MT234P – MULTIVARIABLE CALCULUS – 2022

Fiacre Ó Cairbre

Lecture 1

Chapter 1 – Limits and Partial Derivatives.

Section 1.1 – A little bit of the history of calculus.

Calculus was created independently by Newton and Leibniz in the late 1600s. The ideas in calculus have turned out to be incredibly powerful in solving an abundance of important problems in science, engineering, finance, meteorology, navigation and many other areas.

Section 1.2 – Functions of two or more variables.

Remark 1. Many functions depend on more than one variable. For example, the volume of a cylinder is $\pi r^2 h$ and depends on the two variables, r, h, where r is the radius of the cylinder and h is the height of the cylinder.

Example 1. One can think of the volume above as the function.

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$(r, h) \to \pi r^2 h$$

where the domain of f is $\{(r, h) \in \mathbb{R}^2 : r > 0, h > 0\}$

One sees that $f(2, 3) = 12\pi$

Example 2.

If $f(x, y) = \dfrac{y}{(y - x)^2}$, then the domain of f is

$$\{(x, y) \in \mathbb{R}^2 : x \neq y\}$$

and the range of f is $\mathbb{R}$ because if $t \in \mathbb{R}$, then $t = f(t - 1, t)$

Example 3.

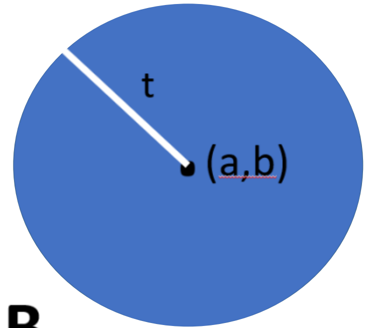
If $g(x, y, z) = e^{\sin(xyz)}$, then the domain of g is $\mathbb{R}^3$ and the range of g is $[e^{-1}, e]$ in $\mathbb{R}$.

Definition 1.

(i) An open ball in $\mathbb{R}^2$ is a set of the form

$$B = \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < t^2\}$$

which has centre (a, b) in $\mathbb{R}^2$ and radius $t > 0$.

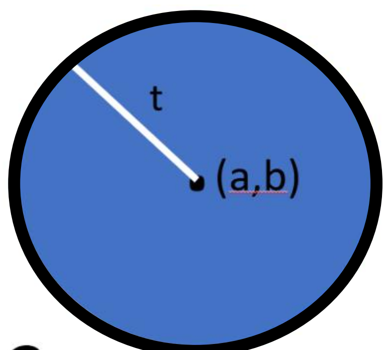


B

(ii) A closed ball in $\mathbb{R}^2$ is a set of the form

$$C = \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b) \leq t^2\}$$

which has centre (a, b) in $\mathbb{R}^2$ and radius $t > 0$.



C

Definition 2.

(i) A point (x, y) in a subset T of $\mathbb{R}^2$ is called an interior point of T if (x, y) is the centre of an open ball that is a subset of T.

(ii) A point (x, y) is a boundary point of a subset W of $\mathbb{R}^2$ if every open ball with centre (x, y) contains points that are not in W and also contains points that are in W. Note that the boundary point (x, y) itself need not be an element of W.

(iii) The interior of a subset X of $\mathbb{R}^2$ is the set of all interior points of X. Denote the set of interior points of X by $\text{Int}(X)$.

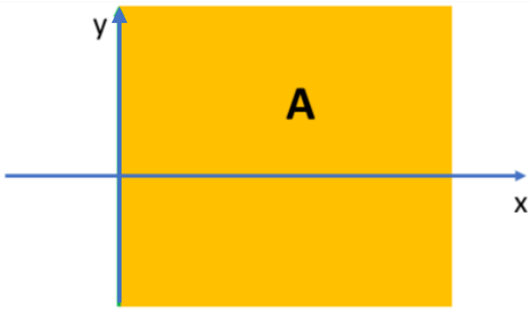
(iv) The boundary of a subset L of $\mathbb{R}^2$ is the set of all boundary points of L. Denote the set of boundary points of L by $\text{Bdy}(L)$.

(v) A subset G of $\mathbb{R}^2$ is called open if and only if $\text{Int}(G) = G$.

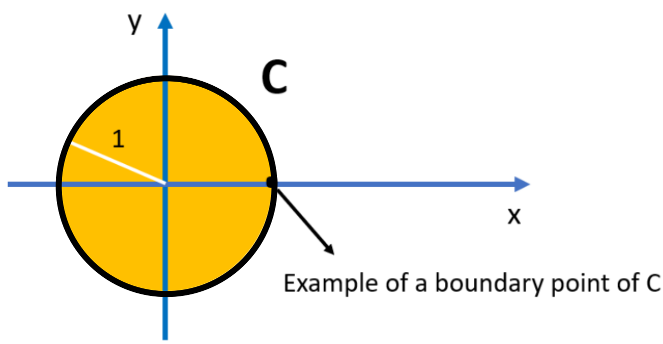
(vi) A subset Z of $\mathbb{R}^2$ is called closed if and only if $\text{Bdy}(Z)$ is a subset of Z.

Example 4.

(a) $A = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ is open because $\text{Int}(A) = A$.



(b) $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is closed because $\text{Bdy}(C)$ is a subset of C.



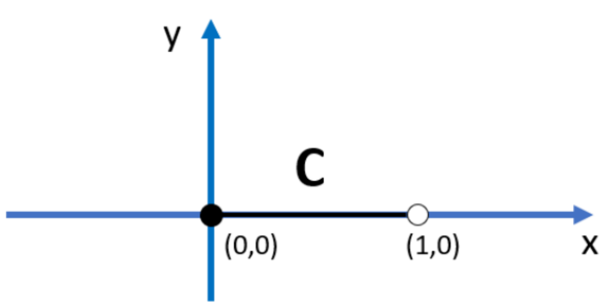
(c) $\mathbb{R}^2$ is open and $\mathbb{R}^2$ is closed.

Example 5.

Give an example of a set in $\mathbb{R}^2$ that is neither open nor closed.

Solution.

$C = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } 0 \leq x < 1\}$ is not open because $(0, 0)$ is an element of C that is not an interior point of C.



C is not closed because $(1, 0)$ is a boundary point of C that is not an element of C. So, C is neither open nor closed.

Definition 3.

A subset T of $\mathbb{R}^2$ is called bounded if it is a subset of an open ball. A subset L of $\mathbb{R}^2$ is called unbounded if it is not bounded.

Example 6.

(i) $C = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 3 \leq y < 8\}$ is bounded because it is a subset of the open ball with centre (0,0) and radius 10.

(ii) $G = \{(x, y) \in \mathbb{R}^2 : y < 4\}$ is unbounded because it is not a subset of an open ball.

Definition 4.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function. Then, the set

$$C_w = \{(x, y) \in \mathbb{R}^2 : f(x, y) = w, \quad \text{for some} \quad w \in \mathbb{R}\}$$

is called a level curve of f. The level curve C_w above is also called the level curve $f(x, y) = w$. Notice that a level curve of f is a subset of the domain of f.

Definition 5.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function. Then, the set

$$G = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \ \text{ is in the domain of } \ f \ \text{ and } \ z = f(x, y)\}$$

is called the graph of f. The graph of f is also called the surface $z = f(x, y)$.

Note that the graph of f is a subset of $\mathbb{R}^3$. Also, note that a level curve of f is a subset of $\mathbb{R}^2$. So, a level curve of f is not a subset of the graph of f.

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Lecture 3

Section 1.3 – Limits and Continuity.

Definition 10.

We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (a, b) and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every $\epsilon > 0$, there exists a corresponding value $\delta > 0$ such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \;\Rightarrow\; |f(x, y) - L| < \epsilon$$

We say that L is the limit of $f(x, y)$ as (x, y) approaches (a, b).

Remark 2.

The intuition behind definition 10 is that L is the limit of $f(x, y)$ as (x, y) approaches (a, b) if we can make $f(x, y)$ as close as we like to L by taking (x, y) sufficiently close to (a, b) but not equal to (a, b).

Remark 3. Note that (a, b) in definition 10 can be any interior point of the domain of f or any boundary point of the domain of f. Also, note that a boundary point of the domain of f need not be in the domain of f. The points (x, y) that approach (a, b) in definition 10 have to be in the domain of f.

Theorem 1.

Suppose $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L_1$ and $\lim\limits_{(x,y)\to(a,b)} g(x, y) = L_2$

Then, the following hold:

(i) $\lim\limits_{(x,y)\to(a,b)} x = a$

(ii) $\lim\limits_{(x,y)\to(a,b)} y = b$

(iii) $\lim\limits_{(x,y)\to(a,b)} k = k, \quad \text{for any} \quad k \in \mathbb{R}$

(iv) $\lim\limits_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L_1 + L_2$

(v) $\lim_{(x,y)\to(a,b)} (f(x,y) - g(x,y)) = L_1 - L_2$

(vi) $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = L_1 L_2$

(vii) $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2}$ if $L_2 \neq 0$

(viii) $\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L_1}$, if $\sqrt[n]{L_1} \in \mathbb{R}$

Example 11.

Find $\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2}$

Solution.

First note that $\lim_{(x,y)\to(3,-4)} x^2 = 9$ and $\lim_{(x,y)\to(3,-4)} y^2 = 16$, by parts (i), (ii) and (vi) in Theorem 1.

So, $\lim_{(x,y)\to(3,-4)} (x^2+y^2) = 25$, by part (iv) in Theorem 1.

Finally, $\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = 5$, by part (viii) in Theorem 1.

Example 12.

Find $\lim_{(x,y)\to(0,0)} f(x,y)$, where $f(x,y) = \dfrac{2x^2-2xy}{\sqrt{x}-\sqrt{y}}$

Solution.

Note that we cannot use Theorem 1(vii) because $\lim_{(x,y)\to(0,0)} (\sqrt{x}-\sqrt{y}) = 0$ from Theorem 1(i), (ii), (v) and (viii).

We have

$$\lim_{(x,y)\to(0,0)} \frac{2x^2-2xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(2x^2-2xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} \qquad (*)$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x(x-y)(\sqrt{x}+\sqrt{y})}{x-y} \qquad (**)$$

$$= \lim_{(x,y)\to(0,0)} 2x(\sqrt{x}+\sqrt{y}) \qquad (***)$$

$$= 0 \qquad (****)$$

Note that we get $(*)$ by multiplying the numerator and denominator of $f(x,y)$ by $\sqrt{x}+\sqrt{y}$ which is allowed because $\sqrt{x}+\sqrt{y}$ is never 0 because we can assume $(x,y) \neq (0,0)$ from definition 10 and remark 2.

Recall from remark 3 that the points (x, y) that approach $(0, 0)$ have to be in the domain of f. So, we get $(***)$ by dividing the numerator and denominator of the function in $(**)$ by $x - y$ which is allowed because $x - y$ is never 0 because the line $x - y = 0$ is not in the domain of f.

Finally, we get $(****)$ by using Theorem 1(i), (ii), (iii), (iv), (vi) and (viii).

Example 13.

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{8xy^2}{x^2 + y^2}$ if it exists.

Solution.

Note that we cannot use Theorem 1(vii) because $\displaystyle\lim_{(x,y)\to(0,0)} (x^2 + y^2) = 0$ from Theorem 1(i), (ii), (iv) and (vi).

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Lecture 2

Example 7.

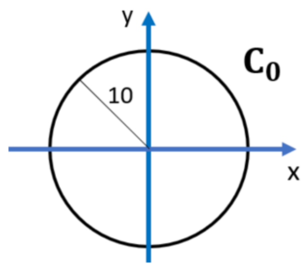
Suppose $f(x,y) = 100 - x^2 - y^2$. Plot the level curves

$$f(x,y) = 0, \quad f(x,y) = 91$$

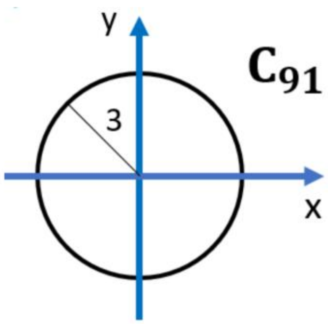
in the domain of f in $\mathbb{R}^2$. Also, draw the graph of the function f.

Solution.

The level curve $f(x,y) = 0$ is the set $C_0 = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 100\}$ which is the circle with centre $(0,0)$ and radius 10.

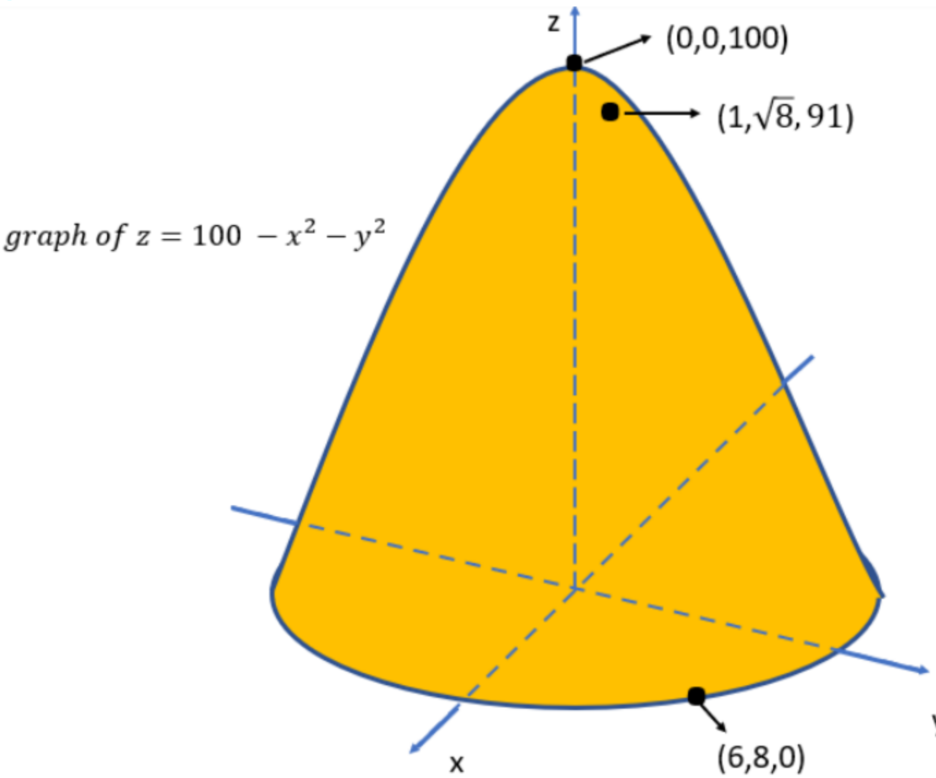


Similarly, the level curve $f(x,y) = 91$ is the set $C_{91} = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 91\}$ which is the circle with centre $(0,0)$ and radius 3.



We will now draw the graph of f in $\mathbb{R}^3$. Note that $f(0,0) = 100$ and so $(0,0,100)$ is on the graph. Note also that $(0,0,100)$ is the top point on the surface $z = f(x,y)$. Also, another example of a point on the graph of f is $(6,8,0)$ because $0 = 100 - 6^2 - 8^2$.

Note that $(6,8)$ is on the level curve C_0 in $\mathbb{R}^2$ and $(6,8,0)$ is on the graph of f. Similarly, note that $(1,\sqrt{8})$ is on the level curve C_{91} and $(1,\sqrt{8},91)$ is on the graph of f.



The domain of $f(x, y)$ is $\mathbb{R}^2$. Note that the level curve $f(x, y) = 100$ is the set consisting of the single point $(0, 0)$. The picture above shows the surface $z = 100 - x^2 - y^2$ for $z \geq 0$. The surface continues on down for $z < 0$. For example, when $z = -21$, we have $x^2 + y^2 = 121$ and so $(1, \sqrt{120}, -21)$ is a point on the surface $z = f(x, y)$. Also, the level curve $f(x, y) = -21$ is the set

$$C_{-21} = \{(x, y) \in \mathbb{R}^2 : f(x, y) = -21\}$$

$$= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 121\}$$

which is the circle with centre $(0, 0)$ and radius 11.

For each fixed $w \leq 100$, the level curve $C_w = \{(x, y) \in \mathbb{R}^2 : f(x, y) = w\}$ is the circle with centre at the origin $(0, 0)$ and radius $\sqrt{100 - w}$.

This explains why the surface looks the way it does – because as w increases up through the z values on the z–axis, the level curves $f(x, y) = w$ are circles with smaller and smaller radius. Note that the points on the graph of f are on the surface in the picture and not inside. For example, $(0, 0, 0)$ is inside but not on the surface because $x = 0,\ y = 0,\ z = 0$ doesn't satisfy the equation $z = 100 - x^2 - y^2$. However, $(1, \sqrt{8}, 91)$ is on the surface (as in the picture) because $x = 1,\ y = \sqrt{8},\ z = 91$ satisfies the equation $z = 100 - x^2 - y^2$.

Example 8.

Suppose $H = \{(x, y) \in \mathbb{R}^2 : y \geq 6\}$. Then $\text{Int}(H) = \{(x, y) \in \mathbb{R}^2 : y > 6\}$ and $\text{Bdy}(H) = \{(x, y) \in \mathbb{R}^2 : y = 6\}$

Definition 6.

(i) An open ball in $\mathbb{R}^3$ is a set of the form

$$S = \{(x, y, z) \in \mathbb{R}^3 : (x - a)^2 + (y - b)^2 + (z - c)^2 < t^2\}$$

which has centre (a, b, c) in $\mathbb{R}^2$ and radius $t > 0$.

(ii) A closed ball in $\mathbb{R}^3$ is a set of the form

$$W = \{(x, y, z) \in \mathbb{R}^3 : (x - a)^2 + (y - b) + (z - c)^2 \leq t^2\}$$

which has centre (a, b, c) in $\mathbb{R}^3$ and radius $t > 0$.

