Delaunay Triangulations on the GPU

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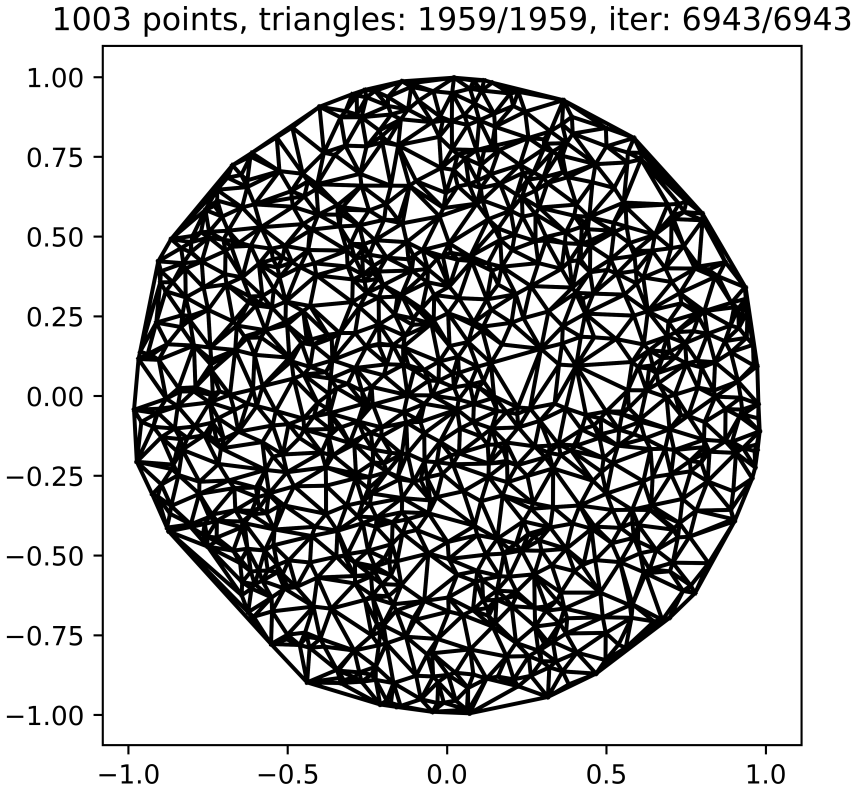


Contents

Abstract

Triangulations of a set of points are a very useful mathematical construction to describe properties of discretised physical systems, such as modelling terrains, cars and wind turbines which are a commonly uses for simulations such as compulational fluid dynamics or other physical properties, and even have use in video games for rendiring and visualising complex geometries. To paint a picure you may think of a triangulation given a set of points P to be a bunch of line segments connecting each point in P in a way such that the edges are non instersecting. A particulary interesting subset of triangulations are Delaunay triangulations (DT). The Delaunay triangulation is a triangulation which maximises all angles in each triangle of the triangulation. Mathematically this gives us an interesting optimization problem which leads to some rich mathematical properties, at least in 2 dimensions, and for the applied size we have a good way to discretize space for the case of simulations with the aid of methods such as Finite Element and Finite Volume methods. Delaunay triangulatinos in particular are a good candiate for these numerical methods as they provide us with fat triangles, as opposed to skinny triangles, which can serve as good elements in the Finite Element method as they tend to improve accuracy [1].

There are many algorithms which compute Delaunay triangulations (cite some overview paper), however alot of them use the the operation of 'flipping' or originally called an 'exchange' [2]. This is a fundamental property of moving through triangulations of a set of points to with the goal of optaining the optimal Delaunay triangulation. This flipping operation involves a configuration of two triangles sharing an edge, forming a quadrilateral with its boundary. The shared egde between these two triangles will be swapped or flipped from the two points at its end to the other two points on the quadrilateral. The original agorithm motivated by ([2]) is hinted to be us this flipping operation to iterate through different triangulations and eventually arrive at the Delaunay trianglution which we desire.

With the flippig operation being at the core of the algorithm, we can notice that is has the possibility of being parallelized. This is desirable as problems which commonly use the DT are run with large datasets and can benefit from the highly parallelisable nature of this algorithm. If we wish to parallize this idea, and start with some initial triangulation, conflicts would only occur if we chose to flip a configuration of triangles which share a triangle. With some care, this is an avoidable situation leads to a highly scalable algorithm. In our case the hardware of choice will be the GPU which is designed with the SIMT model which is particularly well suited for this algorithm as we are mostly performing the same operations in each iteration of the algorithm in parallel.

The goal of this project was to explore the Delaunay triangulations through both serial and parallel algorithms with the goal of presenting a easy to understand, sufficiently complex parallel algorthm designed with with Nvidia's CUDA programming model for running software on their GPUs.

1. Delaunay triangulations

In this section I aim to introduce triangulations and Delaunay triangulations from a mathematical perspective with the foresight to help present the motivation and inspiration for the key algorithms used in this project. For the entirety of this project we only focus on 2 dimensional Delaunay triangulations.

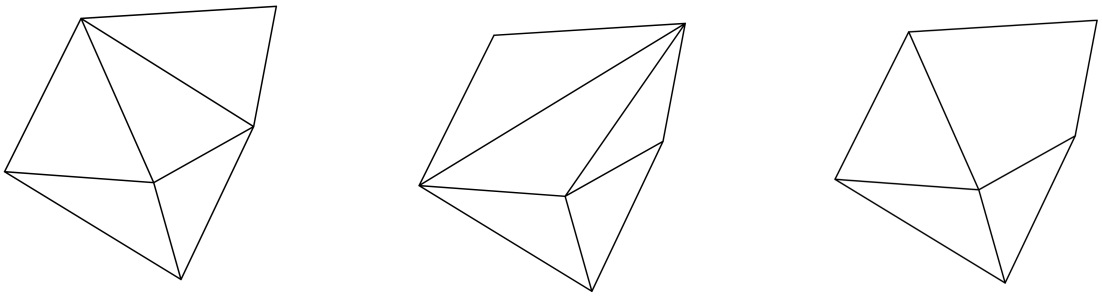
In order to introduce indroduce the Delaunay Traingulation we first must define what we mean mean by a triangultion. In order to create a triangualtion we need a set of points which will make up the vertices of the triangles. But first we want to clarify a possible ambiguity about edges.

Definition 1.1: For a point set P, the term edge is used to indicate any segment that includes precisely two points of S at its endpoints. [3].

Alterantively we could say an edge doesnt contain its enpoints which could be more useful in different contexts. But now we define the triangulation.

Definition 1.2: A *triangulation* of a planar point set P is a subdivision of the plane determined by a maximal set of noncrossing edges whose vertex set is P [3].

This is a somewhat technical but precise definition. The most imortant point in Definition 1.2 is that it is a *maximal* set of noncrossing egdes which for us means that we will not have any other shapes than triangles int this structure.



(a) (b) (c)

Figure 2: Examples of two traingulations (a) (b) on the same set of points. In (c) an illustration of a non maximal set of edges.

A useful fact about triangulations is that we can know how many triangles our triangulation will contain if given a set of points and its convex hull. For our purposes the convex hull will allways be a set which will covering a set of points, in our case the points in our triangulation. This will be usefull when we will be storing triangles as we will allways know the number of triangles that will be created.

