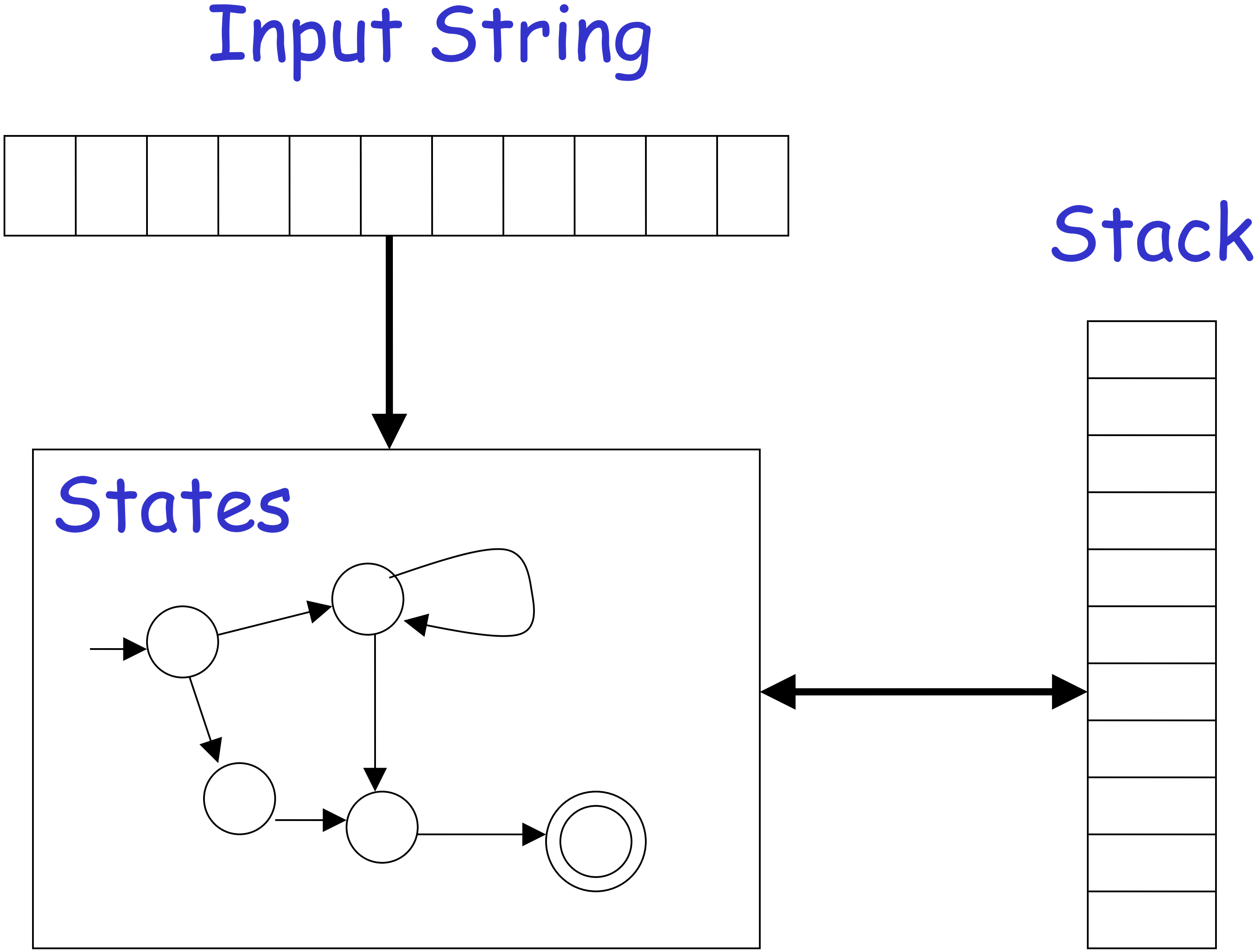
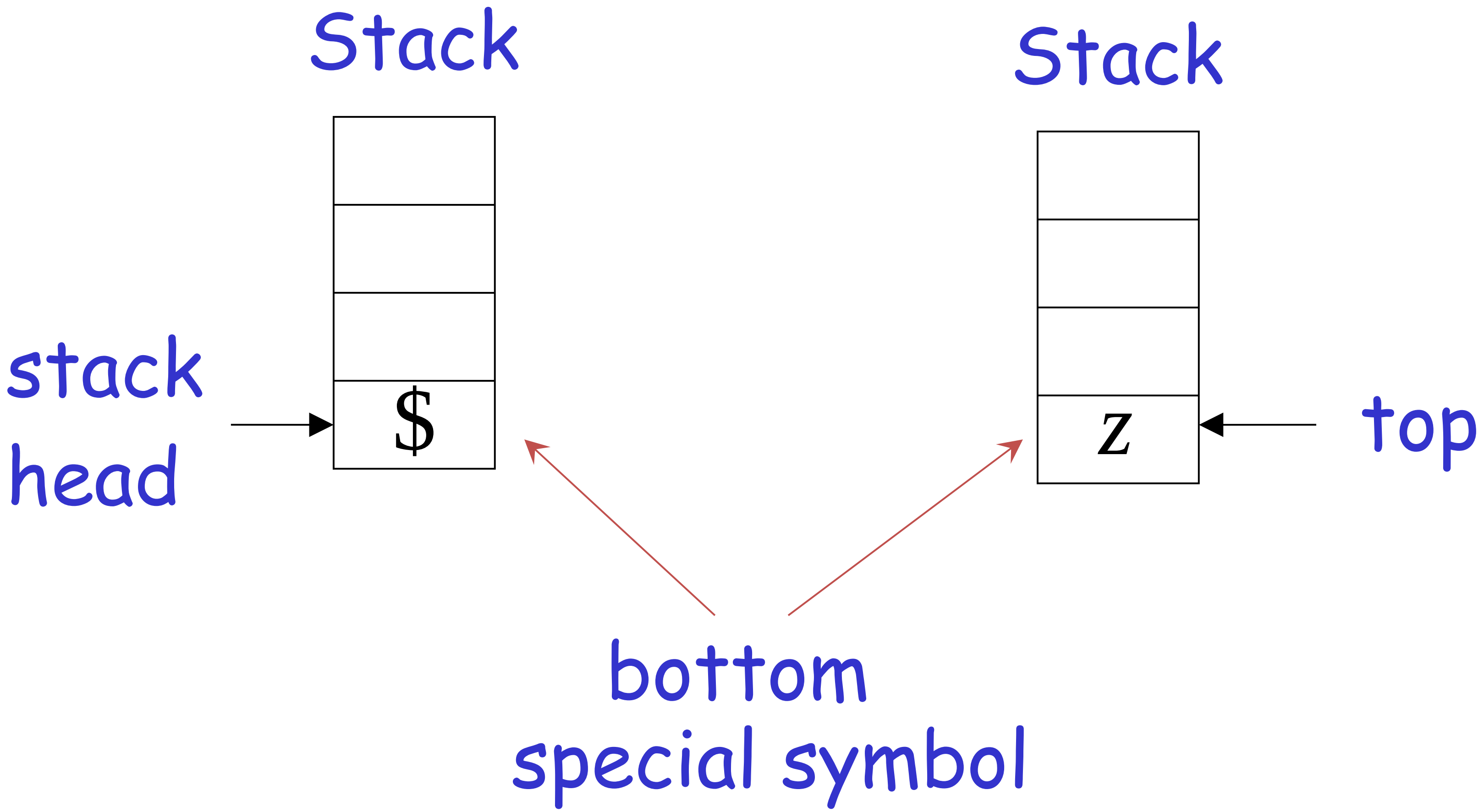
Pushdown Automata
PDAs

Pushdown Automaton -- PDA



Initial Stack Symbol



Stack

Stack

stack head

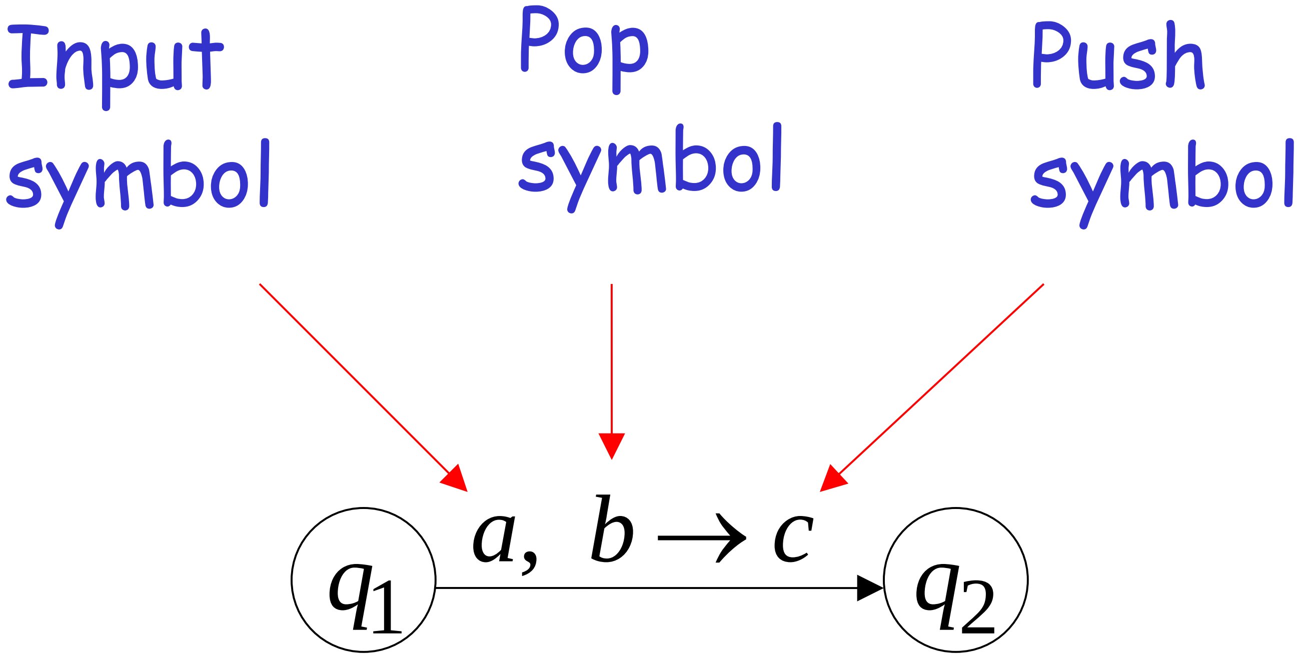
$\$$

z

top

bottom special symbol

The States

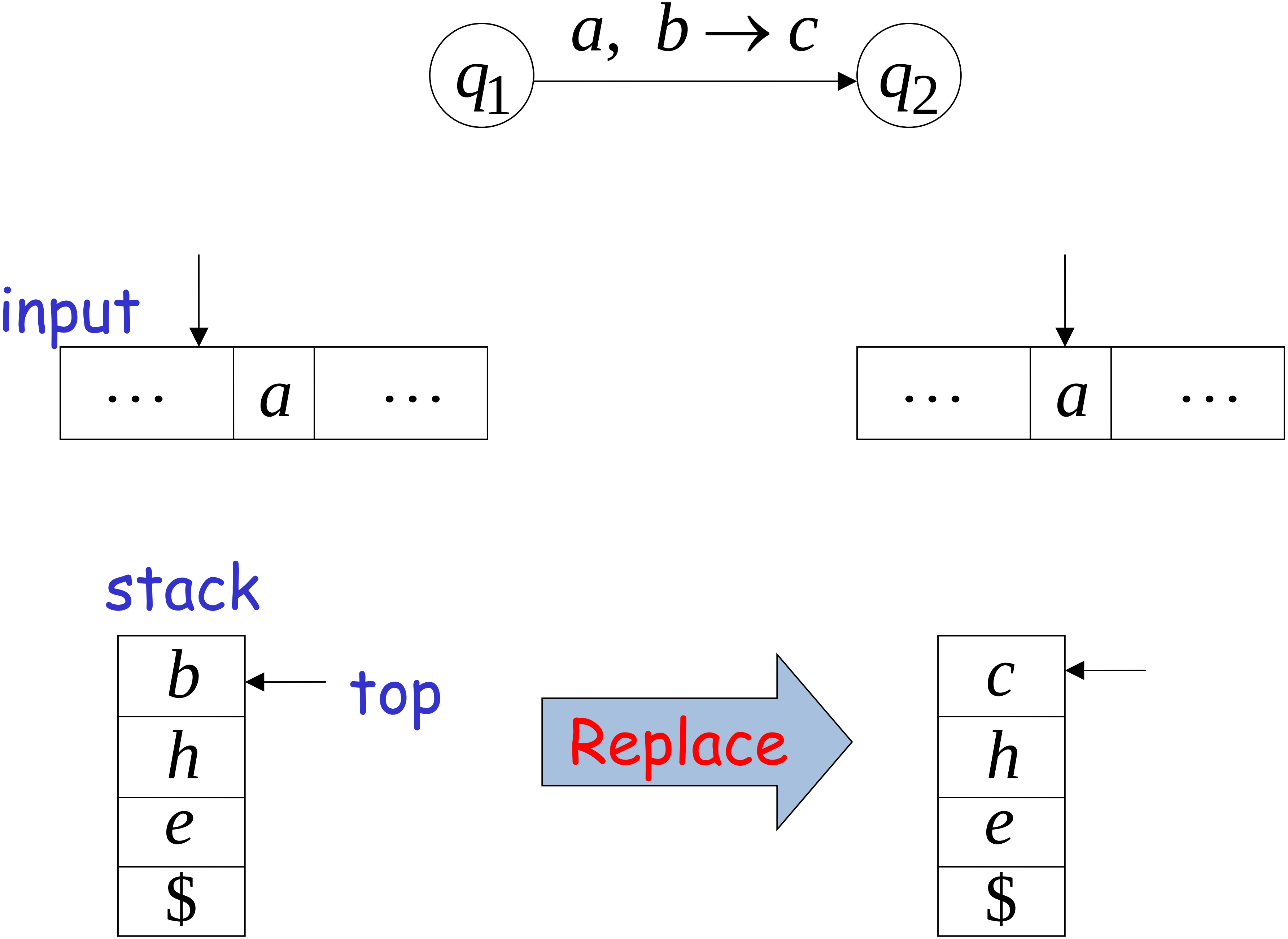


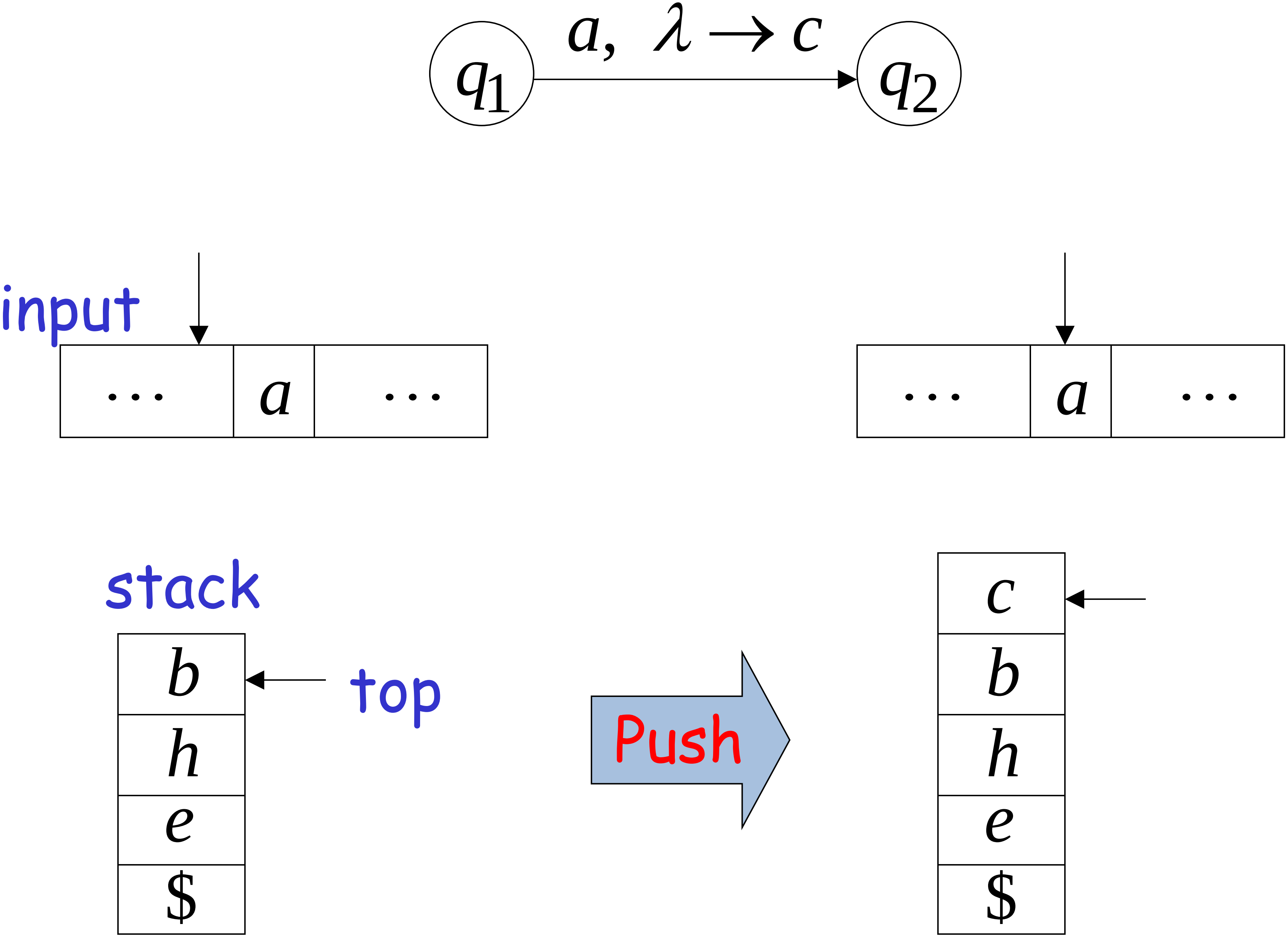
Input symbol

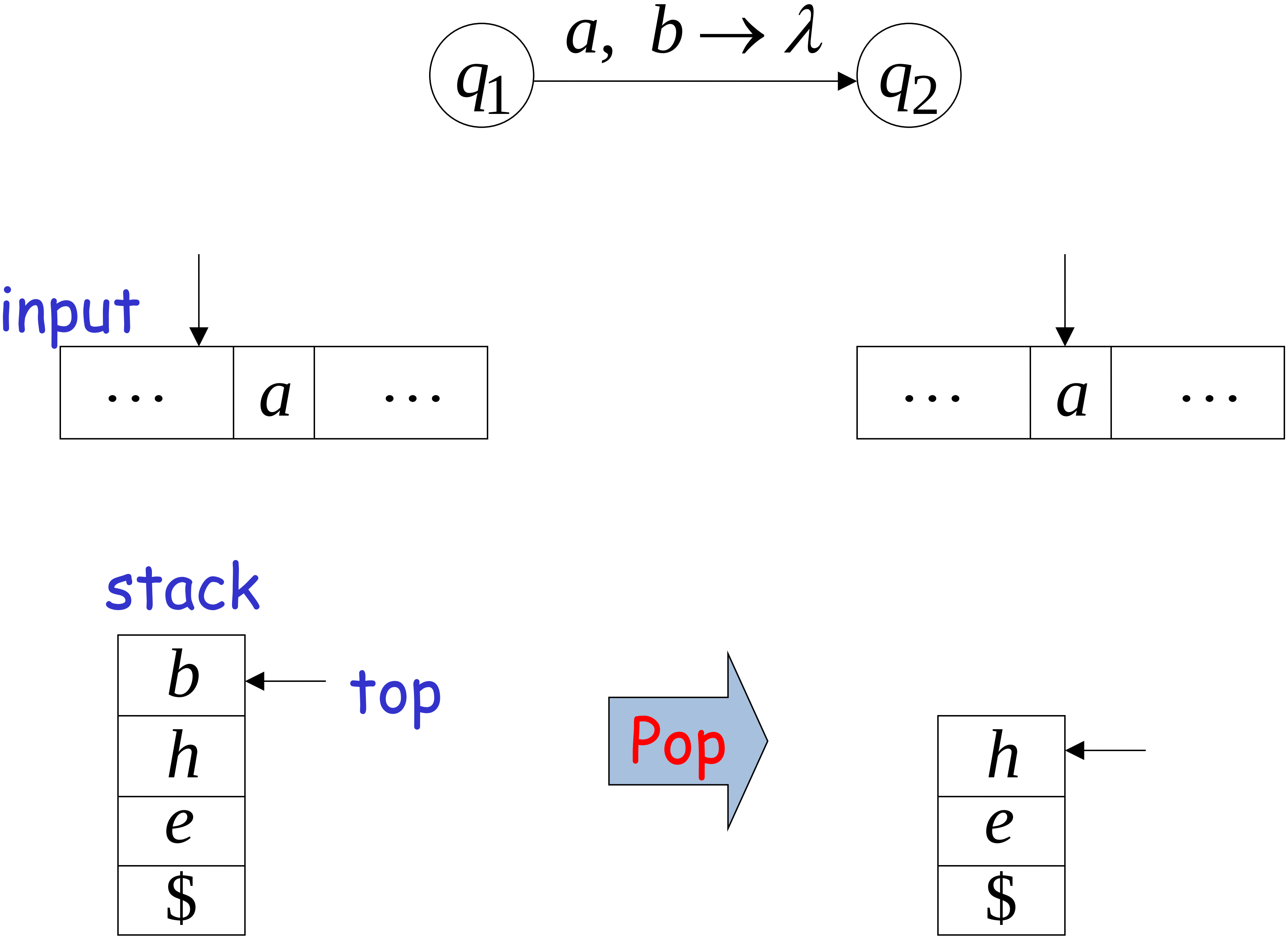
Pop symbol

Push symbol

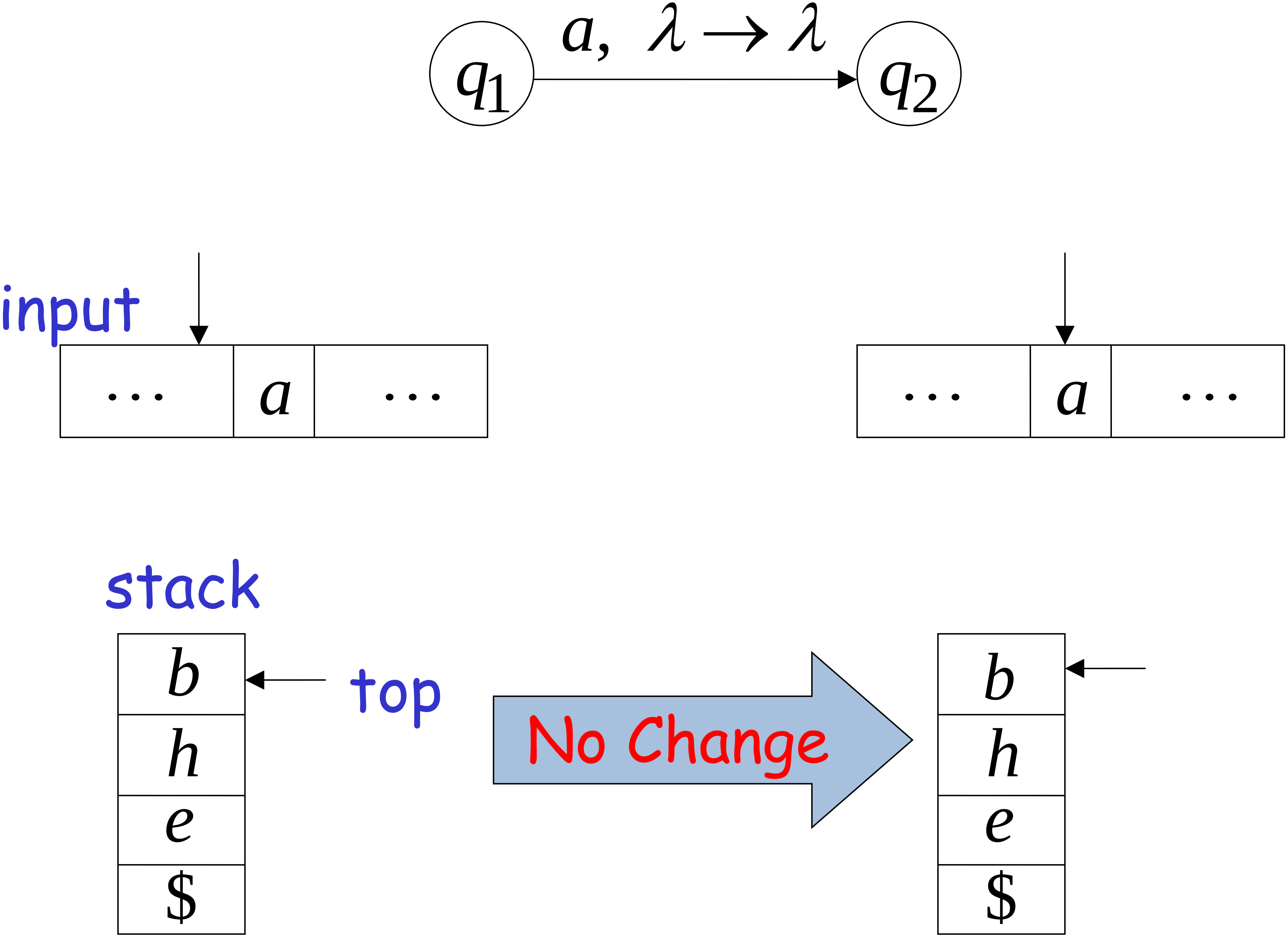
$a, \ b \to c$


q_1
$a, \; b \to c$
q_2
input
$\cdots$
a
$\cdots$
$\cdots$
a
$\cdots$
stack
b
h
e
$\$$
top
Replace
c
h
e
$\$$


q_1
$a, \lambda \to c$
q_2
input
$\cdots$ a $\cdots$
$\cdots$ a $\cdots$
stack
b
h
e
$\$$
top
Push
c
b
h
e
$\$$



q_1
$a, \; b \to \lambda$
q_2
input
$\cdots$
a
$\cdots$
$\cdots$
a
$\cdots$
stack
b
h
e
$\$$
top
Pop
h
e
$\$$



