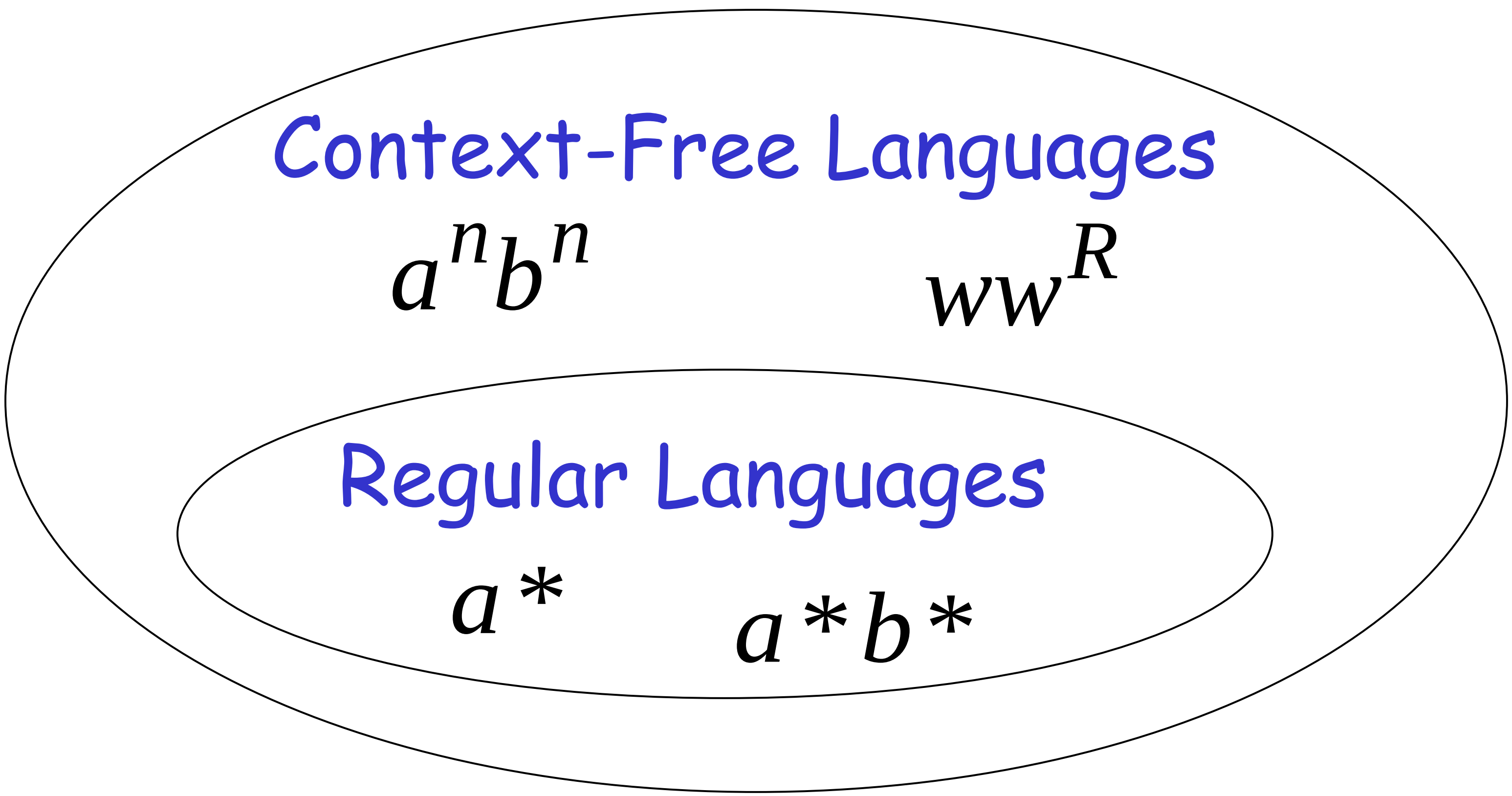
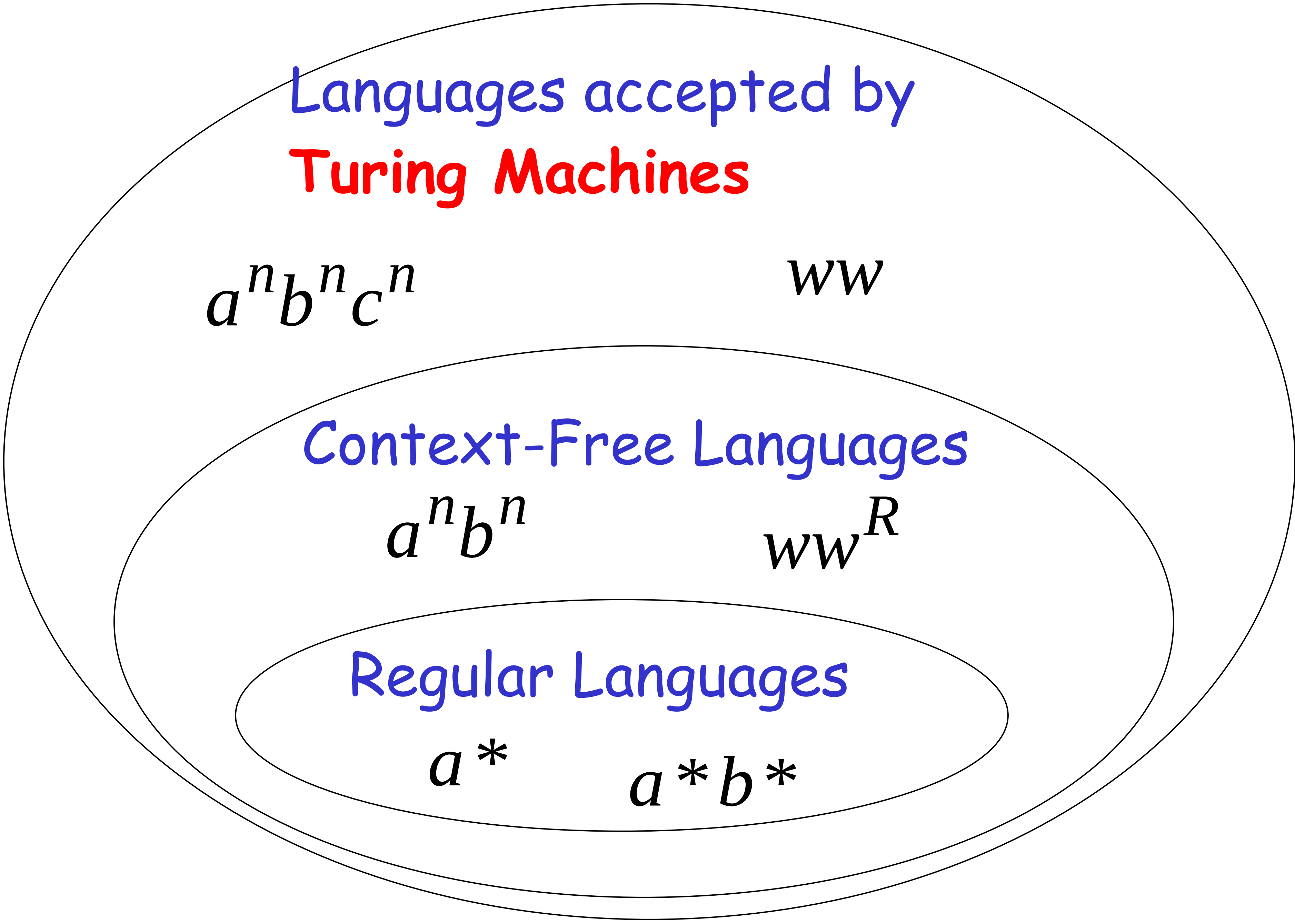
Turing Machines

The Language Hierarchy

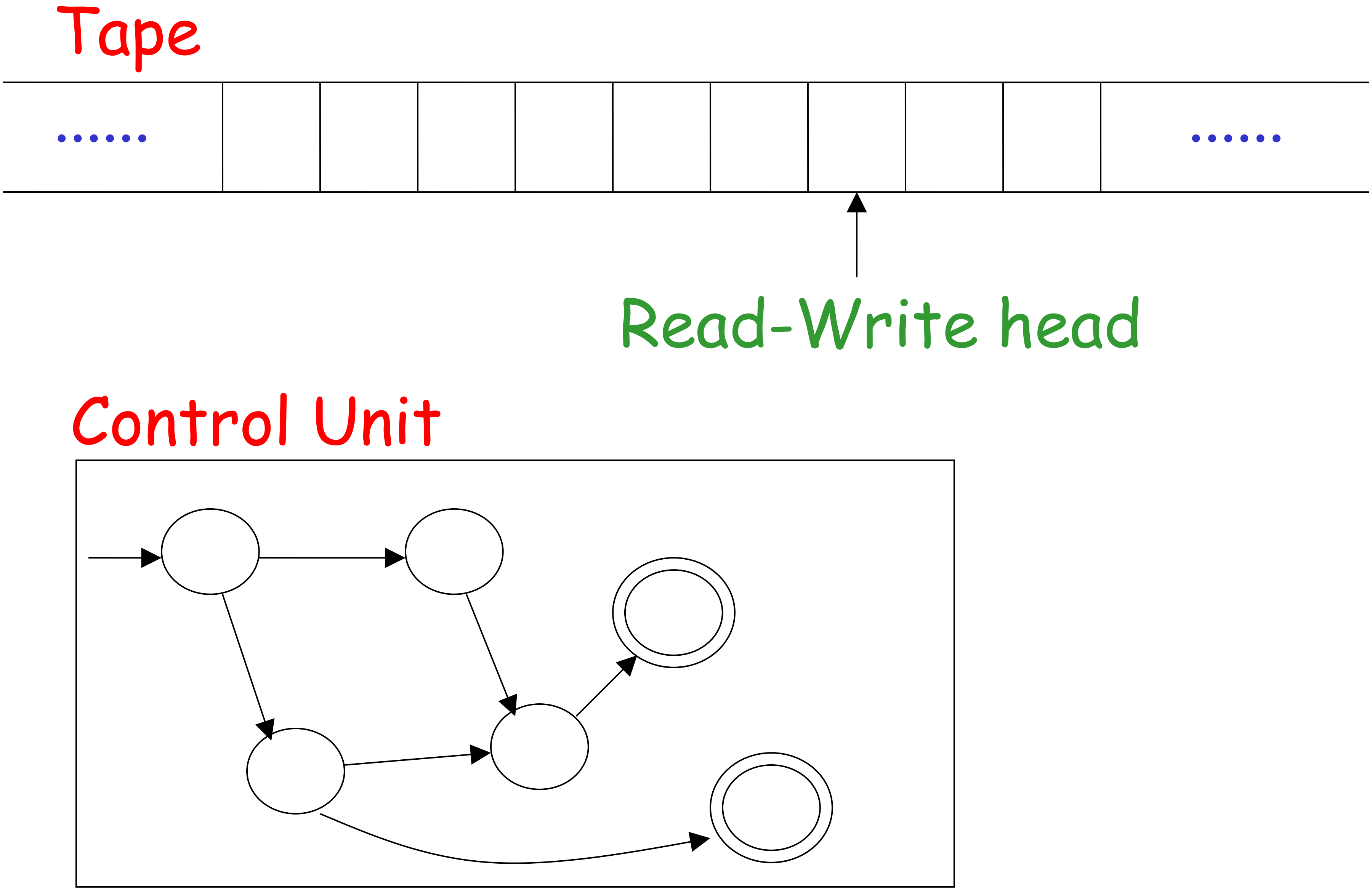
$a^n b^n c^n$?

ww ?



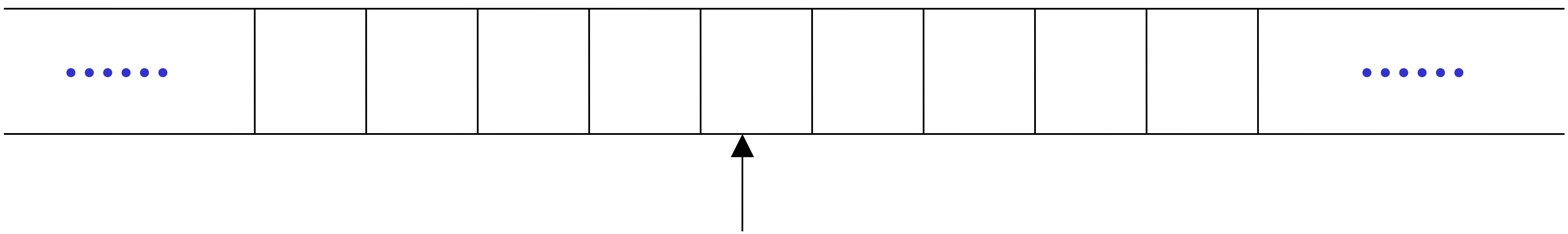

Languages accepted by
Turing Machines
$a^n b^n c^n$
ww
Context-Free Languages
$a^n b^n$
ww^R
Regular Languages
a^*
$a^* b^*$

A Turing Machine



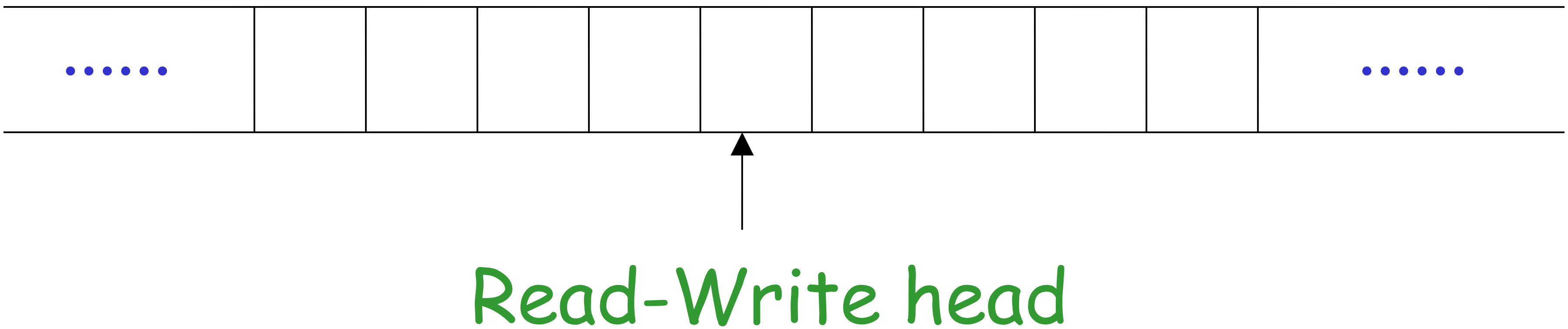
The Tape

No boundaries -- infinite length



Read-Write head

The head moves Left or Right

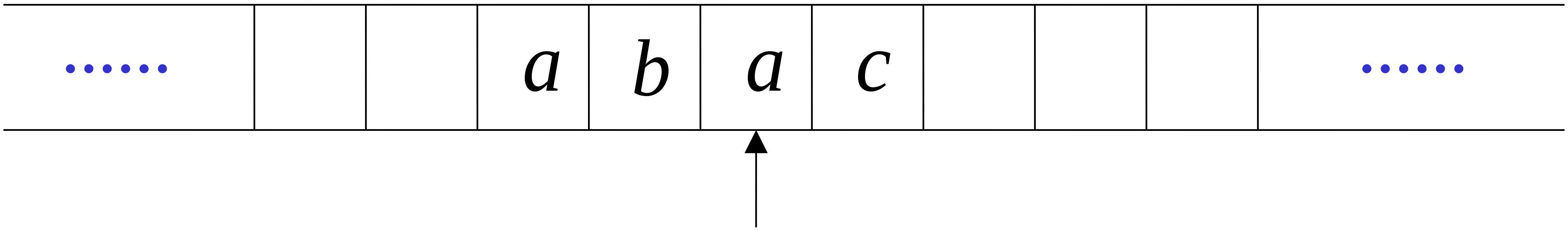


The head at each transition (time step):

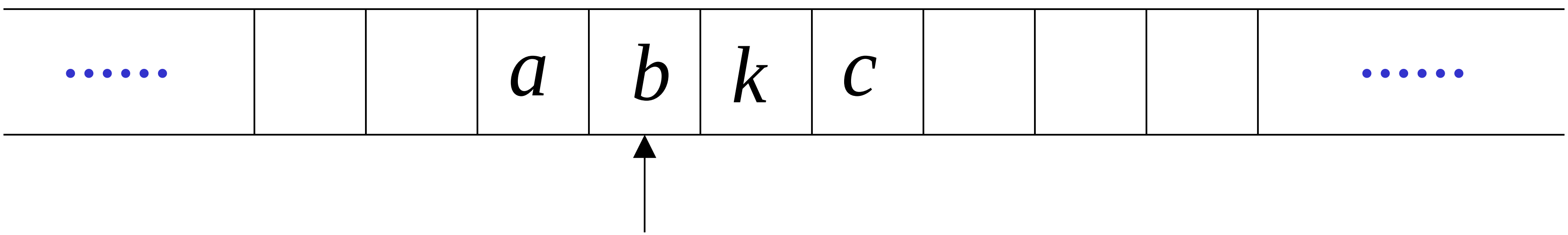
1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

Example:

