# Grammars

# Grammars

Grammars express languages

Example: the English language

$$\langle sentence \rangle \rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle$$

$$\langle noun\_phrase \rangle \rightarrow \langle article \rangle \ \langle noun \rangle$$

$$\langle predicate \rangle \rightarrow \langle verb \rangle$$

$$\langle article \rangle \to a$$

$$\langle article \rangle \to the$$

$$\langle noun \rangle \to cat$$

$$\langle noun \rangle \to dog$$

$$\langle verb \rangle \to runs$$

$$\langle verb \rangle \to walks$$

## A derivation of "the dog walks":

$$
\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun\_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \ dog \ \ \langle verb \rangle \\
&\Rightarrow the \ \ dog \ \ walks
\end{aligned}
$$

## A derivation of "a cat runs":

$$
\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun\_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ \ cat \ \ \langle verb \rangle \\
&\Rightarrow a \ \ cat \ \ runs
\end{aligned}
$$

Language of the grammar:

L = { "a cat runs",
"a cat walks",
"the cat runs",
"the cat walks",
"a dog runs",
"a dog walks",
"the dog runs",
"the dog walks" }

# Notation

Production Rules

$$\underbrace{\begin{array}{l}\langle noun \rangle \to cat \\ \langle noun \rangle \to dog\end{array}}$$

Variable

Terminal

# Another Example

Grammar: $S \to aSb$

$S \to \lambda$

Derivation of sentence $ab$:

$$S \Rightarrow aSb \Rightarrow ab$$

$S \to aSb$ $\qquad$ $S \to \lambda$

Grammar: $S \to aSb$

$S \to \lambda$

Derivation of sentence $aabb$:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$S \to aSb$ $\qquad\qquad$ $S \to \lambda$

## Other derivations:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$
$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb$$

# Language of the grammar

$$S \to aSb$$

$$S \to \lambda$$

$$L = \{a^n b^n : n \geq 0\}$$

# More Notation

Grammar $G = (V, T, S, P)$

$V:$ Set of variables

$T:$ Set of terminal symbols $\Sigma$

$S:$ Start variable

$P:$ Set of Production rules

# Example

Grammar $G$: $S \to aSb$

$S \to \lambda$

$$G = (V, T, S, P)$$

$V = \{S\}$ $\quad$ $T = \{a, b\}$

$P = \{S \to aSb,\ S \to \lambda\}$

# More Notation

**Sentential Form:**

A sentence that contains
variables and terminals

Example:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

Sentential Forms　　　　sentence

We write: $S \overset{*}{\Rightarrow} aaabbb$

Instead of:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

In general we write: $$w_1 \overset{*}{\Rightarrow} w_n$$

If: $$w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \Lambda \Rightarrow w_n$$

By default: $\quad w \stackrel{*}{\Rightarrow} w$

# Example

**Grammar**

$$S \to aSb$$

$$S \to \lambda$$

**Derivations**

$$S \overset{*}{\Rightarrow} \lambda$$

$$S \overset{*}{\Rightarrow} ab$$

$$S \overset{*}{\Rightarrow} aabb$$

$$S \overset{*}{\Rightarrow} aaabbb$$

# Example

**Grammar**

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

**Derivations**

$$S \overset{*}{\Rightarrow} aaSbb$$

$$aaSbb \overset{*}{\Rightarrow} aaaaaSbbbbb$$

# Another Grammar Example

Grammar $G$ :

$$S \to Ab$$

$$A \to aAb$$

$$A \to \lambda$$

Derivations:

$$S \Rightarrow Ab \Rightarrow b$$

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow abb$$

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow aaAbbb \Rightarrow aabbb$$

# More Derivations

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow aaAbbb \Rightarrow aaaAbbbb$$

$$\Rightarrow aaaaAbbbbb \Rightarrow aaaabbbbb$$

$$S \overset{*}{\Rightarrow} aaaabbbbb$$

$$S \overset{*}{\Rightarrow} aaaaaabbbbbbb$$

$$S \overset{*}{\Rightarrow} a^n b^n b$$

# Language of a Grammar

For a grammar $G$
with start variable $S$ :

$$L(G) = \{w : \ S \overset{*}{\Longrightarrow} w\}$$

String of terminals

# Example

For grammar $G$ :

$$S \to Ab$$

$$A \to aAb$$

$$A \to \lambda$$

$$L(G) = \{a^n b^n b : \quad n \geq 0\}$$

Since: $S \overset{*}{\Rightarrow} a^n b^n b$

# A Convenient Notation

$$A \to aAb$$

$$A \to \lambda$$

$$\Longrightarrow \quad A \to aAb \mid \lambda$$

$$\langle article \rangle \to a$$

$$\langle article \rangle \to the$$

$$\Longrightarrow \quad \langle article \rangle \to a \mid the$$

# Linear Grammars

# Linear Grammars

Grammars with
at most one variable at the right side
of a production

Examples:

$$S \to aSb$$

$$S \to \lambda$$

$$S \to Ab$$

$$A \to aAb$$

$$A \to \lambda$$

# A Non-Linear Grammar

Grammar $G$:

$$S \to SS$$

$$S \to \lambda$$

$$S \to aSb$$

$$S \to bSa$$

What is L(G)?

# A Non-Linear Grammar

Grammar $G$:

$$S \to SS$$

$$S \to \lambda$$

$$S \to aSb$$

$$S \to bSa$$

What is L(G)?

$$L(G) = \{w: \quad n_a(w) = n_b(w)\}$$

Number of $a$ in string $w$

# Another Linear Grammar

Grammar $G$ :

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

What is L(G)?

# Another Linear Grammar

Grammar $G$ :

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

What is L(G)?

$$L(G) = \{a^n b^n : n \geq 0\}$$

# Right-Linear Grammars

All productions have form:

$$A \to xB$$

or

$$A \to x$$

string of terminals

Example:

$$S \to abS$$

$$S \to a$$

# Left-Linear Grammars

All productions have form: $A \rightarrow Bx$

or

$A \rightarrow x$

string of terminals

Example:

$$S \rightarrow Aab$$

$$A \rightarrow Aab \mid B$$

$$B \rightarrow a$$

# Regular Grammars

# Regular Grammars

A regular grammar is any
right-linear or left-linear grammar

Examples:

$G_1$

$$S \to abS$$
$$S \to a$$

$G_2$

$$S \to Aab$$
$$A \to Aab \mid B$$
$$B \to a$$

# Observation

Regular grammars generate regular languages

Examples:

$G_1$

$S \to abS$

$S \to a$

$L(G_1) = (ab)^* a$

$G_2$

$S \to Aab$

$A \to Aab \mid B$

$B \to a$

$L(G_2) = aab(ab)^*$

# Regular Grammars Generate Regular Languages

# Theorem

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Grammars}\end{array}\right\} = \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

# Theorem - Part 1

$$\left\{\begin{array}{l}\text{Languages}\\\text{Generated by}\\\text{Regular Grammars}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular}\\\text{Languages}\end{array}\right\}$$

Any regular grammar generates a regular language

# Theorem - Part 2

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

Any regular language is generated by a regular grammar

# Proof - Part 1

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

The language $L(G)$ generated by any regular grammar $G$ is regular

# The case of Right-Linear Grammars

Let $G$ be a right-linear grammar

We will prove: $L(G)$ is regular

**Proof idea:** We will construct NFA $M$ with $L(M) = L(G)$

Grammar $G$ is right-linear

Example:

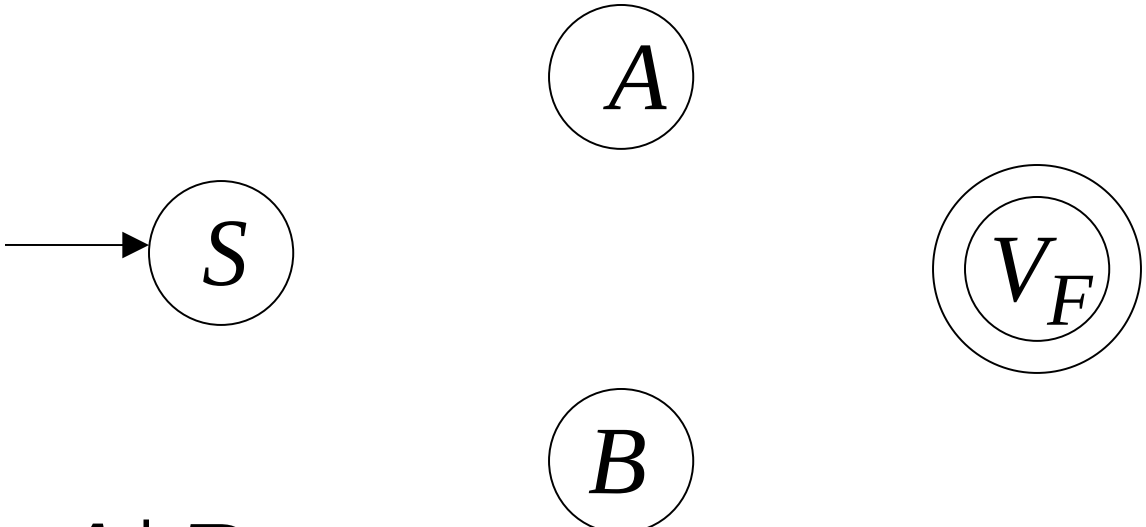
$$S \to aA \mid B$$

$$A \to aa\,B$$

$$B \to b\,B \mid a$$

L(G) ??

Construct NFA $M$ such that every state is a grammar variable:



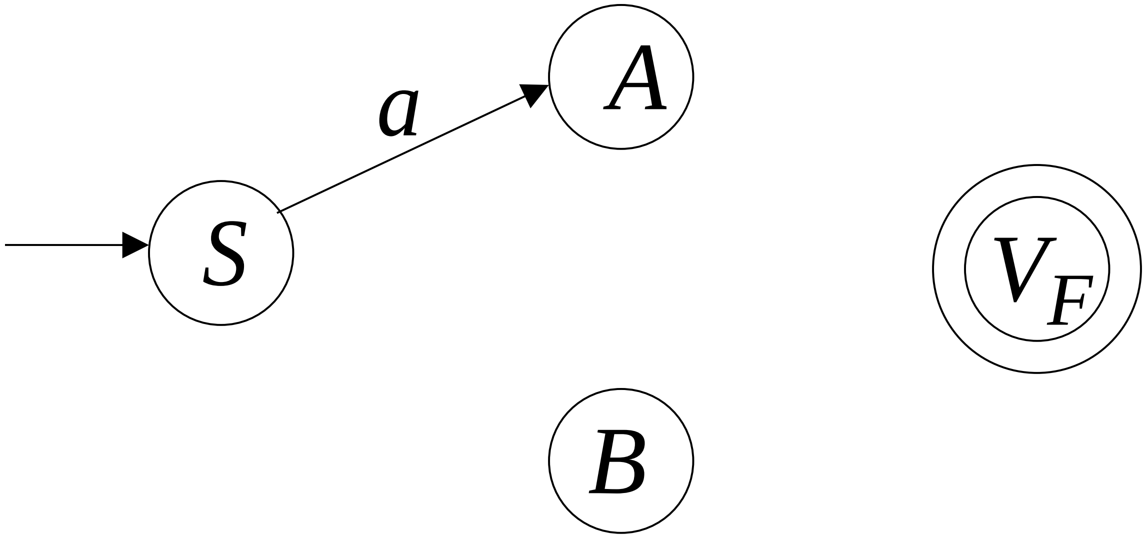

special final state

$$S \to aA \mid B$$

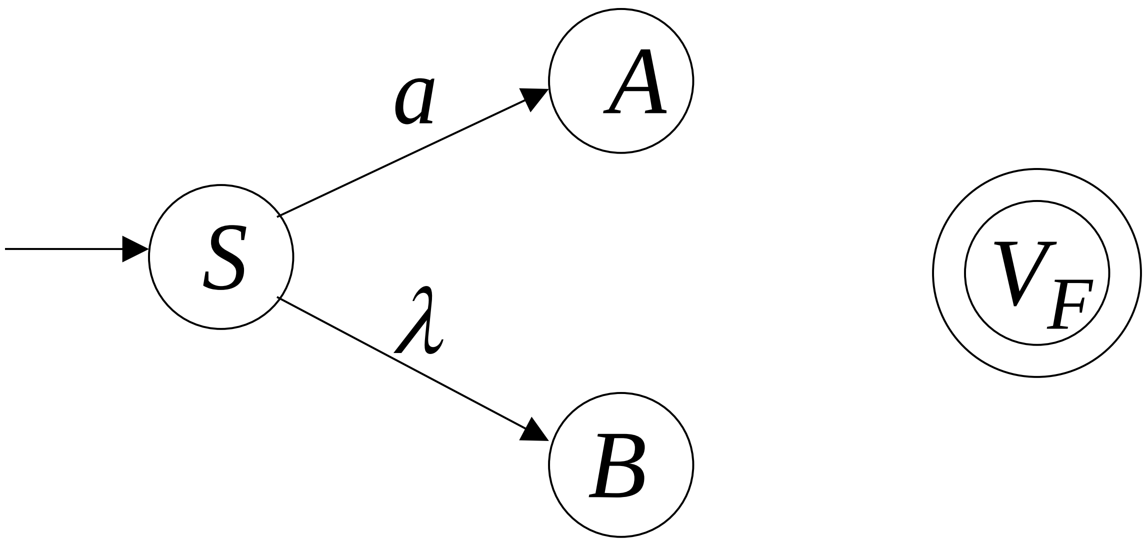
$$A \to aa\,B$$

$$B \to b\,B \mid a$$

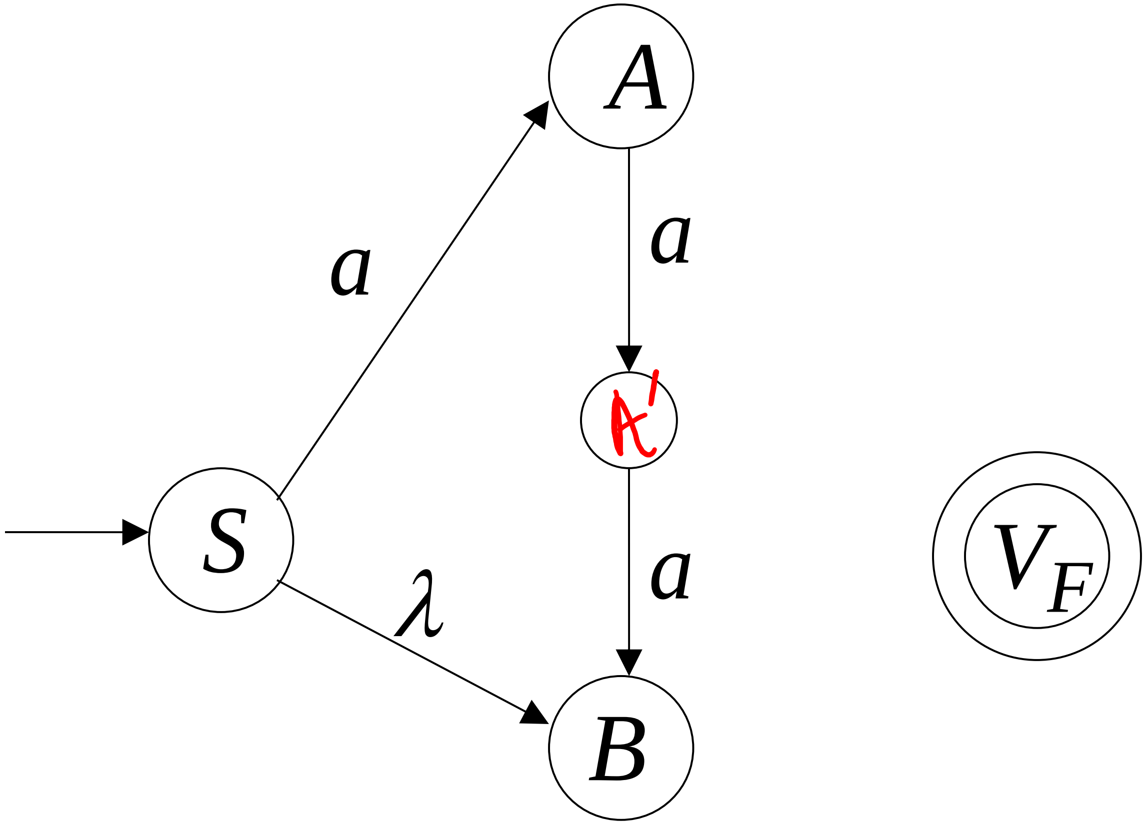
# Add edges for each production:




$$S \to aA$$

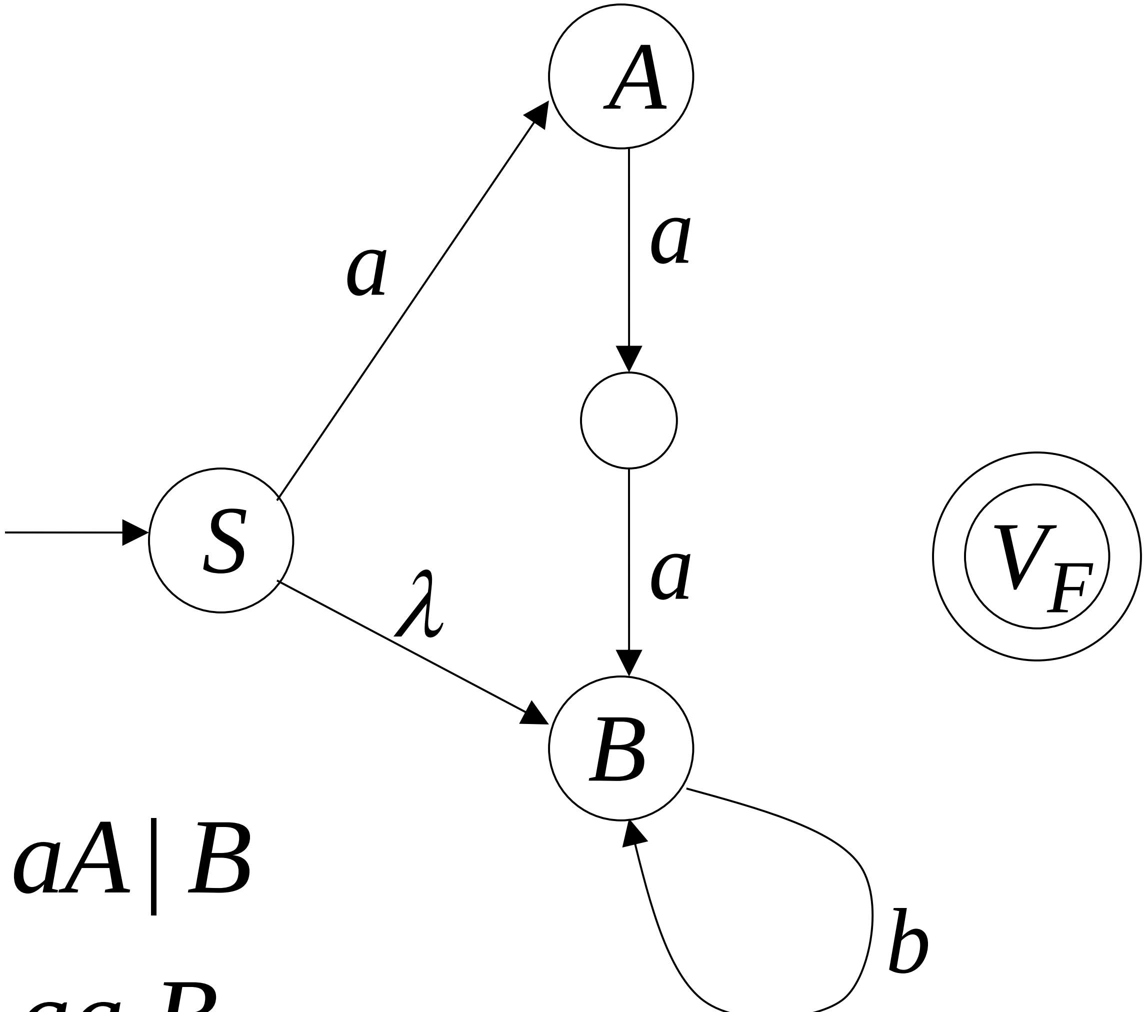



$$S \to aA \mid B$$




$$S \to aA \mid B$$

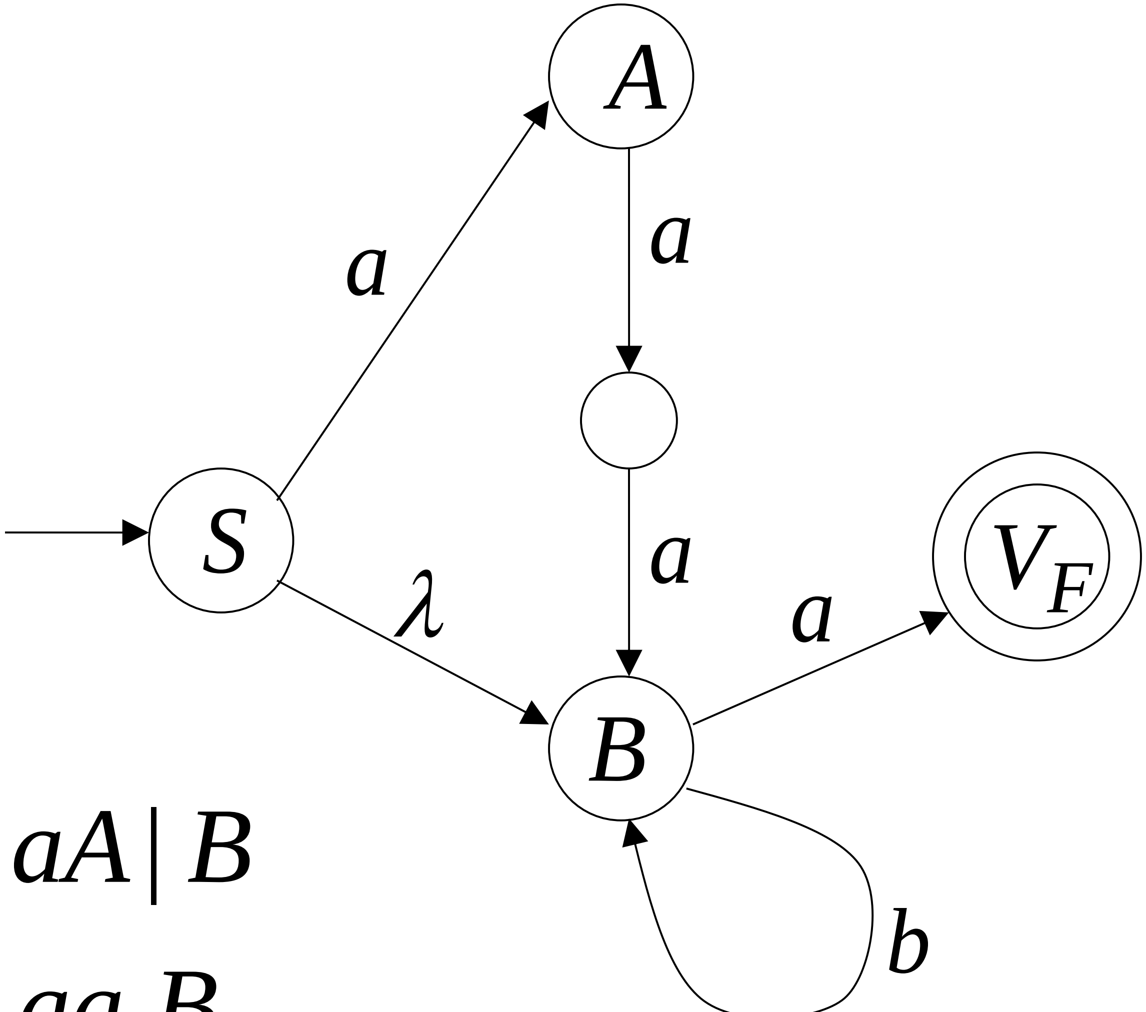
$$A \to aa\, B$$




$$S \rightarrow aA \mid B$$

$$A \rightarrow aa\ B$$

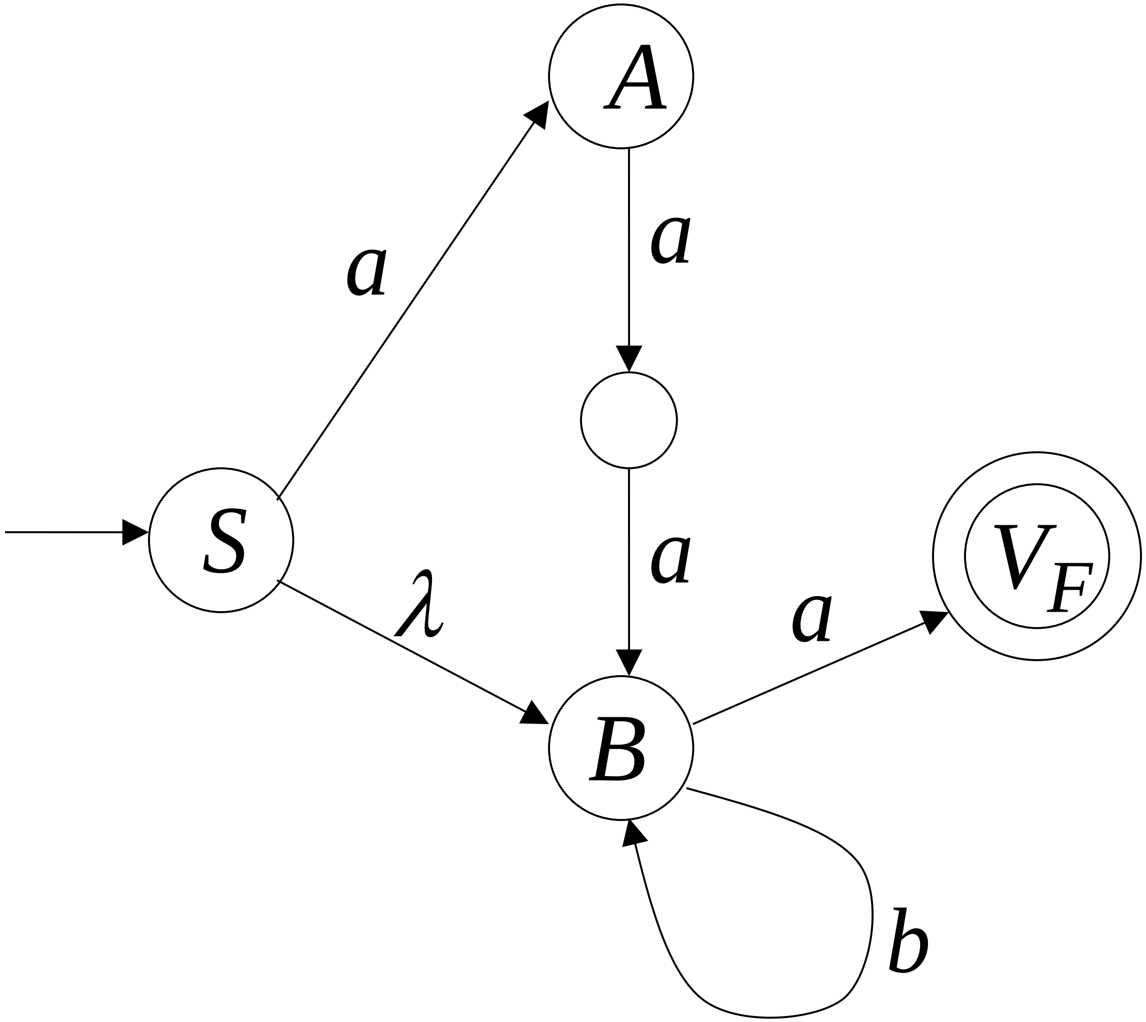
$$B \rightarrow bB$$




$$S \to aA \mid B$$

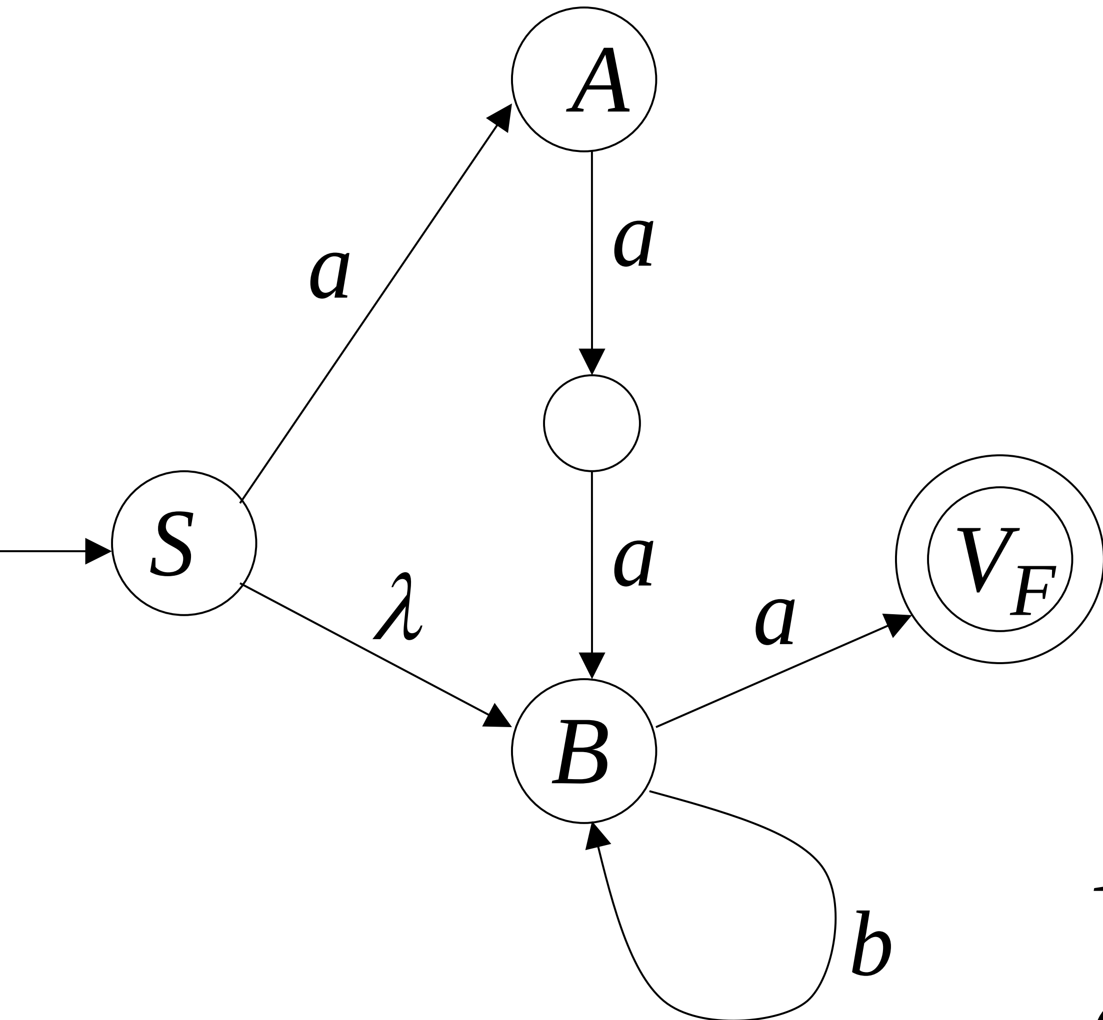
$$A \to aa\,B$$

$$B \to bB \mid a$$




$$S \Rightarrow aA \Rightarrow aaaB \Rightarrow aaabB \Rightarrow aaaba$$

NFA $M$




Grammar

$G$

$$S \to aA \mid B$$

$$A \to aa\,B$$