$q_1 \xrightarrow{a,\ \lambda \to \lambda} q_2$

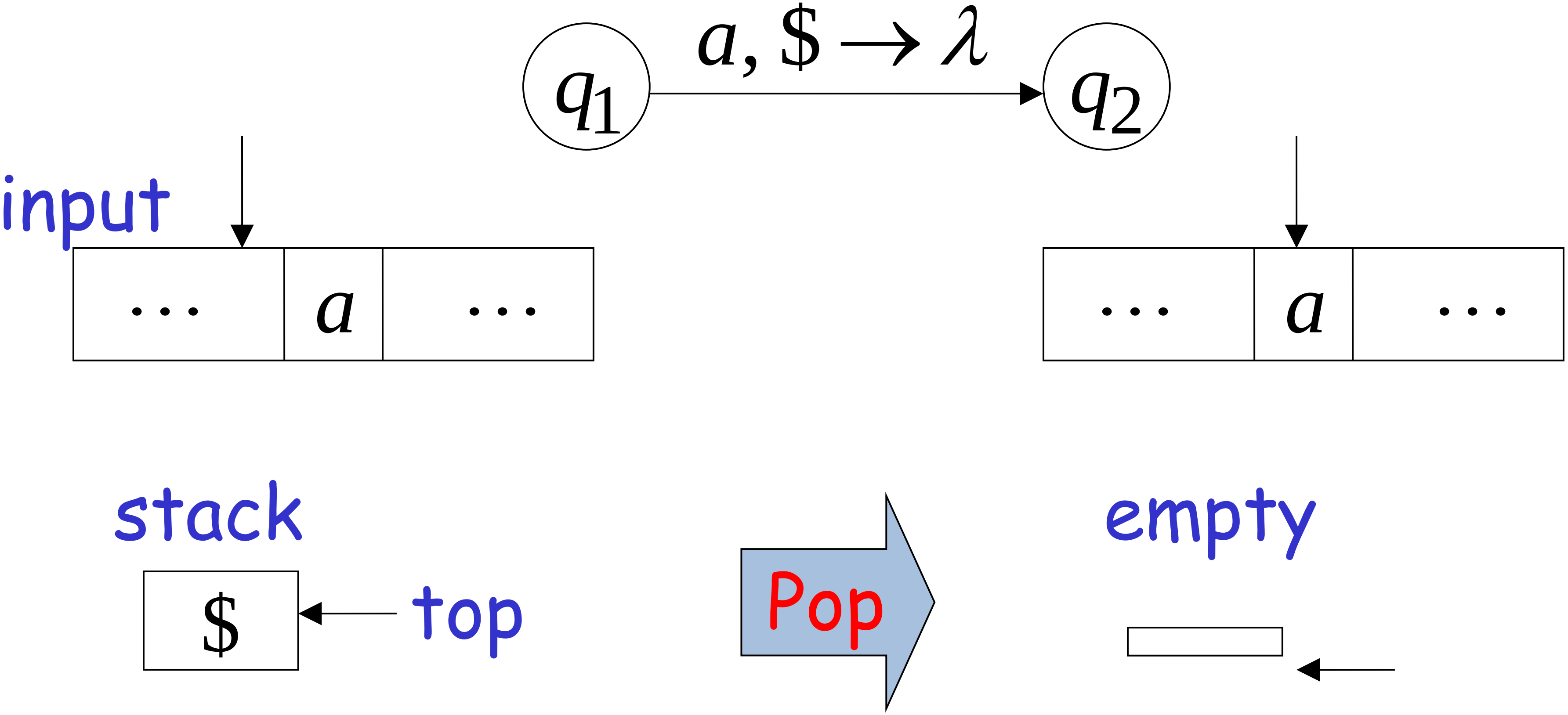
input

stack

top

No Change

A Possible Transition



q_1 $\xrightarrow{a, \$ \to \lambda}$ q_2

input

$\cdots$ | a | $\cdots$

$\cdots$ | a | $\cdots$

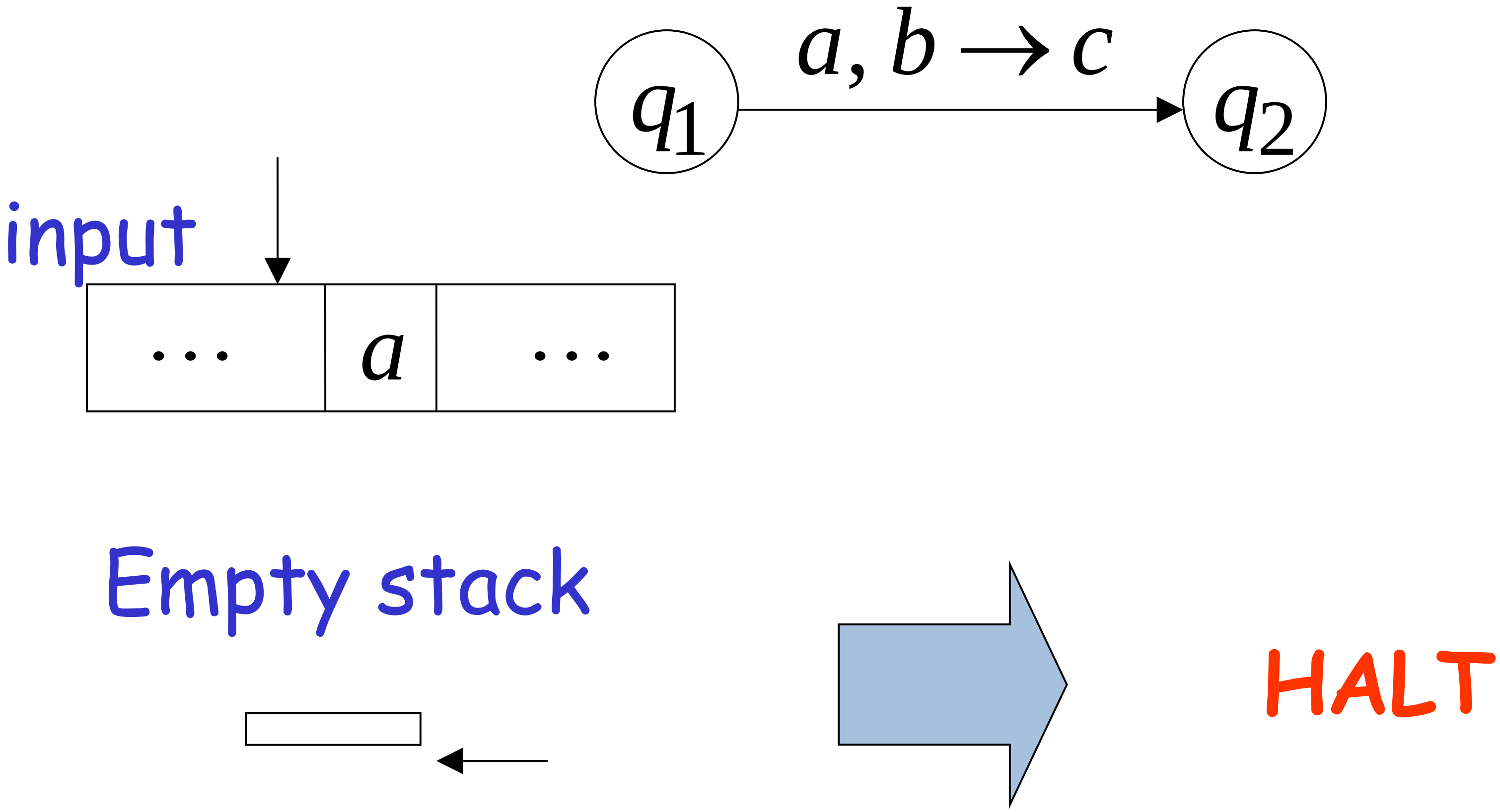
stack

$\$$ ← top

Pop

empty

A Bad Transition



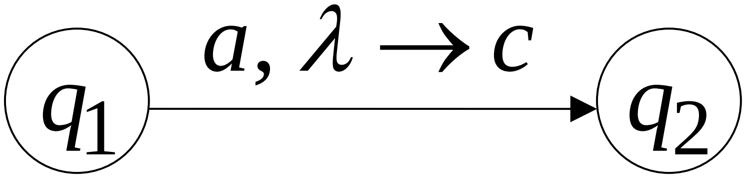
input

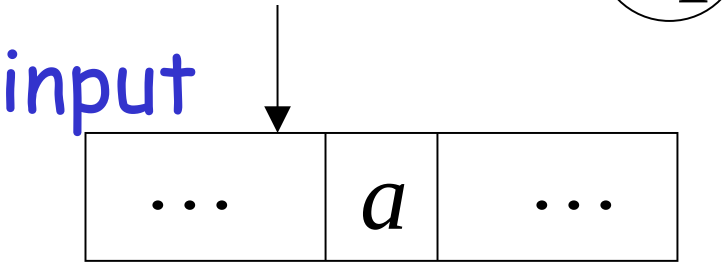
Empty stack

HALT

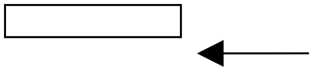
The automaton **Halts** in state q_1
and **Rejects** the input string

A Bad Transition





Empty stack

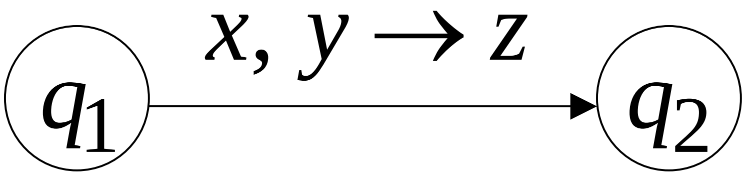


HALT

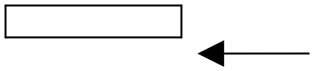
The automaton **Halts** in state q_1

and **Rejects** the input string

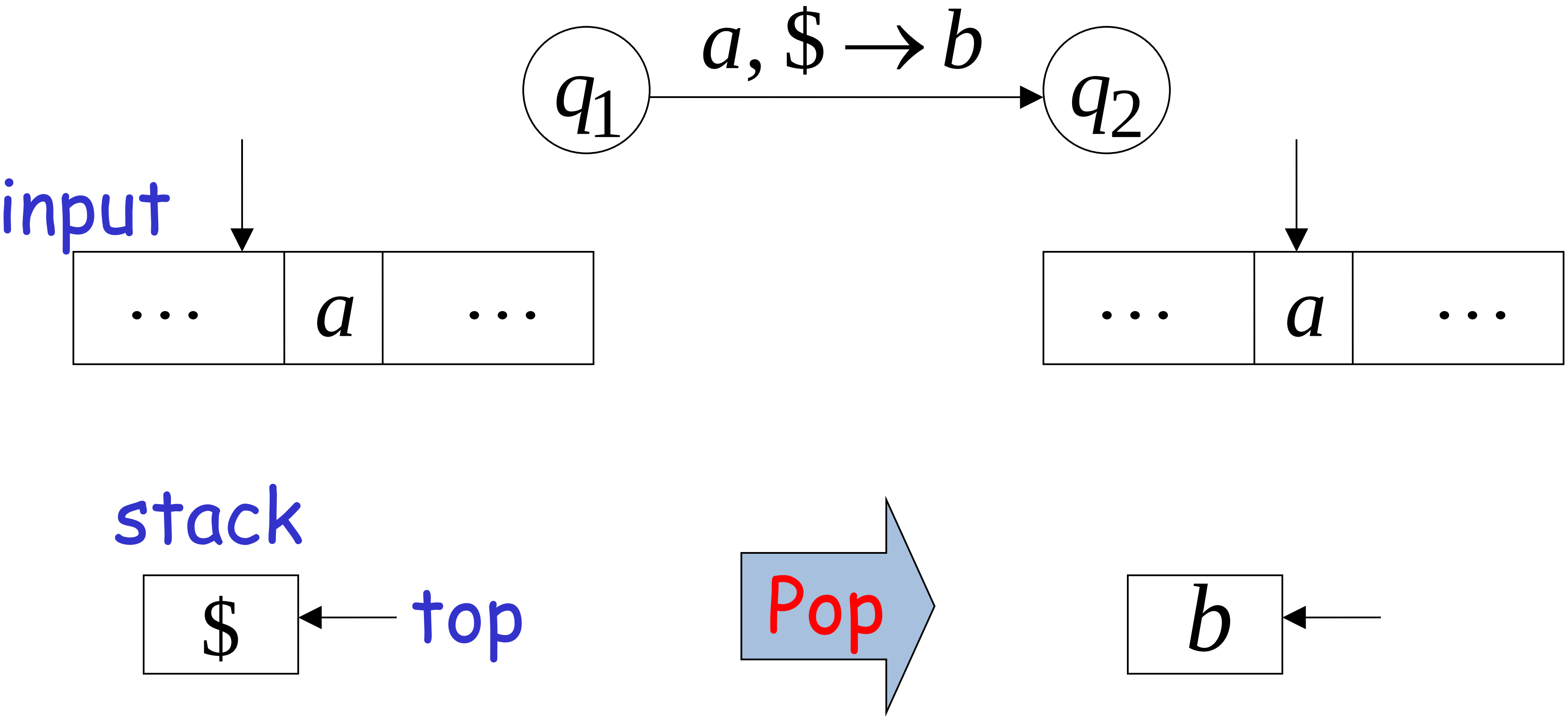
No transition is allowed to be followed
When the stack is empty



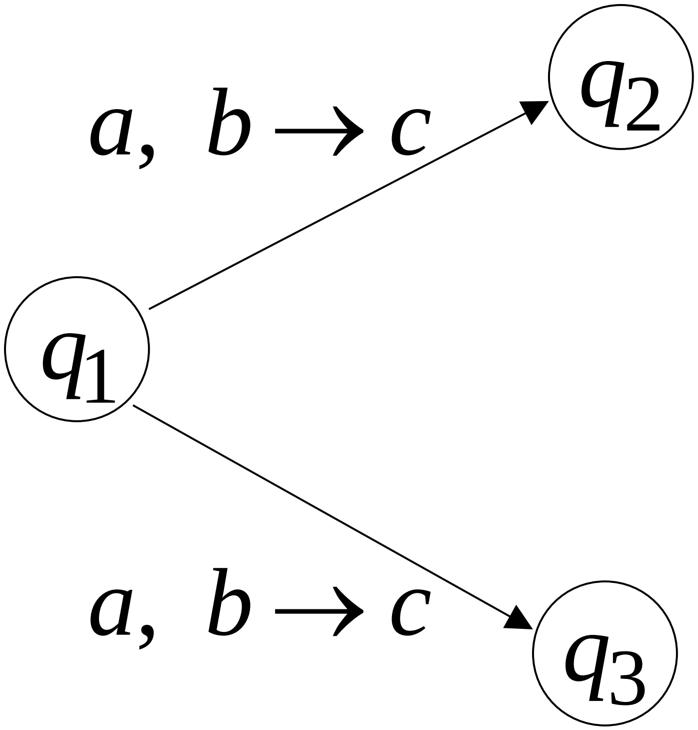
Empty stack

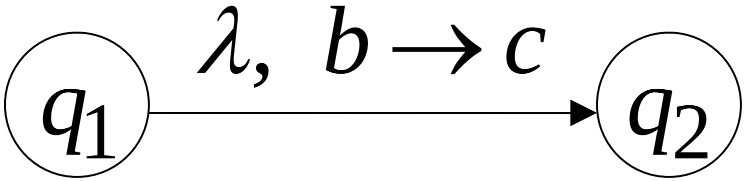


A Good Transition



Non-Determinism



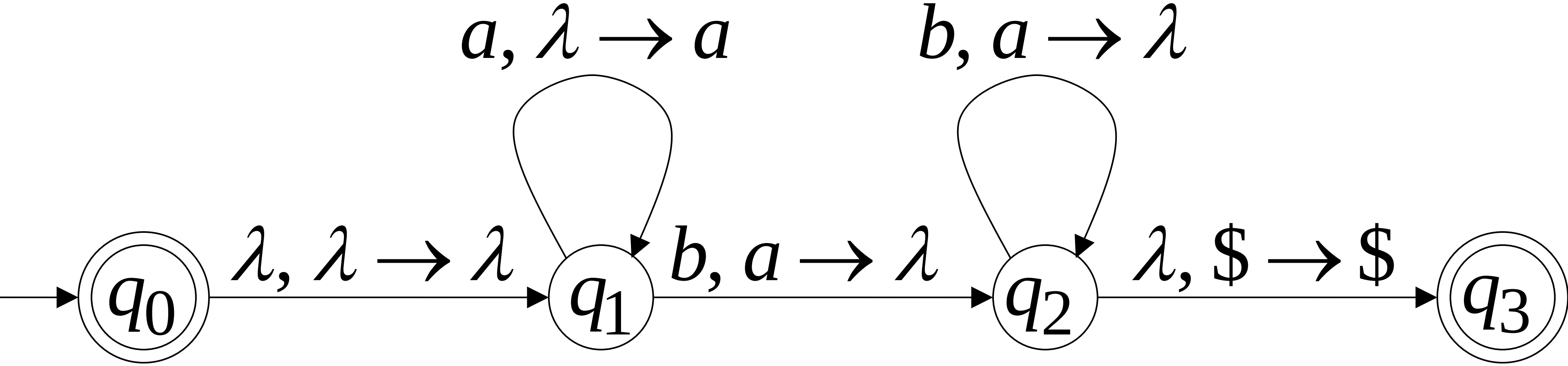


$\lambda -$ transition

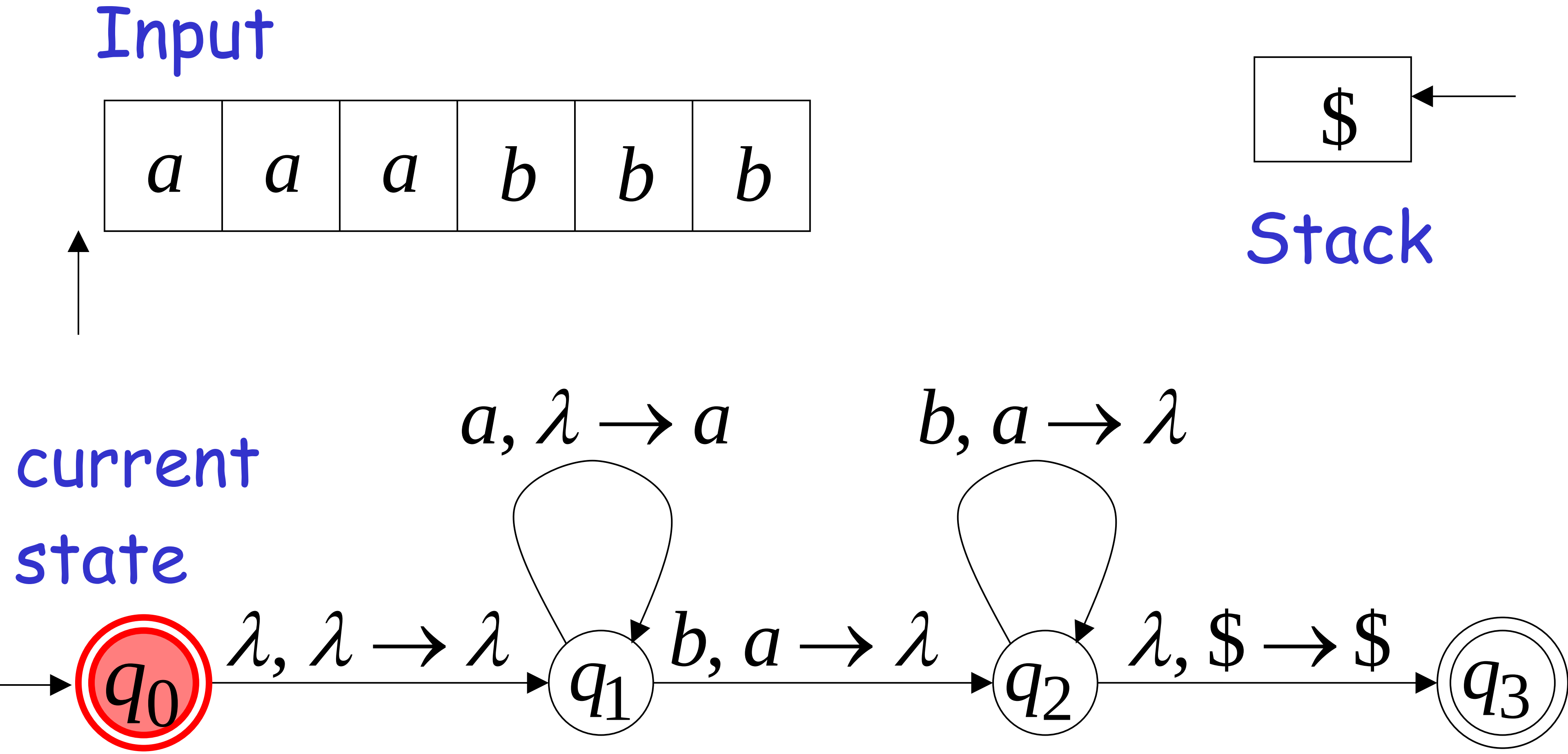
These are allowed transitions in a Non-deterministic PDA (NPDA)

NPDA: Non-Deterministic PDA

Example:


$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
q_0
$\lambda, \lambda \rightarrow \lambda$
q_1
$b, a \rightarrow \lambda$
q_2
$\lambda, \$ \rightarrow \$$
q_3

Execution Example: Time 0



Input

a	a	a	b	b	b

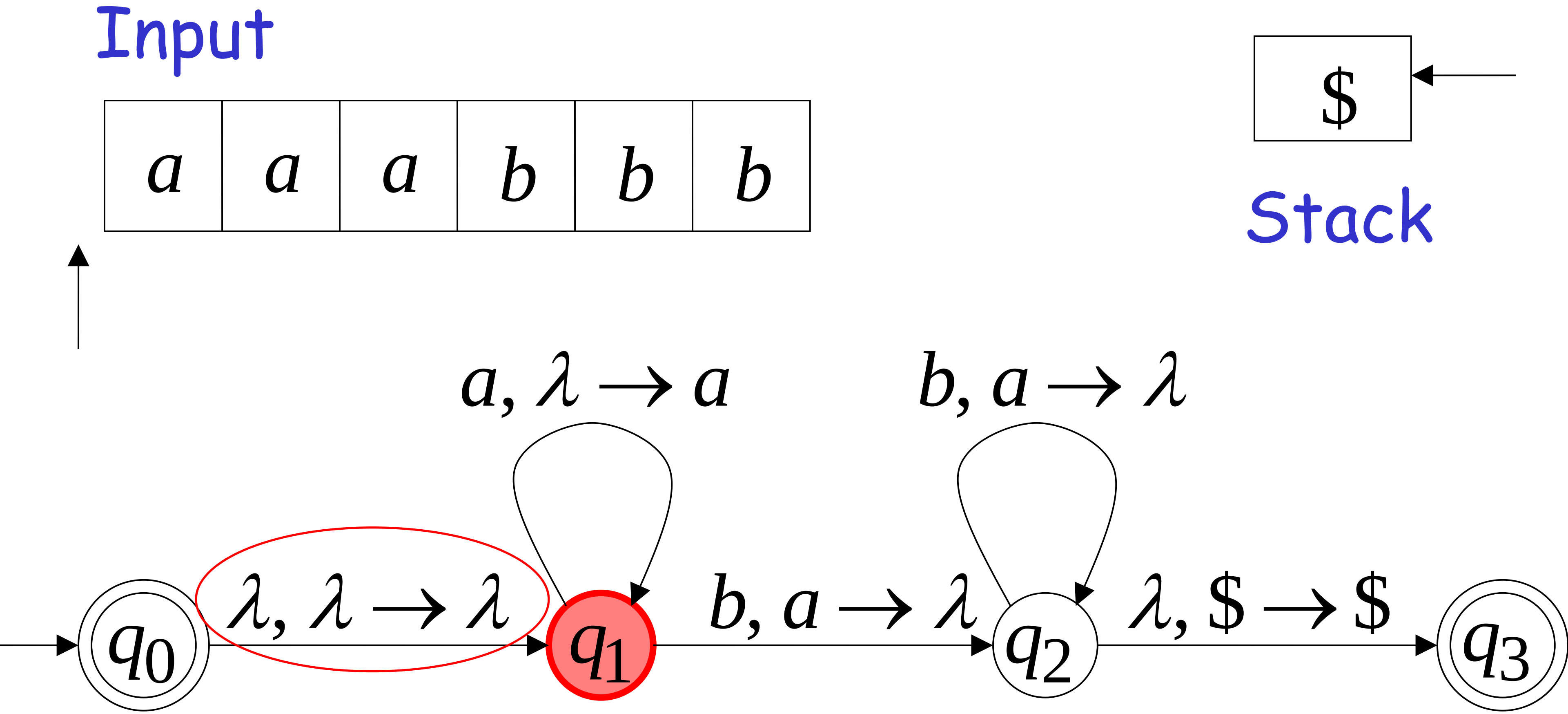
Stack

$\$$

current state

q_0
$\lambda, \lambda \rightarrow \lambda$
q_1
$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
q_2
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_3

Time 1



Input

a a a b b b

$\$$

Stack

$a, \lambda \rightarrow a$

$b, a \rightarrow \lambda$

$\lambda, \lambda \rightarrow \lambda$

$b, a \rightarrow \lambda$

$\lambda, \$ \rightarrow \$$

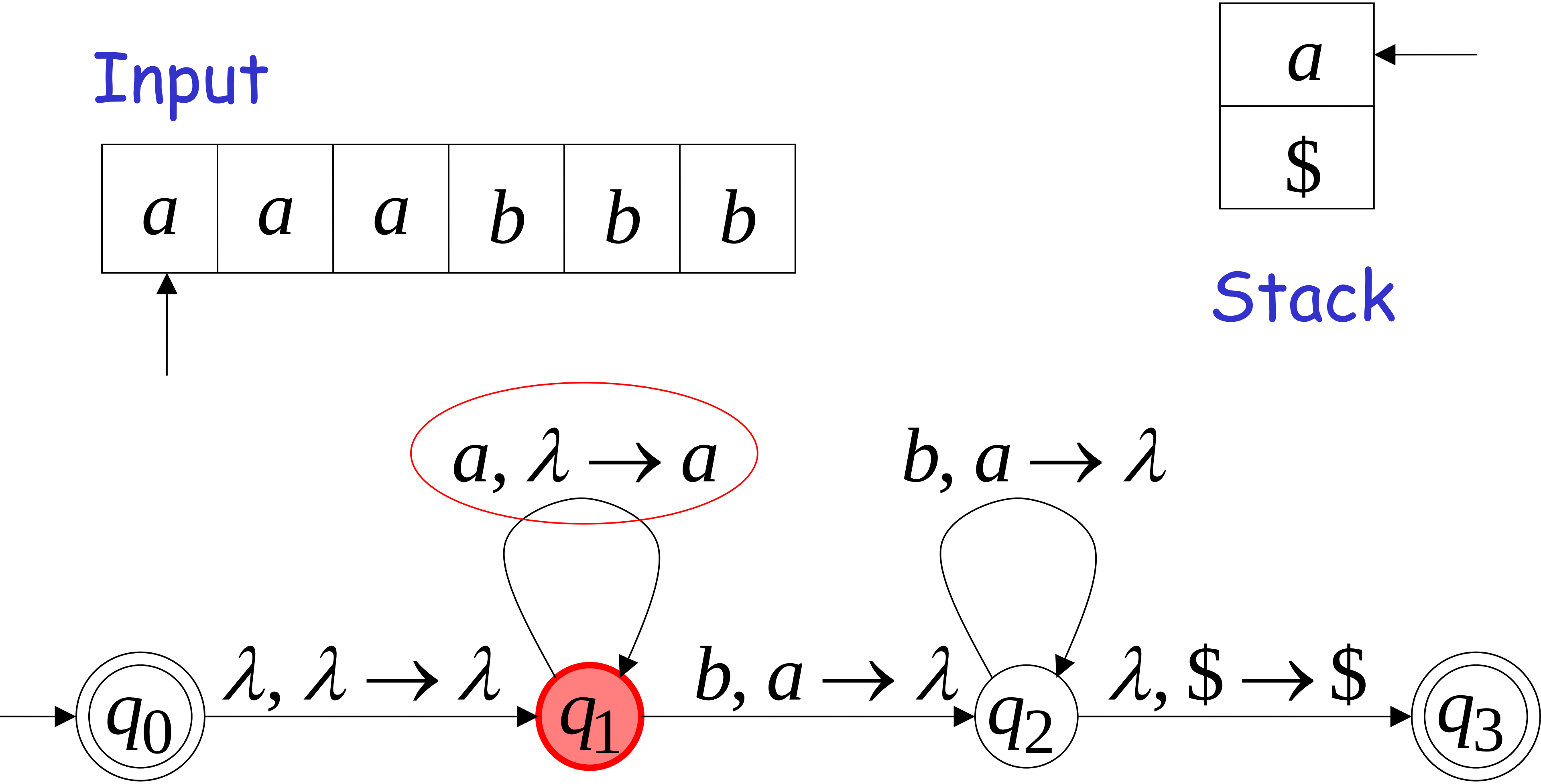
q_0

q_1

q_2

q_3

Time 2


Input
a a a b b b
a
$
Stack
$a, \lambda \to a$
$b, a \to \lambda$
$\lambda, \lambda \to \lambda$
$b, a \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2
q_3

