Poisson Problem 1-D

$$u''(x) = f(x) \quad \text{on } \Omega \quad \text{and} \quad u(x) = 0 \text{ on } \partial\Omega$$

$$\Omega = [0,1]$$

$$u''(x)\varphi(x) = f(x)\varphi(x)$$

$$u''\varphi = f\varphi$$

$$\int_\Omega u''\varphi\,dx = \int_\Omega f\varphi\,dx$$

$$\cancel{u'\varphi\Big|_{\partial\Omega}}^{\,0} - \int_0^1 u'\varphi'\,dx = \int_0^1 f\varphi\,dx$$

$\varphi(0) = 0 = \varphi(1)$

$$u = \sum_i \underline{u_i}\varphi_i \quad , \quad \underline{\varphi} = \varphi_j$$

$$-\int_\Omega \underline{u'}\varphi'\,dx = \int_\Omega f, \varphi\,dx$$

$$\frac{\partial}{\partial x}u = \frac{\partial}{\partial x}\Big(\sum_i u_i\varphi_i(x)\Big) = \sum_i u_i\varphi_i'(x)$$

finite

$$\Rightarrow -\int_\Omega \sum_i u_i\varphi_i'\varphi_j'\,dx = \int_\Omega f\varphi_j\,dx \qquad \forall j$$

$$-\sum_i\Big(u_i\int_\Omega \varphi_i'\varphi_j'\,dx\Big) = \int_\Omega f\varphi_j\,dx \qquad \forall j$$

$$\Rightarrow \begin{pmatrix} \int\varphi_0'\varphi_0' \\ \int\varphi_0'\varphi_p' \\ \vdots \\ \int\varphi_{10} \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ dx \end{pmatrix} = f \qquad A\vec{u} = F \qquad \begin{pmatrix} \int\varphi_0 \\ \int\varphi_1 \\ \vdots \end{pmatrix}$$

N Dim Case

$$\boxed{-\int_\Omega \nabla u \cdot \nabla \varphi_j \, ds = \int_\Omega f \varphi_j \, ds}$$

$$= \sum_i u_i \int_\Omega \nabla \varphi_i \cdot \nabla \varphi_j \, ds = \int_\Omega f \varphi_j \, ds$$

$$\nabla u = \frac{\partial}{\partial x} u(x,y) \varphi_i(x,y) + \frac{\partial}{\partial y} u(x,y) \varphi_i(x,y)$$

$$= u_{x,y} \nabla \varphi_i$$

$$\Rightarrow \sum_i u_i (\nabla \varphi_i, \nabla \varphi_j) = (f, \varphi_j)$$

$$\nabla u = \nabla \sum u_i \varphi_i(x,y)$$

$$= \left(\frac{\partial}{\partial x} \sum_i u_i \varphi_i(x,y), \frac{\partial}{\partial y} \sum_i u_i \varphi_i(x,y) \right)$$

$$= \left(\sum_i u_i \partial_x \varphi_i(x,y), \sum_i u_i \partial_y \varphi_i(x,y) \right) \cdot (a,b)$$

$$= \sum u_i \, ($$

$$\nabla^2 u(x,y) =$$

$$u(x,y) = \sin(\pi x)\sin(\pi y)$$

$$\partial_x^2 u + \partial_y^2 u = -\pi^2 \sin(\pi x)\sin(\pi y) - \pi^2 \sin(\pi x)\sin(\pi y)$$

$$= -2\pi^2 \sin(\pi x)\sin(\pi y)$$

$$u\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \sin\tfrac{\pi}{2}\sin\tfrac{\pi}{2}$$

$$= \sin^2\left(\tfrac{\pi}{2}\right)$$

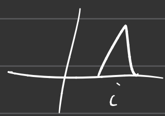
$$= 1$$

$$\frac{\partial}{\partial x}\sin(\pi x) = \frac{\partial \sin(\pi x)}{\partial \pi x}\frac{\partial \pi x}{\partial x}$$

$$= \pi\cos\pi x$$

$$\frac{\partial}{\partial x}\pi\cos\pi x = \pi\frac{\partial\cos(\pi x)}{\partial \pi x}\frac{\pi x}{\partial x} = \pi^2$$

$$A_{ij} = (\varphi_i, \varphi_j)$$

$\varphi_i(x) =$ 

$$\varphi_0(x) = \varphi_N(x)$$

$u(0) = u(1)$ $\Rightarrow$ plc right so

$$u(x) = \sum_{i=0}^{N} u_i \varphi_i(x)$$

$$u(0) = \sum_{i=0}^{N} u_i \varphi_i(0)$$

$$= u(1) = \sum_{i=0}^{N} u_i \varphi_i(1)$$

$$\Rightarrow \boxed{\varphi_i(0) = \varphi_i(1)}$$

$$\sum u_i \int \varphi_i' \varphi_j' \, dx = \int f \varphi_j \, dx$$

Neumann BC's

$$-\nabla^2 u = f \quad \text{on } \Omega$$

$$u = u_D \quad \text{on } \partial\Omega_D$$

$$-\frac{\partial u}{\partial n} = -\nabla u \cdot \hat{n} = h \quad \text{on } \partial\Omega_N$$

So the term that is added for neumann BC's is

$$\int_{\partial\Omega} h \ell$$

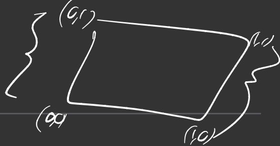
In 1-dim

$$\int_{\partial\Omega} h \ell = h \Big|_0^1 \qquad \text{for } \Omega = [0,1]$$

In 2-dim

$$\int_{\partial\Omega} h \ell = \oint_{\partial\Omega} h \ell \qquad \text{where } \Omega = [0,1]^2$$

Then if we



$$\int_{\partial\Omega} h\varphi_i = \int_0^1 h(0,y)\varphi_i(0,y) + \int_0^1 h(1,y)\varphi_i(0,y)$$

2D poisson with neumann bdry

$$-\nabla^2 u = f$$

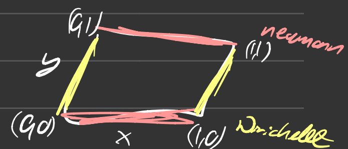
$$u = 1 + x^2 + 2y^2$$

$$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = 4y \quad , \quad \frac{\partial^2 u}{\partial y^2} = 4$$

$$\Rightarrow \quad 2 + 4 = -f \quad \Rightarrow \quad \boxed{f(x,y) = -6}$$

Let the neumann bdry only apply to the sides $y = 0$, $y = 1$



So then

$$-\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = -2x - 4y = n(x,y)$$

Then let $n(x,y) = -4y$

Nope!

Let $u = y^2 \sin(\pi x)$

$$\frac{\partial u}{\partial x} = \pi y^2 \cos(\pi x), \quad \frac{\partial^2 u}{\partial x^2} = -\pi^2 y^2 \sin(\pi x)$$

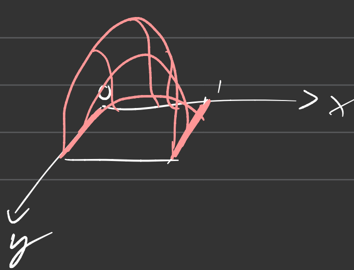
$$\frac{\partial u}{\partial y} = 2y \sin(\pi x), \quad \frac{\partial^2 u}{\partial y^2} = 2\sin(\pi x)$$

$$-\nabla^2 u = \pi^2 y^2 \sin(\pi x) - 2\sin(\pi x)$$

$$= \boxed{(\pi^2 y^2 - 2)\sin(\pi x) = f(x,y)}$$

$$-\frac{\partial u}{\partial n} = \pi y \cos(\pi x) - 2y\sin(\pi x)$$

$$= \boxed{n(x,y) = y(\pi\cos(\pi x) - 2\sin(\pi x))}$$



If

$$u = y\sin(\pi x)$$

$$\partial_x u = \pi y\cos(\pi x) \qquad \partial_y u = \sin \pi x$$

$$\partial_x^2 u = -\pi^2 y \sin(\pi x) \qquad \partial_y^2 u = 0$$

$$-\frac{\partial u}{\partial n} = -\left(\pi y\cos(\pi x) + \sin(\pi x)\right) = n(x,y)$$

$$-\nabla^2 u = -\pi^2 y \sin(\pi x) = f(x,y)$$

Lets not use trig functions cuz it looks like a precision problem

$$u = yx(x-1)$$

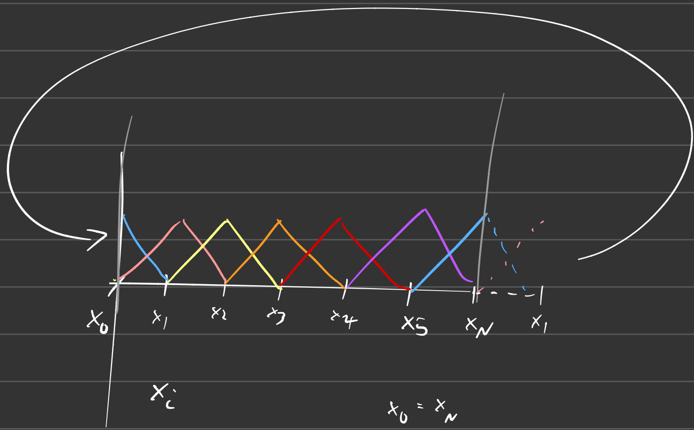
$$\partial_x u = y(x-1+x) \qquad \partial_y u = x(x-1)$$

$$= y(2x-1) \qquad \partial_y^2 u = 0$$

$$\partial_x^2 u = 2y$$

$$-\nabla^2 u = \boxed{2y = f(x,y)} \qquad -\frac{\partial u}{\partial n} = \boxed{n(x,y) = y(2x-1) + x(x-1)}$$

