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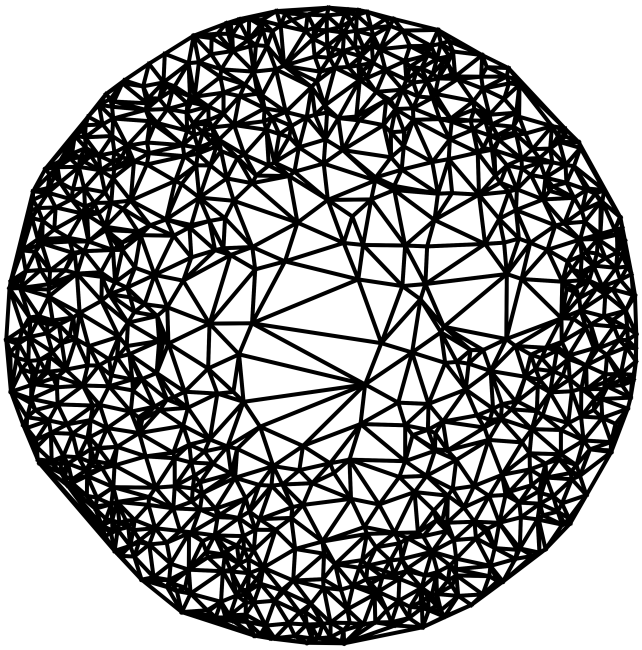
Delaunay Triangulations on the GPU

In partial fulfillment of MSc in High-performance Computing in the School of Mathematics



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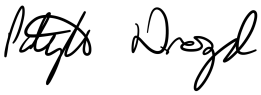

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Abstract

Triangulations of a set of points are a very useful mathematical construction to describe properties of discretized physical systems, such as modelling terrains, cars and wind turbines which are a commonly used for simulations such as computational fluid dynamics and even have use in video games for rendering and visualising complex geometries. To paint a picture, you may think of a triangulation of a set of points P to be a bunch of line segments connecting each point in P in a way such that the edges are non intersecting. A particularly interesting subset of triangulations are Delaunay triangulations (DT). The Delaunay triangulation is a triangulation which maximises all angles in each triangle of the triangulation. Mathematically this gives us an interesting optimization problem which leads to some rich mathematical properties, at least in 2 dimensions, and for alot of applications we are provided with a good way to discretize space for the case of simulations for use in methods such as Finite Element and Finite Volume methods. Delaunay triangulations in particular are a good candidate for these numerical methods as they provide us with fat triangles, as opposed to skinny triangles, which can serve as good elements in the Finite Element method as they tend to improve accuracy of the solvers [1].

There are many algorithms which compute Delaunay triangulations , however a lot of them use the operation of 'flipping' or originally called an 'exchange' [2]. This is a fundamental property of moving through all triangulations of a set of points to with the goal of obtaining the Delaunay triangulation. This flipping operation involves a configuration of two triangles sharing an edge, its boundary forming a quadrilateral. The shared edge between these two triangles will be swapped or flipped from the two points at its end to the other two points on the quadrilateral. The original algorithm, motivated by Lawson[2], hints to us this flipping operation to iterate through different triangulations and eventually arrive at the Delaunay triangulation which we desire.

With the flipping operation being at the core of the algorithm, we can notice that is has the possibility of being parallelized. This is desirable as problems which commonly use the DT are run with large datasets, in this case a large set of points, and can benefit from the highly parallelisable nature of this algorithm. If we wish to parallelize this algorithm, and start with some initial triangulation, conflicts would only occur if we chose to flip a configuration of triangles which share a triangle. With some care, this is an avoidable situation leads to a massively parallelisable algorithm. In our case the hardware of choice will be the GPU which is designed with the SIMT model which is particularly well suited for this algorithm as we are mostly performing the same operations in each iteration of the algorithm in parallel.

The goal of this project was to explore the Delaunay triangulations through both serial and parallel algorithms with the goal of presenting a easy to understand, sufficiently complex parallel algorithm designed with Nvidia's CUDA programming model for running software on their GPUs.

1. Delaunay triangulations

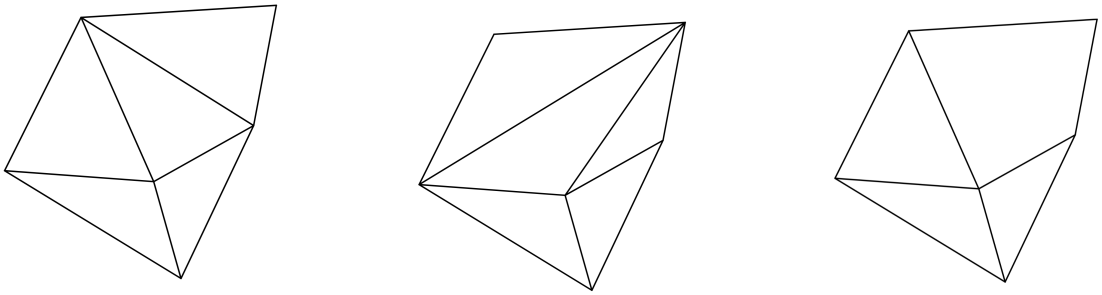
In this section I aim to introduce triangulations and Delaunay triangulations from a mathematical perspective with the foresight to help present the motivation and inspiration for the key algorithms used in this project. In order to introduce the Delaunay Triangulation we first must define what we mean by a triangulation. In order to create a triangulation we need a set of points which will make up the vertices of the triangles. First we want to clarify a possible ambiguity about edges.

Definition 1.1: For a point set P, the term edge is used to indicate any segment that includes precisely two points of S at its endpoints. [3].

Alternatively we could say an edge doesn't contain its endpoints which could be more useful in different contexts. Now we define the triangulation.

Definition 1.2: A *triangulation* of a planar point set P is a subdivision of the plane determined by a maximal set of non-crossing edges whose vertex set is P [3].

This is a somewhat technical but precise definition. The most important point in Definition 1.2 is that it is a *maximal* set of non crossing edges which for us means that we will not have any other shapes than triangles in this structure.



(a) (b) (c)

Figure 1: Examples of two triangulations (a) (b) on the same set of points. In (c) an illustration of a non maximal set of edges.

A useful fact about triangulations is that we can know how many triangles our triangulation will contain if given a set of points and its convex hull. For our purposes the convex hull will always be a set which will covering a set of points, in our case the points in our triangulation. This will be useful when we will be storing triangles as we will always know the number of triangles that will be created and will need to be stored.

Theorem 1.1: Let P be a set of n points in the plane, not all collinear, and let k denote the number of points in P that lie on the boundary of the convex hull of P. Then any triangulation of P has $2n - 2 - k$ triangles and $3n - 3 - k$ edges. [4]

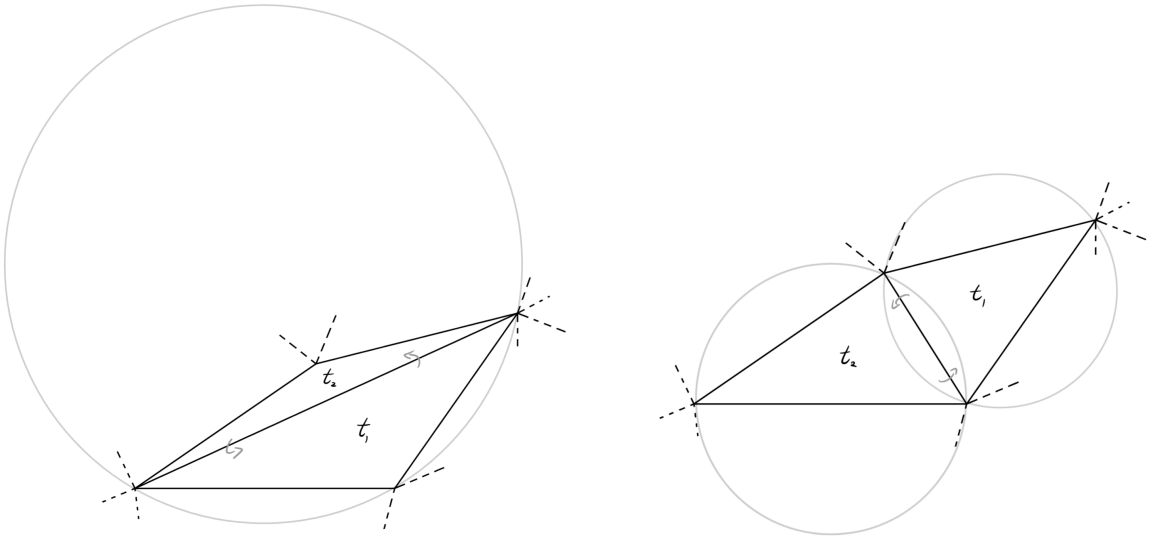
A key feature of all of the Delaunay triangulation theorems we will be considering is that no three points from the set of points P which will make up our triangulation will lie on a line and also that no 4 points like on a circle. Motivation for this definition will become more apparent in Theorem 1.2 and following. Definition 1.3 lets us imagine that our points are distributed randomly enough so that our algorithms will work with no degeneracies appearing. This leads us to the following definition.

Definition 1.3: A set of points P is in *general position* if no 3 points in P are colinear and that no 4 points are cocircular.

From this point onwards we will always assume that the point set P from which we obtain our triangulation will be in *general position*. This is necessary for the definitions and theorems we will define.

In order to define a Delaunay triangulation we would like to establish the motivation for the definition with another, preliminary definition. A Delaunay triangulation is a type of triangulation which in a sense maximizes smallest angles in a triangulation T. This idea is formalized by defining an *angle sequence* $(\alpha_1, \alpha_2, ..., \alpha_{3n})$ of T which is an ordered list of all angles of T sorted from the smallest to largest. With angle sequences we can now compare two triangulations to each other. We can say for two triangulations T_1 and T_2 we write $T_1 > T_2$ (T_1 is fatter than T_2) if the angle sequence of T_1 is lexicographically greater than T_2. Now we can compare triangulations. And by defining Definition 1.4 are able to define a *Delaunay triangulation.*

Definition 1.4: Let e be an edge of a triangulation T_1 , and let Q be the quadrilateral in T_1 formed by the two triangles having e as their common edge. If Q is convex, let T_2 be the triangulation after flipping edge e in T_1. We say e is a *legal edge* if $T_1 \geq T_2$ and e is an *illegal edge* if $T_1 < T_2$ [3]



(a) (b)

Figure 2: Demonstration of the flipping operation for its use in Theorem 1.2. In (a) A configuration that needs to be flipped illustrated by the circumcircle of t_1 containing the auxiliary point of t_2 in its interior. In (b) configuration (a) which has been flipped and no longer needs to be flipped as illustrated by the both circumcirles of t_1 and t_2.

Definition 1.5: For a point set P, a *Delaunay triangulation* of P is a triangulation that only has legal edges. [3]

With Definition 1.5, Delaunay triangulations wish to only contain legal edges and this provides us with a “nice” triangulation with fat triangles.

Theorem 1.2 *(Empty Circle Property)*: Let P be a point set in general position. A triangulation T is a Delaunay triangulation if and only if no point from P is in the interior of any circumcircle of a triangle of T. [3]

Theorem 1.2 is the key ingredient in the Delaunay triangulation algorithms we are going to use. This is because instead of having to compare angles, as would be demanded by Definition 1.5, we are allowed to only perform a computation, involving finding a circumcicle and performing one comparison which would involve determining whether the point not shared by triangles circumcircle is contained inside the circumcircle or not. Algorithms such as initially introduced by Lawson [5] exist which do focus on angle comparisons but are not preferred as they do not introduce desired locality and are more complex.

And finally we present the theorem which guarantees that we will eventually arrive at our desired Delaunay triangluation by stating that we can travel across all possible triangulations of our point set P by using the flipping operation.