Time 0

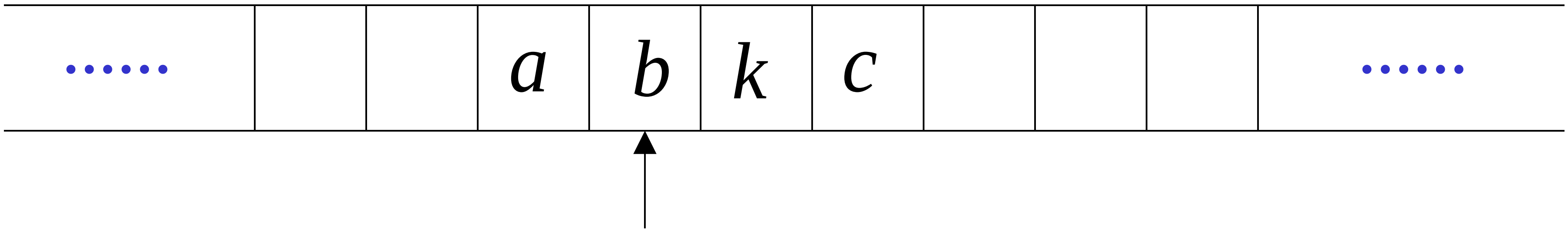


Time 1

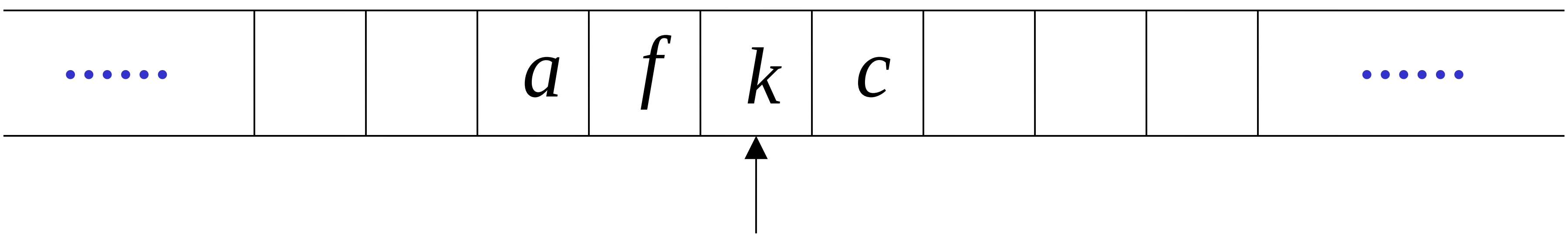


1. Reads a
2. Writes k
3. Moves Left

Time 1

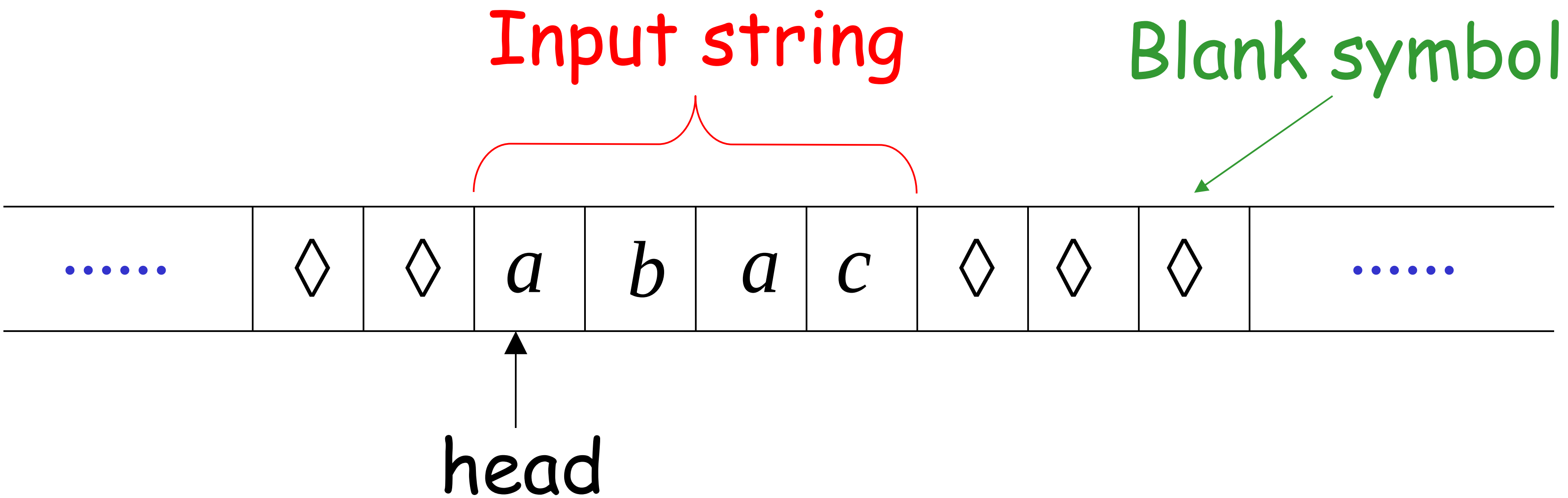


Time 2



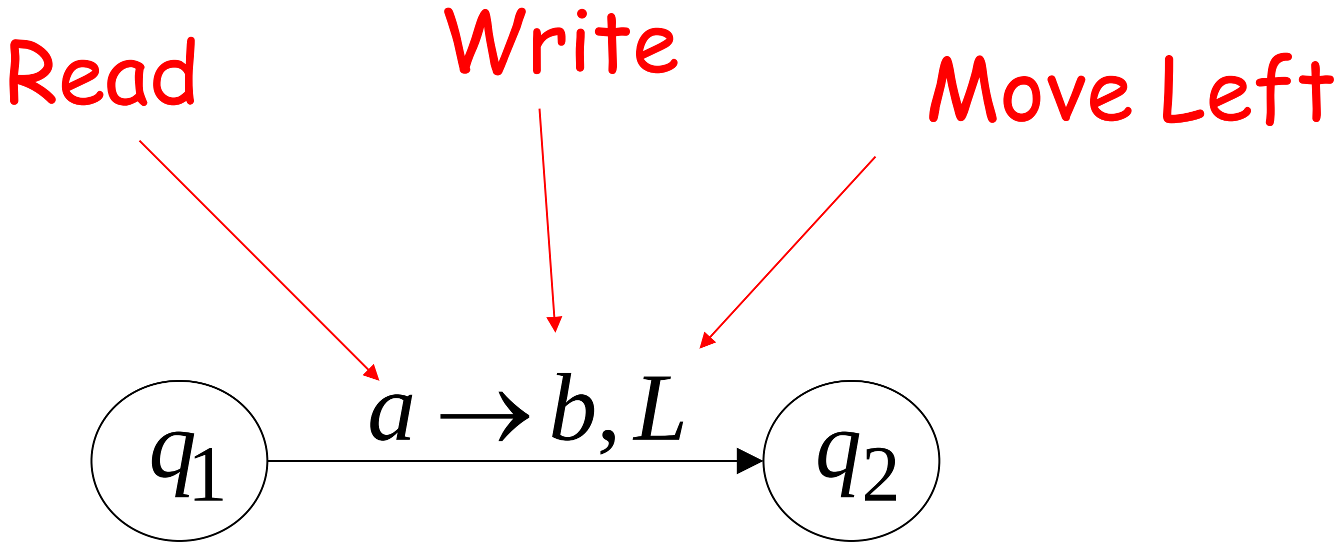
1. Reads b
2. Writes f
3. Moves Right

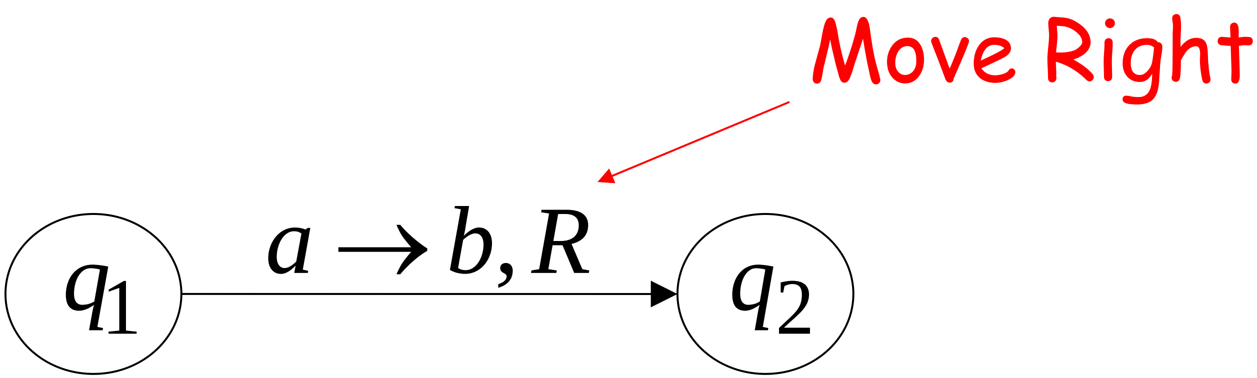
The Input String



Head starts at the leftmost position of the input string

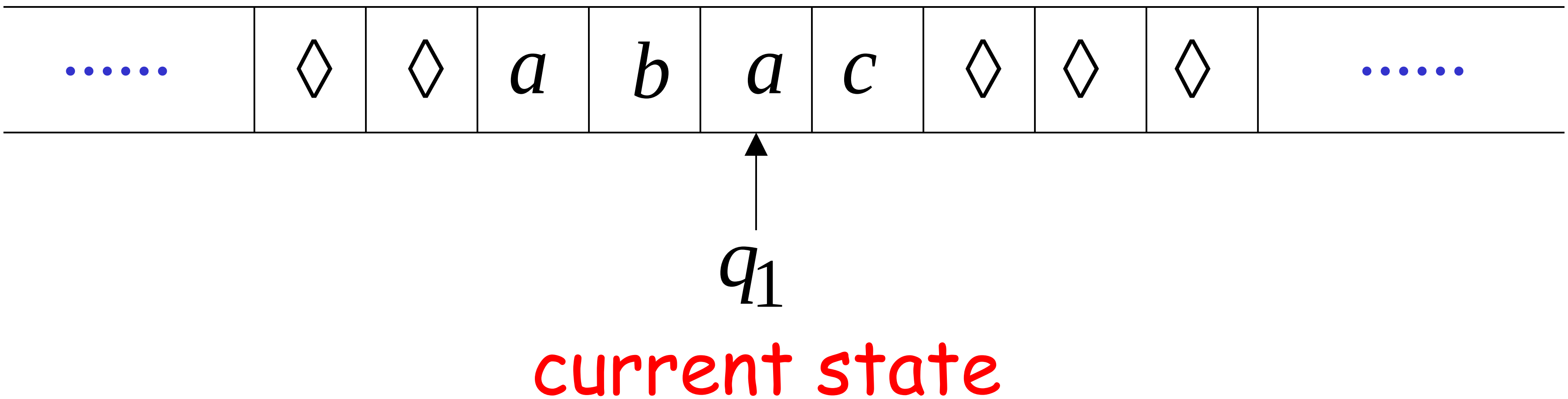
States & Transitions

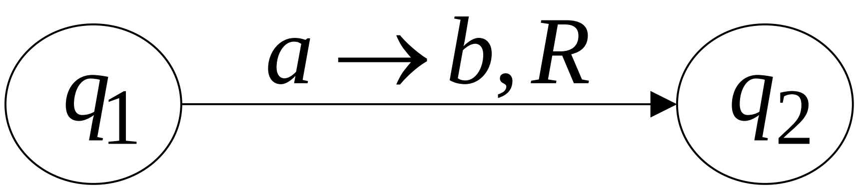


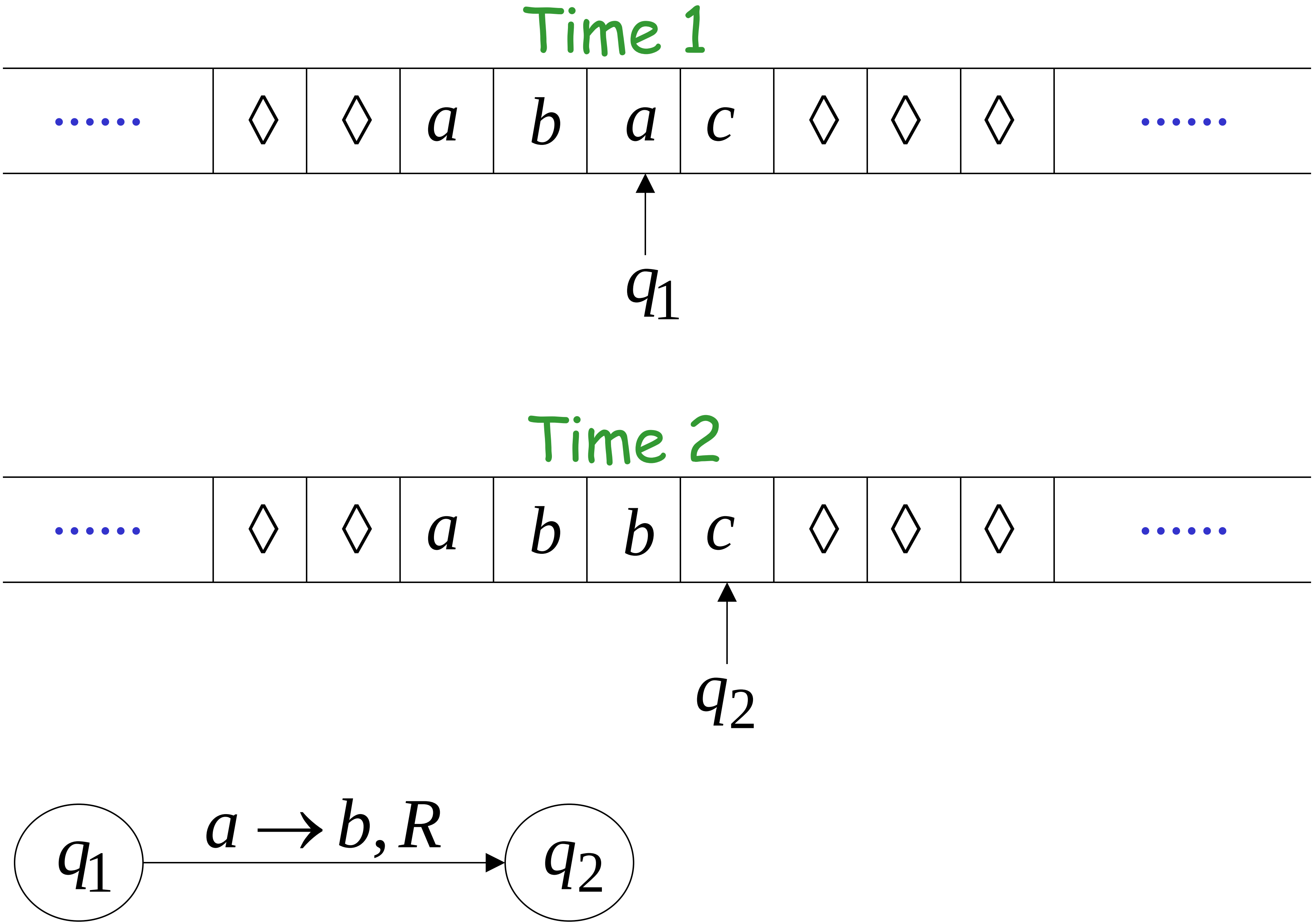


Example:

Time 1



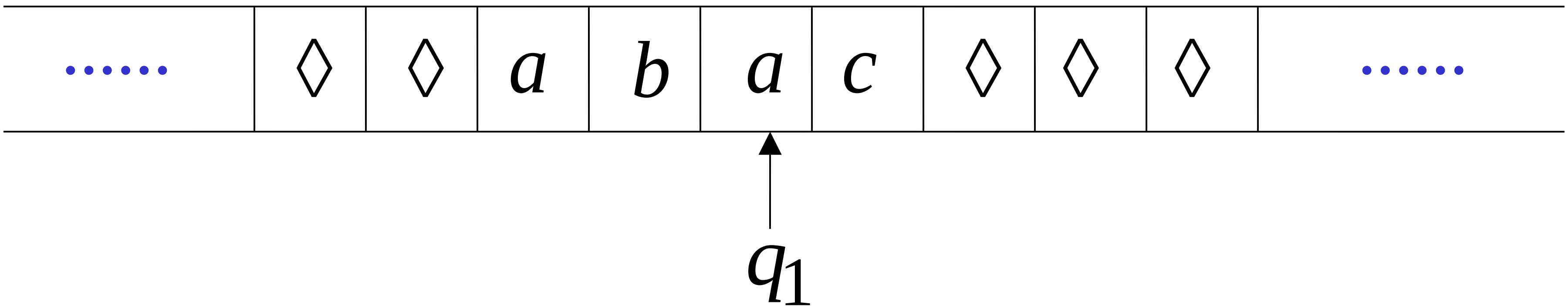




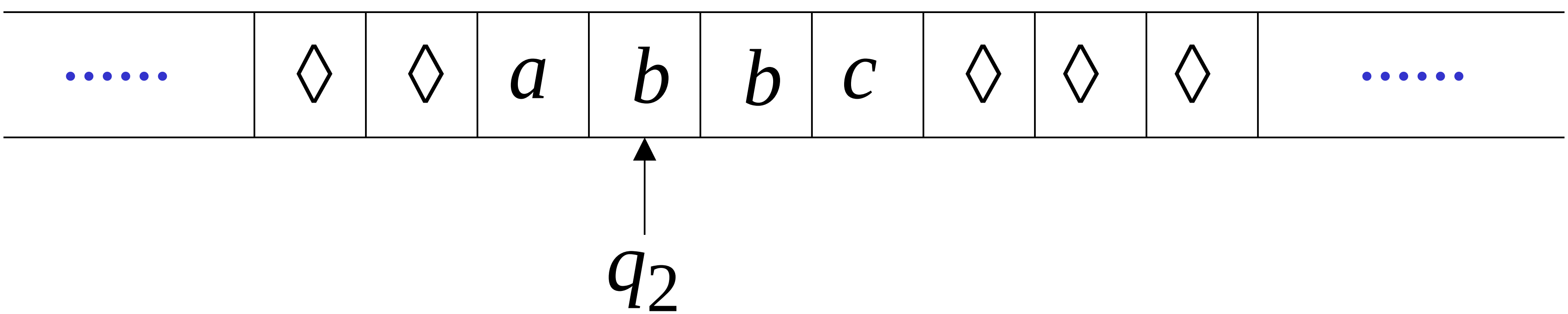
Time 1
◊ ◊ a b a c ◊ ◊ ◊
q_1
Time 2
◊ ◊ a b b c ◊ ◊ ◊
q_2
q_1
$a \rightarrow b, R$
q_2

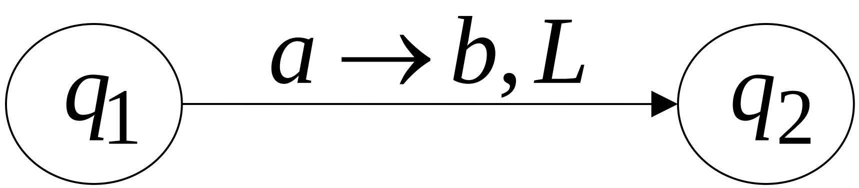
Example:

Time 1



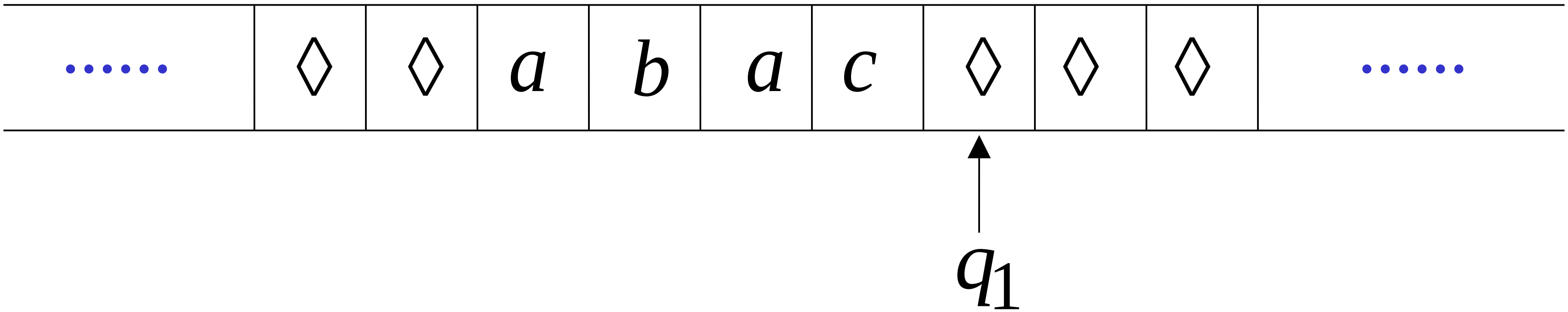
Time 2



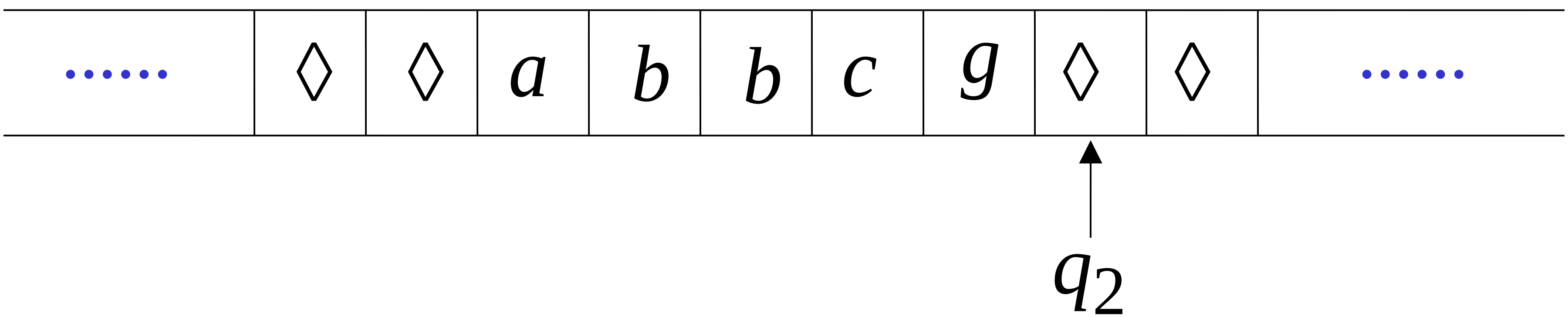


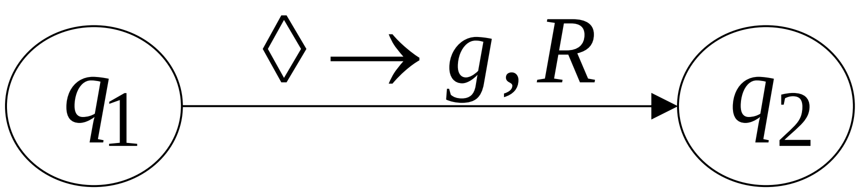
Example:

Time 1



Time 2

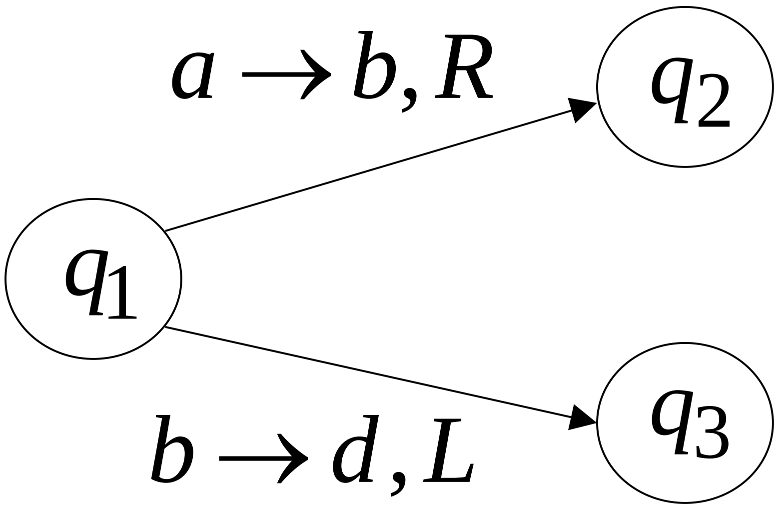




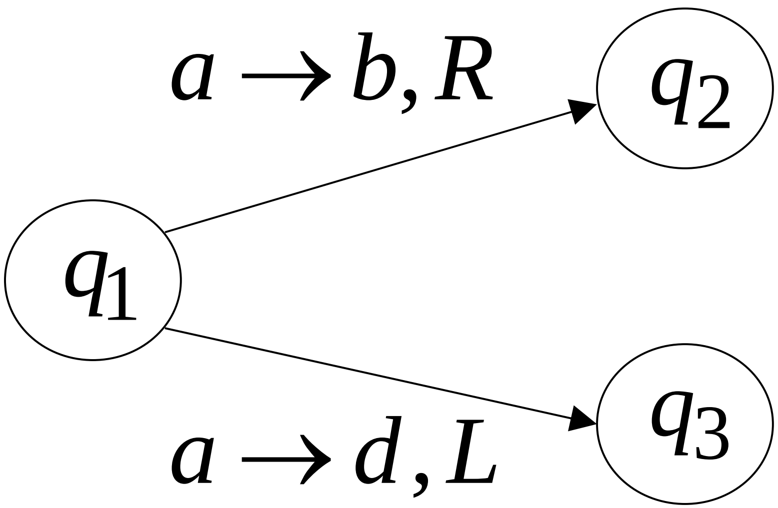
Determinism

Turing Machines are deterministic

<u>Allowed</u>



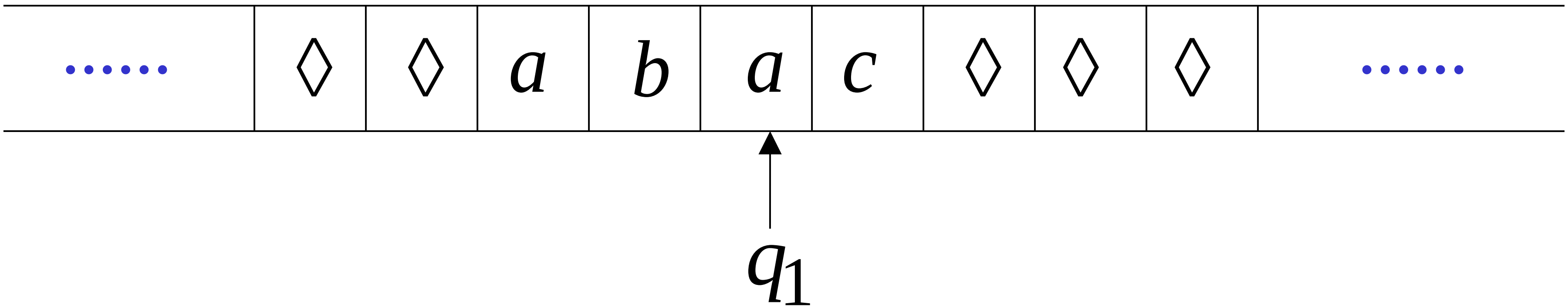
<u>**Not** Allowed</u>

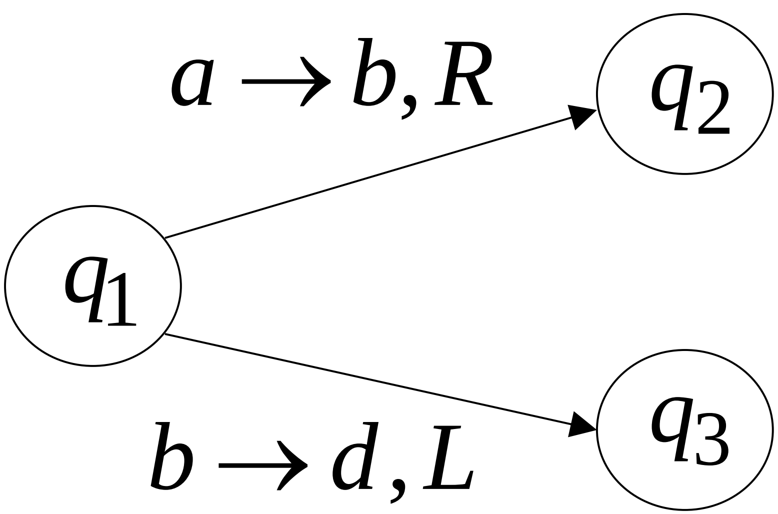


No lambda transitions allowed

Partial Transition Function

Example:





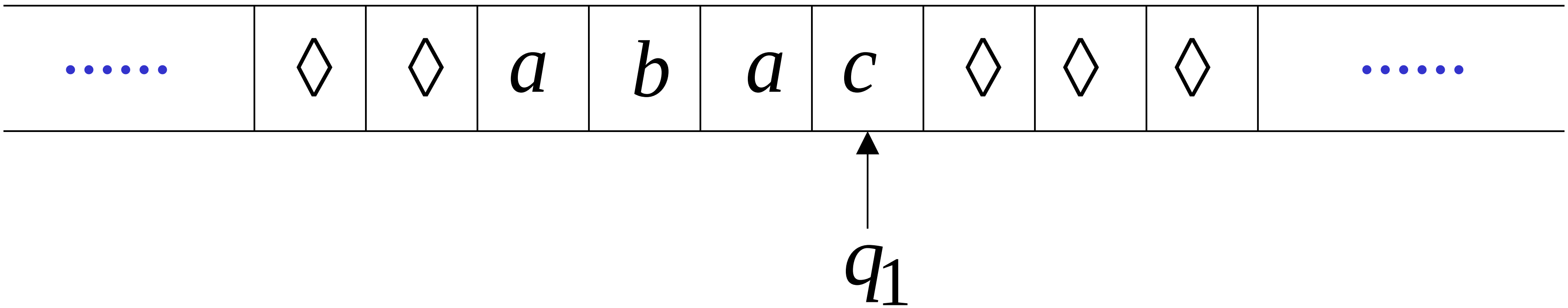
<u>Allowed:</u>

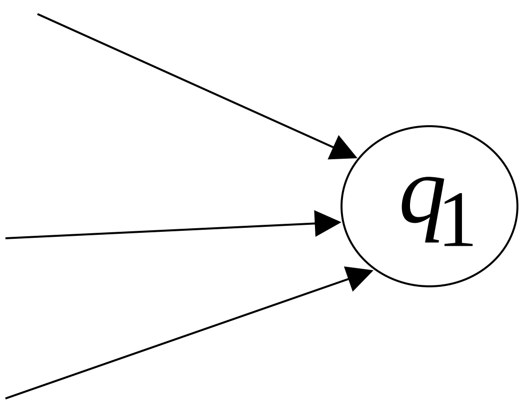
No transition for input symbol c

Halting

The machine ***halts*** in a state if there is no transition to follow

Halting Example 1:

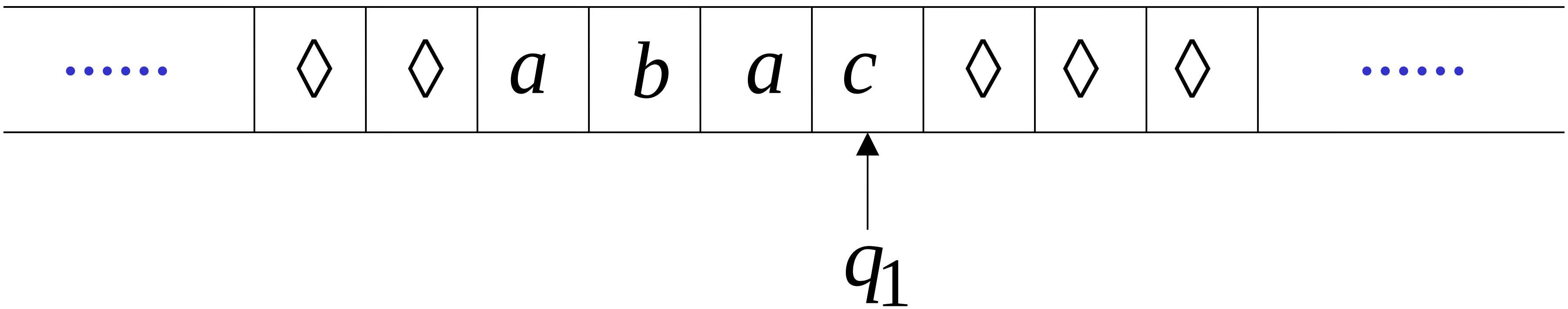


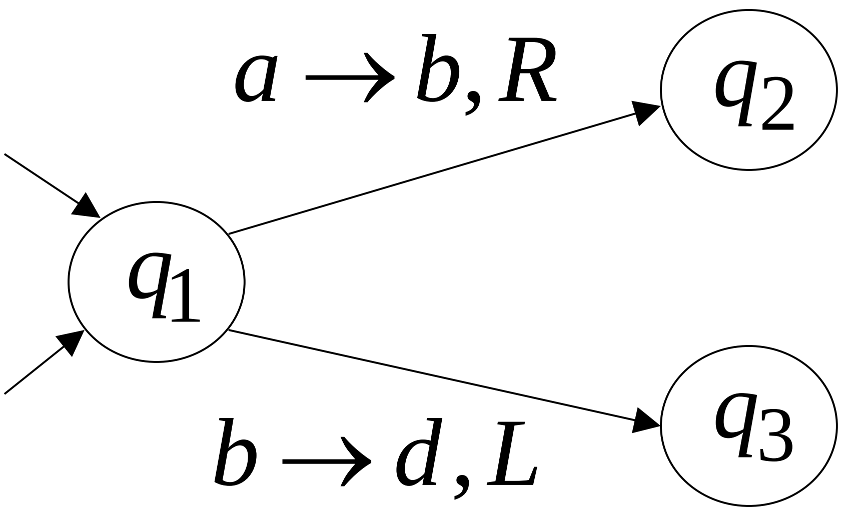


No transition from q_1

HALT!!!

Halting Example 2:

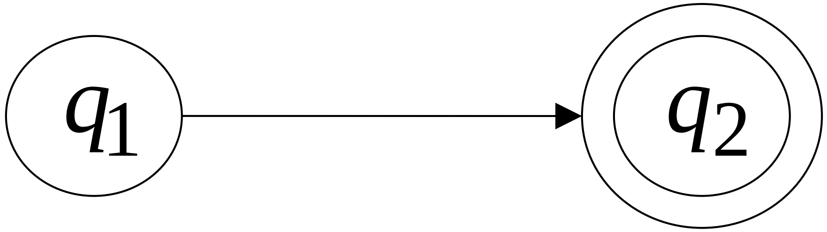




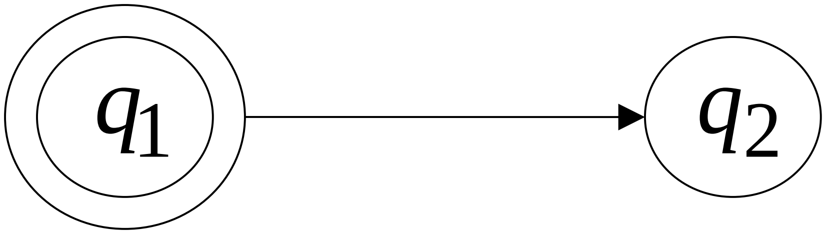
No possible transition from q_1 and symbol c

HALT!!!

