Fourier Transforms of Rectangular Functions

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Consider a rectangular function in continuous time [1]:

$$rect(t) = \begin{cases} 1, & |t| <= T/2 \\ 0, & |t| > T/2 \end{cases} \tag{1}$$

The continuous time, continuous frequency Fourier Transform is given by:

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} rect(t) e^{-j2\pi ft} \\ &= \int_{-T/2}^{T/2} e^{-j2\pi ft} \\ &= \left[\frac{1}{-j2\pi f} e^{-j2\pi ft} \right]_{-T/2}^{T/2} \\ &= \frac{1}{-j2\pi f} \left[e^{-j2\pi f \frac{T}{2}} - e^{j2\pi f \frac{T}{2}} \right] \\ &= \frac{sin(\pi f T)}{\pi f} \end{aligned} \tag{2}$$

Consider a rectangular function in discrete time:

$$rect(n) = \begin{cases} 1, & n < N \\ 0, & otherwise \end{cases} \tag{3}$$

The discrete time, continuous frequency Fourier Transform is given by:

$$
\begin{aligned}
X(w) &= \sum_{n=0}^{N-1} e^{-j\omega n} \\
&= e^{-j\omega 0} + e^{-j\omega 1} + ... + e^{-j\omega(N-1)} \\
e^{-j\omega}X(w) &= e^{-j\omega} + e^{-j\omega 2} + e^{-j\omega 3} + ... + e^{-j\omega N} \\
X(w) - e^{-j\omega}X(w) &= 1 - e^{-j\omega N} \\
X(w) &= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\
\frac{e^{j\frac{\omega N}{2}}}{e^{j\frac{\omega}{2}}}X(w) &= \frac{e^{j\frac{\omega N}{2}}(1 - e^{-j\omega N})}{e^{j\frac{\omega}{2}}(1 - e^{-j\omega})} \\
e^{j\frac{\omega(N-1)}{2}}X(w) &= \frac{e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} \\
X(w) &= e^{-j\frac{\omega(N-1)}{2}} \frac{sin(\frac{\omega N}{2})}{sin(\frac{\omega}{2})}
\end{aligned}
\tag{4}
$$

Which can be interpreted as a complex, unit magnitude phase shift (delay) term multiplied by a real valued magnitude term. For small ω, the denominator term $sin(a) \approx a$, which results in a magnitude spectrum similar to Equation 2.

In deriving Equation 2, a simpler approach is to use the identity for the sum of a geometric series:

$$
\sum_{k=0}^{K-1} ar^k = a\left(\frac{1-r^n}{1-r}\right) \tag{5}
$$

References

[1] Rectangular Function. `https://en.wikipedia.org/wiki/Rectangular_function`.