Theorem 1.1: Let P be a set of n points in the plane, not all collinear, and let k denote the number of points in P that lie on the boundary of the convex hull of P. Then any triangulation of P has $2n - 2 - k$ triangles and $3n - 3 - k$ edges. [4]

A key feature of all of the Delaunay triangulation theorems we will be considering that no three points from the set of points P which will make up our triangulation will lie on a line and alse that no 4 points like on a circle. Motivation for this defintion will become more apparent in Theorem 1.2 and following. Definition 1.3 lets us imagine that our points are distributed randomly enough so that our algorithms will work with no degeneracies appearing. This leads us to the following definition.

Definition 1.3: A set of points P is in *general position* if no 3 points in P are colinear and that no 4 points are cocircular.

From this point onwards we will allways assume that the point set P from which we obtain our triangulation will be in *general position*. This is neccesary for the definitions and theorems we will define.

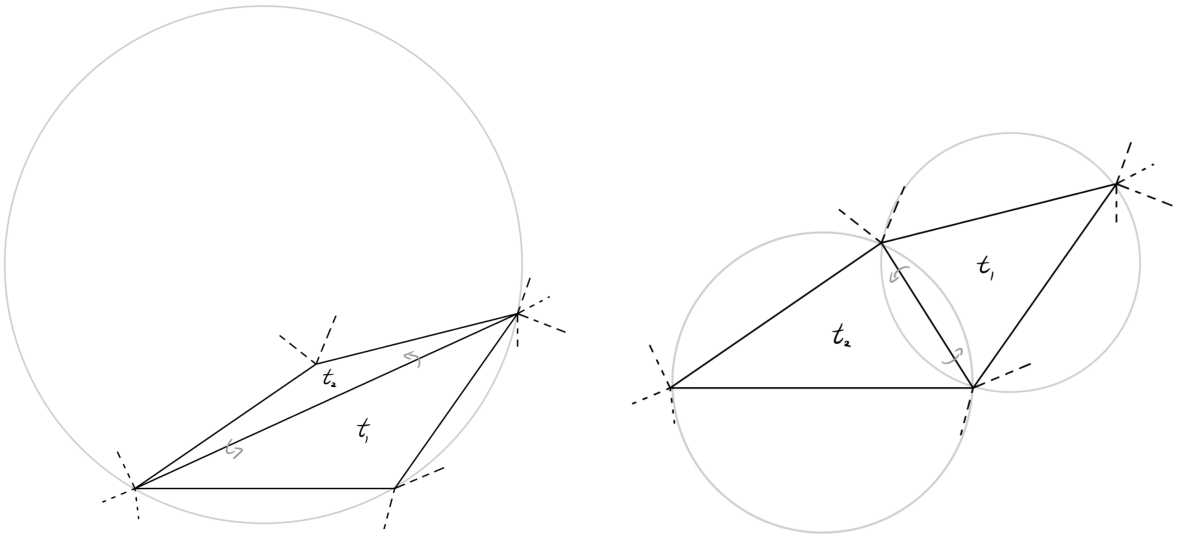
In order to define a Delaunay triangulation we would like to establish the motivation for the definition with another, preliminary definition. A Delaunay triangulation is a type of triangulation which in a sense maximizes smallest angles in a triangulation T. This idea is formalized by defining an *angle sequence* $(\alpha_1, \alpha_2, ..., \alpha_{3n})$ of T which is an ordered list of all angles of T sorted from the smallest to largest. With angle sequences we can now compare two triangulations to eachother. We can say for two triangulations T_1 and T_2 we write $T_1 > T_2$ (T_1 is fatter than T_2) if the angle sequence of T_1 is lexographically greater than T_2. Now we can compare triangulations. And by defining Definition 1.4 are able to define a *Delaunay triangulation.*

Definition 1.4: Let e be an edge of a triangulation T_1 , and let Q be the quadrilateral in T_1 formed by the two triangles having e as their common edge. If Q is convex, let T_2 be the triangulation after flipping edge e in T_1. We say e is a *legal edge* if $T_1 \geq T_2$ and e is an *illegal edge* if $T_1 < T_2$ [3]

Definition 1.5: For a point set P, a *Delaunay triangulation* of P is a triangulation that only has legal edges. [3]

As per Definition 1.5, Delaunay triangulations wish to only contain legal edges and this provides us with a “nice” triangluation with fat triangles.

Theorem 1.2 *(Empty Circle Property)*: Let P be a point set in general position. A triangulation T is a Delaunay triangulation if and only if no point from P is in the interior of any circumcircle of a triangle of T. [3]



(a) (b)

Figure 3: Demonstration of the flipping operation. In (a) A configuration that needs to be flipped illustrated by the circumcircle of t_1 containg the auxillary point of t_2 in its interior. In (b) configuration (a) which has been flipped and no longer needs to be flipped as illustrated by the both circumcirles of t_1 and t_2.

Theorem 1.2 is the key ingredient in the the Delaunay triangulation algorithms we are going to use. This is because instead of having to compare angles, as would be demanded by Definition 1.5, we are allowed to only perform a computation, involving finding a circumcicle and performing one comparison which would involve determining whether the point not shared by triangles circumcircle is contained inside the circumcircle or not. Algorithms such as initially intruduced by Lawson [5] exist which do focus on angle comparisons but are not preferred as they do not introduce desired locality and are more complex.

And finally we present the theorem which guarantees that we will eventually arrive at our desired Delaunay triangluation by stating that we can travel across all possible triangultions of our point set P by using the fliping operation.

Theorem 1.3 *(Lawson)*: Given any two triangulations of a set of points P, T_1 and T_2, there exist a finite sequence of exchanges (flips) by which T_1 can be transformed to T_2. [2]

2. The GPU

The Graphical Processing Unit (GPU) is a type of hardware accelerator originally used to significantly improve running video rendering tasks for example in video games through visualizing the two or three dimensional environments the player would be interacting with or rendering vidoes in movies after the addition of visual effects. Many different hardware accelerators have been tried and tested for more general use, like Intels Xeon Phis, however the more purpose oriented GPU has prevailed in the market and in performance mainly lead by Nvidia in prevoius years, Today, the GPU has gained a more general purporse status with the rise of General Purpose GPU (GPGPU) programming as more and more people have noticed that GPUs are very useful as a general hardware accelerator.

The traditional CPU (based on the Von Neumann architecture) which is built to perform *serial* tasks , the CPU is built to be a general purpose hardware for performing all tasks a user would demand from the computer. In contrast the GPU can't run alone and must be used in conjuction to the CPU. The CPU sends compute intructions for the GPU to perform and data is commonly passes between the CPU and GPU.

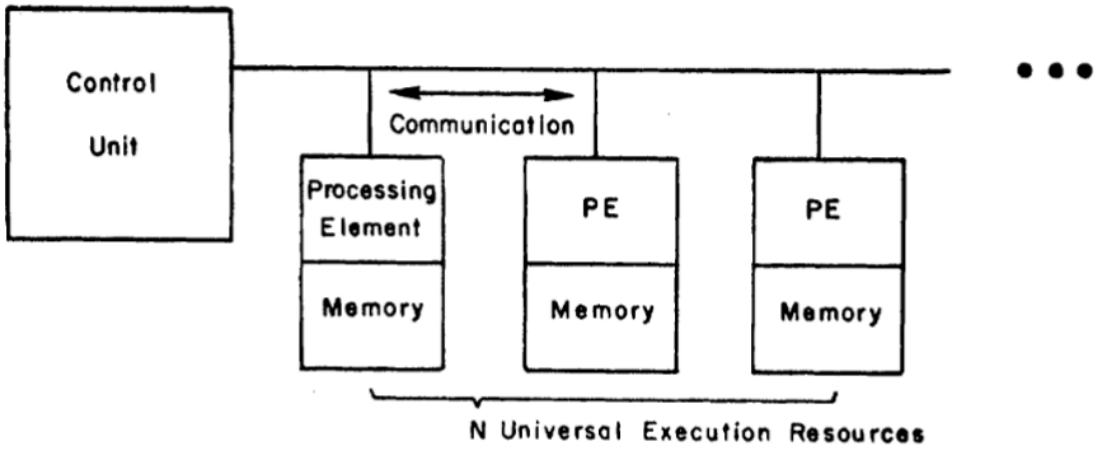


Figure 4: The *Single Instruction Multiple Threads (SIMT)* classification, originally known as an *Array Processor* as illustriated by Michael J. Flynn [6]. The control unit communicates instructions to the N processing element with each processing element having its own memory.