Accepting States



Allowed



Not Allowed

- Accepting states have no outgoing transitions
- The machine halts and accepts

Acceptance

Accept Input string → If machine halts in an accept state

Reject Input string → If machine halts in a non-accept state

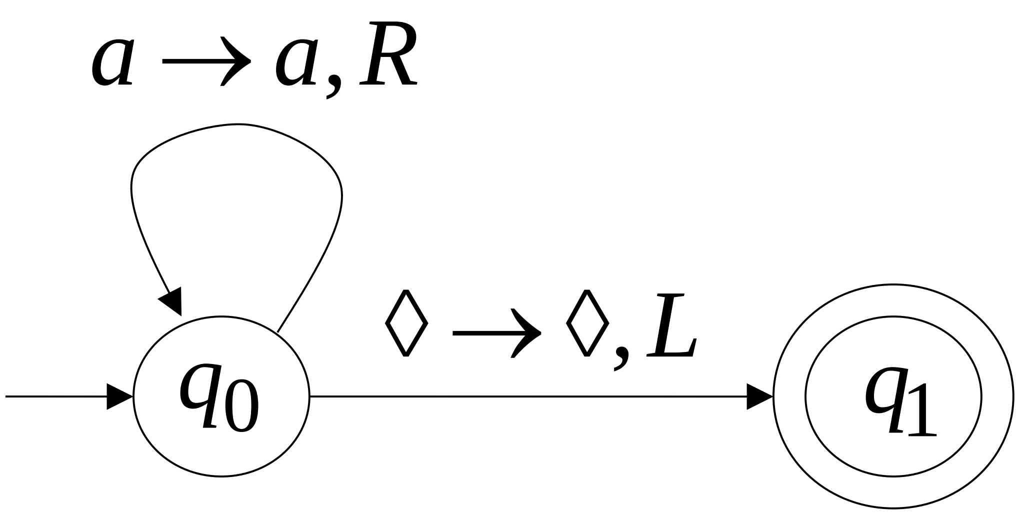
or

If machine enters an *infinite loop*

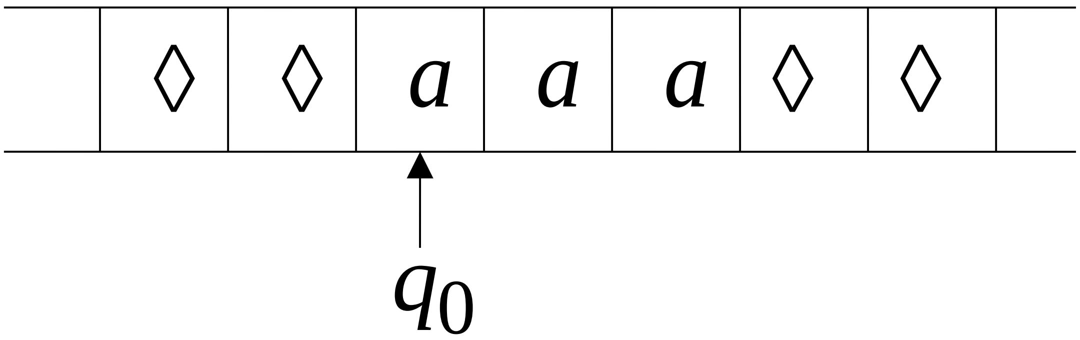
Turing Machine Example

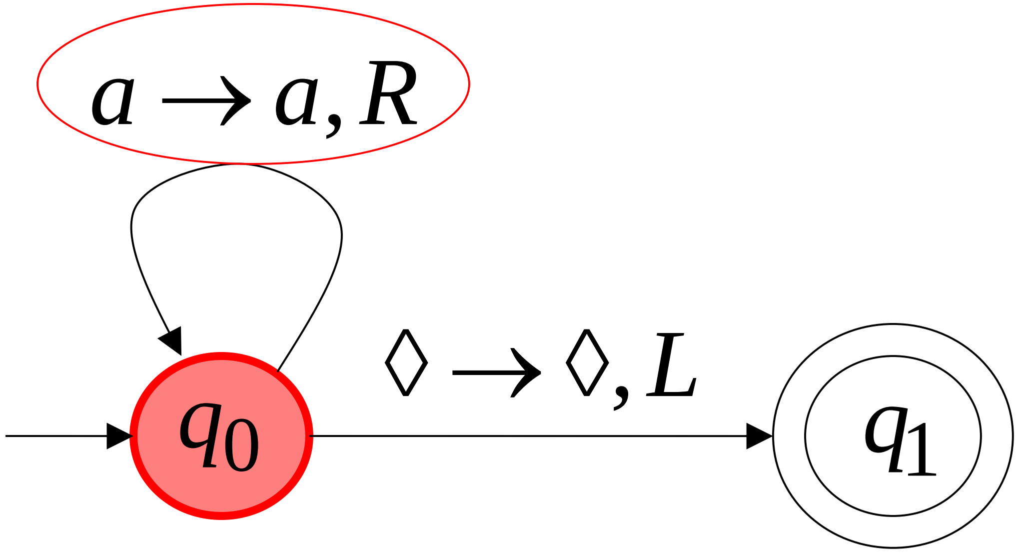
Input alphabet $\Sigma = \{a, b\}$

Accepts the language: a^*

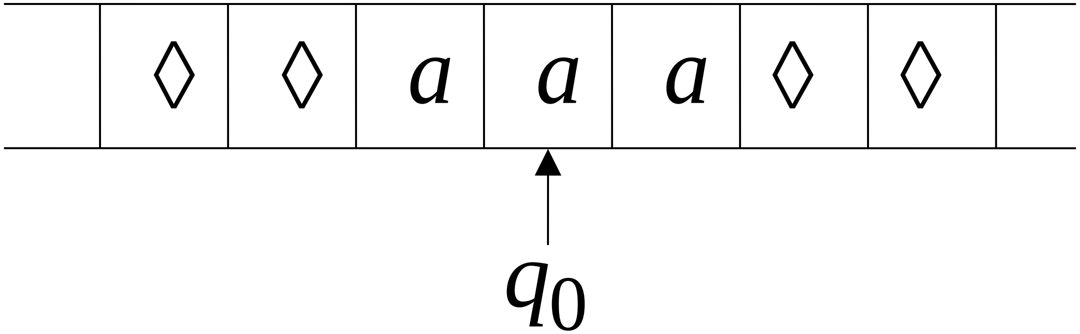


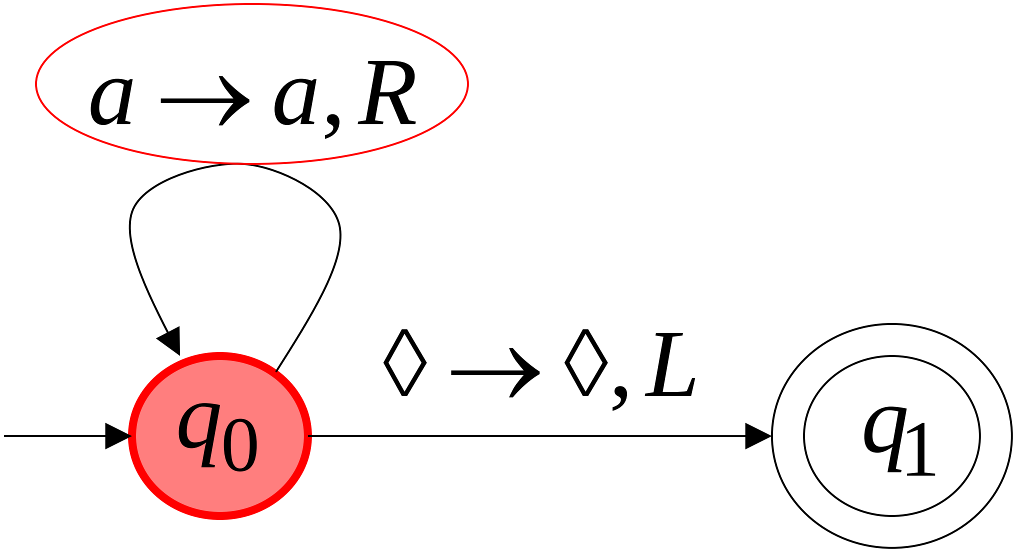
Time 0



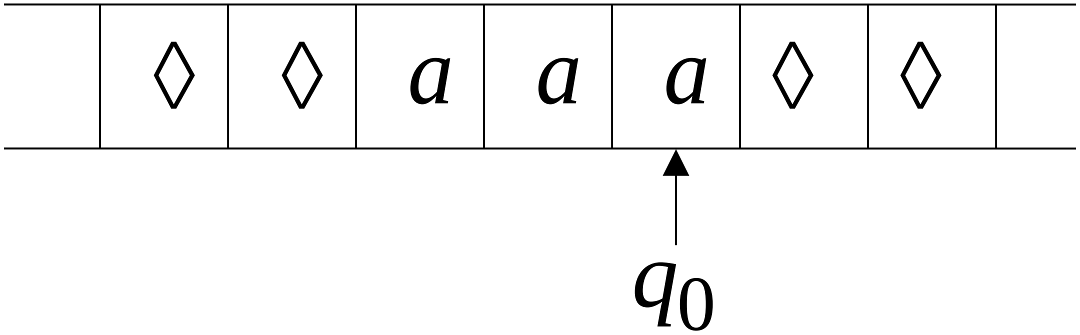


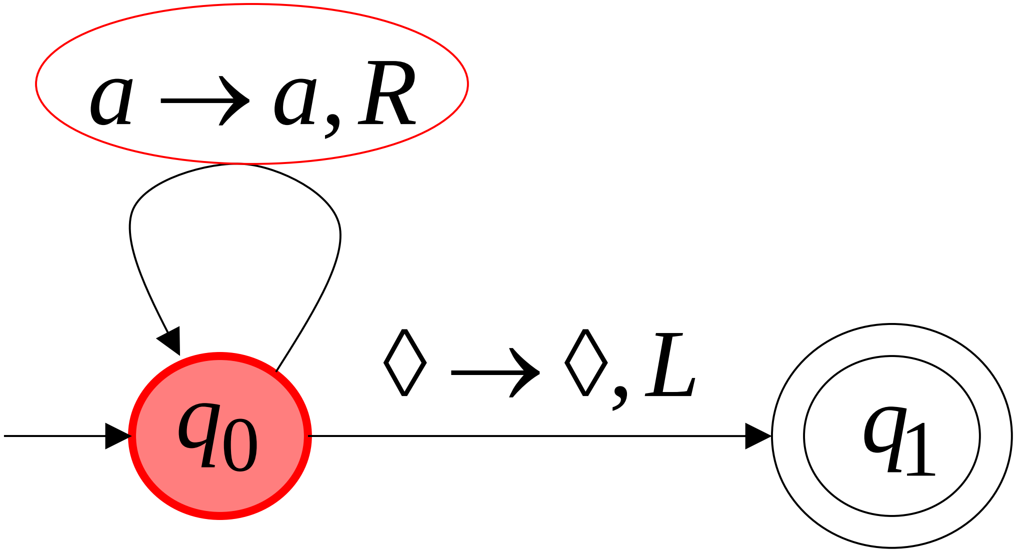
Time 1



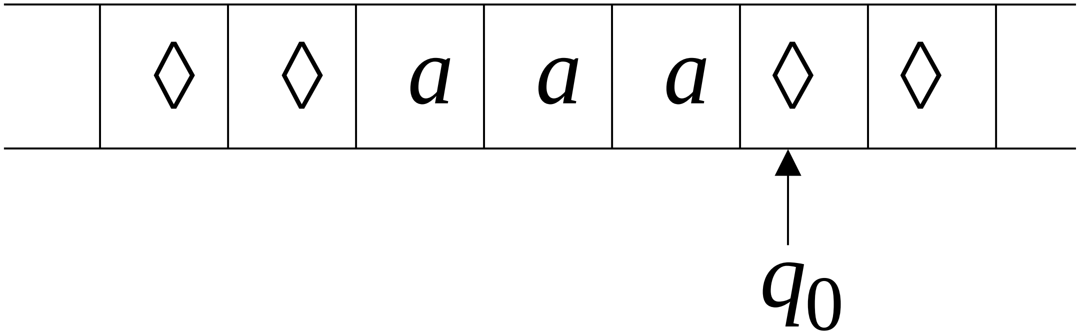


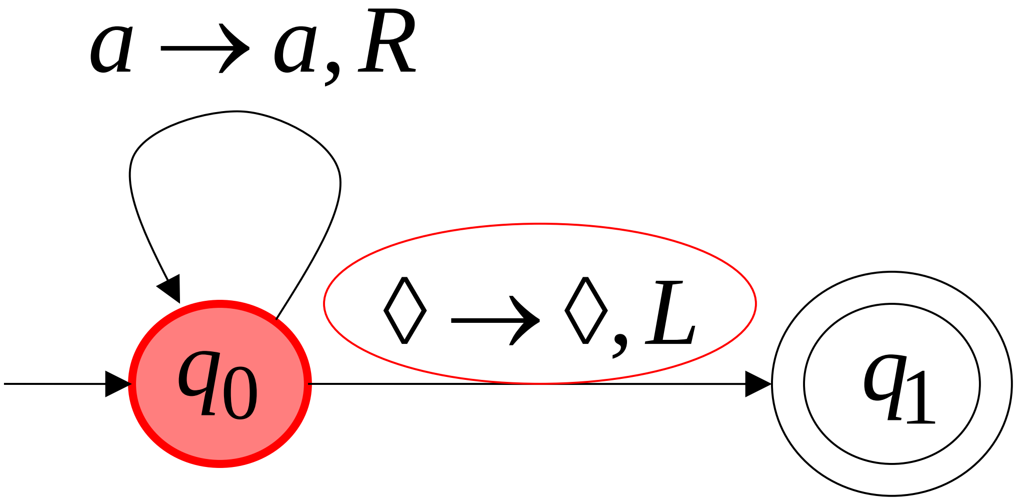
Time 2



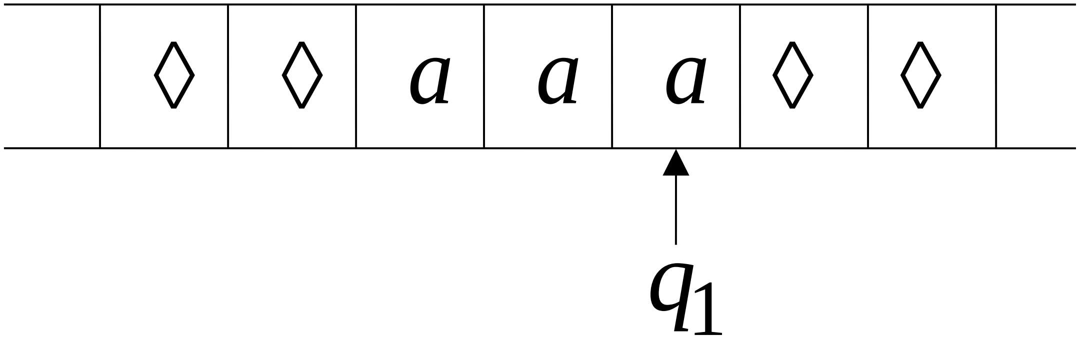


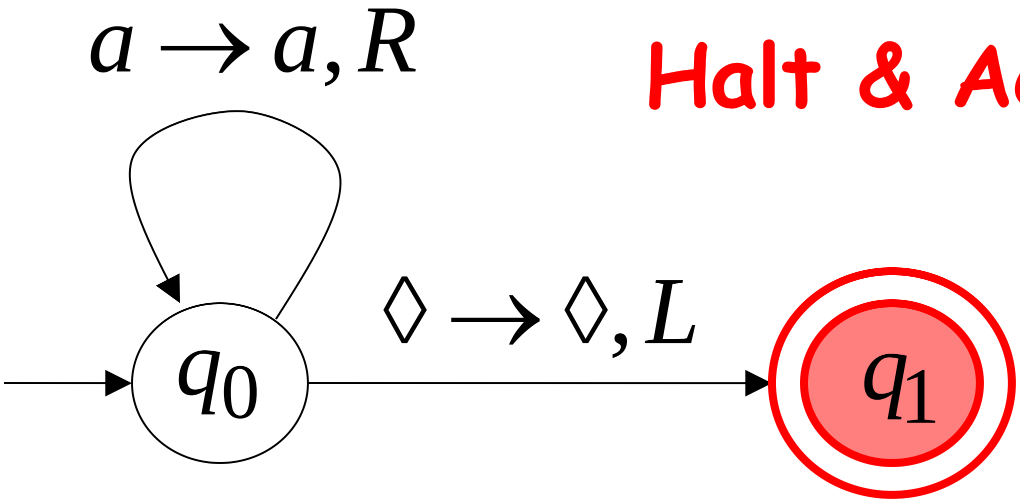
Time 3





Time 4

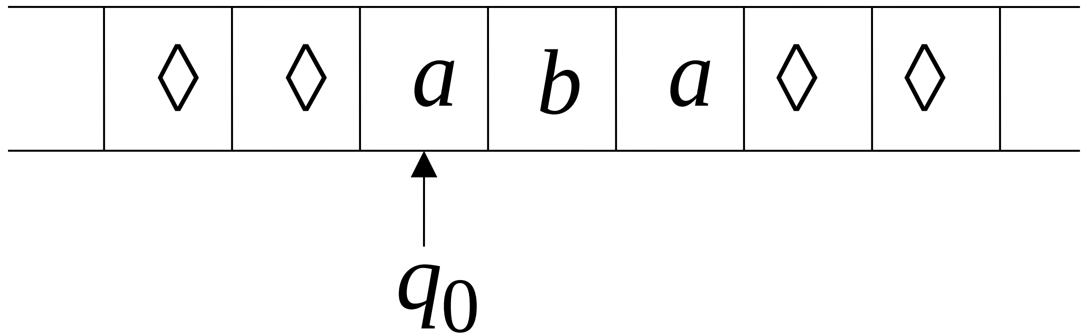


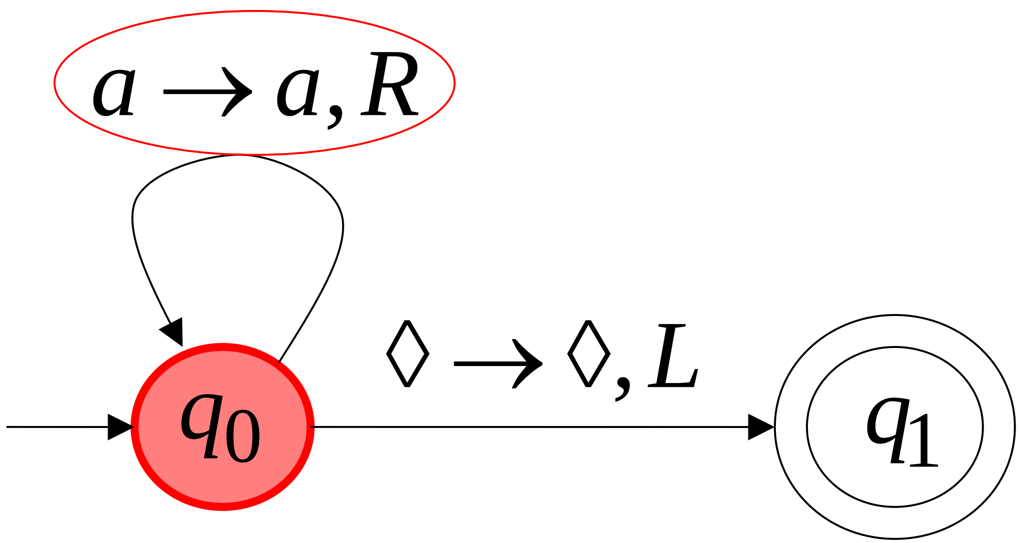


Halt & Accept

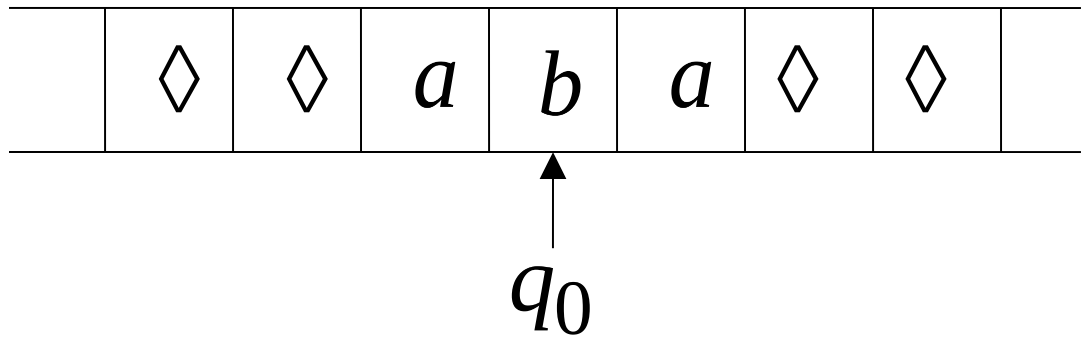
Rejection Example

Time 0



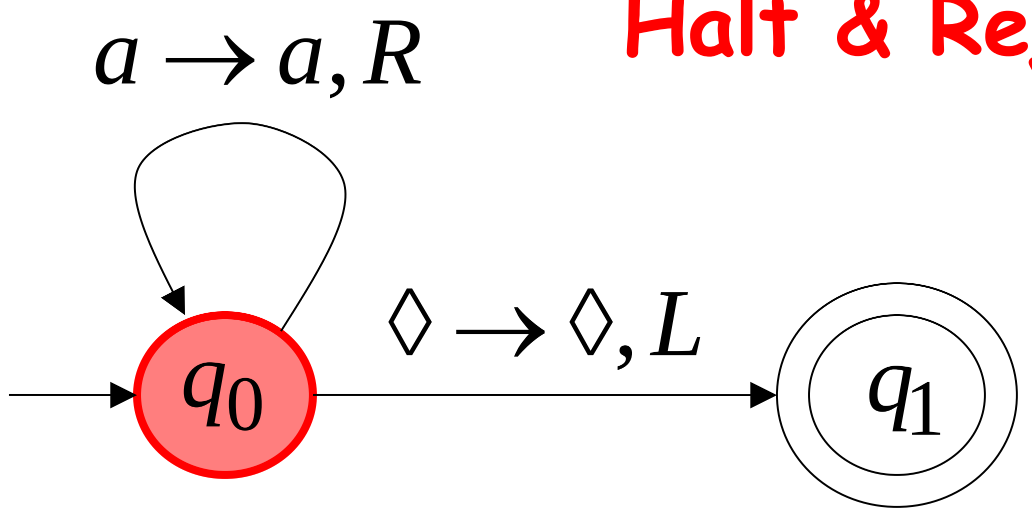