Periodic BC's



Define basis hat functions to be periodic

And do node modulo N when adding to nodes in $A[i,j]$

$$\underbrace{\sum u_i}_{\vec{u}_i} \underbrace{\int \varphi_i' \varphi_j' \, dx}_{A_{ij}} = \underbrace{\int f \varphi_j \, dx}_{\vec{F}_j}$$

A_{ij} doesn't change when $1 \le i,j \le 5$

$A_{0j} =$

$$A = \begin{bmatrix} \int \varphi_0' \varphi_0' & \int \varphi_1' \varphi_0' & \cdots & \int \varphi_N' \varphi_0' \\ \int \varphi_0' \varphi_1' & \int \varphi_1' \varphi_1' & \cdots & \int \varphi_N' \varphi_1' \\ \vdots & \vdots & \ddots & \vdots \\ \int \varphi_0' \varphi_N & \int \varphi_1' \varphi_N & \cdots & \int \varphi_N' \varphi_N' \end{bmatrix}$$

$$\nabla^2 u = \boxed{u'' = -1}$$

$$u(0) = -u(1)$$

$$u'' = -1$$

$$u' = -x + c$$

$$\boxed{u = -\frac{x^2}{2} + xc + d}$$

$$u(0) = d = u(1) = -\frac{1}{2} + c + d$$

$$\Rightarrow d = -\frac{1}{2} + c + d$$

$$\Rightarrow c = \frac{1}{2}$$

So $\quad u = -\frac{x^2}{2} + \frac{x}{2} + d$

Neumann

Let $\quad u' = -x + \frac{1}{2}$

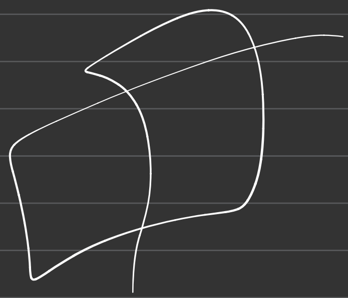
$$-u' = x - \frac{1}{2}$$

$$(\psi_1, \psi_j) = (\varphi_1 + \varphi_{np}, \varphi_j)$$

$$= \int \varphi_1 \varphi_j + \int \varphi_{np} \varphi_j$$

$$= (\varphi_1, \varphi_j) + (\varphi_{np}, \varphi_j)$$

$$= A_{1j} + A_{npj}$$



$\nabla^2 u = -f = -1$

$u(0,y) = u(1,y)$

$n(x,y) = x = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$

$\Rightarrow u = \frac{-x^2}{2} + xc + d$

$u(0) = d = u(1) = \frac{-1}{2} + c + d$

$$\boxed{c = \frac{1}{2}}$$

$u = \frac{-x^2}{2} + xc + d$

Heat Equation

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u + f \qquad u = u(x, y, t)$$
$$f = f(x, y, t)$$

Use finite difference on time step

So we write

$$\left(\frac{\partial u}{\partial t}\right)^{n+1} = \nabla^2 u^{n+1} + f^{n+1}$$

$$\left(\frac{\partial u}{\partial t}\right)^{n+1} \approx \frac{u^{n+1} - u^n}{\Delta t}$$

$$\Rightarrow \quad \frac{u^{n+1} - u^n}{\Delta t} = \nabla^2 u^{n+1} + f^{n+1}$$

Reorder to get

$$u^{n+1} - \Delta t \nabla^2 u^{n+1} = u^n + \Delta t f^{n+1}$$

Now use finite element method on the above

$$u^{n+1}\varphi - \Delta t \nabla^2 u^{n+1}\varphi = u^n \varphi + \Delta t f^{n+1}\varphi$$

$$\int_\Omega u^{n+1}\varphi - \Delta t \nabla^2 u^{n+1}\varphi \, dx = \int_\Omega u^n \varphi + \Delta t f^{n+1}\varphi \, dx$$

$$\vdots$$

$$\boxed{\int_\Omega \left(u^{n+1}\varphi + \Delta t \nabla u^{n+1} \cdot \nabla \varphi \right) dx = \int_\Omega \left(u^n + \Delta t f^{n+1} \right) \varphi \, dx}$$

$$u^{n+1} = \sum_i u_i^{n+1}\varphi_i \quad , \quad \varphi = \varphi_j$$

$$= \int_\Omega \left(\sum_i u_i^{n+1} \varphi_i \varphi_j + \Delta t \sum_i u^{n+1} \nabla \varphi_i \cdot \nabla \varphi_j \right) dx$$

$$= \sum_i u_i^{n+1} \int_\Omega \left(\varphi_i \varphi_j + \Delta t \nabla \varphi_i \cdot \nabla \varphi_j \right) dx$$

$$= \underline{\int_\Omega \left(\sum_i u_i^n \varphi_i + \Delta t f(x,y,t) \right) \varphi_j \, dx}$$

$$u_0(x,y) = e^{-(x^2+y^2)}$$

Then at $t = 0 = \Delta t$ we get

$$\int_\Omega u^1 \varphi \, dx = \int_\Omega u^0 \varphi \, dx \qquad u^0 = u_0$$

Or just pick a nice function like

$$u_0(x, y) = e^{-(x^2+y^2)}$$

$$u_0(x, y) = \sin(\pi y)\sin(\pi y)$$

A selected non-linear Poisson Problem

$$-\nabla \cdot ((u+1)\nabla u) = g \quad \text{on } \Omega$$

$$u = b \quad \text{on } \Gamma$$

Ω, Γ are the domain and boundary respectively

We get a weak form solution of

$$f(u;\varphi) = \int_\Omega ((u+1)\nabla u \cdot \nabla \varphi - g\varphi)\,dx \quad \forall \varphi$$

Then

$$f(u;\varphi_j) = \int_\Omega ((u+1)\nabla u \cdot \nabla \varphi_j - g\varphi_j)\,dx \quad \forall \varphi_j$$

So solve for this u we use a sort of taylor series expansion for whats called the Gateux Derivative of f

Note $f(u+\hat{u};\varphi) \approx f(u;\varphi) + J(u;\varphi,\hat{u})$

and $u^{n+1} = u^n + \hat{u}$

And note that J acts like a derivative for f

$$J(u; \mathcal{Q}, \hat{u}) = \lim_{\varepsilon \to 0} \frac{f(u + \varepsilon \hat{u}; \mathcal{Q}) - f(u; \mathcal{Q})}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\int_\Omega \left[(u + \varepsilon \hat{u} + 1)\nabla(u + \varepsilon \hat{u}) \cdot \nabla \mathcal{Q} - g\mathcal{Q}\right]dx - \int_\Omega \left[(u+1)\nabla u \cdot \nabla \mathcal{Q} - g\mathcal{Q}\right]dx}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0}$$

$$= \int_\Omega \nabla \mathcal{Q} \cdot (\hat{u} \nabla u + (u+1) \nabla \hat{u}\, dx$$

And J is linear in $\hat{u}$ so we can solve for it using the finite element method with some initial guess u and some arbitrary $\mathcal{Q}$

$$J_{ij} = J(u^n; \mathcal{Q}_i, \mathcal{Q}_j) = \int_\Omega \left(\nabla \mathcal{Q}_i \cdot (\mathcal{Q}_j \nabla u^n + (u^n + 1) \nabla \mathcal{Q}_j\right) dx$$

So we essentially just solve

$$J(u^n; \phi_i, \phi_j) = -f(u^n, \phi_j)$$

Where

$$J_{ij} = J(u^n; \phi_i, \phi_j) = \int_\Omega \left(\nabla \phi_i \cdot (\phi_j \nabla u^n + (u^n+1) \nabla \phi_j\right) dx$$

and

$$f_j = f(u^n, \phi_j) = \int_\Omega \left((u^n+1) \nabla u^n \cdot \nabla \phi_j - g\, \phi_j\right) dx$$

For a manufactured solution let

$$u(x,y) = \sin(\pi x)\sin(\pi y)$$ maybe ~~try~~ $u = x(x-1)y(y-1)$

$$\nabla \cdot (u+1)\nabla u = \nabla \cdot \left((u+1)\begin{bmatrix} \pi\cos(\pi x)\sin(\pi y) \\ \pi\sin(\pi x)\cos(\pi y) \end{bmatrix}\right)$$

$$= \pi \nabla \cdot \left((\sin(\pi x)\sin(\pi y)+1)\begin{bmatrix} \cos(\pi x)\sin \pi y \\ \sin(\pi x)\cos \pi y \end{bmatrix}\right)$$

$$= \pi\left(\frac{\partial}{\partial x}(\sin(\pi x)\sin(\pi y)+1)(\cos(\pi x)\sin(\pi y)) + \frac{\partial}{\partial y}(\sin(\pi x)\sin(\pi y)+1)(\sin(\pi x)\cos(\pi y))\right)$$

$$= \pi \left[\frac{\partial}{\partial x}\left(\sin(\pi x)\cos(\pi x)\sin^2(\pi y) + \cos(\pi x)\sin(\pi y) \right) \right.$$

$$\left. + \frac{\partial}{\partial y}\left(\sin^2(\pi x)\sin(\pi y)\cos(\pi y) + \sin(\pi x)\cos(\pi y) \right) \right]$$

$$= \pi \left[\sin^2(\pi y)\left[\pi\cos^2(\pi x) - \pi\sin^2(\pi x)\right] - \pi\sin(\pi x)\sin(\pi y) \right.$$

