# Fourier Transforms of Rectangular Functions
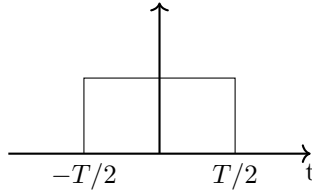
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## 1  Continuous Time Continuous Frequency

Consider a rectangular function in continuous time [1]:

$$rect(t) = \begin{cases} 1, & |t| <= T/2 \\ 0, & otherwise \end{cases} \tag{1}$$

Figure 1: Rectangular Function in Continuous Time



The continuous time, continuous frequency Fourier Transform is given by:

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} rect(t)e^{-j2\pi ft}dt \\
&= \int_{-T/2}^{T/2} e^{-j2\pi ft}dt \\
&= \left[ \frac{1}{-j2\pi f}e^{-j2\pi ft} \right]_{-T/2}^{T/2} \\
&= \frac{1}{-j2\pi f}\left[ e^{-j2\pi f\frac{T}{2}} - e^{j2\pi f\frac{T}{2}} \right] \\
&= \frac{sin(\pi fT)}{\pi f}
\end{aligned} \tag{2}$$

## 2    Discrete Time Continuous Frequency

Consider a rectangular function in discrete time:

$$rect(n) = \begin{cases} 1, & n = 0, 1, \ldots N - 1 \\ 0, & otherwise \end{cases} \tag{3}$$

$$
\begin{aligned}
X(w) &= \sum_{n=-\infty}^{\infty} rect(n)e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} e^{-j\omega n} \\
&= e^{-j\omega 0} + e^{-j\omega 1} + \ldots + e^{-j\omega(N-1)} \\
e^{-j\omega}X(w) &= e^{-j\omega} + e^{-j\omega 2} + e^{-j\omega 3} + \ldots + e^{-j\omega N} \\
X(w) - e^{-j\omega}X(w) &= 1 - e^{-j\omega N} \\
X(w) &= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\
\frac{e^{j\frac{\omega N}{2}}}{e^{j\frac{\omega}{2}}}X(w) &= \frac{e^{j\frac{\omega N}{2}}(1 - e^{-j\omega N})}{e^{j\frac{\omega}{2}}(1 - e^{-j\omega})} \\
e^{j\frac{\omega(N-1)}{2}}X(w) &= \frac{e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} \\
X(w) &= e^{-j\frac{\omega(N-1)}{2}} \frac{sin\left(\frac{\omega N}{2}\right)}{sin\left(\frac{\omega}{2}\right)}
\end{aligned}
\tag{4}
$$

This can be interpreted as a complex term representing a linear phase shift (delay), multiplied by a real valued magnitude term. For small $\omega$, the denominator term $sin(a) \approx a$, which results in a magnitude spectrum similar to the continuous time Fourier Transform (Equation 2).

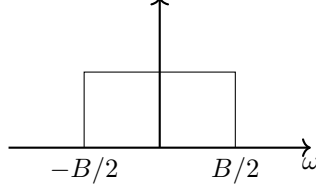Equation 4 can also be derived using the closed form expression for the sum of a geometric series:

$$\sum_{k=0}^{K-1} ar^k = a\left(\frac{1 - r^n}{1 - r}\right) \tag{5}$$

## 3    Low Pass Filter Design

Consider a continuous frequency rectangular function:

$$H(\omega) = \begin{cases} 1, & |\omega| <= B/2 \\ 0, & otherwise \end{cases} \tag{6}$$

Figure 2: Rectangular Function in Continuous Frequency



$H(\omega)$ can be considered an ideal low pass filter for real input signals, with bandwidth $B/2$ radians. The discrete time, continuous frequency inverse Fourier Transform is given by:

$$
\begin{aligned}
h(n) &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} H(\omega)e^{j\omega n} d\omega \\
&= \frac{1}{2\pi} \int_{-B/2}^{+B/2} e^{j\omega n} d\omega \\
&= \frac{1}{2\pi jn} \left[ e^{j\omega B/2} - e^{-j\omega B/2} \right] \\
&= \frac{sin\left(\frac{nB}{2}\right)}{\pi n} \\
&= \left( \frac{\frac{B}{2\pi}}{\frac{B}{2\pi}} \right) \frac{sin\left(\frac{nB}{2}\right)}{\pi n} \\
&= \frac{B}{2\pi} sinc\left(\frac{nB}{2}\right)
\end{aligned}
\tag{7}
$$

The infinite length even sequence $h(n)$ can be considered the time domain impulse response of an ideal low pass filter with cut off frequency $B/2$ radians. In practice filters are limited to $N_{tap}$ taps by applying a tapered window. This results in non-ideal frequency response, rather than an ideal rectangular (brick wall) filter. A suitable length $N_{tap}$ set of Hanning windowed filter coefficients can be found by:

$$
\begin{aligned}
c(i) &= h\left(i - \frac{N_{tap} - 1}{2}\right) w(i) \quad i = 0 \dots N_{tap} - 1 \\
w(i) &= 0.5 - 0.5cos\left(\frac{2\pi}{N_{tap} - 1}\right)
\end{aligned}
\tag{8}
$$

This expression can be used when $N_{tap}$ is odd or even.