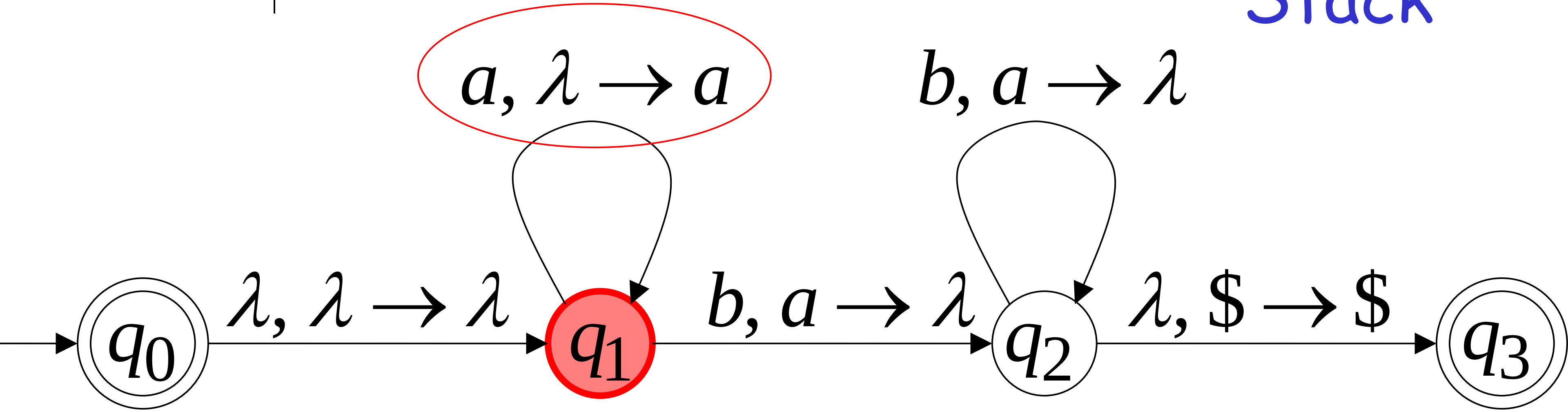
Time 3

Input

a	a	a	b	b	b

a
a
$\$$

Stack



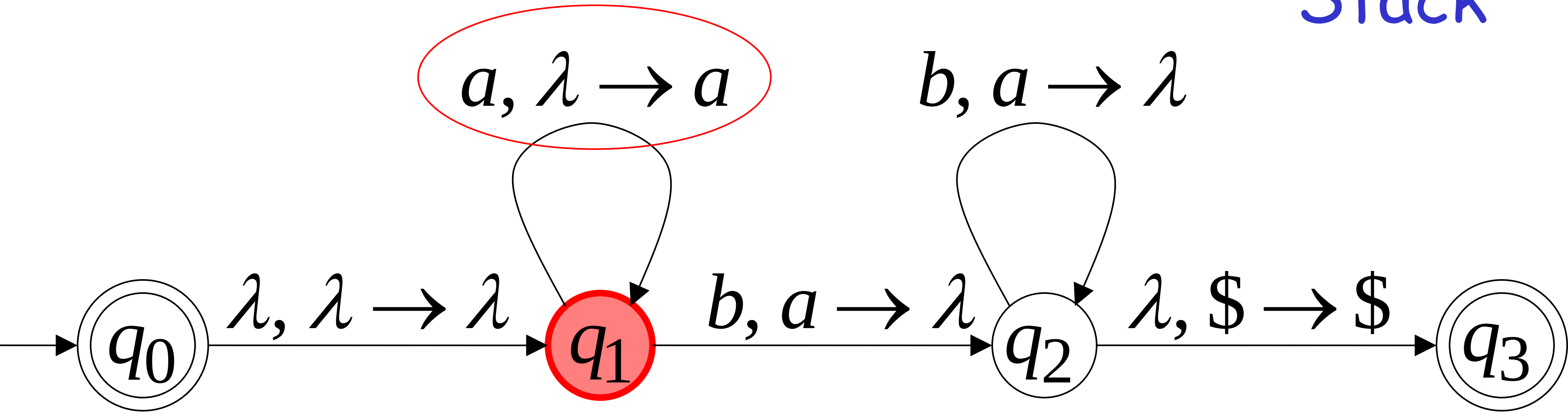
Time 4

Input

a	a	a	b	b	b

a
a
a
$\$$

Stack



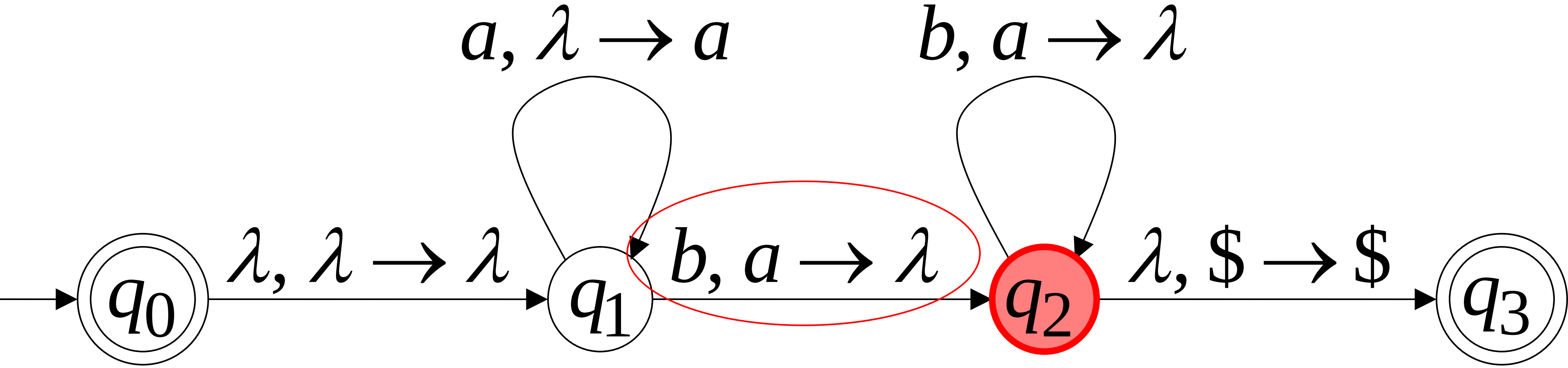
Time 5

Input

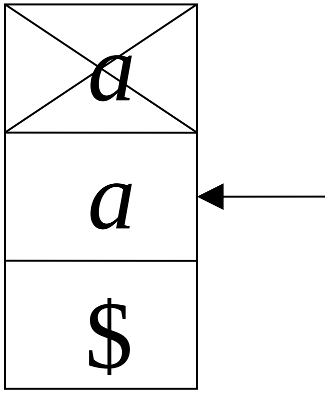
a	a	a	b	b	b

~~a~~
a
a
$\$$

Stack



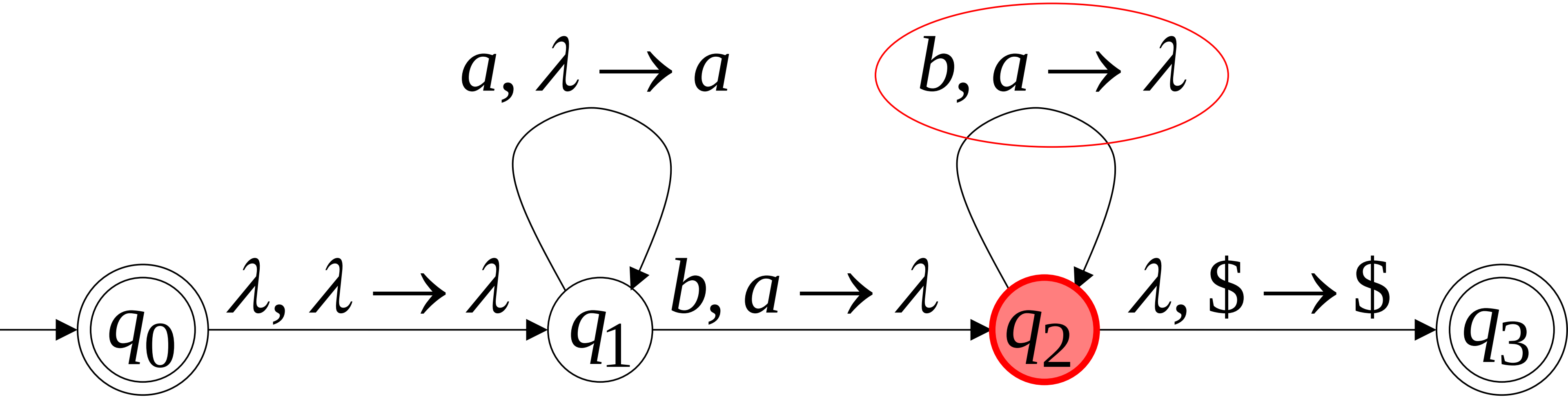
Time 6



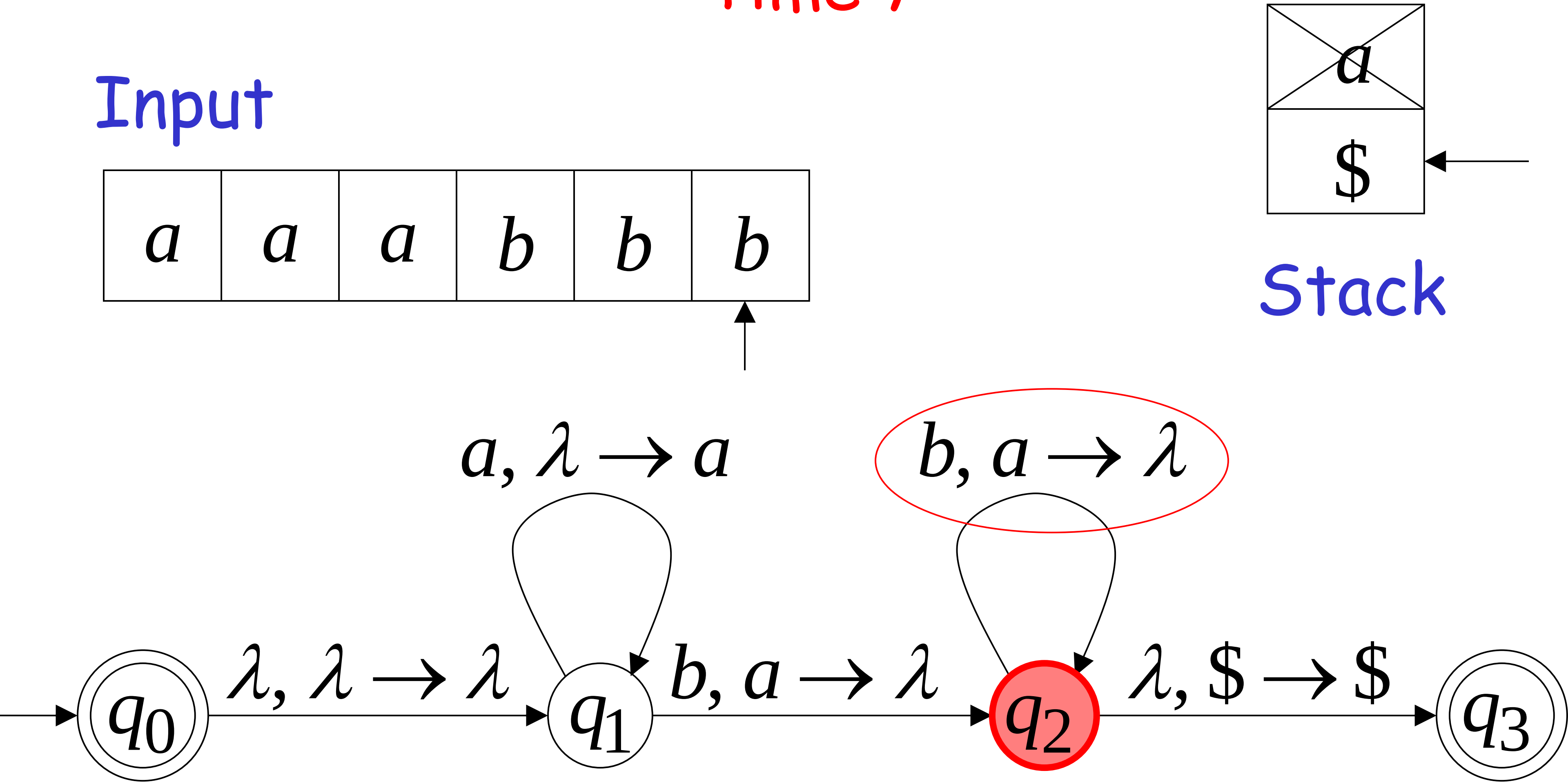
Input

a	a	a	b	b	b

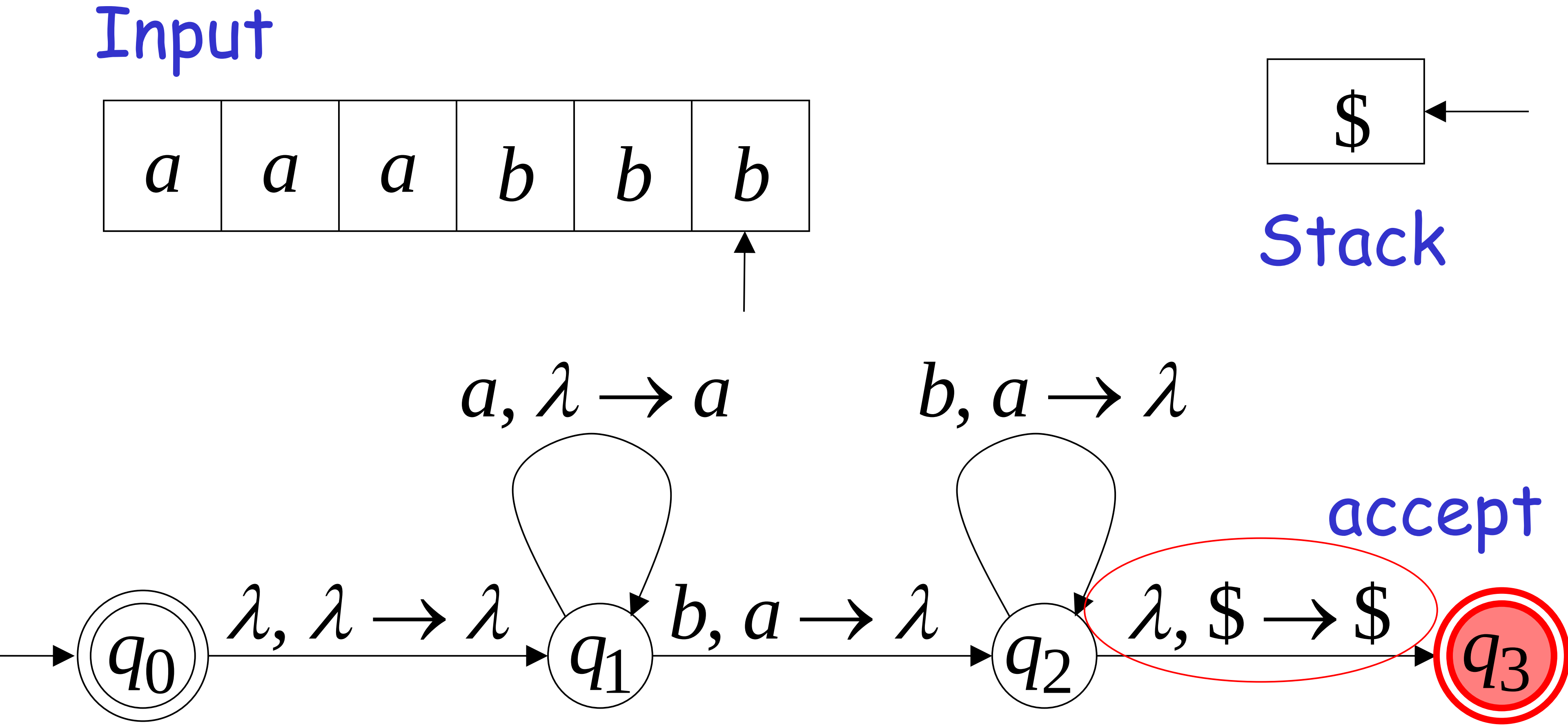
Stack



Time 7



Time 8


Input
a
a
a
b
b
b
$
Stack
$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
accept
q_0
$\lambda, \lambda \rightarrow \lambda$
q_1
$b, a \rightarrow \lambda$
q_2
$\lambda, \$ \rightarrow \$$
q_3

A string is accepted if there is
a computation such that:

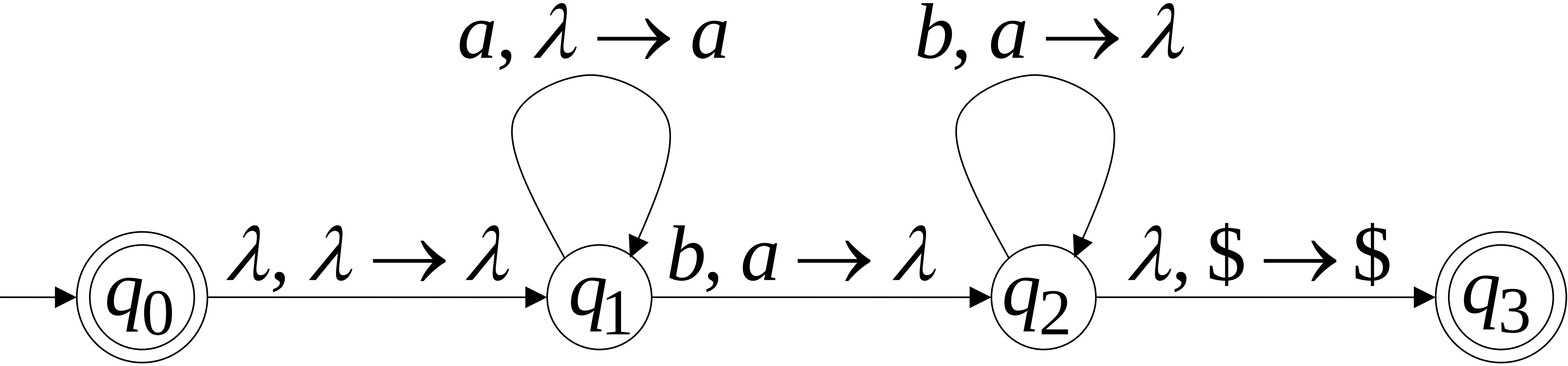
All the input is consumed

AND

The last state is a final state

At the end of the computation,
we do not care about the stack contents

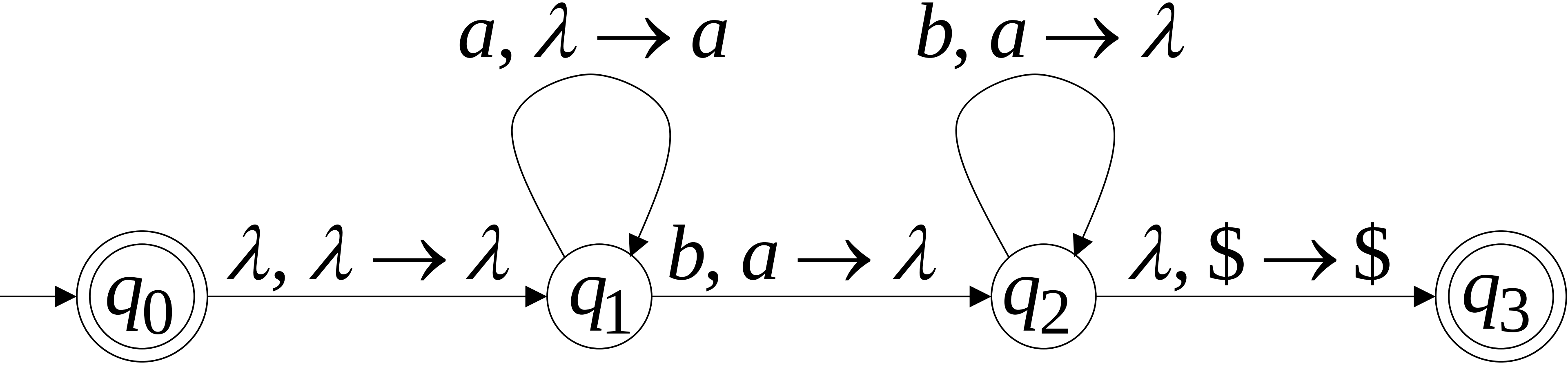
The input string $aaabbb$ is accepted by the NPDA:


$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
q_0
$\lambda, \lambda \rightarrow \lambda$
q_1
$b, a \rightarrow \lambda$
q_2
$\lambda, \$ \rightarrow \$$
q_3

In general,

$$L = \{a^n b^n : n \geq 0\}$$

is the language accepted by the NPDA:


$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
q_0
$\lambda, \lambda \rightarrow \lambda$
q_1
$b, a \rightarrow \lambda$
q_2
$\lambda, \$ \rightarrow \$$
q_3