$$\left. + \sin^2(\pi x)\left[\pi\cos^2(\pi y) - \pi\sin^2(\pi y)\right] - \pi\sin(\pi x)\sin \pi y \right]$$

$$= \pi^2 \left[\sin^2(\pi y)\cos^2(\pi x) - \sin^2(\pi y)\sin^2(\pi x) - \sin(\pi x)\sin(\pi y) \right.$$

$$\left. + \sin^2(\pi x)\cos^2(\pi y) - \sin^2(\pi x)\sin^2(\pi y) - \sin(\pi x)\sin(\pi y) \right]$$

$$= \pi^2 \left[-2\sin(\pi x)\sin(\pi y) - 2\sin^2(\pi y)\sin^2(\pi x) \right.$$

$$\left. + \sin^2(\pi y)\cos^2(\pi x) + \sin^2(\pi x)\cos^2(\pi y) \right]$$

$$u(x,y) = \sin(\pi x)\sin(\pi y)$$

$$\nabla \cdot (u+1)\nabla u$$

$$= \nabla \cdot (u+1)\begin{bmatrix} \partial_x \sin(\pi x)\sin(\pi y) \\ \partial_y \sin(\pi x)\sin(\pi y) \end{bmatrix}$$

$$= \nabla \cdot (u+1)\begin{bmatrix} \pi\cos(\pi x)\sin(\pi y) \\ \pi\sin(\pi x)\cos(\pi y) \end{bmatrix}$$

$$= \pi\,\partial_x\Big(\big(\sin(\pi x)\sin(\pi y)+1\big)\cos(\pi x)\sin(\pi y)\Big)$$

$$+ \pi\,\partial_y\Big(\big(\sin(\pi x)\sin(\pi y)+1\big)\sin\pi x\cos\pi y\Big)$$

$$= \pi\Big[\partial_x\big(\sin\pi x\cos\pi x\sin^2\pi y + \cos\pi x\sin\pi y\big)$$

$$+ \partial_y\big(\sin^2\pi x\sin\pi y\cos\pi y + \sin\pi x\cos\pi y\big)\Big]$$

$$= \pi\Big[\sin^2(\pi y)\big[\pi\cos^2(\pi x) - \pi\sin^2(\pi y)\big)$$

Then to see what the below thing looks like

$$J_{ij} = J(u^n; \varphi_i, \varphi_j) = \int_\Omega \left(\nabla \varphi_i \cdot (\varphi_j \nabla u^n + (u^n+1) \nabla \varphi_j\right) dx$$

$$\nabla \varphi_i \cdot (\varphi_j \nabla u^n + (u^n+1) \nabla \varphi_j)$$

$$= \begin{bmatrix} \partial_x \varphi_i \\ \partial_y \varphi_i \end{bmatrix} \cdot \begin{bmatrix} (\varphi_j \partial_x u^n) + (u^n+1) \partial_x \varphi_j \\ (\varphi_j \partial_y u^n) + (u^n+1) \partial_y \varphi_j \end{bmatrix}$$

$$\nabla \cdot (u+1) \nabla u = \nabla \cdot \begin{bmatrix} (u+1) \partial_x u \\ (u+1) \partial_y u \\ (u+1) \partial_z u \end{bmatrix}$$

$$= \partial_x ((u+1) \partial_x u) + \partial_y ((u+1) \partial_y u)$$

$$\partial_z ((u+1) \partial_z u)$$

Partition of unity

$$x(x-1) y (y-1)$$

If $u = x(x-1)y(y-1)$

$$\nabla \cdot (u+1)\nabla u = \begin{bmatrix} \partial_x \left(x(x-1)y(y-1)+1\right) & \partial_x x(x-1)y(y-1) \\ \partial_y \left(x(x-1)y(y-1)+1\right) & \partial_y x(x-1)y(y-1) \end{bmatrix}$$

$$= \begin{bmatrix} \partial_x \left(x(x-1)y(y-1)+1\right) & (2x-1)y(y-1) \\ \partial_y \left(x(x-1)y(y-1)+1\right) & x(x-1)(2y-1) \end{bmatrix}$$

$$= \begin{bmatrix} \partial_x & (x^2-1)(2x-1)(y^2(y-1)^2) + (2x-1)y(y-1) \\ \partial_y & x^2(x-1)^2(2y-1)y(y-1) + x(x-1)(2y-1) \end{bmatrix}$$

$$= \Big[$$

Non Linear Poisson Problem

The general form of ~~the~~ non linear poisson problem is

$$-\nabla\cdot(q(u)\nabla u) = g$$

with weak formulation of

$$\int_\Omega q(u)\nabla u\cdot\nabla v\, dx = \int_\Omega g v\, dx$$

Consider in 2D

$$-\nabla\cdot(|\nabla u|^2\nabla u) = g$$

$$= -\nabla\cdot\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)\nabla u = g$$

$$\Rightarrow \int_\Omega\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]\nabla u\cdot\nabla v - g v\, dx = f(u;v)$$

Next we calculate J

$$J(u;v,\hat{u}) = \lim_{\varepsilon\to 0}\frac{f(u+\varepsilon\hat{u};v)-f(u;v)}{\varepsilon}$$

$$\left[\left(\frac{\partial(u+\varepsilon\hat{u})}{\partial x}\right)^2 + \left(\frac{\partial u+\varepsilon\hat{u}}{\partial y}\right)^2\right]\nabla(u+\varepsilon\hat{u})\cdot\nabla\varphi - \cancel{g}\varphi$$

$$-\left(\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]\nabla u\cdot\nabla\varphi - \cancel{g}\varphi\right)$$

$$= \left(\partial_x u + \varepsilon\partial_x\hat{u}\right)^2 + \left(\partial_y u + \varepsilon\partial_y\hat{u}\right)^2\left(\nabla u + \varepsilon\nabla\hat{u}\right)\cdot\nabla\varphi$$

$$-\left[(\partial_x u)^2 + (\partial_y u)^2\right]\nabla u\cdot\nabla\varphi$$

$$= \left[\begin{matrix}\cancel{(\partial_x u)^2} + \varepsilon^2(\partial_x u)^2 + 2\varepsilon(\partial_x u)(\partial_x\hat{u}) \\ + \cancel{(\partial_y u)^2} + \varepsilon^2(\partial_y u)^2 + 2\varepsilon(\partial_y u)(\partial_y\hat{u})\end{matrix}\right]\nabla u\cdot\nabla\varphi$$

$$\left[\begin{matrix}(\partial_x u)^2 + \varepsilon^2(\partial_x u)^2 + 2\varepsilon(\partial_x u)(\partial_x\hat{u}) \\ + (\partial_y u)^2 + \varepsilon^2(\partial_y\hat{u})^2 + 2\varepsilon(\partial_y u)(\partial_y\hat{u})\end{matrix}\right]\varepsilon\nabla\hat{u}\cdot\nabla\varphi$$

$$-\left[\cancel{(\partial_x u)^2} + \cancel{(\partial_y u)^2}\right]\nabla u\cdot\nabla\varphi$$

$$= \left[+ \varepsilon^2 (\partial_x u)^2 + 2\varepsilon (\partial_x u)(\partial_x \hat{u}) + \varepsilon^2 (\partial_y u)^2 + 2\varepsilon (\partial_y u)(\partial_y \hat{u}) \right] \nabla u \cdot \nabla \phi$$

$$\left[(\partial_x u)^2 + \varepsilon^2 (\partial_x u)^2 + 2\varepsilon (\partial_x u)(\partial_x \hat{u}) + (\partial_y u)^2 + \varepsilon^2 (\partial_y \hat{u})^2 + 2\varepsilon (\partial_y u)(\partial_y \hat{u}) \right] \varepsilon \nabla \hat{u} \cdot \nabla \phi$$

divide by ε gives

$$\Rightarrow \left[+ \varepsilon (\partial_x u)^2 + 2 (\partial_x u)(\partial_x \hat{u}) + \varepsilon (\partial_y u)^2 + 2 (\partial_y u)(\partial_y \hat{u}) \right] \nabla u \cdot \nabla \phi$$

$$\left[(\partial_x u)^2 + \varepsilon^2 (\partial_x u)^2 + 2\varepsilon (\partial_x u)(\partial_x \hat{u}) + (\partial_y u)^2 + \varepsilon^2 (\partial_y \hat{u})^2 + 2\varepsilon (\partial_y u)(\partial_y \hat{u}) \right] \nabla \hat{u} \cdot \nabla \phi$$

Then take $\varepsilon \longrightarrow 0$

$$\Rightarrow \left[\cancel{\varepsilon (\partial_x u)^2} + 2 (\partial_x u)(\partial_x \hat{u}) + \cancel{\varepsilon (\partial_y u)^2} + 2 (\partial_y u)(\partial_y \hat{u}) \right] \nabla u \cdot \nabla \phi$$

$$+ \left[(\partial_x u)^2 + \cancel{\varepsilon^2 (\partial_x u)^2} + \cancel{2\varepsilon (\partial_x u)(\partial_x \hat{u})} + (\partial_y u)^2 + \cancel{\varepsilon^2 (\partial_y \hat{u})^2} + \cancel{2\varepsilon (\partial_y u)(\partial_y \hat{u})} \right] \nabla \hat{u} \cdot \nabla \phi$$

$$J = \int_\Omega \Big[\left[2 (\partial_x u)(\partial_x \hat{u}) + 2(\partial_y u)(\partial_y \hat{u})\right] \nabla u \cdot \nabla \phi$$
$$+ \left[(\partial_x u)^2 + (\partial_y u)^2\right] \nabla \hat{u} \cdot \nabla \phi \Big] dx$$