Definition 7.

(i) A point (x, y, z) in a subset T of $\mathbb{R}^3$ is called an interior point of T if (x, y, z) is the centre of an open ball that is a subset of T.

(ii) A point (x, y, z) is a boundary point of a subset W of $\mathbb{R}^3$ if every open ball with centre (x, y, z) contains points that are not in W and also contains points that are in W. Note that the boundary point (x, y, z) itself need not be an element of W.

(iii) The interior of a subset X of $\mathbb{R}^3$ is the set of all interior points of X. Denote the set of interior points of X by $\text{Int}(X)$.

(iv) The boundary of a subset L of $\mathbb{R}^3$ is the set of all boundary points of L. Denote the set of boundary points of L by $\text{Bdy}(L)$.

(v) A subset G of $\mathbb{R}^3$ is called open if and only if $\text{Int}(G) = G$.

(vi) A subset Z of $\mathbb{R}^3$ is called closed if and only if $\text{Bdy}(Z)$ is a subset of Z.

Example 9.

(a) $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$ is open because $\text{Int}(A) = A$.

(b) $C = \{(x, y, z) \in \mathbb{R}^3 : z \leq 0\}$ is closed because $\text{Bdy}(C)$ is a subset of C.

Definition 8.

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a function and $w \in \mathbb{R}$. Then, the set

$$S_w = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = w\}$$

is called a level surface of f. The level surface S_w above is also called the level surface $f(x, y, z) = w$. Notice that a level surface of f is a subset of the domain of f.

Definition 9.

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a function. Then, the set

$$G = \{(x, y, z, q) \in \mathbb{R}^4 : (x, y, z) \ \text{ is in the domain of } \ f \ \text{ and } \ q = f(x, y, z)\}$$

is called the graph of f. Note that the graph of f is a subset of $\mathbb{R}^4$.

Example 10.

Describe the level surfaces of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

Solution.

Suppose $w \in \mathbb{R}$ and $w \geq 0$. Then, the level surface $f(x, y, z) = w$ is the set

$$S_w = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = w\}$$

$$= \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = w^2\}$$

which is the sphere with centre $(0, 0, 0)$ and radius w.

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Lecture 4

Example 13 continued.

Suppose $f(x,y) = \dfrac{8xy^2}{x^2+y^2}$. We will prove that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$ by showing that definition 10 is satisfied with $L = 0$ and $(a,b) = (0,0)$.

So, suppose $\epsilon > 0$. Note that

$$\left|\frac{8xy^2}{x^2+y^2}\right| = \frac{8|x|y^2}{x^2+y^2}$$

$$\leq 8|x|$$

$$= 8\sqrt{x^2}$$

$$\leq 8\sqrt{x^2+y^2} \qquad\qquad (*)$$

So, if we choose $\delta = \dfrac{\epsilon}{8}$, then we have that

$$0 < \sqrt{x^2+y^2} < \delta \;\Rightarrow\; |f(x,y) - 0| = \left|\frac{8xy^2}{x^2+y^2}\right|$$

$$\leq 8\sqrt{x^2+y^2} \;\text{ by }\; (*)$$

$$< 8\delta$$

$$= \epsilon$$

So, we have shown that for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that

$$0 < \sqrt{x^2+y^2} < \delta \;\Rightarrow\; |f(x,y) - 0| < \epsilon$$

and we are done because definition 10 is satisfied with $L = 0$ and $(a,b) = (0,0)$ and so $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Example 14.

Suppose $f(x,y) = \dfrac{y}{x}$. Does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution.

We will show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

First note that $f(x,y) = 0$ for any points (x,y) on the line $y = 0$ excluding $(0,0)$. Also, note that $f(x,y) = 1$ for any points (x,y) on the line $y = x$ excluding $(0,0)$.

So, any open ball with centre $(0,0)$ and positive radius will contain points where $f(x,y) = 0$ and will contain points where $f(x,y) = 1$. $\quad (*)$

Now, suppose $L \in \mathbb{R}$. Also, suppose $\epsilon = \dfrac{1}{4}$. Then, there is no corresponding $\delta > 0$ such that

$$0 < \sqrt{x^2 + y^2} < \delta \;\Rightarrow\; |f(x,y) - L| < \epsilon \qquad (**)$$

because $(*)$ and $(**)$ mean that $|0 - L| < \dfrac{1}{4}$ and $|1 - L| < \dfrac{1}{4}$ which implies $|1 - 0| \leq |1 - L| + |L - 0| < \dfrac{1}{2}$ which is impossible.

This means that definition 10 cannot be satisfied for $L \in \mathbb{R}$ and so $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist and we are done.

Remark 4.

The idea behind the solution in example 14 is that the limit doesn't exist because $f(x,y)$ approaches two different values (0 and 1) as (x,y) approaches $(0,0)$ along two different paths ($y = 0$ and $y = x$) in the domain of f. We will now state this idea as a theorem.

Theorem 2 – Two path test for non–existence of a limit.

Suppose a function $f(x,y)$ approaches two different values as (x,y) approaches (a,b) along two different paths in the domain of f. Then, $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example 15.

Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}$ does not exist.

Solution.

Suppose $f(x,y) = \dfrac{2xy}{x^2 + y^2}$

Now, for every $m \in \mathbb{R}$, we have that $f(x,y) = \dfrac{2m}{1 + m^2}$ on the line $y = mx$, excluding $(0,0)$, because if $y = mx$ with $x \neq 0$, then

$$f(x,y) = \frac{2xy}{x^2+y^2} = \frac{2x(mx)}{x^2+(mx)^2} = \frac{2mx^2}{(1+m^2)x^2} = \frac{2m}{1+m^2}$$

So, f approaches two different values (0 and $\frac{4}{5}$) as (x,y) approaches $(0,0)$ along two different paths ($y=0$ and $y=2x$) in the domain of f.

Hence, by Theorem 2, we have that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist and we are done.

Remark 5.

In this chapter, you may assume all functions are real valued unless otherwise stated.

Example 16.

Prove that $\lim\limits_{(x,y)\to(0,0)} \frac{y^2-x^4}{x^4+y^2}$ does not exist.

Solution.

Suppose $f(x,y) = \frac{y^2-x^4}{x^4+y^2}$

Now, for every $m \in \mathbb{R}$, we have that $f(x,y) = \frac{m^2-1}{m^2+1}$ on the path $y = mx^2$, excluding $(0,0)$, because if $y = mx^2$ with $x \neq 0$, then

$$f(x,y) = \frac{m^2x^4-x^4}{x^4+m^2x^4} = \frac{(m^2-1)x^4}{(1+m^2)x^4} = \frac{m^2-1}{m^2+1}$$

So, f approaches two different values (0 and $\frac{3}{5}$) as (x,y) approaches $(0,0)$ along two different paths ($y=x^2$ and $y=2x^2$) in the domain of f.

Hence, by Theorem 2, we have that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist and we are done.

Example 17.

Find $\lim\limits_{(x,y)\to(0,2)} \frac{x-xy+4}{x^3y-5xy-y^2}$ if it exists.

Solution.

Using all parts of Theorem 1, except part (viii), we get that

$$\lim_{(x,y)\to(0,2)} \frac{x-xy+4}{x^3y-5xy-y^2} = \frac{0-0+4}{0-0-4} = -1$$

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Lecture 5

Definition 11.

A function $f(x,y)$ is said to be continuous at (a,b) if the following conditions are satisfied:

(i) $f(a,b)$ is defined.

(ii) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.

(iii) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

A function is called continuous if it's continuous at every point in its domain.

Example 18.

Consider f defined as follows:

$$f(x,y) = \frac{2xy}{x^2+y^2}, \quad \text{for} \quad (x,y) \neq (0,0) \qquad \text{and} \quad f(0,0) = 0$$

Prove that f is continuous at every point in $\mathbb{R}^2$ except $(0,0)$.

Solution.

We know from example 15 that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist and so f is not continuous at $(0,0)$.

If $(a,b) \neq (0,0)$, then by Theorem 1

$$\lim_{(x,y)\to(a,b)} \frac{2xy}{x^2+y^2} = \frac{2ab}{a^2+b^2} = f(a,b)$$

So, f is continuous at (a,b).

Theorem 3.

Suppose $f(x,y)$ is continuous at (a,b) and suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous at $f(a,b)$. Then, $h = g \circ f$ is continuous at (a,b).

Here, $g \circ f$ denotes the composition function g after f, which means that $(g \circ f)(x,y) = g(f(x,y))$.

Example 19.

$h(x,y) = e^{3x-y}$ is continuous at every point $(a,b) \in \mathbb{R}^2$.

Proof.

Let $f(x,y) = 3x - y$ and let $g(x) = e^x$. Then, $h = g \circ f$. Theorem 1 gives

$$\lim_{(x,y)\to(a,b)} (3x - y) = 3a - b$$

So, $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ and hence f is continuous at (a,b). Also, g is continuous and so by Theorem 3 we get that h is continuous at (a,b).

Section 1.4 – Partial Derivatives.

Definition 12.

Suppose $g : \mathbb{R} \to \mathbb{R}$ is a function. We say that L is the limit of $g(x)$ as x approaches a and we write

$$\lim_{x\to a} g(x) = L$$

if for every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that

$$0 < |x - a| < \delta \;\Rightarrow\; |g(x) - L| < \epsilon$$

Remark 6.

Note how definition 12 is similar to definition 10. The intuition behind definition 12 is similar to the intuition behind definition 10, i.e. L is the limit of $g(x)$ as x approaches a if we can make $g(x)$ as close as we like to L by taking x sufficiently close to a but not equal to a.

Example 20.

As an example of limits of functions of one variable, we will discuss the derivative of a function because we will need it for partial derivatives.

Suppose $g : \mathbb{R} \to \mathbb{R}$. Then, the derivative of $g(x)$ is denoted by $g'(x)$ and the value of $g'(x)$ at $a \in \mathbb{R}$ is defined as

$$g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

We also denote $g'(a)$ by $\dfrac{d}{dx} g(x)_{|x=a}$

Now, we will find $f'(x)$ and $f'(3)$, where $f(x) = x^2$.

Well, for $x \in \mathbb{R}$, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} (2x+h), \quad \text{because we can assume} \quad h \neq 0 \quad \text{in the limit as} \quad h \quad \text{approaches} \quad 0$$

$$= 2x$$

So, we get $f'(x) = 2x$ and hence $f'(3) = 6$ and we are done.

Definition 13.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. The partial derivative of $f(x, y)$ with respect to x is denoted by $\dfrac{\partial f}{\partial x}$ and the value of $\dfrac{\partial f}{\partial x}$ at the point (a, b) is defined as

$$\frac{\partial f}{\partial x}_{|(a,b)} = \frac{d}{dx} f(x, b)_{|x=a} = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \qquad (*)$$

Remark 7.

Note that in definition 13, we can consider b as a constant and so $q(x) = f(x, b)$ can be considered as a function from $\mathbb{R}$ to $\mathbb{R}$ and so $(*)$ in definition 13 is saying that

$$\frac{\partial f}{\partial x}_{|(a,b)} = \frac{d}{dx} q(x)_{|x=a}$$

So, the way to find $\dfrac{\partial f}{\partial x}$ is to treat the y variable as a constant and differentiate with respect to x.

Example 21.

If $f(x, y) = x^3 - 3x^2y + 2$, then

$$\frac{\partial f}{\partial x} = 3x^2 - 6xy + 0 = 3x^2 - 6xy \quad \text{and} \quad \frac{\partial f}{\partial x}_{|(2,-3)} = 12 + 36 = 48$$

Definition 14.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. The partial derivative of $f(x, y)$ with respect to y is denoted by $\dfrac{\partial f}{\partial y}$ and the value of $\dfrac{\partial f}{\partial y}$ at the point (a, b) is defined as

$$\frac{\partial f}{\partial y}_{|(a,b)} = \frac{d}{dy} f(a, y)_{|y=b} = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h} \qquad (*)$$

Remark 8.

Note that in definition 14, we can consider a as a constant and so $p(y) = f(a, y)$ can be considered as a function from $\mathbb{R}$ to $\mathbb{R}$ and so $(*)$ in definition 14 is saying that

$$\frac{\partial f}{\partial y}_{|(a,b)} = \frac{d}{dy} p(y)_{|y=b}$$

So, the way to find $\dfrac{\partial f}{\partial y}$ is to treat the x variable as a constant and differentiate with respect to y.

Example 22.

If $f(x, y) = x^3 - 3x^2 y + 2$, then

$$\frac{\partial f}{\partial y} = -3x^2 \quad \text{and} \quad \frac{\partial f}{\partial y}_{|(2,-3)} = -12$$

Example 23.

If $g(x, y) = \arctan(x^2 y) + \arctan(x y^2)$, then

$$\frac{\partial g}{\partial x} = \frac{2xy}{1 + x^4 y^2} + \frac{y^2}{1 + x^2 y^4}$$

and

$$\frac{\partial g}{\partial y} = \frac{x^2}{1 + x^4 y^2} + \frac{2xy}{1 + x^2 y^4}$$

Remark 9.

For functions of more than two variables, we define partial derivatives in a similar way and we evaluate partial derivatives in a similar way as the next example will show.

Example 24.

If $f(x, y, z) = \ln(x + 2y + 3z)$, then treat y and z as constant and differentiate with respect to x to get

$$\frac{\partial f}{\partial x} = \frac{1}{x + 2y + 3z}$$

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Lecture 6

Example 24 continued

Treat x and z as constant and differentiate with respect to y to get

$$\frac{\partial f}{\partial y} = \frac{2}{x+2y+3z}$$

Treat x and y as constant and differentiate with respect to z to get

$$\frac{\partial f}{\partial z} = \frac{3}{x+2y+3z}$$

Remark 10.

We can differentiate twice to get second order partial derivatives. For example,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Example 25.

If $f(x,y) = x\cos y + ye^x$, then

$$\frac{\partial f}{\partial x} = \cos y + ye^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial f}{\partial y} = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

Theorem 4 – Mixed Partial Derivatives Theorem.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose $f(x, y)$ and its partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

are defined in an open set containing (a, b) and are also continuous at (a, b). Then

$$\frac{\partial^2 f}{\partial x \partial y}_{|(a,b)} = \frac{\partial^2 f}{\partial y \partial x}_{|(a,b)}$$

Section 1.5 – Differentiability.

Remark 11.

Suppose $g : \mathbb{R} \to \mathbb{R}$. Suppose g is differentiable at a, i.e. $g'(a)$ exists.

Then $g'(a) = \lim_{h \to 0} \dfrac{g(a+h) - g(a)}{h}$.

Let $\Delta x = h$ and so Δx denotes the change in x as x goes from a to $a + \Delta x$. Similarly, let Δy denote the corresponding change in $g(x)$ as x goes from a to $a + \Delta x$, i.e. $\Delta y = g(a + \Delta x) - g(a)$.

Then,

$$g'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{and so} \quad 0 = \lim_{\Delta x \to 0} (\frac{\Delta y}{\Delta x} - g'(a))$$

Let $\epsilon = \dfrac{\Delta y}{\Delta x} - g'(a)$ and so $\lim_{\Delta x \to 0} \epsilon = 0$

Then

$$\Delta y = g'(a)\Delta x + \epsilon \Delta x, \text{ where } \epsilon \to 0 \text{ as } \Delta x \to 0 \qquad (*)$$

It is this property $(*)$ that we will generalise to functions of two variables.

Theorem 5.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose that the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are defined on an open set W containing the point (a,b). Suppose that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous at (a,b). Define Δz as $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$ so that Δz is the change in f as (x,y) goes from (a,b) to $(a+\Delta x, b+\Delta y)$ in W.

Then, we have the following generalisation of $(*)$:

$$\Delta z = \frac{\partial f}{\partial x}_{|(a,b)} \Delta x + \frac{\partial f}{\partial y}_{|(a,b)} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \text{where } \epsilon_1 \text{ and } \epsilon_2 \to 0 \text{ as } \Delta x \text{ and } \Delta y \to 0 \; (**)$$

Definition 15.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. We say f is differentiable at (a,b) if $\dfrac{\partial f}{\partial x}_{|(a,b)}$ and $\dfrac{\partial f}{\partial y}_{|(a,b)}$ exist and $(**)$ holds for f at (a,b). We say f is differentiable if it's differentiable at every point in its domain.

Theorem 6.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. If $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous on an open set W, then f is differentiable at every point in W.

Theorem 7.

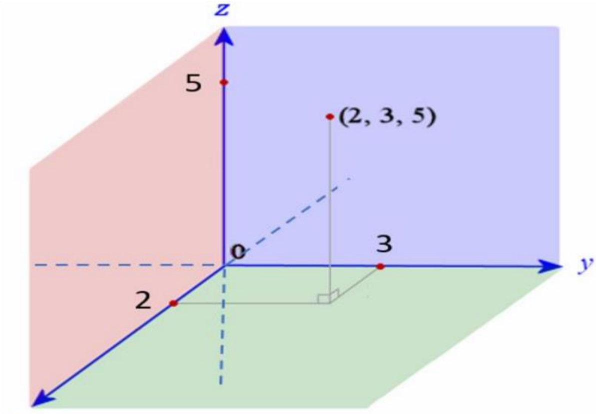
Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. If f is differentiable at (a,b), then f is continuous at (a,b).

Section 1.6 – Quadrics.

In this section we will group some surfaces under the heading of quadric surfaces.