Another NPDA example

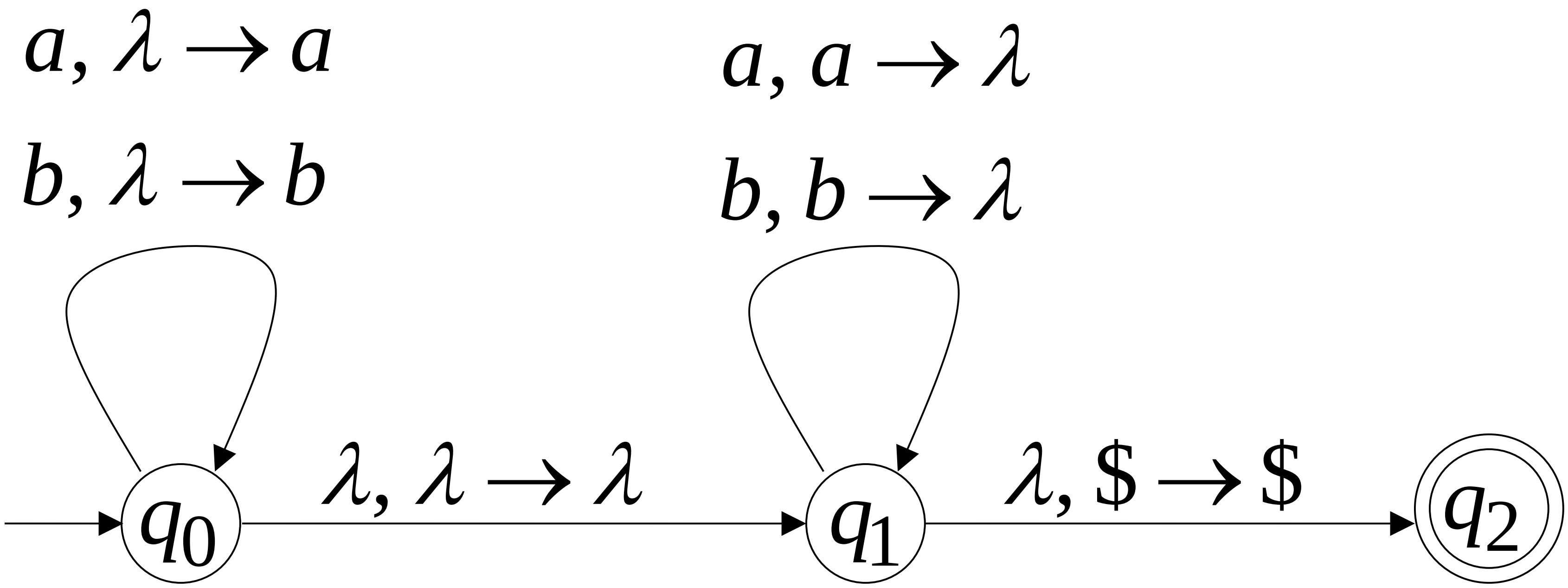
NPDA M

$$L(M) = \{ww^R\}$$

Another NPDA example

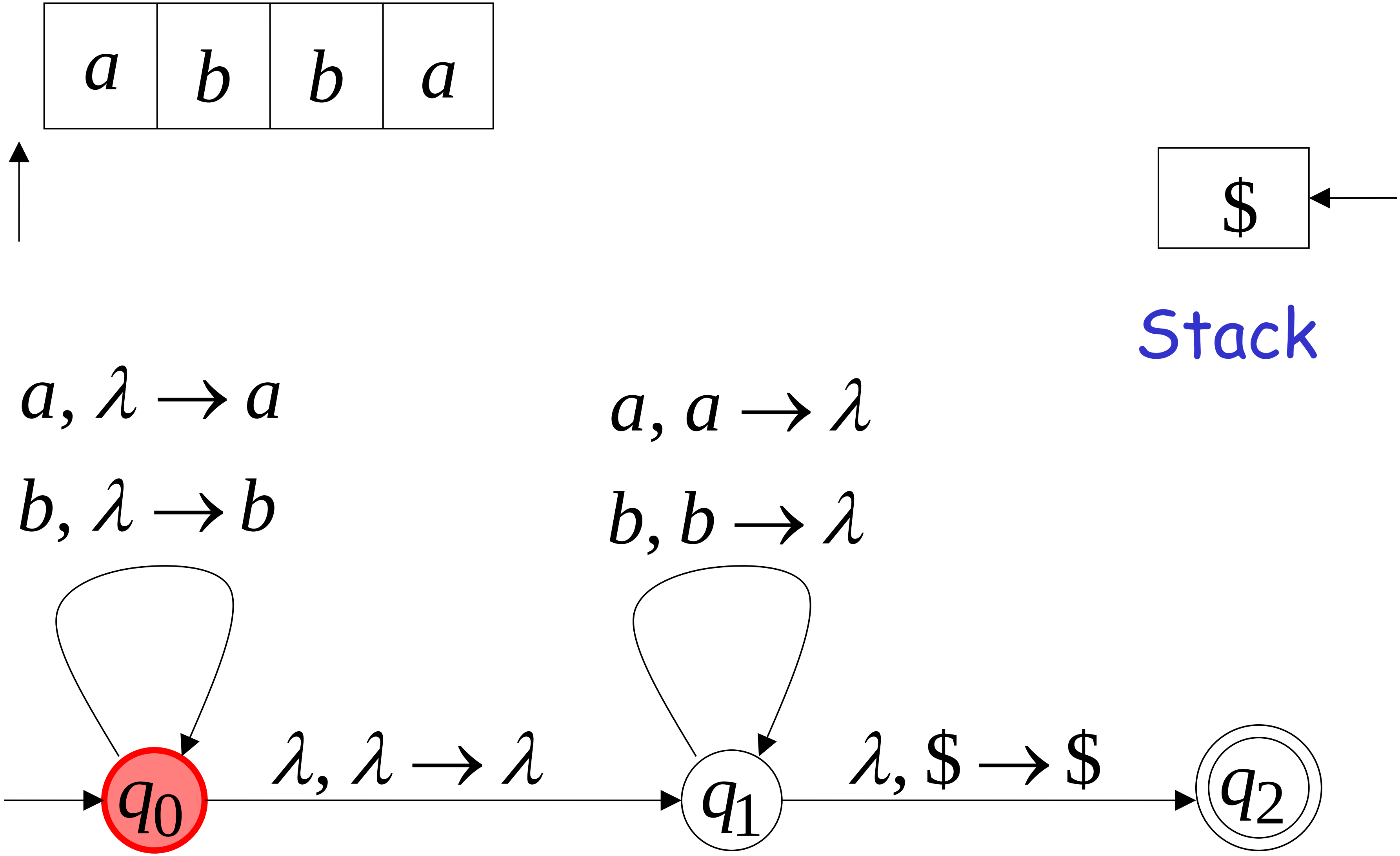
NPDA M

$$L(M) = \{ww^R\}$$



Execution Example: Time 0

Input



a b b a

$\$$

Stack

$a, \lambda \rightarrow a$

$b, \lambda \rightarrow b$

$a, a \rightarrow \lambda$

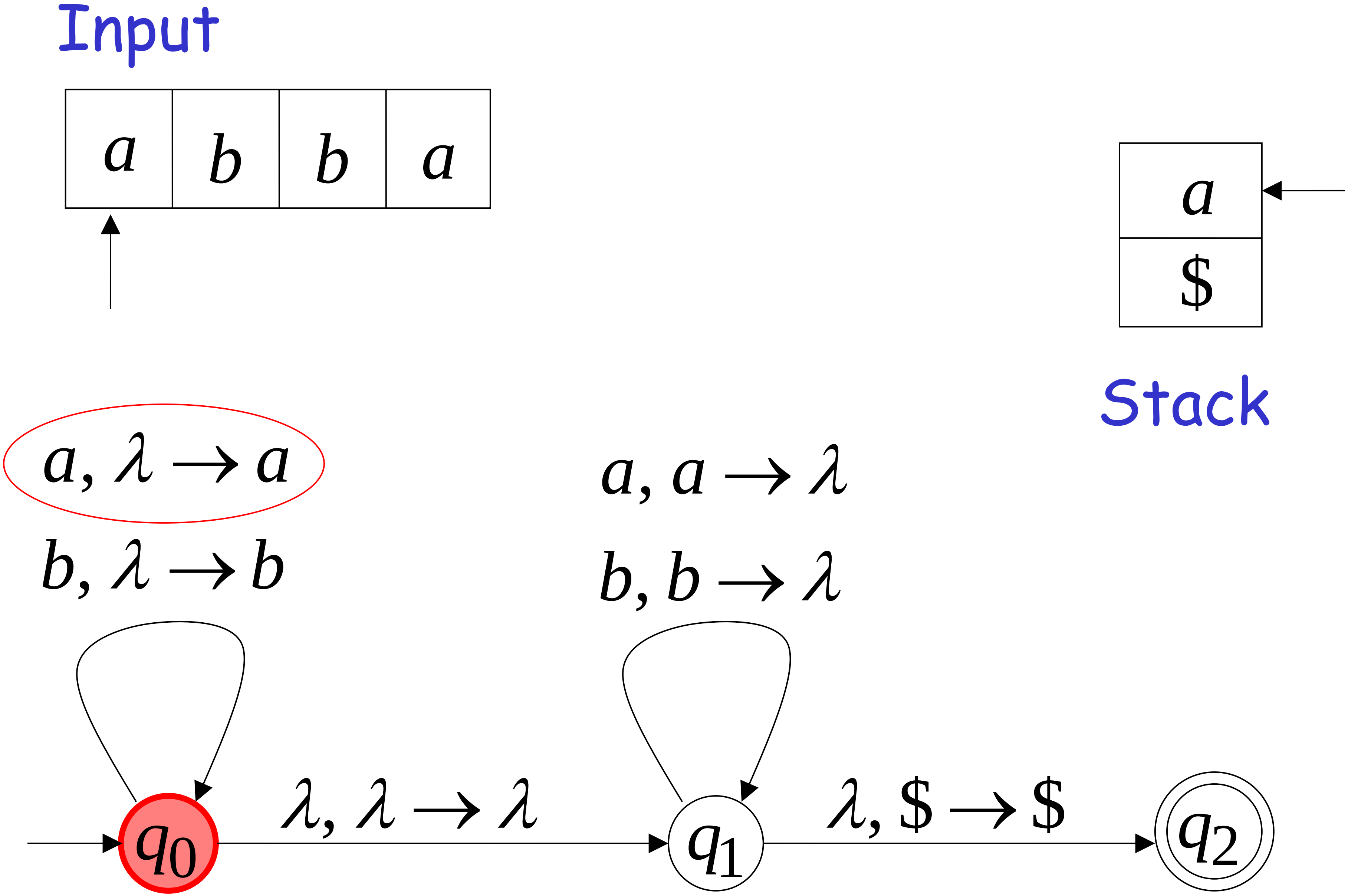
$b, b \rightarrow \lambda$

$\lambda, \lambda \rightarrow \lambda$

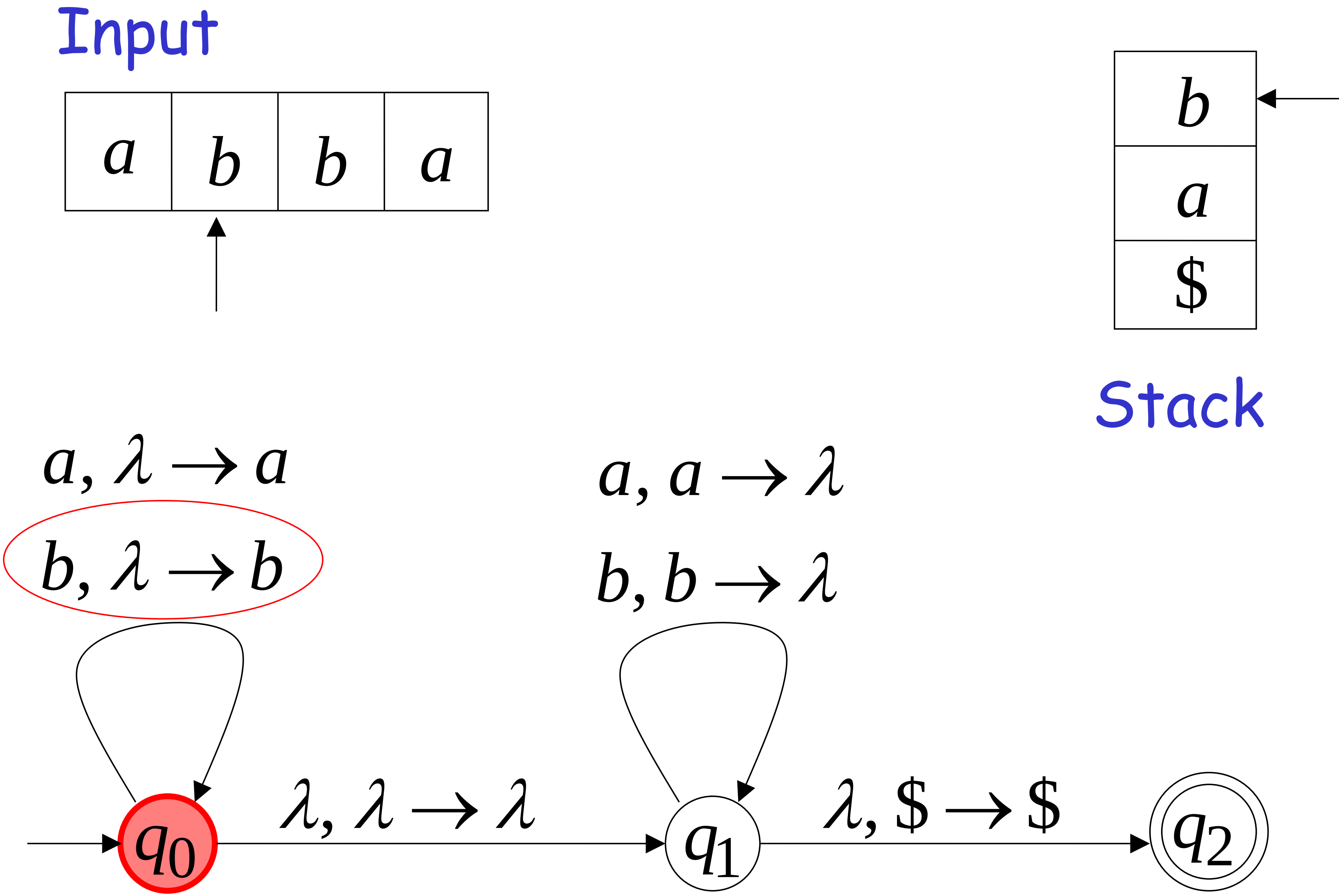
$\lambda, \$ \rightarrow \$$

q_0 q_1 q_2

Time 1


Input
a b b a
a
$\$$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 2


Input
a b b a
b
a
$\$$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 3

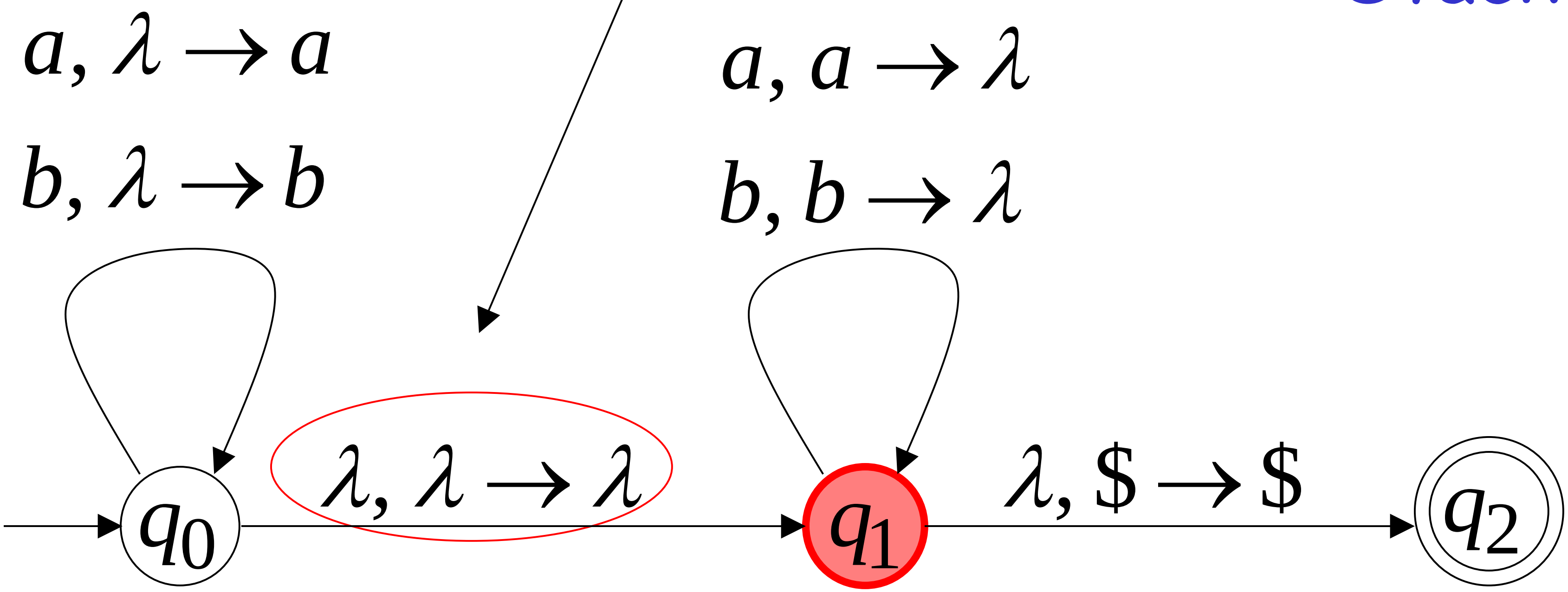
Input

a	b	b	a

Guess the middle of string

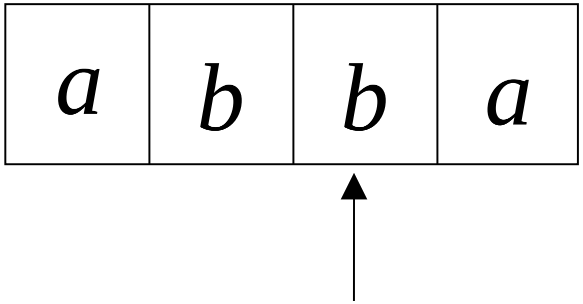
b
a
$\$$

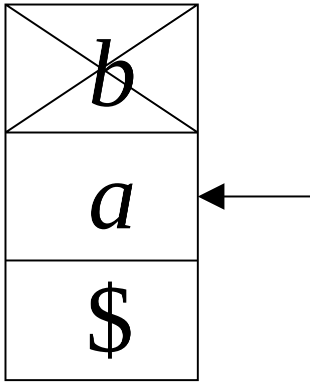
Stack



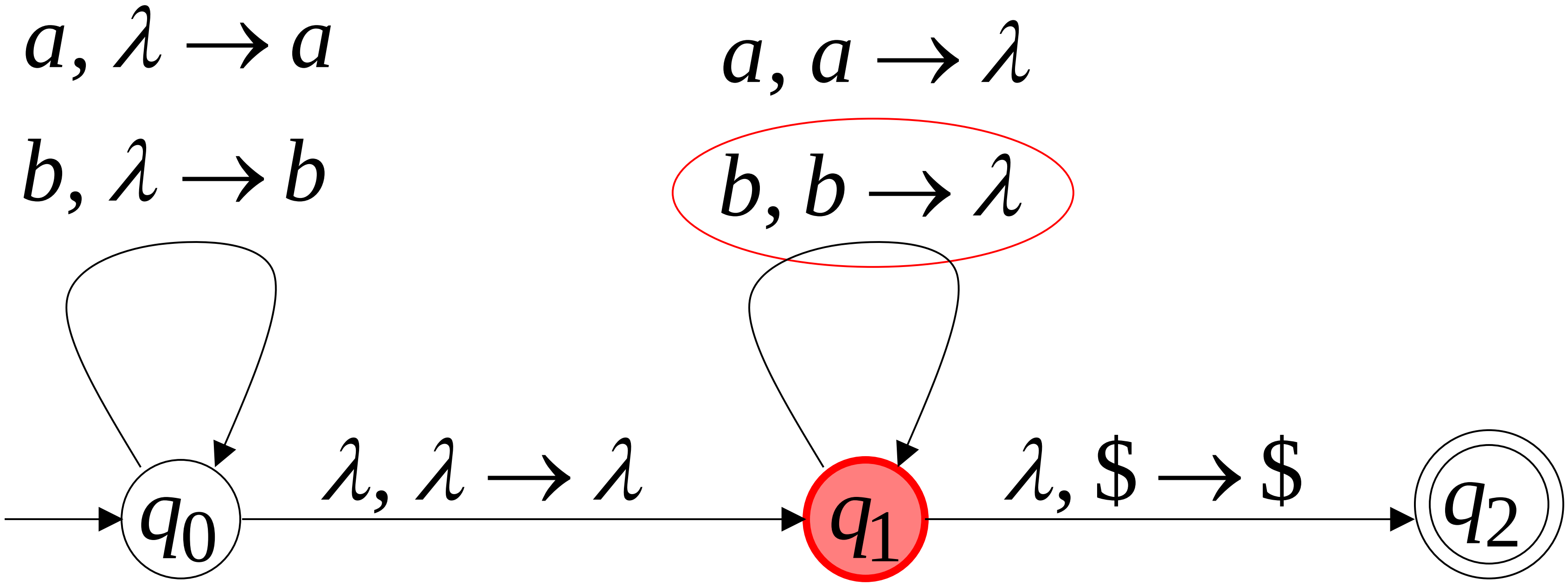
Time 4

Input

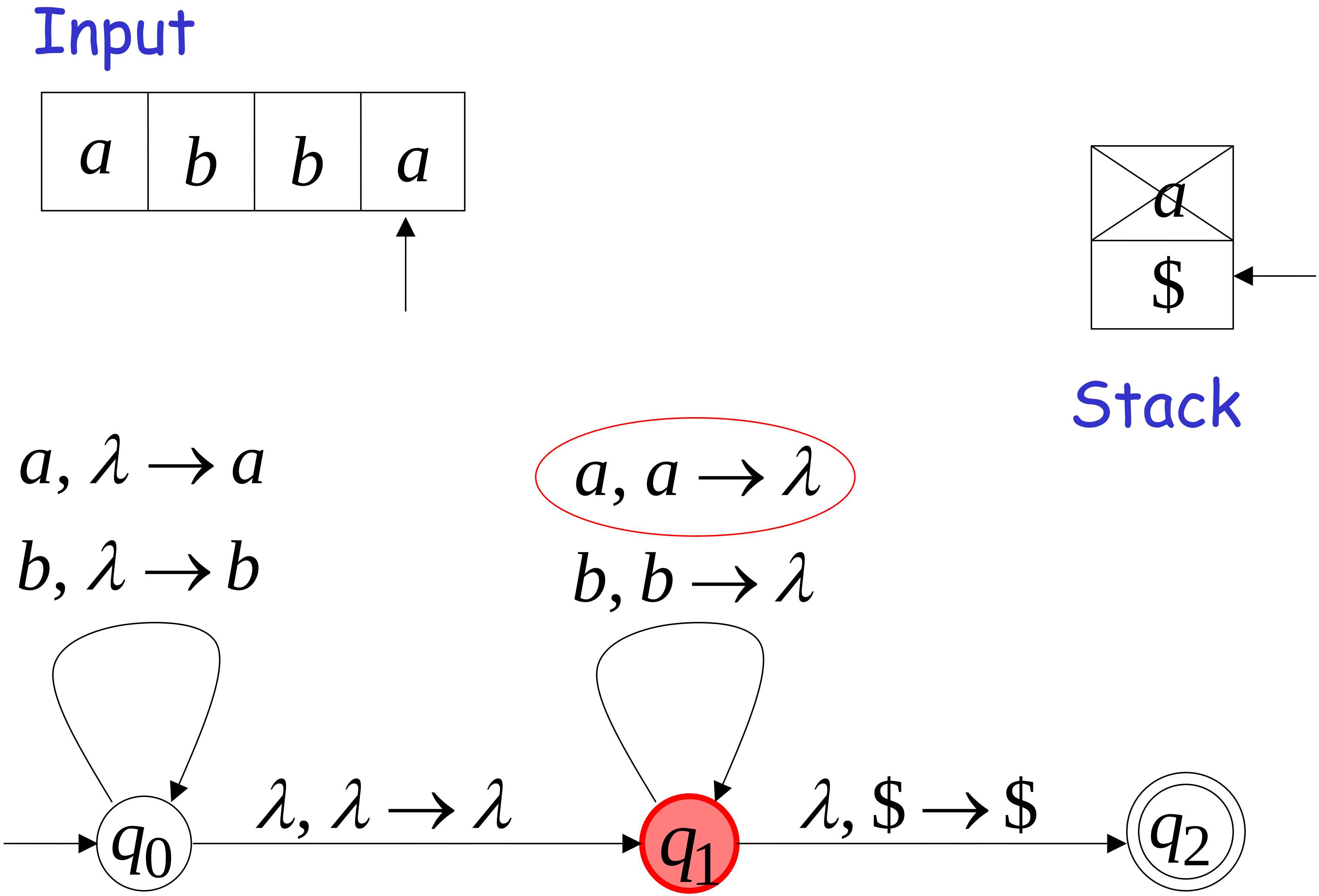




Stack

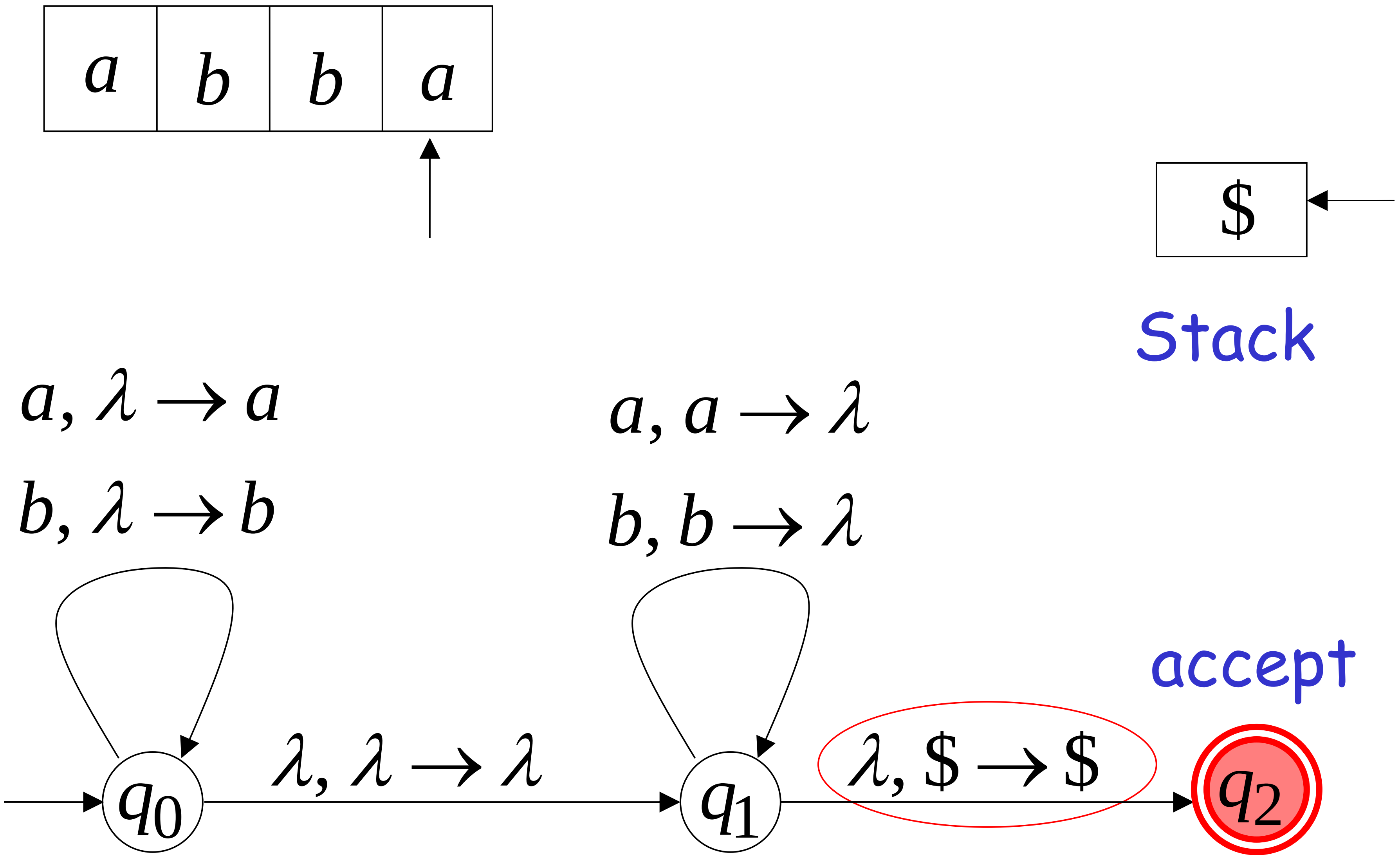

$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 5

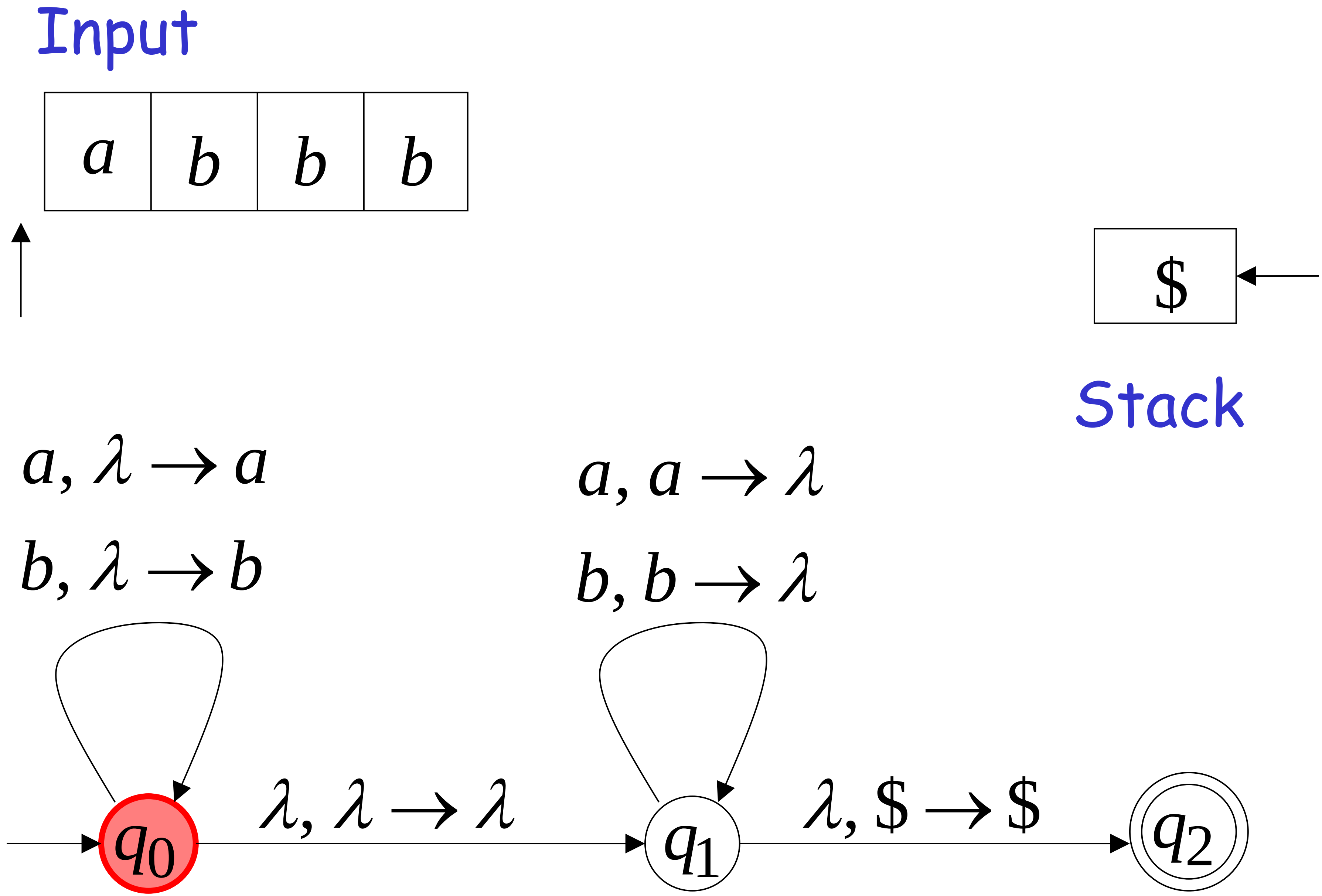

Input
a b b a
$\cancel{a}$
$\$$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 6

Input


a
b
b
a
$\$$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
accept
q_0
q_1
q_2