Swap then $\hat{u}$ for $\hat{u} = \sum \hat{u}_i \phi_i$ and $\phi = \phi_j$

$$J(u; \hat{u}, \phi)$$
$$= \sum_i \hat{u}_i \int_\Omega \Bigg\{ 2\left[\frac{\partial u}{\partial x}\frac{\partial \phi_i}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \phi_i}{\partial y}\right] \nabla u \cdot \nabla \phi_j$$
$$+ \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \nabla \phi_i \cdot \nabla \phi_j \Bigg\} dx$$

Lets compute a forcing function g for $u = e^{xy}$

$$-\nabla \cdot (|\nabla u|^2 \nabla u) = g$$

$$= -\nabla \cdot \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} e^{xy} = y e^{xy}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} e^{xy} = x e^{xy}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = e^{xy}(x+y)$$

$$= -\nabla \cdot e^{2xy}(x^2+y^2)\begin{bmatrix} y e^{xy} \\ x e^{xy} \end{bmatrix}$$

$$= -\nabla \cdot \begin{bmatrix} (x^2+y^2) y e^{3xy} \\ (x^2+y^2) x e^{3xy} \end{bmatrix}$$

$$= -\frac{\partial}{\partial x}\left((x^2+y^2) y e^{3xy}\right)$$

$$- \frac{\partial}{\partial y}\left((x^2+y^2) x e^{3xy}\right)$$

$$= -\frac{\partial}{\partial x} x^2 y e^{3xy} - \frac{\partial}{\partial x} y^3 e^{3xy}$$

$$-\frac{\partial}{\partial y} x^3 e^{3xy} - \frac{\partial}{\partial y} y^2 x e^{3xy}$$

$$= -2xye^{3xy} - 3y^2x^2e^{3xy} - 3y^4e^{3xy}$$

$$- 3x^4e^{3xy} - 2yxe^{3xy} - 3y^2x^2e^{3xy}$$

$$= -e^{3xy}\left(2xy - 3y^2x^2 - 3y^4 - 3x^4\right)$$

$$\left(\frac{\partial u}{\partial x}\right)^2 = y^2 e^{2xy}$$

$$\left(\frac{\partial u}{\partial y}\right)^2 = x^2 e^{2xy}$$

$$= -\frac{\partial}{\partial x}\left(xye^{2xy} + y^2e^{2xy}\right)$$

$$-\frac{\partial}{\partial y}\left(x^2e^{2xy} + xye^{2xy}\right)$$

$$= -ye^{2xy} - 2y^2xe^{2xy} - 2y^3e^{2xy}$$

$$- 2x^3e^{2xy} - xe^{2xy} - 2x^2ye^{2xy}$$

$$= e^{2xy}(-y - 2y^2x - 2y^3 - 2x^3 - x - 2x^2y)$$

$$= -e^{2xy}\left(y[1+2(yx+y^2)] + x[1+2(xy+x^2)]\right)$$

So our forcing function is

$$g(x,y) = -e^{2xy}\left(y[1+2(yx+y^2)] + x[1+2(xy+x^2)]\right)$$

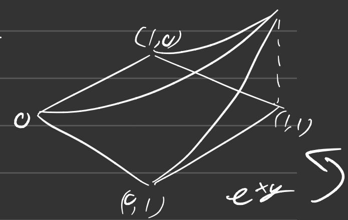
$$-\nabla\cdot(1-w)\nabla u = g$$

Let $u = e^{-((x-1)^2+(y-1)^2)}$

To do

Rewrite Basis functions to allow any boundary conditions

Like this



Vector Partial differential equations

The Navier Stokes equations

$$\rho \frac{\partial \vec{u}}{\partial t} - \nu \nabla^2 \vec{u} + \rho(\vec{u}\cdot\vec{\nabla})\vec{u} + \nabla P = \vec{f}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot \rho\vec{u} = 0$$

In 2D with $\vec{u} = (u_1, u_2)$

$$\rho \frac{\partial u_1}{\partial t} - \nu\left(\frac{\partial^2 u_1}{\partial^2 x} + \frac{\partial^2 u_1}{\partial^2 y}\right) + \rho u_1 \frac{\partial u_1}{\partial x} + \rho u_2 \frac{\partial u_1}{\partial y} + \frac{\partial P}{\partial x} = 0$$

$$\rho \frac{\partial u_2}{\partial t} - \nu\left(\frac{\partial^2 u_2}{\partial^2 x} + \frac{\partial^2 u_2}{\partial^2 y}\right) + \rho u_1 \frac{\partial u_2}{\partial x} + \rho u_2 \frac{\partial u_2}{\partial y} + \frac{\partial P}{\partial y} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

Some simplifications

$\vdots$

$$-\nu\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right) + \frac{\partial p}{\partial x} = 0$$

$$-\nu\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right) + \frac{\partial p}{\partial y} = 0$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0$$

Or more compactly

$$\begin{cases} \nu \nabla^2 u_1 = \dfrac{\partial p}{\partial x} \\[2ex] \nu \nabla^2 u_2 = \dfrac{\partial p}{\partial y} \end{cases} \qquad \nabla \cdot u = 0$$

Now set

$$u_1 = \sum_i u_{1_i} \varphi_i \qquad p = \sum_i p_i \varphi_i$$

$$u_2 = \sum_i u_{2_i} \varphi_i$$

Then if $q = (q_1, q_2, q_3)$

$$\nu \nabla^2 u_1 = \frac{\partial p}{\partial x}$$

$$\Rightarrow \int_\Omega \nu \nabla^2 u_1 \, q_1 = \int_\Omega \frac{\partial p}{\partial x} q_1$$

$$\Rightarrow \nu \int_\Omega \nabla u_1 \cdot \nabla q_1 = \int_\Omega p \frac{\partial q_1}{\partial x}$$

and similarly

$$\nu \int_\Omega \nabla u_2 \cdot \nabla q_2 = \int p \frac{\partial q_2}{\partial y}$$

and

$$\nabla \cdot u = 0$$

$$\Rightarrow \int_\Omega \nabla \cdot u \, q_3 = 0$$

$$\nu \int_\Omega \nabla u_1 \cdot \nabla \phi_1 = \int_\Omega p \frac{\partial \phi_1}{\partial x}$$

$\phi = (\phi_1, \phi_2, \phi_3)$

$\phi_i = (\phi_{1_i}, \phi_{2_i}, \phi_{3_i})$

$\phi_{ijk} = (\phi_{1_i}, \phi_{2_j}, \phi_{3_k})$

$(i,j,k) \quad i,j,k \in [0,+0.1,1] \times [0,+0.1,1]$

$10 \cdot 10 \cdot 10 = 10^3 = 1000$

with $\quad u_1 = \sum u_{1_i} \phi_{1_i} \qquad \phi = \phi_j$

$$\boxed{\sum_i u_{1_i} \left(\nu \int_\Omega \nabla \phi_{1_i} \cdot \nabla \phi_{1_j} \right) = \sum_i p_i \int_\Omega \phi_{3_i} \frac{\partial \phi_{1_j}}{\partial x}} + \int_\Omega f_1 \phi_{1_j}$$

Similarly for u_2 we get

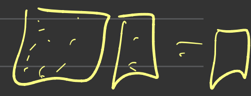
$$\boxed{\sum_i u_{2_i} \left(\nu \int_\Omega \nabla \phi_{2_i} \cdot \nabla \phi_{2_j} \right) = \sum_i p_i \int_\Omega \phi_{3_i} \frac{\partial \phi_{2_j}}{\partial y}}$$

and $\quad \int_\Omega \nabla \cdot u \, \phi_3 = \int_\Omega \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right) \phi_{3_j} = 0$

$$= \boxed{\sum_i u_{1_i} \int_\Omega \frac{\partial \phi_{1_i}}{\partial x} \phi_{3_j} + \sum_i u_{2_i} \int_\Omega \frac{\partial \phi_{2_i}}{\partial y} \phi_{3_j} = 0}$$

$$\underbrace{u_{1_1}(\ ) + u_{1_2}(\ ) + \dots + u_{1_n}(\ )}_{u_1} + \underbrace{u_{2_1}(\ ) + \dots + u_{2_n}(\ )}_{u_2} + \underbrace{0 + \dots + 0}_{p} = \underbrace{0}_{f}$$

$$\overbrace{0+\dots+0}^{u_1} + \overbrace{u_{2_1}(\ )+\dots+u_{2_n}(\ )}^{u_2} - \overbrace{p_1(\ )-\dots-p_n(\ )}^{p} = \overbrace{0}^{b}$$



$$\begin{bmatrix} u_1 & & \\ & u_2 & \\ & & p \end{bmatrix} \begin{bmatrix} u_{1_1} \\ \vdots \\ u_{1_n} \\ u_{2_1} \\ \vdots \\ u_{2_n} \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = \left.\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}\right\} \text{force bits}$$

For the next part

Block Matrix Kinda

$$p\hat{n} = \sin(\pi x)\sin(\pi y)\,\hat{n}$$

Manufacturing a solution

$$-\nu\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right) + \frac{\partial p}{\partial x} = 0$$

$$-\nu\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right) + \frac{\partial p}{\partial y} = 0$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0$$