Theorem 1.3 *(Lawson)*: Given any two triangulations of a set of points P, T_1 and T_2, there exist a finite sequence of exchanges (flips) by which T_1 can be transformed to T_2. [2]

2. The GPU

The Graphical Processing Unit (GPU) is a type of hardware accelerator originally used to significantly improve running video rendering tasks for example in video games through visualizing the two or three dimensional environments the player would be interacting with or rendering videos in movies after the addition of visual effects. Many different hardware accelerators have been tried and tested for more general use, like Intel's Xeon Phis, however the more purpose oriented GPU has prevailed in the market and in performance mainly lead by Nvidia in previous years. Today, the GPU has gained a more general purpose status with the rise of General Purpose GPU (GPGPU) programming as more and more people have noticed that GPUs are very useful as a general hardware accelerator.

The traditional CPU, based on the Von Neumann architecture, which is built to perform *serial* tasks , the CPU is built to be a general purpose hardware for performing all tasks a user would demand from the computer. In contrast the GPU can't run alone and must be used in conjunction to the CPU. The CPU sends compute instructions for the GPU to perform and data commonly passes between the CPU and GPU when performing computations.

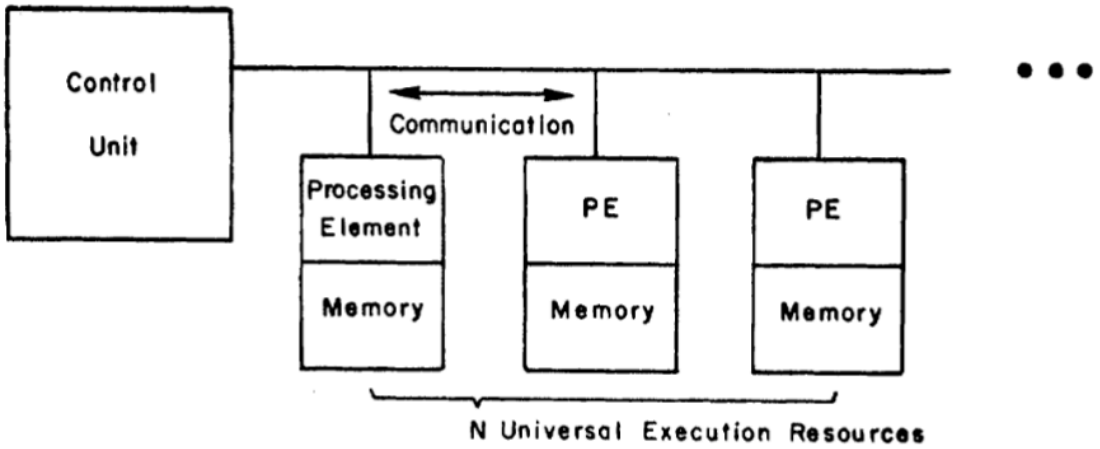


Figure 3: The *Single Instruction Multiple Threads (SIMT)* classification, originally known as an *Array Processor* as illustrated by Michael J. Flynn [6]. The control unit communicates instructions to the N processing element with each processing element having its own memory.

What makes the GPU incredibly useful in certain use cases, like the one of this thesis, is its architecture which is build to enable massively parallelizable tasks. In Flynn's Taxonomy [6], the GPUs architecture is based a subcategory of the Single Instruction Multiple Data (SIMD) classification known as Single Instruction Multiple Threads (SIMT) also known as an Array Processor. The SIMD classification allows for many processing units to perform the same tasks on a shared dataset with the SIMT classification additionally allowing for each processing unit having its own memory allowing for more diverse processing of data.

Nvida's GPUs take the SIMT model and further develop it. There are three core abstractions which allow Nvidia's GPU model to be successful; a hierarchy of thread groups, shared memories and synchronization [7]. The threads, which represents a theoretical processes which encode programmed instructions, are launched together in groups of 32 known as *warps*. This is the smallest unit of physical instructions that is executed on the GPU, in contrast to a single thread of execution which can also be executed but must be run alongside 31 other processes. The threads are further grouped into *thread blocks* which are used as a way of organising the *shared memory* to be used by each thread in this thread block. And once more the *thread blocks* grouped into a *grid*. This

hierarchy of memories and units of instructions allows the GPU be significantly faster for suitable algorithms than their CPU equivalent. Along side the compute and memory hierarchies mentioned the GPU code can also be run asynchronously, to allow for instruction coalescence and contains many more other types of memory storage options which are suitable for more specific tasks .*texture* and *surface* memory whose names are derived from their applications in video game programming applications have built in interpolation features and *texture* is read only which allows the developer of code to really increase the performance of their code.

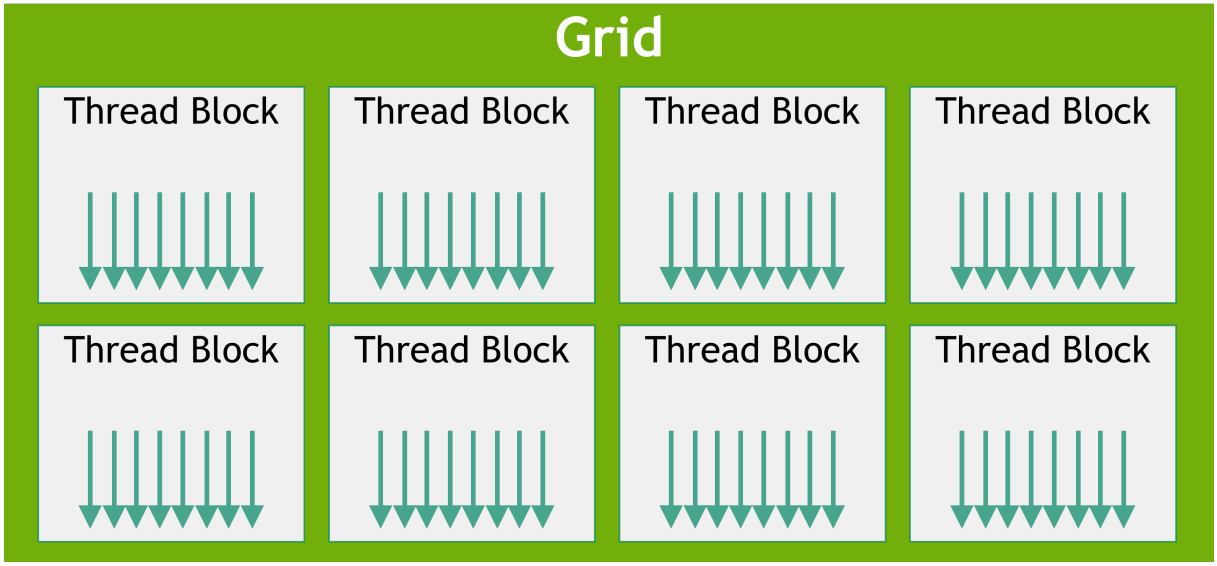


Figure 4: An illustration of the structure of the GPU programming model. As the lowest compute instruction we have a thread block consisting of a number threads ≤ 1024. The thread blocks are contained in a grid. [7]

Unlike parallel CPU programming models such as OMP and MPI, CUDA which is Nvidia's programming API for developing software for their GPUs, most of the time creating modified algorithms for the GPUs architecture is necessary if we begin with a serialized algorithm. Even though programming parallel CPU code also requires the development of a modified algorithm, in my experience, most of the time the existing serial algorithm is divided among CPU cores and message passing between these cores is the most performance critical aspect of the code. Most of the skill in developing GPU code is in making efficient use of the correct memory locations on the GPU and keeping in mind the SIMT programming model. In the case of the GPU, code is run in lock step which means if the kernel has multiple possible execution paths, which can be introduced by programming language features such as *if* statements or variable lenght *for* loops, the cores in a streaming multiprocessor will only execute one the of the if statements which others will lay doing nothing, which defeats the entire purpose of the parallel execution of threads. Because of these features, programming for GPUs is more restrictive but also allows for very large speedups. Some common good practices for programming for GPUs include using short *kernel* calls (the *kernel* is a function which runs on the GPU) but extremely spread out problems over the cores of the GPU. Making use of the closest memory locations and not using *global* memory for reading large chunks of memory and using the locality of memory reads. And when applicable use asynchronous *kernel* calls so that the GPU is using its compute and not waiting for memory transfers.

3. Algorithms

In this section we focus on two types of algorithms, serial and parallel, but with a major focus on the parallel algorithm. Commonly algorithms are first developed with a serialized version and only later optimized into parallelized versions if possible. This is how I will be presenting my chosen Delaunay triangulation algorithms in order to portray a chronological development of ideas used in all algorithms. And so we first begin by explaining the chosen serial version of the DT algorithm.

3.1. Serial

The simplest type of DT algorithm can be stated as follows in Algorithm 1

Algorithm 1: Lawson Flip algorithm

Let P be a point set in general position. Initialize T as any triangulation of P. If T has an illegal edge, flip the edge and make it legal. Continue flipping illegal edges, moving through the flip graph of P in any order, until no more illegal edges remain. [3]

This algorithm presents with a bit of ambiguity however I believe its a good algorithm to keep in mind when progressing to more complex algorithms as it presents the most important feature in a DT algorithm, that is, checking if an edge in the triangulation is legal, and if its not, we flip it. Most DT algorithms take this core concept and build a more optimized versions of it with as Algorithm 1 has a complexity of $O(n^2)$ [8].

The next best serial algorithm commonly presented by popular textbooks [4], [9] is the *randomized incremental point insertion* Algorithm 2. When implemented properly this algorithm should have a complexity of $O(n \log(n))$ [4]. This algorithm is favoured for its relative low complexity and ease of implementation . The construction this algorithm is a bit mathematically involved however the motivation behind the construction of the algorithm is to perform point insertions, and after each point insertion we perform necessary flips to transform the current triangulation into a DT. This in turn reduces the number of flips we need to perform and this is reflected in the runtime complexity.

Algorithm 2: Randomized incremental point insertion

Data: point set P
Out: Delaunay triangulation T

1 Initialize T with a triangle enclosing all points in P
2 Compute a random permutation of P
3 **for** $p \in P$
4 Insert p into the triangle t containing it
5 **for** each edge $e \in t$
6 LegalizeEdge(e, t)
7 return T

A significant part of this algorithms the FlipEdge function in Algorithm 3. This function performs the necessary flips, both number of and on the correct edges, for the triangulation in the current iteration of point insertion to become a DT.

Algorithm 3: LegalizeEdge

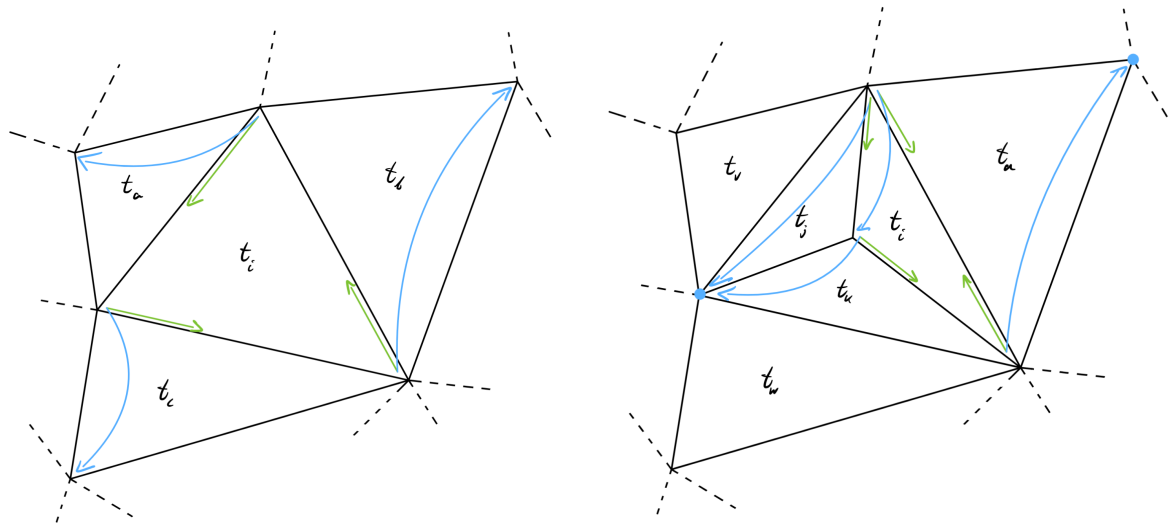
Data: egde e, triangle t_a

1 **if** e is illegal
2 *flip* with triangle t_b across edge e
3 let e_1, e_2 be the outward facing edges of t_b
4 LegalizeEdge(e_1, t_b)
5 LegalizeEdge(e_2, t_b)

The constructions necessary to explain why the *LegalizeEdge* routine created a DT is again slightly mathematically involved but is discussed in [4]. In the following sections we will discuss the point insertion and flipping steps in more detail.