One use case is filtering an OFDM modem signal which has significant side lobe energy at frequencies outside of the carriers. In this use case, we wish to have the filter -1dB point at a frequency just outside the carriers, in order to minimise distortion to the PSK constellation. For a given $N_{tap}$, the smallest

$B/2$ can be found by experiment that produces acceptable distortion of the scatter diagram.

## 3.1 Complex Band Pass Filter

Equation 6 can be considered a low pass filter of bandwidth $B/2$ radians for real valued signals, or a bandpass filter with centre frequency 0 radians and bandwidth $B$ radians for analytic (single sided complex valued) signals. This suggests the low pass prototype filter can be shifted in frequency to construct a bandpass filter for analytic signals.

Given the low pass prototype coefficients $h(n)$, we wish to design a band pass filter:

$$H(\omega) = \begin{cases} 1, & \alpha - B/2 <= \omega <= \alpha + B/2 \\ 0, & otherwise \end{cases} \tag{9}$$

Where $\alpha$ is the centre frequency of the bandpass filter of bandwidth $B$.
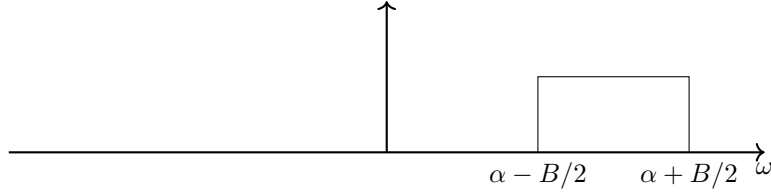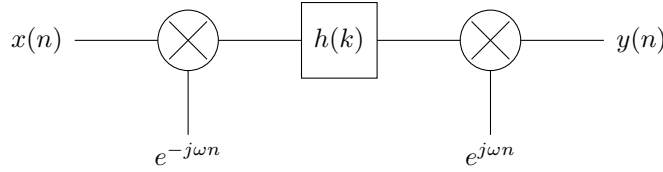
Figure 3: Bandpass Filter



Figure 4: Bandpass Filter Signal Processing



Consider the time domain sequence of operations in Figure 4:

$$y(n) = e^{j\alpha n} \sum_{k=0}^{N_{tap}-1} h(k)x(n-k)e^{-j\alpha(n-k)} \tag{10}$$

Here we frequency shift the input energy in $x(n)$ centred on $\alpha$ down to 0, filter it, and shift the result back up to a centre frequency of $\alpha$, effectively implementing

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the band pass filter in Equation 9. Re-arranging Equation 10:

$$y(n) = e^{j\alpha n} \sum_{k=0}^{N_{tap}-1} h(k)x(n-k)e^{-j\alpha n}e^{j\alpha k}$$

$$= \sum_{k=0}^{N_{tap}-1} h(k)e^{j\alpha k}x(n-k)$$

$$= \sum_{k=0}^{N_{tap}-1} h_\alpha(k)x(n-k)$$

$$h_\alpha(k) = h(k)e^{j\alpha k}$$

(11)

Note $h_\alpha(k)$ is complex, and can be pre-computed from $h(n)$ at initialisation time. An alternative approach to deriving Equation 11 would be to find $h_\alpha(k)$ directly from the inverse FT of (9).

Each operation in the summation of 11 is complex so requires $4N_{tap}$ Multiply Accumulates (MACs) to compute each $y(n)$, or for a block of $N$ samples, $4N_{tap}N$ MACs.

In contrast, Equation 10 has real coefficients, so for complex $x(n)$ requires $2N_{tap}$ MACs to compute each $y(n)$, plus 4 MACs each for the up and down frequency shifts. We assume $e^{-j\alpha n}$ is stored as a look up table and requires no FOPs. Alternatively it can be computed using a recursion with an additional 4 MACs/sample. The up and down conversions are conjugates so only one needs to be computed, the other can then be obtained by a sign change. For a block of $N$ samples, this is a total of $(2N_{tap}+8)N$ MACs.

Table 10 presents the MACs required to process a 100ms ($N = 800$ samples at $Fs = 8000Hz$) block of samples. In this example the real coefficient algorithm in Equation 10 is the most efficient.

| Algorithm | MACs |
|---|---|
| Real $h(k)$ | 166,400 |
| Complex $h_\alpha(k)$ | 320,000 |

Table 1: Multiply Accumulates (MACs) to bandpass filter a $N = 800$ block of samples using a $N_{tap} = 100$ filter

## 3.2   Further Work

MACs analysis - Note on shifts at end of block rather than sample by sample updates of pointers.

Octave script to compare and plot fir1() to 7. Show effect of windowing on frequency response.

Rather than selecting $B/2$ by trial and error, we could find an expression for the windowed FT $X_w(\omega)$, and solve for $B/2$ given $|X_w(\omega)| = 10^{-1/20}$, where

$\omega$ is the frequency of the desired -1dB point (e.g. just outside of the outermost OFDM carrier). This could be evaluated for any $N_t$, trading off filter performance (and hence the OFDM signal spurious mask) for CPU load.

# References

[1] Rectangular Function. `https://en.wikipedia.org/wiki/Rectangular_function`.