For zero boundary conditions

Let $u = (\sin(\pi x)\sin(\pi y),\ \sin(\pi x)\sin(\pi y))$

$p = \sin(\pi x)\sin(\pi y)$

$$-\nu\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right) + \frac{\partial p}{\partial x} = 0$$

$$2\nu\pi^2 \sin(\pi x)\sin(\pi y) + \pi\cos(\pi x)\sin(\pi y) = f_1$$

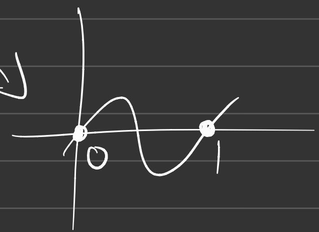
$$2\nu\pi^2 \sin(\pi x)\sin(\pi y) + \pi\sin(\pi x)\cos(\pi y) = f_2$$

But

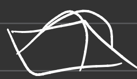
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = 0$$

Complex differentiable function ?!!!!

$$\Rightarrow \boxed{\frac{\partial u_1}{\partial x} = -\frac{\partial u_2}{\partial y}}$$



$$u_1 = \sin(\pi x)\sin(\pi y)$$



$$\frac{\partial u_1}{\partial x} = \pi\cos(\pi x)\sin(\pi y)$$

$$-u_2 = \int \pi\cos(\pi x)\sin(\pi y)\,dy$$

$$= \pi\cos(\pi x)\int \sin \pi y \; dy$$

$$= \pi\cos(\pi x)\left(-\frac{1}{\pi}\cos(\pi y)\right)$$

$$\Rightarrow u_2 = \cos(\pi x)\cos(\pi y) + f(x)$$

But we need $u_2(\{0,1\},\{0,1\}) = 0$ so let $f(x) = -1$

$u_2(0,0) = 0 = 1 \cdot 1 + f(0) =$

$u_2(0,1) = 0 =$

$u_2(1,0) = 0 = 0 + f(1)$

$u_2(1,1) = 1 + f(1) = 0$

$f(0) = -1$

$f(0) = 0$

$1 + f(1) = 1 + f(0) = f(0) = f(1)$

$u_2 = X(x)Y(y)$

If $u_2(a,b) = \begin{cases} 0 & \text{if } a=0 \text{ or } b=0 \\ & \text{else} \end{cases}$

$u_2($

Maybe

$$u_1 = x(x-1)y(y-1)$$

$$\frac{\partial u_1}{\partial x} = y(y-1)\left[(x-1) + x\right]$$

$$= y(y-1)(2x-1)$$

$$-u_2 = (2x-1)\int y(y-1)\,dy$$

$$= (2x-1)\int y^2 - y\,dy$$

$$= (2x-1)\left(\frac{y^3}{3} - \frac{y^2}{2}\right) + f(x)$$

$$= (2x-1)\left(y^2\left(\frac{y}{3} - \frac{1}{2}\right) + f(x)\right)$$

$$u_2 = -(2x-1)y^2\left(\frac{y}{3} - \frac{1}{2}\right) + f(x)$$

$$u_2(0,y) = u_2(1,y) = u_2(x,0) = u_2(x,1)$$

$$u_2(0,y) = y^2\left(\frac{y}{3} - \frac{1}{2}\right) + f(0) = 0$$

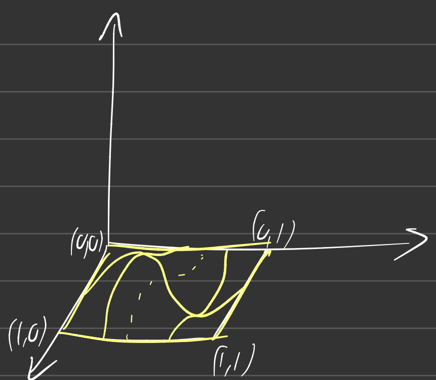
$$u_2(1,y) = -y^2\left(\frac{y}{3} - \frac{1}{2}\right) + f(1) = 0$$

$u_2(x,0) = 0 + f(x)$!

$u_2(x,1) = \frac{1}{6}(2x-1) + f(x) = 0$!

$= \frac{1}{6}$

$$\frac{\partial u_1}{\partial x} = -\frac{\partial u_2}{\partial y}$$

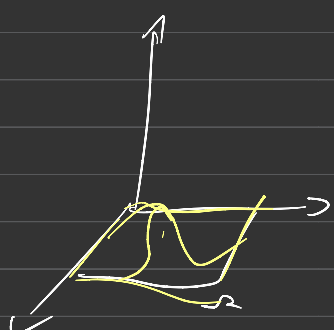


$$\begin{cases} -\nu\left(\dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\partial^2 u_1}{\partial y^2}\right) + \dfrac{\partial p}{\partial x} = f_1 \\[2ex] -\nu\left(\dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{\partial^2 u_2}{\partial y^2}\right) + \dfrac{\partial p}{\partial y} = f_2 \\[2ex] \dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial y} = 0 = \nabla \cdot u \end{cases}$$

$$u(x,y) = (u_1(x,y), u_2(x,y))$$

$$f(x,y) = (f_1(x,y), f_2(x,y))$$

$$\rightarrow \left\{ \frac{\partial u_1}{\partial x} = -\frac{\partial u_2}{\partial y} \right. \quad \text{with} \quad \partial\Omega = 0$$



Another Solution for Stokes equation

$$\nabla \cdot \nabla \gamma = 0$$

$$\gamma(x,y) = (1-\cos(2\pi x))(1-\cos(2\pi y))$$

$$\nabla \gamma = u = \begin{bmatrix} (2\pi \sin(2\pi x))(1-\cos(2\pi y)) \\ (1-\cos(2\pi x))(2\pi \sin(2\pi y)) \end{bmatrix}$$

Also need

$$\int_\Omega p = 0$$

$$\int_0^1\int_0^1 p(x,y)\,dx\,dy = 0$$

$$\int_0^1\int_0^1 p(x)\,p(y)\,dx\,dy = \int_0^1 p(x)\,dx \int_0^1 p(y)\,dy$$

$$\int_0^1 p(x)\,dx = P(1) - P(0) \qquad p(x,y) = (2x-1)(2y-1)$$

$$P(x) = x(x-1) \qquad \frac{\partial}{\partial x} P(x) = 2x - 1$$

$$\int_0^1\int_0^1 p(x,y)\,dx\,dy = \int_0^1 2x-1\,dx \int_0^1 (2y-1)\,dy$$

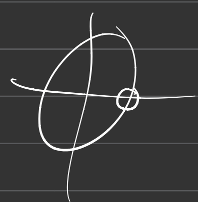
$$= (x^2-x)\Big|_0^1 (y^2-y)\Big|_0^1$$

$$= x(x-1)\Big|_0^1 y(y-1)\Big|_0^1$$

$$= 0\cdot 0 = 0$$

$\int_0^1 p(x)\,dx = 0$ and $p(1) = p(0) = 0$

if $p(x) = \sin(2\pi x)$

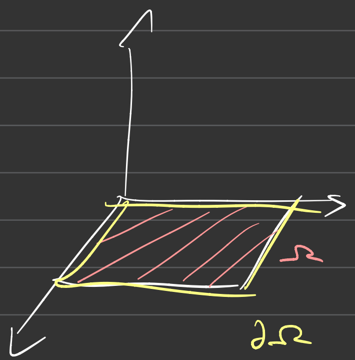


$$\int_0^1 \sin(2\pi x)\,dx = -\frac{\cos 2\pi x}{2\pi}\Big|_0^1$$

$$= \frac{1}{2\pi} - \frac{1}{2\pi} = 0$$

$$\begin{cases} \nabla \cdot ((1+u)\nabla u) = f(x,y) \end{cases}$$

$u(x,y) = (u_1(x,y), u_2(x,y))$



$\Omega = [0,1] \times [0,1]$

Unsteady Stokes

$$\rho \frac{\partial \vec{u}}{\partial t} + \nu \nabla^2 \vec{u} - \nabla p = \vec{f}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0$$

Time discretisation gives

$$\left(\frac{\partial \vec{u}}{\partial t}\right)^{n+1} \approx \frac{\vec{u}^{n+1} - \vec{u}^{n}}{\Delta t}$$

So

$$\rho \left(\frac{\partial \vec{u}}{\partial t}\right)^{n+1} + \nu \nabla^2 \vec{u}^{n+1} - \nabla p^{n+1} = \vec{f}$$

$$= \rho \frac{\vec{u}^{n+1} - \vec{u}^{n}}{\Delta t} + \nu \nabla^2 \vec{u}^{n+1} - \nabla p^{n+1} = \vec{f}$$

$$= \rho \vec{u}^{n+1} - \rho \vec{u}^{n} + \Delta t \nu \nabla^2 \vec{u}^{n+1} - \Delta t \nabla p^{n+1} = \Delta t \vec{f}$$

$$\Rightarrow \rho \vec{u}^{n+1} + \Delta t \nu \nabla^2 \vec{u}^{n+1} - \Delta t \nabla p^{n+1} = \Delta t \vec{f} + \rho \vec{u}^{n}$$

$\Rightarrow$

$$\sum_i u_i^{n+1} \int \left(\rho \phi_i \phi_j + \Delta t \nu \nabla \phi_i \cdot \nabla \phi_j \right) dx - \sum_i P_i \int \left(\phi_j \nabla \psi_i \right) dx$$

$$= \int \left(\Delta t \vec{f} + \rho \vec{u}^n \right) \phi_j \, dx$$

Then

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho u = 0$$

$$\Rightarrow \left(\frac{\partial \rho}{\partial t} \right)^{n+1} + \nabla \cdot \left(\rho^{n+1} u^{n+1} \right) = 0$$

$$= \frac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla \cdot \rho^{n+1} u^{n+1} = 0$$

$$\nabla \cdot \rho \vec{u}$$
$$= \rho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \rho$$

$$\Rightarrow \rho^{n+1} + \Delta t \, \nabla \cdot \left(\rho^{n+1} \vec{u}^{n+1} \right) = \rho^n$$

$$\rho^{n+1} + \Delta t \left[\left(\rho^{n+1} \nabla \cdot \vec{u}^{n+1} \right) + \left(\nabla \rho^{n+1} \cdot \vec{u}^{n+1} \right) \right]$$

$$= \rho^{n+1} + \Delta t \left(\rho^{n+1} \nabla \cdot \vec{u}^{n+1} \right) + \Delta t \left(\nabla \rho^{n+1} \cdot \vec{u}^{n+1} \right) = \rho^n$$