3.1.1. Point insertion

Point insertion procedure goes as follows. An initial triangulation is necessary to begin advance in the point insertion procedure. This is commonly done by adding 3 extra points to our triangulation from which we will construct a *supertriangle* which will contain all of the point in the set we wish to construct the DT. These extra 3 points will later be removed. In our triangulation if there is a point not yet inserted we choose to use it to split the existing triangle in which this point lies in into 3 new triangles. This process is repeated until no more points are left to insert. The point insertion step would be followed by the *LegaiseEdge* procedure. Figure 5 illustrates this process.



(a) Before insertion. (b) After insertion.

Figure 5: An illustration of the point insertion in step 4 of Algorithm 2. In figure (a) the center most triangle t_i will be then triangle in which a point will be chosen for insertion. Triangle t_i knows its neighbours across each edge represented by the green arrows and knows the points opposite each of these edges. After the point it inserted (b), t_i is moved and two new triangles t_j, t_k are created to accommodate the new point. Each new triangle t_i, t_j, t_k can be fully constructed from the previously existing t_i and each neighbour of t_i in (a) has its neighbours updated to reflect the insertion. The neighbouring triangles opposite points are updated by accessing the opposite point across the edge of the neighbouring triangle and obtaining the index of the edge which has the triangle currently being split. The index of the opposite point will always be 0 by construction. The neighbouring triangle is also updated similarly but with the appropriate index which will be the one of the triangle who's modifying the neighbouring triangle.

It might be nice to see results from just running the point insertion algorthim by itself, without the flipping which would take place in between which will be further explored in the next section. In Figure 6 we see the result after the super triangle points and their corresponding triangles have been removed. It is good to note that the point insertion algorithm is in general a triangulation algorithm. The state of the triangulation in Figure 6 is not particularly useful in any applications I seen but I thought I must include it in order to show an intermediate step in the process of the construction of the complete algorithm.

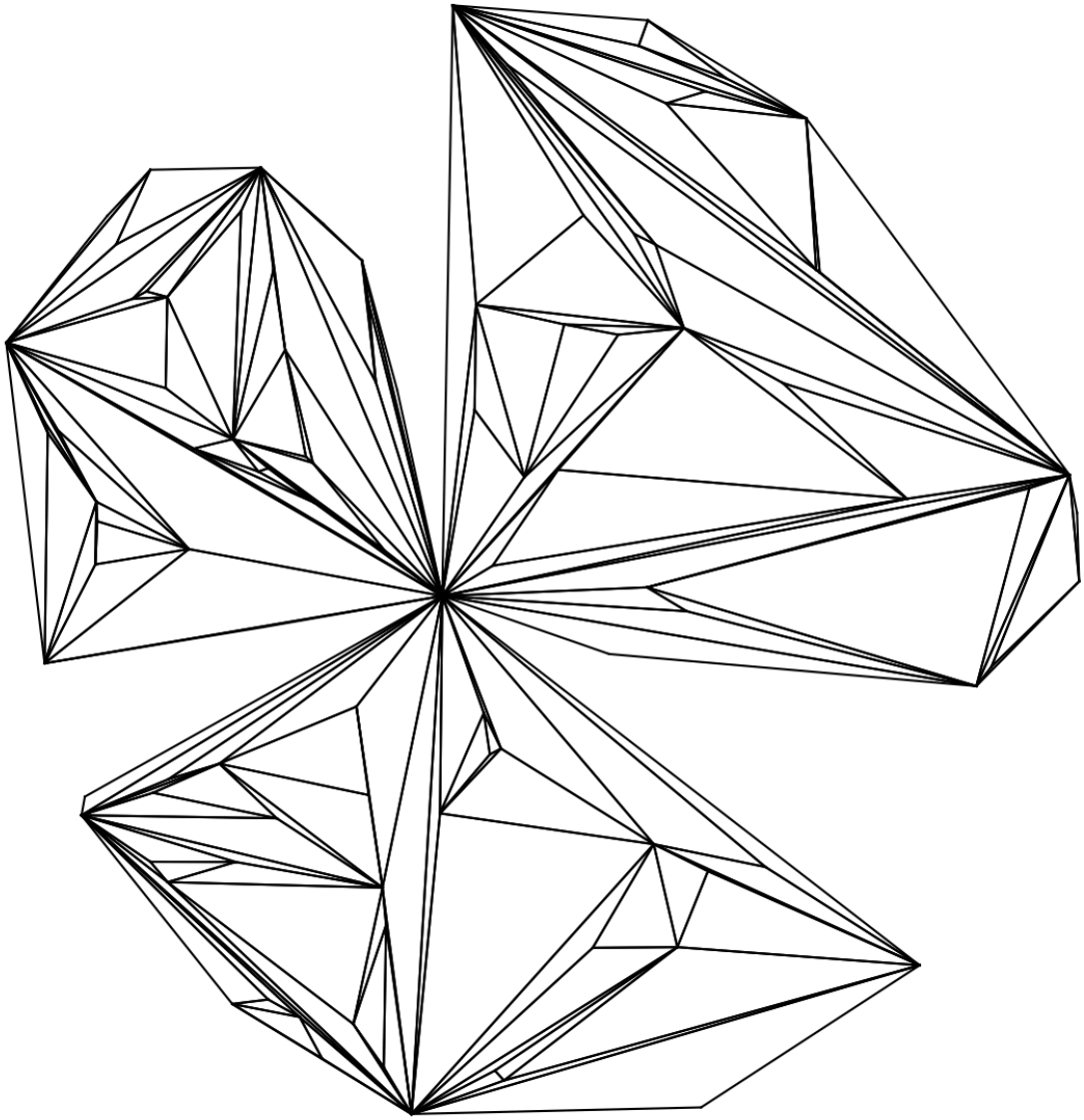
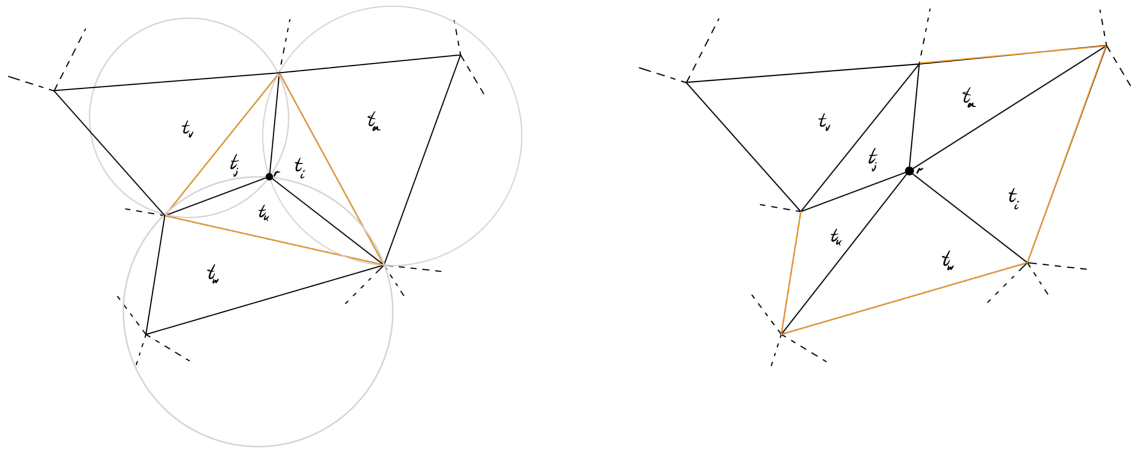


Figure 6: Output from only running the point insertion triangulation algorithm. The additional points added to form the super triangle and triangles containing these points are removed from this uniform distribution of points on a disk.

3.1.2. Flipping

Once a point insertion step is complete, appropriate flipping operations are then performed. Figure 7 illustrates this procedure. One can observe that the new edges introduced by the point insertion do not need to be flipped as they their circumcircles will not contain the points opposite the edge by construction [10] and also would interfere with other triangles if flipped as the configurations are

not convex. New edges are chosen to be ones which have not been previously flipped surrounding the point insertion and only need to be checked once.



(a) Before flipping. (b) After flipping.

Figure 7: Illustrating the flipping operation. In figure (a), point r has just been inserted and the orange edges are have been marked to be checked for flipping. Two of these end edges end up being flipped in (b). The edges inside would not qualify for flipping as any quadrilateral would not form a convex region. In order to perform the edge flipping algorithm we choose to construct the two new triangles which would form after the edge flip and the overwrite the two triangles which should no longer exist. A useful construction for ease of implementation and readability is to create a temporary *quad* data structure which contains the necessary information for constructing the new triangles. The existing edge can be thought of a being rotated counter clockwise which lets us know where the indexes of the previous triangles are being overwritten to in the array of *tri* structures in later described in Listing 1. Most of the triangles can be constructed internally but the neighbouring triangles also need to have their neighbour information and points are opposite the neighbouring edge updated.

The implementation was written in C++ and was not written with a large amount of object oriented programming (OOP) techniques for an gentler transition to a CUDA implementation as CUDA heavily relies on pointer semantics and does not support some of the more convenient OOP features. However as CUDA does support OOP features on the host side so the I chose to write a *Delaunay* class which holds most of the important features of the computation as methods which are executed in the constructor of the *Delaunay* object.

3.1.3. Analysis

The analysis in this section will be brief but I hope succinct as the majority of the work done was involved in the parallelized versions of this algorithm showcased in the following sections.

In Figure 8 below we can observe the time complexity of the serial algorithm. This algorithm can theoretically achieve a complexity of $O(n\log(n))$ however my naive implementation does not achieve this and we have a $O(n^2)$ scaling as seen by the straight line in the log plot. Even though this is not the result I have hoped for, this is still a useful piece of code to compare the future GPU implementation with. I believe that a $O(n\log(n))$ complexity can be achieved by using a directed acyclic graph structure (DAG) for faster memory access in finding in which triangles points are contained in.

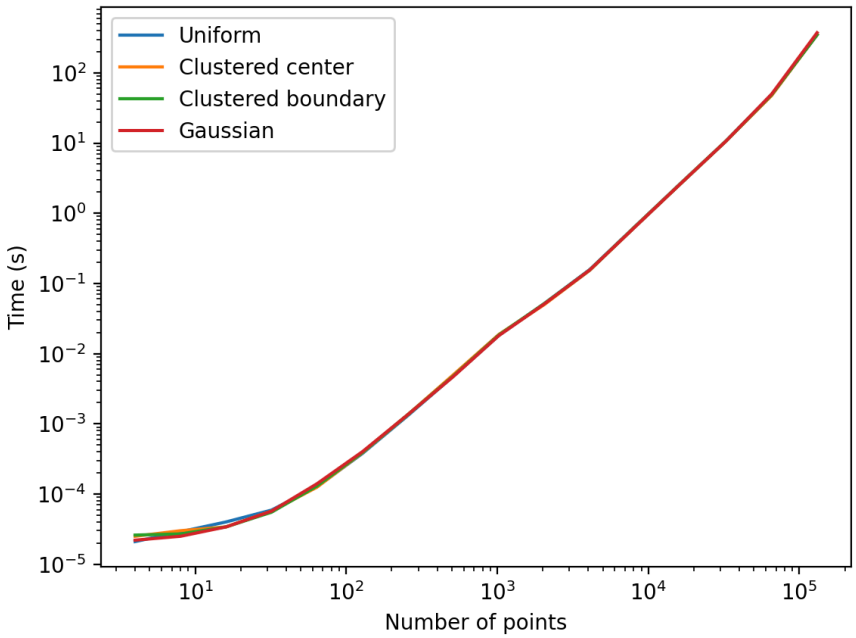


Figure 8: Plot showing the amount of time it took serial code to run with respect to the number of points in the triangulation. From this plot we can see that when run with different point distributions the algorithm takes essentially the same amount of time to complete. The nature of the algorithm is to pick a point each iteration and perform the same operations around it so it is not surprising that different point distributions don't affect the runtime. The slight increase in change of runtime noticed between 10 and 10^2 is due to the increased memory demands of the executable and is reflected by the increased time accessing memory from other cache locations.