$$B \to bB \mid a$$

$$L(M) = L(G) =$$

$$aaab^*a + b^*a$$

# In General

A right-linear grammar $G$

has variables: $V_0, V_1, V_2, \ldots$

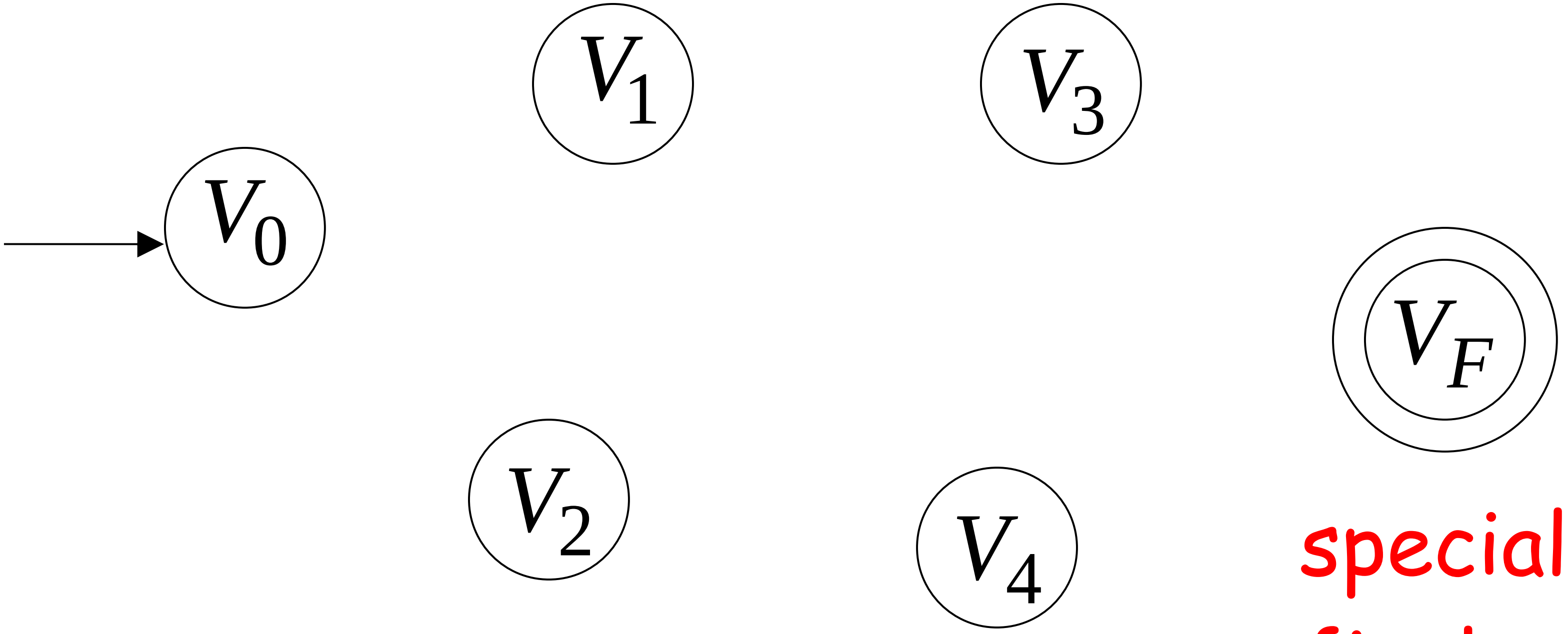
and productions: $V_i \rightarrow a_1 a_2 \ldots a_m V_j$

or

$V_i \rightarrow a_1 a_2 \ldots a_m$

We construct the NFA $M$ such that:

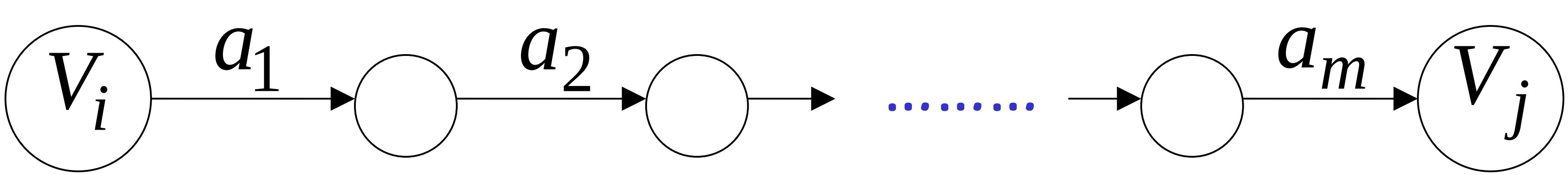
each variable $V_i$ corresponds to a node:




special final state

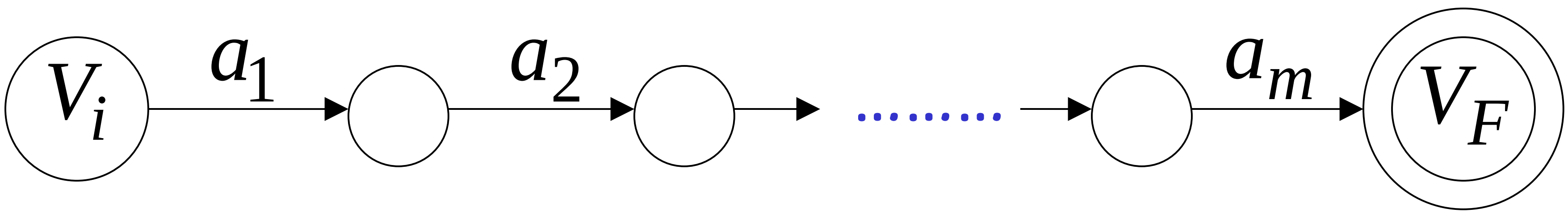
For each production: $V_i \to a_1 a_2 \Lambda\ a_m V_j$