Weak form

$$\rho^{n+1} \phi + \Delta t \left(\rho^{n+1} \nabla \cdot \vec{u}^{n+1} \right) \phi + \Delta t \left(\nabla \rho^{n+1} \cdot \vec{u}^{n+1} \right) \phi = \rho^n \phi$$

$$\rho^n \rightsquigarrow \sum_i \rho_i^n \alpha_i \quad , \quad \alpha \rightsquigarrow \alpha_j$$

$$\int \sum_i \rho_i^n \Big(\alpha_j +$$

Doing time discretisation for NS

$$\rho\frac{\partial\vec{u}}{\partial t} - \nu\nabla^2\vec{u} + \rho(\vec{u}\cdot\vec{\nabla})\vec{u} + \nabla p = \vec{f}$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{u} = 0$$

$$\left(\frac{\partial u}{\partial t}\right)^{n+1} \approx \frac{u^{n+1} - u^n}{\Delta t}$$

$$\rho\frac{\partial u_1}{\partial t} - \nu\nabla^2 u_1 + \rho(\vec{u}\cdot\vec{\nabla})u_1 + \vec{\nabla}p = \vec{f}$$

time discretisation

$u \to u_1^{n+1}$

$\left(\frac{\partial u}{\partial t}\right)^{n+1} \to \frac{u^{n+1} - u^n}{\Delta t}$

$$\rho\left(\frac{u_1^{n+1} - u_1^n}{\Delta t}\right) - \nu\nabla^2 u_1^{n+1} + \rho(\vec{u}^{n+1}\cdot\vec{\nabla})u_1^{n+1} + \vec{\nabla}p^{n+1} = f^{n+1}$$

$$\rho u_1^{n+1} - \rho u_1^n - \nu\Delta t\nabla^2 u_1^{n+1} + \rho\Delta t(\vec{u}^{n+1}\cdot\vec{\nabla})u_1^{n+1} + \Delta t\vec{\nabla}p^{n+1} = \Delta t f^{n+1}$$

$$\rho u_1^{n+1} - \rho u_1^n - \nu\Delta t\nabla^2 u_1^{n+1} + \rho\Delta t(\vec{u}^{n+1}\cdot\vec{\nabla})u_1^{n+1} + \Delta t\vec{\nabla}p^{n+1} = \Delta t f^{n+1}$$

$$\rho u_1^{n+1} - \nu \Delta t \nabla^2 u_1^{n+1} + \rho \Delta t (\vec{u}^{n+1} \cdot \vec{\nabla}) u_1^{n+1} + \Delta t \vec{\nabla} p^{n+1} = \Delta t f^{n+1} + \rho u_1^n$$

<u>Weak form</u>

$$\int_\Omega u_1 \varphi_{1j} - \nu \Delta t \int_\Omega \nabla u_1^{n+1} \cdot \nabla \varphi_{1j} + \int_\Omega \rho \Delta t (\vec{u}^{n+1} \cdot \vec{\nabla}) u_1^{n+1} \varphi_{1j}$$

$$+ \Delta t \int_\Omega p^{n+1} \nabla \varphi_{1j} = \int_\Omega (\Delta t f^{n+1} + \rho u_1^n) \varphi_{1j}$$

$$\Rightarrow \boxed{\begin{aligned} \rho \frac{D\vec{u}}{Dt} &= \rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] \\ &= \rho\vec{f} - \vec{\nabla}P + \eta\nabla^2\vec{u} + (\zeta + \tfrac{\eta}{3})\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) \end{aligned}}$$

Navier Stokes equations

Big Daddy Equation

along with the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

and some equation of state

$$\underbrace{\rho\frac{\partial \vec{u}}{\partial t}}_{\text{time}} + \underbrace{\rho(\vec{u}\cdot\vec{\nabla})\vec{u}}_{\text{non-linear}} = \underbrace{\rho\vec{f} - \nabla P + \eta\vec{\nabla}^2\vec{u} + (\zeta + \tfrac{\eta}{3})\nabla(\vec{\nabla}\cdot\vec{u})}_{\text{linear}}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{u} = 0$$

$$\nabla(\vec{\nabla}\cdot\vec{u}) = \nabla(\partial_x u_x + \partial_y u_y + \partial_z u_z)$$

$$= \begin{bmatrix} \partial_x(\partial_x u_x + \partial_y u_y + \partial_z u_z) \\ \partial_y(\quad \cdot \quad - \quad \cdot \quad) \\ \partial_z(\quad \cdot \quad - \quad \cdot \quad) \end{bmatrix}$$

$$= \begin{bmatrix} \partial_x^2 u_x + \partial_x(\partial_y u_y + \partial_z u_z) \\ \partial_y^2 u_y + \partial_y(\partial_x u_x + \partial_z u_z) \\ \partial_z^2 u_z + \partial_z(\partial_x u_x + \partial_y u_y) \end{bmatrix}$$

$$\partial_x^2 u_x + \partial_x(\partial_y u_y + \partial_z u_z)$$

Fix $u_x = \sum u_{x_i} \phi_i$

$$\partial_x^2 \sum u_{x_i} \phi_i + \partial_x\left(\partial_y \sum_i u_{y_i} \phi_i + \partial_z \sum_i u_{z_i} \phi_i\right)$$

$$= \sum u_{x_i} \partial_x^2 \phi_i + \sum_i u_{y_i} \partial_x \partial_y \phi_i + \sum_i u_{z_i} \partial_x \partial_z \phi_i$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

time discretization

$\rho \to \rho^{n+1}$

$\left(\frac{\partial \rho}{\partial t}\right)^{n+1} \to \frac{\rho^{n+1} - \rho^{n}}{\Delta t}$

$$\frac{\rho^{n+1} - \rho^{n}}{\Delta t} + \vec{\nabla} \cdot \rho^{n+1} \vec{u} = 0$$

$$\rho^{n+1} - \rho^{n} + \Delta t \, \vec{\nabla} \cdot \rho^{n+1} \vec{u} = 0$$

$$\rho^{n+1} + \Delta t \, \vec{\nabla} \cdot (\rho^{n+1} \vec{u}) = \rho^{n}$$

$$\int_\Omega \rho^{n+1} \varphi + \Delta t \, \vec{\nabla} \cdot (\rho^{n+1} \vec{u}) \varphi = \int_\Omega \rho^{n} \varphi$$

$\rho^{n+1} = \sum_i \rho_i^{n+1} \varphi_i$

$u = u^{n+1}$, $\varphi = \varphi_j$

$$\int_\Omega \sum_i \rho_i^{n+1} \varphi_i \varphi_j + \Delta t \left(\vec{\nabla} \cdot \left(\sum \rho_i^{n+1} \varphi_i \vec{u}^{n+1} \right) \right) \varphi_j = \int_\Omega \rho^{n} \varphi_j$$

$$= \int_\Omega \sum_i \rho_i^{n+1} \varphi_i \varphi_j + \Delta$$

$$\vec{\nabla} \cdot \sum \rho_i^{n+1} \mathcal{Q}_i \vec{u} = \vec{\nabla} \cdot \begin{bmatrix} \sum \rho_i^{n+1} (\mathcal{Q}_i u_1) \\ \sum \rho_i^{n+1} (\mathcal{Q}_i u_2) \\ \sum \rho_i^{n+1} (\mathcal{Q}_i u_3) \end{bmatrix}$$

$$\frac{\partial}{\partial x}\left(\sum_i \rho_i^{n+1} (\mathcal{Q}_i u_1) \right)$$

$$= \sum_i \rho_i^{n+1} \frac{\partial}{\partial x} (\mathcal{Q}_i u_1)$$

$$= \sum_i \rho_i^{n+1} (\mathcal{Q}_i' u_1 + \mathcal{Q}_i u_1')$$

$$u_x = x^2(1-x)^2 y(1-y)(1-2y)$$

$$u_y = -x(1-x)(1-2x)\,y^2(1-y)^2$$

$$\frac{\partial u_y}{\partial y} = -x(1-x)(1-2x)\left(2y(1-y)^2 - 2y^2(1-y)\right)$$

$$= -2xy\,(1-x)(1-2x)\left((1-y)^2 - y(1-y)\right)$$

$$= -2xy\,(1-x)(1-2x)(1-y)(1-y-y)$$

$$= -2xy\,(1-x)(1-2x)(1-y)(1-2y)$$

$$\frac{\partial u_x}{\partial x} =$$

$$u_x = x^2(1-x)^2 y(1-y)(1-2y)$$

$$u_y = -x(1-x)(1-2x)\, y^2(1-y)^2$$

$$\frac{\partial u_x}{\partial x} = \left[x^2\left(\partial_x(1-x)^2\right) + \left(\partial_x x^2\right)(1-x)^2\right] \quad y(1-y)(1-2y)$$