3.2. Parallel

The parallelisation of the DT is conceptually not very different than its serial counterpart. We will be considering only parallelisation with a GPU here which lends itself to algorithms which are created with a GPUs architecture in mind. This means that accessing data will be largely done by accessing global arrays which all threads of execution have access to. Methods akin to divide and conquer [11] would be useful if we consider multi CPU or multi GPU systems. However that is not in the scope of this project but would be particularly interesting to see a multi GPU systems implementation for this algorithm made publicly available. An overview of the parallelized algorithm is in Algorithm 4 mostly adapted from [12] which is to my understanding as of this moment the fastest GPU DT algorithm.

Algorithm 4: Parallel point insertion and flipping

```
   Data: A point set P
   Out: Delaunay Triangluation T
1  Initialize T with a triangle t enclosing all points in P
2  Initialize locations of p ∈ P to all lie in t
3  while there are p ∈ P to insert
4      for each p ∈ P do in parallel
5          choose p_t ∈ P to insert if any
6      for each t ∈ T with p_t to insert do in parallel
7          split t
8      while there are illegal edges
9          for each triangle t ∈ T do in parallel
10             mark whether it should be flipped
11         for each triangle t ∈ T in a configuration marked to flip do in parallel
12             flip t
13     update locations of p ∈ P
14 return T
```

Algorithm 4 takes as input a point set P for the triangulation to be constructed from and return the DT from the transformed triangulation T. *(line 1)* The triangulation is initialized as a triangle enclosing all points in P by adding 3 new points to the triangulation and is constructed in a way such that all of the other points lie inside this triangle which is noted in *(line 2)*. These extra three points will later be removed. *(line 3)* Tells us to keep performing the main work of the algorithm as long as there are points to be inserted into T. *(lines 4-5)* We pick out points in parallel which can be inserted into T by checking in which triangle each point not yet inserted, if any, is closest to the circumcenter of the triangle. This point will be inserted in the *(lines_6-7)* in which for every triangle which has a point inside it to be inserted we split the existing triangle t into 3 new triangles which all contain the inserted point p. Now in *(lines 8-12)* at this point, we have a non Delaunay mesh which needs to be transformed and so we perform necessary flipping operations in order for this to be a DT. For each triangle we first check whether we should flip with any 3 any of its neighbours by checking if each edge is illegal. If an edge is found to be illegal the first neighbouring triangle is marked to be flipped with. Following this we check whether any triangles marked for flipping would be conflicting with any other configuration flipping, and if so, it is discarded for this iteration of the while loop. In *(lines 11-12)* we perform the flipping operation for each triangle which wont have any conflicts. At the end of the outermost while loop in *(line 13)* we update our knowledge of where

points which have not yet been inserted not lie after the changes by the point insertion creating new triangles and flipping changing the triangles themselves.

Algorithm 4 exploits the most parallelisable aspects of the point insertion Algorithm 2, which are the point insertion, for which only one triangle is involved in at a time, and the flipping operation, which can be parallised but some book keeping needs to be taken care of in order for conflicting flip to not be performed. With a large point set this parallelisation allows for a massively algorithm as a large number of point insertions and flips can be performed in parallel. Flipping conflicts can happen when two different configurations of neighbouring triangles want to flip and these two configurations share a triangle, as illustrated in Figure 10.

3.2.1. Constructing the super triangle

In order to be able to begin our DT algorithm, a *supertriangle* needs to be constructed. This needs to be done only once throughout the duration of the algorithm. Two routines in this algorithm deserve to be parallelized, computing the average point and computing the largest distance between two points. Computing the average point involves calculating the total sum of all points by a reduction which is followed by a division in each coordinate by the number of points in the set. When computing the maximum distance between two points a CUDA kernel is launched which spawns a thread for each point which then compares every other point to it by calculating the distance between them and stores the maximum distance within the memory in each thread. Within this computation each point is compared to itself once which is conscious decision since compute on the GPU is cheap and otherwise each thread would be receiving different instructions which is not friendly to the SIMT programming model on the GPU. Once these calculations are finished an atomic max operation is performed to shared memory and then another atomic max to global memory which gives us our final value of the maximum distance. These two quantities are then used to construct a *supertriagle* which will encompass all points in the set of points we provide. The maximum distance is the radius and the average point is center to a circle which will be the incircle of an equilateral triangle which becomes our constructed *supertriagle*. Algorithm 5 outlines this process.

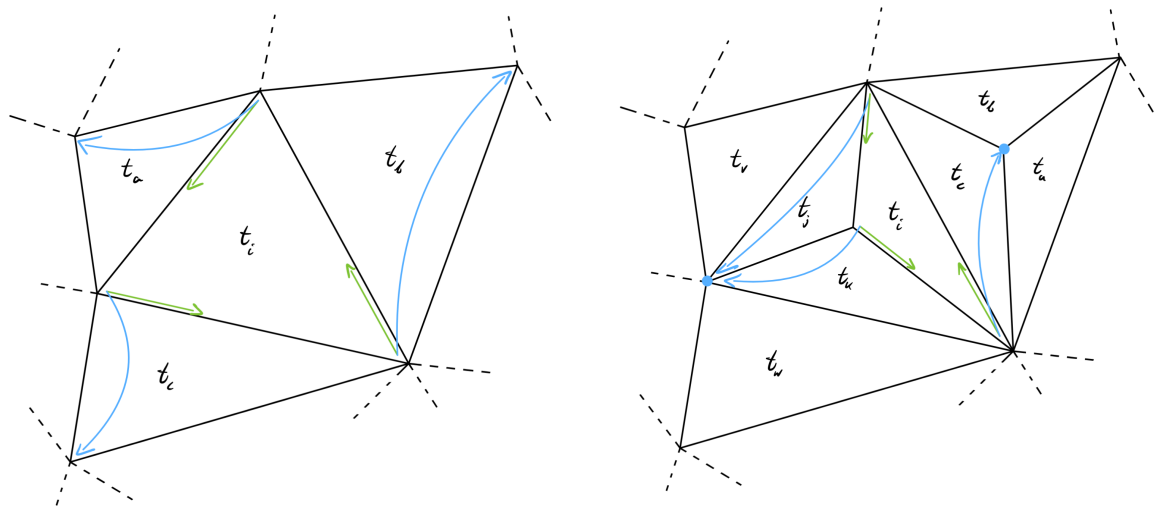
Algorithm 5: Parallel super triangle construction

Data: point set P

1 Compute the average point
2 Compute the largest distance between two points in P
3 Set center to average point
4 Set radius to largest distance
5 Construct triangle from circle as incircle

3.2.2. Point insertion

The point insertion step is very well suited for parallelisation. Parallel point insertion can be performed with minimal interference with their neighbours. This procedure is performed independently for each triangle with a point to insert. The only complication arises in the updating of neighbouring triangles information about their newly updated neighbours and opposite points. This must be done after all new triangles have been constructed and saved to memory. Only then you can exploit the data structure and traverse the neighbouring triangle to a update the correct triangles appropriate edge.



(a) Before insertion. (b) After insertion.

Figure 9: Parallel point insertion

The implementation of the parallel point insertion algorithm relies on two steps, preparation of points to be inserted and the insertion of points. If only the point insertion procedure is performed we also need to update point locations which is normally done after the flipping operations needed.

The preparation step involves a handful of checks or verifications to find out which point should be inserted into each triangle. In this algorithm we wish to find the most suitable point for each triangle to have inserted into it. We do this by finding out which point, which is not yet inserted into the triangulation lies in which triangle. The point closest to the circumcenter of the triangle is chosen to be inserted. Two CUDA kernels are used in this procedure, one to calculate the distances of each point to their corresponding circumcentres and another to find the minimum distance. This procedure relies on computing the distance twice as compute is cheap on GPU as opposed to copying memory of the triangle structures. In between all of these arrays which contain information about uninserted points *ptsUninsterted* are used throughout in order to not waste resources in the form of threads which would obtain instructions to do nothing. The *ptsUninsterted* array is sorted in order to launch the minimum number of threads needed. A few other kernels are used for book keeping purposes which consist of resetting certain values, for example the smallest distance between two points in each triangle is set to the maximum value as there are atomic min operations performed for which this is necessary in the next iteration of point insertion. We also keep an array which holds the indexes of triangles triangles which how hold points to be inserted which again prevents unnecessary thread launches.

Algorithm 6: prepForInsert

1 Reset index of the point to insert in each triangle
2 Set counter for number of points uninserted to be 0
3 Writes uninserted point index to ptsUninserted
4 Calculates and writes the smallest distance to circumcenter of triangle
5 Finds and writes the index of point with smallest distance to circumcenter of triangle
6 Resets counter of the number of points to insert
7 Counts the number of triangles which are marked for insertion
8 Sorts the array triWithInsert for efficient thread launches
9 Resets the value of the distance of point to circumcenter in each triangle

Once the preparation step is completed, which makes up the majority of the compute for point insertion procedure Figure 16 we can now actually insert the points which have been pick out. The logic is mostly consistent as in Figure 5 but needs to be adapted in order for it to be parallelized.

For the creation and rewriting involved in making the 3 new triangles stays the same except two things. First of which the locations in which the new triangles are written in need cannot be simply written to the next unwritten location in the list of triangle structs. A simple map can be created once we know how many triangles will need to split. We use the index of each thread to identify where we will place each newly created two triangles as we still overwrite the existing triangle with one of the new triangles that it is split into. We can use the following expression to know where to start writing the two new triangles $\text{nTri} + 2 * \text{idx}$ where nTri represents the current number of triangles in the triangulation and idx the index of the thread.

Secondly, the updating of the neighbouring triangles also need some extra care. The splitting or point insertion step is written as two CUDA kernels. One which writes the internal structure of the 3 new triangles and another kernel takes care of updating the relevant neighbours of the 3 new triangles. It is necessary to split up this procedure since if it was not split up the external neighbouring triangles could be overwritten while they are being created. The algorithm relies on the neighbouring triangles already existing to find the relevant neighbour to update which is done so by traversing the split triangles counter clockwise in order to the relevant neighbouring triangle. It is also important to note that the nTri variable, should only be updated after the parallel point insertion procedure is complete as the updating it during this process have consequences on the locations of the newly created triangles storage location.