Rejection Example: Time 0



Input

a b b b

$\$$

Stack

$a, \lambda \to a$

$b, \lambda \to b$

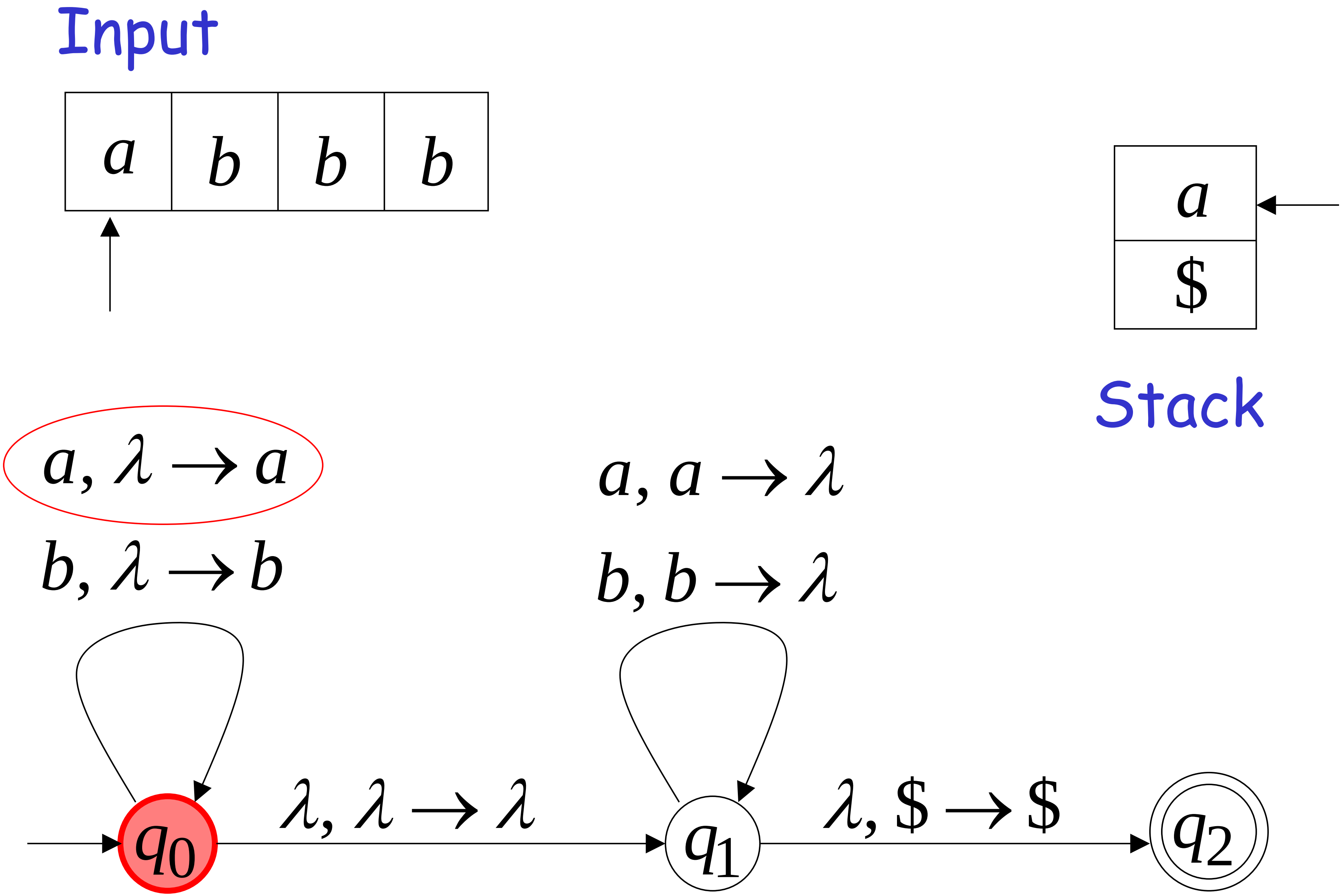
$a, a \to \lambda$

$b, b \to \lambda$

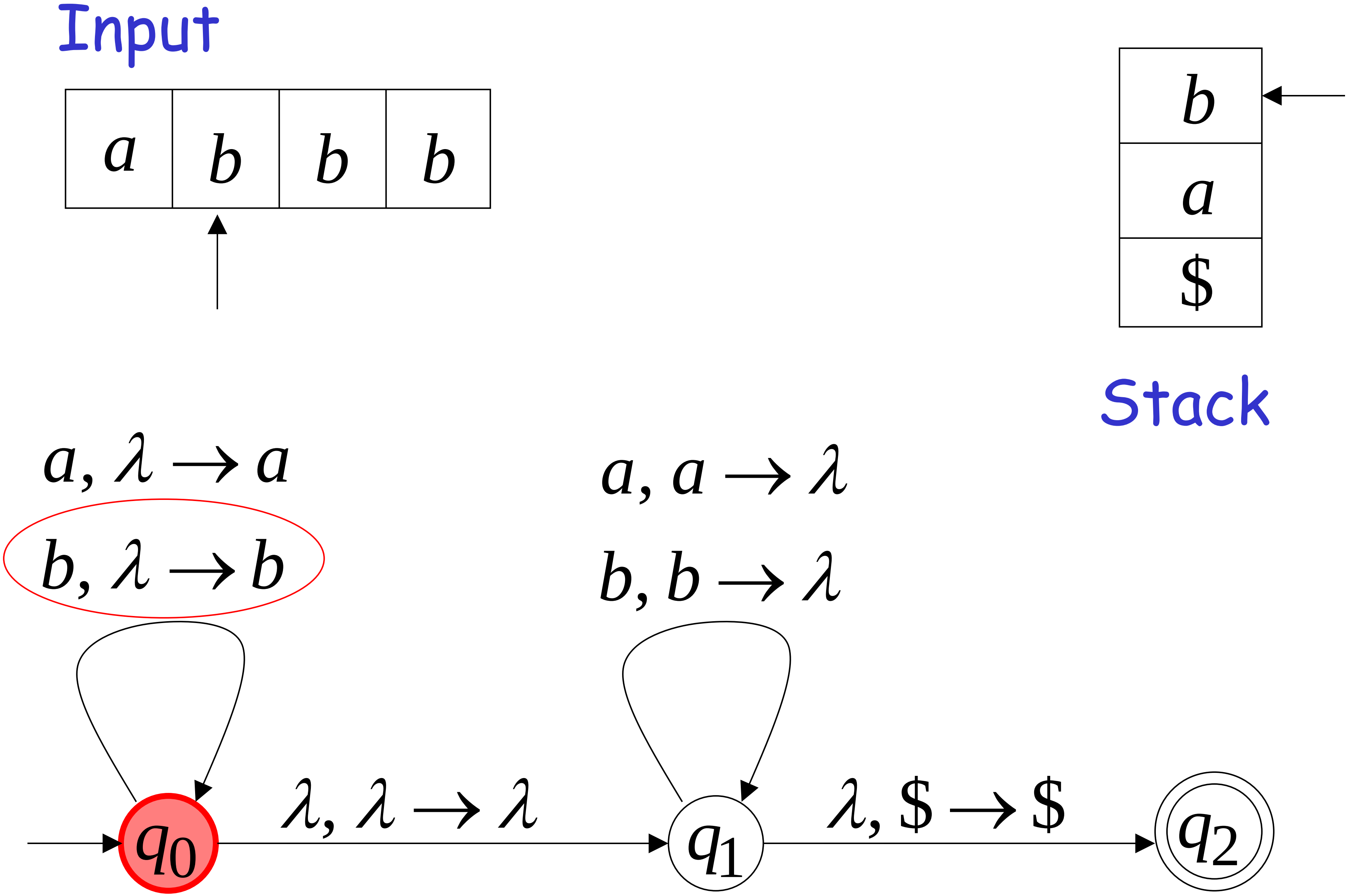
$\lambda, \lambda \to \lambda$

$\lambda, \$ \to \$$

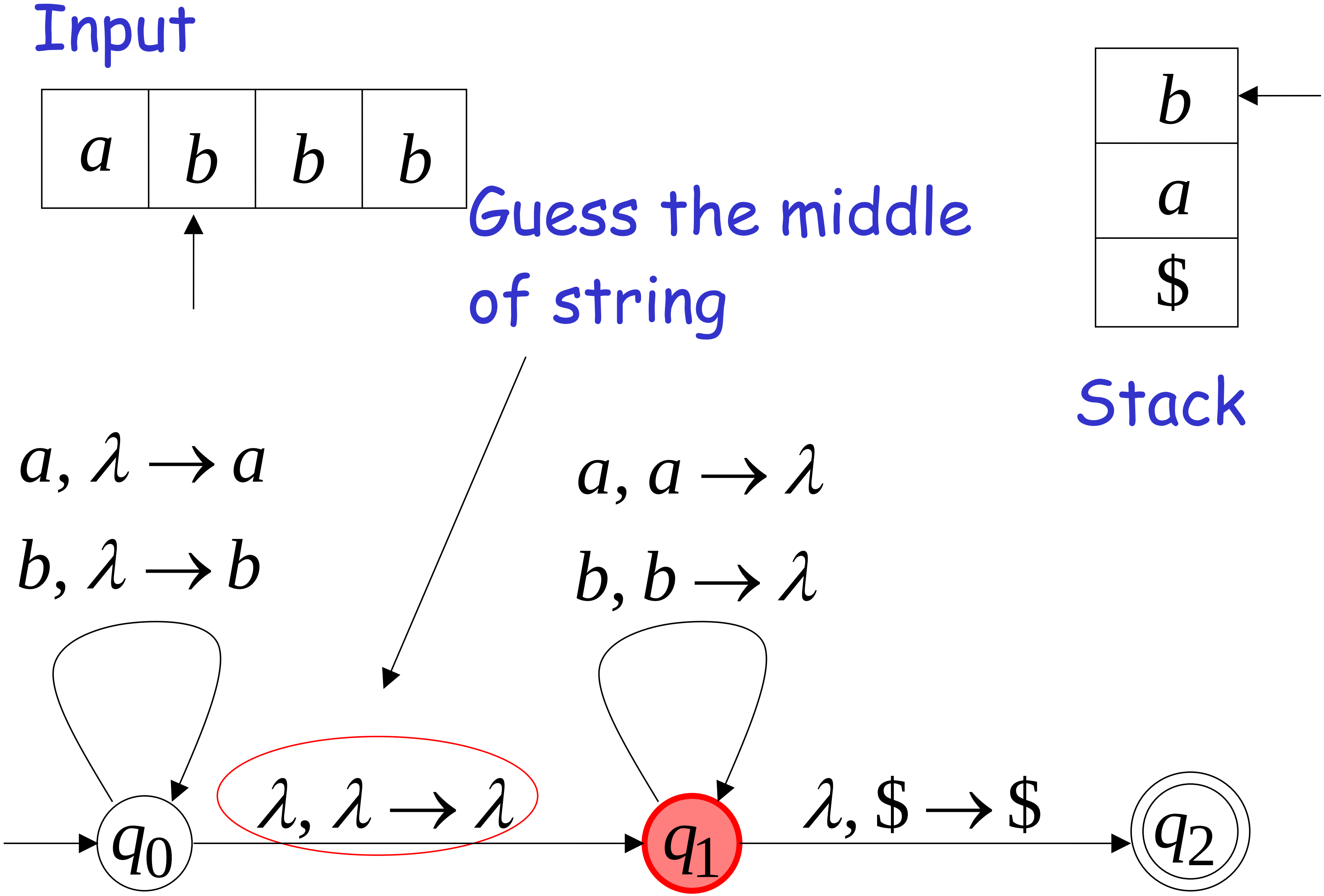
Time 1


Input
a b b b
a
$\$$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2

Time 2


Input
a b b b
b
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 3



Input

a	b	b	b

b
a
$\$$

Stack

Guess the middle of string

$a, \lambda \rightarrow a$

$b, \lambda \rightarrow b$

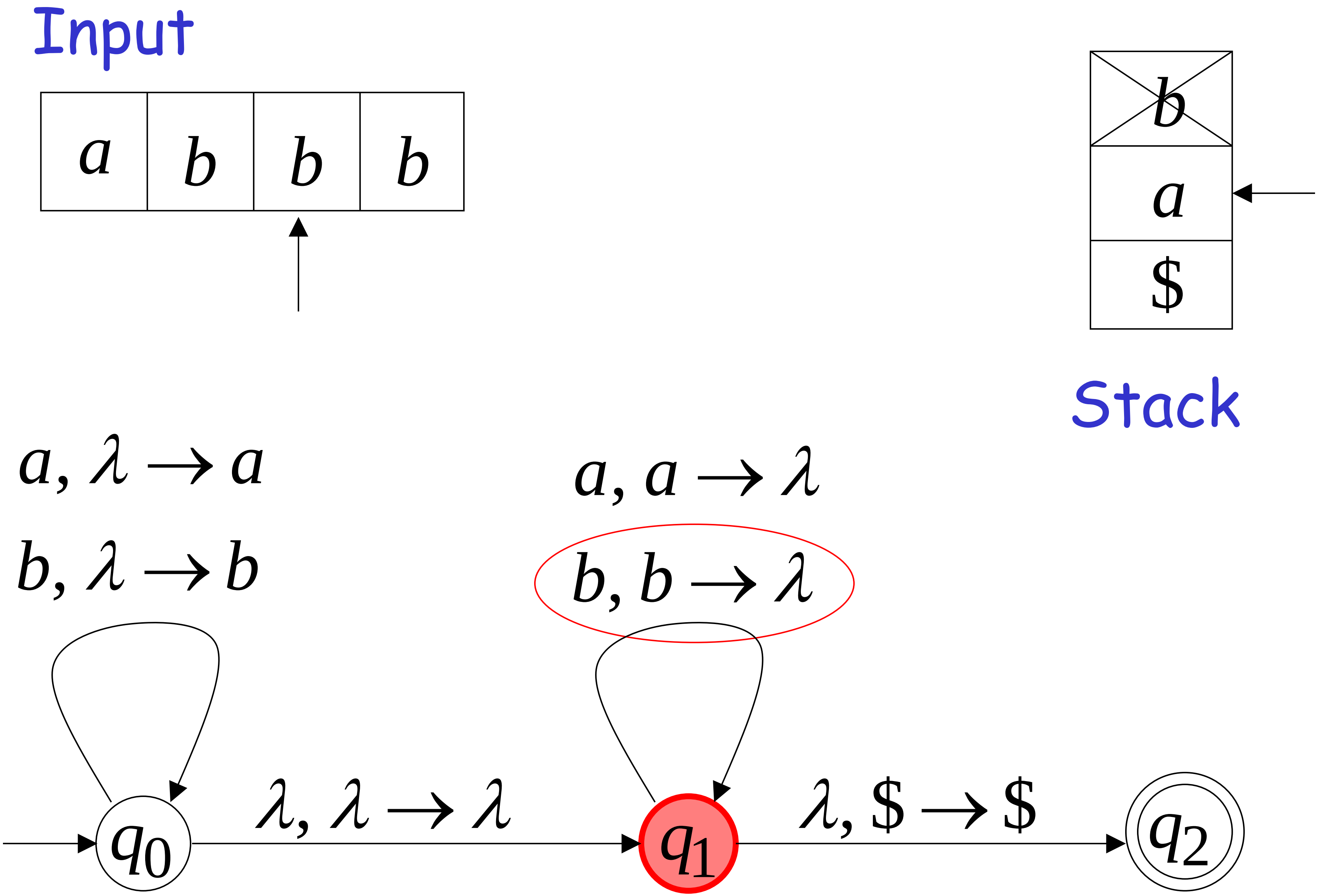
$a, a \rightarrow \lambda$

$b, b \rightarrow \lambda$

$\lambda, \lambda \rightarrow \lambda$

$\lambda, \$ \rightarrow \$$

Time 4


Input
a
b
b
b
~~b~~
a
$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2

Time 5

There is no possible transition.

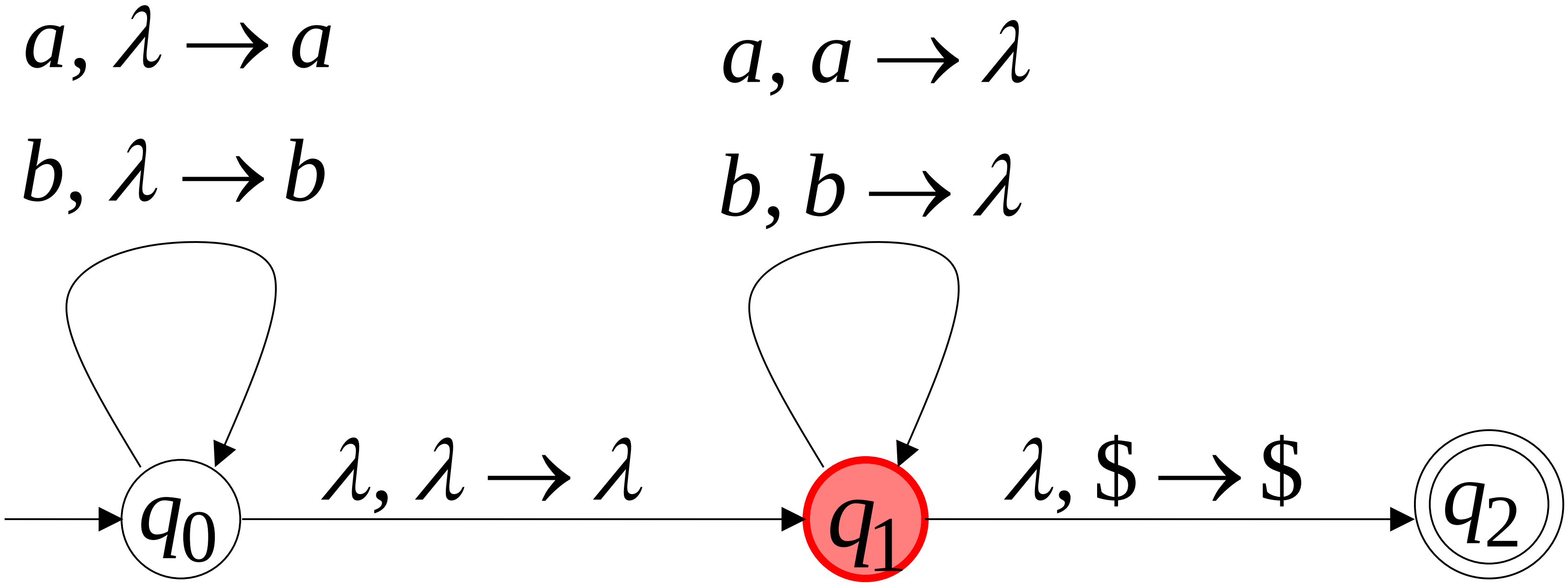
Input

a	b	b	b

Input is not consumed

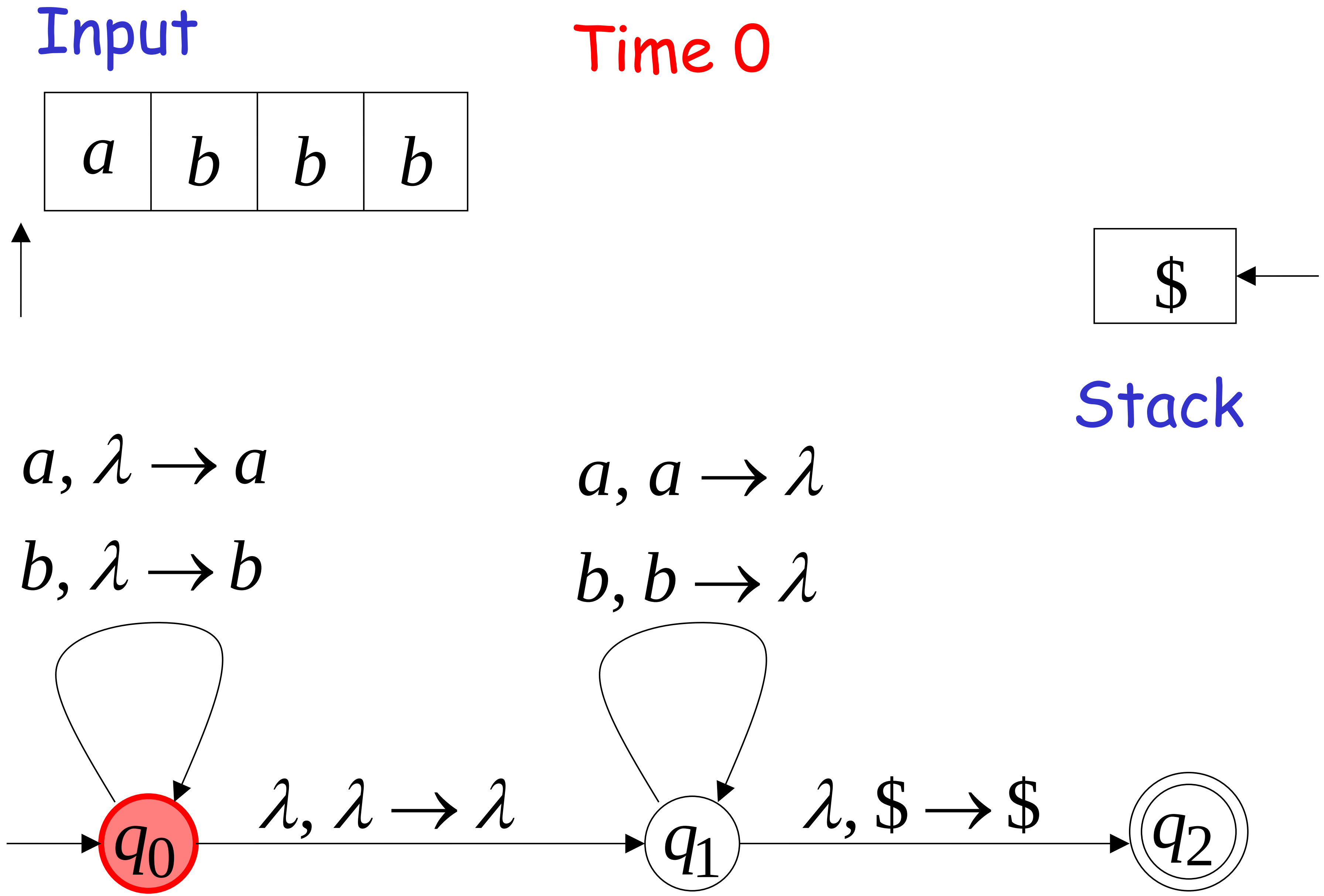
a
$\$$

Stack



$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Another computation on same string:



Input

Time 0

a	b	b	b

$\$$

Stack

$a, \lambda \rightarrow a$

$b, \lambda \rightarrow b$

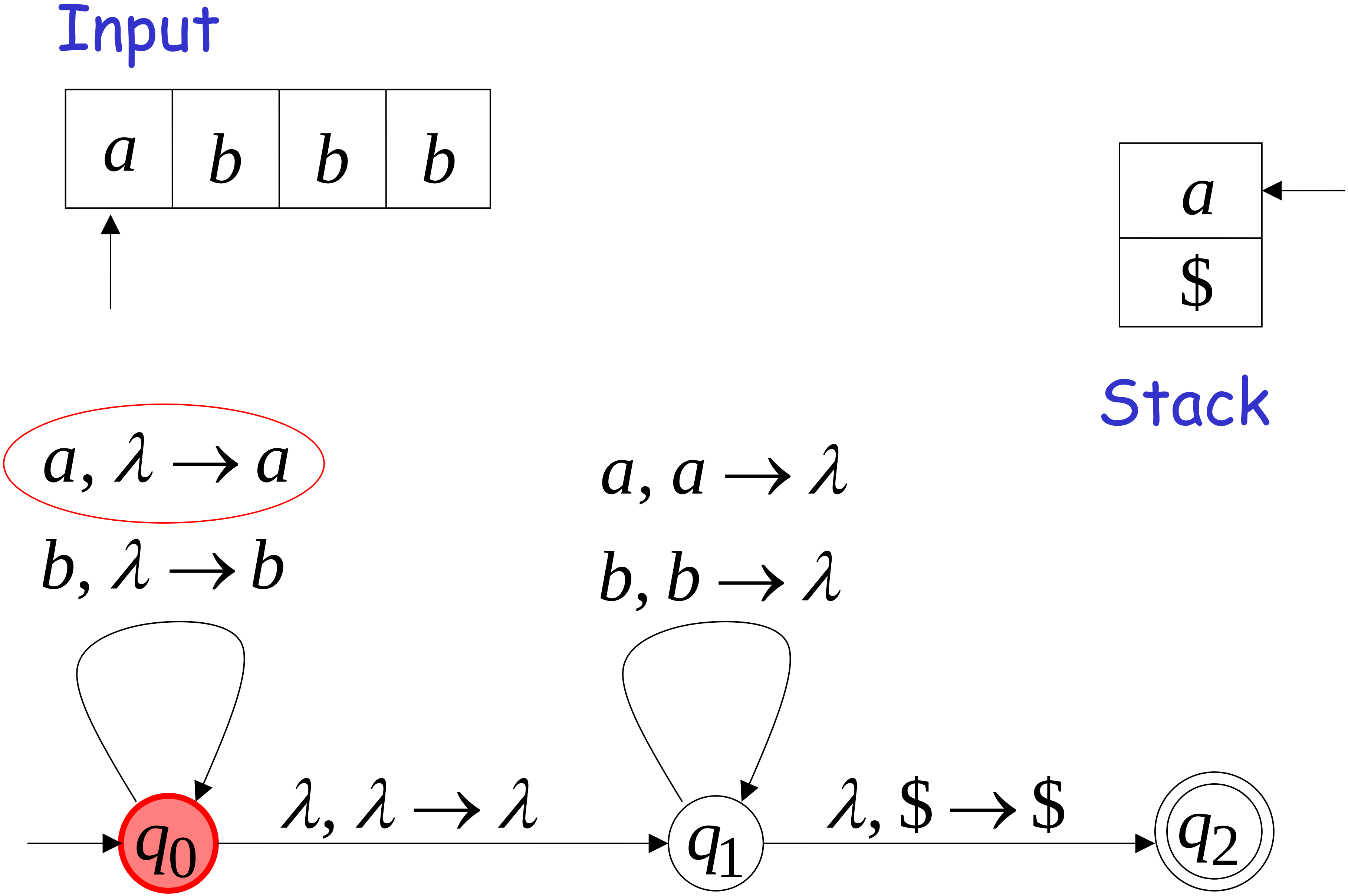
$a, a \rightarrow \lambda$

$b, b \rightarrow \lambda$

$\lambda, \lambda \rightarrow \lambda$

$\lambda, \$ \rightarrow \$$

Time 1



Input

a	b	b	b

a
$\$$

Stack

$a, \lambda \rightarrow a$

$b, \lambda \rightarrow b$

$a, a \rightarrow \lambda$

$b, b \rightarrow \lambda$

$\lambda, \lambda \rightarrow \lambda$

$\lambda, \$ \rightarrow \$$

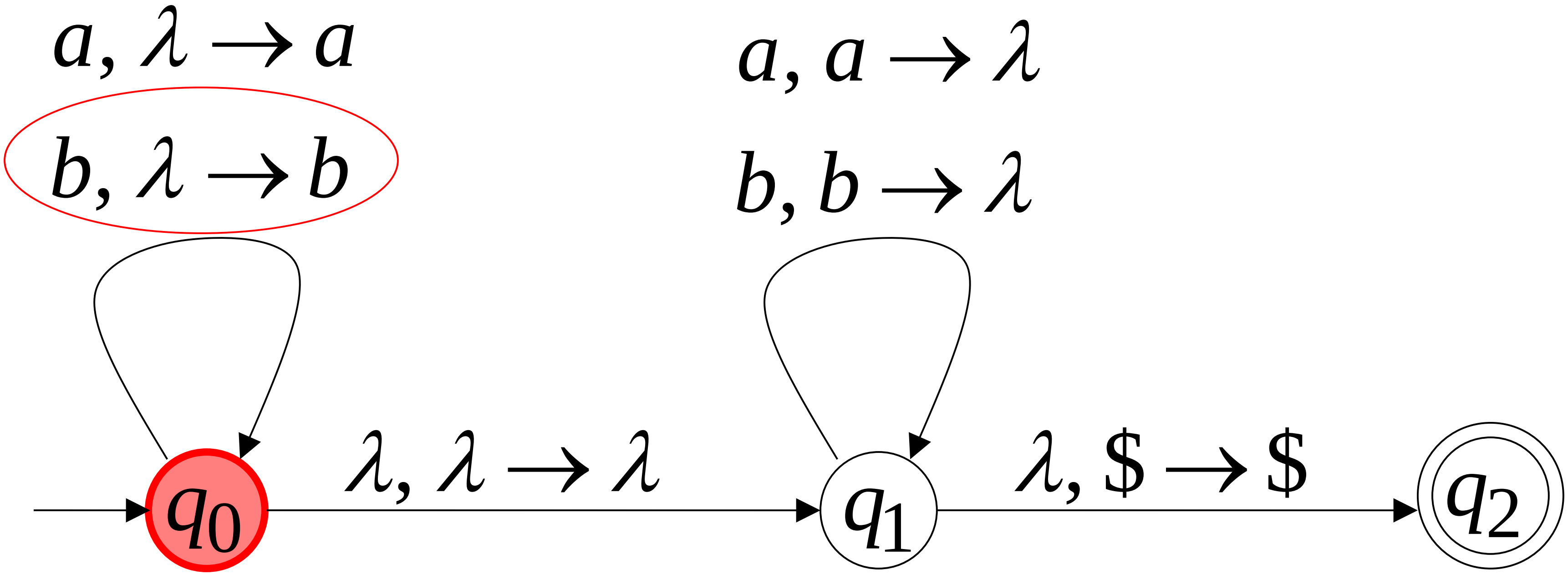
Time 2

Input

a	b	b	b

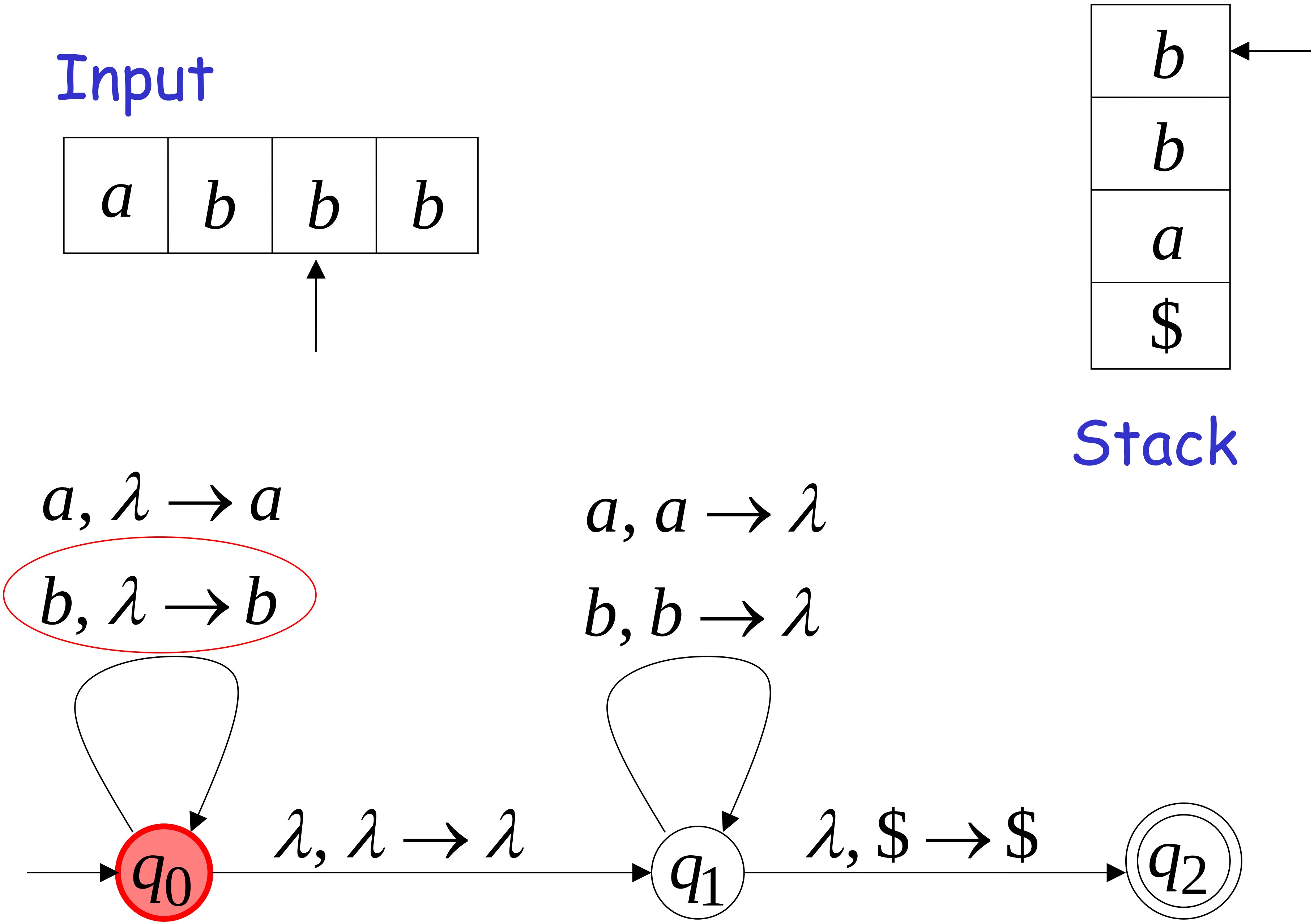
b
a
$\$$

Stack



$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 3



Input

a	b	b	b

b
b
a
$\$$

Stack

$a, \lambda \rightarrow a$

$b, \lambda \rightarrow b$

$\lambda, \lambda \rightarrow \lambda$

$a, a \rightarrow \lambda$

$b, b \rightarrow \lambda$

$\lambda, \$ \rightarrow \$$

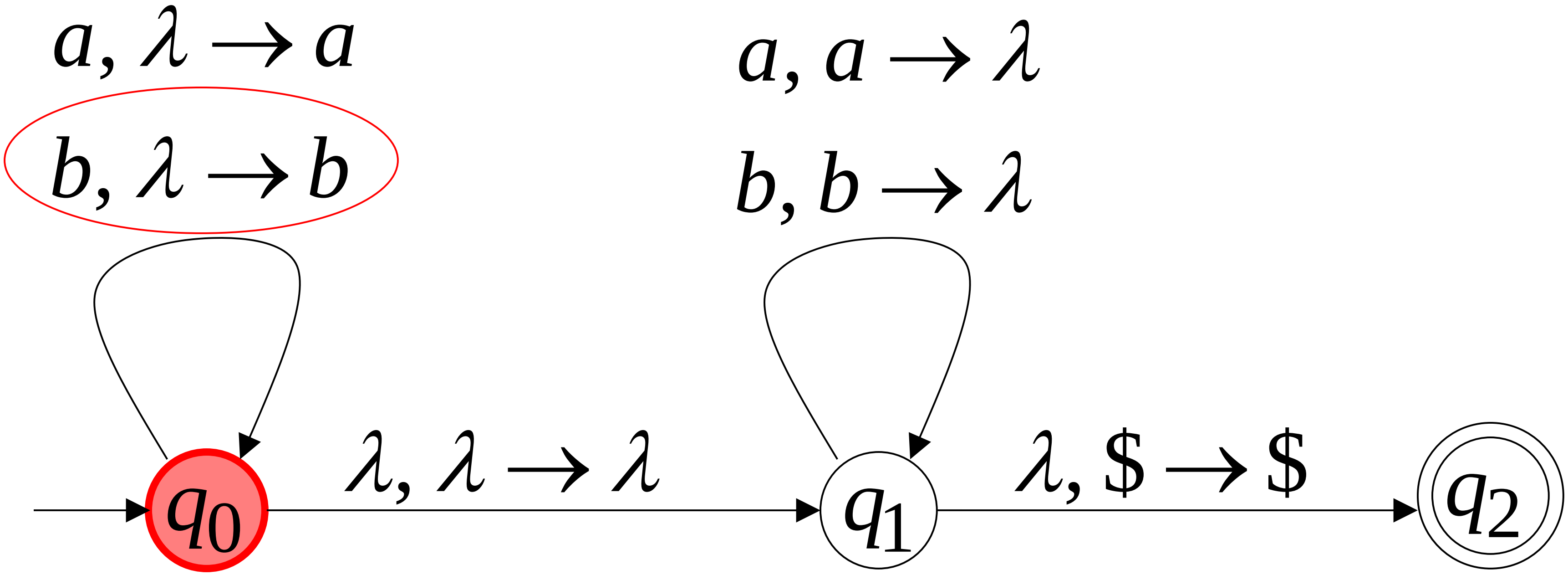
Time 4

Input

a	b	b	b

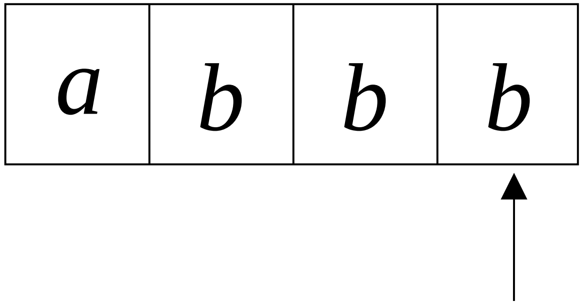
Stack

b
b
b
a
$\$$

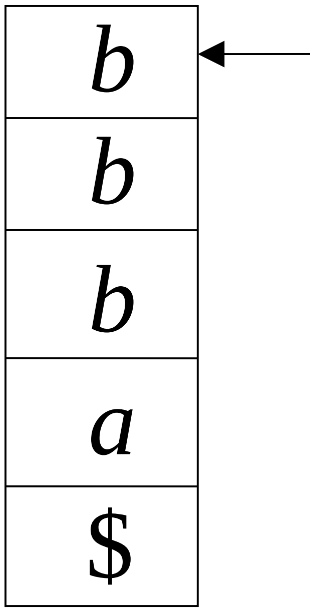

$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 5

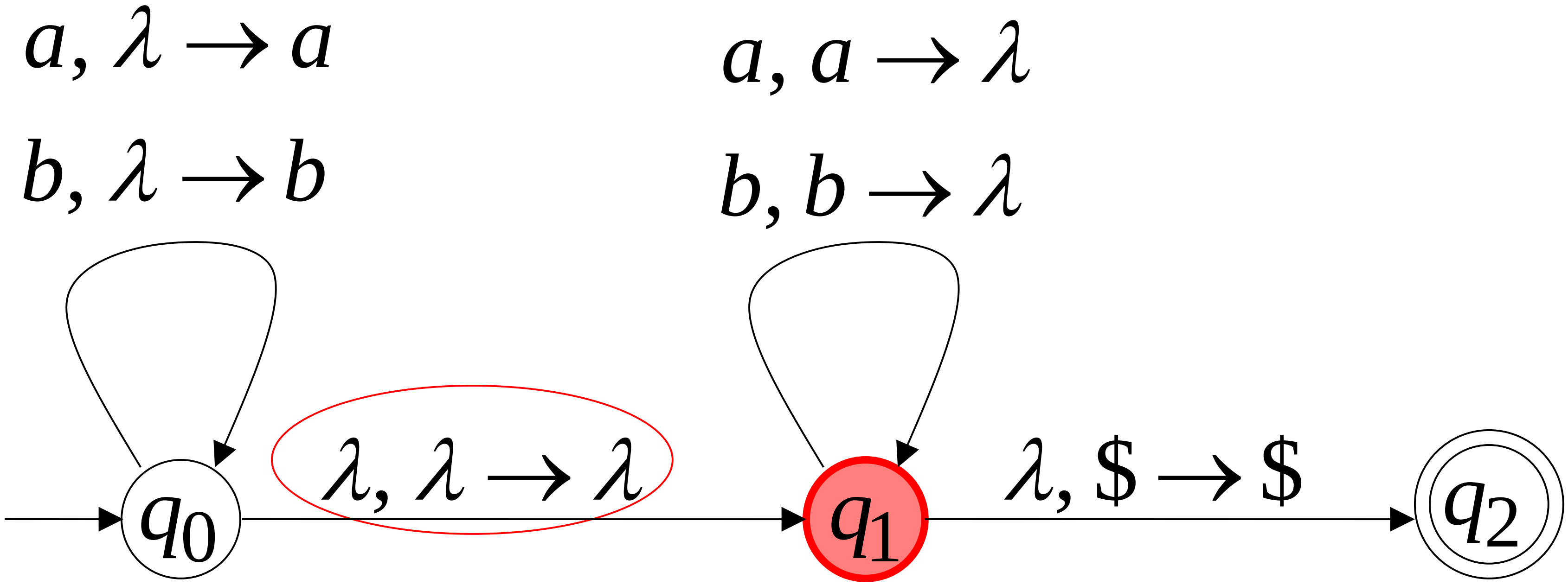
Input



No final state
is reached

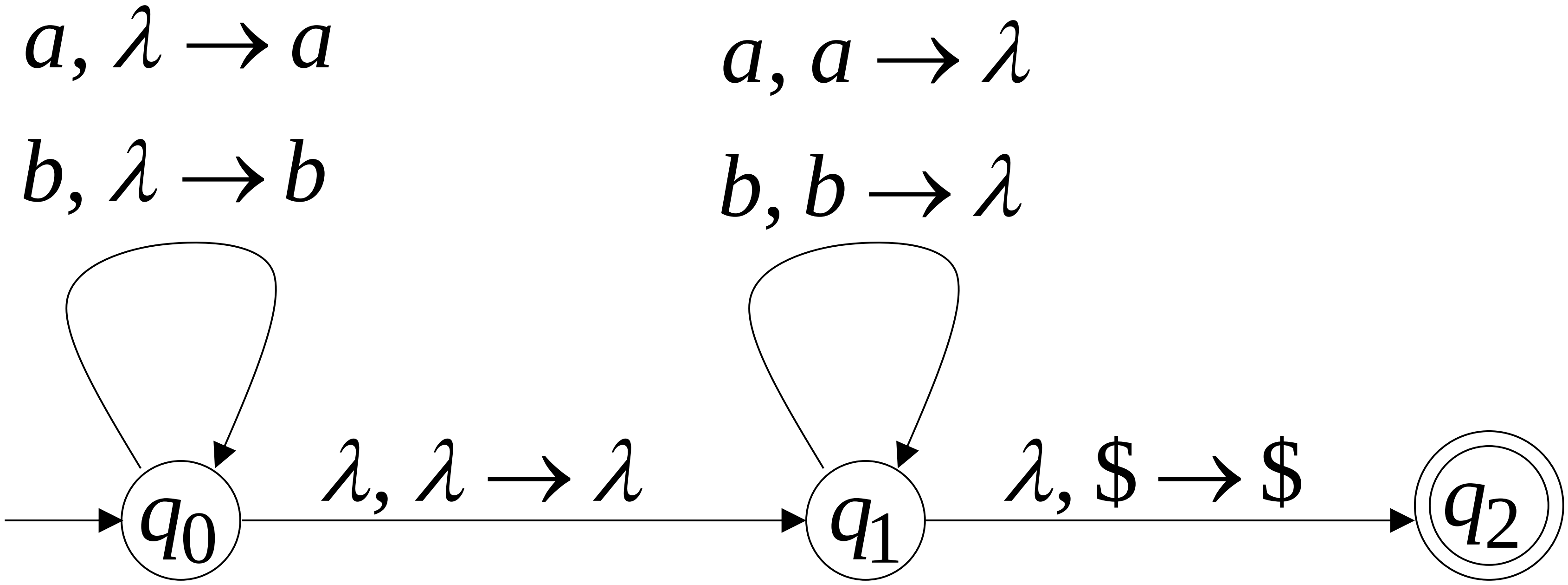


Stack



There is no computation that accepts string $abbb$

$$abbb \notin L(M)$$



A string is rejected if there is NO computation such that:

All the input is consumed

AND

The last state is a final state

At the end of the computation, we do not care about the stack contents

In other words, a string is rejected if in every computation with this string:

The input cannot be consumed

OR

The input is consumed and the last state is not a final state

OR

The stack head moves below the bottom of the stack

Another NPDA example

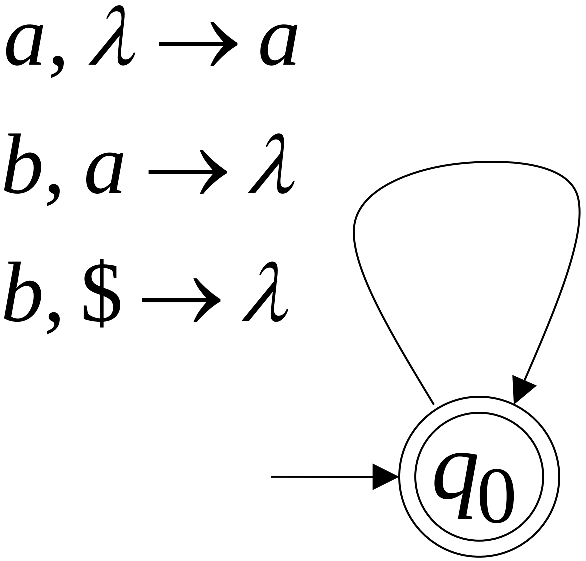
NPDA M

$$L(M) = \{a^n b^m \ : \ n \geq m - 1\}$$

Another NPDA example

NPDA M

$$L(M) = \{a^n b^m \ : \ n \geq m-1\}$$


$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$b, \$ \rightarrow \lambda$
q_0

Execution Example: Time 0

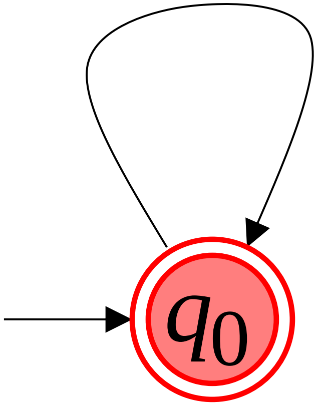
Input

a	a	b

$$a, \lambda \rightarrow a$$

$$b, a \rightarrow \lambda$$

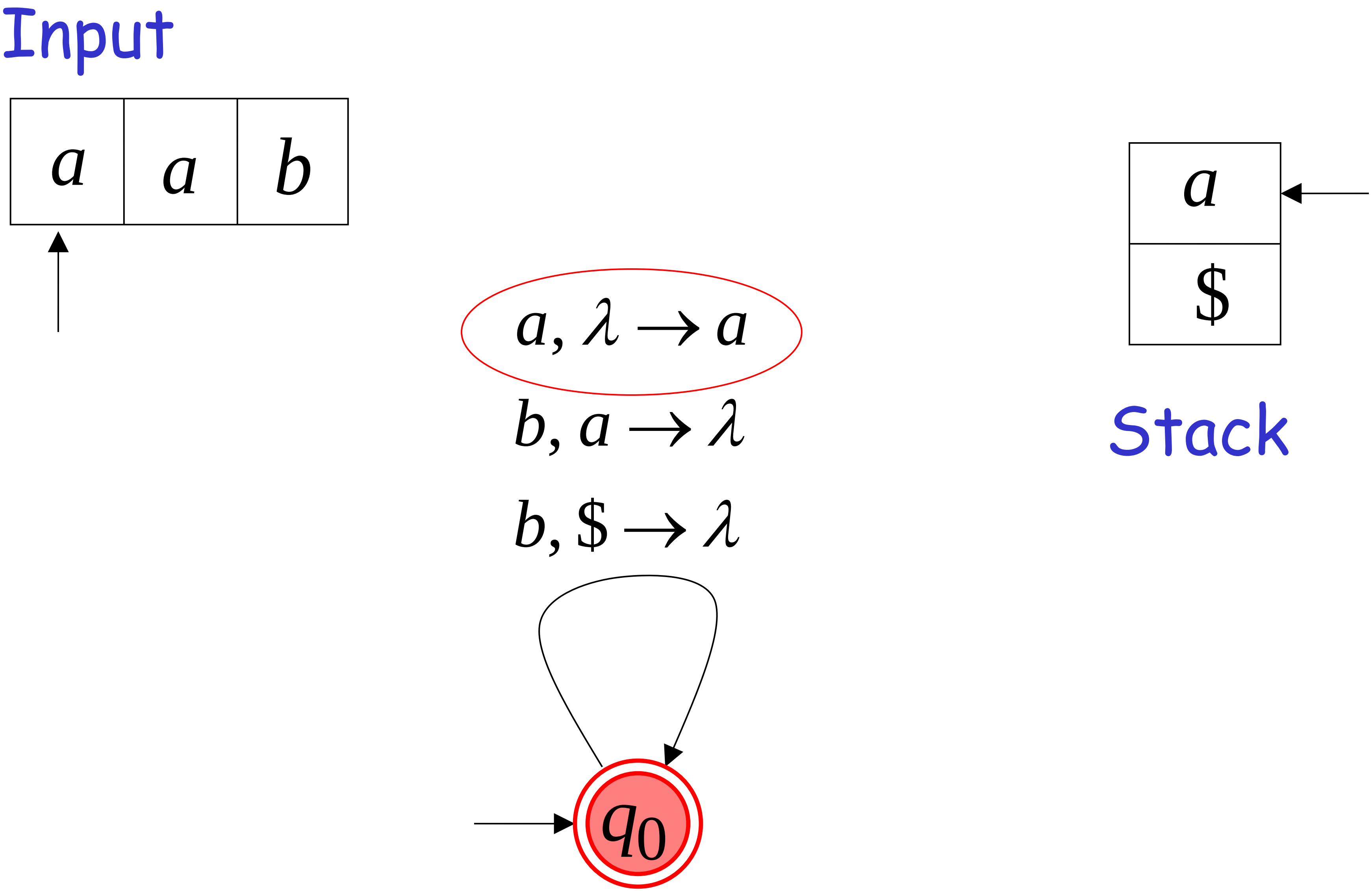
$$b, \$ \rightarrow \lambda$$



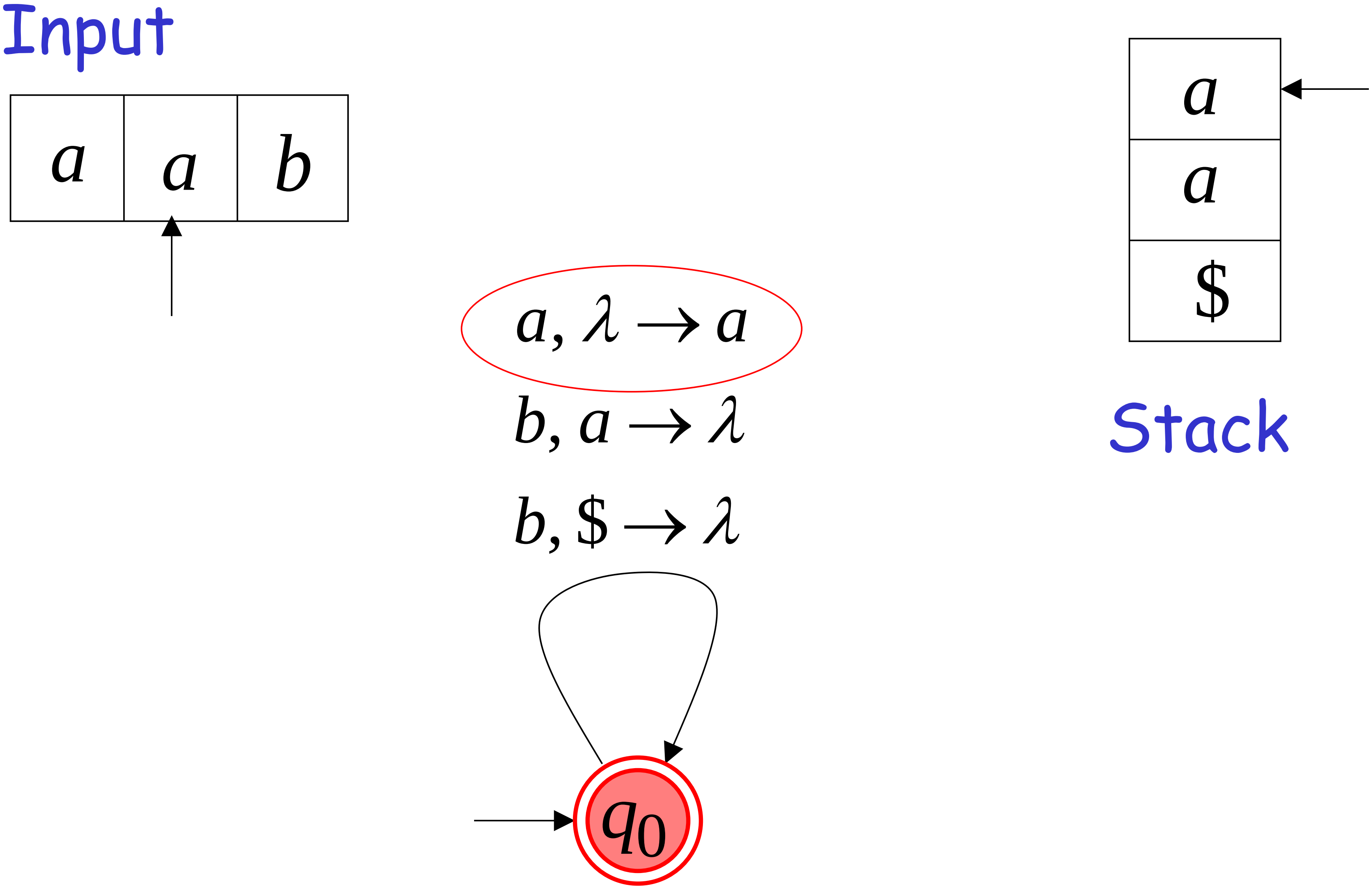
$\$$

Stack

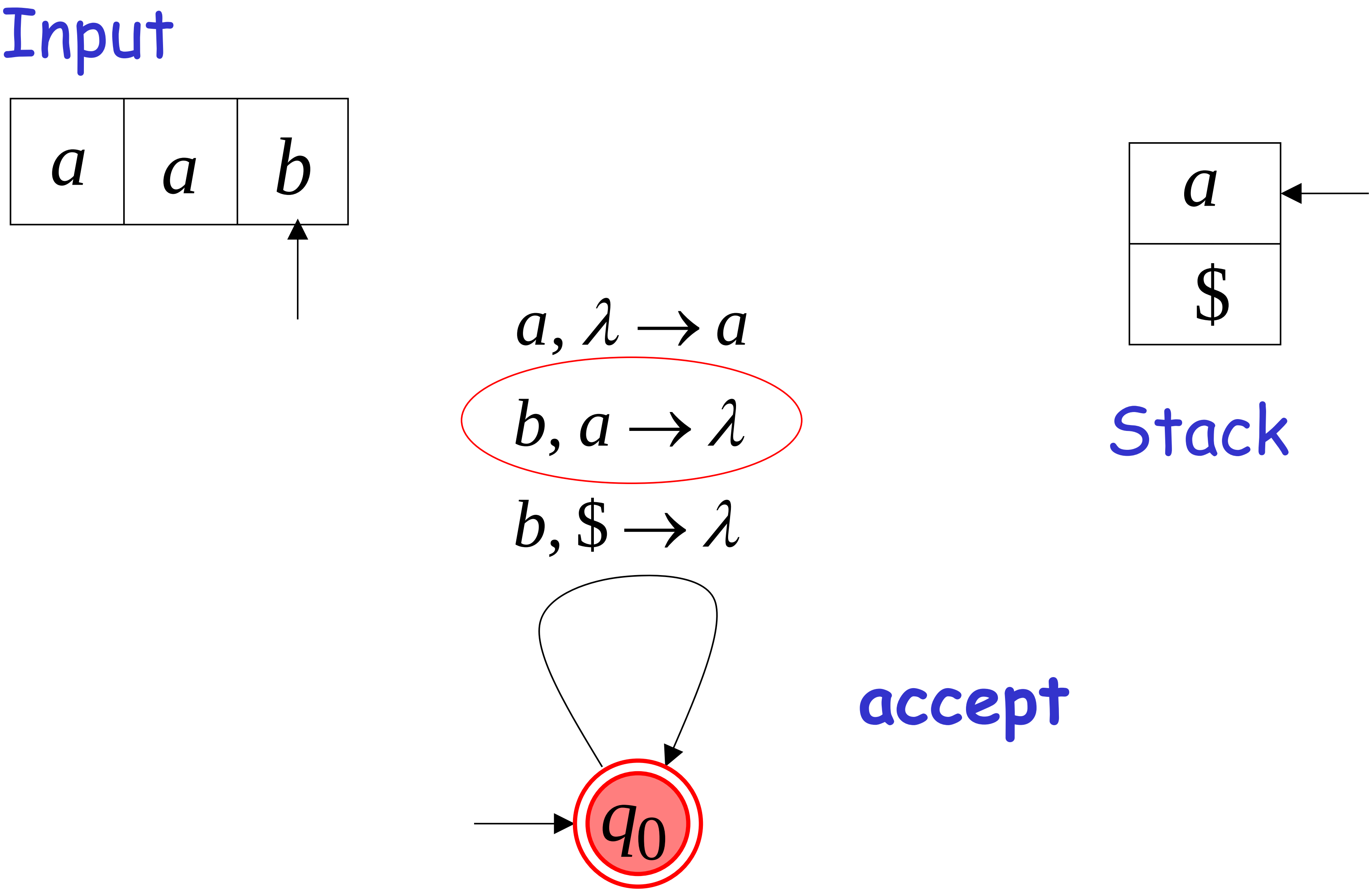
Time 1


Input
a
a
b
$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$b, \$ \rightarrow \lambda$
q_0
a
$\$$
Stack

Time 2


Input
a
a
b
a
a
$\$$
Stack
$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$b, \$ \rightarrow \lambda$
q_0

Time 3



Input

a	a	b

$a, \lambda \to a$

$b, a \to \lambda$

$b, \$ \to \lambda$

q_0

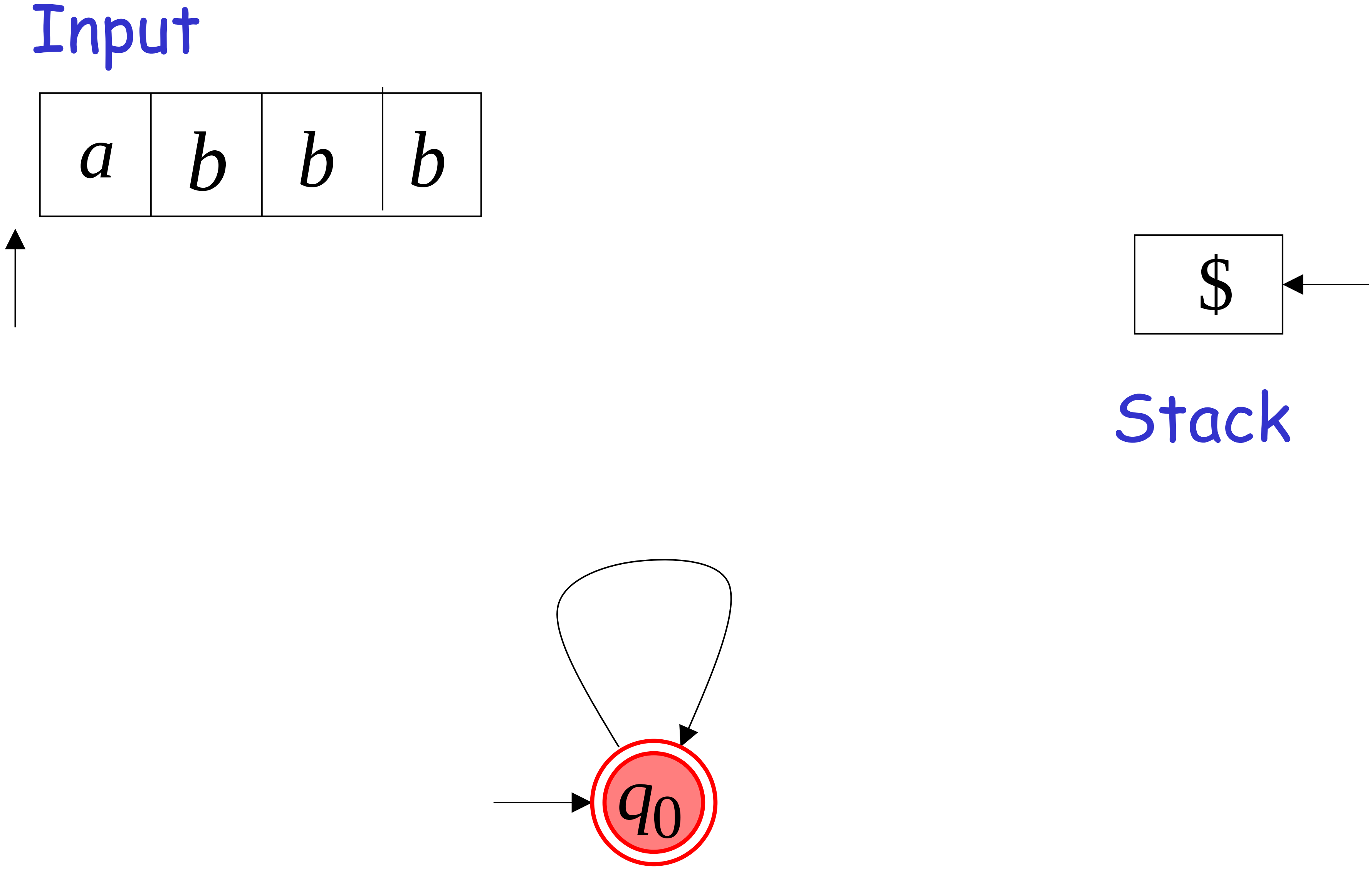
accept

a
$\$$

Stack

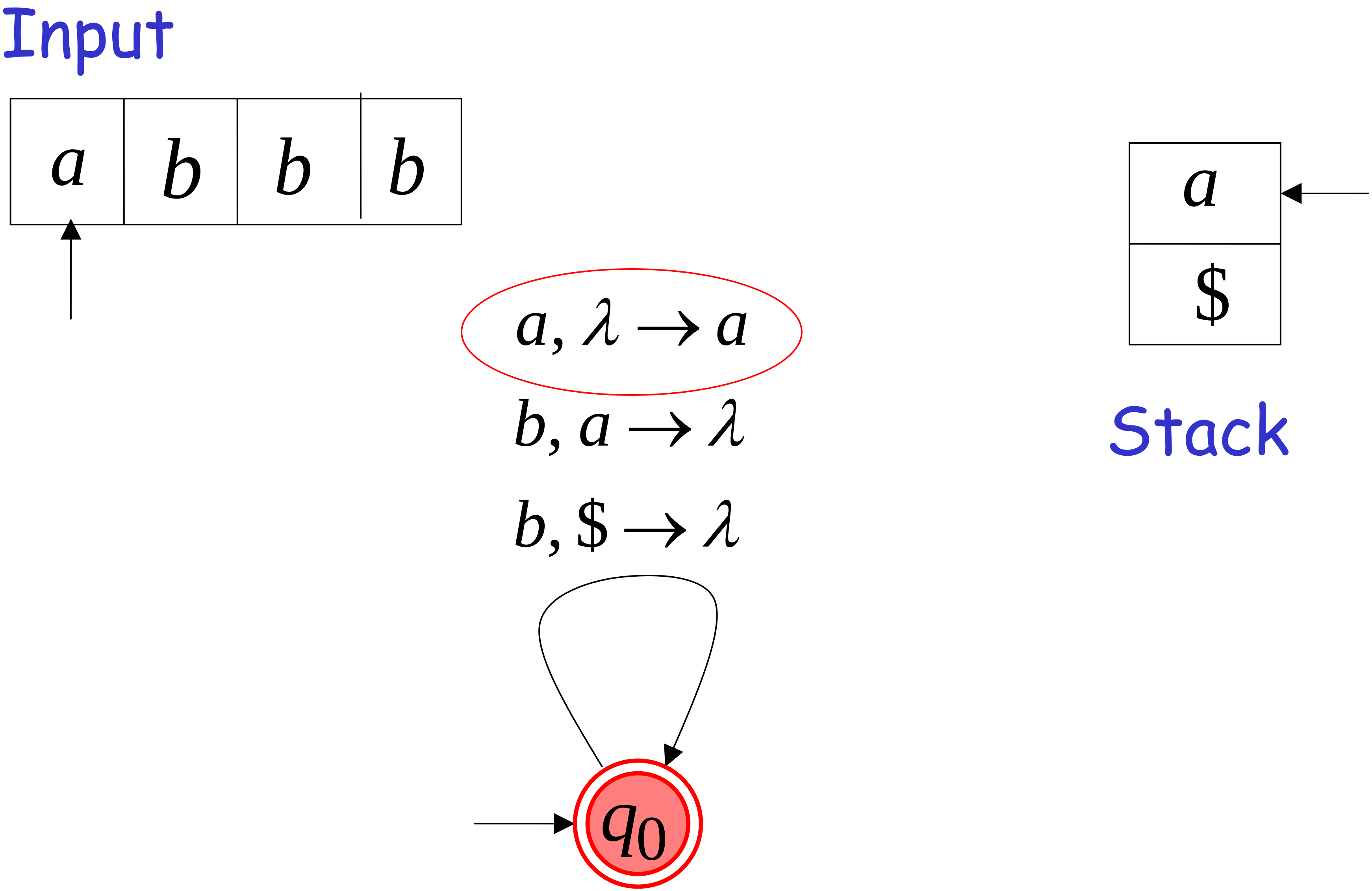
Rejection example:

Time 0


Input

Stack

Time 1



Input

a b b b

$a, \lambda \rightarrow a$

$b, a \rightarrow \lambda$

$b, \$ \rightarrow \lambda$

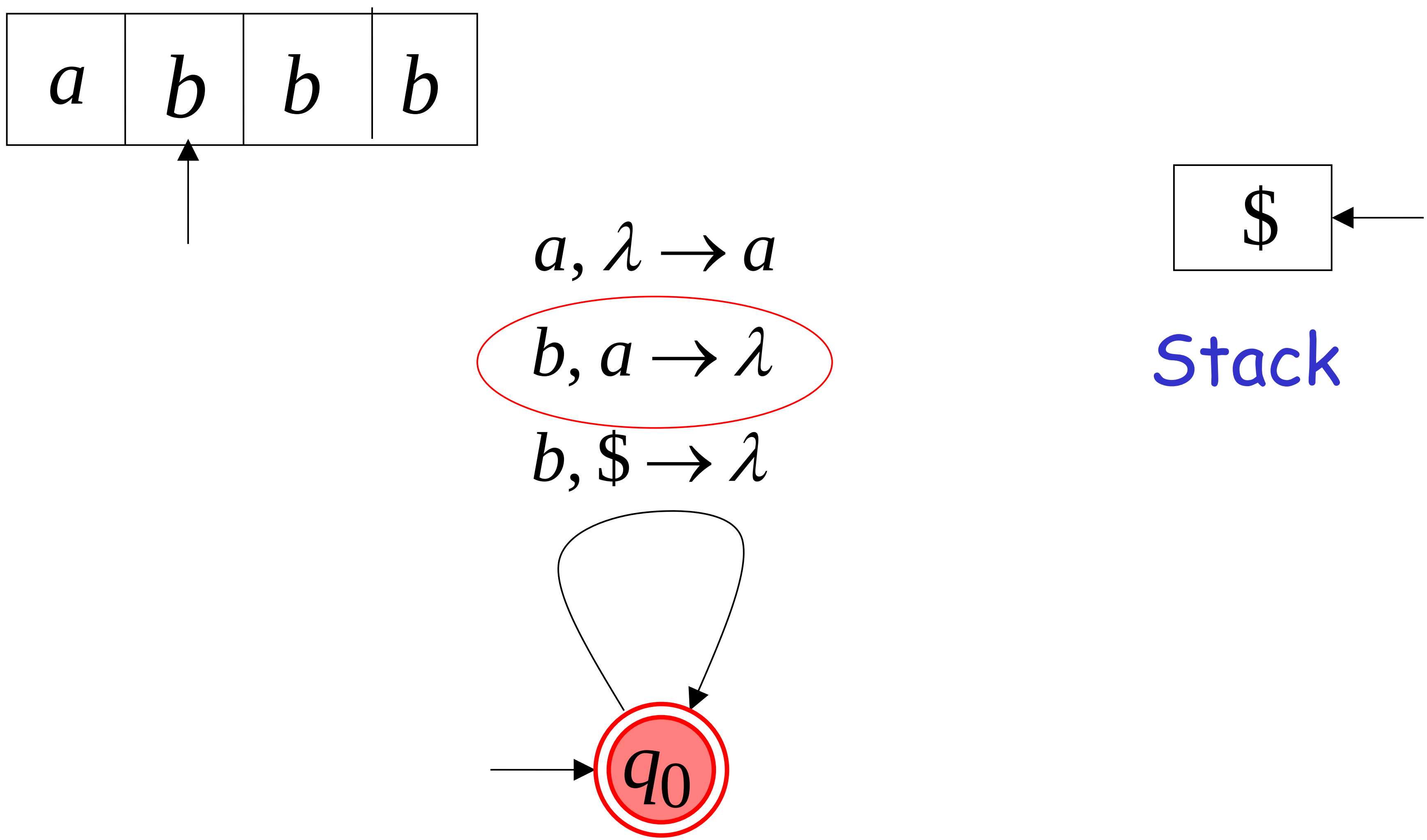
q_0

a

$\$$

Stack

Time 2



Input

a	b	b	b

$a, \lambda \rightarrow a$

$b, a \rightarrow \lambda$

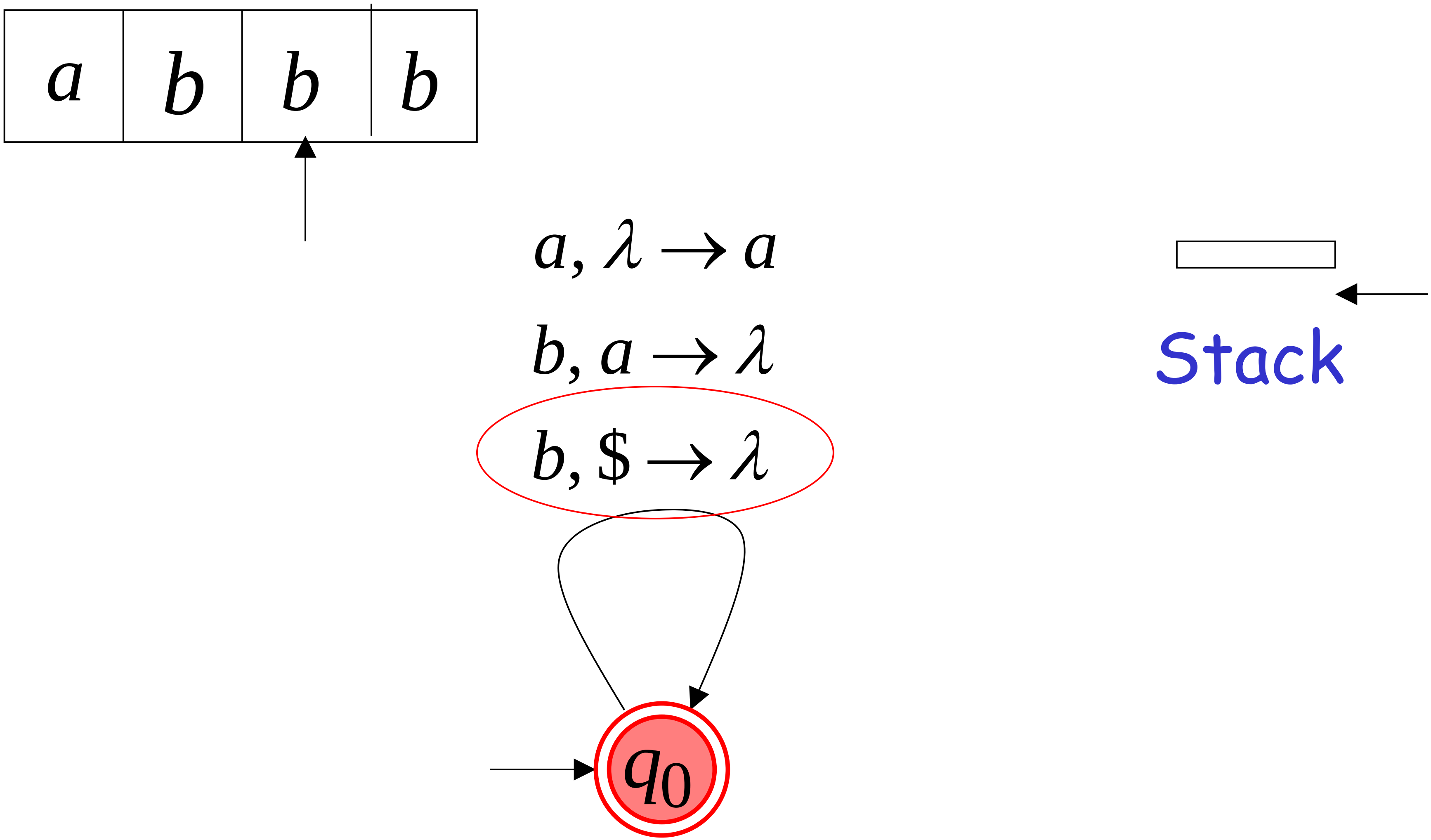
$b, \$ \rightarrow \lambda$

$\$$

Stack

Time 3

Input



a	b	b	b

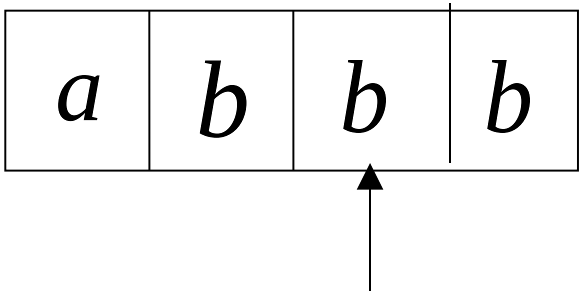
$a, \lambda \rightarrow a$

$b, a \rightarrow \lambda$

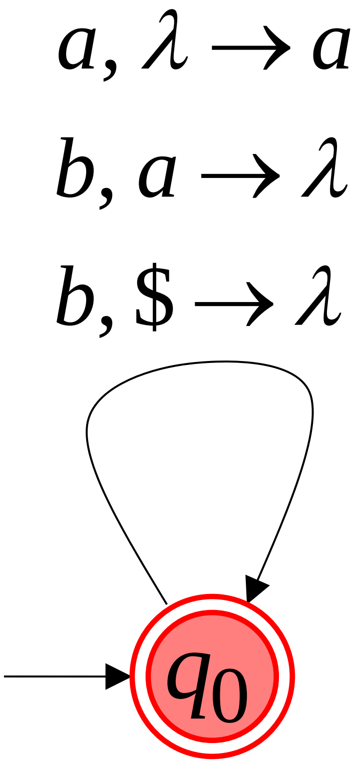
$b, \$ \rightarrow \lambda$

Time 4

Input



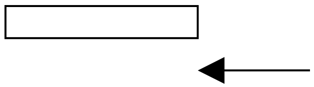
a	b	b	b



$a, \lambda \rightarrow a$

$b, a \rightarrow \lambda$

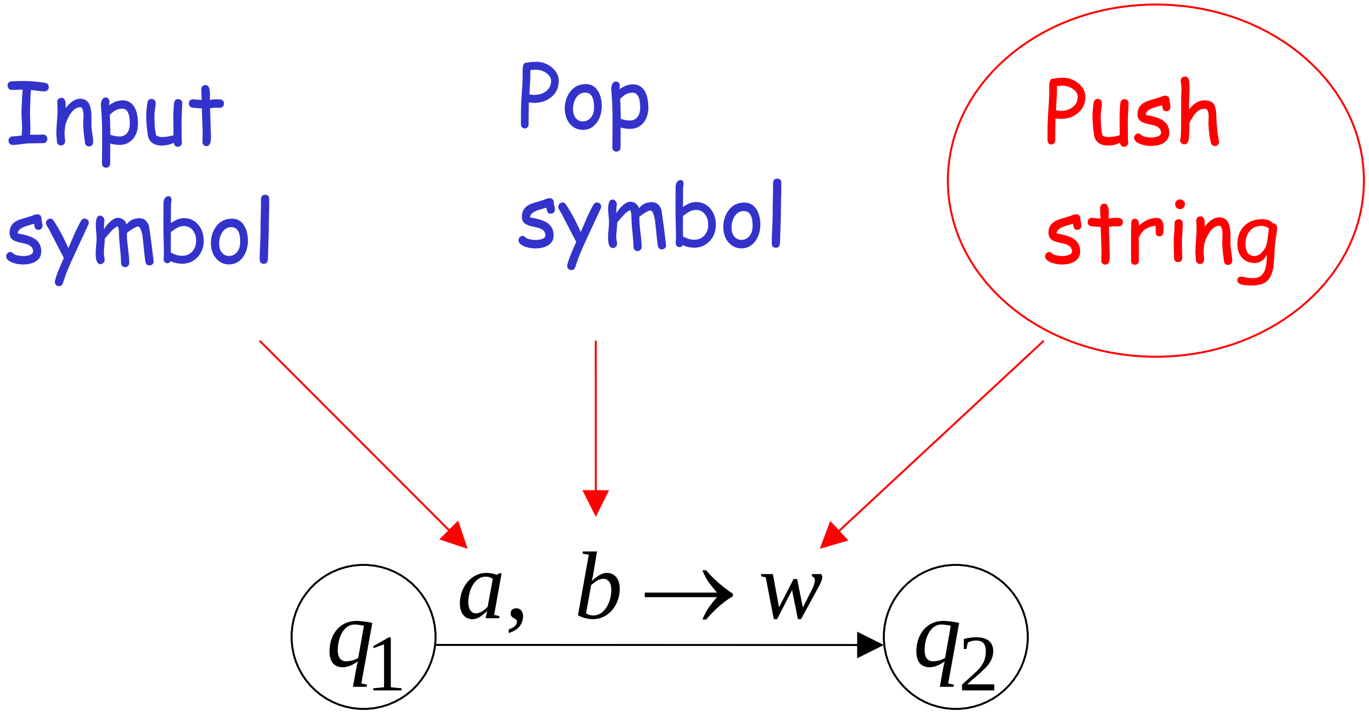
$b, \$ \rightarrow \lambda$



Stack

Halt and Reject

Pushing Strings

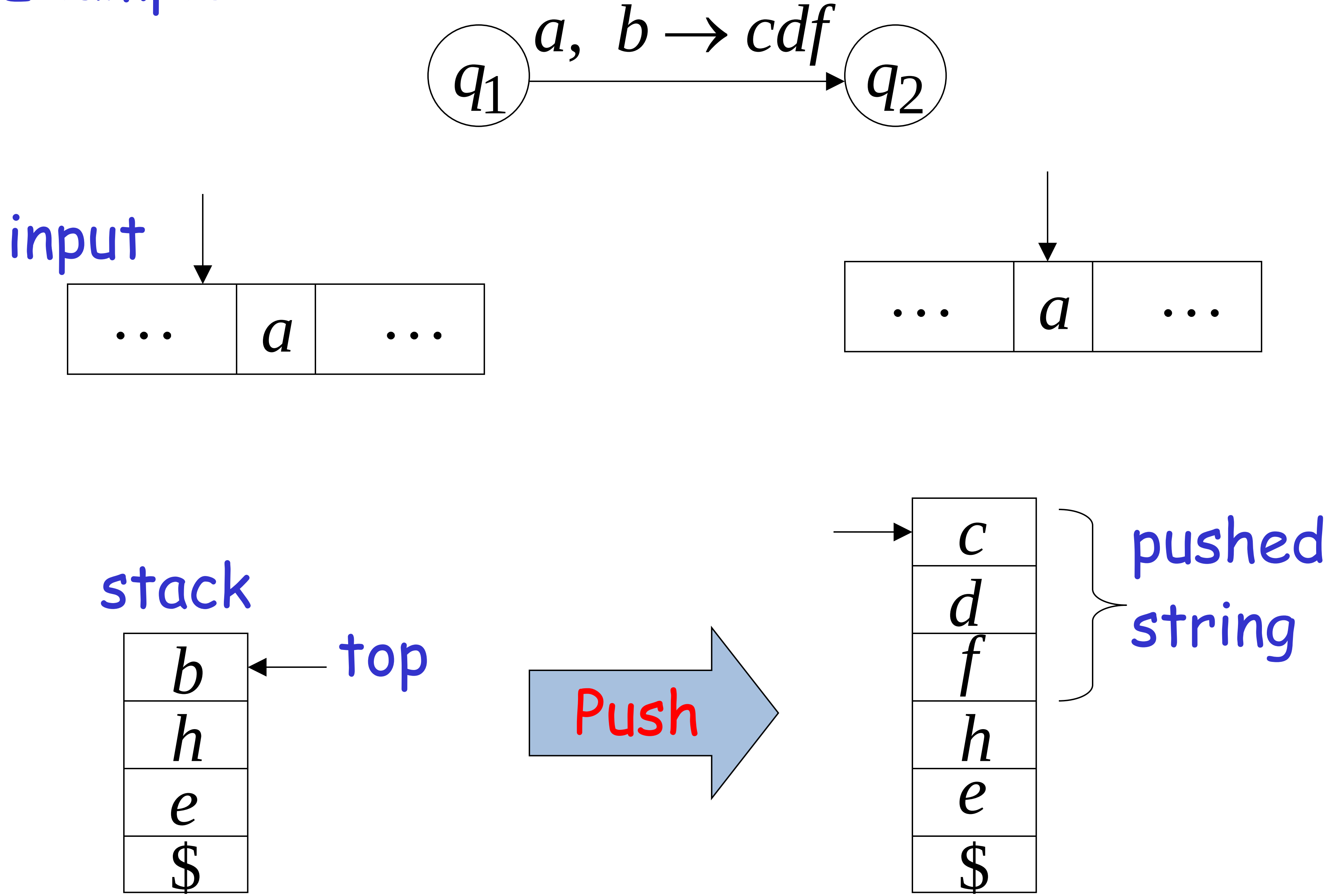


Input symbol

Pop symbol

Push string

Example:


q_1
$a, \; b \rightarrow cdf$
q_2
input
$\cdots$
a
$\cdots$
$\cdots$
a
$\cdots$
stack
b
h
e
$\$$
top
Push
c
d
f
h
e
$\$$
pushed
string

Another NPDA example

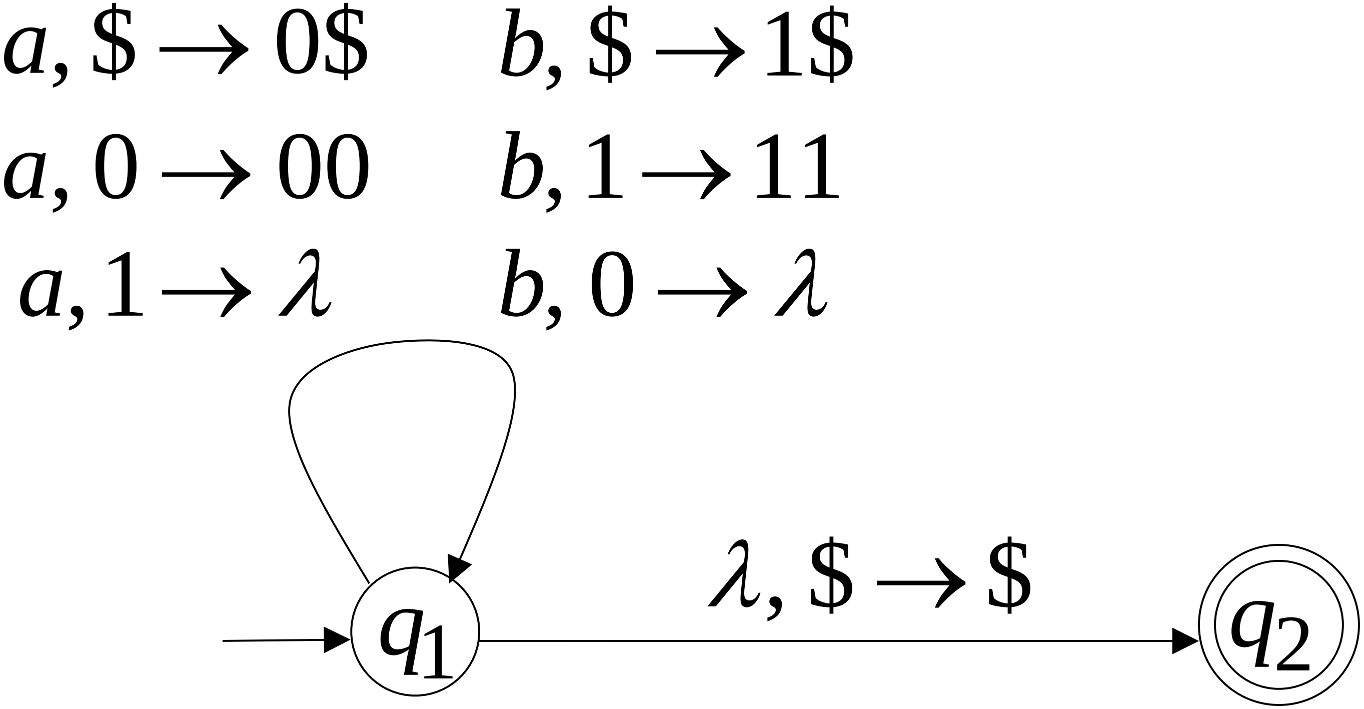
NPDA M

$$L(M) = \{w : \quad n_a = n_b\}$$

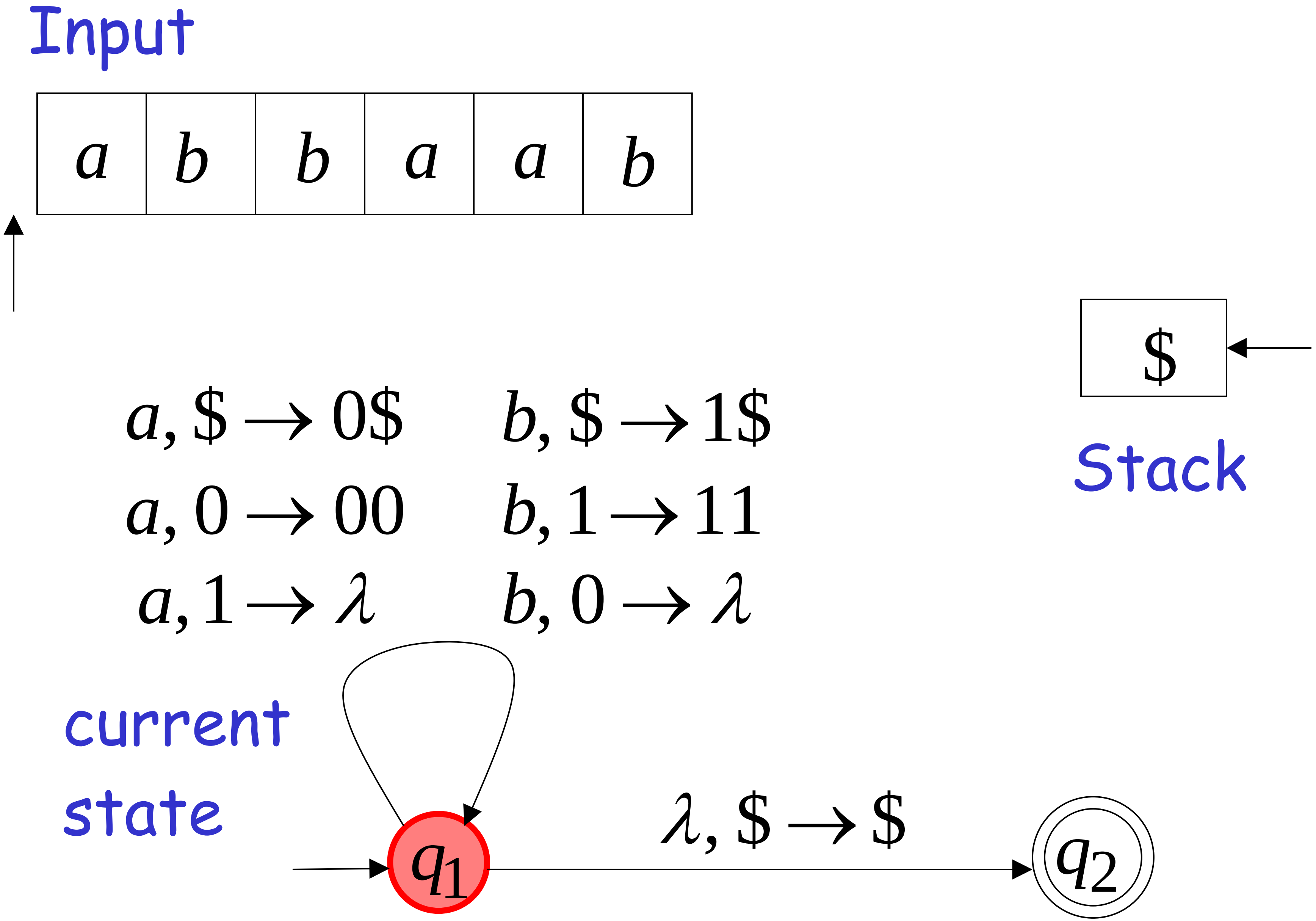
Another NPDA example

NPDA M

$$L(M) = \{w : \; n_a = n_b\}$$



Execution Example: Time 0



Input

a	b	b	a	a	b

$\$$

Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$

current state

Time 1

Input

a	b	b	a	a	b

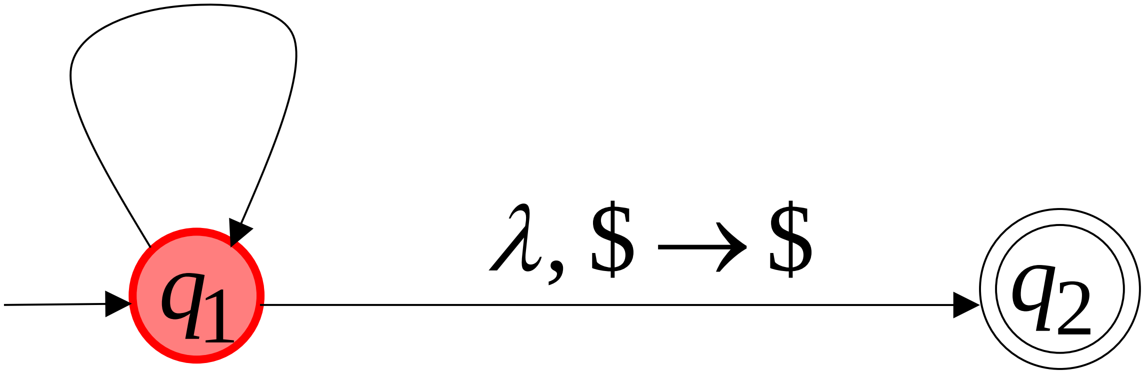
0
\$

Stack

$a, \$ \rightarrow 0\$$ $\quad b, \$ \rightarrow 1\$$

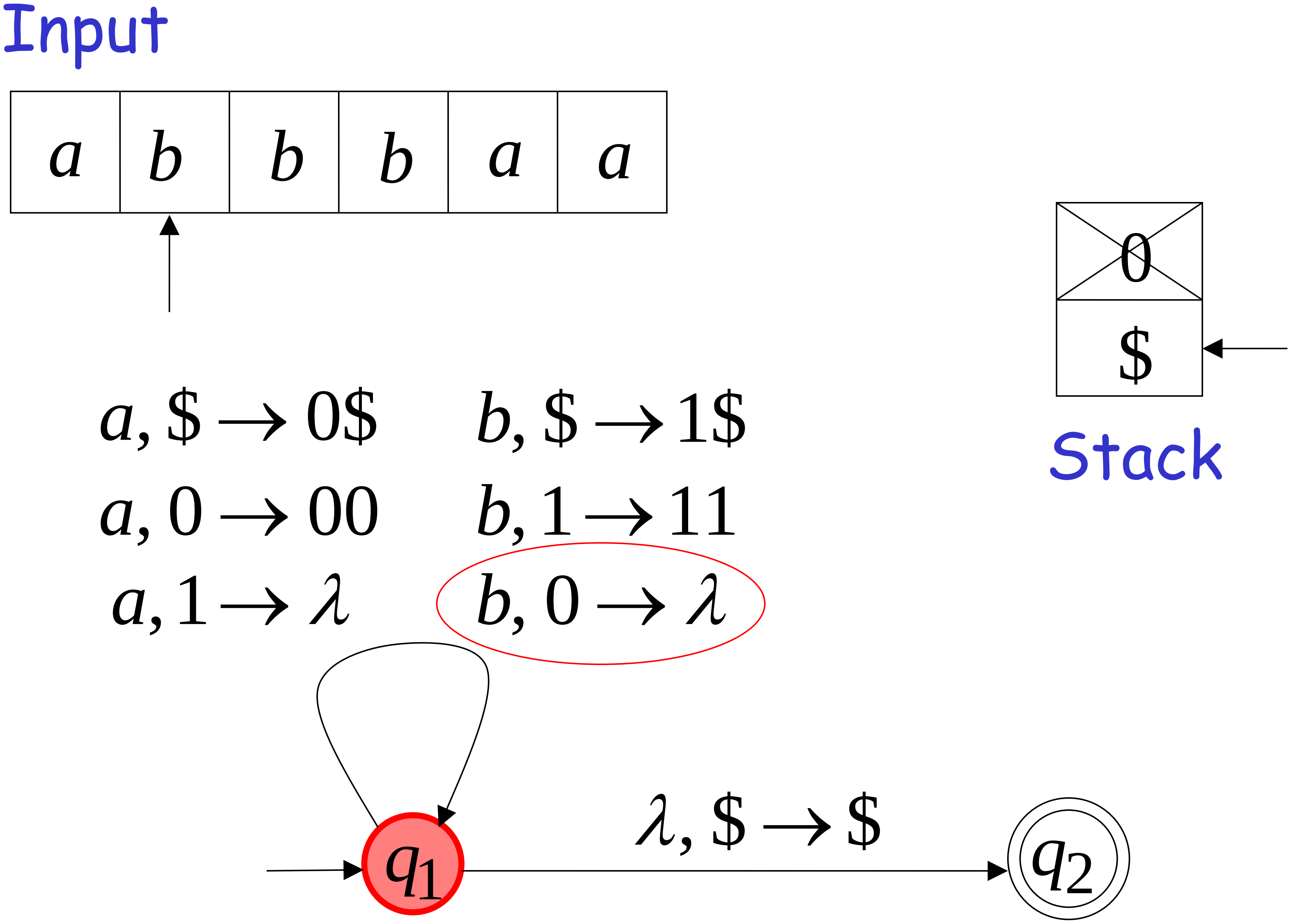
$a, 0 \rightarrow 00$ $\quad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad b, 0 \rightarrow \lambda$