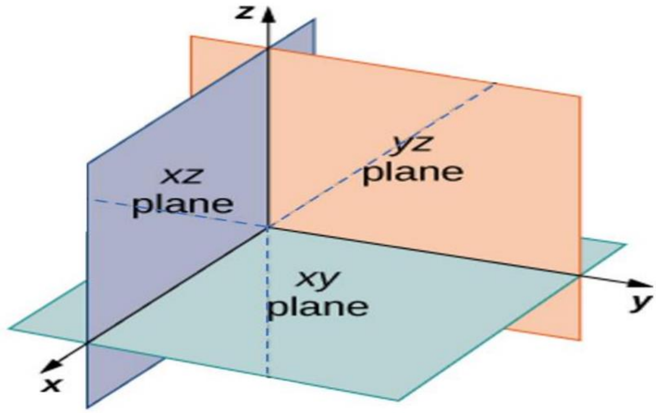
Remark 12.

Recall how to plot points in $\mathbb{R}^3$.



Remark 13.

Recall planes in $\mathbb{R}^3$.



Remark 14.

The equation of a quadric surface in $\mathbb{R}^3$ is a second degree equation of the form

$$kx^2 + ly^2 + mz^2 + nxy + pxz + qyz + rx + sy + tz + u = 0$$

where $k, l, m, n, p, q, r, s, t, u \in \mathbb{R}$

So, the quadric surface is the set of points:

$$\{(x, y, z) \in \mathbb{R}^3 : kx^2 + ly^2 + mz^2 + nxy + pxz + qyz + rx + sy + tz + u = 0\}$$

The intersection of the surface with a plane is called the trace of the surface in the plane. In order to visualise the surface, it can be useful to find the traces in planes parallel to the xy-plane, planes parallel to the xz–plane and planes parallel to the yz-plane. Note that a plane parallel to the xy–plane has the equation $z = t$, for some constant $t \in \mathbb{R}$. Similarly, a plane parallel to the xz–plane has the equation $y = s$, for some constant $s \in \mathbb{R}$. Finally, a plane parallel to the yz–plane has the equation $x = w$, for some constant $w \in \mathbb{R}$.

Example 26 – Elliptic paraboloid.

The equation is

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

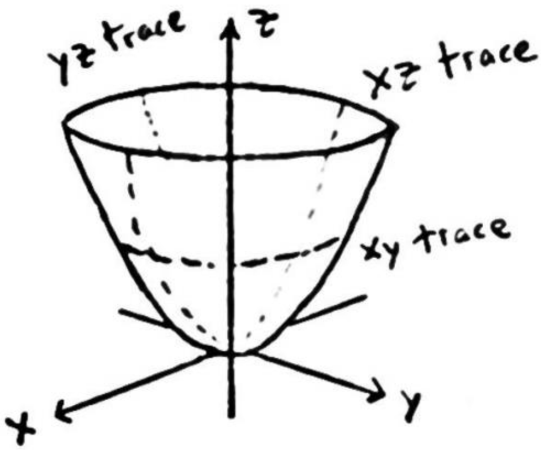
where $a, b \in \mathbb{R}$. The relevant traces in a plane

parallel to the xy–plane are ellipses $\quad (*)$

parallel to the xz–plane are parabolas $\quad (**)$

parallel to the yz–plane are parabolas

The reason for $(*)$ above is because when $z = t$, for some constant $t \in \mathbb{R}$, then $t = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ is an ellipse. The reason for $(**)$ above is because when $y = s$, for some constant $s \in \mathbb{R}$, then $z = \dfrac{x^2}{a^2} + \dfrac{s^2}{b^2}$ is a parabola. See the picture below.



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Lecture 7

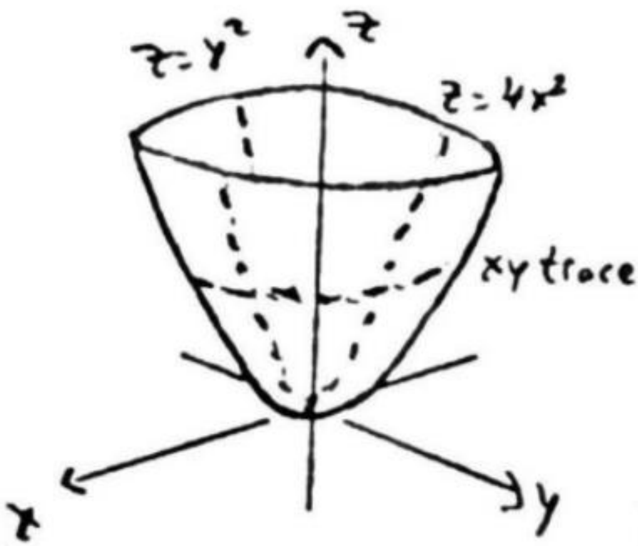
Example 27.

Sketch the surface given by $z - y^2 - 4x^2 = 0$.

We first write the equation in the form

$$z = 4x^2 + y^2$$

and so we have an elliptic paraboloid with the picture below. Note that the trace in the $z = 4$ plane (parallel to the xy–plane) is the ellipse $x^2 + \frac{y^2}{4} = 1$. The trace in the xz–plane is the parabola $z = 4x^2$. The trace in the yz–plane is the parabola $z = y^2$. See the picture below.



Example 28 – Ellipsoid.

The equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

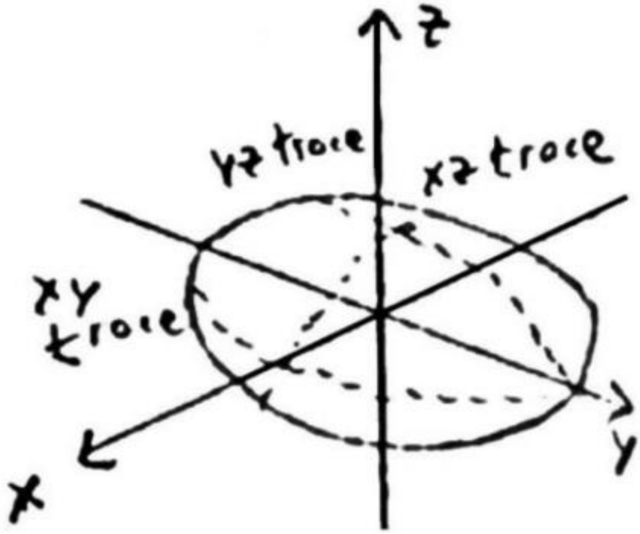
where $a, b, c \in \mathbb{R}$. The relevant traces in a plane

parallel to the xy–plane are ellipses

parallel to the xz–plane are ellipses

parallel to the yz–plane are ellipses

See the picture below.



Example 29 – Elliptic Cone.

The equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

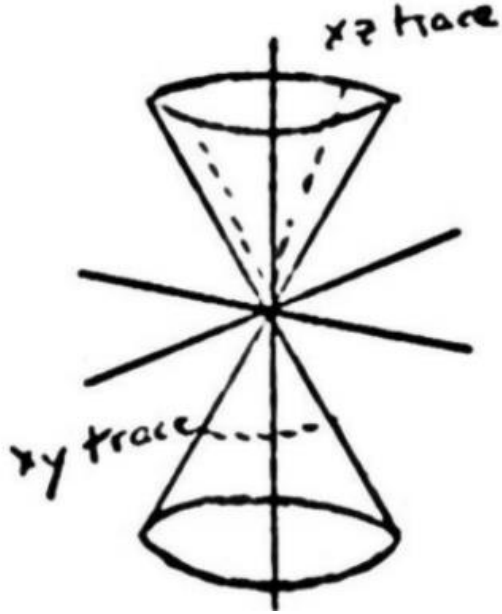
where $a, b, c \in \mathbb{R}$. The relevant traces in a plane

parallel to the xy–plane are ellipses

parallel to the xz–plane are hyperbolas

parallel to the yz–plane are hyperbolas

See the picture below.



Example 30 – Hyperbolic Paraboloid.

The equation is

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

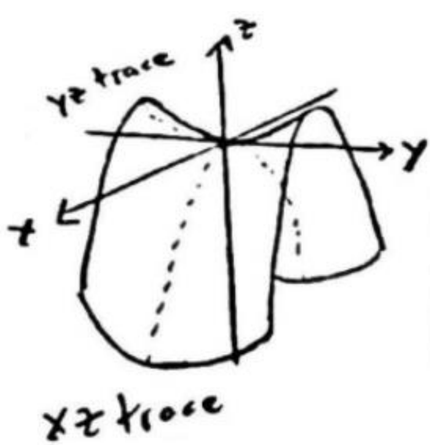
where $a, b \in \mathbb{R}$. The relevant traces in a plane

parallel to the xy–plane are hyperbolas

parallel to the xz–plane are parabolas

parallel to the yz–plane are parabolas

See the picture below.



Example 31 – Hyperboloid of one sheet.

The equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

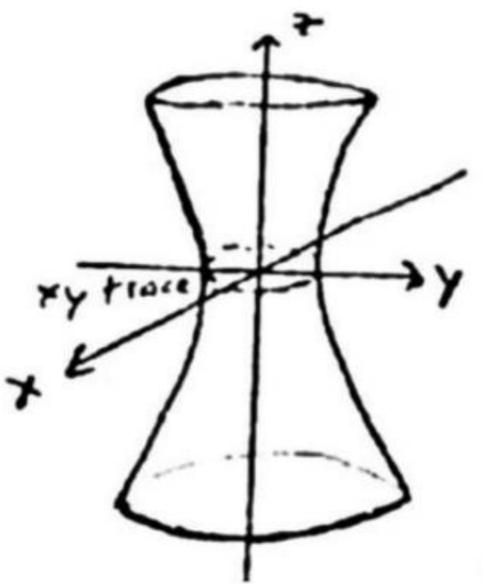
where $a, b, c \in \mathbb{R}$. The relevant traces in a plane

$$\text{parallel to the} \quad xy\text{–plane} \quad \text{are ellipses}$$

$$\text{parallel to the} \quad xz\text{–plane} \quad \text{are hyperbolas}$$

$$\text{parallel to the} \quad yz\text{–plane} \quad \text{are hyperbolas}$$

See the picture below.



Example 32 – Hyperboloid of two sheets.

The equation is

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

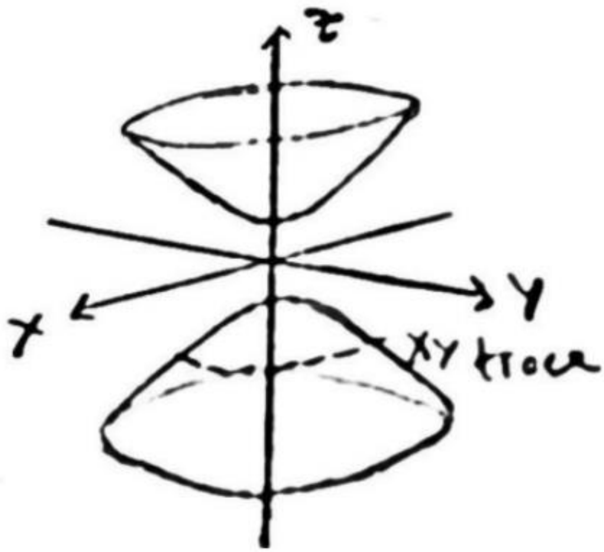
where $a, b, c \in \mathbb{R}$. The relevant traces in a plane

$$\text{parallel to the} \quad xy\text{–plane} \quad \text{are ellipses}$$

$$\text{parallel to the} \quad xz\text{–plane} \quad \text{are hyperbolas}$$

parallel to the yz–plane are hyperbolas

See the picture below.



Section 1.6 – Chain Rule.

Remark 15.

We will first discuss some notation. Suppose $w = f(x)$ is a differentiable function of one variable. Then we can denote the derivative $w'(x)$ by $\frac{dw}{dx}$.

Recall the chain rule for functions of one variable which states that if $w = f(x)$ is a differentiable function and $x = g(t)$ is a differentiable function of one variable, then $w(t) = (f \circ g)(t)$ is the composition of f after g and $w(t)$ is differentiable and

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

We will now discuss various chain rules for functions of more than one variable.

Theorem 8 – Chain Rule.

Suppose $w = f(x, y)$ is a differentiable function of two variables and also suppose x and y are both differentiable functions of the one variable t. Then w is a differentiable function of the one variable t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Example 33.

If $w = \ln(x^2 + y^2)$, $x = e^{-t}$, $y = e^t$, then

$$(i) \qquad \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{2x}{x^2+y^2}(-e^{-t}) + \frac{2y}{x^2+y^2}(e^{t})$$

$$= \frac{2(e^{2t}-e^{-2t})}{e^{2t}+e^{-2t}}$$

(ii) When $t = 1$ we get $\dfrac{dw}{dt} = \dfrac{2(e^2-e^{-2})}{e^2+e^{-2}}$

Theorem 9 – Chain Rule for three variables.

Suppose $w = f(x, y, z)$ is a differentiable function of three variables and also suppose x, y and z are all differentiable functions of the one variable t. Then w is a differentiable function of the one variable t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example 34.

If $w = 2xy + z$ and $x = \cos t$, $y = \sin t$, $z = t$, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= -2y(\sin t) + 2x(\cos t) + 1$$

$$= -2\sin^2 t + 2\cos^2 t + 1$$

Theorem 10 – Another Chain Rule.

Suppose $w = f(x, y)$ is a differentiable function of two variables and also suppose $x = g(r, s)$, $y = h(r, s)$ are both differentiable functions of the two variables r, s. Then w is a differentiable function of the two variables r, s and

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

Example 35.

If $w = x^2 + y^2$ and $x = 2r - 3s$, $y = r + 5s$, then

$$(i)\quad \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = 2x(2) + 2y(1) = 10r - 2s$$

$$(ii)\quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = 2x(-3) + 2y(5) = -2r + 68s$$

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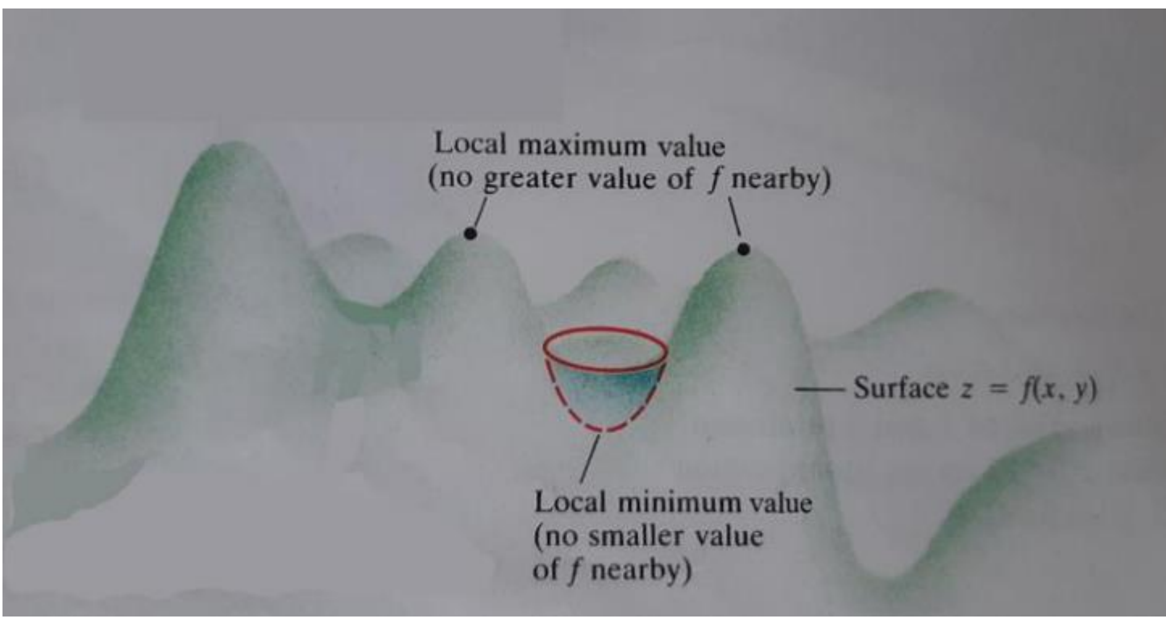
Lecture 10

Remark 1 continued.

Intuitively, one can think of an example of a surface $z = f(x, y)$ like a mountain range in the picture below. Note that, in the picture, a local maximum value of f is like a single mountain peak but is not necessarily the highest mountain peak in the whole mountain range. Similarly, if (a, b) is a local minimum of f, then it may not necessarily be the case that $f(a, b) \leq f(x, y)$, for all points (x, y) in the domain of f.



Theorem 1 – First Derivative test for local maxima and local minima.

Suppose $f(x, y)$ has a local maximum or local minimum at an interior point (a, b) of its domain. Also, suppose that $\frac{\partial f}{\partial x}_{|(a,b)}$ and $\frac{\partial f}{\partial y}_{|(a,b)}$ both exist. Then,

$$\frac{\partial f}{\partial x}_{|(a.b)} = 0 = \frac{\partial f}{\partial y}_{|(a.b)}$$

Definition 2.

Suppose (a,b) is an interior point of the domain of $f(x,y)$ with $\frac{\partial f}{\partial x}_{|(a,b)} = 0 = \frac{\partial f}{\partial y}_{|(a,b)}$ or where one or both of $\frac{\partial f}{\partial x}_{|(a,b)}$, $\frac{\partial f}{\partial y}_{|(a,b)}$ don't exist. Then, (a,b) is called a critical point of f.

Remark 2.

Theorem 1 says that the only candidates for local maxima and local minima of $f(x,y)$ are critical points of f and boundary points of the domain of f.

Definition 3.

Suppose (a,b) is a critical point of a differentiable function $f(x,y)$. Then (a,b) is called a saddle point of f if in every open ball, with centre (a,b), there are domain points (x,y) where $f(x,y) > f(a,b)$ and there are also domain points (z,w) where $f(a,b) > f(z,w)$.

Example 1.

