathcal{S} and $\tau \in \mathcal{T}_p$ be an optimal trajectory of minimal size p from s_0 . If $\forall k \in \{1, \dots, p-1\}$, $\overline{U}^*(s_k) = \overline{U}^*(s_0)$, then, using Lemma C.4, $\forall k \in \{1, \dots, p-1\}$, $\bar{\delta}^*(s_k) := \delta_0^{(s_k)}(s_k) = \delta_k^{(s_0)}(s_k)$, and then the criterion in s_0 is maximized with $(\bar{\delta}^*)$ since it is maximized with $(\delta_t^{(s_0)})_{t=0}^{p-1}$: the optimality of the strategy is shown.

Otherwise, let k be the smallest integer $\in \{1, \dots, p-1\}$ such that $\overline{U}^*(s_k) > \overline{U}^*(s_0)$: the definition of k and Lemma C.3 implies that $\overline{U}^*(s_k) > \overline{U}^*(s_i) = \overline{U}^*(s_0)$, $\forall i \in \{0, \dots, k-1\}$.

Reiterate beginning with $s_0^{(1)} = s_k$: let $p^{(1)}$ be the number of iterations until variable $\overline{U}^*(s^{(1)})$ of Algorithm 10 converges *i.e.* the smallest integer such that $\overline{U}^*(s_0^{(1)}) = u_{p^{(1)}}^*(s_0^{(1)})$. Let $\tau^{(1)} \in \mathcal{T}_{p^{(1)}}$ be a trajectory which maximizes $\min \left\{ \min_{i=0}^{p^{(1)}-1} \pi \left(s_{i+1} \mid s_i, \delta_i^{(s_0^{(1)})}(s_i) \right), \Psi(s_{p^{(1)}}^{(1)}) \right\}$ ($\tau^{(1)}$ is an optimal trajectory of minimal size from $s_k = s_0^{(1)}$). We select $k^{(1)}$ in the same way as previously and reiterate beginning with $s_0^{(2)} = s_{k^{(1)}}^{(1)}$ which is such that $\overline{U}^*(s_{k^{(1)}}^{(1)}) > \overline{U}^*(s_0^{(1)})$, and $\overline{U}^*(s_{k^{(1)}}^{(1)}) > \overline{U}^*(s_i^{(1)}) \forall i \in \{0, \dots, k^{(1)}-1\}$ etc. Lemma C.5 below shows that all selected states $(s_0, \dots, s_{k-1}, s_0^{(1)}, \dots, s_{k^{(1)}-1}^{(1)}, s_0^{(2)}, \dots, s_{k^{(2)}-1}^{(2)}, s_0^{(3)}, \dots)$, are different. Thus this selection process ends since $\#\mathcal{S}$ is a finite set. The total number of selected states is denoted by $p^* = k + \sum_{i=1}^{q-1} k^{(i)} + p^{(q)}$ with $q \geq 0$ the number of new selected trajectories. Then the strategy

$$(\delta') = (\delta_0, \dots, \delta_{k-1}, \delta_0^{(s_0^{(1)})}, \dots, \delta_{k^{(1)}-1}^{(s_0^{(1)})}, \dots, \delta_{p^{(q)}}^{(s_0^{(q)})})$$

corresponds to (δ^*) over $\tau' = (s'_1, \dots, s'_{p^*}) = (s_0, s_1, \dots, s_{k-1}, s_0^{(1)}, \dots, s_{k^{(1)}-1}^{(1)}, \dots, s_{p^{(q)}-1}^{(m)})$ and this strategy is optimal because $\overline{U}^*(s_0) = \overline{U}(s_0, (\delta^*))$: indeed,

$$\begin{aligned} \overline{U}^*(s_0) &= \min \left\{ \min_{i=0}^{k-1} \pi(s'_{i+1} \mid s'_i, \delta'(s'_i)), u_{p-k}^*(s_k) \right\} \\ &\leq \min \left\{ \min_{i=0}^{k-1} \pi(s'_{i+1} \mid s'_i, \delta'(s'_i)), \overline{U}^*(s_k) \right\} \\ &= \min \left\{ \min_{i=0}^{k^{(1)}-1} \pi(s'_{i+1} \mid s'_i, \delta'(s'_i)), u_{p^{(1)}-k^{(1)}}^*(s_{k^{(1)}}) \right\} \\ \dots &\leq \min \left\{ \min_{i=0}^{p^*-1} \pi(s'_{i+1} \mid s'_i, \delta'(s'_i)), \Psi(s_{p^*}) \right\}. \end{aligned}$$