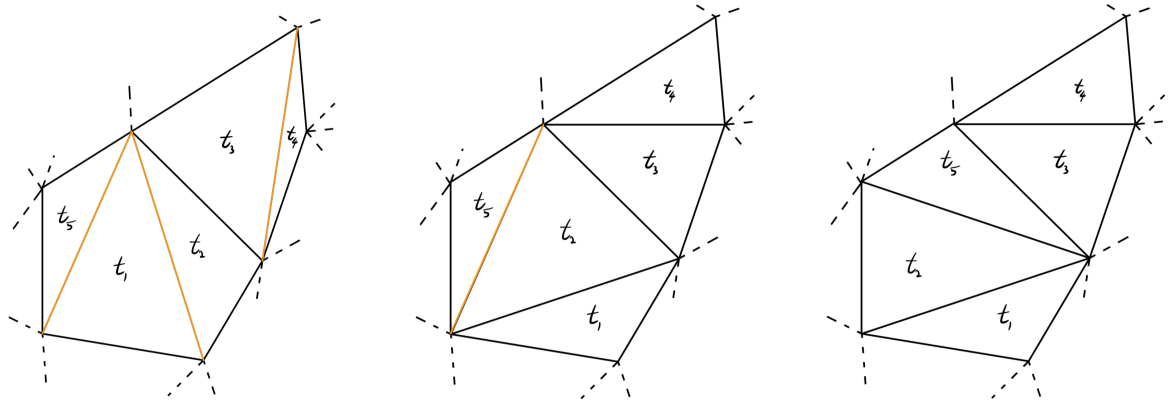
Algorithm 7: Parallel insert

1 Insert point in marked triangles
2 Update neighbours
3 Update number of triangles and number of points inserted
4 Reset triWithInsert for next iteraiton

3.2.3. Flipping

As briefly mentioned earlier, flipping can be performed in a highly parallel manner however some book keeping needs to be taken care of. The logic within the flipping operation is split up into three main steps. The first one is the writing of triangles to be flipped each configuration into a *Quad* Listing 2 data structure which here is mainly created for the purpose of keeping steps in the whole

procedure to be non conflicting more importantly stores relevant information about the previous state of the triangulation. This *Quad* struct will aid us in constructing the flipped configuration. The two new triangles created from the flip are the written by one kernel and appropriate neighbours are then updated in a separate kernel. Splitting the writing of the new flipped triangles is once again important as updating the neighbours relies on writing to the correct index of triangle since neighbouring triangles could also be involved in a flip. Figure 10 showcases the parallel flipping procedure.



(a) (b) (c)

Figure 10: Illustration of parallel flipping while accounting for flipping non conflicting configurations. Edges colored orange are marked for flipping. For each configuration marked for flipping by each orange edge the triangle with the smallest index will be the one performing the flipping operation, and the configuration with the smallest index (min of both indexes of triangles the configuration) will have priority to flip first in each round of parallel flipping. In the first figure (a) 3 edges are marked for flipping. Only configurations of triangles t_1t_2 and t_3t_4, with configuration indexes 1 and 3 respectively, will flip. Configuration t_5, t_1 with a configuration index of 1 will not flip in the first parallel flipping iteration (b) as it is not the minimum index in its configuration. (c) Showcases the final outcome of the parallel flipping.

However before the we can perform our parallel flipping we need to know which triangles need to be flipped and which triangles should be flipped in order for there to be no conflicts between flips. In order to know which triangles should be flipped a kernel is launched to perform an *incircle* test on each edge of each triangle currently in the triangulation. The *incircle* test whether the point opposite each edge of each triangle is contained inside the circumcircle created by the triangle associated with the thread of computation. This test directly follows from Theorem 1.2. Following this test, some configurations of triangles may have been marked in a way that two configurations will share a triangle they want to flip with. In order to avoid we give each configuration of triangles a configuration index obtained by using the minimum index of both triangles and we write this to both triangles using an atomic min operation given a single triangles can be involved in more than one configuration. This is done by one CUDA kernel and is followed by another kernel which stores indexes of triangles which should perform a flipping operation into an auxiliary array. Only triangles which are the smallest index of triangles which will be involved in a flip and whose neighbour and itself both still hold the same configuration index are allowed to flip in a given parallel flipping pass. Once this performing and *incircle* test and making sure none of our flips will conflict with each other we can proceed to the parallel flipping procedure described previously.

Algorithm 8: Parallel flipping

```
Set array of triangles which should be flipped to −1
Perform incircle checks on all triangles and mark successful triangles for flipping
Check for possible flip conflicts and mark successful triangles for flipping
while there are configurations to flip
    Write relevant quadrilaterals
    Overwrites new triangles internal structure
    Updates neighbours information
    Perform incircle checks on all triangles and mark sucessful triagles for flipping
    Check for possible flip conflicts and mark sucessful triagles for flipping
Reset mark for flipping in tri struct
```

3.2.4. Updating point locations

The final part of Algorithm 4 is the updating of point locations. This process involves finding points lie in which triangle and noting the index of this triangle to an auxiliary array. This is a necessary step for the calculation of the nearest point to the circumcenter of the triangle for preparing the point insertion procedure. This is done by one CUDA kernel which spawns a thread for each point and in each of these threads loops through all triangles which triangle this point lies in. When the triangle is found, the index of the triangle which contains this point is saved to an auxiliary array which maps indexes of points to indexes of triangles.

Algorithm 9: Update point locations

```
for each uninserted p ∈ P do in parallel
    for t ∈ T
        check if p is contained in t
        if p lies in t
            mark p to lie in t
            break
```

3.2.5. Analysis

In this section we will analyse and visualize some results and which we have produces for our DT algorithm. All tests were run with a *NVIDIA GeForce RTX 3090* as the GPU alongside an *AMD Ryzen Threadripper 3960X 24-Core Processor* CPU, with the exception of some results in Figure 21.

We shall begin with some visualization of the algorithm. Figure 11 displays the raw evolution of the algorithm. We can follow the figures from left to right in alphabetical order to see the history of the procedure. This series of visualization confirms to us that our algorithm actually performs the tasks we designed it to perform. The super triangle enveloping all points is created and the algorithm proceeds to insert points and flip necessary configurations without any intersecting edges. These pictures don't present every single iteration saved by the algorithm as sometimes nothing happens for example when there are no configurations to flip in the early stages of the algorithms execution.

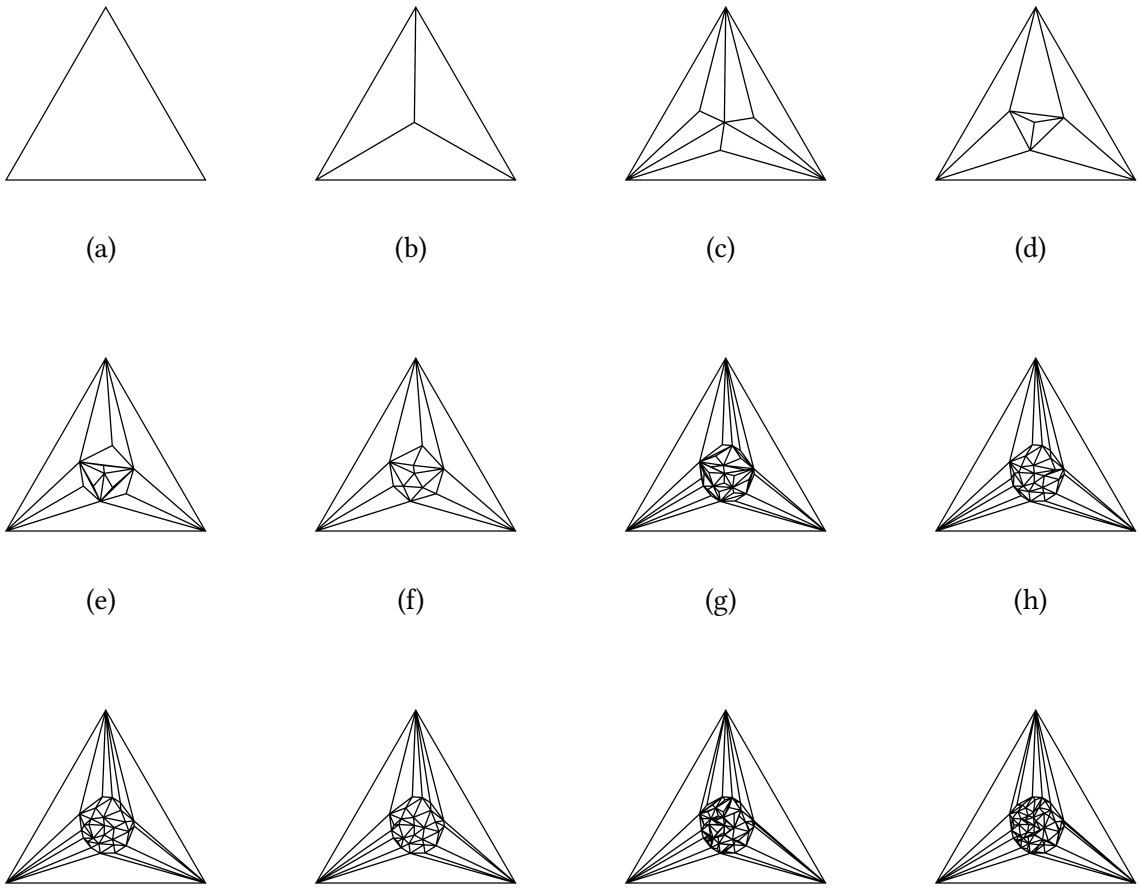


Figure 11: These figures show the history of the DT algorithm. The algorithm begins by initializing a super triangle (a) which is constructed to contain each point desired by the user. Here a uniform point distribution on a unit disk is used. In (b) and (c) a point insertion is performed and in (c) certain edges are marked for parallel flipping for which the result is displayed in (d). The algorithm proceeds in following subfigures with a series of point insertion followed by the required number of parallel flipping operations. In the final result, triangles which contain the initialized supertriangle points are removed and we are left with the desired triangulation as can be seen in Figure 17.

In the two figures, Figure 12 and Figure 13, we see how our algorithms performs in time. These exclude the construction of the supertriangle as it is performed only one and does not contribute to

the significant parts of the algorithm. Both plots are logarithmic in both axes to suit the number of points tested. In Figure 12 we notice that for a number of points less than 10^3 the rate of change of runtime algorithm is constant after which a threshold is passed for which the runtime begins to increase by a large amount. One key bottleneck in our implementation is the updating of point locations which is currently done the most naive way possible. We check for each triangle each point which is extremely inefficient and is reflected in this graph. A better analysis would involve improving this procedure with a purpose built data structure for accessing point locations and then noting the overall memory locations of relevant information for this routine.

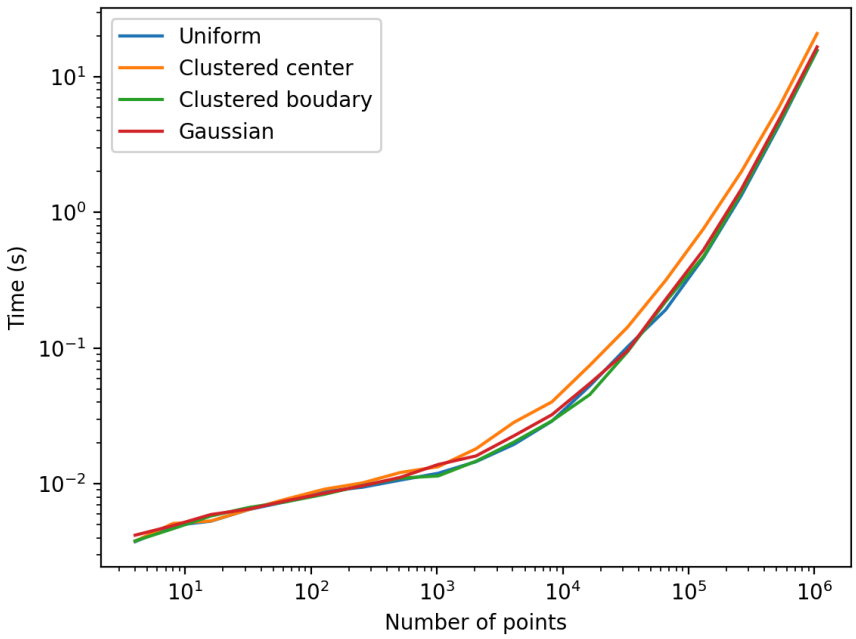


Figure 12: Plot showing the amount of time it took the GPU code to run with respect to the number of points in the triangulation. Different line colors show the code run with a different underlying point distribution.