Find the local maxima and local minima (if any) of $f(x,y) = x^2 + y^2 - 4y + 9$.

Solution.

The domain of f is $\mathbb{R}^2$ and so there are no boundary points of the domain of f. So, by remark 2, the only candidates for local maxima and local minima are critical points.

Note that $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y - 4 \quad (*)$.

So, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere and so the only critical points are where $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$. So, by $(*)$ we have that $(0,2)$ is the only critical point.

Note that $f(x,y) = x^2 + (y-2)^2 + 5$ and so $f(x,y)$ is never less than $5 = f(0,2)$. Hence, $(0,2)$ is indeed a local minimum and it's the only local minimum of f. Also, there are no local maxima of f and we are done.

Example 2.

Find the local maxima and local minima (if any) of $f(x,y) = 3x^2 - y^2$.

Solution.

The domain of f is $\mathbb{R}^2$ and so there are no boundary points of the domain of f. So, the only candidates for local maxima and local minima are critical points.

Note that $\frac{\partial f}{\partial x} = 6x$ and $\frac{\partial f}{\partial y} = -2y \quad (*)$.

So, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere and so the only critical points are where $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$. So, by $(*)$ we have that $(0,0)$ is the only critical point.

However, every open ball, with centre $(0,0)$, contains points $(x,0)$, with $x \neq 0$, where $f(x,0) = 3x^2 > f(0,0)$ and also contains points $(0,y)$, with $y \neq 0$, where $f(0,y) = -y^2 <$

$f(0,0)$, So, $(0,0)$ is neither a local maximum nor local minimum. Consequently, f has no local maxima and no local minima and we are done.

Note that $(0,0)$ is actually a saddle point of f.

Theorem 2 – Second derivative test for local maxima and local minima.

Suppose $f(x,y)$ and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are continuous on an open ball with centre (a,b). Also, suppose that $\frac{\partial f}{\partial x}_{|(a,b)} = 0 = \frac{\partial f}{\partial y}_{|(a,b)}$

Then,

(i) f has a local maximum at (a,b) if

$$\frac{\partial^2 f}{\partial x^2}_{|(a,b)} < 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}_{|(a,b)} \frac{\partial^2 f}{\partial y^2}_{|(a,b)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(a,b)} > 0$$

(ii) f has a local minimum at (a,b) if

$$\frac{\partial^2 f}{\partial x^2}_{|(a,b)} > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}_{|(a,b)} \frac{\partial^2 f}{\partial y^2}_{|(a,b)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(a,b)} > 0$$

(iii) f has a saddle point at (a,b) if

$$\frac{\partial^2 f}{\partial x^2}_{|(a,b)} \frac{\partial^2 f}{\partial y^2}_{|(a,b)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(a,b)} < 0$$

(iv) The test is inconclusive if

$$\frac{\partial^2 f}{\partial x^2}_{|(a,b)} \frac{\partial^2 f}{\partial y^2}_{|(a,b)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(a,b)} = 0$$

Remark 3.

In Theorem 2, the expression

$$\frac{\partial^2 f}{\partial x^2}_{|(a,b)} \frac{\partial^2 f}{\partial y^2}_{|(a,b)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(a,b)}$$

is called the Hessian of f.

Remark 4.

Look back at example 30 in chapter 1 where $z = \frac{y^2}{b^2} - \frac{x^2}{a^2} = f(x,y)$ and you will see that $(0,0)$ is a saddle point because of theorem 2(iii) above. You will also see how it looks like $(0,0)$ is on a saddle in the picture in example 30 in chapter 1.

Example 3.

Find the local maxima, local minima and saddle points (if any) of $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution.

The domain of f is $\mathbb{R}^2$ and so there are no boundary points of the domain of f. So, the only candidates for local maxima, local minima and saddle points are critical points.

Note that $\dfrac{\partial f}{\partial x} = -6x + 6y$ and $\dfrac{\partial f}{\partial y} = 6y - 6y^2 + 6x \qquad (*)$.

So, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist everywhere and so the only critical points are where $\dfrac{\partial f}{\partial x} = 0 = \dfrac{\partial f}{\partial y}$. So, by $(*)$ above we have that (x,y) is a critical point $\iff$

$$-6x + 6y = 0 = 6y - 6y^2 + 6x$$

$$\iff x = y \;\text{ and }\; 2y - y^2 = 0$$

$$\iff x = y \;\text{ and }\; y(y-2) = 0$$

$$\iff x = y \;\text{ and }\; y = 0, 2$$

So, the only critical points are $(0,0)$ and $(2,2)$.

Note that $\dfrac{\partial^2 f}{\partial x^2} = -6$, $\dfrac{\partial^2 f}{\partial y^2} = 6 - 12y$ and $\dfrac{\partial^2 f}{\partial x \partial y} = 6$

and so

$$\frac{\partial^2 f}{\partial x^2}_{|(0,0)} \frac{\partial^2 f}{\partial y^2}_{|(0,0)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(0,0)} = -72$$

and

$$\frac{\partial^2 f}{\partial x^2}_{|(2,2)} \frac{\partial^2 f}{\partial y^2}_{|(2,2)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(2,2)} = 72$$

Hence, by theorem 2, $(0,0)$ is a saddle point of f and $(2,2)$ is a local maximum of f. Also, there are no other local maxima, local minima or saddle points of f.

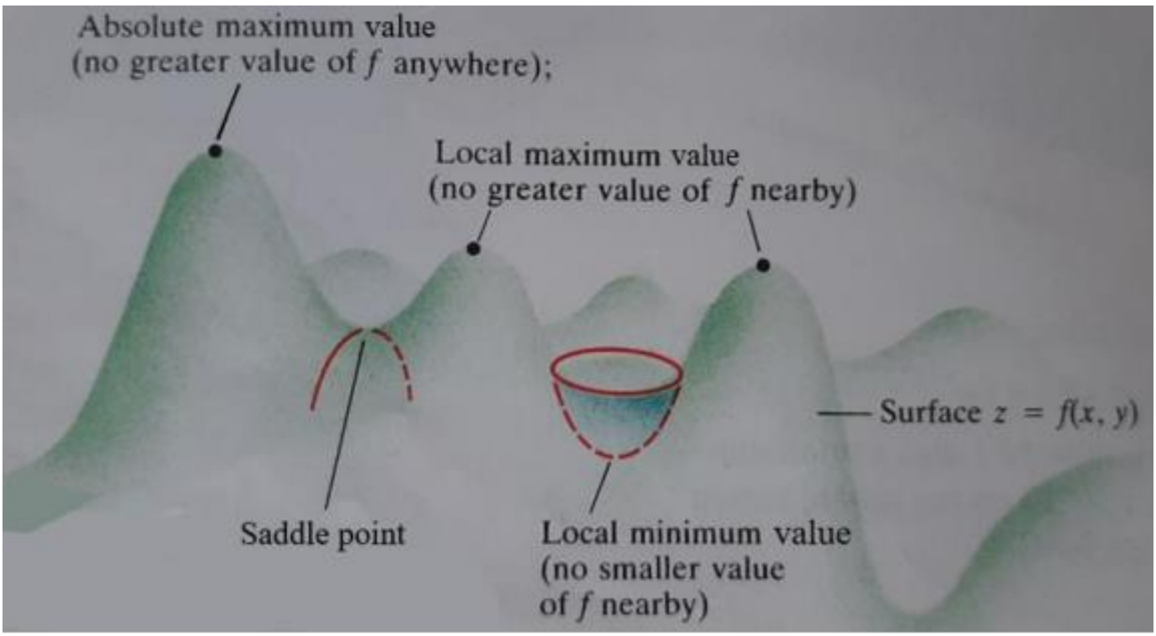
Definition 4.

(i) $f(a,b)$ is an absolute maximum value of f if $f(a,b) \geq f(x,y)$, for all (x,y) in the domain of f. We say that (a,b) is an absolute maximum of f.

(ii) $f(a,b)$ is an absolute minimum value of f if $f(a,b) \leq f(x,y)$, for all (x,y) in the domain of f. We say that (a,b) is an absolute minimum of f.

Remark 5.

We will now add to remark 1. Intuitively, one can think of an example of a surface $z = f(x,y)$ like a mountain range in the picture below (like after remark 1). Note that, in the picture below, an absolute maximum value of f is the highest mountain peak in the whole mountain range.



Remark 6.

If $g(x)$ is a continuous function of one variable on a closed bounded interval $[a,b]$ in $\mathbb{R}$, then g has at least one absolute maximum in $[a,b]$ and g has at last one absolute minimum in $[a,b]$.

The following states something similar for functions of two variables.

Theorem 3.

Suppose $f(x,y)$ is a continuous function on a closed bounded subset T of $\mathbb{R}^2$. Then f has at least one absolute maximum in T and f has at least one absolute minimum in T.

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Lecture 11

Remark 7 – Strategy for finding absolute maxima and absolute minima of a continuous function f on a closed bounded set T in $\mathbb{R}^2$.

Step 1.

List the interior points of T that are candidates for local maxima and local minima of f. These points are the critical points of f. Then, evaluate f at these points.

Step 2.

Check the boundary points of T and evaluate f at the candidates for absolute maxima and absolute minima of f.

Step 3.

Select the greatest value of f from steps 1 and 2 above. This will give the absolute maximum value of f on T. The points where f takes on this absolute maximum value, will be the absolute maxima of f on T.

Select the smallest value of f from steps 1 and 2 above. This will give the absolute minimum value of f on T. The points where f takes on this absolute minimum value, will be the absolute minima of f on T.

Remark 8.

Before we do an example using remark 7, we will discuss the case for functions of one variable on a closed interval $[p, q]$ in $\mathbb{R}$ because we will need that case.

Suppose $g(x)$ is a function of one variable on a closed interval $[p, q]$ in $\mathbb{R}$. Then, $g(w)$ is a local maximum value of g if $g(w) \geq g(x)$, for all domain points x in an open interval with centre w. We call w a local maximum. $g(v)$ is a local minimum value of g if $g(v) \leq g(x)$, for all domain points x in an open interval with centre v. We call v a local minimum.

Suppose $g(x)$ has a local maximum or local minimum at $t \in (p, q)$. Also, suppose $g'(t)$ exists. Then, $g'(t) = 0$.

Suppose z is in (p, q) where $g'(z) = 0$ or where $g'(z)$ doesn't exist. Then, z is called a critical point of g.

$g(k)$ is an absolute maximum value of g if $g(k) \geq g(x)$, for all domain points x. We call k an absolute maximum. $g(m)$ is an absolute minimum value of g if $g(m) \leq g(x)$, for all domain points x. We call m an absolute minimum.

Remark 9.

Using the definitions in remark 8, we now give the three steps for finding absolute maxima and absolute minima of a continuous function g on a closed interval $[p, q]$ in $\mathbb{R}$.

Step 1.

Evaluate g at the critical points.

Step 2.

Evaluate g at the endpoints p, q of $[p, q]$.

Step 3.

Select the greatest value of g from steps 1 and 2 above. This will give the absolute maximum value of g on $[p, q]$. The points where g takes on this absolute maximum value, will be the absolute maxima of g on $[p, q]$.

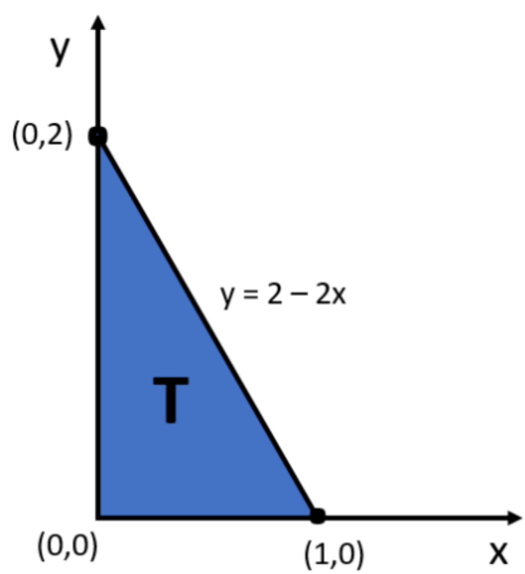
Select the smallest value of g from steps 1 and 2 above. This will give the absolute minimum value of g on $[p, q]$. The points where g takes on this absolute minimum value, will be the absolute minima of g on $[p, q]$.

Example 4.

Find the absolute maxima and absolute minima of $f(x, y) = x^2 + y^2$ on the closed triangular region T bounded by the lines $x = 0$, $y = 0$ and $y + 2x = 2$ in the first quadrant. Also, find the absolute maximum value and the absolute minimum value of f on T.

Solution.

Denote the closed triangular region by T. It is helpful to draw a picture of T below.



We see that $T = \{(x, y) \in \mathbb{R}^2 : y \leq 2 - 2x,\ x \geq 0,\ y \geq 0\}$. Note that T is a closed bounded subset of $\mathbb{R}^2$ and that f is continuous on T and so we can use the strategy in remark 7.

Step 1 – Interior points of T.

Note that $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$. So,

$$\frac{\partial f}{\partial x}_{|(a,b)} = 0 = \frac{\partial f}{\partial y}_{|(a,b)} \quad \Longleftrightarrow \quad (a,b) = (0,0)$$

However, $(0,0)$ is not an interior point of T and so step 1 produces no critical points.

Step 2 – Boundary points of T.

The boundary of T consists of the three sides of the triangle and we will take one side at a time.

(i) On the line segment joining $(0,0)$ to $(1,0)$ we have $y = 0$ and so $f(x,y) = f(x,0) = x^2$, which may be considered as a function of one variable on $[0,1]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0,1$ and the x–values in $(0,1)$, where

$$0 = \frac{d}{dx}(x^2) \Rightarrow x = 0$$

and so there are no such x–values in $(0,1)$. So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0,1$. So, the candidates for the absolute maxima and absolute minima for $f(x,y)$ are $(0,0)$ and $(1,0)$ and we evaluate f at these points to get

$$f(0,0) = 0 \quad \text{and} \quad f(1,0) = 1 \qquad\qquad (*)$$

(ii) On the line segment joining $(0,0)$ to $(0,2)$ we have $x = 0$ and so $f(x,y) = f(0,y) = y^2$, which may be considered as a function of one variable on $[0,2]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0,2$ and the y–values in $(0,2)$, where

$$0 = \frac{d}{dy}(y^2) \Rightarrow y = 0$$

and so there are no such y–values in $(0,2)$. So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0,2$. So, the candidates for absolute maxima and absolute minima for $f(x,y)$ are $(0,0)$ and $(0,2)$ and we evaluate f at these points to get

$$f(0,0) = 0 \quad \text{and} \quad f(0,2) = 4 \qquad\qquad (**)$$

(iii) On the line segment joining $(1,0)$ to $(0,2)$ we have $y = 2 - 2x$ and so $f(x,y) = x^2 + (2-2x)^2 = x^2 + 4 - 8x + 4x^2 = 5x^2 - 8x + 4$, which may be considered as a function of one variable on $[0,1]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0,1$ and the x–values in $(0,1)$, where

$$0 = \frac{d}{dx}(5x^2 - 8x + 4) = 10x - 8 \Rightarrow x = \frac{4}{5}$$

So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0, 1$ and $\frac{4}{5}$. So, the candidates for absolute maxima and absolute minima for $f(x, y)$ are $(0, 2)$, $(1, 0)$ and $(\frac{4}{5}, \frac{2}{5})$ and we evaluate f at these points to get

$$f(0,2) = 4, \quad f(1,0) = 1 \quad \text{and} \quad f(\frac{4}{5}, \frac{2}{5}) = \frac{4}{5} \qquad (***)$$

Step 3.

The relevant values from steps 1 and 2 are in $(*)$, $(**)$, $(***)$ and so the relevant values are

$$f(0,0) = 0, \quad f(1,0) = 1, \quad f(0,2) = 4, \quad f(\frac{4}{5}, \frac{2}{5}) = \frac{4}{5} \qquad (****)$$

Select the greatest value of f from $(****)$ to get that the absolute maximum value of f on T is 4. The points where f takes on this absolute maximum value, will be the absolute maxima of f on T and so $(0, 2)$ is the only absolute maximum of f on T.

Select the smallest value of f from $(****)$ to get that the absolute minimum value of f on T is 0. The points where f takes on this absolute minimum value, will be the absolute minima of f on T and so $(0, 0)$ is the only absolute minimum of f on T.

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Lecture 13

Remark 11.

Many important and powerful applications of mathematics to science, engineering and other areas involve finding absolute maxima and absolute minima of functions of more than one variable. The theory in our last few lectures can help to find such absolute maxima and absolute minima in certain situations. The next section will discuss another concept that can help find absolute maxima and absolute minima in certain situations.

Section 3.2 – Lagrange Multipliers.

Remark 12 – The method of Lagrange Multipliers.

Suppose $f(x, y)$ and $g(x, y)$ are differentiable functions. The candidates for the absolute maxima and absolute minima of $f(x, y)$ subject to the constraint $g(x, y) = 0$ are the points (a, b) which satisfy

$$\nabla f_{|(a,b)} = \lambda \nabla g_{|(a,b)} \quad \text{and} \quad g(a, b) = 0 \quad \text{for some} \quad \lambda \in \mathbb{R}$$

Example 7.

Use Lagrange multipliers to find the greatest value and smallest value of $f(x, y) = xy$ on the ellipse $x^2 + 2y^2 = 1$. Also, find the points where f has this greatest value and this smallest value.

Solution.

We need to find the absolute maxima and absolute minima of $f(x, y) = xy$ subject to the constraint $g(x, y) = 0$, where $g(x, y) = x^2 + 2y^2 - 1$.