The “ \leq ” signs are in fact “=” since otherwise we would find a strategy such that $\overline{U}(s_0, (\delta')) > \overline{U}^*(s_0)$, while $\overline{U}^*(s_0)$ is the maximal value. Thus (δ^*) is optimal: $\overline{U}^*(s_0) = \min \left\{ \min_{i=0}^{p^*-1} \pi(s'_{i+1} \mid s'_i, \delta^*(s'_i)), \Psi(s_{p^*}) \right\}$. ■

Lemma C.5

The process described in the previous proof in order to construct a trajectory maximizing the criterion with (δ^) always selects different system states.*

Proof: First, two equal states in the same selected trajectory $\tau^{(m)}$ would contradict the hypothesis that $p^{(m)}$ is the smallest integer such that $u_{p^{(m)}}^*(s_0^{(m)}) = \overline{U}^*(s_0^{(m)})$. Indeed let k and l be such that $0 \leq k < l \leq p^{(m)}$ and suppose that $s_k^{(m)} = s_l^{(m)}$. For clarity in the next calculations, we omit “ (m) ”: $p = p^{(m)}$ and $\forall i \in \{0, \dots, l\}$, $s_i = s_i^{(m)}$. We can write

$$u_{p-k}^*(s_k) = \min \left\{ \min_{i=k}^{l-1} \pi(s_{i+1} \mid s_i, \delta_i^{(s_0)}(s_i)), u_{p-l}^*(s_l) \right\} \leq u_{p-l}^*(s_l) = u_{p-l}^*(s_k),$$

as $s_l = s_k$. However $u_{p-k}^*(s_k) \geq u_{p-l}^*(s_k)$ (non-decreasing sequence and $p - k > p - l$). We finally get $u_{p-k}^*(s_k) = u_{p-l}^*(s_k)$, thus

$$\begin{aligned} \overline{U}^*(s_0) &= \min \left\{ \min_{i=0}^{k-1} \pi(s_{i+1} \mid s_i, \delta_i^{(s_0)}(s_i)), u_{p-k}^*(s_k) \right\} \\ &= \min \left\{ \min_{i=0}^{k-1} \pi(s_{i+1} \mid s_i, \delta_i^{(s_0)}(s_i)), u_{p-l}^*(s_l) \right\} \\ &= \min \left\{ \min_{i=0, \dots, k-1, l, \dots, p-1} \pi(s_{i+1} \mid s_i, \delta_i^{(s_0)}(s_i)), \Psi(s_p) \right\}. \end{aligned}$$

Consequently, a $(p^{(m)} - l + k)$ -sized horizon is good enough to reach the optimal value: it is a contradiction. Finally, if we suppose that a state \bar{s} appears two times in the sequence of selected states, then this state belongs to two different selected trajectories $\tau^{(m)}$ and $\tau^{(m')}$ (with $m < m'$). Lemma C.3, and the definition of $k^{(m)}$ which implies that $\overline{U}^*(s_0^{(m+1)})$ is strictly greater than the criterion's optimal values in each of the states $s_0^{(m)}, \dots, s_{k^{(m)}-1}^{(m)}$, lead to the inequalities $\overline{U}^*(s_0^{(m')}) \leq \overline{U}^*(\bar{s}) < \overline{U}^*(s_0^{(m+1)})$, as \bar{s} is in $\tau^{(m)}$ and in $\tau^{(m')}$. It is a contradiction. Indeed, as $m < m'$, the following inequality holds: $\overline{U}^*(s_0^{(m+1)}) \leq \overline{U}^*(s_0^{(m')})$. ■