Figure 13 displays the speedup by comparing the serial implementation with our GPU implementation. This comparison is quite unfair to the serial implementation as we are not comparing the same algorithms exactly. The GPU algorithm needed to be rewritten with a deep understanding of the GPU programming model. By the end of the rewrite it is not the same algorithm we started with. It is still a useful benchmark since it does show us that with a bit of work converting a simple implementation into a highly parallelized version can give immense amounts of speedup. The speedup here is comparing the runtime of the serial code with for a given number of points and with the runtime of the GPU code with the same number of points. Both implementations are run with single precision floating point arithmetic. We can notice in Figure 13 that we only begin to get an improvement in performance once we cross 10^3 points, but as we do we get a drastic increase in performance of 1000 times in the case of 10^5.

In Figure 14 we see what the result would look like in each iteration if no flipping operations were performed. This figure aids to portray the DT as the angle maximizing triangulation. In these figures we don't have any angle maximising work done which can be seen in the last few figures. The lines drawn appear to be thick. The triangles in these figures also appear, for the most part, a lot more narrow than their counterparts in Figure 11.

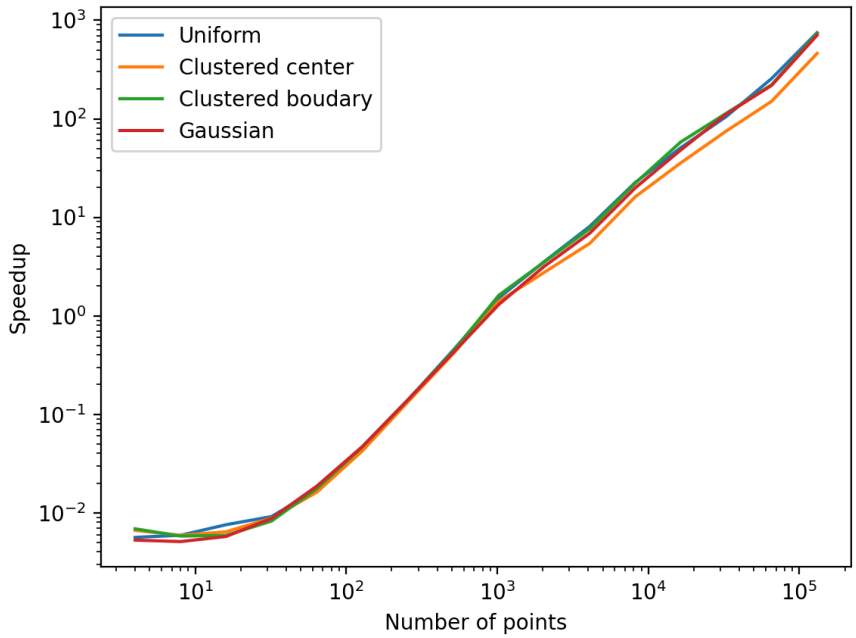
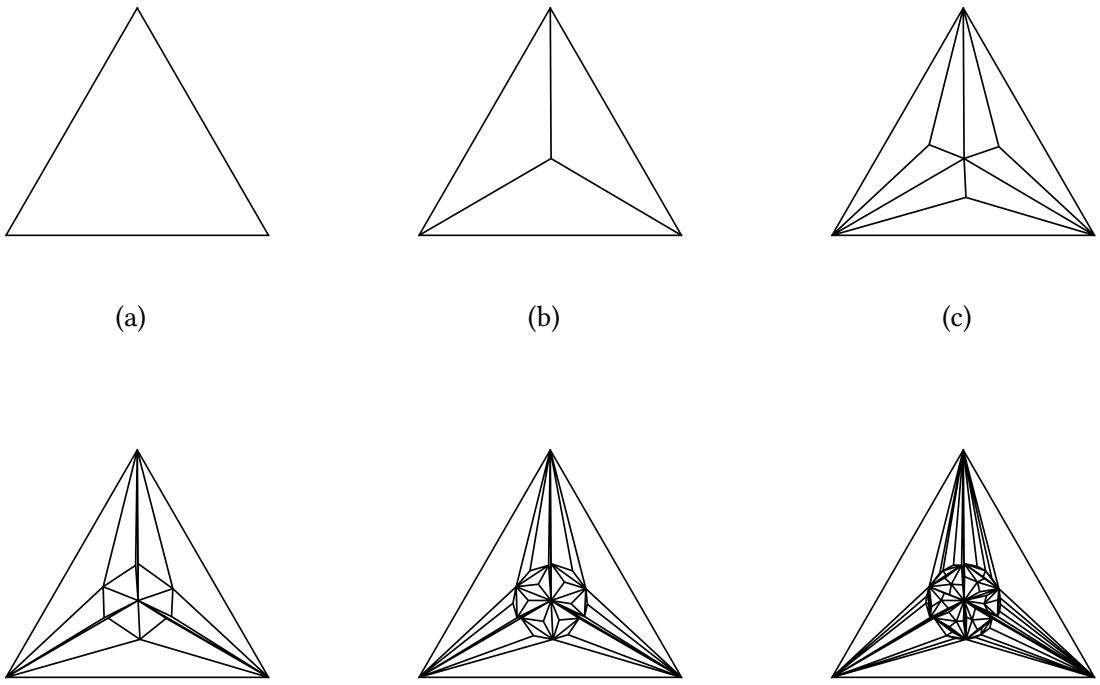


Figure 13: Plot showing speedup of the GPU code with respect to the serial implementation of the incremental point insertion Algorithm 2. Speedup here is defined as the ratio of time the serial code took to run with the time the GPU code took to run.



(a) (b) (c)

(d) (e) (f)

Figure 14: These figures show the evolution of the only the point insertion algorithm without any flipping of configurations. The point insertion proceeds in alphabetical order noting the labels of each subfigure.

A key metric which needs to be considered during the use of a GPU algorithm is the *block size* also known by a more descriptive name, the number of threads per block. This property of the algorithm determines how many threads share a particular part of memory which the *block size* determines. In the case of our algorithm, changing this quantity mainly affects some atomic operations which act on this *shared memory*.

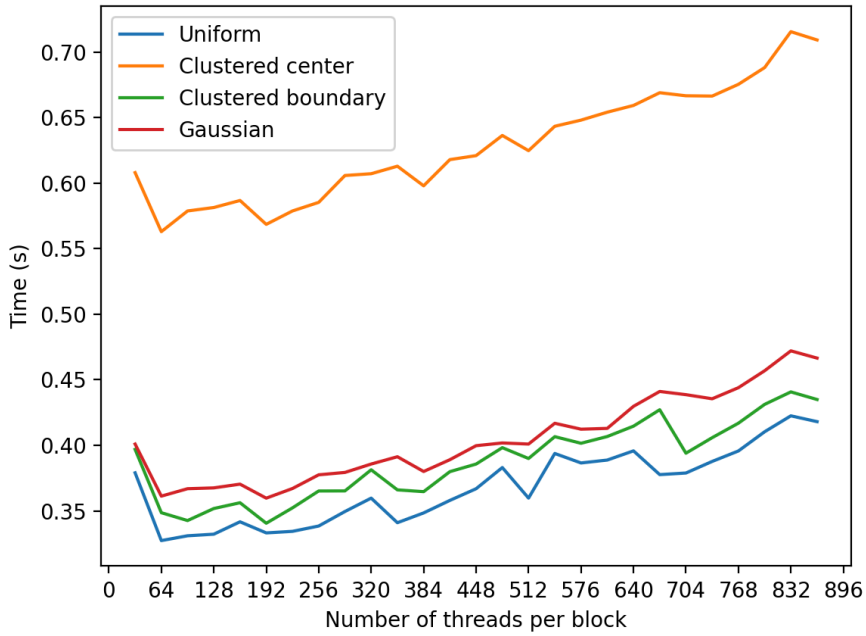


Figure 15: Showing the time it took for the GPU DT code to run with 10^5 points while varying the number of threads per block which is also known as the block size. We clearly see increasing the number of threads per block decreases performance. This is due to the way we implemented some features of the code using shared memory for which in this case with a larger block size, more atomic operations are trying to act on the same memory locations and this will lead to serialized behaviour with the exception of very small block sizes. Hence by observing the figure we can deduce that the most effective block sizes are in between 64 and 192. A block size of 128 was for performing all experiments shown in each figure in this report.

In order to profile the code we decide to measure how long each significant logical part of the algorithm takes to complete in each pass of insertion and flipping Figure 16. We add these values to obtain the amount of time it took to run each logical chunk over the total runtime of the algorithm. By a quick glance we can see that the updating of point locations take up the majority of the runtime as we increase the number of points. This is mainly because of a naive implementation of this procedure for which each point checks whether each triangle for whether this point is contained in which triangle. We can see that this implementation works well for a small number of points but as we increase the number of points it begins to dominate the runtime and severely affect the performance of the algorithm. The profiling of code as done in this way is in general a very useful way of inspecting the performance of your code as this narrows down what the developer should focus on improving. This set of graphs changed a handful of times during development of this code with initially the *flip* procedure taking up a majority of the runtime. What made the *flip* procedure take so long was that it initially saves the state of the triangulation for each pass of parallel flipping. Toggling the incremental saving of data, which is used to plot Figure 14 and Figure 11, unsurprisingly increased performance of the code which is a reason why I didn't notice

how bad the performance of updating points locations was until late in the development and analysis of the code.

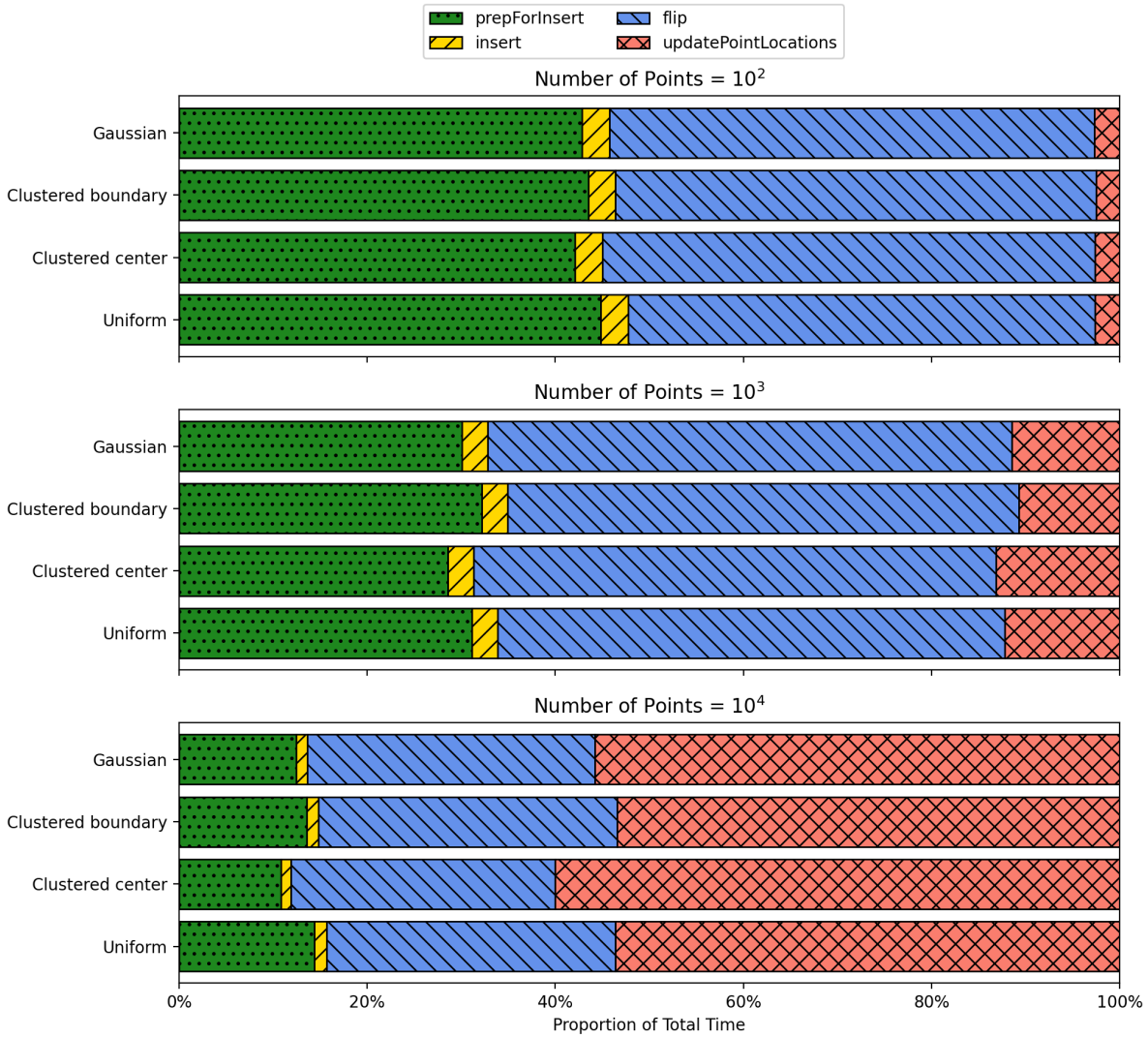


Figure 16: Showing the proportions of time each function took as a percentage of the total runtime for a given number of points. Each color represents a different set operation which perform a task. The *prepForInsert* routine performs necessary steps for to be followed up by the insertion step. This involves the calculation and writing of which point is nearest the circumcenter of each triangle and other necessary resetting of values. *insert* simply inserts the points which were chosen in the previous step. The *flip* procedure performs passes of parallel flipping by calculating which configurations should be flipped and prevents and flipping conflicts from occurring. Finally *upadtePointLocations* checks for each uninserted point for which index of triangle it lies in. The algorithm for this plot was performed on a uniform distribution of for 3 different numbers of points.