$$= \left[x^2 2(1-x)(-1) + 2x(1-x)^2\right] y(1-y)(1-2y)$$

$$= \left[-2x^2(1-x) + 2x(1-x)^2\right] y(1-y)(1-2y)$$

$$= 2x(1-x)\left[-x + (1-x)\right] y(1-y)(1-2y)$$

$$= 2x(1-x)(1-2x)y(1-y)(1-2y)$$

$$u_y = -x(1-x)(1-2x)\, y^2(1-y)^2$$

$$\frac{\partial u_y}{\partial y} = -x(1-x)(1-2x)\left[\left(\partial_y y^2\right)(1-y)^2 + y^2\partial_y(1-y)^2\right]$$

$$= -x(1-x)(1-2x)\left[2y(1-y)^2 + y^2 2(1-y)(-1)\right]$$

$$= -x(1-x)(1-2x)\, 2y(1-y)\left[(1-y) - y\right]$$

$$= -x(1-x)(1-2x)2y(1-y)(1-2y)$$

$$\frac{\partial^2 u_y}{\partial y^2} = \frac{\partial}{\partial y}\left(-x(1-x)(1-2x)2y(1-y)(1-2y)\right)$$

$$= -2x(1-x)(1-2x)\left[(\partial_y y)(1-y)(1-2y) + y\,\partial_y(1-y)(1-2y)\right]$$

$$= -2x(1-x)(1-2x)\left[(1-y)(1-2y) + y\left[(\partial_y(1-y))(1-2y) + (1-y)(\partial_y(1-2y))\right]\right]$$

$$= -2x(1-x)(1-2x)\left[(1-y)(1-2y) + y(-1)(1-2y) + y(1-y)(-2)\right]$$

$$= -2x(1-x)(1-2x)\left[(1-y)(1-2y) - y(1-2y) - 2y(1-y)\right]$$

$$\frac{\partial^2 u_x}{\partial x^2} = \frac{\partial}{\partial x}\left(2x(1-x)(1-2x)y(1-y)(1-2y)\right)$$

$$= 2y(1-y)(1-2y)\frac{\partial}{\partial x}\left(x(1-x)(1-2x)\right)$$

$$= 2y(1-y)(1-2y)\left[(1-x)(1-2x) + x(\partial_x(1-x))(1-2x) + x(1-x)(\partial_x(1-2x))\right]$$

$$= 2y(1-y)(1-2y)\left[(1-x)(1-2x) - x(1-2x) - 2x(1-x)\right]$$

$$u_x = x^2(1-x)^2 y(1-y)(1-2y)$$

$$u_y = -x(1-x)(1-2x)\,y^2(1-y)^2$$

$$\frac{\partial u_x}{\partial y} = x^2(1-x)^2\Big[(\partial_y y)(1-y)(1-2y) + y\big(\partial_y((1-y)(1-2y))\big)\Big]$$

$$= x^2(1-x)^2\big[(1-y)(1-2y) - y(1-2y) - 2y(1-y)\big]$$

$$\frac{\partial^2 u_x}{\partial y^2} = x^2(1-x)^2\big[-2(1-y) - (1-2y) - (1-2y) - y(-2) - 2(1-y)$$
$$- 2(-1)y\big]$$

$$= x^2(1-x)^2\big[-2(1-y) - (1-2y) - (1-2y) + 2y - 2(1-y) + 2y\big]$$

$$= x^2(1-x)^2\big[-2(1-2y) - 4(1-y) + 4y\big]$$

$$\frac{\partial u_y}{\partial x} = -y^2(1-y)^2\frac{\partial}{\partial x}\big(x(1-x)(1-2x)\big)$$

$$= -y^2(1-y)^2\big[(1-x)(1-2x) + x(-1)(1-2x) + x(-2)(1-x)\big]$$

$$= -y^2(1-y)^2\big[(1-x)(1-2x) - x(1-2x) - 2x(1-x)\big]$$

$$\frac{\partial^2 u_y}{\partial x^2} = -y^2(1-y)^2\big[-(1-2x) - 2(1-x) - (1-2x) + 2x$$
$$- 2(1-x) + 2x\big]$$

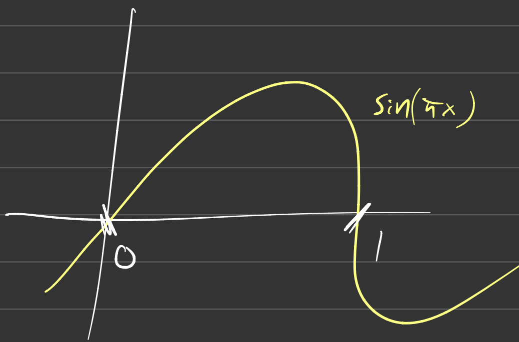
$$= y^2(1-y)^2\big[2(1-2x) + 4(1-x) - 4x\big]$$

$$p(x,y) = \sin(2\pi x)\sin(2\pi y)$$

$$\frac{\partial p}{\partial x} = 2\pi\cos(2\pi x)\sin(2\pi y)$$

$$\frac{\partial p}{\partial y} = 2\pi\sin(2\pi x)\cos(2\pi y)$$

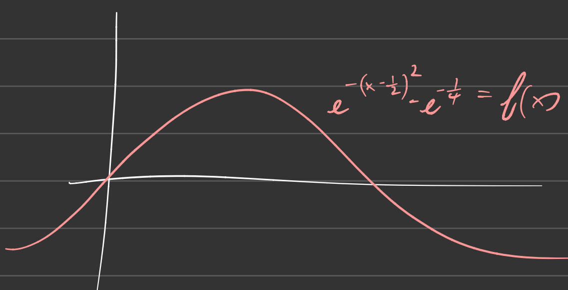
$\sin(\pi x)$ ⟵ for making use of zero bc's



2dim

$\sin(\pi x)\sin(\pi y)$

$\exp(-(x-\frac{1}{2})^2) - a = 0$ for $x = 0, 1$



2dim

$\left(e^{-(x-\frac{1}{2})^2} - e^{-\frac{1}{4}}\right)\left(e^{-(y-\frac{1}{2})^2} - e^{-\frac{1}{4}}\right)$

$e^{-(x-\frac{1}{2})^2-(y-\frac{1}{2})^2} +$

$f(0) = f(1) = e^{-\frac{1}{4}} = a$

$$u_1 = \left(e^{-(x-\frac{1}{2})^2} - e^{-\frac{1}{4}}\right)\left(e^{-(y-\frac{1}{2})^2} - e^{-\frac{1}{4}}\right)$$

$$\frac{\partial u_1}{\partial x} = -\left(e^{-(y-\frac{1}{2})^2} - e^{-\frac{1}{4}}\right)2\left(x-\frac{1}{2}\right)\left(e^{-(x-\frac{1}{2})^2}\right)$$

$$-u_2 = -2\left(x-\frac{1}{2}\right)\left(e^{-(x-\frac{1}{2})^2}\right)\int\left(e^{-(y-\frac{1}{2})^2} - e^{-\frac{1}{4}}\right)dy$$

$$= -2\left(x-\frac{1}{2}\right)\left(e^{-(x-\frac{1}{2})^2}\right)\left(-\frac{1}{2}\sqrt{\pi}\,\mathrm{erf}\left(\frac{1}{2}-x\right) + f(x)\right)$$

Basis functions

$$\varphi_j(x) = \begin{cases} 0 & , x \leq j-h \\ \frac{x}{h} + \left(1 - \frac{j}{h}\right) & , j-h \leq x \leq j \\ \frac{-x}{h} + \left(1 + \frac{j}{h}\right) & , j \leq x \leq j+h \\ 0 & , x \geq j+h \end{cases}$$

$$\varphi_j'(x) = \begin{cases} 0 & , x \leq j-h \\ \frac{1}{h} & , j-h \leq x \leq j \\ -\frac{1}{h} & , j \leq x \leq j+h \\ 0 & , x \geq j+h \end{cases}$$

$$\varphi_{i,j}(x,y) = \varphi_i(x)\varphi_j(y)$$

$$\nabla \varphi_{ij}(x,y) = \left(\frac{\partial}{\partial x} \varphi_{i,j}(x,y) \; , \; \frac{\partial}{\partial y} \varphi_{ij}(x,y) \right)$$

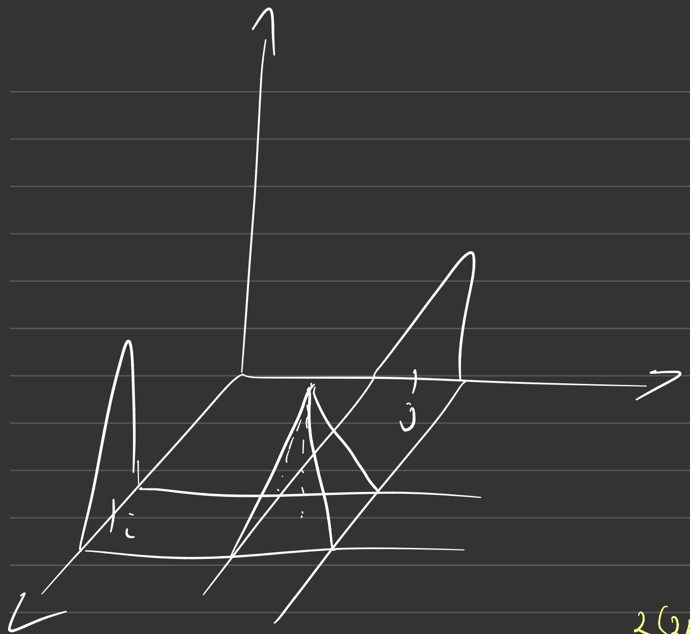
$$= \left(\varphi_i'(x)\varphi_j(y) \; , \; \varphi_i(x)\varphi_j'(y) \right)$$

$$\nabla \ell_{i,j} \cdot \nabla \ell_{a,b} = \ell_i'(x)\,\ell_j(y)\,\ell_a'(x)\,\ell_b(y) + \ell_i(x)\,\ell_j'(y)\,\ell_a(x)\,\ell_b'(y)$$

$$\nabla \ell_{i,j} \cdot \left(\ell_{a,b} \nabla u + (u+1) \nabla \ell_{a,b} \right)$$

$$= \begin{bmatrix} \ell_i'(x)\,\ell_j(y) \\ \ell_i(x)\,\ell_j'(y) \end{bmatrix} \cdot \left(\begin{bmatrix} \ell_{a,b}\, \partial_x(u) \\ \ell_{a,b}\, \partial_y(u) \end{bmatrix} + \begin{bmatrix} (u+1)\,\ell_a'(x)\,\ell_b(y) \\ (u+1)\,\ell_a(x)\,\ell_b'(y) \end{bmatrix} \right)$$

$$= \left(\ell_i'(x)\,\ell_j(y)\right)\left(\ell_a(x)\,\ell_b(y)\,\frac{\partial u}{\partial x} + (u+1)\,\ell_a'(x)\,\ell_b(y)\right) + \left(\ell_i(x)\,\ell_j'(y)\right)\left(\ell_a(x)\,\ell_b(y)\,\frac{\partial u}{\partial y} + (u+1)\,\ell_a(x)\,\ell_b'(y)\right)$$