What makes the GPU increadibly usefull in certain usecases (like the one of this report) is its architecture which is build to enable massively parallelisable tasks. In Flynn's Taxonomy [6], the GPUs architecture is based a subcategory of the Single Instruction Multiple Data (SIMD) classification known as Single Instruction Multiple Threads (SIMT) also known as an Array Processor. The SIMD classification allows for many proccessing units to perform the same tasks on a shared dataset with the SIMT classification additionally allowing for each processing unit having its own memory allowing for more diverse proccessing of data.

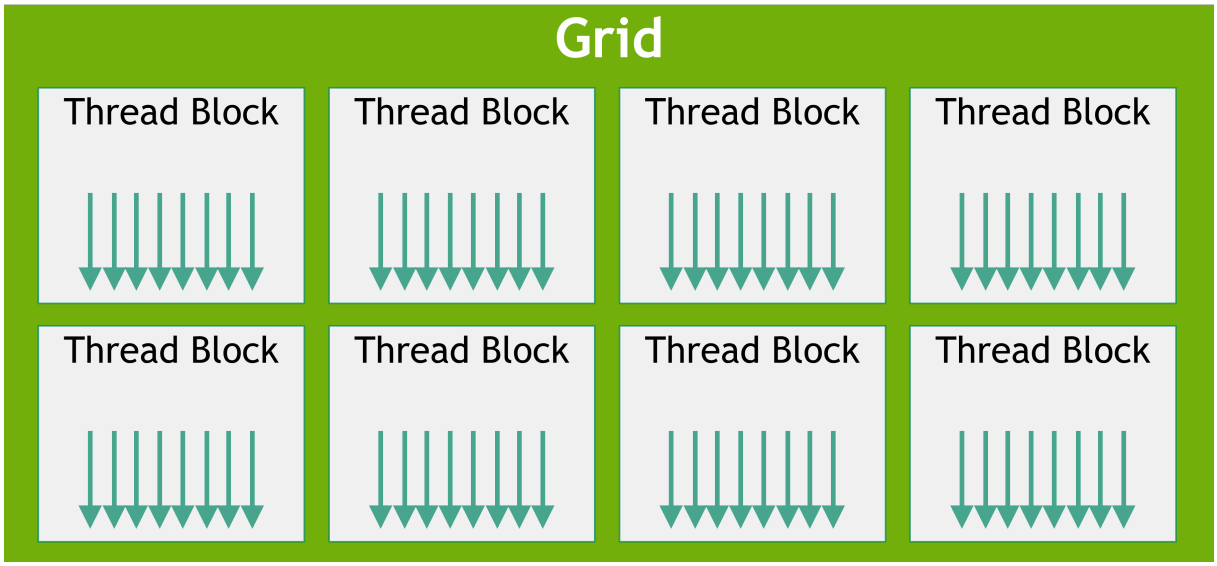


Figure 5: An illustration of the structure of the GPU programming model. As the lowest compute instrcution we have a thread block consisting of a number threads ≤ 1024. The thread blocks are contained in a grid. [7]

Nvida's GPUs take the SIMT model and further develop it. There are three core abstractions which allow Nvidia's GPU model to be succesfull; a hierarchy of thread groups, shared memories and synchronization [7]. The threads, which represent the a theoretical proccesses which encode programmed instrcutions, are launched together in groups of 32 known as *warps*. This is the smallest unit of instructions that is executed on the GPU. The threads are further grouped into *thread blocks* which are used as a way of organising the shared memory to be used by each thread in this thread block. And once more the *thread blocks* grouped into a *grid*.

3. Algorithms

In this section we focus on two types of algorithms, serial and parallel, but with a focus on the parallel algorithm. Commonly algorithms are first developed with a serialized verion and only later optimized into parallelized versions if possible. This is how I will be presenting my chosen Delanay Triangulation algorithms in order to portray a chronolgical development of ideas used in all algorithms.

3.1. Serial

The simplest type of DT algorithm can be stated as follows in Algorithm 1

Algorithm 1: Lawson Flip algorithm

Let P be a point set in general position. Initialize T as any trianulation of P. If T has an illegal edge, flip the edge and make it legal. Continue flipping illegal edges, moving through the flip graph of P in any order, until no more illegal edges remain. [3]

This algorithm presents with a bit of ambiguity however I believe its a good algorithm to keep in mind when progressing to more complex algorithms as it presents the most important feature in a DT algorithm, that is, checking if an edge in the traingultion is legal and if not flipping it. Most DT algorithms take this core concept and build a more optimized verions of it with as Algorithm 1 has a complexity of $O(n^2)$ [8].

The next best serial algorithm commonly presented by popular textbooks [4], [9] is the *randomized incrimental point insertion* Algorithm 2. When implemented properly this algorithm should have a complexity of $O(n \log(n))$ [4]. This algorithm is favoured for its relative low complexity and ease of implementaion. The construction this algorithm is a bit mathematically involved however the motivation behind the construction of the algorithm is to perform point instertions, and after each point insertion we perform necessary flips to transform the current triangulation into a DT. This in turn reduces the number of flips we need to perform and this is reflected in the runtume complexity.

Algorithm 2: Randomized incremental point insertion

```
Data: point set P
1 | Initialize T with a triangle enclosing all points in P
2 | Compute a random permutation of P
3 | for p ∈ P
4 | Insert p into the triangle t containing it
5 | for each edge e ∈ t
6 | | LegalizeEdge(e, t)
7 | | return T
```

A signficant part of this algorithmis the FlipEdge function in Algorithm 3. This function performs the necessary flips, both number of and on the correct edges, for the triangulation in the current iteration of point insertion to become a DT.