D PROOFS OF CHAPTER III

D.1 Proof of Property III.2.1

Proof:

$$\begin{aligned}
q^a &= \max_{(s'_v, \beta') \in \mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}} \min \left\{ \pi(s'_v, \beta' | s_v, \beta, a), \overline{U}^*(s'_v, \beta') \right\} \\
&= \max_{X' \in \mathcal{S}_v \times \Pi_{\mathcal{L}}^{\mathcal{S}_h}} \min \left\{ \min_{i=1}^n \pi(X'_i | \text{parents}(X'_i), a), \overline{U}^*(X') \right\} \\
&= \max_{X'_n \in \{\top, \perp\}} \min \left\{ \pi(X'_n | \text{parents}(X'_n), a), \dots \right. \\
&\quad \max_{X'_2 \in \{\top, \perp\}} \min \left\{ \pi(X'_2 | \text{parents}(X'_2), a), \right. \\
&\quad \left. \left. \max_{X'_1 \in \{\top, \perp\}} \min \left\{ \pi(X'_1 | \text{parents}(X'_1), a), \overline{U}^*(X') \right\} \right\} \dots \right\}
\end{aligned}$$

where the last equation is due to the fact that, for any variables $x, y \in \mathcal{X}, \mathcal{Y}$ finite spaces, and any functions $\varphi : \mathcal{X} \rightarrow \mathcal{L}$ and $\psi : \mathcal{Y} \rightarrow \mathcal{L}$, we have:

$$\max_{y \in \mathcal{Y}} \min \{ \varphi(x), \psi(y) \} = \min \{ \varphi(x), \max_{y \in \mathcal{Y}} \psi(y) \},$$

see the equation I.30 of Property I.2.1. ■

D.2 Proof of Theorem 24

Proof: First $S_{h,0}^1, \dots, S_{h,0}^l$ are initially NI-independent, see Definition I.2.6, *i.e.* $\forall (s^1, \dots, s^l) \in \mathcal{S}_h^1 \times \dots \times \mathcal{S}_h^l$,

$$\Pi(S_{h,0}^1 = s^1, \dots, S_{h,0}^l = s^l) = \min \{ \Pi(S_{h,0}^1 = s^1), \dots, \Pi(S_{h,0}^l = s^l) \}.$$

Then $\exists (\beta_0^j)_{j=1}^l \in \prod_{j=1}^l \Pi_{\mathcal{L}}^{\mathcal{S}_h^j}$ such that $\forall s_h = (s^1, \dots, s^l) \in \mathcal{S}_h^1 \times \dots \times \mathcal{S}_h^l$, $\beta_0(s_h) = \min_{j=1}^l \beta_0^j(s^j)$.

As discussed in Section III.3.2, the d-Separation criterion can be used to prove NI-independences on the DBN of Figure III.4: we show now that hidden variables of time step $t+1$ are NI-independent conditional on the information i_{t+1} . In fact, as shown in Figure III.4, given $1 \leq i < j \leq l$, $S_{h,t+1}^i$ and $S_{h,t+1}^j$ are d-separated by the evidence $I_{t+1} = i_{t+1}$.

Thus, $\forall s = (s^1, \dots, s^l) \in \mathcal{S}_h$, $\Pi(S_{h,t+1} = s | I_{t+1} = i_{t+1}) = \min_{j=1}^l \Pi(S_{h,t+1}^j = s^j | I_{t+1} = i_{t+1})$ *i.e.* variables $(S_{h,t+1}^j)_{j=1}^l$ are NI-independent conditional on information I_{t+1} . It shows that $\beta_{h,t+1}(s) = \min_{j=1}^l \beta_{h,t+1}^j(s^j)$.

Let us denote by $X \rightarrow Y$ the fact that there is an arrow from X to Y in the DBN. Note that the proved independence would not hold if the same observation variable O_{t+1}^k , $k \in \{1, \dots, l\}$ concerned two different hidden state variables $S_{h,t+1}^i$ and $S_{h,t+1}^j$, *i.e.* if $S_{h,t+1}^i \rightarrow O_{t+1}^k$: and $S_{h,t+1}^j \rightarrow O_{t+1}^k$: indeed O_{t+1}^k is part of information I_{t+1} , thus there would be a convergent (towards O_{t+1}^k) relationship between $S_{h,t+1}^i$ and $S_{h,t+1}^j$ *i.e.* the hidden state variables would have been dependent (because d-connected) conditioned on information I_{t+1} . Moreover if the next hidden state variable $S_{h,t+1}^i$ depended on the current hidden state variable $S_{h,t}^j$, ($S_{h,t}^j \rightarrow S_{h,t+1}^i$) then $S_{h,t+1}^i$ and $S_{h,t+1}^j$ would have been dependent conditioned on information I_{t+1} because d-connected through $S_{h,t}^j$ ($S_{h,t}^j \rightarrow S_{h,t+1}^i$ is also true). ■