we add transitions and intermediate nodes

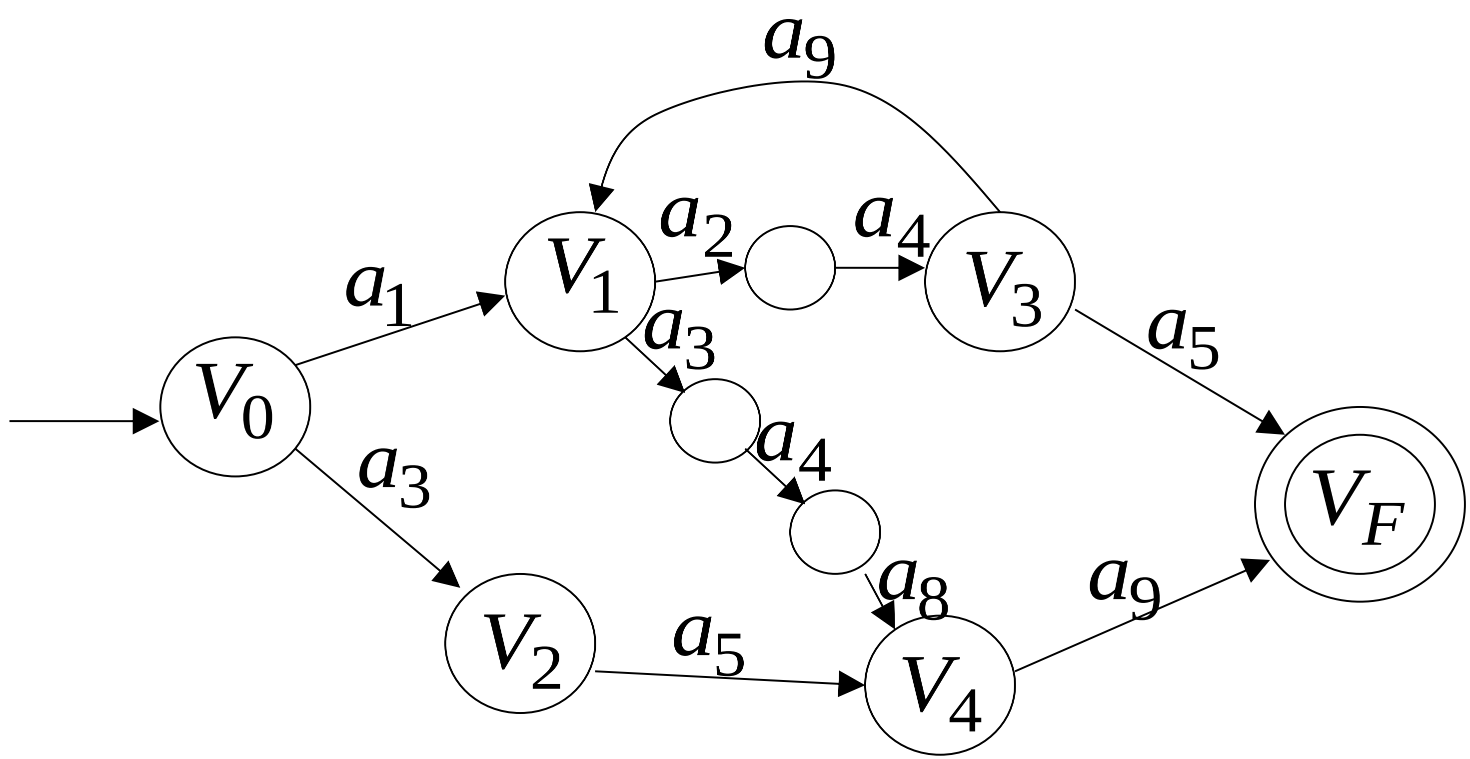


For each production: $V_i \to a_1 a_2 \Lambda\ a_m$

we add transitions and intermediate nodes



Resulting NFA $M$ looks like this:




It holds that: $L(G) = L(M)$

# The case of Left-Linear Grammars

Let $G$ be a left-linear grammar

We will prove: $L(G)$ is regular

**Proof idea:**

We will construct a right-linear grammar $G'$ with $L(G) = L(G')^R$

Since $G$ is left-linear grammar the productions look like:

$$A \to B a_1 a_2 \Lambda\ a_k$$

$$A \to a_1 a_2 \Lambda\ a_k$$

Construct right-linear grammar $G'$

Left linear $G$

$$A \to Ba_1a_2\Lambda\ a_k$$

$$A \to Bv$$

Right linear $G'$

$$A \to a_k\Lambda\ a_2a_1B$$

$$A \to v^R B$$

Construct right-linear grammar $G'$

Left linear $G$

$$A \to a_1 a_2 \Lambda \; a_k$$

$$A \to v$$

Right linear $G'$

$$A \to a_k \Lambda \; a_2 a_1$$

$$A \to v^R$$

It is easy to see that: $L(G) = L(G')^R$

Since $G'$ is right-linear, we have:

$L(G') \Rightarrow L(G')^R \Rightarrow L(G)$

Regular Language | Regular Language | Regular Language

# Proof - Part 2

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

Any regular language $L$ is generated by some regular grammar $G$

Any regular language $L$ is generated
by some regular grammar $G$

**Proof idea:**

Let $M$ be the NFA with $L = L(M)$.

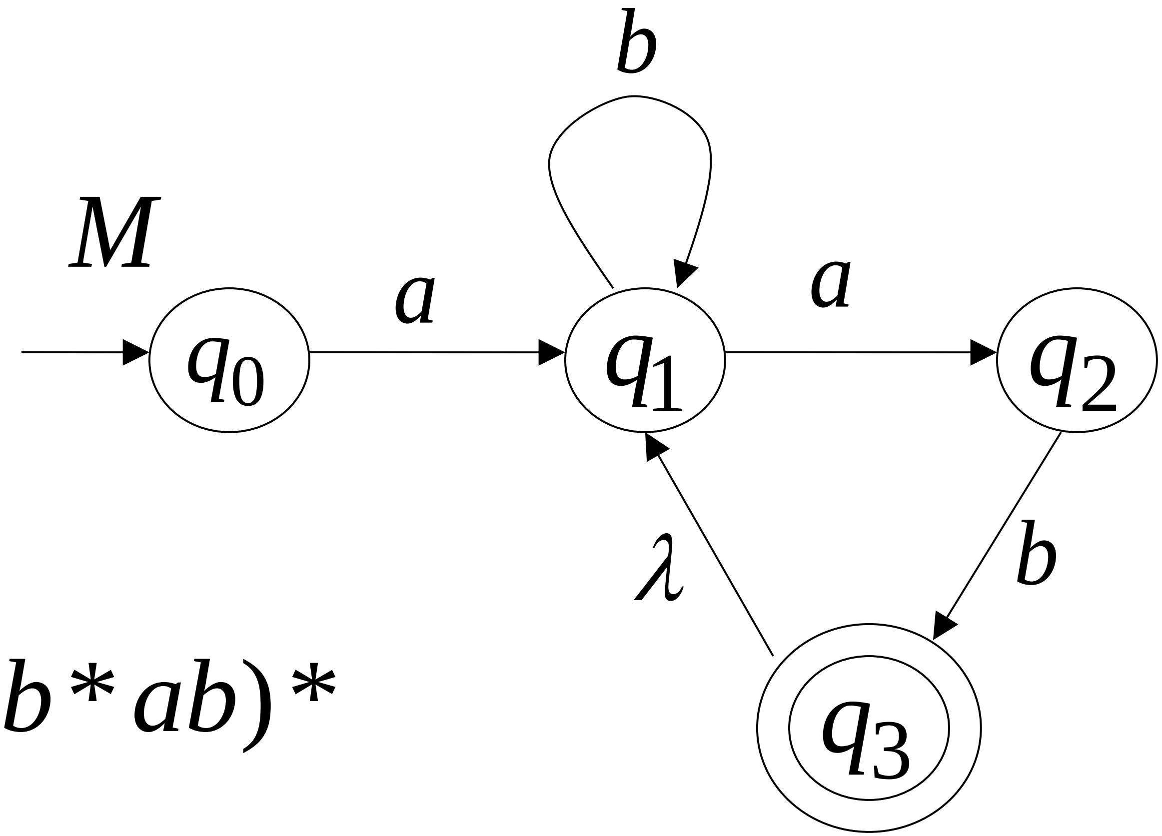
Construct from $M$ a regular grammar $G$
such that