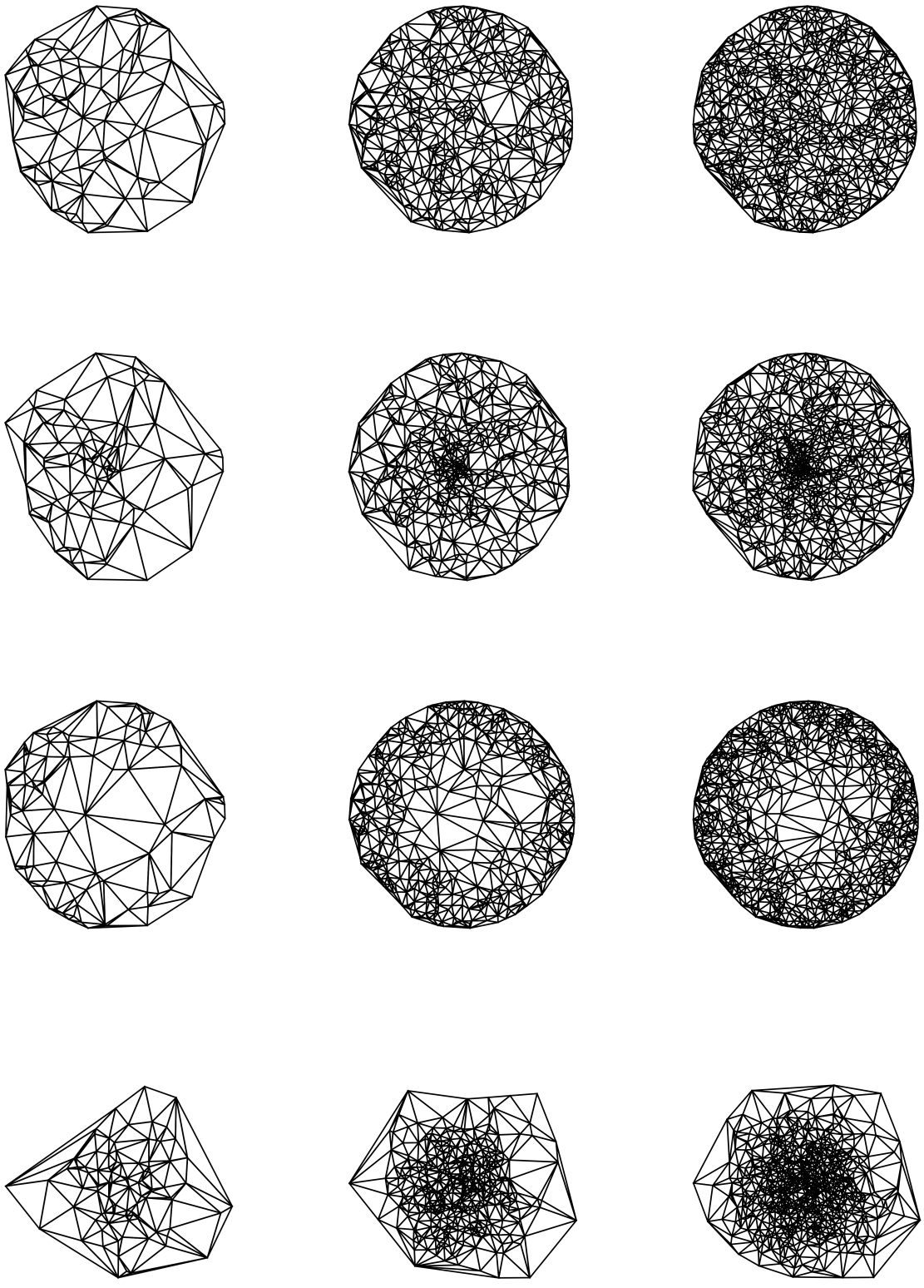


Figure 17: Visualisations of Delaunay triangulations of various point distributions. The grid should be read as follows. Along the horizontal the number of points involved increases gradually and with 100, 500, 1000 points in the first second and third column respectively. In each row we draw from different point distributions. The rows draw from a uniform unit disk distribution, a distribution on a disk with points clustered in the center, a distribution on a disk with points clustered near the boundary and a Gaussian distribution with mean 0 and variance 1, in rows 1, 2, 3 and 4 respectively.

Depending on the desired application of this algorithm one may use it to obtain only a near Delaunay triangulation. How close a triangulation is to being a DT can be calculated by counting all of the non Delaunay edges. The sum of these edges can be compared with the total number of edges in the triangulation. This is calculated and printed at the end of running our code alongside checking if the triangulation produced is indeed a DT. Taking a look at Figure 18 we have the number of configurations of triangles flipped for every iteration of the algorithm on the left and the total number of flips for each pass within the *flip* function. The majority of flips being performed during the execution of the algorithm are performed in the later stages of the algorithm but what can be noticed in the rightmost figure is that, for each pass of parallel flipping, only the first two or three passes contribute significantly to the triangulation performing the majority the flips. Following these flips we are left flipping a relatively tiny number of configurations. One could choose to only perform 2 passes of parallel flipping and the algorithm would process the majority of flipable configurations. This would in turn significantly reduce the amount of time spent on the flipping procedures as each pass of parallel flipping lasts about the same amount of time for a given iteration.

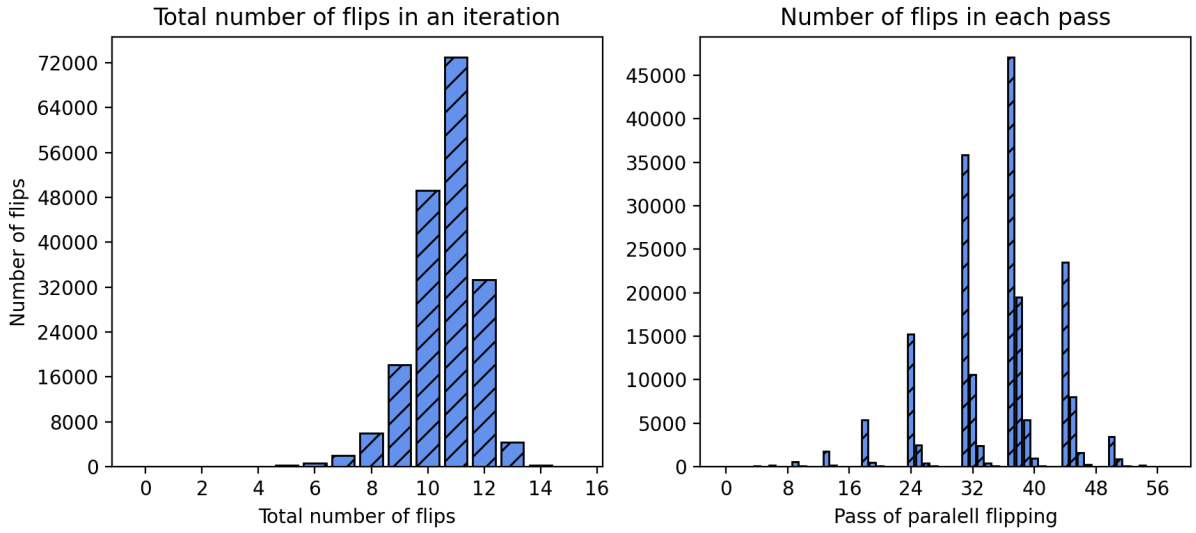


Figure 18: Figures showing numbers of flips performed during during each call to the parallel flipping function *flip* on the left and on the right the number of flips performed during each pass of parallel flipping performed within the *flip* function. Algorithm performed on 10^5 points and a uniform distribution of points.

After counting how many flips are performed in each pass of flipping and iteration, another similar question to ask is how many point insertions there are per iteration. Results for the same number of points and the same point distribution as in Figure 18 can be seen in Figure 19. We can immediately notice that we have a incredibly similar looking graphs, with the exception of the numbers on the y axis. The number of flips in each iteration appears to be proportional to the number of point insertions preceding the passes of parallel flipping. This should not be unexpected as when we perform point insertions we are creating three new edges that can be potentially marked for flipping. From the red line showing us by how much the number of points increase per iteration we can notice that the increase number of point insertions per iteration stays roughly around 3 times until one step before we reach the peak. The 3 times rate can be explained by the underlying uniform point distribution from which we likely insert into most triangles and the fact that when we do perform a point insertion we create 3 new triangles after destroying the triangle which had a point inserted into it.

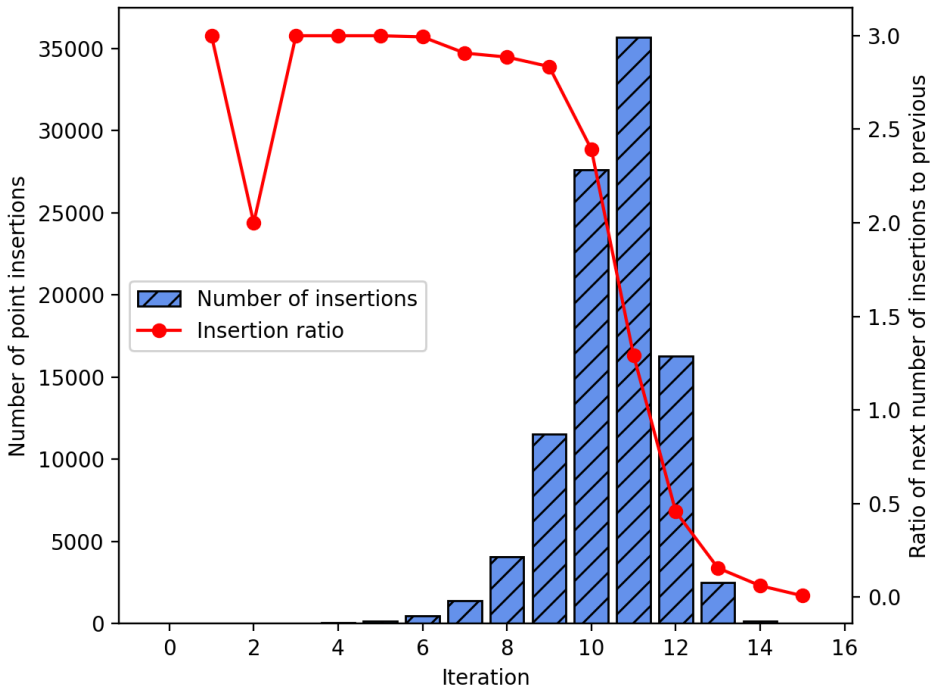


Figure 19: This figure shows the number of points inserted into the existing triangulation during each pass of the algorithm as blue bars with quantity noted on the left y axis. In red a line is shown to represent the ratio of number of points inserted to the previous number of points inserted. From this we can see by how much points the triangulation increases in each iteration shown on the right y axis. Algorithm performed on 10^5 points and a uniform distribution of points.