D.3 Proof of Lemma III.3.1

Proof: Let j be an integer in $\{1, \dots, l\}$. Using Theorem 24, and M-independence assumptions of the DBN of Figure III.4 leading to the distributions (III.3), (III.4) and (III.5),

$$\begin{aligned}
& \Pi \left(O_{t+1}^j = o'_j, S_{h,t+1}^j = s'_j \mid I_t = i_t, a_t \right) \\
&= \max_{s_h^j \in S_h^j} \min \left\{ \Pi \left(O_{t+1}^j = o'_j, S_{h,t+1}^j = s'_j \mid S_{h,t}^j = s_h^j, I_t = i_t, a_t \right), \Pi \left(S_{h,t}^j = s_h^j \mid I_t = i_t, a_t \right) \right\} \\
&= \max_{s_h^j \in S_h^j} \min \left\{ \Pi \left(O_{t+1}^j = o'_j, S_{h,t+1}^j = s'_j \mid S_{h,t}^j = s_h^j, I_t = i_t, a_t \right), \beta_t^j(s_h^j) \right\} \\
&= \max_{s_h^j \in S_h^j} \min \left\{ \Pi \left(O_{t+1}^j = o'_j \mid S_{h,t+1}^j = s'_j, I_t = i_t, a_t \right), \Pi \left(S_{h,t+1}^j = s'_j \mid I_t = i_t, a_t \right), \beta_t^j(s_h^j) \right\} \\
&= \max_{s_h^j \in S_h^j} \min \left\{ \pi \left(o'_j \mid s_{v,t}, s'_j, a_t \right), \Pi \left(s'_j \mid s_{v,t+1}, s_h^j, a_t \right), \beta_t^j(s_h^j) \right\}
\end{aligned}$$

denoted then by $\pi \left(o'_j, s'_j \mid s_{v,t}, \beta_t^j, a_t \right)$. The belief update can be then computed from this joint possibility distribution using the qualitative possibilistic conditioning (Definition I.2.7). ■

D.4 Proof of Theorem 25

Proof: First, as visible state variables $(S_{v,t+1}^i)_{i=1}^m$ are d-separated by the evidence $I_t = i_t$, they are NI-independent: $\forall s_{v,t+1} = (s_{v,t+1}^1, \dots, s_{v,t+1}^m), \Pi(S_{v,t+1} = s_{v,t+1} \mid I_t = i_t, a_t) = \min_{i=1}^m \pi(s_{v,t+1}^i \mid s_{v,t}, a_t)$. The proof of Lemma III.3.1 shows that $\Pi(O_{t+1}^j = o_{t+1}^j \mid I_t = i_t, a_t) = \Pi(O_{t+1}^j = o_{t+1}^j \mid S_{v,t} = s_{v,t}, B_t^j = \beta_t^j, a_t)$, where β_t^j is the marginal belief state constructed from i_t . This distribution is then denoted by $\pi(o_{t+1}^j \mid s_{v,t}, \beta_t^j, a_t)$. Moreover, observation variables $(O_{t+1}^j)_{j=1}^l$ are d-separated by the evidence $I_t = i_t$, and are then NI-independent: $\forall o_{t+1} = (o_{t+1}^1, \dots, o_{t+1}^l), \Pi(O_{t+1} = o_{t+1} \mid I_t = i_t, a_t) = \min_{j=1}^l \pi(o_{t+1}^j \mid s_{v,t}, \beta_t^j, a_t)$. Finally, $\forall i \in \{1, \dots, m\}$ and $\forall j \in \{1, \dots, l\}$, $S_{v,t+1}^i$ and O_{t+1}^j are d-separated by the evidence $I_t = i_t$, and then NI-independent. Thus, $\forall s_{v,t+1} = (s_{v,t+1}^1, \dots, s_{v,t+1}^m) \in S_v$, $\forall o_{t+1} = (o_{t+1}^1, \dots, o_{t+1}^l) \in \mathcal{O}$,

$$\begin{aligned}
& \Pi(S_{v,t+1} = s_{v,t+1}, O_{t+1} = o_{t+1} \mid I_t = i_t, a_t) \\
&= \min \left\{ \min_{i=1}^m \pi(s_{v,t+1}^i \mid s_{v,t}, a_t), \min_{j=1}^l \pi(o_{t+1}^j \mid s_{v,t}, \beta_t^j, a_t) \right\}.
\end{aligned}$$