So, by remark 12, in order to find candidates (a, b) for the absolute maxima and absolute minima of xy subject to the constraint $x^2 + 2y^2 - 1 = 0$, we look for points (a, b) and real numbers λ that satisfy

$$\nabla f_{|(a,b)} = \lambda \nabla g_{|(a,b)} \quad \text{and} \quad g(a, b) = 0 \qquad (*)$$

Now, $(*)$ gives us

$$(y\vec{i} + x\vec{j})_{|(a,b)} = \lambda(2x\vec{i} + 4y\vec{j})_{|(a,b)} \quad \text{and} \quad a^2 + by^2 - 1 = 0$$

$$\Rightarrow b = 2\lambda a, \quad a = 4\lambda b, \quad a^2 + 2b^2 - 1 = 0$$

$$\Rightarrow b = 8\lambda^2 b$$

$$\Rightarrow b(1 - 8\lambda^2) = 0$$

$$\Rightarrow b = 0 \quad \text{or} \quad \lambda = \pm\frac{1}{\sqrt{8}}$$

We now look at the two cases separately:

CASE 1: Suppose $b = 0$. Then $a = 4\lambda b = 0$ which is impossible because $a^2 + 2b^2 = 1$.

CASE 2: Suppose $\lambda = \pm\frac{1}{\sqrt{8}}$. Then $b = 2\lambda a$ implies that

$$b = \pm\frac{a}{\sqrt{2}} \qquad (**)$$

Now, $a^2 + 2b^2 = 1$ and so $(**)$ implies that $a^2 + a^2 = 1$ which gives $a = \pm\frac{1}{\sqrt{2}}$.

Now, using $(**)$ we get that the candidates for the required absolute maxima and absolute minima are

$$(\frac{1}{\sqrt{2}}, \frac{1}{2}), \quad (\frac{1}{\sqrt{2}}, -\frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, \frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, -\frac{1}{2})$$

We evaluate f at these candidates to get

$$f(\frac{1}{\sqrt{2}}, \frac{1}{2}) = \frac{1}{\sqrt{8}}, \quad f(\frac{1}{\sqrt{2}}, -\frac{1}{2}) = -\frac{1}{\sqrt{8}}, \quad f(-\frac{1}{\sqrt{2}}, \frac{1}{2}) = -\frac{1}{\sqrt{8}}, \quad f(-\frac{1}{\sqrt{2}}, -\frac{1}{2}) = \frac{1}{\sqrt{8}}$$

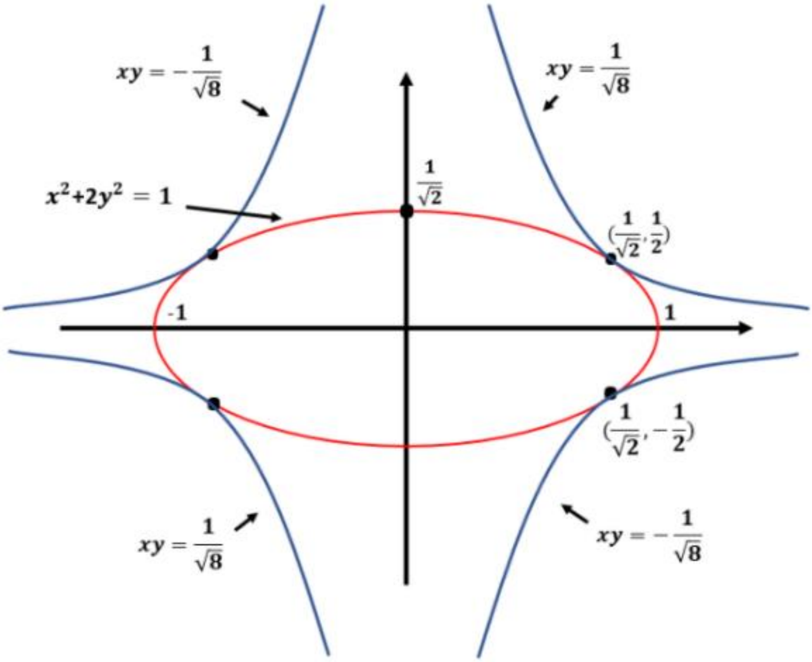
So, the greatest value of xy on the ellipse $x^2 + 2y^2 = 1$ is $\frac{1}{\sqrt{8}}$ and the points where f has this greatest value are

$$(\frac{1}{\sqrt{2}}, \frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, -\frac{1}{2})$$

Similarly, the smallest value of xy on the ellipse $x^2 + 2y^2 = 1$ is $-\frac{1}{\sqrt{8}}$ and the points where f has this smallest value are

$$(\frac{1}{\sqrt{2}}, -\frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, \frac{1}{2})$$

We have answered the question fully. We will now do something extra and discuss some geometry related to our work above. In the picture below, we see the ellipse $x^2 + 2y^2 = 1$ and some level curves of $f(x, y) = xy$. Note that the level curves of xy are the hyperbolas $xy = c$, for some $c \in \mathbb{R}$.



Notice that we were looking for the greatest value of xy and the smallest value of xy subject to the constraint that (x, y) also lies on the ellipse $x^2 + 2y^2 = 1$. Now, note that the farther the level curves $xy = c$ are away from the origin $(0,0)$, then the larger the absolute value of xy is. So, geometrically, we can say that we were looking for the level curves (i.e. hyperbolas $xy = c$) that intersect the ellipse above and also lie the farthest away from the origin.

So, which hyperbolas $xy = c$ (that intersect the ellipse above) will lie the farthest away from the origin? We see that the hyperbolas that just graze the ellipse are the ones that will achieve this condition of intersecting the ellipse above and being the farthest away from the origin. These hyperbolas will be $xy = \frac{1}{\sqrt{8}}$ and $xy = -\frac{1}{\sqrt{8}}$ and furthermore the points of intersection between these hyperbolas and the ellipse will be

$$(\frac{1}{\sqrt{2}}, \frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, -\frac{1}{2}), \quad (\frac{1}{\sqrt{2}}, -\frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, \frac{1}{2})$$

This all corresponds to the two facts (proved earlier)

(i) that the greatest value of xy on the ellipse $x^2 + 2y^2 = 1$ is $\frac{1}{\sqrt{8}}$ and the points where f has this greatest value are

$$(\frac{1}{\sqrt{2}}, \frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, -\frac{1}{2})$$

(ii) that the smallest value of xy on the ellipse $x^2 + 2y^2 = 1$ is $-\frac{1}{\sqrt{8}}$ and the points where f has this smallest value are

$$(\frac{1}{\sqrt{2}}, -\frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, \frac{1}{2})$$

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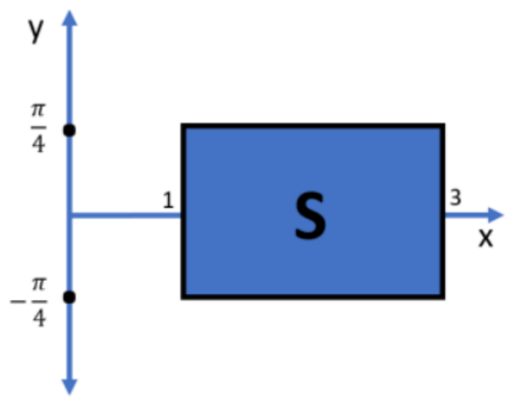
Lecture 12

Example 5.

Find the absolute maxima and absolute minima of $f(x, y) = (4x - x^2)\cos y$ on the closed rectangular region S given by $1 \le x \le 3$, $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$. Also, find the absolute maximum value and the absolute minimum value of f on S.

Solution.

Denote the rectangular region by S. It is helpful to draw a picture of S below.



We see that $S = \{(x, y) \in \mathbb{R}^2 : 1 \le x \le 3,\ -\frac{\pi}{4} \le y \le \frac{\pi}{4}\}$ Note that S is a closed bounded subset of $\mathbb{R}^2$ and that f is continuous on S and so we can use the strategy in remark 7.

Step 1 – Interior points of S.

Note that $\dfrac{\partial f}{\partial x} = (4 - 2x)\cos y$ and $\dfrac{\partial f}{\partial y} = -(4x - x^2)\sin y$. So,

$$\frac{\partial f}{\partial x}_{|(a,b)} = 0 = \frac{\partial f}{\partial y}_{|(a,b)} \quad \text{and} \quad (a, b) \in \text{Int}(S) \iff (a, b) = (2, 0)$$

So, $(2, 0)$ is the only critical point and $f(2, 0) = 4$

Step 2 – Boundary points of S.

The boundary of S consists of the four sides of the rectangle and we will take one side at a time.

(i) On the line segment joining $(1, -\frac{\pi}{4})$ to $(3, -\frac{\pi}{4})$ we have $f(x, y) = f(x, -\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(4x - x^2)$, which may be considered as a function of one variable on $[1,3]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $1, 3$ and the x–values in $(1, 3)$, where

$$0 = \frac{d}{dx}\frac{1}{\sqrt{2}}(4x - x^2) \Rightarrow 4 - 2x = 0 \Rightarrow x = 2$$

So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $1, 3$ and 2. So, the candidates for the absolute maxima and absolute minima for $f(x, y)$ are $(1, -\frac{\pi}{4})$, $(3, -\frac{\pi}{4})$ and $(2, -\frac{\pi}{4})$ and we evaluate f at these points to get

$$f(1, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(3, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(2, -\frac{\pi}{4}) = \frac{4}{\sqrt{2}} \qquad (*)$$

(ii) On the line segment joining $(1, \frac{\pi}{4})$ to $(3, \frac{\pi}{4})$ we have $f(x, y) = f(x, \frac{\pi}{4}) = \frac{1}{\sqrt{2}}(4x - x^2)$, which may be considered as a function of one variable on $[1, 3]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $1, 3$ and the x–values in $(1, 3)$, where

$$0 = \frac{d}{dx}\frac{1}{\sqrt{2}}(4x - x^2) \Rightarrow 4 - 2x = 0 \Rightarrow x = 2$$

So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $1, 3$ and 2. So, the candidates for the absolute maxima and absolute minima for $f(x, y)$ are $(1, \frac{\pi}{4})$, $(3, \frac{\pi}{4})$ and $(2, \frac{\pi}{4})$ and we evaluate f at these points to get

$$f(1, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(3, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(2, \frac{\pi}{4}) = \frac{4}{\sqrt{2}} \qquad (**)$$

(iii) On the line segment joining $(3, -\frac{\pi}{4})$ to $(3, \frac{\pi}{4})$ we have $f(x, y) = f(3, y) = 3\cos y$, which may be considered as a function of one variable on $[-\frac{\pi}{4}, \frac{\pi}{4}]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $-\frac{\pi}{4}, \frac{\pi}{4}$ and the y–values in $(-\frac{\pi}{4}, \frac{\pi}{4})$, where

$$0 = \frac{d}{dy}3\cos y \Rightarrow -3\sin y = 0 \Rightarrow y = 0$$

So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $-\frac{\pi}{4}, \frac{\pi}{4}$ and 0. So, the candidates for the absolute maxima and absolute minima for $f(x, y)$ are $(3, -\frac{\pi}{4})$, $(3, \frac{\pi}{4})$ and $(3, 0)$, and we evaluate f at these points to get

$$f(3, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(3, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(3, 0) = 3 \qquad (***)$$

(iv) On the line segment joining $(1,-\frac{\pi}{4})$ to $(1,\frac{\pi}{4})$ we have $f(x,y) = f(1,y) = 3\cos y$, which may be considered as a function of one variable on $[-\frac{\pi}{4},\frac{\pi}{4}]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $-\frac{\pi}{4},\frac{\pi}{4}$ and the y–values in $(-\frac{\pi}{4},\frac{\pi}{4})$, where

$$0 = \frac{d}{dy}3\cos y \Rightarrow -3\sin y = 0 \Rightarrow y = 0$$

So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $-\frac{\pi}{4},\frac{\pi}{4}$ and 0. So, the candidates for the absolute maxima and absolute minima for $f(x,y)$ are $(1,-\frac{\pi}{4})$, $(1,\frac{\pi}{4})$ and $(1,0)$, and we evaluate f at these points to get

$$f(1,-\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(1,\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(1,0) = 3 \qquad (****)$$

Step 3.

The relevant values from steps 1 and 2 are in step 1 and $(*)$, $(**)$, $(***)$, $(****)$ and so the relevant values are

$$f(2,0) = 4, \quad f(1,-\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(3,-\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(2,-\frac{\pi}{4}) = \frac{4}{\sqrt{2}}, \quad f(1,\frac{\pi}{4}) = \frac{3}{\sqrt{2}}$$

$$f(3,\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(2,\frac{\pi}{4}) = \frac{4}{\sqrt{2}}, \quad f(3,0) = 3, \quad f(1,0) = 3 \qquad (*****)$$

Select the greatest value of f from $(*****)$ to get that the absolute maximum value of f on T is 4. The points where f takes on this absolute maximum value, will be the absolute maxima of f on T and so $(2,0)$ is the only absolute maximum of f on T.

Select the smallest value of f from $(*****)$ to get that the absolute minimum value of f on T is $\frac{3}{\sqrt{2}}$. The points where f takes on this absolute minimum value, will be the absolute minima of f on T and so the absolute minima of f of T are

$$(1,-\frac{\pi}{4}), \quad (1,\frac{\pi}{4}), \quad (3,-\frac{\pi}{4}), \quad (3,\frac{\pi}{4})$$

Remark 10.

Our next example will give an application of our theory.

Example 6.

A flat rectangular plate is given by $1 \le x \le 3$, $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$. The plate is heated so that the temperature at the point (x,y) is $(4x - x^2)\cos y$. Find the temperatures at the hottest and coldest points on the plate. Also, find the points on the plate where the hottest and coldest temperatures occur.

Solution.

Suppose $f(x, y) = (4x - x^2)\cos y$. Then, we can use the results of step 3 in example 5 to get that the temperature at the hottest point on the plate is 4 this occurs at the point $(2, 0)$ on the plate.

Also, we can use the results of step 3 in example 5 to get that the temperature at the coldest point on the plate is $\frac{3}{\sqrt{2}}$ and this occurs at the points

$$(1, -\frac{\pi}{4}), \quad (1, \frac{\pi}{4}), \quad (3, -\frac{\pi}{4}), \quad (3, \frac{\pi}{4})$$

on the plate.

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Lecture 14

Remark 13.

The method of Lagrange Multipliers only produces candidates for absolute maxima and absolute minima of f subject to the constraint $g(x,y)=0$. There may no such absolute maxima and there may be no such absolute minima and the next example will illustrate this.

Example 8.

(i) Use Lagrange multipliers to find the candidates for absolute maxima and absolute minima of $f(x,y)=x+y$ subject to the constraint $xy=16$.

(ii) Show that there are no absolute maxima and no absolute minima of $f(x,y)=x+y$ subject to the constraint $xy=16$.

Solution.

(i) Let $g(x,y)=xy-16$ so that the constraint $xy=16$ can be written in the form $g(x,y)=0$.

So, by remark 12, in order to find candidates (a,b) for the absolute maxima and absolute minima of $f(x,y)=x+y$ subject to the constraint $g(x,y)=0$, we look for points (a,b) and real numbers λ that satisfy

$$\nabla f_{|(a,b)} = \lambda \nabla g_{|(a,b)} \quad \text{and} \quad g(a,b)=0 \qquad (*)$$

Now, $(*)$ gives us

$$(\vec{i}+\vec{j})_{|(a,b)} = \lambda(y\vec{i}+x\vec{j})_{|(a,b)} \quad \text{and} \quad ab-16=0$$

$$\Rightarrow 1=\lambda b, \quad 1=\lambda a, \quad ab=16$$

$$\Rightarrow \lambda a = \lambda\left(\frac{16}{a}\right)$$

$$\Rightarrow \lambda\left(a-\frac{16}{a}\right)=0$$

$$\Rightarrow \lambda = 0 \quad \text{or} \quad a=\pm 4$$

We now look at the two cases separately:

CASE 1: Suppose $\lambda = 0$. Then $1 = \lambda a = 0$ which is impossible.

CASE 2: Suppose $a = \pm 4$. This implies $b = \pm 4 \quad (**)$.

Now, using $(**)$ we get that the candidates for the required absolute maxima and absolute minima are $(4, 4)$, $(-4, -4)$. So, we have finished (i).

(ii) We will now show that there are no absolute maxima and no absolute minima of $f(x, y) = x + y$ subject to the constraint $xy = 16$.

Notice that the farther you go from the origin on the hyperbola $xy = 16$ in the first quadrant, the larger the sum $x + y$ is. This means that there are no absolute maxima for $x + y$ subject to the constraint $xy = 16$.

Similarly, notice that the farther you go from the origin on the hyperbola $xy = 16$ in the fourth quadrant, the smaller the sum $x + y$ is. This means that there are no absolute minima for $x + y$ subject to the constraint $xy = 16$. We are now finished with (ii).

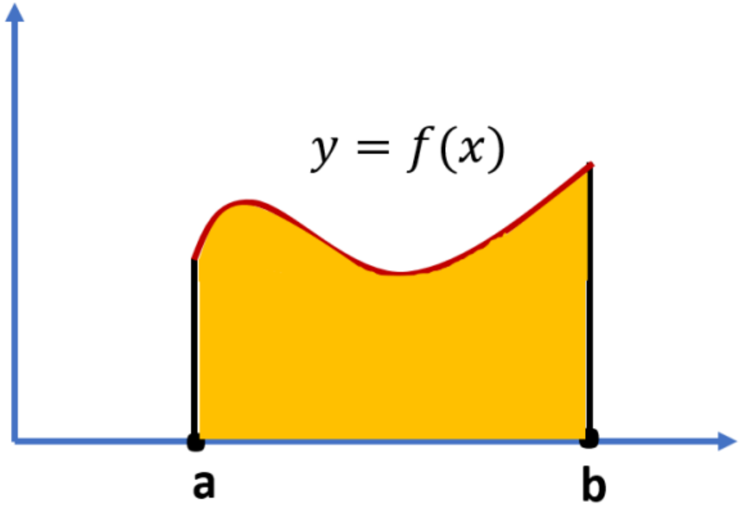
Chapter 4 – Multiple Integration.