Time 1



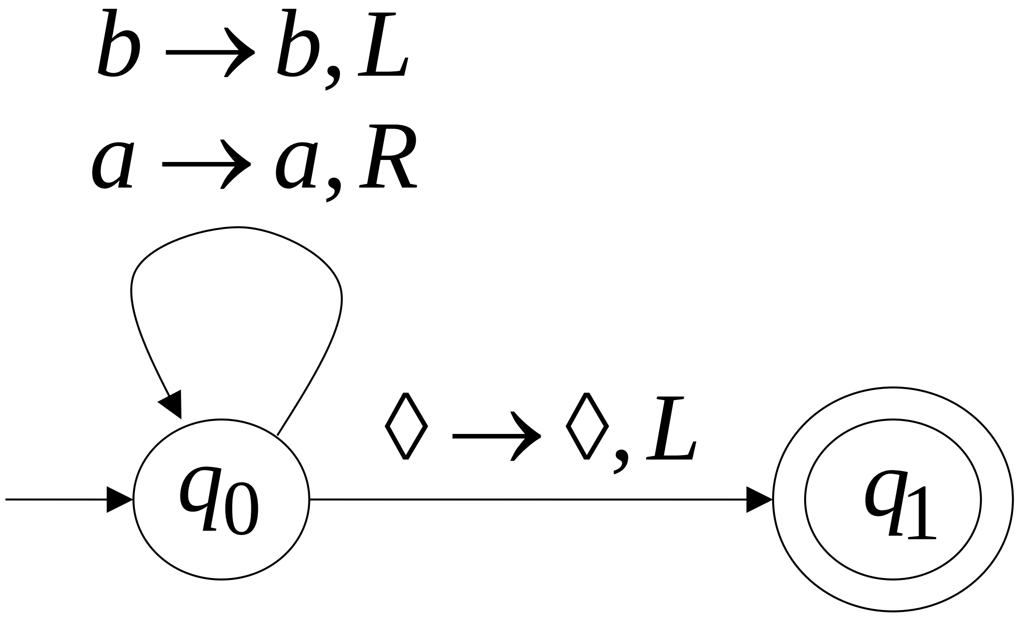
No possible Transition

Halt & Reject

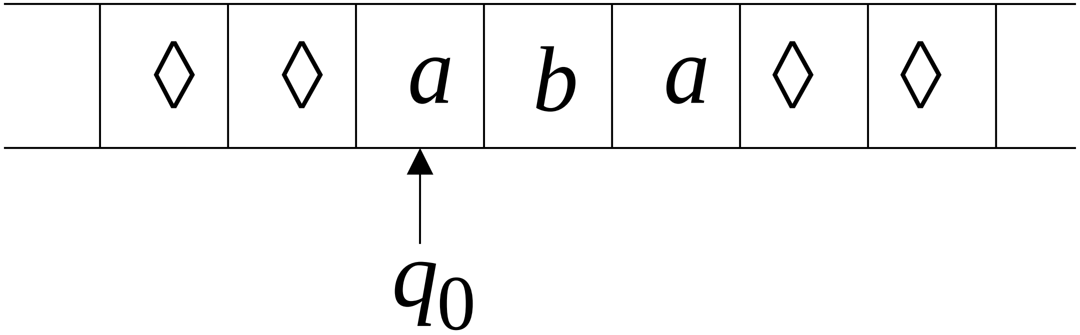


Infinite Loop Example

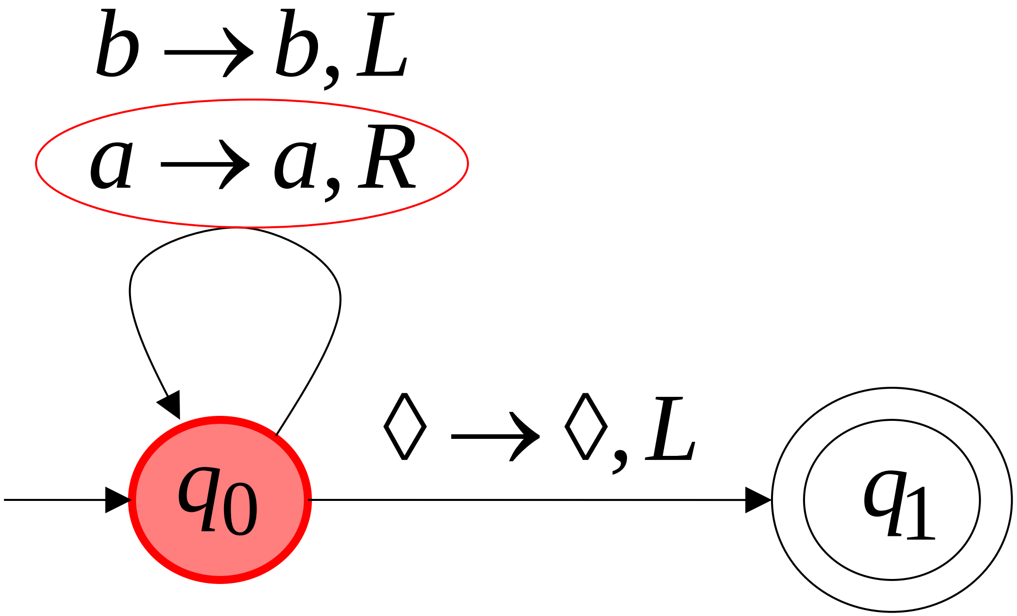
A Turing machine for language $a^* + b(a+b)^*$



Time 0

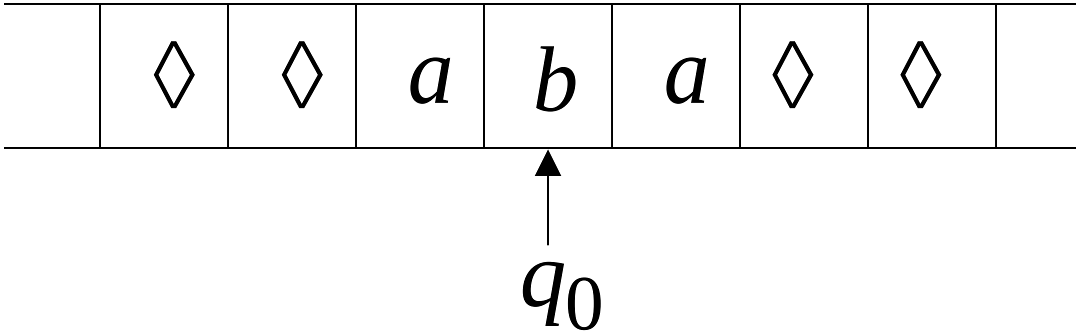


◊
◊
a
b
a
◊
◊
q_0

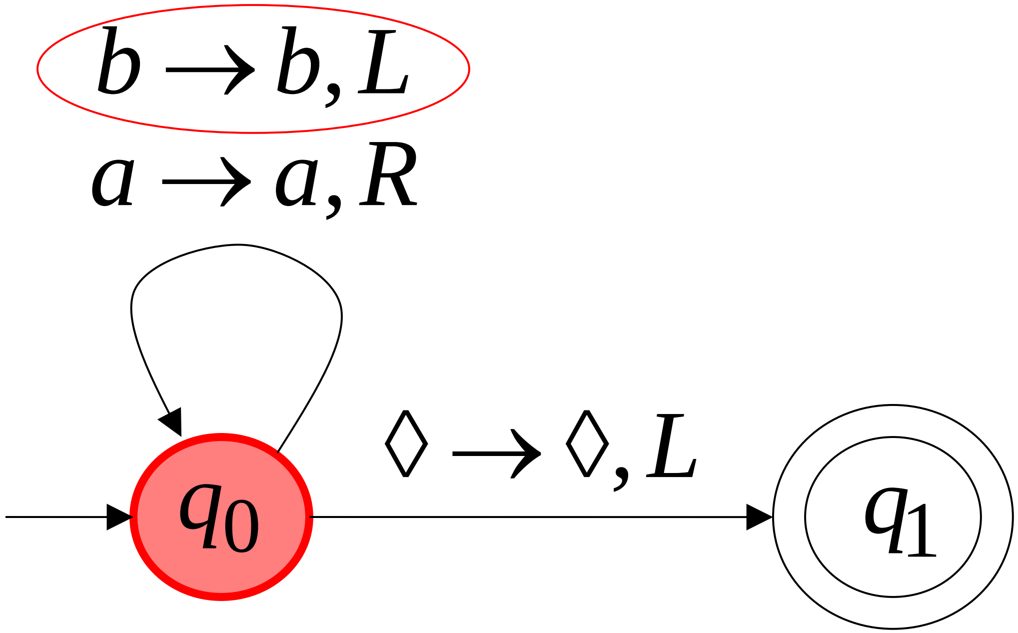


$b \rightarrow b, L$
$a \rightarrow a, R$
$\Diamond \rightarrow \Diamond, L$
q_0
q_1

Time 1

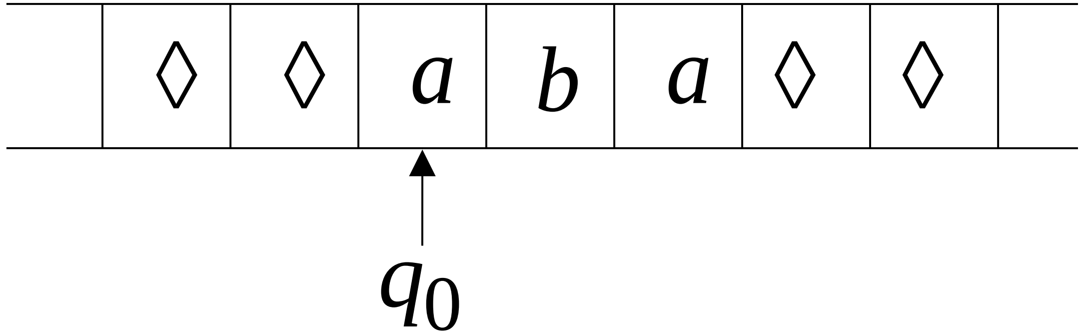


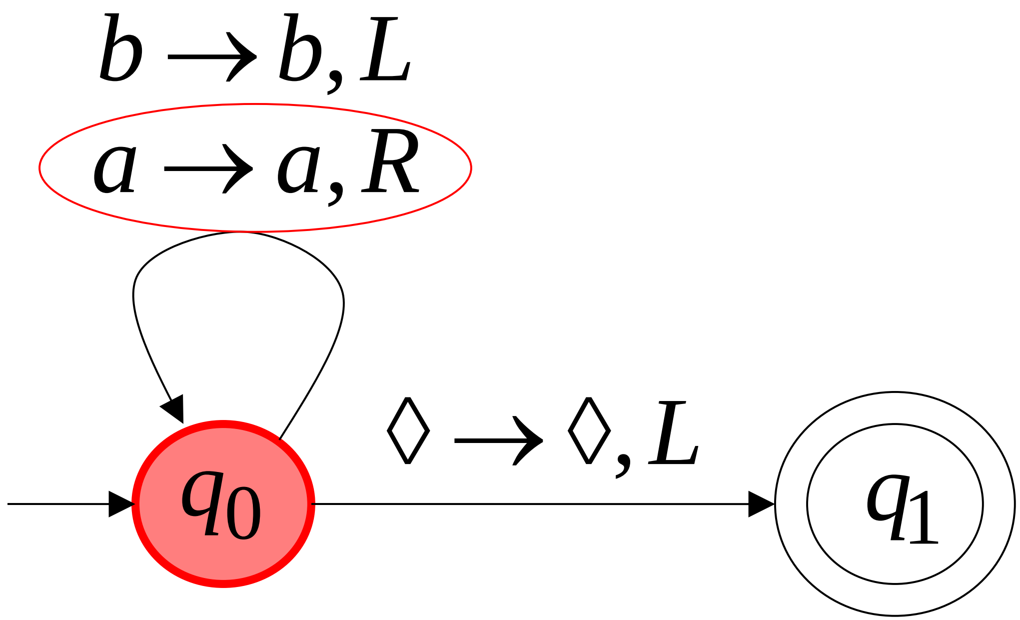
◊ ◊ a b a ◊ ◊
q_0

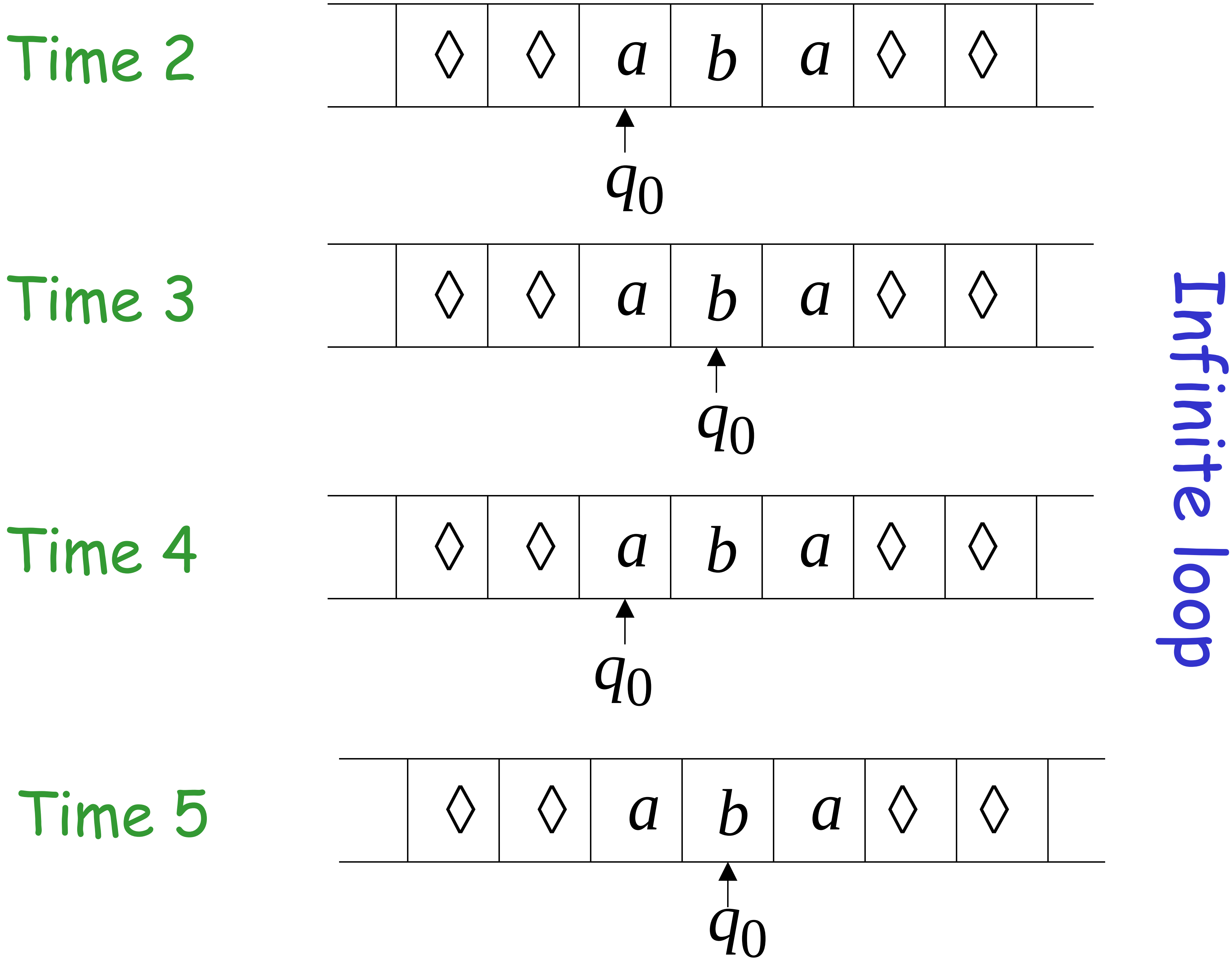


$b \rightarrow b, L$
$a \rightarrow a, R$
$\lozenge \rightarrow \lozenge, L$
q_0
q_1

Time 2







Time 2
◊ ◊ a b a ◊ ◊
q0
Time 3
◊ ◊ a b a ◊ ◊
q0
Time 4
◊ ◊ a b a ◊ ◊
q0
Time 5
◊ ◊ a b a ◊ ◊
q0
Infinite loop

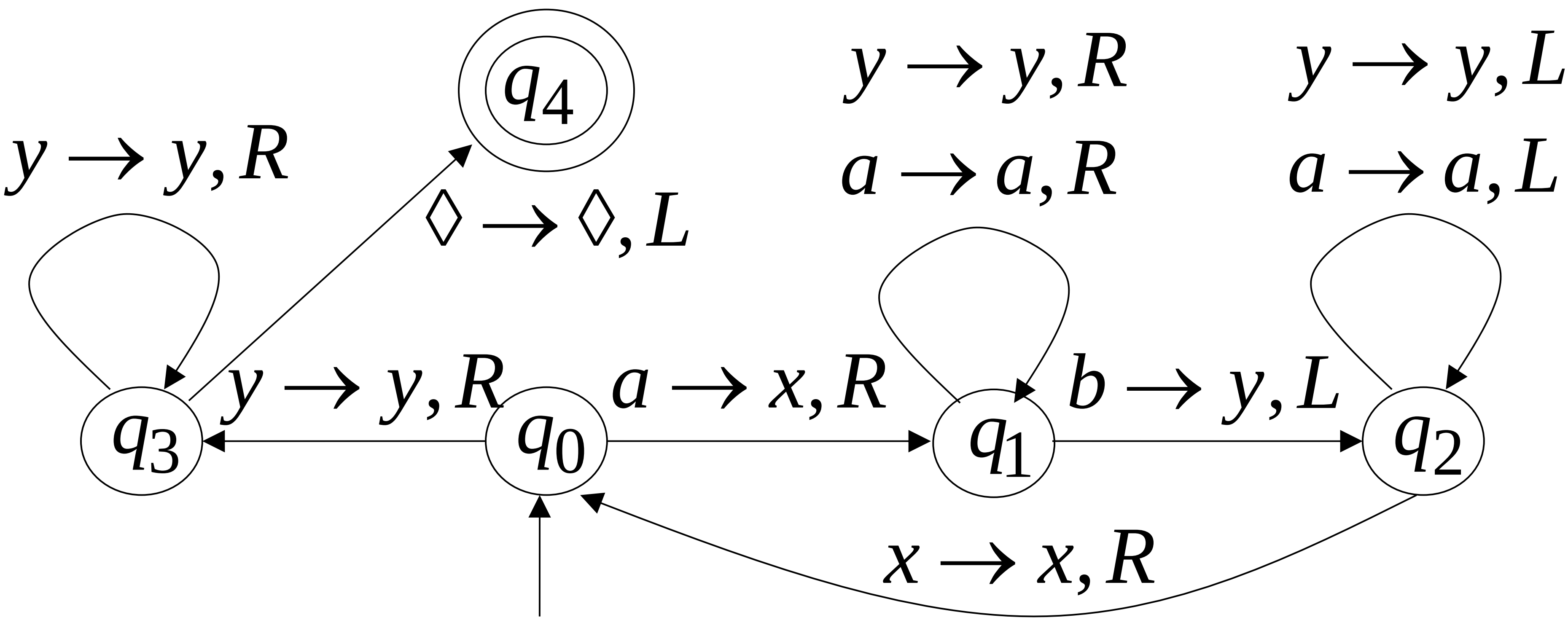
Because of the infinite loop:

- The accepting state cannot be reached
- The machine never halts
- The input string is rejected

Another Turing Machine Example

Turing machine for the language $\{a^n b^n\}$

$n \geq 1$



Basic Idea:

Match a's with b's:

Repeat:

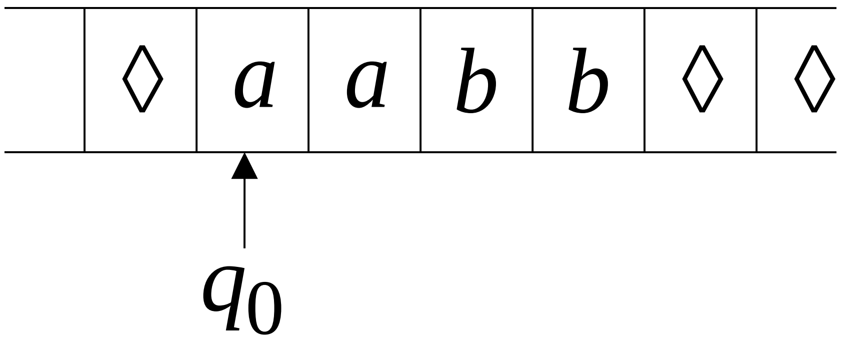
replace leftmost a with x

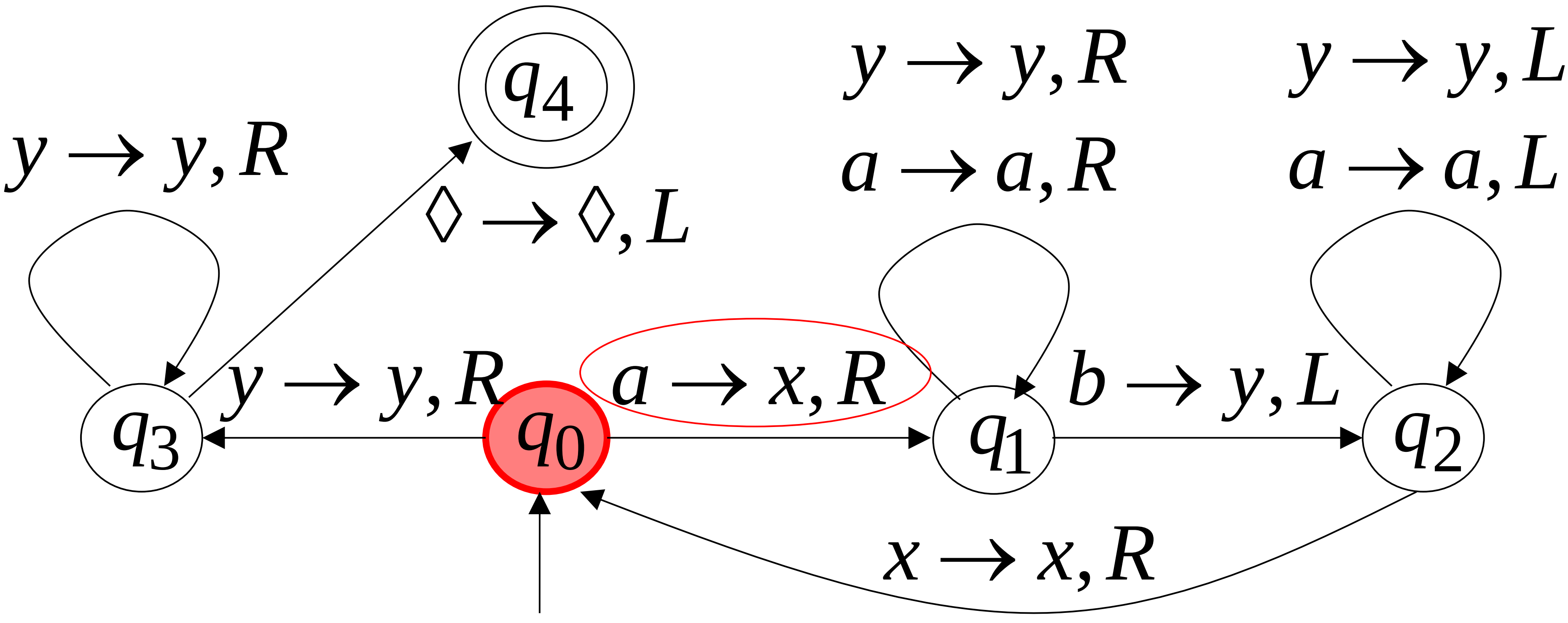
find leftmost b and replace it with y

Until there are no more a's or b's

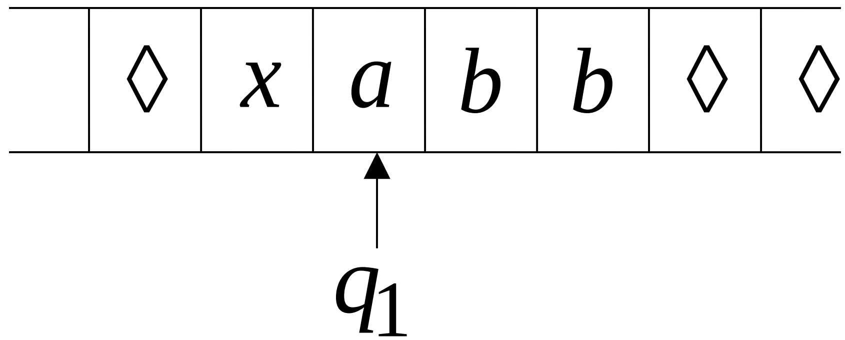
If there is a remaining a or b reject

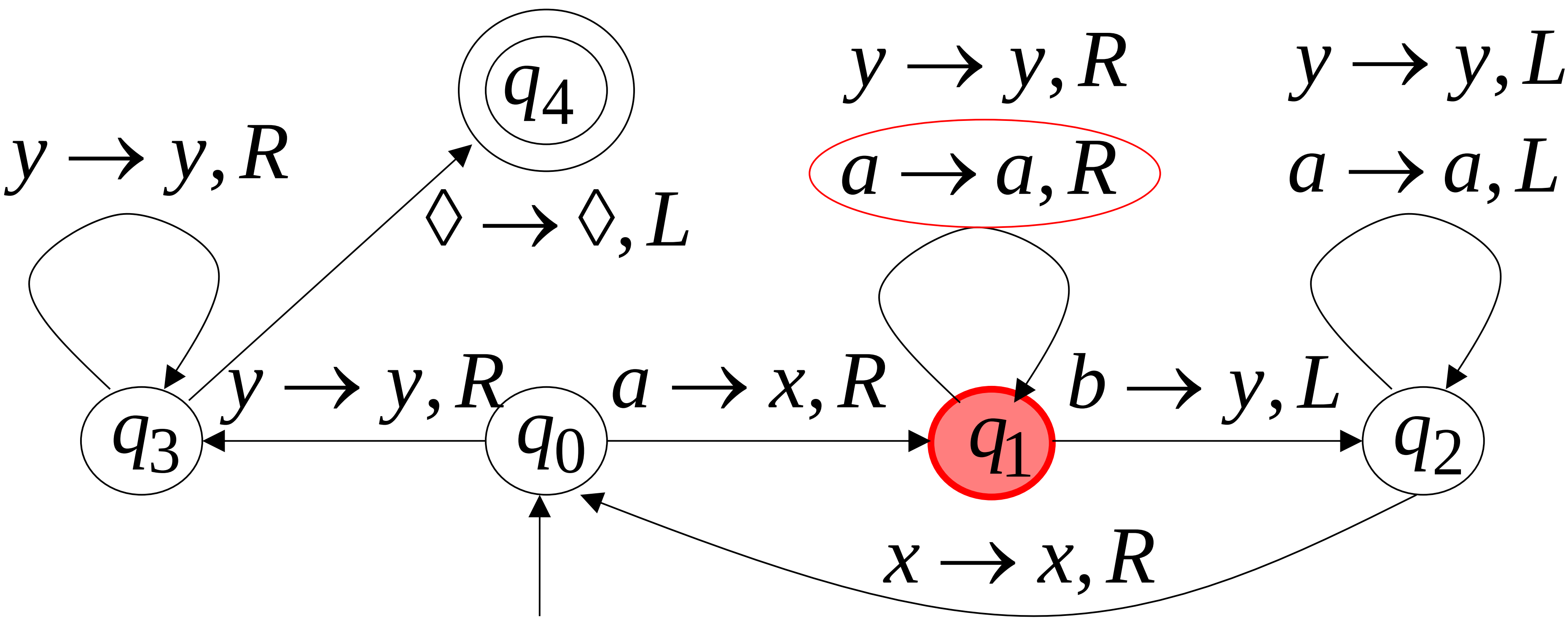
Time 0


◊
a
a
b
b
◊
◊
q_0

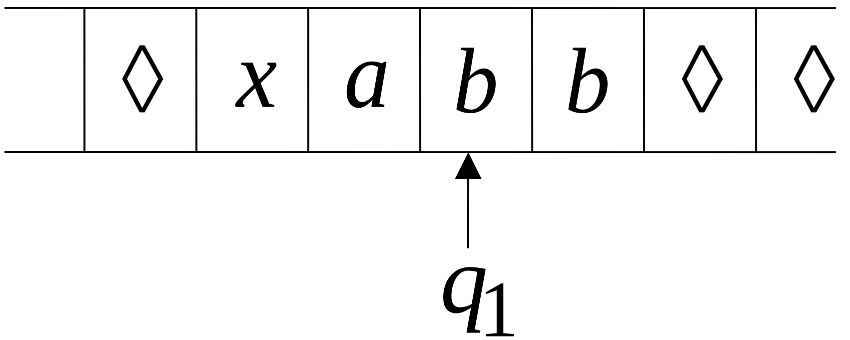

q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 1

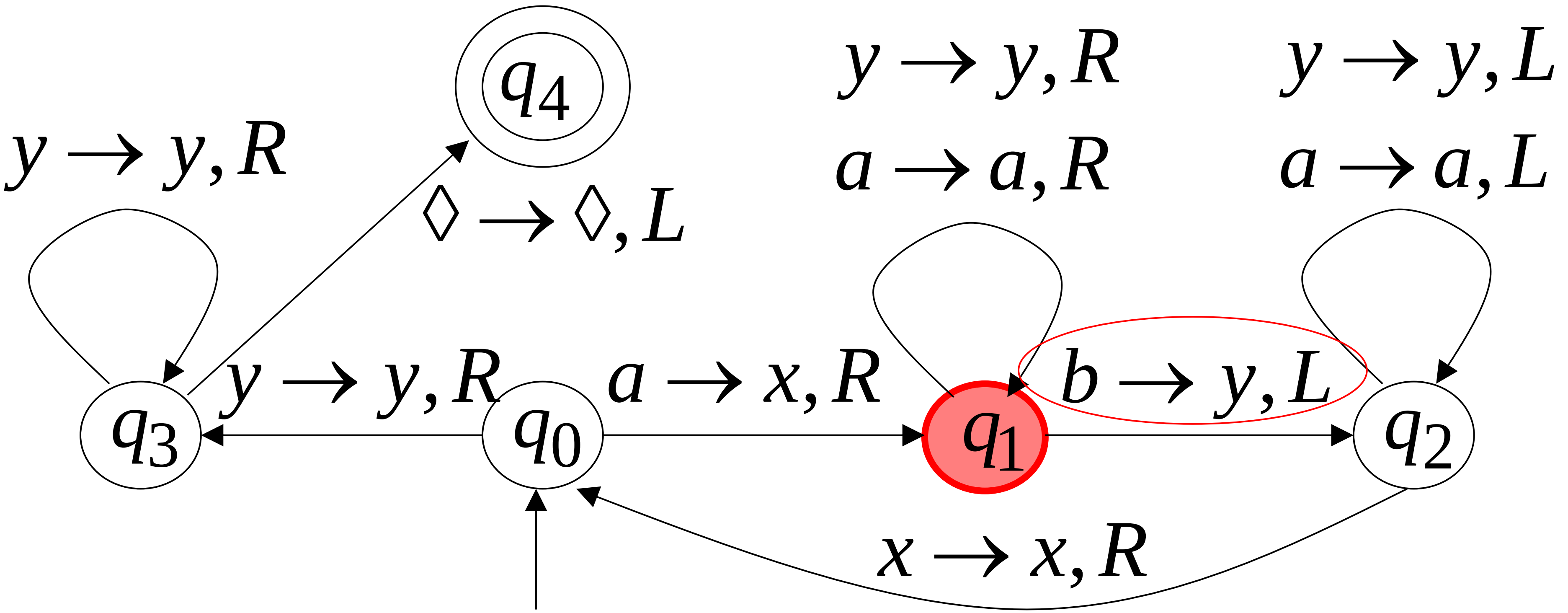

◊
x
a
b
b
◊
◊
q_1


q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 2

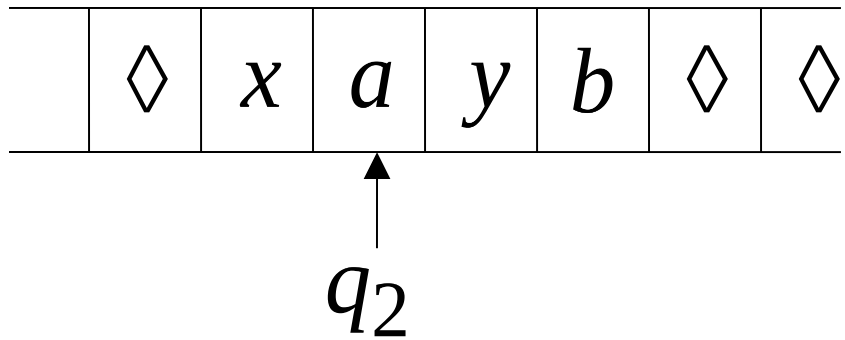


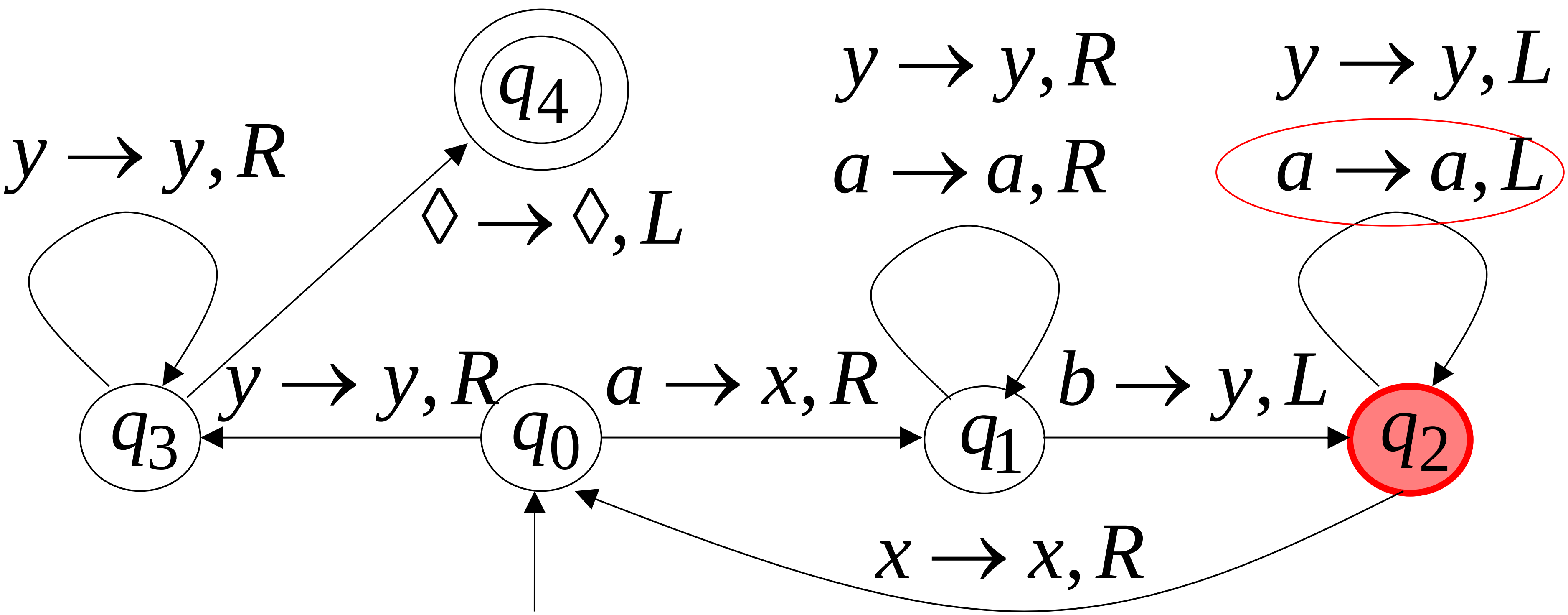
◊
x
a
b
b
◊
◊
q_1



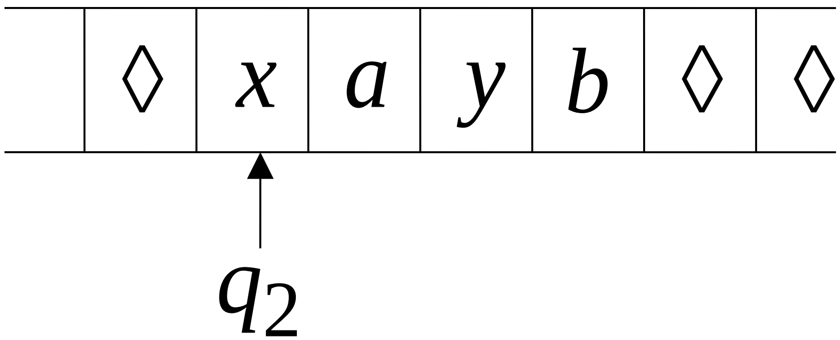
q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