$2(2h)^2 = c^2$

$8h^2 = c^2$

$c = 2\sqrt{2}h$

$(i-h,\ i+h)$ $(j-h,\ j+h)$

$2h$

$$-\nabla\cdot\left(g(u)\nabla u\right)$$

$$= -\nabla\cdot\left(g(u)\left(\partial_x u,\ \partial_y u,\ \partial_z u\right)\right)$$

$$= -\nabla\cdot\left(g(u)\partial_x u,\ g(u)\partial_y u,\ g(u)\partial_z u\right)$$

$$= -\left(\partial_x, \partial_y, \partial_z\right)\cdot\left(g(u)\partial_x u,\ g(u)\partial_y u,\ g(u)\partial_z u\right)$$

$$= -\ \partial_x\left(g(u)\partial_x u\right) + \partial_y\left(g(u)\partial_y u\right) + \partial_z\left(g(u)\partial_z u\right)$$

$$= -\ \partial_x q(u)\,\partial_x u + q(u)\,\partial_x^2 u +$$

$$\partial_y q(u)\,\partial_y u + q(u)\,\partial_y^2 u +$$

$$\partial_z q(u)\,\partial_z u + q(u)\,\partial_z^2 u$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2$$

$$\frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial y}{\partial x}$$

$$(\nabla u)^2 + \boxed{u\,\nabla^2 u}$$

$u_1u_2 + u_3u_8$

$$\boxed{\vec{\nabla} u = \vec{f}} \qquad u \quad \text{scalar}$$

$$\begin{bmatrix} \partial_x u \\ \partial_y u \end{bmatrix} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$$

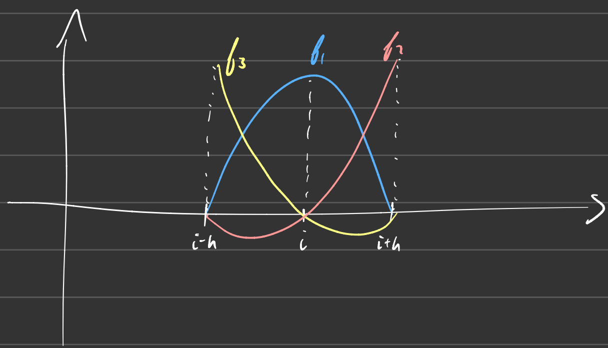
$V = \{ \text{~~arterial~~ basis} \}$

$V_h = \{ \varphi_i(x) \mid \quad \}$

$V_h = \{ (\varphi_i(x), \varphi_j(x)) \mid \quad \} \ni \psi(x)$

$$\vec{\nabla} u \cdot \psi = \vec{f} \cdot \psi$$

Quadratic Basis functions



$$b_1(x) = b(x - (i-h))(x - (i+h))$$

$$b_1(i) = 1 \Rightarrow b(i) = b(i - i + h)(i - i - h)$$

$$= b(h)(-h) = -bh^2 = 1$$

$$b = -\frac{1}{h^2}$$

$$b_1(x) = -\frac{1}{h^2}(x - (i-h))(x - (i+h))$$

$$b_2(x) = b(x - i)(x - (i-h))$$

$$b_2(i+h) = b(i + h - i)(i + h - i + h)) = 1$$

$$= 2bh^2 = 1 \Rightarrow b = \frac{1}{2h^2}$$

$$\begin{cases} f_1(x) = -\frac{1}{h^2}\left(x-(i-h)\right)\left(x-(i+h)\right) \\ f_2(x) = \frac{1}{2h^2}(x-i)\left(x-(i-h)\right) \\ f_3(x) = \frac{1}{2h^2}(x-i)\left(x-(i+h)\right) \end{cases}$$

The function is

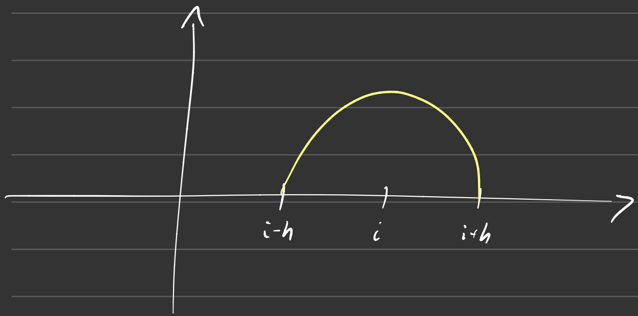
$$\text{guard\_func}(x) = 1 - \left(f_2(x) + f_3(x)\right)$$

$$= 1 - \left(\frac{x-i}{2h^2}\right)\left(\left(x-(i-h)\right) + \left(x-(i+h)\right)\right)$$

$$= 1 - \left(\frac{x-i}{2h^2}\right)\left(x-i+h + x-i-h\right)$$

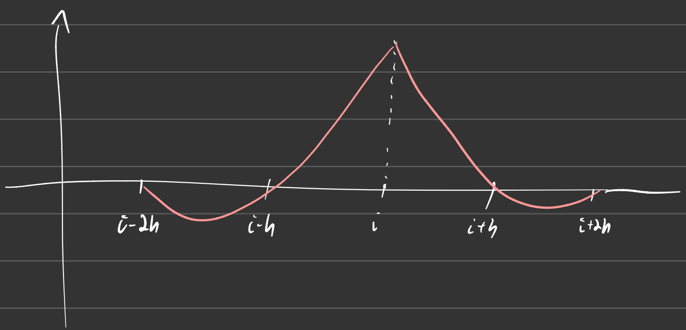
$$= 1 - \left(\frac{x-i}{2h^2}\right)\left(2x-2i\right)$$

$$\boxed{= 1 - \left(\frac{x-i}{h}\right)^2} = f_1(x)$$



$$f_1(x) = 1 - \left(\frac{x-i}{h}\right)^2 = -\frac{1}{h^2}\left(x-(i-h)\right)\left(x-(i+h)\right)$$

$$f(x) = \begin{cases} 0 & x \leq i-h \\ 1 - \left(\frac{x-i}{h}\right)^2 & i-h < x < i+h \\ 0 & x \geq i+h \end{cases}$$



$$g_1(x) = b(x-(i-2h))(x-(i-h))$$

$$g_1(i) = b(i-i+2h)(i-i-h) = 1$$

$$= b(2h)(-h) = 1$$

$$\Rightarrow b = -\frac{1}{2h^2}$$

$$g_2(x) = b(x-(i+h))(x-(i+2h))$$

$$g_2(i) = 1 = b(i-i-h)(i-i-2h)$$

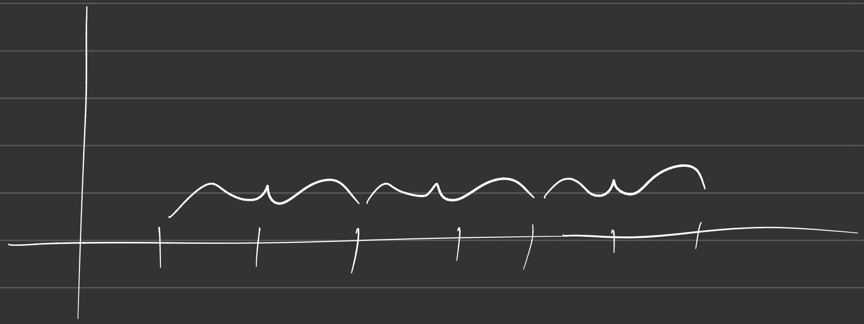
$$= b(-h)-2h = b2h^2$$

$$\Rightarrow b = \frac{1}{2h^2}$$

$$g(x) = \begin{cases} 0 & x \leq i-2h \\ \frac{1}{2h^2}(x-(i-2h))(x-(i-h)) & i-2h < x \leq i \\ \frac{1}{2h^2}(x-(i+h))(x-(i+2h)) & i < x < i+2h \\ 0 & x \geq i+2h \end{cases}$$

Num of elements vertices allowed with quadratic

$H = 3 + 2n \quad , n \in \mathbb{N}$



Things to include in report

- 1 dim Poisson
- A comment on basis functions and 2D poisson
- time dependence - Heat equation
- ~~Non linear~~
- Stokes flow

- A comment on putting things into weak form
- Navier Stokes