Section 4.1 – Multiple Integrals.

Remark 1.

It will be helpful to recall the situation for functions of one variable before we motivate the definition of integrals of functions of two variables. There are many important and powerful applications of integrals to science, engineering and other areas.

Recall that definite integrals of the form $\int_a^b f(x)\,dx$ were used to find the area of the shaded region below.



The shaded region above is bounded by the curve $y = f(x)$, the x–axis and the vertical lines $x = a$, $x = b$. The definition of $\int_a^b f(x)\,dx$ can be motivated as follows:

We will first motivate the definition of the area of the shaded region above (when f is continuous and non–negative on $[a, b]$). We partition the closed interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ (for $1 \leq i \leq n$), each of length $\dfrac{b-a}{n}$, by selecting points $x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n$ where $x_0 = a$ and $x_n = b$. The vertical lines $x = x_i$ divide the shaded region above into vertical strips. We approximate each strip by a rectangle with base $[x_{i-1}, x_i]$ and height $f(w_i)$ where $f(w_i)$ is the absolute minimum value of f on $[x_{i-1}, x_i]$. Let $\Delta x = \dfrac{b-a}{n}$, so that the sum of the areas of these rectangles is

$$\sum_{i=1}^{n} f(w_i)\Delta x \qquad (*)$$

We say that $(*)$ approximates the area of the shaded region above. Note that this approximation may not necessarily be a good approximation if n is small. However, we expect the approximation to improve as n increases (i.e. as Δx approaches zero). With this as motivation, we define the area of the shaded region to be

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(w_i)\Delta x$$

We now consider any function f on $[a, b]$. Motivated by the discussion above regarding area, we partition the closed interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ (for $1 \leq i \leq n$), by selecting points $x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n$ where $x_0 = a$ and $x_n = b$. $P = \{x_0, x_1, \ldots, x_n\}$ is called the corresponding partition of $[a, b]$. Let $\Delta x_i = x_i - x_{i-1}$ denote the length of the i^{th} subinterval. Select a random point c_i in $[x_i, x_{i-1}]$ and consider the so called Riemann sum:

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

The norm of the partition P above is the length of the longest subinterval and is denoted by $||P||$. We are now ready to define the definite integral $\int_a^b f(x)\, dx$:

We say

$$\lim_{||P||\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = L \qquad (**)$$

if given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$||P|| < \delta \;\Rightarrow\; \left| \sum_{i=1}^{n} f(c_i)\Delta x_i - L \right| < \epsilon$$

for any choice of the c_i in $[x_{i-1}, x_i]$.

If L exists in (**) above, then we define the definite integral of f over $[a, b]$ by

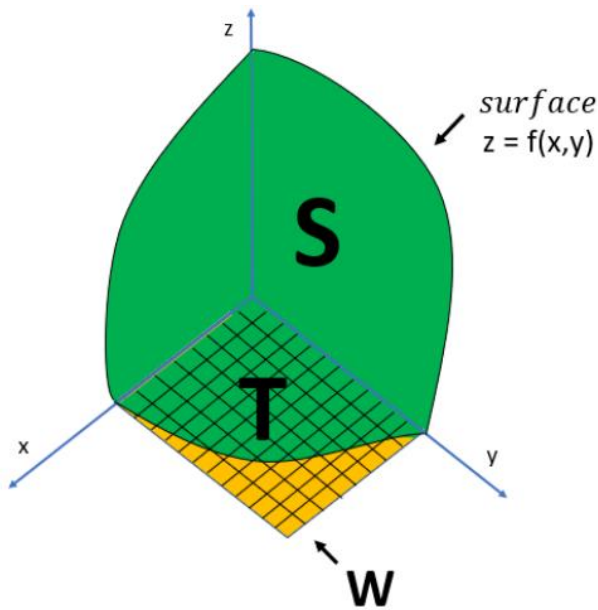
$$\int_a^b f(x)\, dx = L$$

Remark 2.

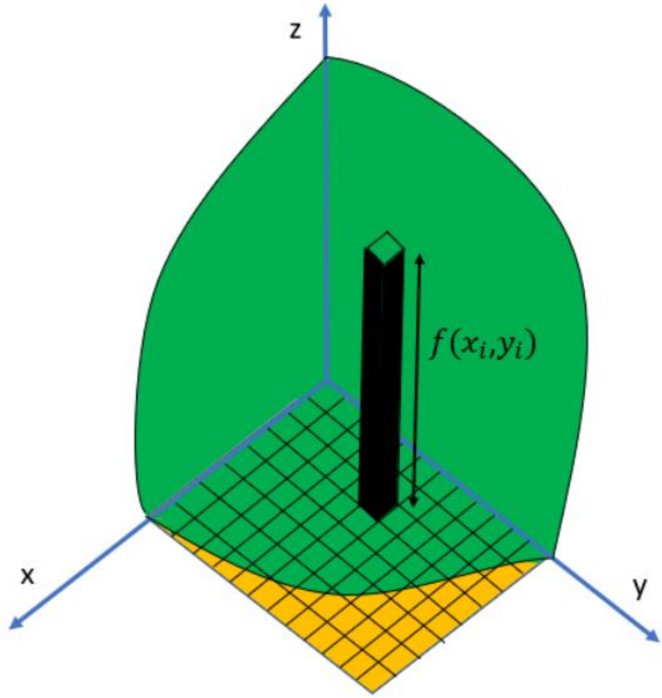
Suppose f is a continuous function on $[a, b]$ and suppose $f(x) \geq 0$, for all $x \in [a, b]$. Then, based on the above discussion it is no surprise that $\int_a^b f(x)\, dx$ is the area of the region bounded by the curve $y = f(x)$, the x–axis and the vertical lines $x = a,\ x = b$.

Remark 3.

The approach for defining the integral of a function of two variables will be similar in some ways to the one variable case in remark 1. Suppose f is a continuous function and non–negative on a region T in the xy–plane. We will first try to find the volume of a three–dimensional solid region S bounded above by the surface $z = f(x, y)$ and bounded below by the region T in the xy–plane in the picture below.



We start by superimposing a rectangular grid W over T. The n rectangles lying entirely in T form an inner partition Δ, whose norm $||\Delta||$ is defined as the length of the longest diagonal of the n rectangles. We then select a point (x_i, y_i) in each rectangle and form the rectangular prism with height $f(x_i, y_i)$ as in the picture below.


z
$f(x_i, y_i)$
x
y

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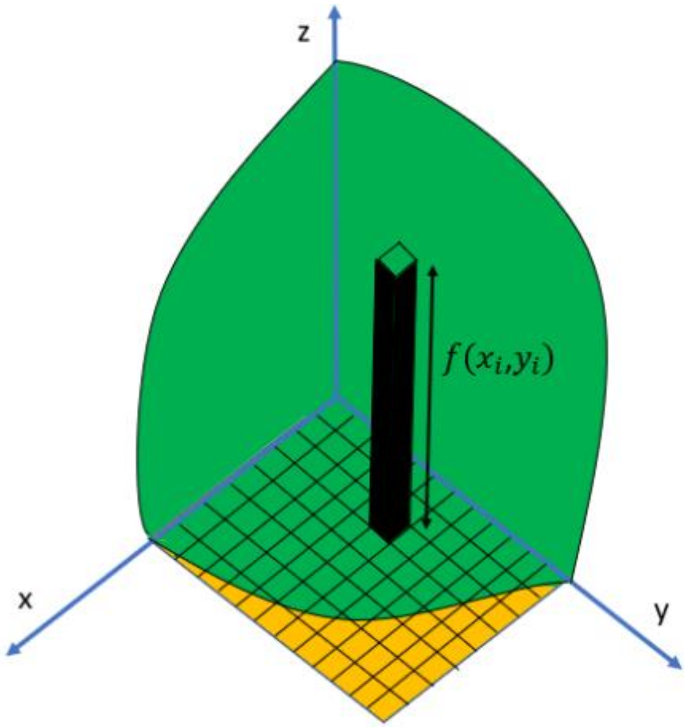


We denote the area of the i^{th} rectangle by ΔA_i so that the volume of the i^{th} prism is $f(x_i, y_i)\Delta A_i$. We say that the so called Riemann sum

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

approximates the volume of S. Note that this approximation may not necessarily be a good approximation if $||\Delta||$ is big. However, we expect the approximation to improve as $||\Delta||$ approaches zero. With this as motivation, we define the volume of S to be

$$\lim_{||\Delta||\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

where this limit is defined in a similar way as $(**)$ in remark 1.

With the above as motivation (and using similar notation as above), we are now ready to define the so called double integral of f over T as follows:

Definition 1 – Double integral.

Suppose $f(x, y)$ is defined on a closed, bounded region T in the xy–plane. The double integral of f over T is defined as

$$\int_T \int f(x,y)\, dA = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i$$

If this limit exists, we say that f is integrable over T.

On account of the discussion above, it is not surprising that a double integral is used to find the volume of a solid region. Here is the definition of the volume of a solid region:

Definition 2 – Volume of a solid region.

Suppose f is integrable over a region T in the xy–plane and suppose $f(x, y) \geq 0$ for all $(x, y) \in T$. Then, the volume of the solid region lying above T and below the surface $z = f(x, y)$ is given by

$$V = \int_T \int f(x,y)\, dA$$

Theorem 1 – Properties of Double Integrals.

Suppose f and g are continuous functions on a closed bounded region T in the xy–plane and suppose k is a constant. Then,

(i) $\displaystyle\int_T \int k f(x,y)\, dA = k \int_T \int f(x,y)\, dA$

(ii) $\displaystyle\int_T \int (f(x,y) + g(x,y))\, dA = \int_T \int f(x,y)\, dA + \int_T \int g(x,y)\, dA$

(iii) $\displaystyle\int_T \int f(x,y)\, dA \geq 0$, if $f(x,y) \geq 0$

(iv) $\displaystyle\int_T \int f(x,y)\, dA = \int_{T_1} \int f(x,y)\, dA + \int_{T_2} \int f(x,y)\, dA$

where T is the union of two disjoint sets T_1 and T_2.

Remark 4.

A definite integral of a function of one variable, $\int_a^b g(x)\, dx$, will now be called a single integral (to distinguish it from a double integral).

Theorem 2 – Fubini's Theorem.

Suppose f is a continuous function on a region T in the xy–plane.

(i) If $T = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$, where g_1 and g_2 are continuous on $[a, b]$, then

$$\int_T \int f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dydx \qquad (*)$$

We will discuss the expression on the right hand side of $(*)$ in remark 5 below.

(ii) If $T = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$, where h_1 and h_2 are continuous on $[c,d]$, then

$$\int_T \int f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dxdy \qquad (**)$$

We will discuss the expression on the right hand side of $(**)$ in remark 5 below.

Remark 5.

The two integrals on the right in $(*)$ and $(**)$ above are called iterated integrals. Fubini's theorem says that if T is horizontally simple (i.e. $a \leq x \leq b$), then the double integral in (i) can be calculated by performing two single integrals, one after the other – you integrate with respect to (w.r.t.) y first (treating x as constant, like we did with partial derivatives) and then you integrate w.r.t. x last (treating y as constant). This will be made clear in example 1 below.

Similarly, if T is vertically simple (i.e. $c \leq y \leq d$), then the double integral in (ii) can be calculated by performing two single integrals, one after the other – you integrate w.r.t. x first (treating y as constant) and then you integrate w.r.t. y last (treating x as constant). This will be made clear in remark 6 below.

In the examples below the notation

$$[w(x)]_a^b$$

is the usual notation for $w(b) - w(a)$.

Example 1. Find the volume of the solid lying below $z = 4 - x - y$ and above the square T given $0 \leq x \leq 1$ and $1 \leq y \leq 2$.

Solution.

By definition 2, the required volume is

$$V = \int_T \int (4 - x - y)\, dA$$

$$= \int_{x=0}^1 \left(\int_{y=1}^2 (4 - x - y)\, dy \right) dx \qquad \text{by Fubini's Theorem (i)}$$

$$= \int_{x=0}^1 \left[4y - xy - \frac{y^2}{2} \right]_1^2 dx$$

$$= \int_0^1 \left[(8 - 2x - 2) - (4 - x - \frac{1}{2}) \right] \, dx$$

$$= \int_0^1 \left(\frac{5}{2} - x \right) \, dx$$

$$= \left[\frac{5x}{2} - \frac{x^2}{2} \right]_0^1 = 2$$

Remark 6.

In example 1 we could use Fubini's Theorem (ii) and integrate w.r.t. y last, as follows

$$V = \int_T \int (4 - x - y) \, dA$$

$$= \int_{y=1}^2 \left(\int_{x=0}^1 (4 - x - y) \, dx \right) dy \qquad \text{by Fubini's Theorem (ii)}$$

$$= \int_{y=1}^2 \left[4x - \frac{x^2}{2} - yx \right]_0^1 \, dy$$

$$= \int_1^2 \left[(4 - \frac{1}{2} - y) - (0 - 0 - 0) \right] \, dy$$

$$= \int_1^2 \left(\frac{7}{2} - y \right) \, dy$$

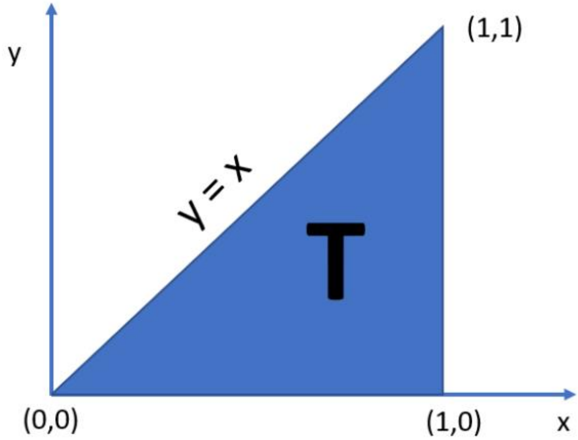
$$= \left[\frac{7y}{2} - \frac{y^2}{2} \right]_1^2 = 2$$

Example 2.

Find $\int_T \int xy \, dA$, where T is the triangle with vertices $(0,0), (1,0), (1,1)$.

Solution.

It's important to draw a good picture here.



Consequently, we see that $T = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq x\}$. Using Fubini's theorem (i) we get that:

$$\begin{aligned}
\int_T\int xy\ dA &= \int_{x=0}^{1}\left(\int_{y=0}^{x} xy\ dy\right)\ dx \\
&= \int_0^1 \left[\frac{xy^2}{2}\right]_0^x\ dx \\
&= \int_0^1 \frac{x^3}{2}\ dx \\
&= \left[\frac{x^4}{8}\right]_0^1
\end{aligned}$$

Remark 7.

In example 2, we also see that from the picture, $T = \{(x,y) : 0 \leq y \leq 1,\ y \leq x \leq 1\}$. and so using Fubini's theorem (ii) we get that:

$$\begin{aligned}
\int_T\int xy\ dA &= \int_{y=0}^{1}\left(\int_{x=y}^{1} xy\ dx\right)\ dy \\
&= \int_0^1 \left[\frac{x^2y}{2}\right]_y^1\ dy \\
&= \int_0^1 \left(\frac{y}{2} - \frac{y^3}{2}\right)\ dy \\
&= \left[\frac{y^2}{4} - \frac{y^4}{8}\right]_0^1 \\
&= \frac{1}{8}
\end{aligned}$$

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Example 3.

Find $\int_T \int 5y\, dA$, where T is the region on the xy–plane bounded by the curves $y = x^2 - 3$ and $y = -2x$.

Solution.

It's important to draw a good picture. We first find the points of intersection of the two curves $x^2 - 3$ and $-2x$.

$$x^2 - 3 = -2x$$

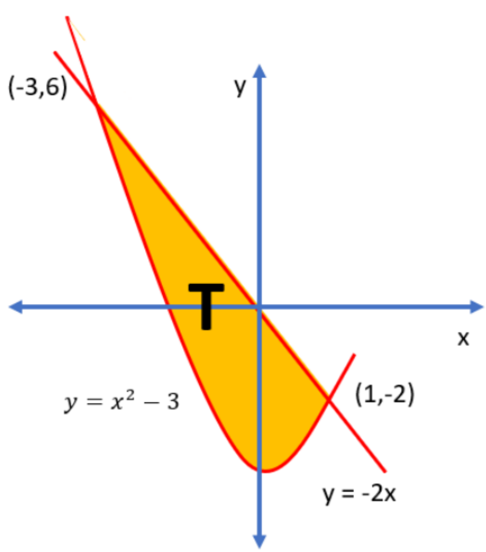
$$\Rightarrow x^2 + 2x - 3 = 0$$

$$\Rightarrow (x+3)(x-1) = 0$$

$$\Rightarrow x = -3, 1$$

.

So, $(-3, 6)$ and $(1, -2)$ are the two points of intersection.



So, $T = \{(x, y) : -3 \leq x \leq 1,\ x^2 - 3 \leq y \leq -2x\}$.

