How to generate equidistributed points on the surface of a sphere

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There are two qualitatively different ways in which we could try to achieve equidistribution of points on a surface. One is to randomly place them in such a way that the probability of ending up in some particular region is proportional to the area of that region (two-dimensional Poisson statistics). This gives equidistribution *on average*. The second is to regularly place points such that their distance in two orthogonal directions is locally always the same. This gives typically a better result (no fluctuations and no accidental overlap), but the emerging partial crystallinity could sometimes be undesirable. For the case of a sphere an example for both strategies is presented.

I. SPHERICAL COORDINATES

The most straightforward way to create points on the surface of a sphere are classical spherical coordinates, in which a point is addressed via its two angular coordinates, the polar angle $\vartheta \in [0;\pi]$ and the azimuthal angle $\varphi \in [0,2\pi]$. If the sphere has radius r, the Cartesian coordinates of that point are given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \sin\vartheta\,\cos\varphi \\ \sin\vartheta\,\sin\varphi \\ \cos\vartheta \end{pmatrix} . \tag{1}$$

II. RANDOM PLACEMENT

When one wants to evenly place points on the surface of a sphere, it is important to realize that it is *not* correct to simply choose the spherical angles ϑ and φ equidistributed from their permissible intervals. In order to find out what has to be done instead, it is helpful to have a look back at the area element. Notice that it can be written as

$$\mathrm{d}A = r^2 \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi = r\,\mathrm{d}(r\cos\vartheta)\,\mathrm{d}\varphi = r\,\mathrm{d}z\,\mathrm{d}\varphi . \tag{2}$$

We thus see that it *is* correct to choose φ equidistributed from $[0;2\pi]$ and to also choose z equidistributed from $[-r;r]$. This then gives the following algorithm for placing N randomly equidistributed points on the surface of a sphere of radius r:

```
repeat N times {
   Choose z equidistributed from [−r;r].
   Choose φ equidistributed from [0;2π].
   Set x = √(r² − z²) cos φ.
   Set y = √(r² − z²) sin φ.
}
```

The result of such an algorithm is illustrated in Fig. 1a.

III. REGULAR PLACEMENT

Regular equidistribution can be achieved by choosing circles of latitude at constant intervals d_ϑ and on these circles points with distance d_φ, such that $d_\vartheta \simeq d_\varphi$ and that $d_\vartheta d_\varphi$ equals the average area per point. This then gives the following algorithm:

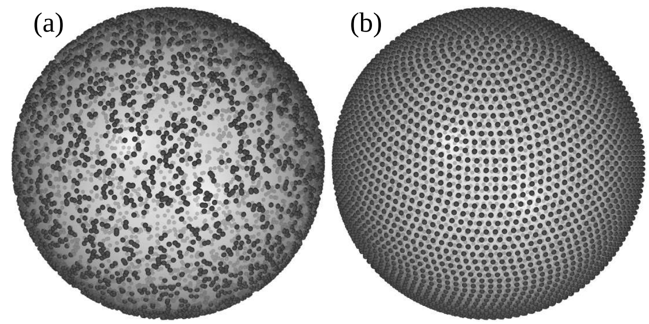

FIG. 1: Illustration of the two algorithms for putting points equidistributed onto the surface of a sphere discussed in these notes. In both cases $N = 5000$, case (a) is the random placement, case (b) the regular one. In the latter the algorithm actually could only place 4999 points.

```
Set N_count = 0.
Set a = 4πr²/N and d = √a.
Set M_ϑ = round[π/d].
Set d_ϑ = π/M_ϑ and d_φ = a/d_ϑ.
For each m in 0...M_ϑ − 1 do {
   Set ϑ = π(m + 0.5)/M_ϑ.
   Set M_φ = round[2π sin ϑ/d_φ].
   For each n in 0...M_φ − 1 do {
      Set φ = 2πn/M_φ.
      Create point using Eqn. (1).
      N_count += 1.
   }
}
```

At the end of this algorithm N_{count} points have been placed, with N_{count} very close to N. The result of such an algorithm is illustrated in Fig. 1b.

Note that the regular placement is far more even: neither local clustering (points almost sitting on top of each other) nor global $\sqrt{N}$ fluctuations occur. For setting up a liquid state the second algorithm thus appears more favorable.