$$L(M) = L(G)$$

Since $L$ is regular

there is an NFA $M$ such that $L = L(M)$

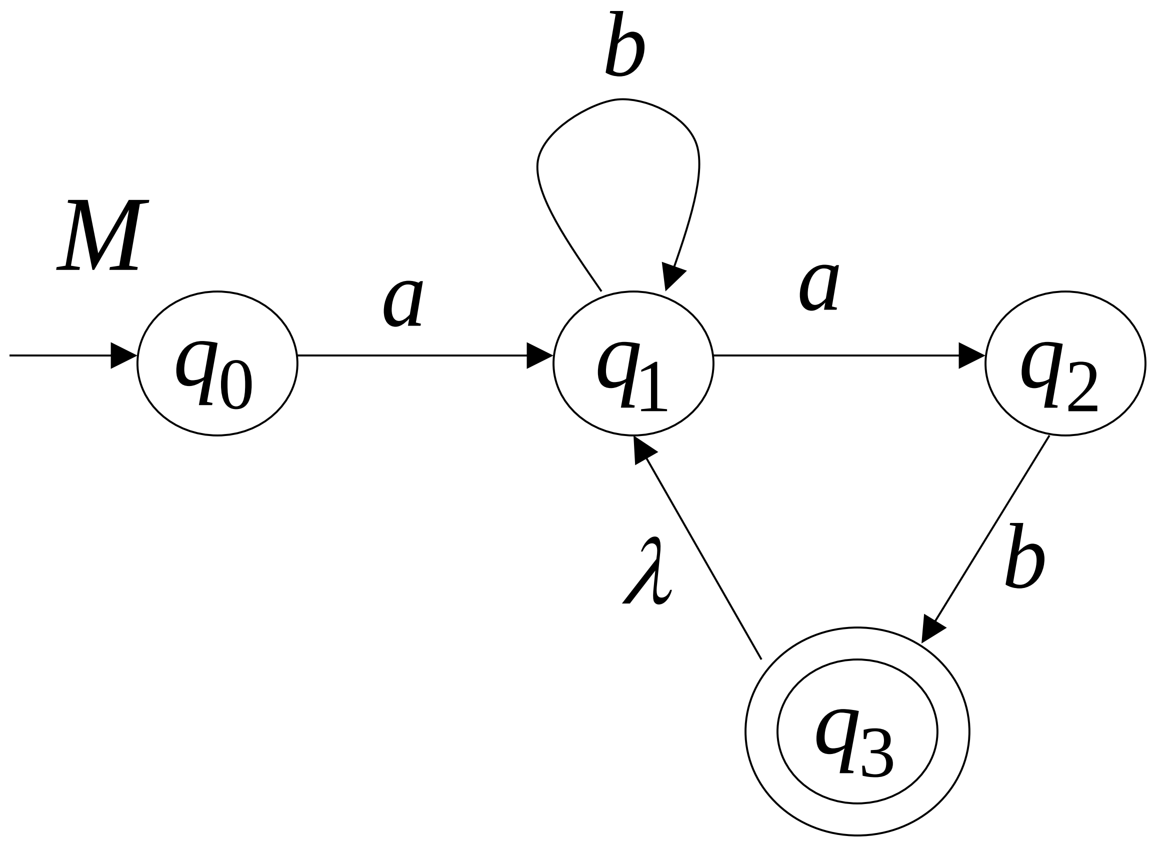
Example:



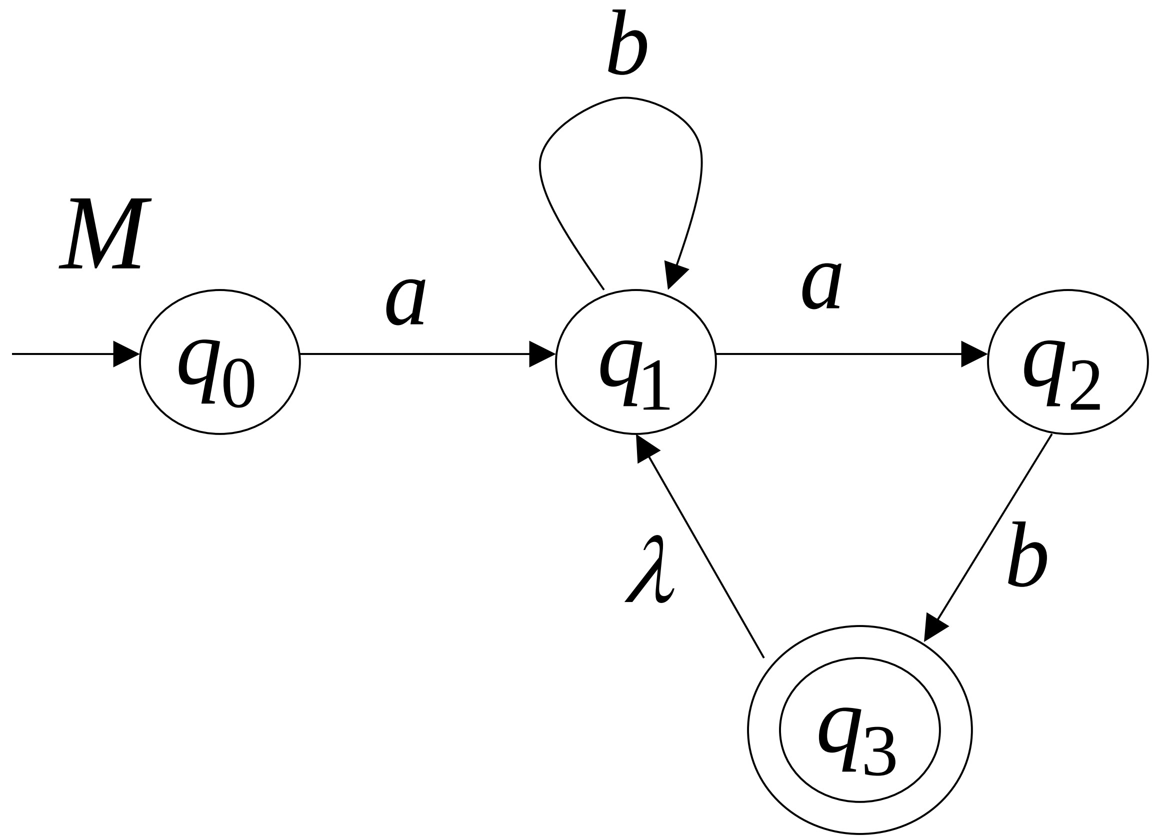

$L = ab^*ab(b^*ab)^*$

$L = L(M)$

Convert $M$ to a right-linear grammar




$$q_0 \to aq_1$$




$$q_0 \to aq_1$$

$$q_1 \to bq_1$$

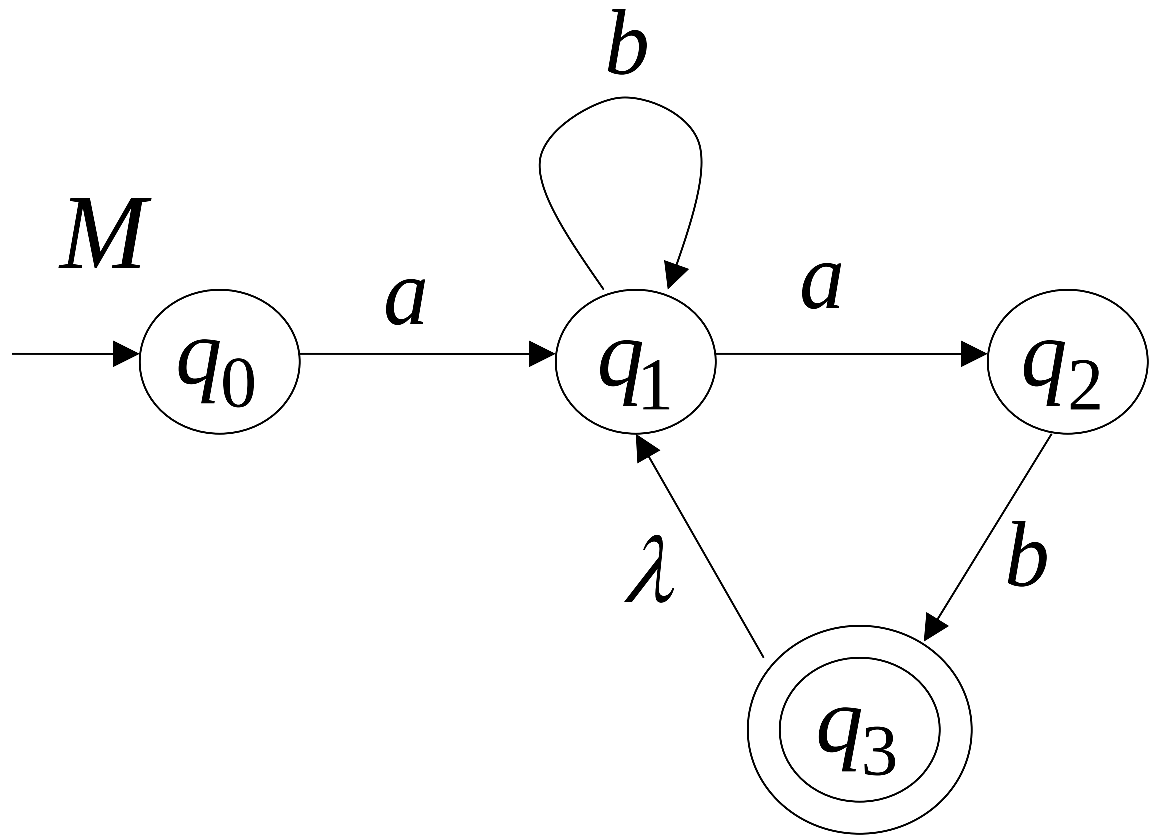
$$q_1 \to aq_2$$

$$q_0 \to aq_1$$

$$q_1 \to bq_1$$

$$q_1 \to aq_2$$

$$q_2 \to bq_3$$



$$L(G) = L(M) = L$$

$G$

$$q_0 \to aq_1$$

$$q_1 \to bq_1$$

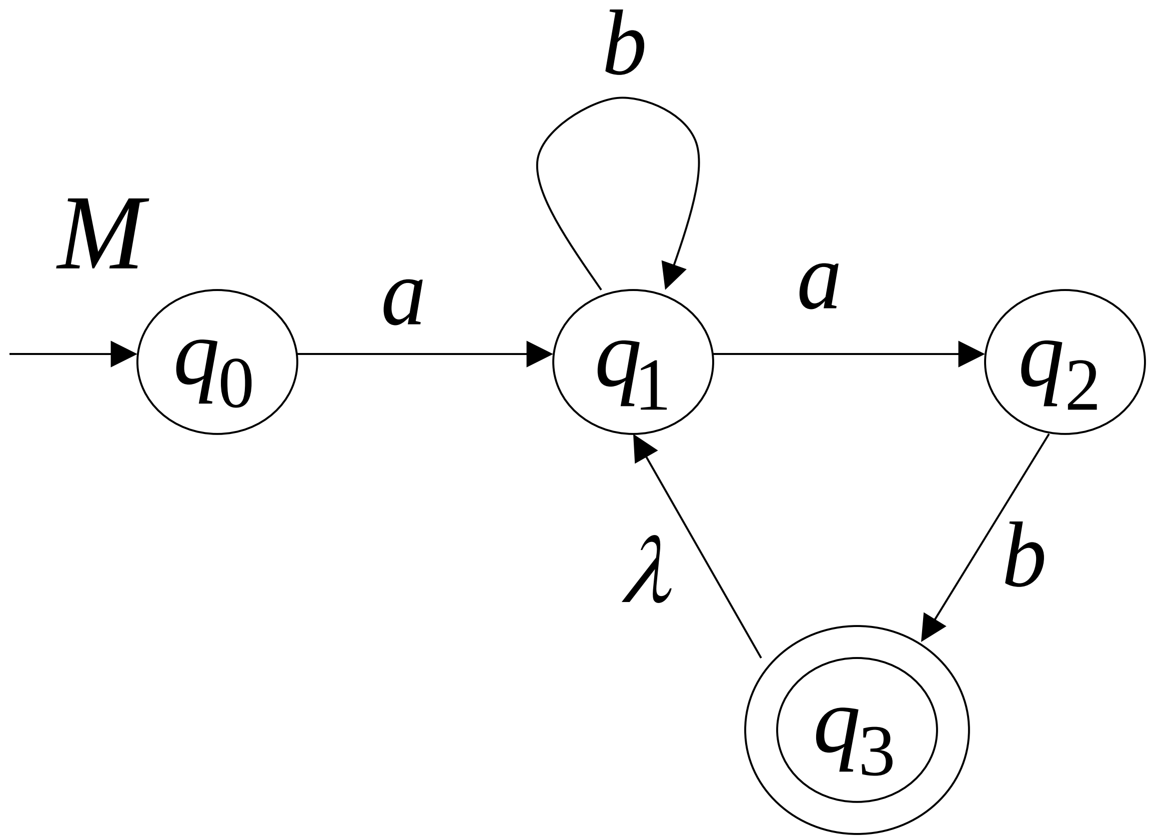
$$q_1 \to aq_2$$

$$q_2 \to bq_3$$

$$q_3 \to q_1$$

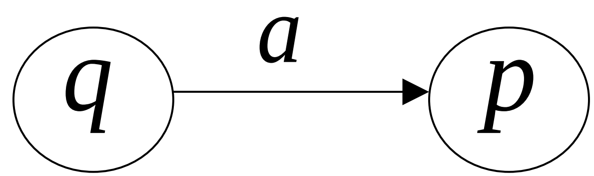
$$q_3 \to \lambda$$

$M$



# In General

For any transition:

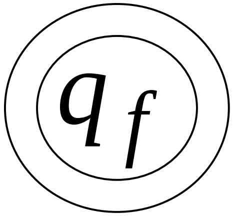



Add production:

$$q \to ap$$

variable terminal variable

For any final state:




Add production:

$$q_f \to \lambda$$

Since $G$ is right-linear grammar

$G$ is also a regular grammar

with $L(G) = L(M) = L$