Hence, by Fubini's theorem (i) we have

$$\int_T\int 5y\,dA = \int_{-3}^{1}\left(\int_{x^2-3}^{-2x} 5y\,dy\right)dx$$

$$= \int_{-3}^{1}\left[\frac{5y^2}{2}\right]_{x^2-3}^{-2x} dx$$

$$= \int_{-3}^{1}\left(10x^2 - \frac{5}{2}(x^4 - 6x^2 + 9)\right)dx$$

$$= \left[\frac{10x^3}{3} - \frac{x^5}{2} + \frac{15x^3}{3} - \frac{45x}{2}\right]_{-3}^{1}$$

$$= \left[\frac{25x^3}{3} - \frac{x^5}{2} - \frac{45x}{2}\right]_{-3}^{1}$$

$$= \left((\frac{25}{3} - \frac{1}{2} - \frac{45}{2}) - (-225 + \frac{243}{2} + \frac{135}{2})\right)$$

$$= \frac{25}{3} - \frac{424}{2} + 225$$

$$= \frac{25}{3} + 13$$

$$= \frac{64}{3}$$

Remark 8.

In some cases you need to reverse the order of integration (when using Fubini's Theorem) because it may not be possible to integrate without reversing the order. The next example illustrates this.

Example 4. Find $\int_0^1 \int_x^1 e^{y^2}\,dydx$

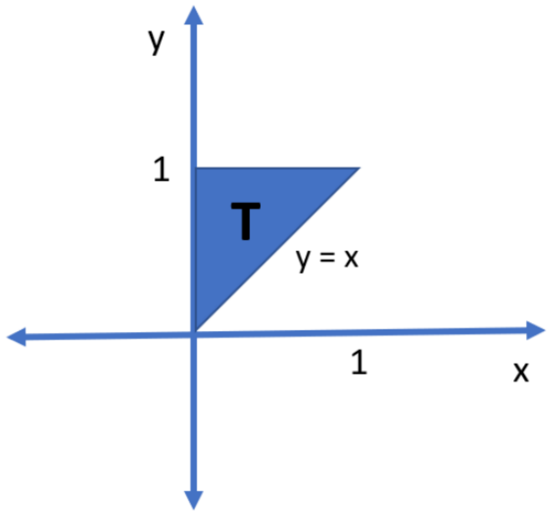
Solution.

There is no elementary function whose derivative is e^{y^2} and so one cannot integrate w.r.t. y first in the above iterated integral. Elementary functions are essentially the type of functions you have studied so far. In order to try to reverse the order of integration we need to look at the relevant region T in Fubini's theorem.

By Fubini's theorem (i) we have:

$$\int_0^1\int_x^1 e^{y^2}\,dydx = \int_T\int e^{y^2}\,dA$$

where $T = \{(x, y) : 0 \leq x \leq 1,\ x \leq y \leq 1\}$. Here is a picture of T.



Note that we can also write T as $T = \{(x, y) : 0 \leq y \leq 1,\ 0 \leq x \leq y\}$. So, by Fubini's theorem (ii) we can integrate w.r.t. x first to get

$$\int_T\int e^{y^2}\,dA = \int_0^1 \int_0^y e^{y^2}\,dx dy$$

$$= \int_0^1 \left[xe^{y^2}\right]_0^y dy$$

$$= \int_0^1 ye^{y^2}\,dy$$

For this single integral, use the substitution rule with $u = y^2$ so

$$\int_0^1 ye^{y^2}\,dy = \frac{1}{2}\int_0^1 e^u\,du$$

$$= \frac{1}{2}\left[e^u\right]_0^1$$

$$= \frac{1}{2}(e - 1)$$

Definition 3.

The average value of a function f over the region T is

$$\frac{1}{S}\int_T\int f(x, y)\,dA$$

where S is the area of T.

Example 5.

Find the average value of $f(x,y) = \sin(x+y)$ over the region T given by $0 \le x \le \pi,\ 0 \le y \le \pi$.

Solution.

The average value is

$$\frac{1}{\pi^2}\int_T\int \sin(x+y)\,dA$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\left(\int_0^{\pi}\sin(x+y)\,dy\right)dx$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\left[-\cos(x+y)\right]_0^{\pi}\,dx$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\left(-\cos(x+\pi)+\cos x\right)\,dx$$

$$= \frac{1}{\pi^2}\left[-\sin(x+\pi)+\sin x\right]_0^{\pi}$$

$$= 0$$

Example 6.

Find the average value of $g(x,y) = x\cos(xy)$ over the region T given by $0 \le x \le \pi,\ 0 \le y \le 1$.

Solution.

The average value is

$$\frac{1}{\pi}\int_T\int x\cos(xy)\,dA$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\int_0^{1}x\cos(xy)\,dy\right)dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(xy)\right]_0^{1}\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\sin x - 0)\,dx$$

$$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$= \frac{1}{\pi}(1+1)$$

$$= \frac{2}{\pi}$$

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Lecture 17

Section 4.2 – Surface Area.

Definition 4.

Suppose $f(x,y)$ and its partial derivatives are continuous on the closed region T in the xy–plane. Then, the area of the surface given by $z = f(x,y)$ over T is given by

$$\int_T\int \sqrt{1+f_x^2+f_y^2}\,dA$$

where

$$f_x = \frac{\partial f}{\partial x},\quad f_x^2 = (\frac{\partial f}{\partial x})^2,\quad f_y = \frac{\partial f}{\partial y},\quad f_y^2 = (\frac{\partial f}{\partial y})^2$$

Example 7.

Find the surface area of the portion of the surface $z = \frac{1}{2}(x^2-2y)$ lying above the triangular region T in the xy-plane bounded by the lines $y = 0, x = 2$ and $y = 3x$.

Solution.

We see that $T = \{(x,y) : 0 \leq x \leq 2,\ 0 \leq y \leq 3x\}$.

Let $f(x,y) = \frac{1}{2}(x^2-2y)$. Then

$$f_x = x \quad \text{and} \quad f_y = -1$$

So, the required surface area is

$$S = \int_T\int \sqrt{1+f_x^2+f_y^2}\,dA$$

$$= \int_0^2 \left(\int_0^{3x} \sqrt{2+x^2}\,dy\right) dx$$

$$= \int_0^2 \left[y\sqrt{2+x^2}\right]_0^{3x} dx$$

$$= \int_0^2 \left(3x\sqrt{2+x^2}\right)\, dx \qquad (*)$$

Now use substitution on $(*)$ with $u = 2 + x^2$ to get

$$(*) = \frac{3}{2}\int_2^6 \sqrt{u}\, du$$

$$= \left[u^{\frac{3}{2}}\right]_2^6$$

$$= 6\sqrt{6} - 2\sqrt{2}$$

Example 8.

Find the surface area of the portion of the surface $z = 2x + 3y$ lying above the rectangular region T given by $-1 \leq x \leq 2,\ 0 \leq y \leq 2$.

Solution.

Let $f(x, y) = 2x + 3y$. Then

$$f_x = 2 \quad \text{and} \quad f_y = 3$$

So, the required surface area is

$$S = \int_T\int \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= \int_{-1}^2 \left(\int_0^2 \sqrt{1+4+9}\, dy\right) dx$$

$$= \int_{-1}^2 \left[y\sqrt{14}\right]_0^2\, dx$$

$$= \int_{-1}^2 \left(2\sqrt{14}\right)\, dx$$

$$= \left[2x\sqrt{14}\right]_{-1}^2\, dx$$

$$= 6\sqrt{14}$$

Section 4.3 – Line Integrals.

Remark 9.

We will first discuss what a piecewise smooth curve is. Suppose C is a curve in the xy-plane given by the parametric equations $x = x(t), y = y(t)$ for $a \leq t \leq b$. An example of such a curve is the circle with centre $(0, 0)$ and radius 1 which has parametric equations $x = \cos t,\ y = \sin t$ for $0 \leq t \leq 2\pi$.

We now say that C is smooth if

$$\frac{dx}{dt} \quad \text{and} \quad \frac{dy}{dt}$$

are continuous on $[a, b]$. Similarly, suppose B is a curve in $\mathbb{R}^3$ given by the parametric equations $x = x(t), y = y(t), z = z(t)$ for $a \leq t \leq b$. We say that B is smooth if

$$\frac{dx}{dt}, \frac{dy}{dt} \quad \text{and} \quad \frac{dz}{dt}$$

are continuous on $[a, b]$.

A curve W is piecewise smooth if the interval $[a, b]$ can be subdivided into a finite number of subintervals, on each of which W is smooth.

Example 9.

Consider the curve C given by $x = \cos t,\ y = \sin t$, for $0 \leq t \leq 2\pi$. Then, C is smooth. C is the circle of radius 1 with centre at the origin (as mentioned above).

Consider the curve W given by $x = \cos t,\ y = \sin t,\ z = t$ for $0 \leq t \leq 4\pi$. Then, W is smooth. W is a helix.

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Definition 5.

We will now define the length of a curve. Suppose C is a curve in the xy-plane given by the parametric equations $x = x(t),\ y = y(t)$ for $a \le t \le b$. Suppose

$$\frac{dx}{dt} \quad \text{and} \quad \frac{dy}{dt}$$

are continuous on $[a, b]$. If C is traced once as t moves from a to b, then the length of C is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Suppose B is a curve in $\mathbb{R}^3$ given by the parametric equations $x = x(t),\ y = y(t),\ z = z(t)$ for $a \le t \le b$. Suppose

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt}$$

are continuous on $[a, b]$. If B is traced once as t moves from a to b, then the length of B is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example 10.

Suppose B is the helix given by

$$x = \cos t,\ y = \sin t,\ z = t, \quad \text{for} \quad 0 \le t \le 2\pi$$

The length of B is

$$\int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt$$

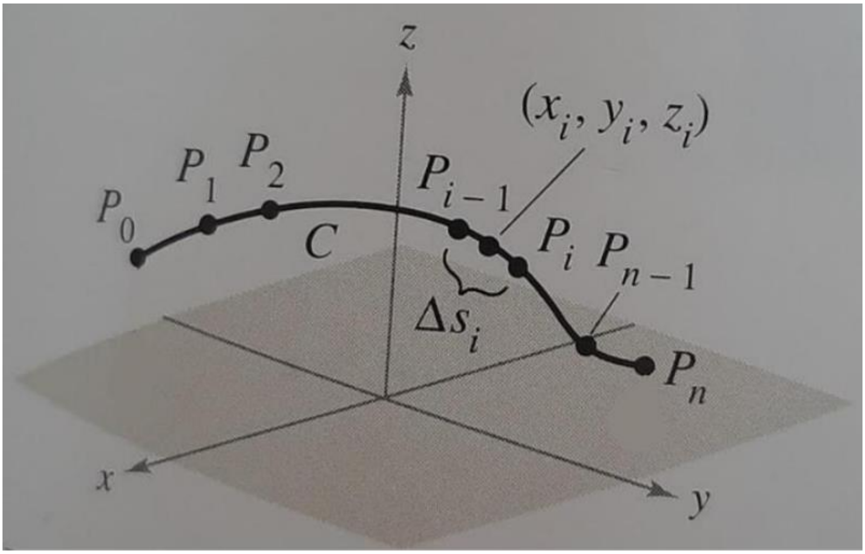
$$= \int_0^{2\pi} \sqrt{2}\, dt$$

$$= 2\pi\sqrt{2}$$

Remark 10.

We previously defined single integrals and double integrals. We will now define the so called line integral. Recall that we motivated the definition of a single integral by looking at the notion of area and we motivated the definition of a double integral by looking at volume. Here we will motivate the definition of a line integral by looking at the mass of a wire of finite length given by a curve C in $\mathbb{R}^3$.

Suppose the density (i.e. mass per unit length) of the wire at the point (x, y, z) is given by $f(x, y, z)$. Subdivide the curve C by the points $P_0, P_1, \ldots, P_n$ giving n subarcs as in the picture below.



Denote the length of the i^{th} subarc by Δs_i and let $||\Delta||$ denote the length of the longest subarc. Now, select a point (x_i, y_i, z_i) in the i^{th} subarc. We say that the sum

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i$$

approximates the total mass of the wire. Note that this approximation may not necessarily be a good approximation if $||\Delta||$ is big. However, we expect the approximation to improve as $||\Delta||$ approaches zero. With this as motivation, we define the mass of the wire to be

$$\lim_{||\Delta||\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i$$

where this limit is defined in a similar way as (**) in remark 1

With the above as motivation (and using similar notation as above), we are now ready to define the so called line integral of f along C as follows:

Definition 6.

Suppose f is defined in a region containing a smooth curve C of finite length. The line integral of f along C is defined as

$$\int_C f(x,y,z)\ ds = \lim_{||\Delta||\to 0} \sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta s_i$$

provided this limit exists.

We can similarly define the line integral for a function of two variables. So, suppose $g(x,y)$ is defined in a region containing a smooth curve C of finite length. The line integral of g along C is defined as

$$\int_C g(x,y)\ ds = \lim_{||\Delta||\to 0} \sum_{i=1}^{n} g(x_i,y_i)\Delta s_i$$

provided this limit exists.

Remark 11.

The following theorem shows how to calculate line integrals by using single integrals.

Theorem 3.

(i) Suppose g is continuous on a set containing a smooth curve C in $\mathbb{R}^2$. Suppose C has parametric equations

$$x = x(t),\ y = y(t) \quad \text{for} \quad t \in [a,b]$$

Then

$$\int_C g(x,y)\,ds = \int_a^b g(x(t),y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

Note that here $x'(t) = \dfrac{dx}{dt}$ and $y'(t) = \dfrac{dy}{dt}$

(ii) Suppose h is continuous on a set containing a smooth curve C in $\mathbb{R}^3$. Suppose C has parametric equations

$$x = x(t),\ y = y(t), z = z(t) \quad \text{for} \quad t \in [a,b]$$

Then

$$\int_C h(x,y,z)\,ds = \int_a^b h(x(t),y(t),z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

Note that here $z'(t) = \dfrac{dz}{dt}$

Remark 12.

In the special case where $g(x,y) = 1$, for all (x,y) in the domain of g above, then we get that the line integral

$$\int_C g(x,y)\,ds$$

is the length of the curve C because

$$\int_C g(x,y)\,ds = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

Example 11.

There are many applications of line integrals. For example, line integrals can be used to calculate masses. A coil spring lies along the helix C given by

$$x = \cos 4t,\ \ y = \sin 4t,\ \ z = t, \quad \text{for} \quad t \in [0, 2\pi]$$

The spring's density is the constant function $f(x,y,z) = 1$. Find the mass of the spring where the mass M of the spring is given by the line integral

$$M = \int_C f(x,y,z)\,ds$$

Solution.

$$M = \int_C f(x,y,z)\,ds$$

$$= \int_0^{2\pi} f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt \qquad (*)$$

Now

$$x'(t) = -4\sin 4t,\ \ y'(t) = 4\cos 4t,\ \ z'(t) = 1$$

.

Hence

$$(*) = \int_0^{2\pi} \sqrt{17}\,dt$$

$$= 2\pi\sqrt{17}$$

So, the mass of the spring is $2\pi\sqrt{17}$.

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Lecture 21

Example 16.

Find

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_x^{\sqrt{\frac{\pi}{2}}} \int_1^3 \sin(y^2)\, dz dy dx$$

Solution.

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_x^{\sqrt{\frac{\pi}{2}}} \int_1^3 \sin(y^2)\, dz dy dx = \int_0^{\sqrt{\frac{\pi}{2}}} \int_x^{\sqrt{\frac{\pi}{2}}} \left[z \sin(y^2)\right]_1^3 \, dy dx$$

.

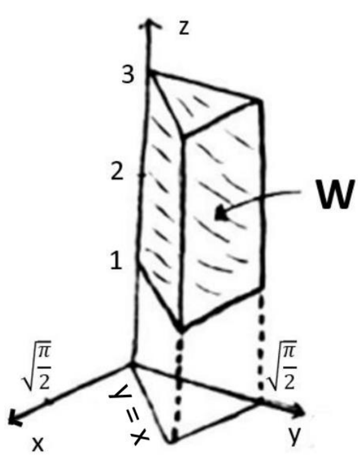
$$= \int_0^{\sqrt{\frac{\pi}{2}}} \int_x^{\sqrt{\frac{\pi}{2}}} 2 \sin(y^2)\, dy dx$$

.

We cannot proceed here because there is no elementary function h such that $h'(y) = \sin(y^2)$. So, we need to change the order of integration. The relevant set W (for Fubini's theorem 7) is

$$W = \{(x, y, z) : 0 \le x \le \sqrt{\frac{\pi}{2}},\ x \le y \le \sqrt{\frac{\pi}{2}},\ 1 \le z \le 3\}$$

and so by looking at the picture below we have that W can also be written as



$$W = \{(x, y, z) : 0 \leq y \leq \sqrt{\frac{\pi}{2}},\ 0 \leq x \leq y,\ 1 \leq z \leq 3\}$$

We now get that

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_x^{\sqrt{\frac{\pi}{2}}} \int_1^3 \sin(y^2)\, dz dy dx = \int\int_W \int g(x, y, z)\, dV, \quad \text{by Fubini's theorem 7 (i)}$$

$$= \int_0^{\sqrt{\frac{\pi}{2}}} \int_0^y \int_1^3 \sin(y^2)\, dz dx dy, \quad \text{by Fubini's theorem 7 (iii)}$$

$$= \int_0^{\sqrt{\frac{\pi}{2}}} \int_0^y \left[z \sin(y^2)\right]_1^3 dx dy$$

$$= 2 \int_0^{\sqrt{\frac{\pi}{2}}} \int_0^y \sin(y^2)\, dx dy$$

$$= 2 \int_0^{\sqrt{\frac{\pi}{2}}} \left[x \sin(y^2)\right]_0^y dy$$

$$= 2 \int_0^{\sqrt{\frac{\pi}{2}}} y \sin(y^2)\, dy \qquad (*)$$

Use the substitution rule on $(*)$ with $u = y^2$ to get that

$$(*) = \int_0^{\frac{\pi}{2}} \sin u\, du = [-\cos u]_0^{\frac{\pi}{2}} = 1$$

Definition 7.