Algorithm 3: LegalizeEdge

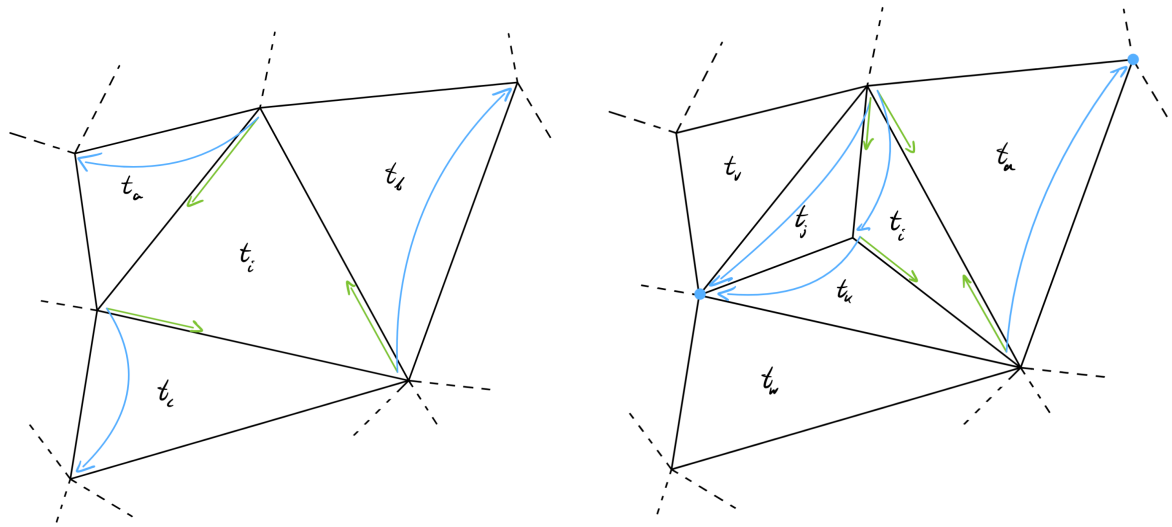
Data: egde e, triangle t_a

1 **if** e is illegal
2 *flip* with triangle t_b across edge e
3 let e_1, e_2 be the outward facing egdes of t_b
4 LegalizeEdge(e_1, t_b)
5 LegalizeEdge(e_2, t_b)

In the following sections we will discuss the point insertion and flipping steps in more detail.

3.1.1. Point insertion

In the point instertion procedure there a point not yet insterted in the point set making up the final triagulation is chosen and then the triangle in which it lies in is found and selected for insertion. Figure 6 illustrates this process.



(a) Before insertion. (b) After insertion.

Figure 6: An illustration of the point insertion in step 4 of Algorithm 2. In figure (a) the center most triangle t_i will be then triangle in which a point will be chosen for insertion. Triangle t_i knows its neighbours across each edge represented by the green arrows and knows the points opposite each of these edges. After the point it inserted (b), t_i is moved and two new triangles t_j, t_k are created to accomodate the new point. Each new trianlge t_i, t_j, t_k can be fully constructed from the previously existing t_i and each neighbour of t_i in (a) has its neighbours updated to reflect the insertion. The neighbouring triangles opposite points are updated by accesing the opposite point across the edge of the neighbouring triangle and obtaining the index of the edge which has the triangle currently being split. The index of the opposite point will allways be 0 by construction. The neighbouring triangle is also updated similarly but with the appropriate index which will be the one of the triangle who's modifying the neighbouring triangle.

It might be nice to see results from just running the point insertion algorthim by itself, whithout the flipping which would take place inbetween which will be further explored in the next section. In Figure 7 we see the result after the super triangle points and their corresponding triangles have been removed.

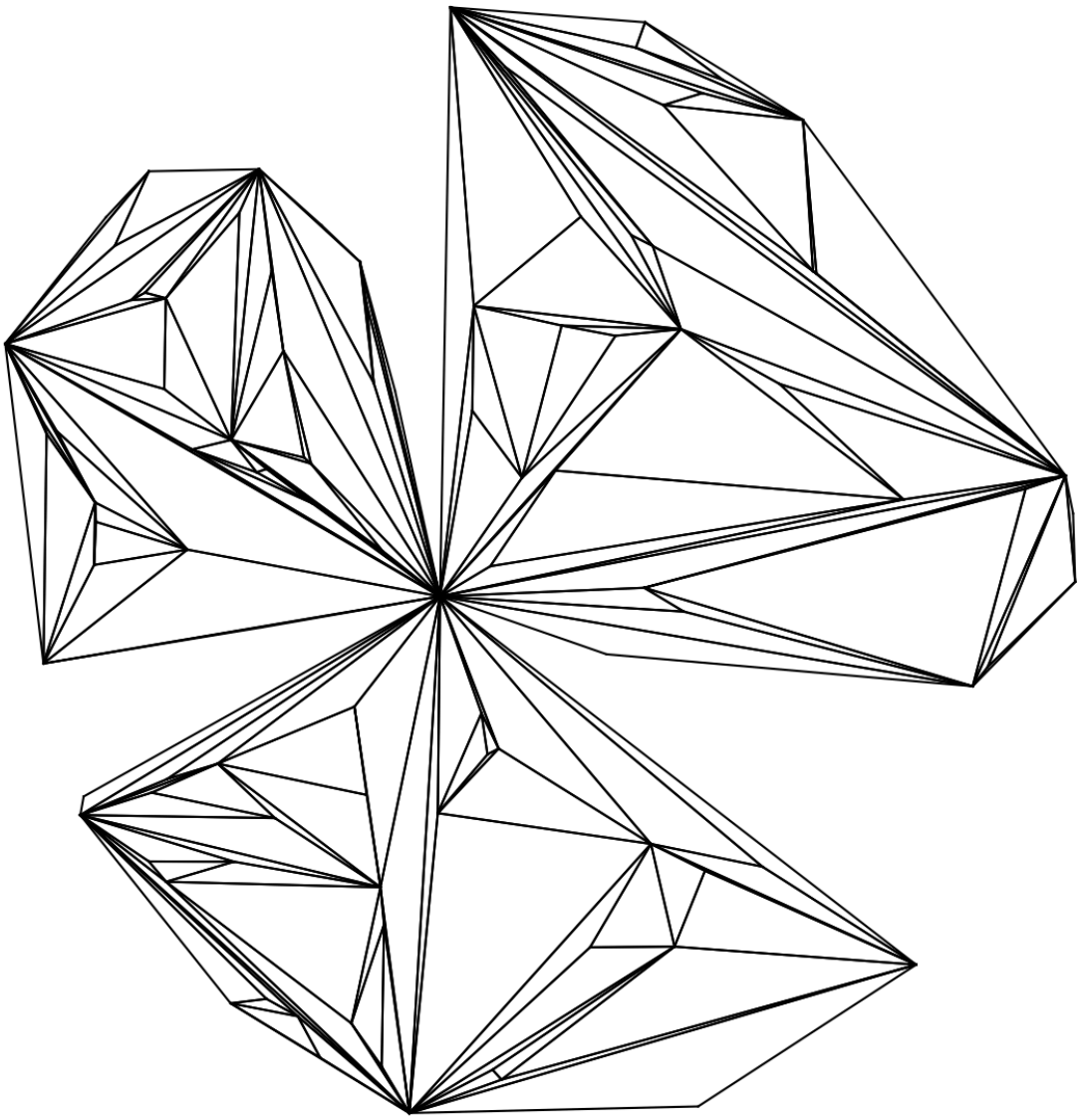
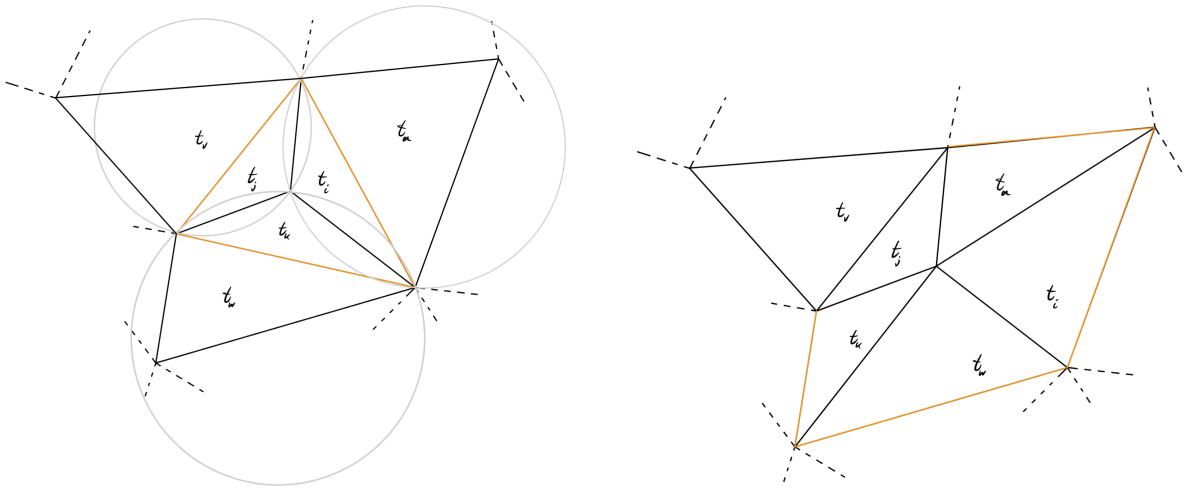