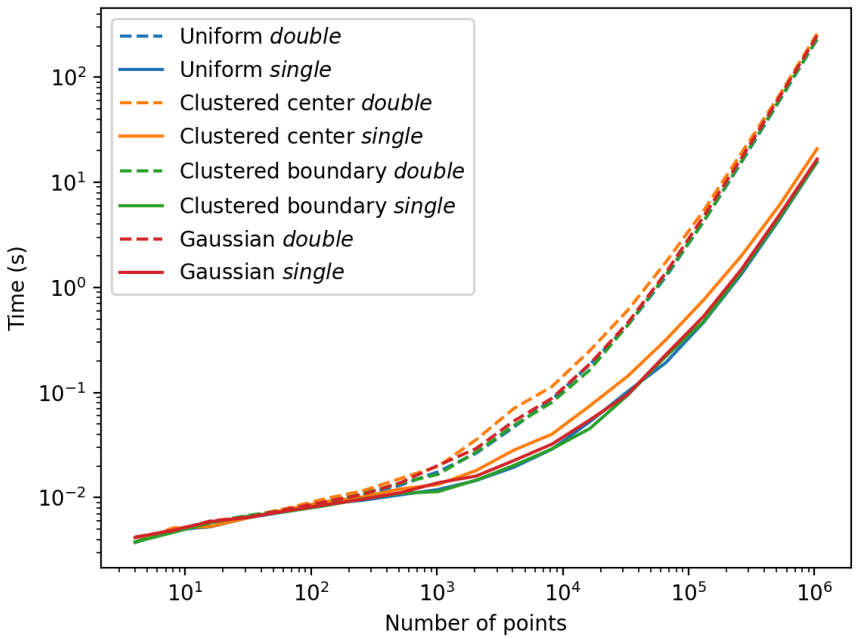


Figure 20: This figure displays the difference in runtime between the same GPU code in single and in double precision. Solid lines show the run time with their respective point distribution in single precision and dashed lines of the same color show the run time of the same distribution but in double precision instead.

When reaching sizes of around 10^6 points, our DT algorithm begins to get stuck in flipping operations. This is due to the single precision floating point arithmetic used. This flaw is amended by tracking if the algorithm is repeating the same flipping operations but this leaves us without being certain that what we create is indeed a Delaunay triangulation. Hence the need for double precision arithmetic. Other approaches is adaptive methods to change the precision of the incircle checks when needed [12]. We implemented a way of changing the precision of the whole algorithm which allows the user to choose between calculating in single or double precision. In Figure 20 we compare the runtime of single and double precision codes with the number of points which construct the triangulation. Unsurprisingly double precision arithmetic takes longer than single precision however it could be advantageous to run with double with a larger number of points if the precise nature of the Delaunay triangulation is desired.

When comparing how scalable an algorithm is in the world of parallel CPU programming, with concepts such as strong and weak scaling, there is no standardized way of doing so for a single GPU code. The strong and weak scaling approaches of analysis can be useful for GPUs when we have a multi GPU code however we have not created a multi GPU code. The next best approach we found, used by [12], is to instead compare run time on different GPUs. Alongside the run time we also calculate the normalized run time defined by the run time divided by the product of the number of cores and the base clock frequency of the respective GPU. The normalized runtime is a reasonable metric to consider as the divisor is a measure of how often a computation is performed.

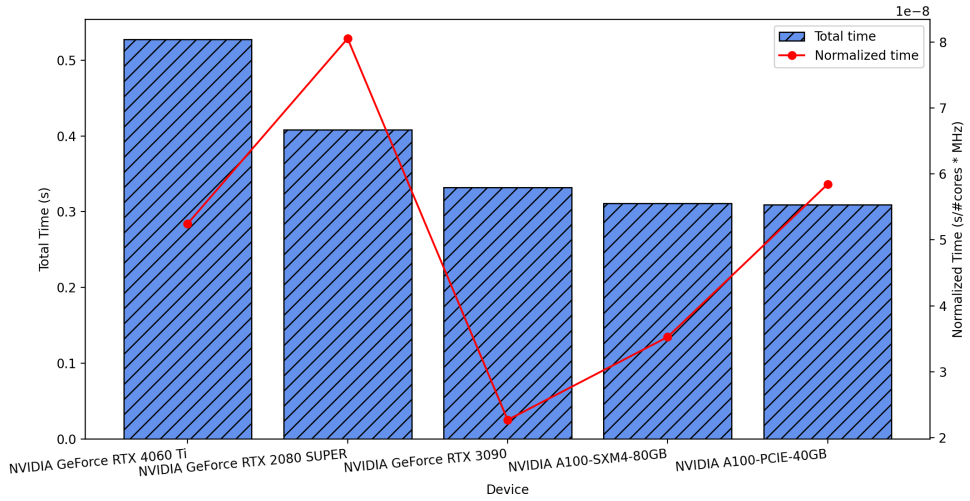


Figure 21: A comparison of the algorithm running on a variety of Nvidia GPUs which I had access to at the time. This benchmark is performed by averaging 5 runs of the DT algorithm on a uniform set of 10^5 points. From this figure we can see that this algorithm doesn't scale well as we would like for the red line (normalized time) to decrease along with the bars (real time). What we can deduce from this plot is that our algorithm scales well on RTX GPUs but not so well on the A100 GPUs since we can see the red line decreasing for the RTX GPUs and not for the A100s.

Both types of GPU series RTX and A100 architectures are designed for different purposes. RTX GPUs are mainly designed for real time tasks such as playing video games where it is important for the user to see the results of computations resonably quickly. While the A100s are specifically designed to be run in data centers or supercomputers which don't necessarily demand the ease of access of data being processed by the GPU. This leads us to the fast compute which we see on the A100s being around as fast as the RTX 3090 but they don't scale as well in comparison with core count and clock frequency since our algorithm doesn't massively rely on passing massive amounts of data between the host and device.

3.3. Data Structures

The core data structure that is needed in this algorithm is one to represent the triangulation itself. There are a handful of different approaches to this problem including representing edges by the qaud edge data structure [10] however we choose to represent the triangles in our triangulation by explicit triangle structures [13] which hold necessary information about their neighbours for the construction of the triangulation and for performing point insertion and flipping operations.

```
struct __align__(64)  Tri {
  int p[3]; // indexes of points in pts list
  int n[3]; // idx to Tri neighbours of this triangle
  int o[3]; // index in neigbouring tri of point opposite the egde

  // takes values 0 or 1 for marking if it shouldn't or should be inserted into
  int insert;
  // the index of the point to insert
  int insertPt;
  // entry for the minimum distance between point and circumcenter
  REAL insertPt_dist;
  // marks an edge to flip 0,1 or 2
  int flip;
  // mark whether this triangle should flip in the current iteration of flipping
  int flipThisIter;
  // the minimum index for both triangles which could be involved in a flip
  int configIdx;
};
```

Listing 1: Data structure needed for Point insertion algorithm. Its main features are that it holds a pointer to an array of points which will be used for the triangulation, the index of those points as ints which form this triangle, its daughter triangles which are represented as ints which belong to an array of all triangle elements and whether this triangle is used in the triangulation constructed so far. Aligned to 64 bytes for more efficient accessing of memory.

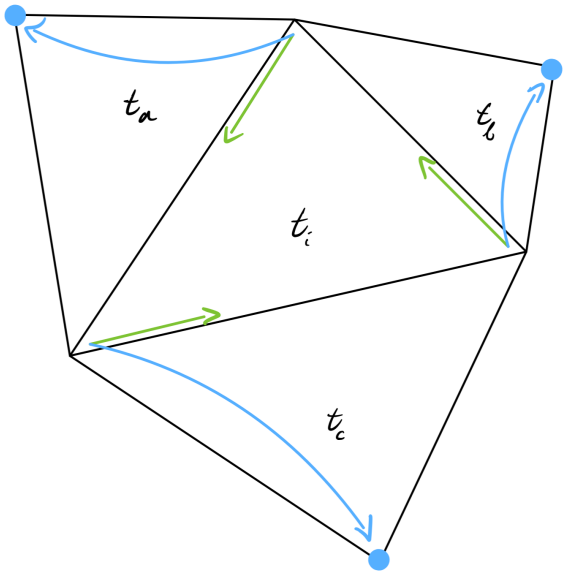


Figure 22: An illustration of the *Tri* data structures main features. We describe the triangle t_i int the figure. Oriented counter clockwise points are stored as indexes an array containing two dimensional coordinate representing the point. The neighbours are assigned by using the right hand side of each edge using and index of the point as the start of the edge and following the edge in the counter clockwise direction. The neighbours index will by written in the corresponding entry in the structure.

This data structure was chosen for the ease of implementation and as whenever we want to read a triangle we will be a significant amount of data about it and this locality theoretically helps with memory reads, as opposed to storing separate parts of date about the triangle in different structures, for example, separating point and neighbour information into two different structs.

The Listing 2 below is used in the flipping step of the algorithm and is only used as an intermediate representation of the triangles which will be created and the data needed to update its neighbours

```
struct __align__(64) Quad  {
  int p[4]; // indexes of points in pts list
  int n[4]; // idx to Tri neighbours across the edge
  int o[4]; // index in neigbouring tri of point opposite the egde
};
```

Listing 2: Data structure used in the flipping algorithm. This quadrilateral data structure holds information about the intermediate state of two triangles involved in a configuration currently being flipped. This struct is used in the construction of the two new triangles created and in the updating of neighbouring triangles data. Aligned to 64 bytes for more efficient accessing of memory.

4. Further work

In this section I hope to describe some of the next steps I would take in this project if I had more time.

A better algorithm for the updating of point locations is necessary to be implemented as in its current state it is extremely inefficient. The best candidate would be to implement a Directed Acyclic Graph (DAG) data structure which is commonly don't in applications such as this one. This DAG structure would allow a much faster and more efficient finding of point locations as we would save the structure of the history of triangle locations nested through flipping operation and point insertions which would avoid a lot of unnecessary calculations and memory fetches.

In this report I have mostly only performed an analysis on the runtime of each algorithm but I haven't considered to plot how much each memory location is occupied in the GPU. This could further give insight into the inner workings of the algorithm and possibly provide better profiling opportunities which could help in the optimisation of the code.

During the process of preparing for the point insertion step we always use the same criterion for picking which point to insert, that is, picking the point nearest the circumcenter of the triangles. There are other possible candidate for this procedure some of which being choosing a random point inside the triangle, picking the point nearest the incenter of the triangle or the point nearest the average of the three vertices of the triangle.

Another test I would have like to perform is to see if we could improve the runtime by restricting the maximum number of passes of flipping in each call of the *flip* function. With similar thinking for this test it would be interesting to see how much close to a DT the final triangulation would be after applying different restrictions to the algorithm and how much the total runtime would be improved.

5. Conclusion

The Delaunay Triangulation is a complex algorithm with multiple valid approaches to reach the same result. Its foundation in rich mathematical theory, particularly the edge-flipping operation, has inspired both efficient serial algorithms and their highly parallelisable counterparts. Our work demonstrated that significant reductions in runtime can be achieved through parallelisation, provided the algorithm lends itself well to this transformation.

We explored the mathematical principles underlying the algorithm and examined the challenges of adapting CPU-based implementations to run on GPUs. This process involved a detailed analysis of the transition from serial to parallel code. While the parallel version showed promising performance improvements, there is still room for further optimisation and analysis.

In conclusion, although GPU programming can offer substantial speedups, it is also a time-intensive process that requires careful algorithm design and performance tuning to maximise efficiency.

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