As it only depends on the current visible state and belief state, we can denote it by $\Pi(S_{v,t+1} = s_{v,t+1}, O_{t+1} = o_{t+1} \mid B_t^\pi = \beta_t, a_t) = \pi(s_{v,t+1}, o_{t+1} \mid \beta_t, a_t)$. Then, as B_{t+1}^π is equal to $\beta_{t+1} = (\beta_{t+1}^1, \dots, \beta_{t+1}^l)$ with the possibility degree of the observations leading to it,

$$\begin{aligned}
& \pi(s_{v,t+1}, \beta_{t+1} \mid \beta_t, a_t) \\
&= \max_{\substack{(o^1, \dots, o^l) \in \mathcal{O}^j \text{ s.t. } \forall j, \\ \nu^j(s_{v,t}, \beta_t^j, a_t, o^j) = \beta_{t+1}^j}} \pi(s_{v,t+1}, o_{t+1} \mid \beta_t, a_t) \\
&= \max_{\substack{(o^1, \dots, o^l) \in \mathcal{O} \text{ s.t. } \forall j, \\ \nu^j(s_{v,t}, \beta_t^j, a_t, o^j) = \beta_{t+1}^j}} \min \left\{ \min_{i=1}^m \pi(s_{v,t+1}^i \mid s_{v,t}, a_t), \min_{j=1}^l \pi(o_{t+1}^j \mid s_{v,t}, \beta_t^j, a_t) \right\} \\
&= \min \left\{ \min_{i=1}^m \pi(s_{v,t+1}^i \mid s_{v,t}, a_t), \max_{\substack{(o^1, \dots, o^l) \in \mathcal{O} \text{ s.t. } \forall j, \\ \nu^j(s_{v,t}, \beta_t^j, a_t, o^j) = \beta_{t+1}^j}} \min_{j=1}^l \pi(o_{t+1}^j \mid s_{v,t}, \beta_t^j, a_t) \right\} \\
&= \min \left\{ \min_{i=1}^m \pi(s_{v,t+1}^i \mid s_{v,t}, a_t), \min_{j=1}^l \pi(\beta_{t+1}^j \mid s_{v,t}, \beta_t^j, a_t) \right\},
\end{aligned}$$

using the equation (I.30) of Property I.2.1. ■

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NOTATIONS

$f : \mathcal{X} \rightarrow \mathcal{Y}$	f is a function from \mathcal{X} to \mathcal{Y}
$f : x \mapsto y$	the function f maps the element $x \in \mathcal{X}$ to the element $y \in \mathcal{Y}$
$f(., x)$	function $f^x : y \mapsto f(x, y)$ if f is defined on $\mathcal{X} \times \mathcal{Y}$ and $x \in \mathcal{X}$
$\mathbb{1}_A$	indicator function of the set A
$\mathbb{1}_{F(s)}$	function equal to one if the formula $F(s)$ is true, and zero otherwise (Kronecker delta)
$\#A$	size of set A
\overline{A}	complementary set of A : if the working set is Ω , $A \cup \overline{A} = \Omega$ and $A \cap \overline{A} = \emptyset$
\mathbb{N}, \mathbb{N}^*	set of non negative integers, of positive integers
\mathbb{R}, \mathbb{R}_+	set of real numbers, non negative real numbers
\mathbb{R}^d	real vector space with d dimensions
$\mathcal{P}(\Omega)$	the set of all subsets of the set Ω
$parents(S)$	parents and descendants of a variable (node) S in a Bayesian Network
$descend(S)$	
\mathbb{P}, \mathbb{E}	probability and expectation
$\mathcal{F}_B(\mathcal{S}, \mathbb{R})$	the set of the bounded functions from the set \mathcal{S} to the set \mathbb{R}
$X \perp\!\!\!\perp Y Z$	X is independent from Y conditional on Z
$S \sim b$	b is the probability distribution of the random variable S : $\forall s, \mathbb{P}(S = s) = b(s)$
$\bar{b} \propto b$	\bar{b} is the probabilistic normalization of b : $\bar{b}(s) = b(s) / \sum_{\tilde{s}} b(\tilde{s})$
$b^\pi \propto^\pi b$	b^π is the possibilistic normalization of b : $b^\pi(s) = 1$ if $b(s) = \max_{s'} b(s')$, $b(s)$ otherwise
$\langle x, y \rangle_{\mathbb{R}^S}$	the scalar product of two vectors x and y from \mathbb{R}^S
\circ	function composition operator
$\mathbb{P}^{\mathcal{S}}$	set of the probability distributions over \mathcal{S}
$\Pi_{\mathcal{L}}^{\mathcal{S}}$	set of the possibilistic distributions over \mathcal{S} when the possibilistic scale is \mathcal{L}