Figure 7: Output from only running the point insertion triangulation algorithm. The additional points added to form the super triangle and triangles containing these points are removed from this uniform distribution of points on a disk.

3.1.2. Flipping

Once the point insertion step is complete, appropirate flipping operations are then performed. Figure 8 illustrates this procedure. One can observe that the new edges introduced by the point insertion do not need to be flipped as they their circumcircles will not contain the points opposite the edge by construction [10] and also would interfere with other triangles if flipped as the configurations are not convex. New edges are chosen to be ones which have not been previously flipped surrounding the point insertion and only need to be checked once [10].



(a) Before flipping. (b) After flipping.

Figure 8: Illustrating the flipping operation. In figure (a), point r has just been inserted and the orange edges are have been marked to be checked for flipping. Two of these end edges end up being flipped in (b). The edges inside would not qualify for flipping as any quadrilateral would not form a convex region.

3.1.3. Implementation

The implementaion was written in a C++ style, without excessive use of object oriented programmingn (OOP) techniques for an gentler trinsition to a CUDA implementaion as CUDA heavily relies on pointer semantics and doesnt not support some of the more convenient OOP. However as CUDA does support OOP features on the host side so the I chose to write a *Delaunay* class which holds most of the important features of the computation as methods which are exectued in the constructor of the *Delaunay* object.

```
__device__ void circumcircle(Point a, Point b, Point c, Point* center, float* r) {

  float ba0 = b.x[0] - a.x[0];
  float ba1 = b.x[1] - a.x[1];
  float ca0 = c.x[0] - a.x[0];
  float ca1 = c.x[1] - a.x[1];

  float det = ba0*ca1 - ca0*ba1;

  det = 0.5 / det;
  float asq = ba0*ba0 + ba1*ba1;
  float csq = ca0*ca0 + ca1*ca1;
  float ctr0 = det*(asq*ca1 - csq*ba1);
  float ctr1 = det*(csq*ba0 - asq*ca0);

  *r = sqrt(ctr0*ctr0 + ctr1*ctr1);
  center->x[0] = ctr0 + a.x[0];
  center->x[1] = ctr1 + a.x[1];
}
```

Listing 1: wow

```
__device__ float incircle(Point d, Point a, Point b, Point c){
  // +: inside  | flip
  // 0: on      |
  // -: outside | dont flip

  Point center;
  float rad;
  circumcircle(a, b, c, &center, &rad);

  // distance from center to d
  float dist_sqr = (d.x[0] - center.x[0])*(d.x[0] - center.x[0])
          + (d.x[1] - center.x[1])*(d.x[1] - center.x[1]);

  return (rad*rad - dist_sqr);
}
```

Listing 2: wow

3.1.4. Analysis

The analysis in this section will be brief but I hope succint as the majority of the work done was involved in the paralleliezd verions of this algorithm showcased in the following sections.

In Figure 9 below we can observe the time complexity of the serial algorithm. This algorithm can theoretically achieve a complexity of $O(n\log(n))$ however my naive implementaion does not achieve this and we have a $O(n^2)$ scaling as seen by the non straight line in the log plot. Even though this is not the result I have expected, this is still a uselfull piece of code to compare the future GPU implementaion with. I believe that a $O(n\log(n))$ complexity can be achived by using a directed acyclic graph structure (DAG) for faster memory access in finding in which triangles points are contained in.

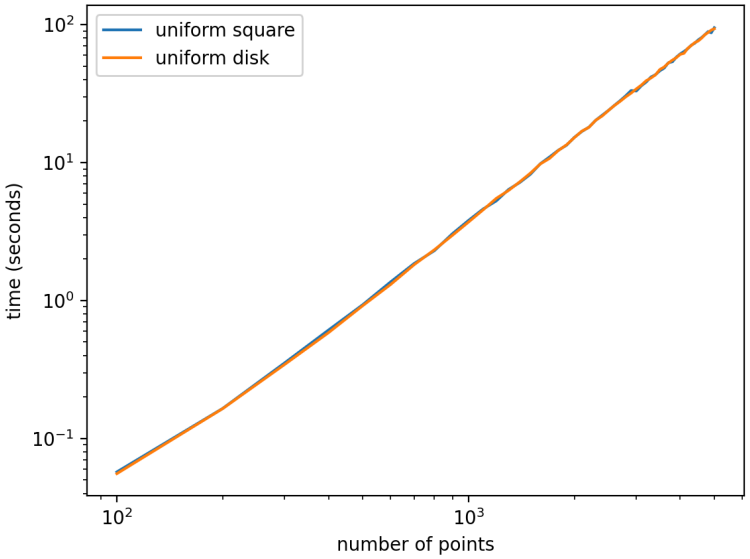


Figure 9: Plot showing the amount of time it took serial code to run with respect to the number of points in the triangulation. **NEED LOG PLOT**

3.2. Parallel

The parallelization of the DT is conceptually not very different than its serial counterpart. We will be considering only parallelization with a GPU here which lends itself to algorithms which are

created with their arichitecures in mind. This means that accesing data will be largely done by accessing global arrays which all threads of execution have access to. Methods akin to divide and conquer [11] would be useful if we consider multi CPU or multi GPU systems but that is not in the scope of this project. An overview of the parallelized algorithm is in Algorithm 4 mostly adapted from [12] which is as of this moment the fastest GPU delaunay triangluation algorithm.