$\lambda, \$ \rightarrow \$$

Time 3



Input

a	b	b	b	a	a

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$

$\lambda, \$ \rightarrow \$$

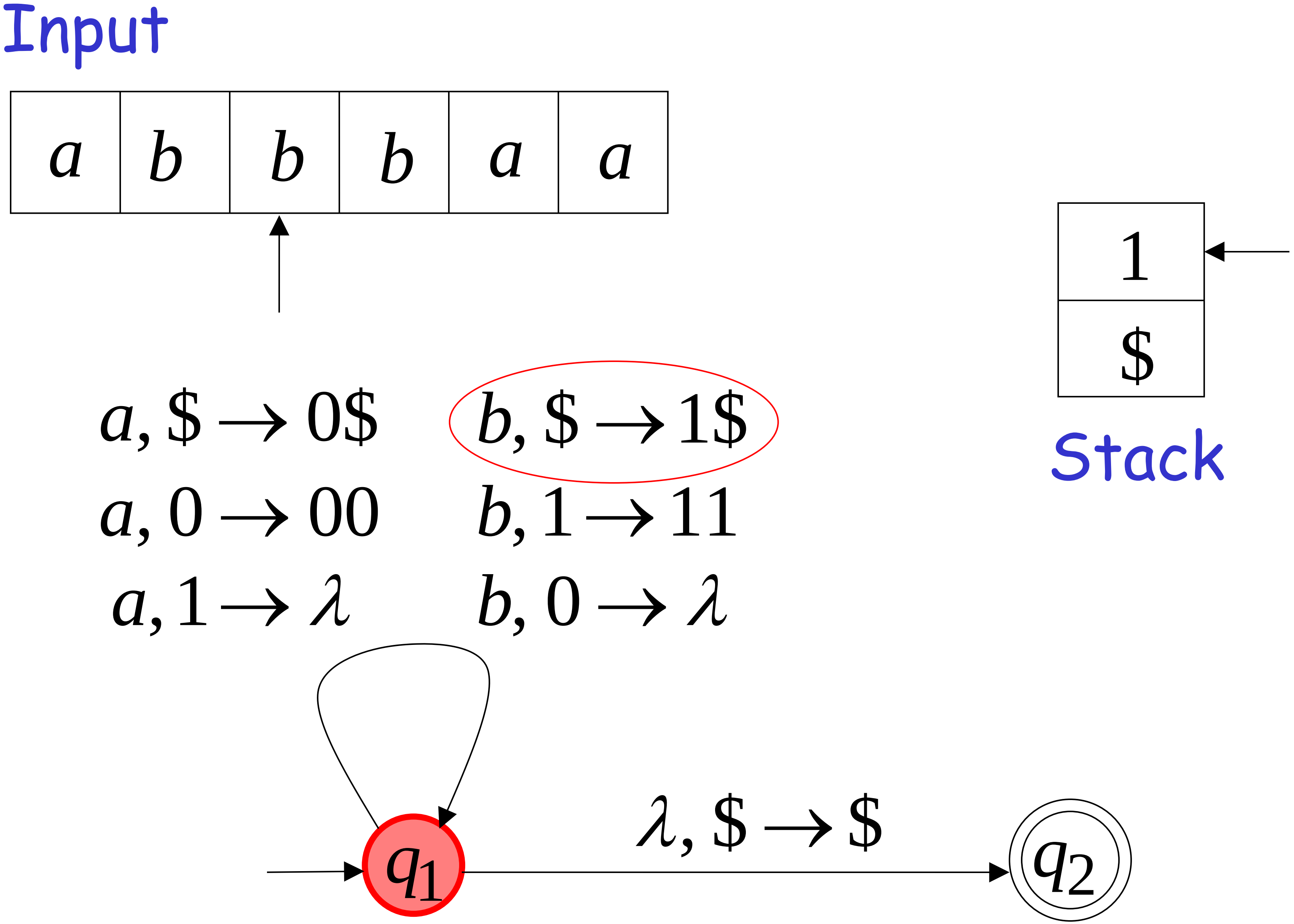
q_1 q_2

$\cancel{0}$

$\$$

Stack

Time 4



Input

a	b	b	b	a	a

Stack

1
$

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$

$\lambda, \$ \rightarrow \$$

Time 5

Input

a	b	b	b	a	a

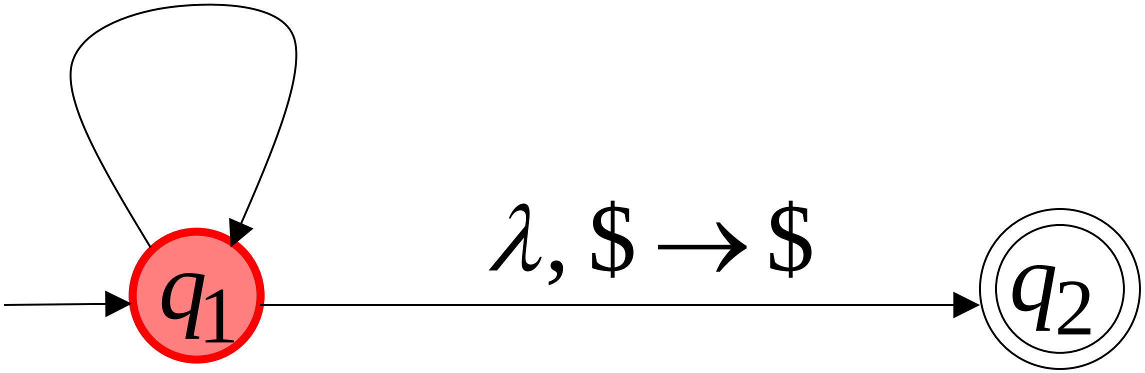
1
1
$

Stack

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



$$\lambda, \$ \rightarrow \$$$

Time 6

Input

a	b	b	b	a	a

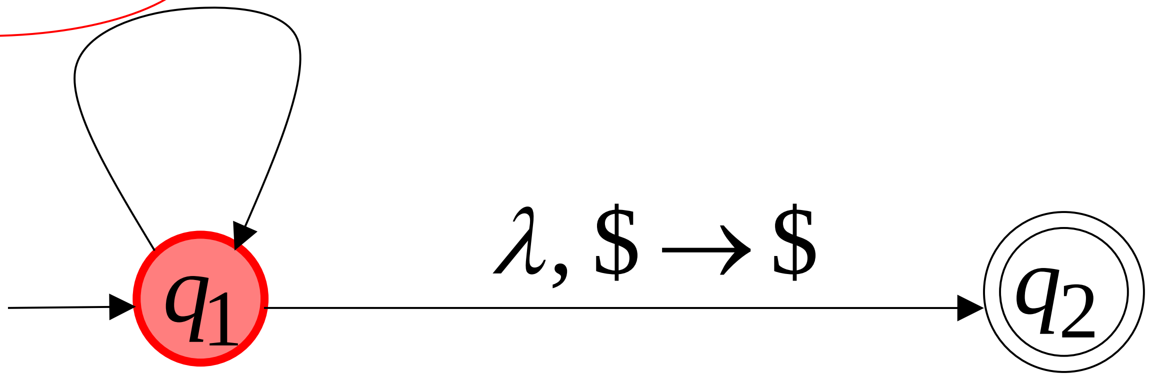
~~1~~
1
$

Stack

$$a, \$ \to 0\$ \qquad b, \$ \to 1\$$$

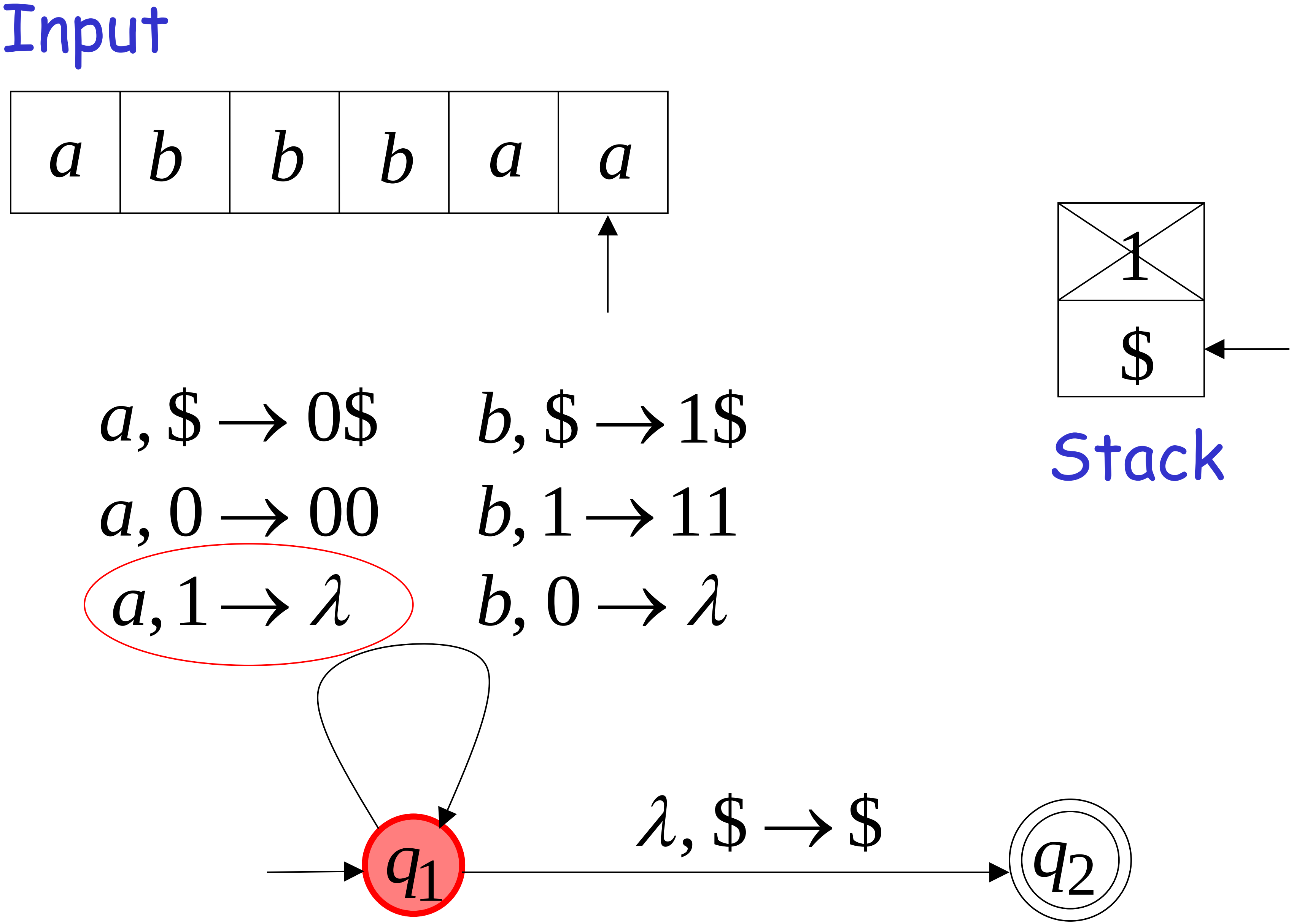
$$a, 0 \to 00 \qquad b, 1 \to 11$$

$$a, 1 \to \lambda \qquad b, 0 \to \lambda$$



$$\lambda, \$ \to \$$$

Time 7



Input

a	b	b	b	a	a

$\cancel{1}$

$\$$

Stack

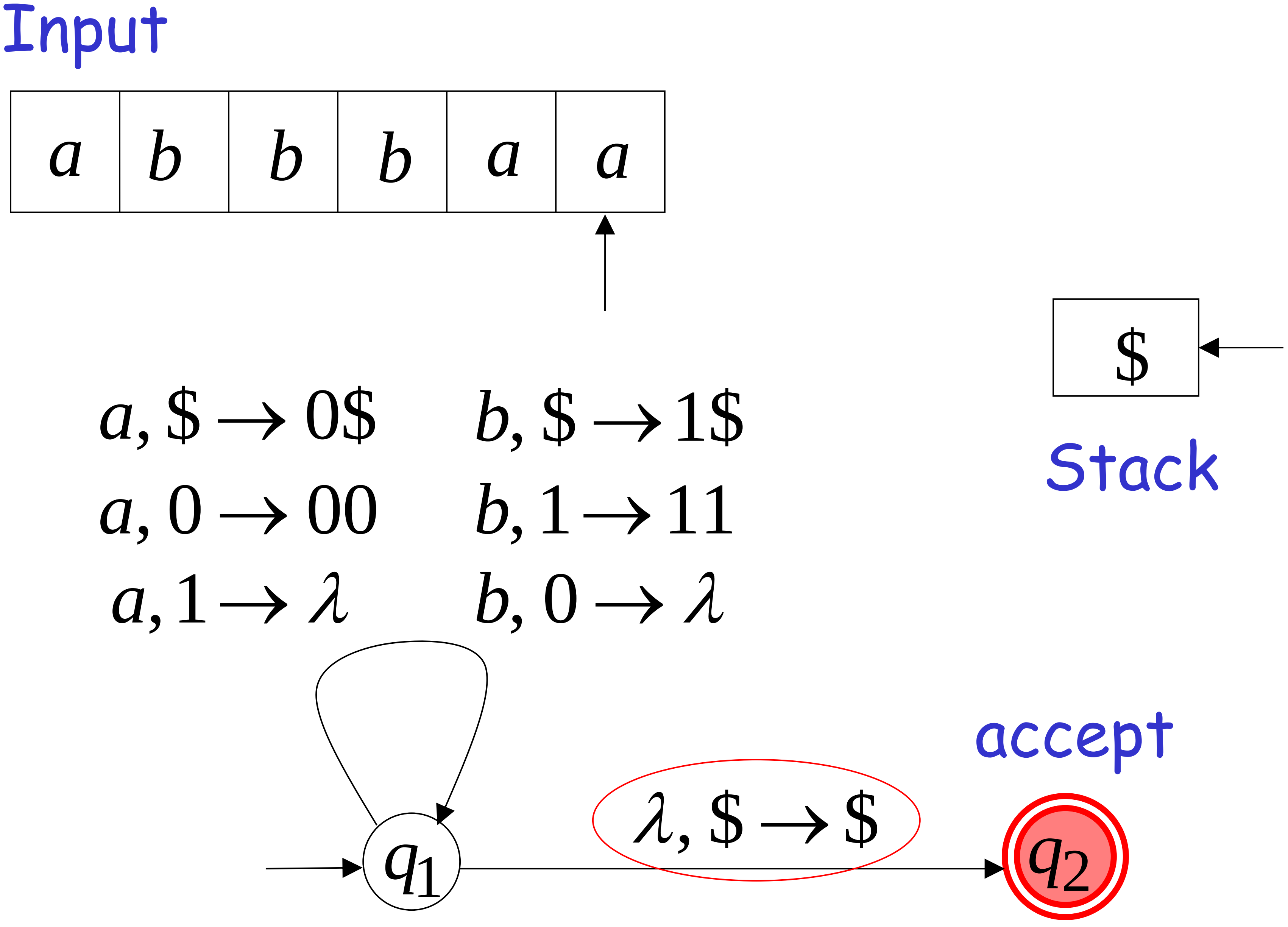
$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$

$$\lambda, \$ \rightarrow \$$$

Time 8

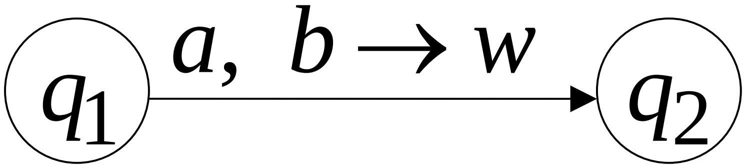


$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

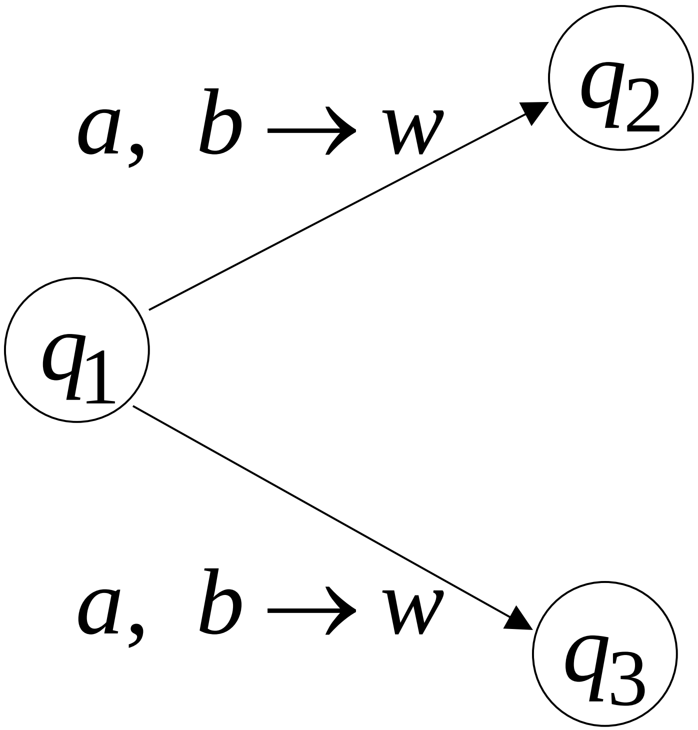
$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$

Formalities for NPDAs



Transition function:

$$\delta(\underline{q_1}, \underline{a}, \underline{b}) = \{(\underline{q_2}, \underline{w})\}$$



Transition function:

$$\delta(q_1, a, b) = \{(q_2, w), \ (q_3, w)\}$$

Formal Definition

Non-Deterministic Pushdown Automaton
NPDA

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$

States

Input alphabet

Stack alphabet

Transition function

Initial state

Stack start symbol

Final states

Formal Definition

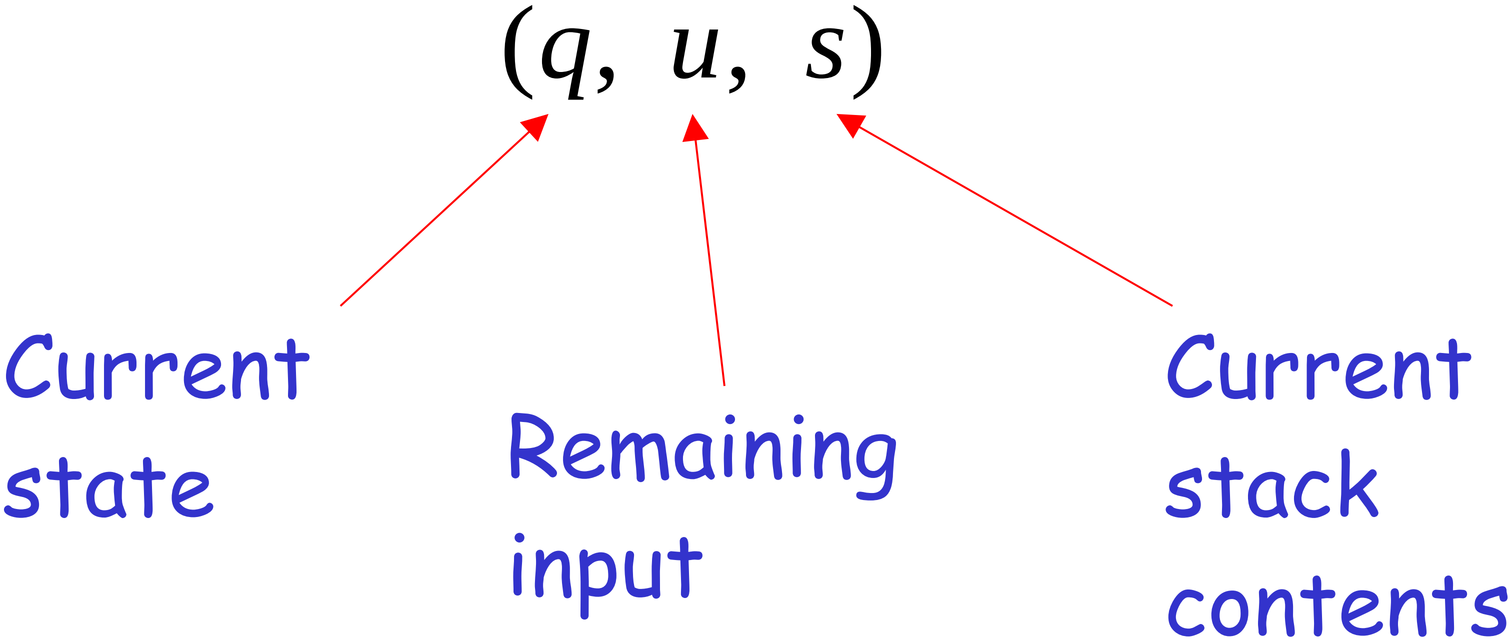
Non-Deterministic Pushdown Automaton

NPDA

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$

$$\boxed{\delta : Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow \text{finite subsets of } \mathrm{Q} \times \Gamma^*}$$

Instantaneous Description



$$(q,\ u,\ s)$$

Current state

Remaining input

Current stack contents

Example: Instantaneous Description

$$(q_1, bbb, aaa\$)$$

Time 4:

Input

a	a	a	b	b	b

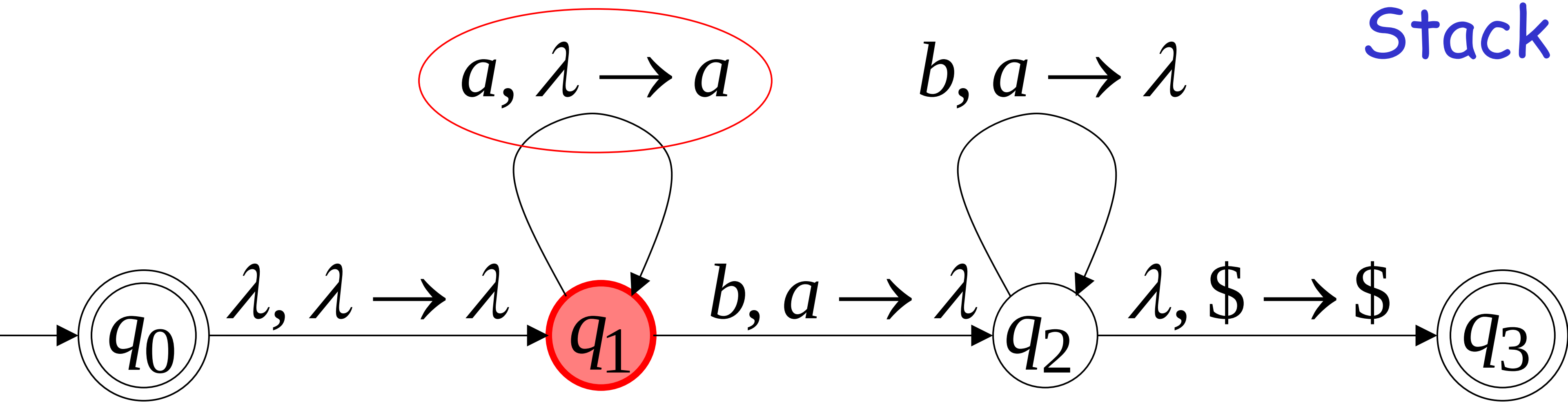
a

a

a

$\$$

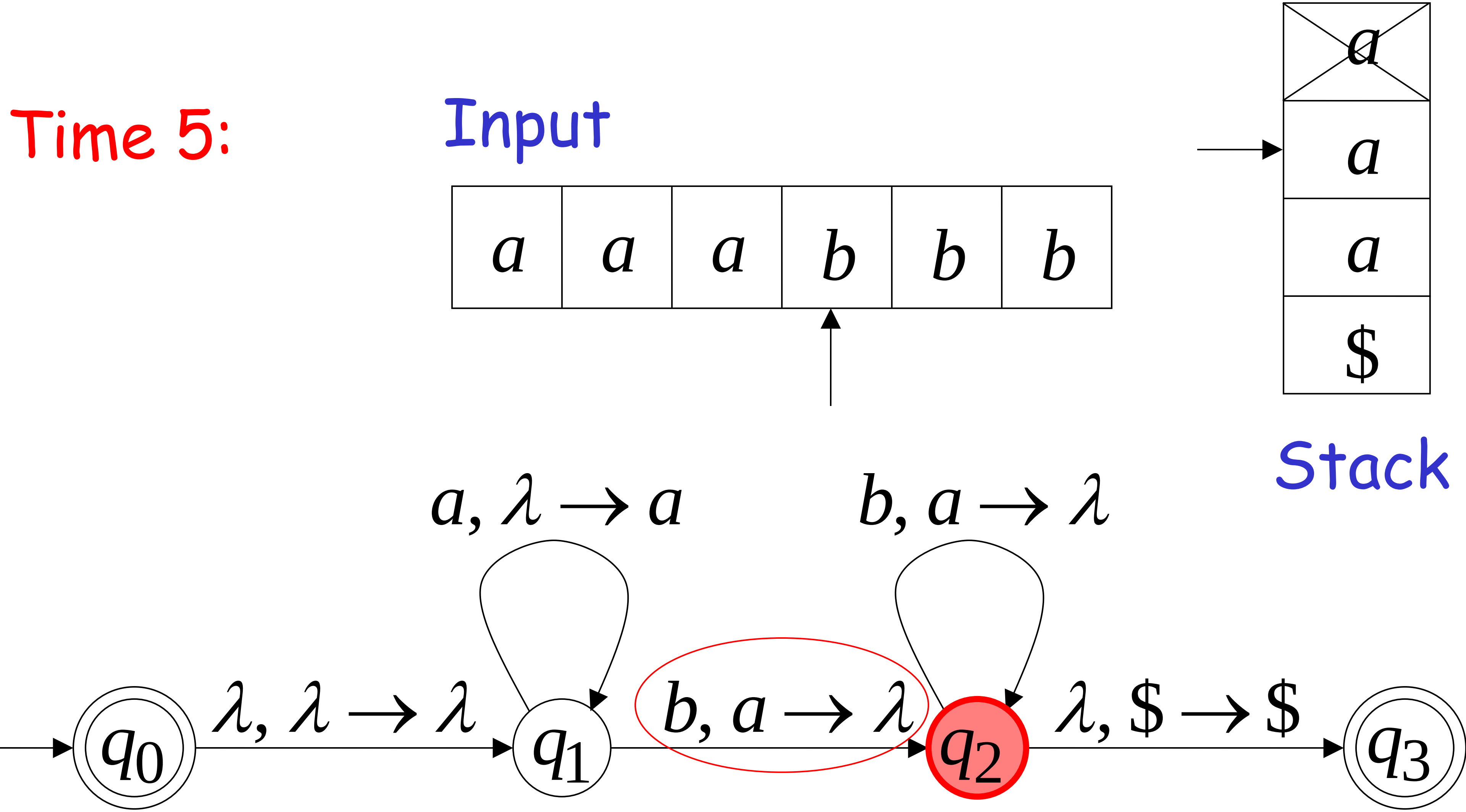
Stack



Example: Instantaneous Description

$$(q_2, bb, aa\$)$$

Time 5:


Input
a
a
a
b
b
b
~~a~~
a
a
$
Stack
$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

We write:

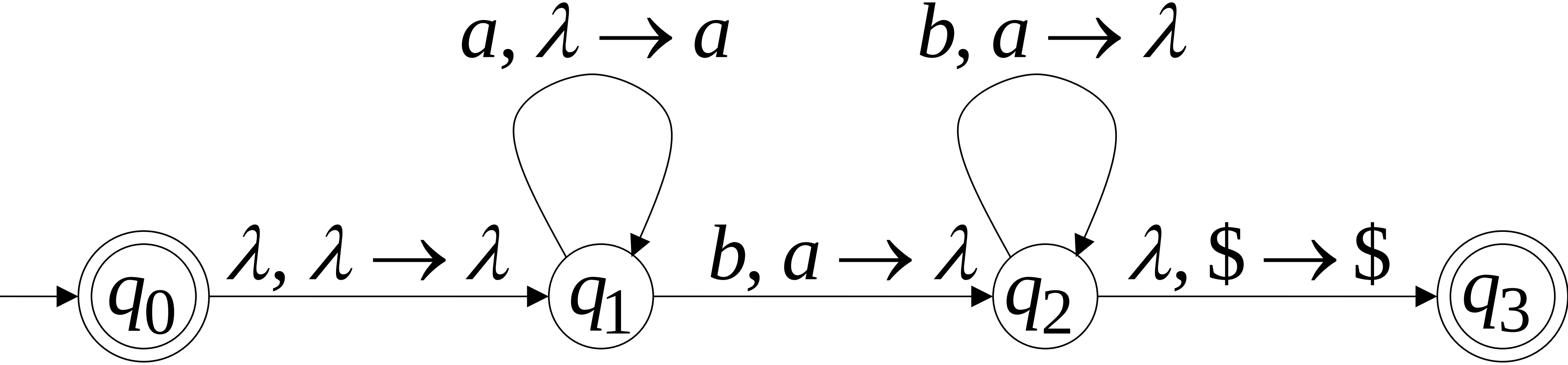
$$(q_1, bbb, aaa\$) \succ (q_2, bb, aa\$)$$

Time 4

Time 5

A computation:

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$
$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$
$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$



$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$

$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$

$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$

For convenience we write:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

Formal Definition

Language $L(M)$ of NPDA M:

$$L(M) = \{w: \ (q_0, w, s) \overset{*}{\succ} (q_f, \lambda, s')\}$$

Initial state

Final state

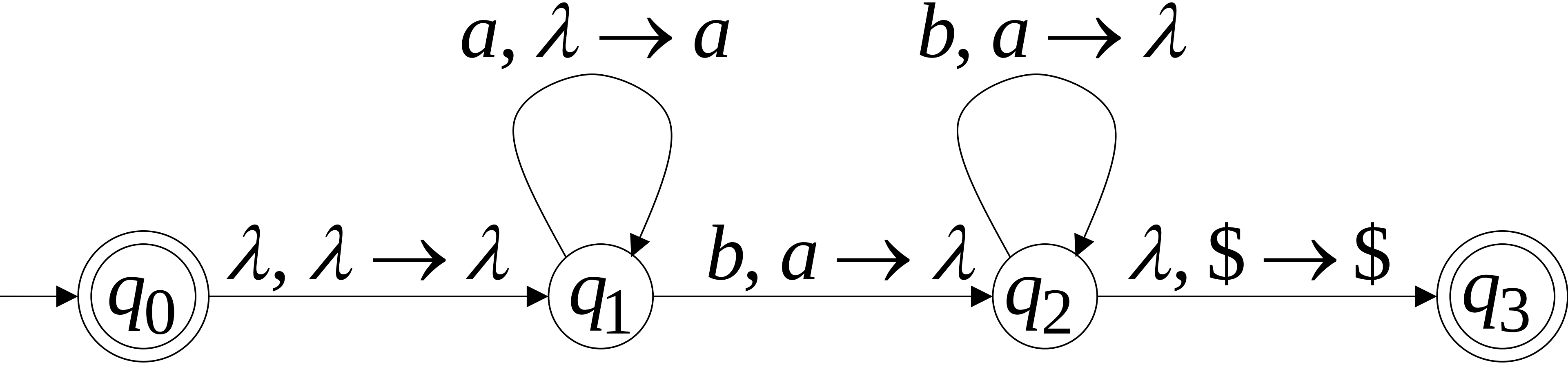
Example:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

⇩

$$aaabbb \in L(M)$$

NPDA M:



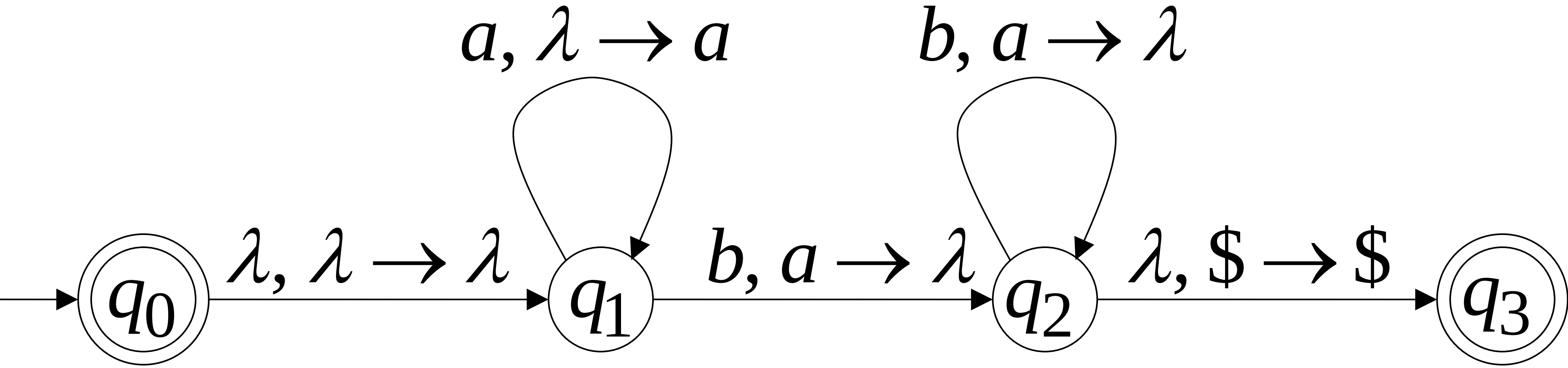
$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

$$(q_0, a^n b^n, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

$$\Downarrow$$

$$a^n b^n \in L(M)$$

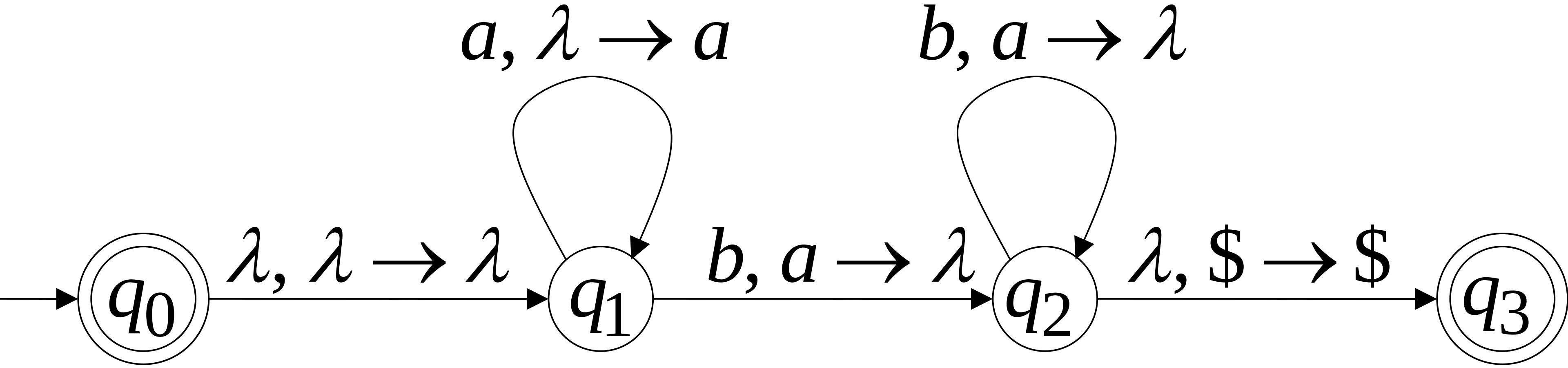
NPDA M:



$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

Therefore: $L(M) = \{a^n b^n : n \geq 0\}$

NPDA M:



$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3