RELATED RESEARCH MATERIAL

This PhD thesis has led to the development of following materials:

Conference papers

- *Qualitative Possibilistic Mixed-Observable MDPs*, Proceedings of the Twenty-Ninth Annual Conference on Uncertainty in Artificial Intelligence (UAI-13) with Florent Teichteil-Königsbuch, Jean-Loup Farges, and Didier Dubois, (2013) [41]. Presentation: poster and 2 minute speech in english.
- *Structured Possibilistic Planning Using Decision Diagrams*, Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence, (AAAI-14) with Florent Teichteil-Königsbuch, Jean-Loup Farges and Didier Dubois, (2014) [42]. Presentation: poster and 20 minute speech in english.
- *Planning in Partially Observable Domains with Fuzzy Epistemic States and Probabilistic Dynamics*, Proceedings of the Ninth International Conference on Scalable Uncertainty Management (SUM-15) with Didier Dubois, Jean-Loup Farge and Florent Teichteil-Königsbuch, (2015). Presentation in english.
- *Processus Décisionnels de Markov Possibilistes à Observabilité Mixte*, Eighth Journées Francophones sur la Planification, la Décision et l'Apprentissage pour la conduite de systèmes (JFPDA-13), with Florent Teichteil-Königsbuch, Jean-Loup Farges, and Didier Dubois, (2013). Presentation: poster and 30 minute speech in french.
- *Planification dans des domaines partiellement observables avec des états épistémiques flous et une dynamique probabiliste.*, Tenth Journées Francophones sur la Planification, la Décision et l'Apprentissage pour la conduite de systèmes (JFPDA-15), with Didier Dubois, Jean-Loup Farge and Florent Teichteil-Königsbuch, (2015). Presentation: 30 minute speech in french.

Workshop paper

- *Structured Possibilistic Planning Using Decision Diagrams*, Workshop “Models and Paradigms for Planning under Uncertainty: a Broad Perspective” of the Twenty-Fourth International Conference on Automated Planning and Scheduling (ICAPS-14). Presentation by Florent Teichteil-Königsbuch.

Journal papers

- *A Possibilistic Estimation of Human Attentional Errors*, in the IEEE Transactions on Fuzzy Systems (TFS), with Sergio Pizzoli, Catherine Tessier, Jean-Loup Farges, Didier Dubois and Frédéric Dehais.
- *Processus Décisionnels de Markov Possibilistes à Observabilité Mixte*, in a special edition of the Revue d'Intelligence Artificielle (RIA), with Florent Teichteil-Königsbuch, Jean-Loup Farges, and Didier Dubois.

Oral presentations

- “Possibilistic MDP and POMDP”. Seminar in english of the SequeL team at INRIA Lille, France (2012).
- “Modèles possibilistes pour la décision séquentielle dans l’incertain”. Seminar in french of the MAD team at GREYC Caen, France (2013).
- EDSYS days, doctoral school conference: 30 minute speech in french at ISAE-SUPAERO, Toulouse France (2014).
- PhD student days (JDD), 20 minute speech in french at Onera – The French Aerospace Lab, Toulouse France (2014 and 2015).
- Onera-DLR Aerospace Symposium, (ODAS 2015), 30 minute speech in english, at Onera – The French Aerospace Lab, Toulouse France (2015).

Code

- Participation in the International Probabilistic Planning Competition 2014 – MDP track, with the algorithm PPUDD.
- Github repository <http://www.github.com/drougui/ppudd>.