Algorithm 4: Parallel point insertion and flipping

```
   Data: A point set P
1  | Initialize T with a triangle t enclosing all points in P
2  | Initialize locations of p ∈ P to all lie in t
3  | while there are p ∈ P to insert
4  |   for each p ∈ P do in parallel
5  |     | choose p_t ∈ P to insert if any
6  |   for each t ∈ T with p_t to insert do in parallel
7  |     | split t
8  |   while there are illegal edges
9  |     for each triangle t ∈ T do in parallel
10 |       | mark whether it should be flipped
11 |     for each triangle t ∈ T in a configuration marked to flip do in parallel
12 |       | flip t
13 |   update locations of p ∈ P
14 | return T
```

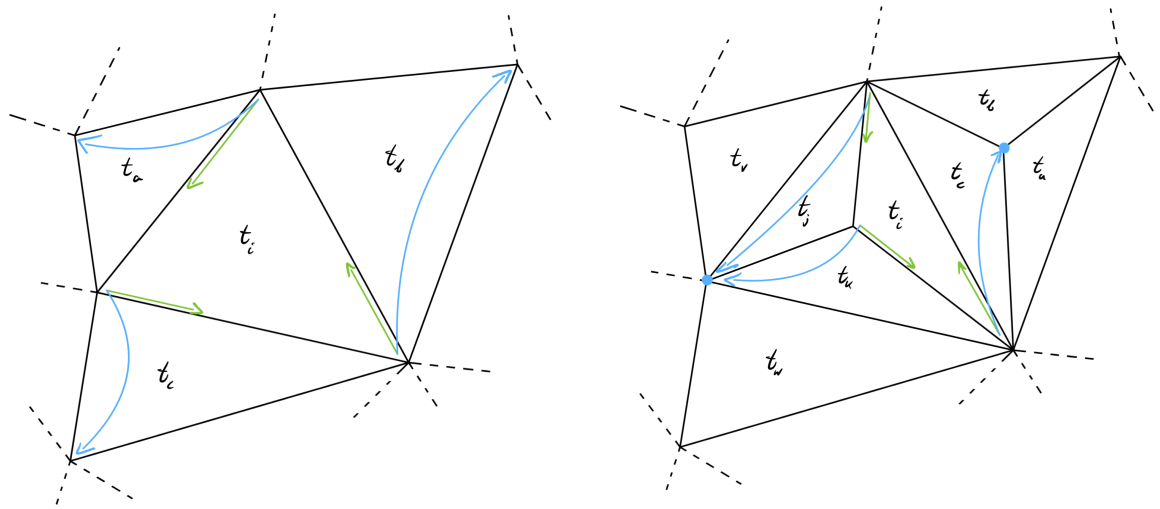
Algorithm 4 is takes as input a point set P for the triangluation to be constructed from and return the DT from the transformed triangulation T. *(line 1)* The triangultion is initialized as a triangle enclosing all points in P by adding 3 new points to the triangulation. These extra three points will later be removed. *(line 2)* Tells us to keep peforming the main work of the algorithm as long as there are points to be inserted into T. *(lines 3-4)* We pick out points in parellel which can be inserted into T by checking in which triangle each point not yet inserted, if any, is closest to the circumcenter of the triangle. This point will be inserted in the *(lines_5-6)* in which for every triangle which has a point inside it to be inserted we split the existing triangle t into 3 new triangles which all contain the inserted point p. Now in *(lines 7-10)* at this point, we have a non Delaunay mesh which needs to be transformed and so we perform neccesary flipping operations in order for this to be a DT. For each triangle we first check whether we should flip with any 3 any of its neighbours by checking if each edge is illegal. If an edge is found to be illegal the first neighbouring triangle is marked to be flipped with. Following this we check whether any triangles marked for flipping would be conlflicting with any other configuration flipping, and if so, it is discarded for this iteration of the while loop. In *(lines 11-12)* we perform the flipping operation for each triangle which wont have any conflicts. At the end of the outermost while loop in *(line 13)* we update our knowlege of where points which have not yet been instered not lie after the chages by the point insertion creating new triangles and flipping changing the triangles themselves.

Algorithm 4 exploits the most parallizable aspects of the point insertion Algorithm 2, which are the point insertion, for which only one triangle is involved in at a time, and the flipping operation, which can be parallized but some book keeping needs to be taken care of in order for confilicting flip to not be performed. With a large point set this parallelization allows for a massively alogorithm as a

large number of point insertions and flips can be performed in parallel. Flipping conlficts can happen when two different configurations of neighbouring triangles want to flip and these two configurations share a triangle, as illustrated in Figure 11.

3.2.1. Insertion

The parellel point insertion step is very well suited for parallelization. Parallel point insertion can be performed without any interfernce with their neighbours. This procedure is performed independantly for each triangle with a point to insert. The only complication arises in the updating of neighbouring triangles information about their newly updated neighbours and opposite points. This must be done after all new triangles have been constructed and saved to memory. Only then you can exploit the data structure and traverse the neighbouring triangle to a update the correct triangles appropirate edge.



(a) Before insertion. (b) After insertion.

Figure 10: Parallel point insertion

Implementation

The implemenation of the parallel point insertion algorithm relies on two steps, preparation of points to be inserted and the insertion of points. If only the point insertion procedure is performed we also need to update point locations which is normally done after the flipping operations needed.

The preparation step ivolves a handful of checks or verifications to find out which point should be inserted into each triangle. In this algorithm we wish to find the most suitable point for each triangle to have inserted into it. We do this by finding out which point, which is not yet inserted into the triangulation lies in which triangle. The point closest to the circumcenter of the triangle is chosen to be inserted. Two CUDA kernels are used in this procedure, one to calculate the distances of each point to their corresponding circumcentres and another to find the minimum distance. This procedure relies on computing the distance twice as compute is cheap on GPU as opposed to copying memory of the triangle structures. In between all of this arrays which contain information about uninserted points *ptsUninsterted* are used throughout in order to not waste resources in the form of threads which would obtain instructions to do nothing. The *ptsUninsterted* array is sorted in order to launch the minimum number of threads needed. A few other kerels are used for book keeping purposes which **arent?** explained.

Once the preperation step is completed, which makes up the majority of the compute for point insertion procedure (Figure 14) we can now actually insert the points which have been pickout out. The logic in the parallel point insertion step is as follows. The logic is mostly consistent as in

Figure 6 but needs to be adapted in order for it to be parallelized. For the creation and rewriting involved in making the 3 new traingles stays the same except for the updating of the neighbouring triangles before insertion. If a neighbouring triangle will not perform a point insertion

Algorithm 5: prepForInsert

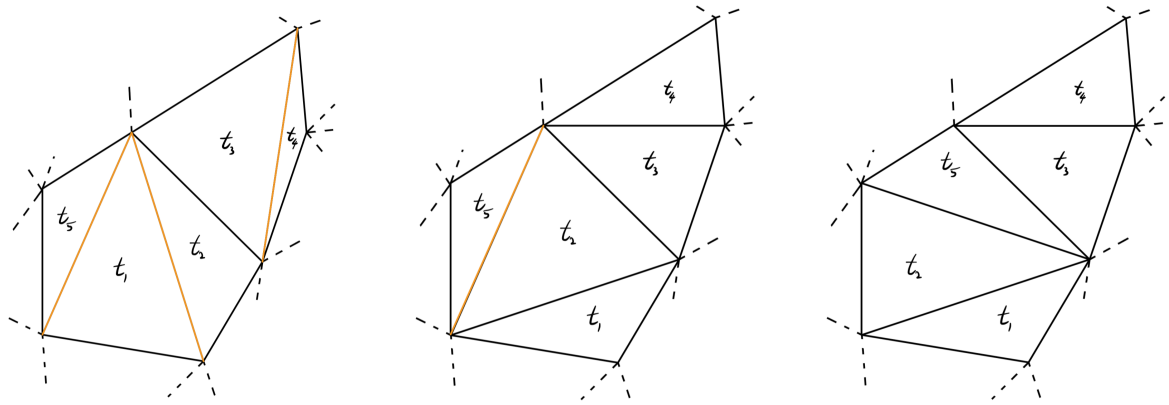
```
1 updatePtsUninsterted
2 gpuSort ptsUninserted_d;
3 setInsertPtsDistance
4 setInsertPts
5 prepTriWithInsert
6 gpuSort triWithInsert_d nTriMax_d;
```

Algorithm 6: insert

```
1 insertKernel
2 updateNbrsAfterIsertKernel
3 update num tri
4 update num pts inserted
```

3.2.2. Flipping

As briefly mentiond earlier, flippig can be performed in a highly parallel manner however some care needs to be taken. The logic within the flippig operation is split up into three main steps. The first one is the reading of triangles to be flipping in the corresponding configuraion into a *Quad* data structure which here is mainly created for the purpose of an easier implementation. This *Quad* strut will aid us in constructing flipped cofiguartion. The the two new triagles are the written by one kernel and appropriate neighbours are then updated in a separate kerel.