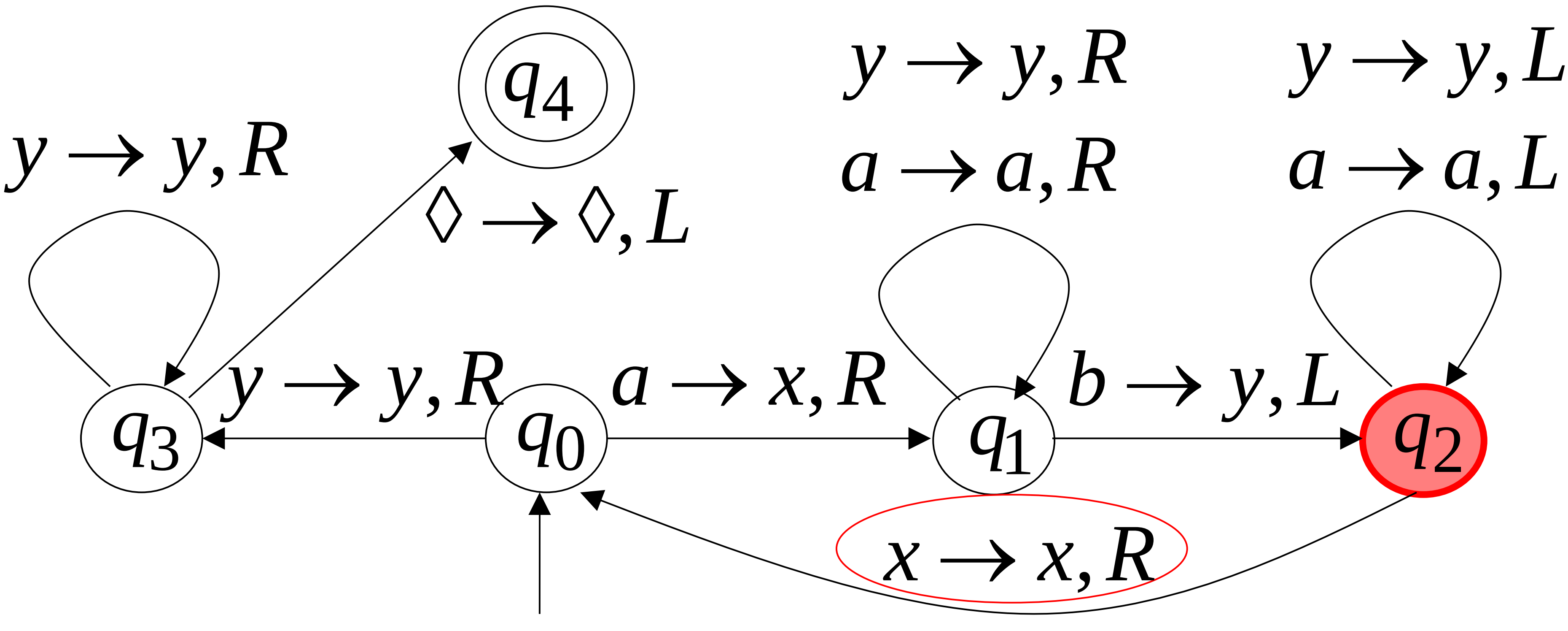
Time 3



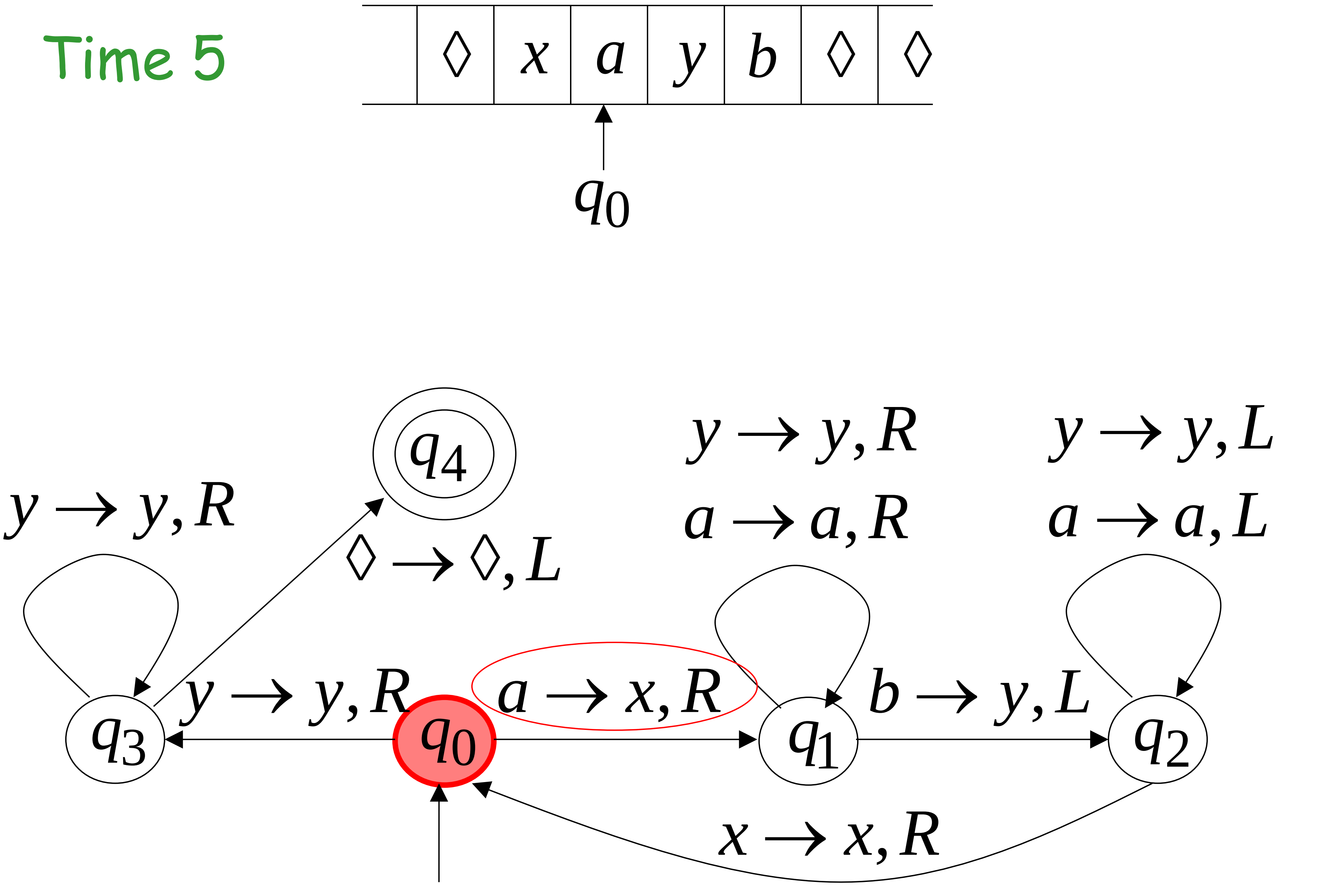


Time 4

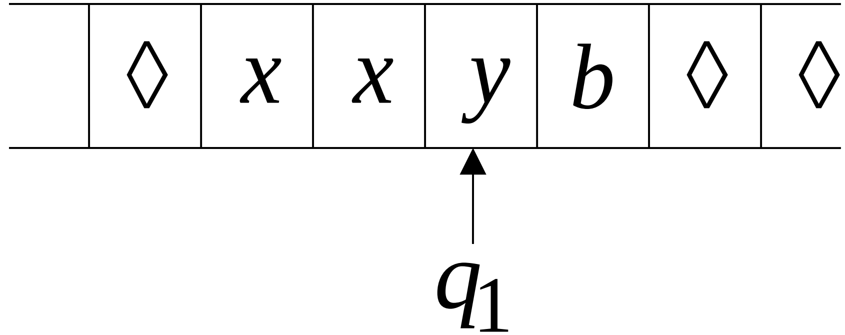

◊
x
a
y
b
◊
◊
q_2

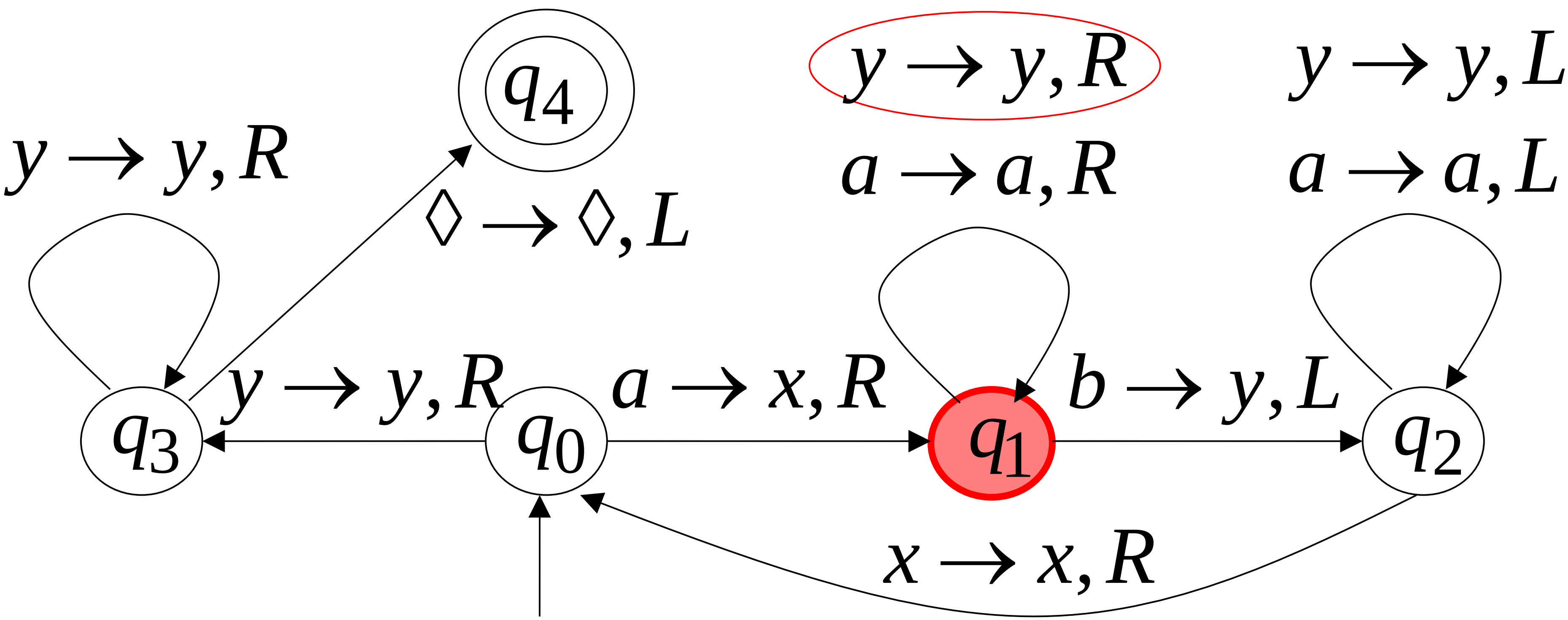

q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 5

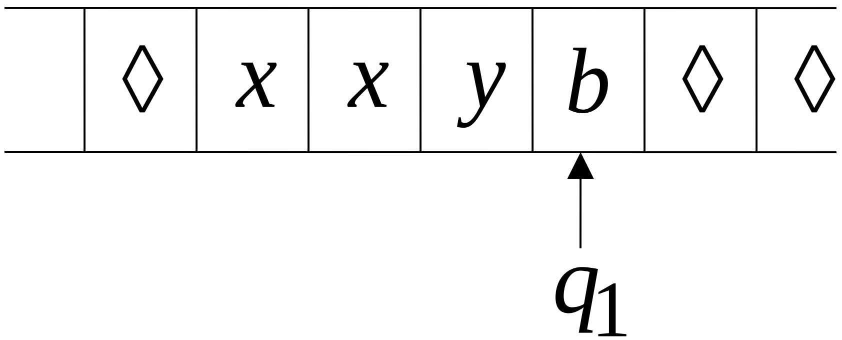


Time 6

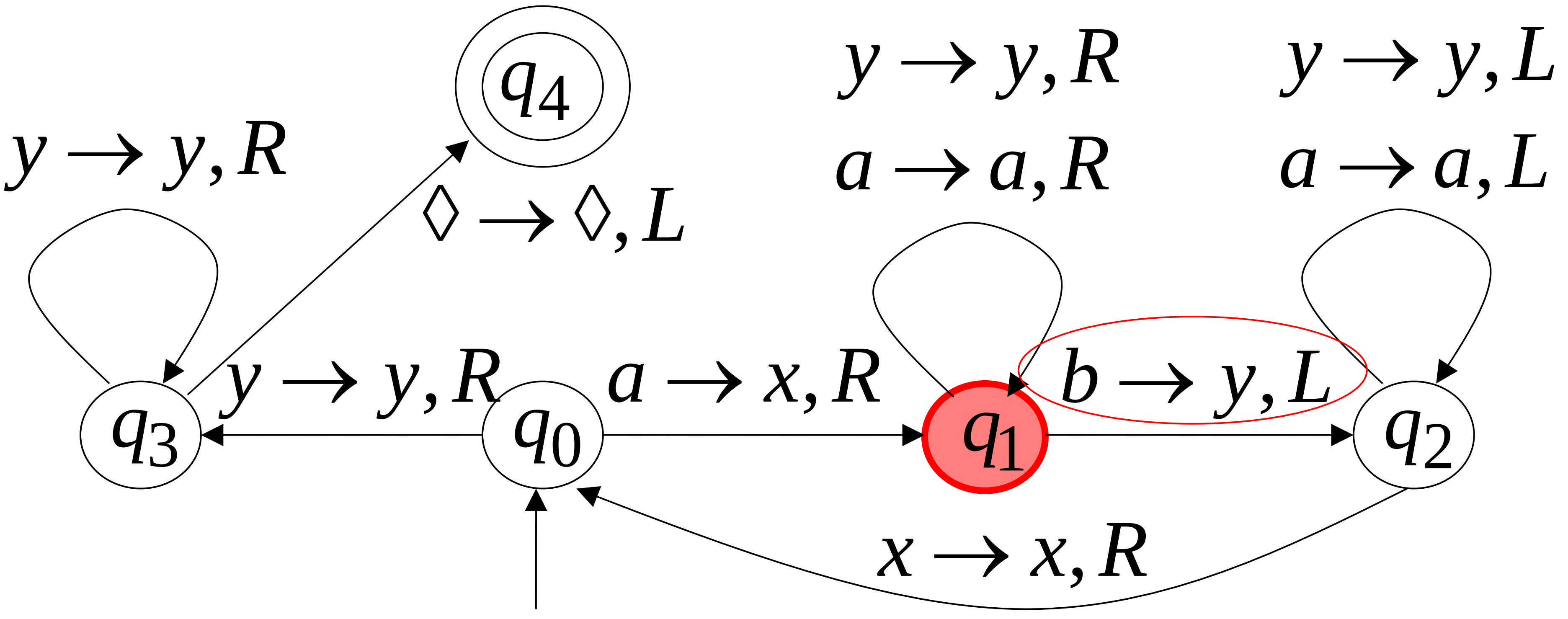

◊
x
x
y
b
◊
◊
q_1


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 7

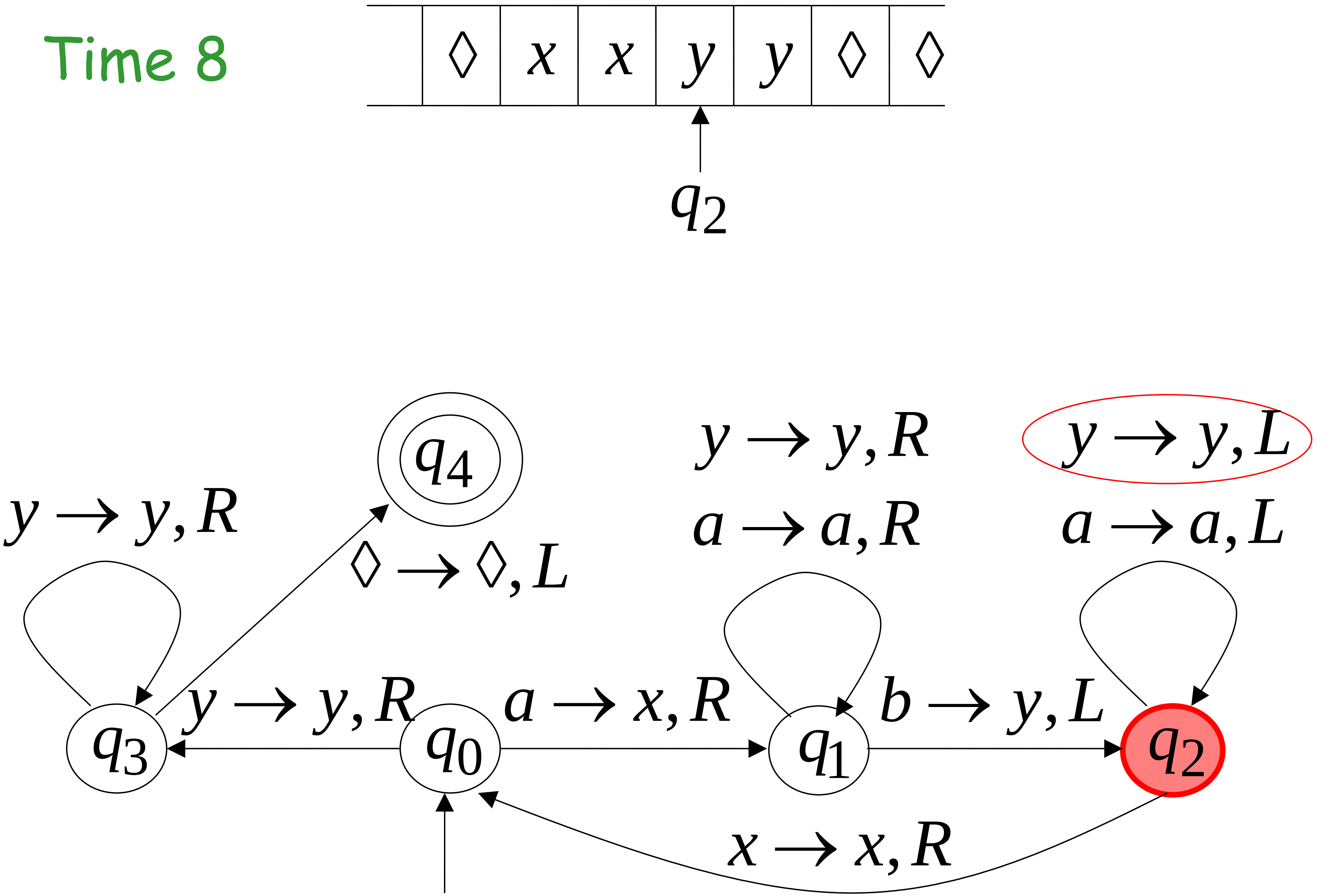


◊
x
x
y
b
◊
◊
q_1



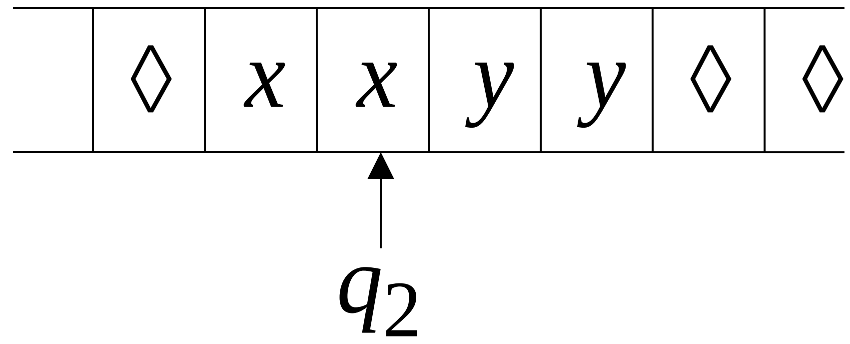
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