Title: Exploiting Imprecise Information Sources for Sequential Decision Making under Uncertainty

Abstract: Partially Observable Markov Decision Processes (POMDPs) define a useful formalism to express probabilistic sequential decision problems under uncertainty. When this model is used for a robotic mission, the *system* is defined as the features of the robot and its environment, needed to express the mission. The system state is not directly seen by the agent (the robot). Solving a POMDP consists thus in computing a strategy which, on average, achieves the mission best *i.e.* a function mapping the information known by the agent to an action. Some practical issues of the POMDP model are first highlighted in the robotic context: it concerns the modeling of the agent ignorance, the imprecision of the observation model and the complexity of solving real world problems. A counterpart of the POMDP model, called π -POMDP, simplifies uncertainty representation with a qualitative evaluation of event plausibilities. It comes from Qualitative Possibility Theory which provides the means to model imprecision and ignorance. After a formal presentation of the POMDP and π -POMDP models, an update of the possibilistic model is proposed. Next, the study of factored π -POMDPs allows to set up an algorithm named PPUDD which uses Algebraic Decision Diagrams to solve large structured planning problems. Strategies computed by PPUDD, which have been tested in the context of the competition IPPC 2014, can be more efficient than those produced by probabilistic solvers when the model is imprecise or for high dimensional problems. We show next that the π -Hidden Markov Processes (π -HMP), *i.e.* π -POMDPs without action, produces useful diagnosis in the context of Human-Machine interactions. Finally, a hybrid POMDP benefiting from the possibilistic and the probabilistic approach is built: the qualitative framework is only used to maintain the agent's knowledge. This leads to a strategy which is pessimistic facing the lack of knowledge, and computable with a solver of fully observable Markov Decision Processes (MDPs). This thesis proposes some ways of using Qualitative Possibility Theory to improve computation time and uncertainty modeling in practice.

Keywords: POMDP, Planning under Uncertainty, Possibility Theory, Autonomous Robotics, Imprecise Knowledge

Titre: Tirer Profit de Sources d'Information Imprécises pour la Décision Séquentielle dans l'Incertain

Résumé: Les Processus Décisionnels de Markov Partiellement Observables (PDMPOs) permettent de modéliser facilement les problèmes probabilistes de décision séquentielle dans l'incertain. Lorsqu'il s'agit d'une mission robotique, les caractéristiques du robot et de son environnement nécessaires à la définition de la mission constituent le *système*. Son état n'est pas directement visible par l'*agent* (le robot). Résoudre un PDMPO revient donc à calculer une stratégie qui remplit la mission au mieux en moyenne, *i.e.* une fonction prescrivant les actions à exécuter selon l'information reçue par l'agent. Ce travail débute par la mise en évidence, dans le contexte robotique, de limites pratiques du modèle PDMPO: elles concernent l'ignorance de l'agent, l'imprécision du modèle d'observation ainsi que la complexité de résolution. Un homologue du modèle PDMPO appelé π -PDMPO, simplifie la représentation de l'incertitude: il vient de la Théorie des Possibilités Qualitatives qui définit la plausibilité des événements de manière qualitative, permettant la modélisation de l'imprécision et de l'ignorance. Une fois les modèles PDMPO et π -PDMPO présentés, une mise à jour du modèle possibiliste est proposée. Ensuite, l'étude des π -PDMPOs factorisés permet de mettre en place un algorithme appelé PPUDD utilisant des Arbres de Décision Algébriques afin de résoudre plus facilement les problèmes structurés. Les stratégies calculées par PPUDD, testées par ailleurs lors de la compétition IPPC 2014, peuvent être plus efficaces que celles des algorithmes probabilistes dans un contexte d'imprécision ou pour certains problèmes à grande dimension. Nous montrons ensuite que les Processus de Markov Cachés possibilistes (π -PMCs), *i.e.* les π -PDMPOs sans les actions, produisent de bons diagnostics dans le contexte de l'interaction Homme-Machine. Enfin, un PDMPO hybride tirant profit des avantages des modèles probabilistes et possibilistes est présenté: seule la connaissance de l'agent est maintenue sous forme qualitative. Ce modèle mène à une stratégie qui réagit de manière pessimiste au défaut de connaissance, et calculable avec des algorithmes de résolution des Processus Décisionnels de Markov entièrement observables (PDM). Cette thèse propose d'utiliser les possibilités qualitatives dans le but d'obtenir des améliorations en termes de temps de calcul et de modélisation de l'incertitude en pratique.

Mots-clés: PDMPO, Planification dans l'Incertain, Théorie des Possibilités, Robotique Autonome, Connaissance Imprecise