(a) (b) (c)

Figure 11: Illustration of parallel flipping while accounting for flipping non conflicting configurations. Edges colored orange are marked for flipping. For each configuration marked for flipping by each orange edge the triangle with the smallest index will be the one performing the flipping operation, and the configuration with the smallest index (min of both indexes of triangles the configuration) will have priority to flip first in each round of parallel flipping. In the first figure (a) 3 edges are marked for flipping. Only configurations of triangles t_1t_2 and t_3t_4, with configuration indexes 1 and 3 respectively, will flip. Configuration t_5, t_1 with a configuration index of 1 will not flip in the first parallel flipping iteration (b) as it is not the minimum index in its configuration. (c) Showcases the final outcome of the parallel flipping.

3.2.3. Implementation

Flipping

3.2.4. Analysis

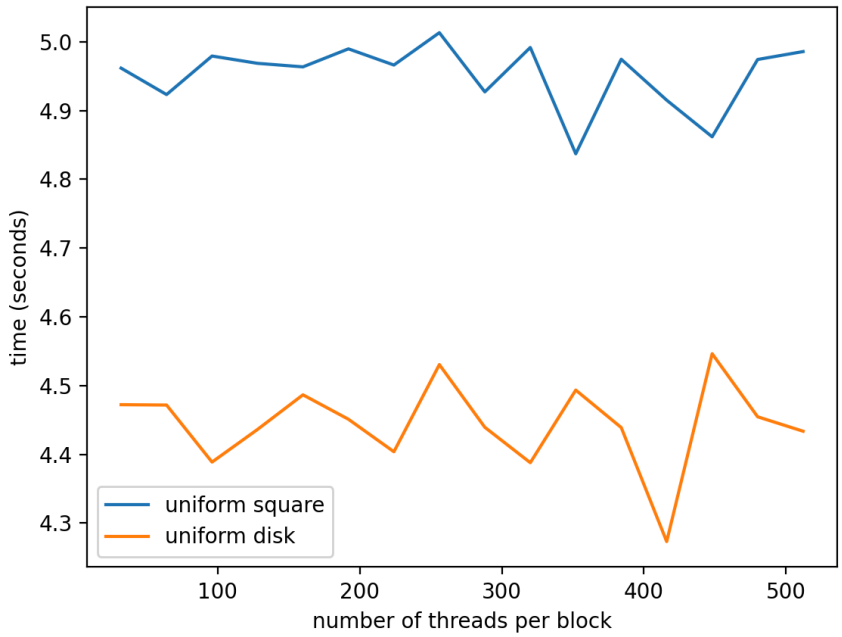


Figure 12: Showing the time it took for the GPU DT code to run with 10000 points while varying the number of threads per block also know as the block size. This is a rather naiive way of finding the optimal number of threads per block as a better analysis **TODO** would involve logically similar block of code to have their own block size. Currently the block size doesn't rationally affect the runtime.

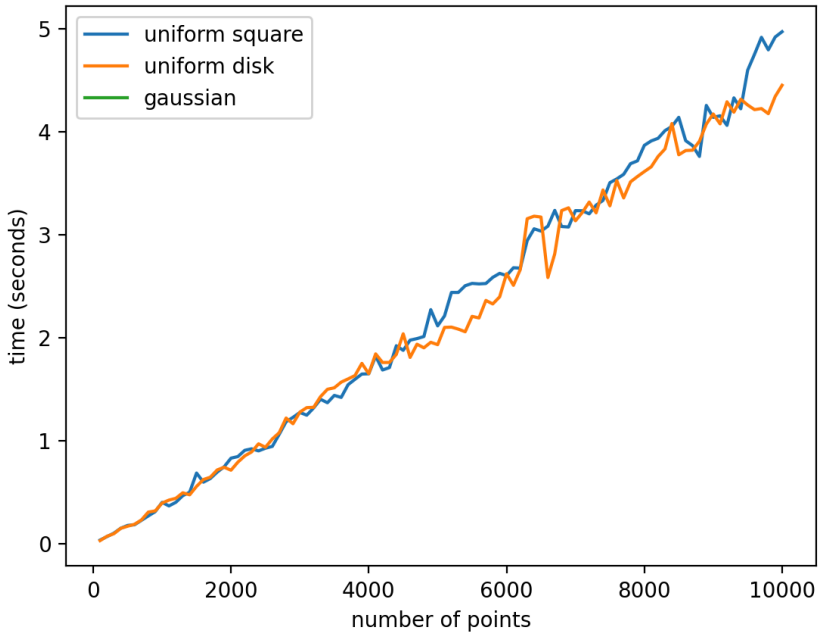


Figure 13: Plot showing the amount of time it took the GPU code to run with respect to the number of points in the triangulation.

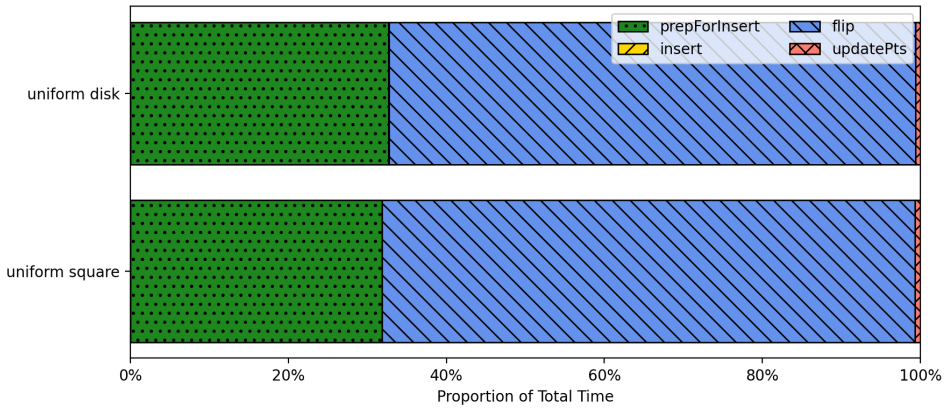


Figure 14: Showing the proportions of time each function took as a percentage of the total runtime.