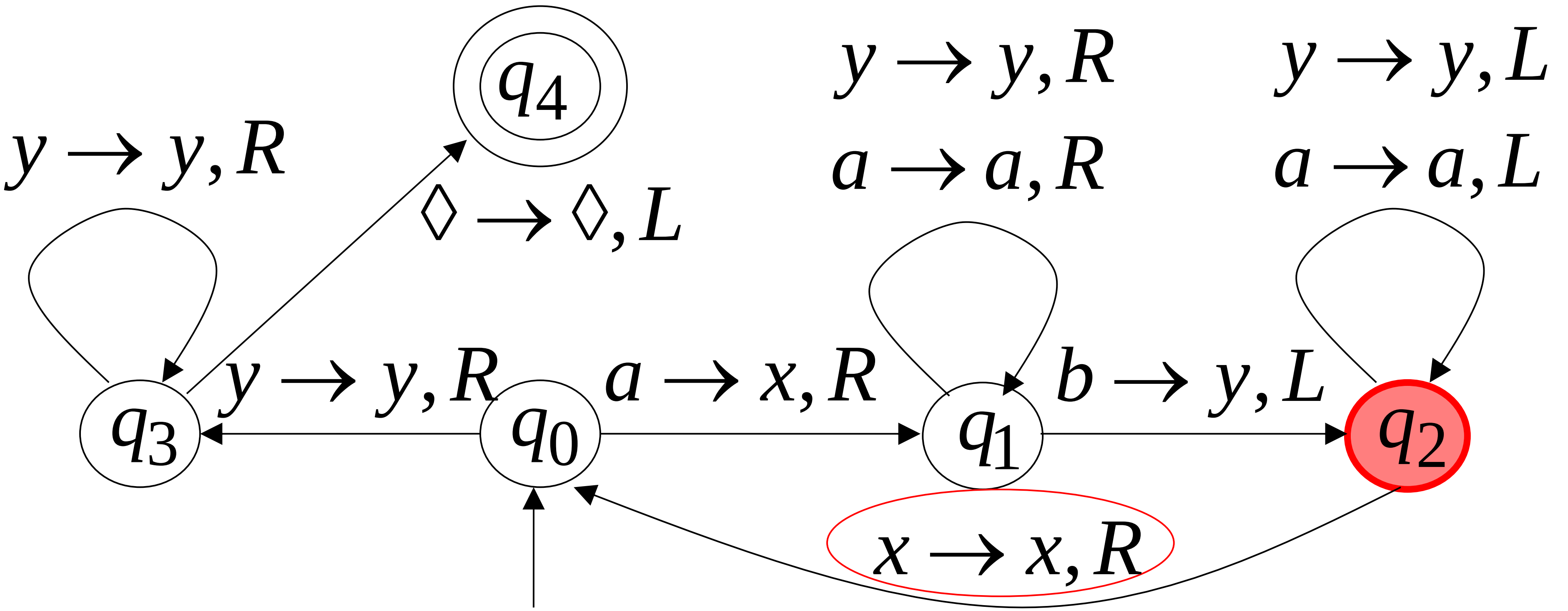
Time 8



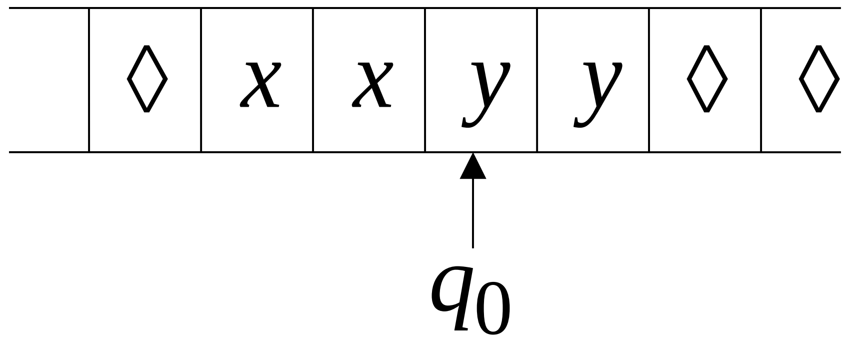
◊
x
x
y
y
◊
◊
q_2
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
q_4
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 9

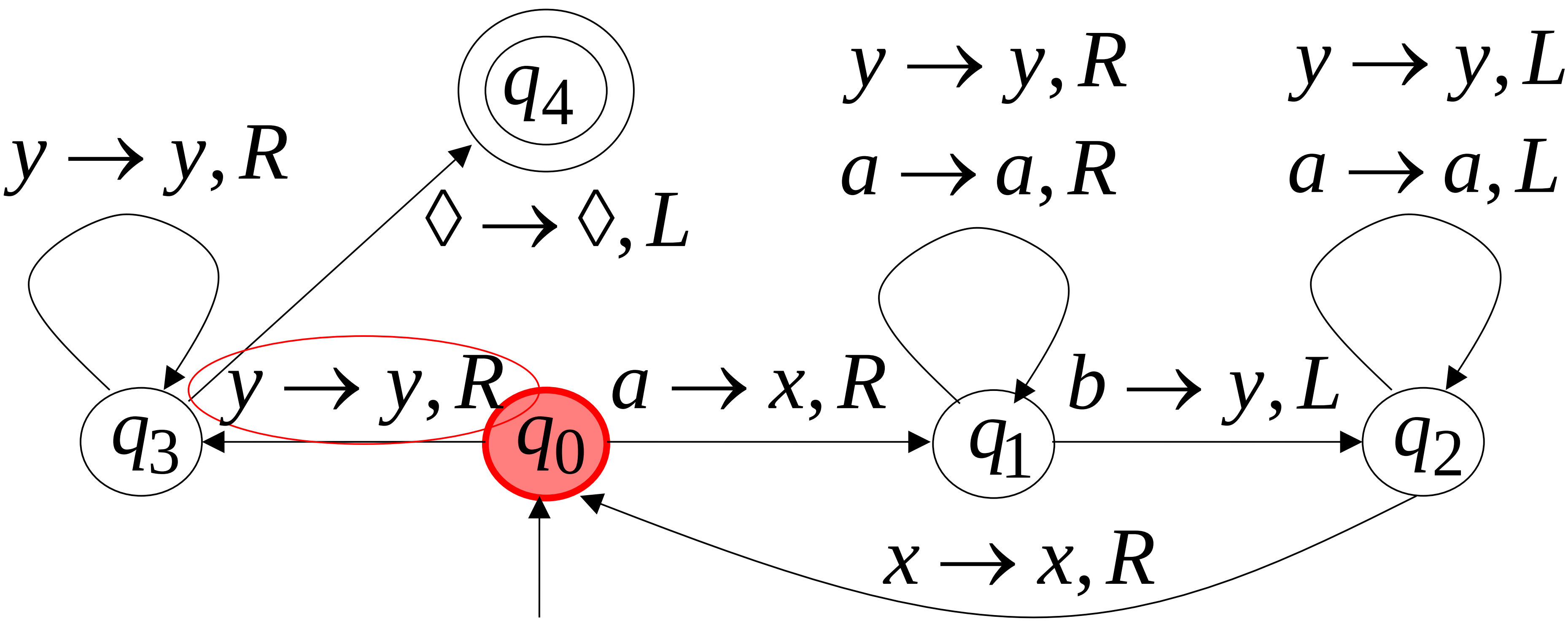

◊ x x y y ◊ ◊
q_2


q_4
$\Diamond \to \Diamond, L$
$y \to y, R$
q_3
$y \to y, R$
q_0
$a \to x, R$
$y \to y, R$
$a \to a, R$
q_1
$b \to y, L$
$y \to y, L$
$a \to a, L$
q_2
$x \to x, R$

Time 10

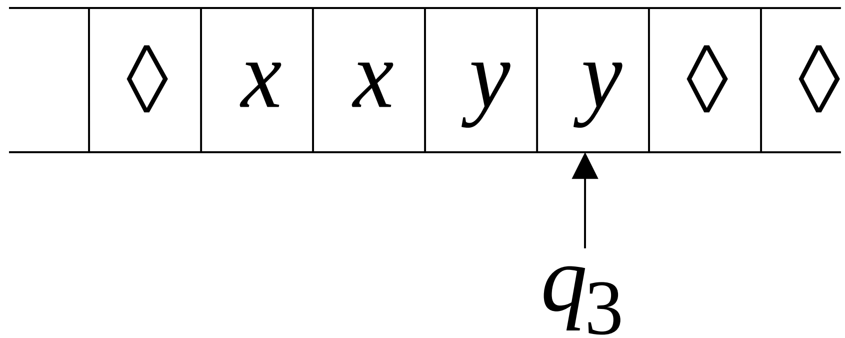


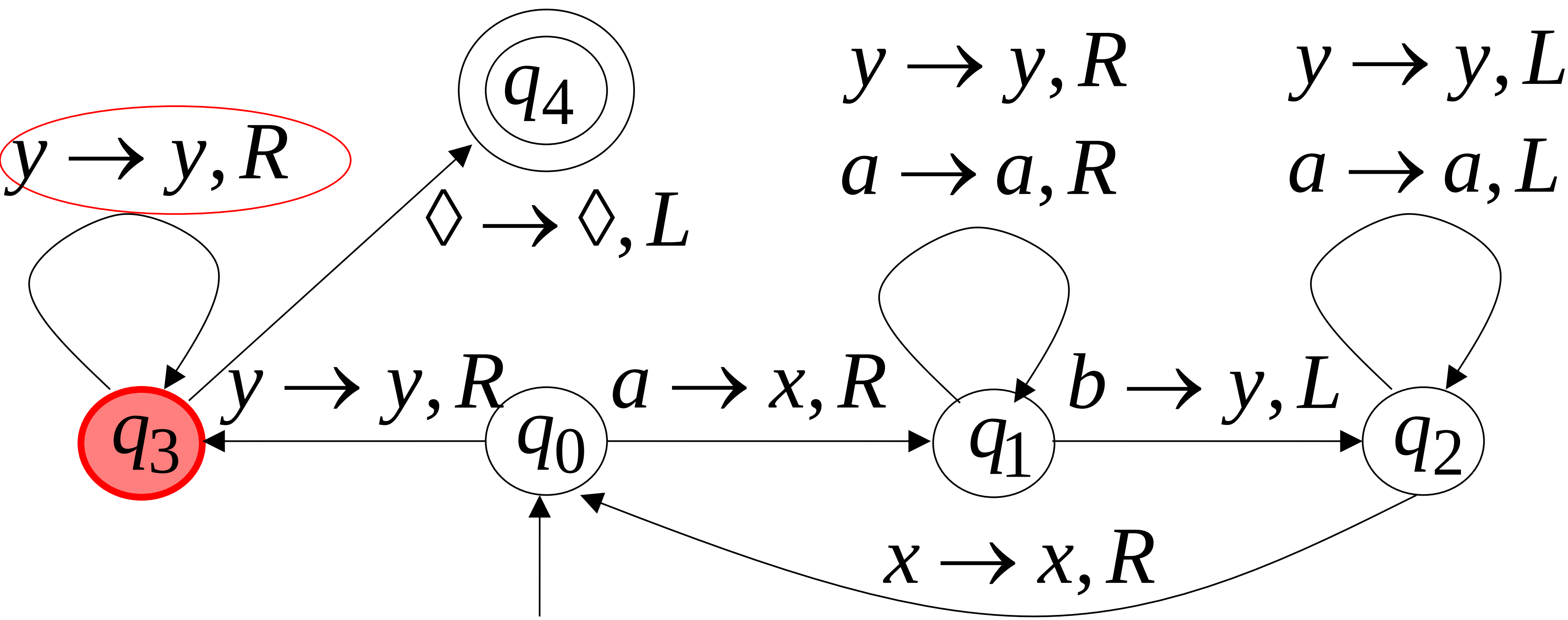
◊ x x y y ◊ ◊
q_0



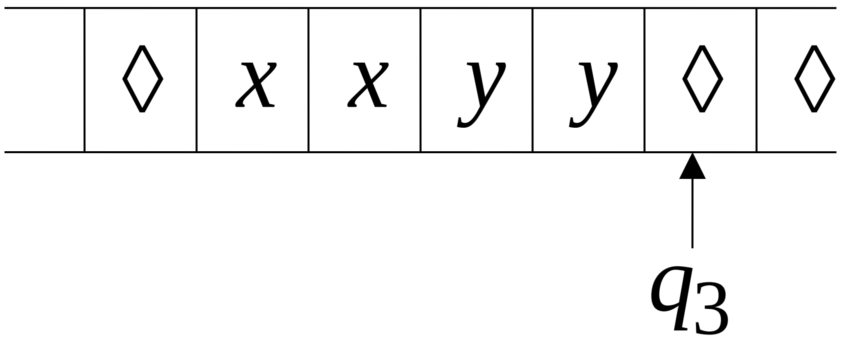
q_0
q_1
q_2
q_3
q_4
$a \rightarrow x, R$
$b \rightarrow y, L$
$x \rightarrow x, R$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$\lozenge \rightarrow \lozenge, L$

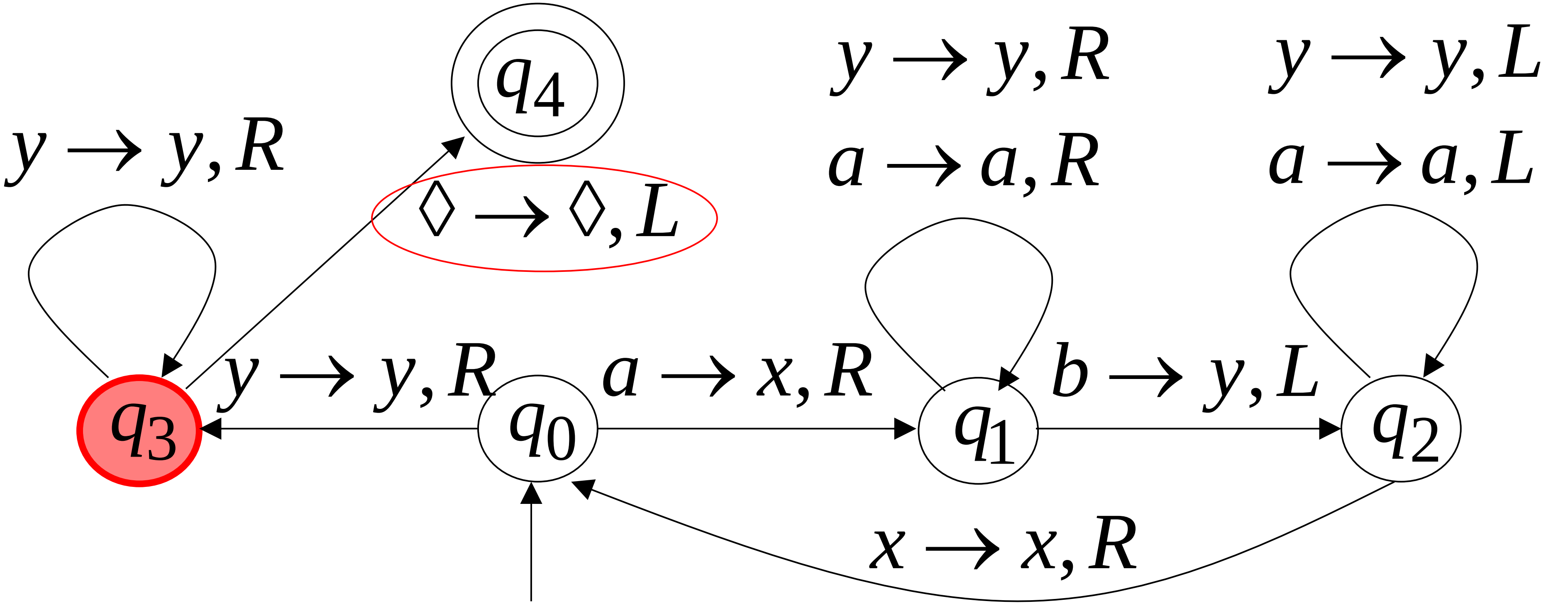
Time 11



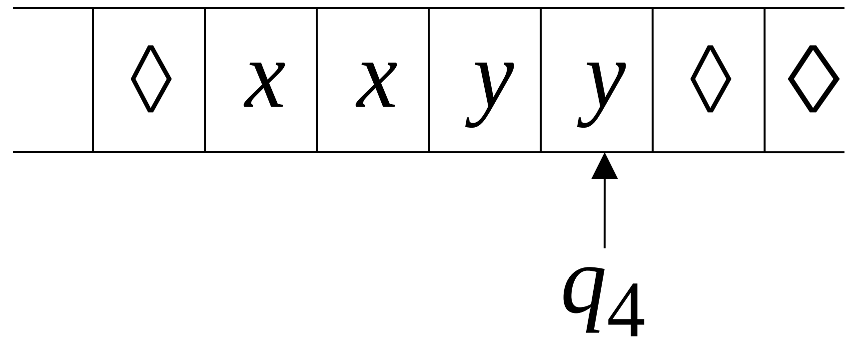


Time 12

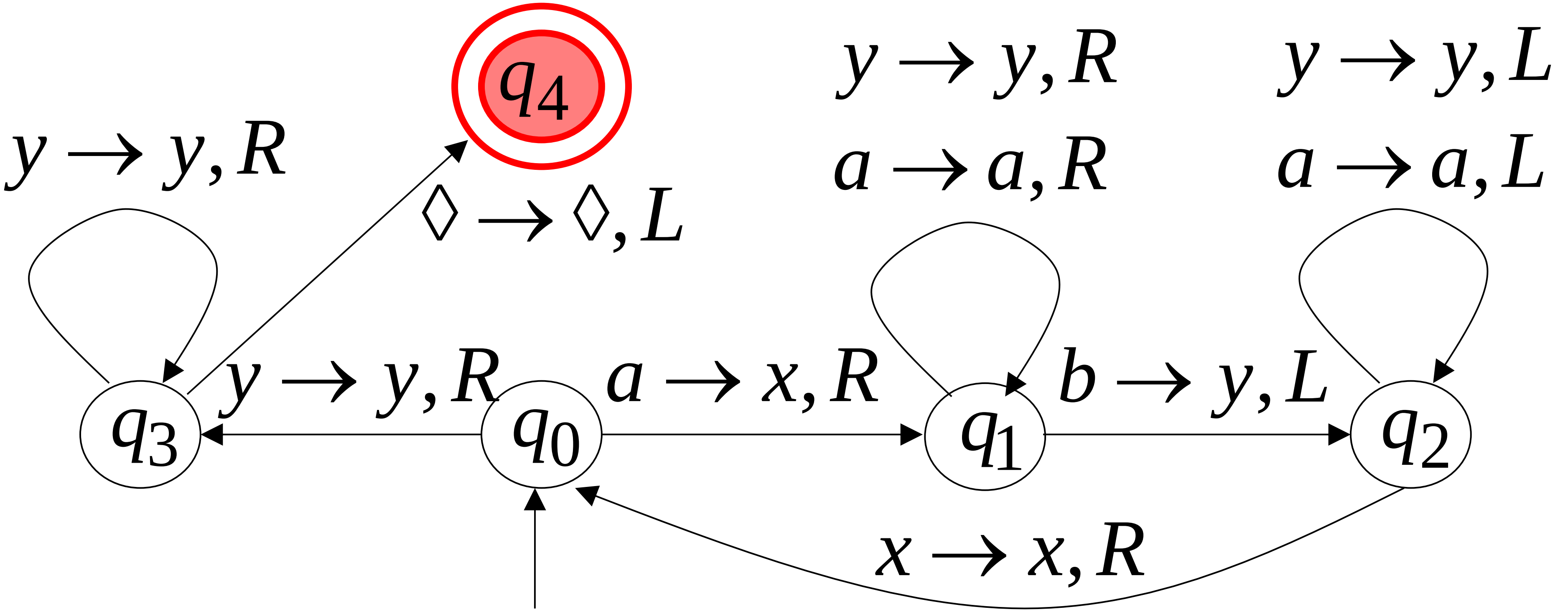

◊
x
x
y
y
◊
◊
q_3


q_4
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 13



Halt & Accept



Observation:

If we modify the machine for the language $\{a^n b^n\}$

we can easily construct a machine for the language $\{a^n b^n c^n\}$