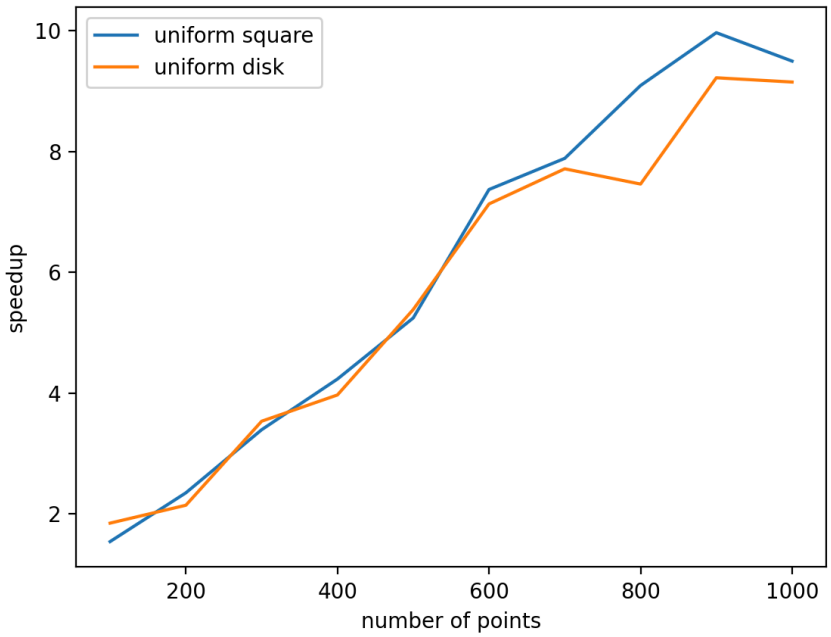
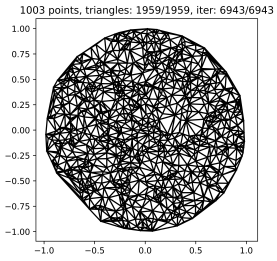
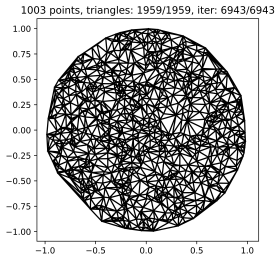


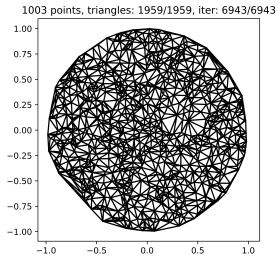
Figure 15: Plot showing speedup of the GPU code with respect to the serial implemenation of the incremental point insertion Algorithm 2. The speedup here is comparing the runtime of the serial code with for a given number of points and with the runtime of the GPU code with the same number of points. Both implementaions are run with single precision floating point arithmetic. Speedup here is defined as the ratio $\frac{\text{timeCPU}}{\text{timeGPU}}$.



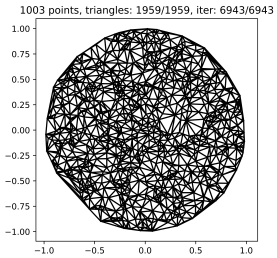
(a) Uniform distribution of unit square.



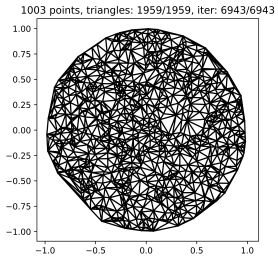
(b) Uniform distribution of unit square.



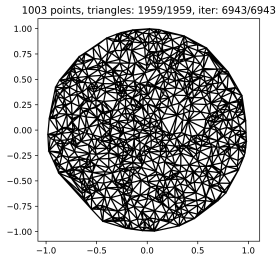
(c) Uniform distribution of unit square.



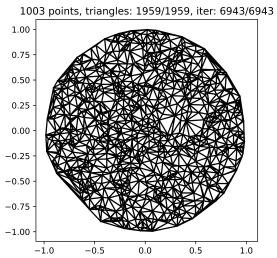
(d) Uniform distribution of unit Disk.



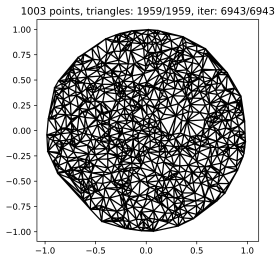
(e) Uniform distribution of unit Disk.



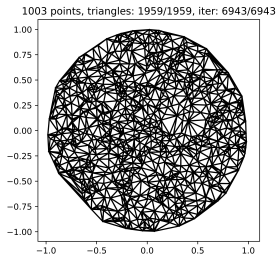
(f) Uniform distribution of unit Disk.



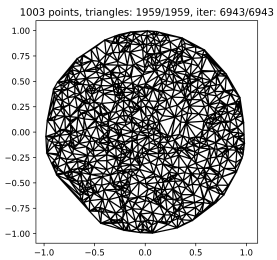
(g) Unit Sphere projected onto unit disk.



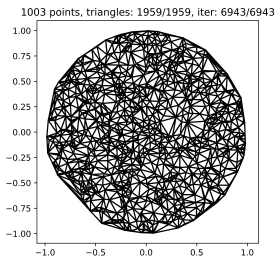
(h) Unit Sphere projected onto unit disk.



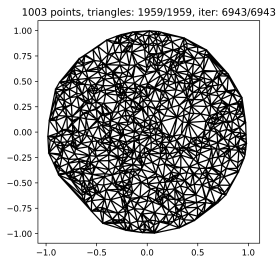
(i) Unit Sphere projected onto unit disk.



(j) Gaussian distribution with mean 0 and variance 1.



(k) Gaussian distribution with mean 0 and variance 1.



(l) Gaussian distribution with mean 0 and variance 1.

Figure 16: Visualisations of Delaunay triangluations of various point distributions.

3.3. Data Structures

The core data structure that is needed in this algorithm is one to represent a the triangulation itself. There are a handful of different approaches to this problem inculding representing edges by the qaud edge data structure [10] however we choose to represent the triangles in our triangulation by explicit triangle structures [13] which hold neccesary information about their neighbours for the construction of the trianulation and for performing point insertion and flipping operations.

```
struct Tri {
  int p[3]; // points
  int n[3]; // neighours
  int o[3]; // opposite points
};
```

Listing 3: Triangle data structure. Stores point index info, the indexes of its neighbouring triangles and the points opposite its edges by the index of that point in the corresponding neighbour.

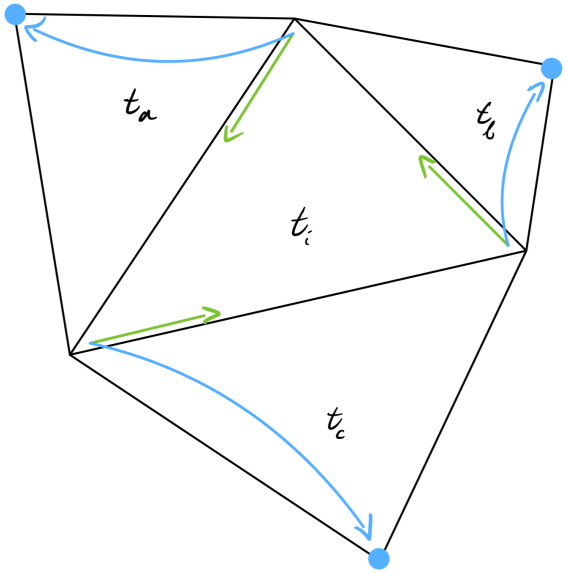


Figure 17: An illustration of the *Tri* data structures main features. We describe the triangle t_i int the figure. Oriented counter clockwise points are stored as indexes an array containing two dimensional coordinate represeting the point. The neighbours are assigned by using the right hand side of each edge using and index of the point as the start of the edge and following the edge in the CCW direction. The neighbours index will by written in the corresponding entry in the structure.

This data structure was chosen for the ease of implementation and as whenever we want to read a traigle we will be a significant amount of data about it and this locality theoreitcally helps with memory reads, as opposed to storing separate parts of date about the triangle in different structures, ie separating point and neighbour information into two different structs.

The Listing 4 below

```
struct Quad {
  int p[4];
  int n[4];
  int o[4];
};
```

Listing 4: Quad data structure. Stores point index info, the indexes of its neighbouring triangles and the points opposite its edges by the index of that point in the corresponding neighbour.

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