The average value of a function $f(x, y, z)$ over a region W in $\mathbb{R}^3$ is defined to be

$$\frac{1}{S} \int\int_W \int f(x, y, z)\, dV$$

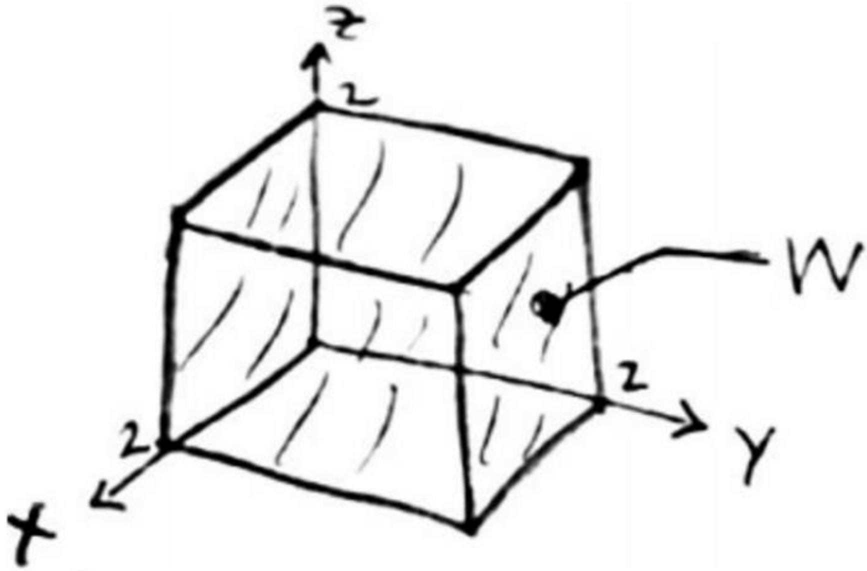
where S is the volume of W.

Example 17.

Find the average value of $f(x, y, z) = xyz$ over the cube W bounded by the xy-plane, the xz-plane, the yz-plane and the planes $x = 2$, $y = 2$, $z = 2$ in the first octant.

Solution.

See the picture below for $W = \{(x, y, z) : 0 \leq x \leq 2,\ 0 \leq y \leq 2,\ 0 \leq z \leq 2\}$



The volume of W is 8 and the required average value is

$$\frac{1}{8}\int\int_{W}\int xyz\,dV$$

$$=\frac{1}{8}\int_{0}^{2}\int_{0}^{2}\int_{0}^{2} xyz\,dxdydz$$

$$=\frac{1}{8}\int_{0}^{2}\int_{0}^{2}\left[\frac{x^2}{2}yz\right]_{0}^{2} dydz$$

$$=\frac{1}{8}\int_{0}^{2}\int_{0}^{2} 2yz\,dydz$$

$$=\frac{1}{8}\int_{0}^{2}\left[y^2 z\right]_{0}^{2} dz$$

$$=\frac{1}{8}\int_{0}^{2} 4z\,dz$$

$$=\frac{1}{8}\left[2z^2\right]_{0}^{2}$$

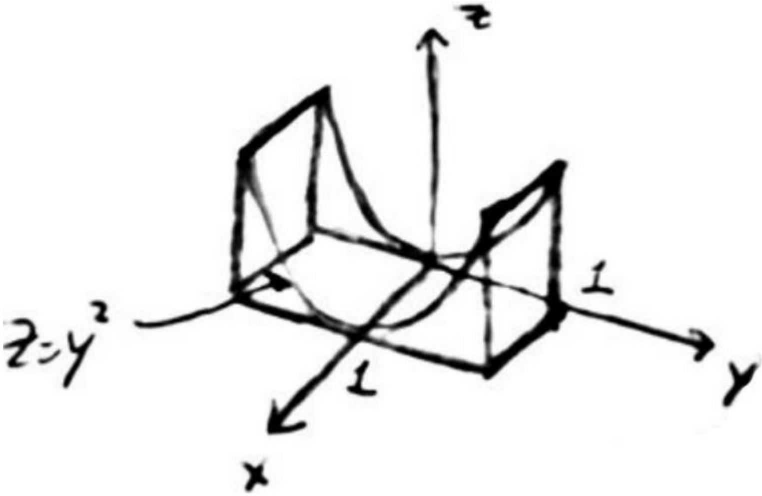
$$=1$$

Example 18.

Suppose T is the solid region in $\mathbb{R}^3$ that is bounded by the surface $z=y^2$ and the planes $z=0$, $x=0$, $x=1$, $y=1$, $y=-1$. Find the volume of T.

Solution.

See the picture below.



Note that

$$T = \{(x, y, z) : 0 \le x \le 1,\ -1 \le y \le 1,\ 0 \le z \le y^2\}$$

The volume W of T is given by the triple integral

$$W = \int\int_T\int dV$$

So,

$$W = \int_0^1 \int_{-1}^1 \int_0^{y^2} dzdydx, \quad \text{by Fubini's theorem 7 (i)}$$

$$= \int_0^1 \int_{-1}^1 [z]_0^{y^2}\, dydx$$

$$= \int_0^1 \int_{-1}^1 y^2\, dydx$$

$$= \int_0^1 \left[\frac{y^3}{3}\right]_{-1}^1 dx$$

$$= \int_0^1 \frac{2}{3}\, dx$$

$$= \left[\frac{2x}{3}\right]_0^1$$

$$= \frac{2}{3}$$

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Lecture 19

Example 12.

Find $\int_C (x^2+y^2-2z)\,ds$, where C is the curve with parametric equations

$$x=t,\ \ y=-3t,\ \ z=2t, \quad \text{for} \quad t\in[0,1]$$

Solution.

Let $f(x,y,z)=x^2+y^2-2z$. Now

$$x(t)=t \Rightarrow x'(t)=1$$

$$y(t)=-3t \Rightarrow y'(t)=-3$$

$$z(t)=2t \Rightarrow z'(t)=2$$

So,

$$\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}$$

$$=\sqrt{1+9+4}$$

$$=\sqrt{14}$$

Hence,

$$\int_C (x^2+y^2-2z)\,ds$$

$$=\int_0^1 (t^2+9t^2-4t)\sqrt{14}\,dt$$

$$=\sqrt{14}\int_0^1 (10t^2-4t)\,dt$$

$$= \sqrt{14}\left[\frac{10t^3}{3} - 2t^2\right]_0^1$$

$$= \sqrt{14}\left(\frac{4}{3}\right)$$

$$= \frac{4\sqrt{14}}{3}$$

Example 13.

Find $\int_C (x + \sqrt{y} - z^2)\, ds$, where C is the curve with parametric equations

$$x = t,\ y = t^2,\ z = 0, \quad \text{for} \quad t \in [0, 1]$$

Solution.

Let $f(x, y, z) = x + \sqrt{y} - z^2$. Now,

$$x(t) = t \Rightarrow x'(t) = 1$$

$$y(t) = t^2 \Rightarrow y'(t) = 2t$$

$$z(t) = 0 \Rightarrow z'(t) = 0$$

So,

$$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

$$= \sqrt{1 + 4t^2}$$

Hence,

$$\int_C (x + \sqrt{y} - z^2)\, ds$$

$$= \int_0^1 (t + \sqrt{t^2} - 0)\sqrt{1 + 4t^2}\, dt$$

$$= 2\int_0^1 t\sqrt{1 + 4t^2}\, dt \qquad (*)$$

Use the substitution rule on $(*)$ with $u = 1 + 4t^2$ to get

$$(*) = \frac{1}{4}\int_1^5 \sqrt{u}\,du$$

$$= \frac{1}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^5$$

$$= \frac{1}{4}\left[\frac{2}{3}5^{\frac{3}{2}} - \frac{2}{3}\right]$$

$$= \frac{1}{6}(5\sqrt{5} - 1)$$

Section 4.4 – Surface Integrals.

Remark 13.

We have already met a variety of different types of integrals in previous lectures. Now we will meet what is called a surface integral which will relate to integrating a function over a surface. All these types of integrals have many important and powerful applications in Science, Engineering and other areas.

We will now discuss how to integrate a function over surface. To motivate the idea we can consider an electric charge distributed over a surface S given by $z = f(x, y)$. Now suppose that the function $h(x, y, z)$ gives the electric charge per unit area (i.e. charge density) at each point (x, y, z) on S. Then, the total charge on S can be calculated in the following way:

Suppose R is the vertical projection of S onto the xy-plane (i.e. $R = \{(x, y, 0) : (x, y, z) \in S\}$). We partition R into small rectangles A_k, $1 \leq k \leq n$, like we did before in the definition of a double integral. Denote the area of A_k by ΔA_k. Directly above A_k lies a patch B_k of S with area ΔB_k and this patch can be approximated with a parallelogram shaped piece E_k of the tangent plane that has area ΔE_k. One can show that ΔE_k can be approximated by

$$\sqrt{1 + f_x^2(x_k, y_k) + f_y^2(x_k, y_k)}\Delta A_k$$

where $(x_k, y_k, 0)$ is a point in A_k. The total charge over B_k can then be approximated by

$$h(x_k, y_k, z_k)\sqrt{1 + f_x^2(x_k, y_k) + f_y^2(x_k, y_k)}\Delta A_k$$

The total charge over S can then be approximated by

$$\sum_{k=1}^{n} h(x_k, y_k, z_k)\sqrt{1 + f_x^2(x_k, y_k) + f_y^2(x_k, y_k)}\Delta A_k \qquad (*)$$

The approximation gets better as the rectangles A_k get smaller. So, we now define the surface integral of h over S to be the limit of $(*)$ as the length of the longest diagonal of the n rectangles A_k goes to zero. We denote the surface integral of h over S by

$$\int_S\int h(x,y,z)\,dS$$

The limit above is also the double integral

$$\int_R\int h(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\,dA$$

This motivates the following theorem which is useful for calculating surface integrals.

Theorem 6.

Suppose S is a surface given by $z=f(x,y)$ and suppose R is the vertical projection of S onto the xy-plane as above. If f, f_x, f_y are continuous on R and $g(x,y,z)$ is continuous on S, then the surface integral

$$\int_S\int g(x,y,z)\,dS$$

is given by

$$\int_S\int g(x,y,z)\,dS = \int_R\int g(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\,dA$$

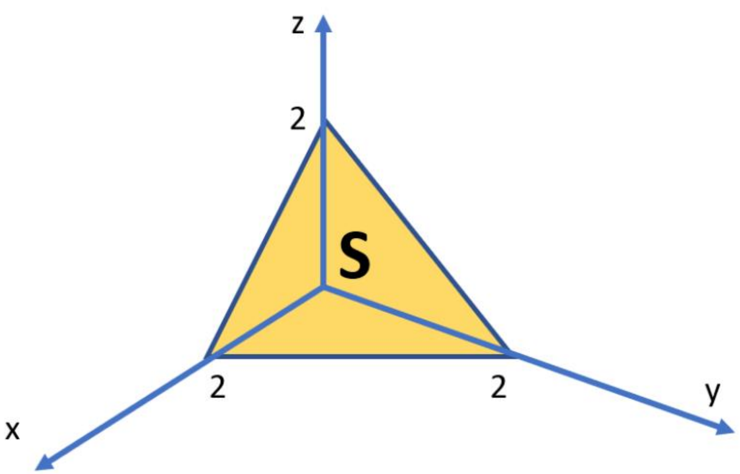
where the integral over R is a double integral as before.

Example 14.

Find the surface integral, $\int_S\int(xy+z)\,dS$, where S is that part of the plane $x+y+z=2$ in the first octant (i.e. $x\geq 0, y\geq 0, z\geq 0$).

Solution.

See the picture for S below.



We have that $z=2-x-y$ and so we let $f(x,y)=2-x-y$ and $g(x,y,z)=xy+z$.

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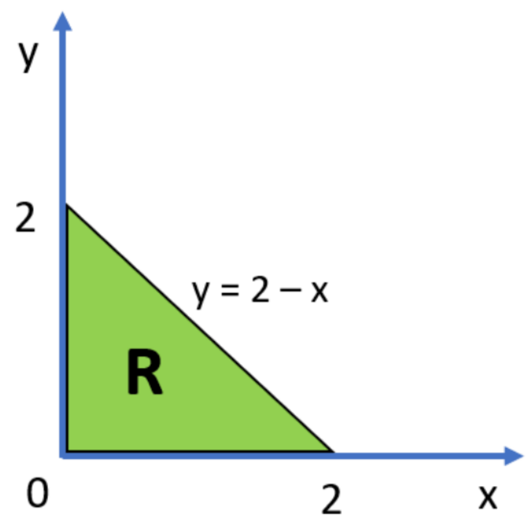
Lecture 20

Example 14 continued.

We have that $z = 2 - x - y$ and so we let $f(x, y) = 2 - x - y$ and $g(x, y, z) = xy + z$. Hence

$$\int_S\int (xy + z)\, dS = \int_R\int g(x, y, f(x, y))\sqrt{1 + f_x^2 + f_y^2}\, dA \quad \text{by theorem 6} \qquad (*)$$

where R is the triangle in the picture below.



So, $R = \{(x, y) : 0 \leq x \leq 2,\ 0 \leq y \leq 2 - x\}$

Now, $f_x = -1$ and $f_y = -1$ and so $(*)$ is

$$= \int_R\int (xy + 2 - x - y)\sqrt{1 + (-1)^2 + (-1)^2}\, dA$$

$$= \sqrt{3}\int_R\int (xy + 2 - x - y)\, dA$$

$$= \sqrt{3}\int_0^2\int_0^{2-x} (xy + 2 - x - y)\, dydx \quad \text{by Fubini's theorem}$$

$$= \sqrt{3}\int_0^2 \left[\frac{xy^2}{2} + 2y - xy - \frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \sqrt{3}\int_0^2 \left(\frac{x(2-x)^2}{2} + 2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\right) dx$$

$$= \sqrt{3}\int_0^2 \left(\frac{x(2-x)^2}{2} + \frac{(2-x)^2}{2}\right)\, dx$$

$$= \sqrt{3}\int_0^2 \left(\frac{4x-4x^2+x^3}{2} + \frac{4-4x+x^2}{2}\right)\, dx$$

$$= \sqrt{3}\int_0^2 \left(\frac{4-3x^2+x^3}{2}\right)\, dx$$

$$= \sqrt{3}\left[2x - \frac{x^3}{2} + \frac{x^4}{8}\right]_0^2$$

$$= \sqrt{3}(4-4+2)$$

$$= 2\sqrt{3}$$

Example 15.

Suppose a flat sheet of metal has the shape given by the surface $z = 1 + x + 2y$ that lies above the rectangle R given by $0 \leq x \leq 4$, $0 \leq y \leq 2$. If the density of the sheet is given by $g(x, y, z) = x^2yz$, then find the mass M of the sheet, where M is given by the surface integral

$$M = \int_S\int g(x, y, z)\, dS$$

where S is the surface given by $z = 1 + x + 2y$.

Solution.

Let $f(x, y) = 1 + x + 2y$. Then,

$$M = \int_S\int g(x, y, z)\, dS$$

$$= \int_R\int g(x, y, 1+x+2y)\sqrt{1+f_x^2+f_y^2}\, dA, \quad \text{by theorem 6}$$

$$= \int_0^4\int_0^2 x^2y(1+x+2y)\sqrt{1+1+4}\, dydx, \quad \text{by Fubini's theorem}$$

$$= \sqrt{6}\int_0^4\int_0^2 (x^2y + x^3y + 2x^2y^2)\, dydx$$

$$= \sqrt{6}\int_0^4 \left[\frac{x^2y^2}{2} + \frac{x^3y^2}{2} + \frac{2x^2y^3}{3}\right]_0^2 dx$$

$$= \sqrt{6} \int_0^4 \left(2x^2 + 2x^3 + \frac{16x^2}{3} \right) dx$$

$$= \sqrt{6} \int_0^4 \left(\frac{22x^2}{3} + 2x^3 \right) dx$$

$$= \sqrt{6} \left[\frac{22x^3}{9} + \frac{x^4}{2} \right]_0^4$$

$$= \sqrt{6} \left(\frac{1408}{9} + 128 \right)$$

$$= \frac{2560\sqrt{6}}{9}$$

Section 4.5 – Triple Integrals.

Remark 14.

In definition 1 we defined the double integral of a function of two variables, $f(x, y)$, over a set in $\mathbb{R}^2$. In this section we will discuss (what will be called) the triple integral of a function of three variables, $g(x, y, z)$, over a set in $\mathbb{R}^3$. Recall how useful Fubini's theorem was for finding double integrals by performing two single integrals (one after the other). Well, we will also have a Fubini's theorem for triple integrals which will show how a triple integral can be evaluated by performing three single integrals (one after the other).

We will now motivate the definition of a triple integral. Suppose, $g(x, y, z)$ is a function defined on a closed bounded set in T in $\mathbb{R}^3$. We partition a rectangular region about T into n rectangular cells by planes parallel to the coordinate planes (i.e. the xy-plane, xz-plane and yz-plane). Suppose the k^{th} cell has volume ΔV_k. Pick a point (x_k, y_k, z_k) in the k^{th} cell and consider the sum

$$W = \sum_{k=1}^{n} g(x_k, y_k, z_k)\Delta V_k \qquad (*)$$

Then, the triple integral of g over T is denoted by

$$\int\int_T\int g(x, y, z)\, dV$$

and is defined to be the limit of $(*)$ above as the length of the longest diagonal of the n cells goes to zero. We say that g is integrable over T if this limit exists.

Remark 15.

If g is the constant function 1 in $(*)$ above, then $(*)$ will approximate the volume of T. This approximation will get better as the cells get smaller. So, we can define the volume of T to be the limit above and hence

$$\text{Volume of } T = \int\int_T\int dV$$

Remark 16 – Some properties of triple integrals.

Suppose all the integrals below exist. Then

$$\int\int_W\int kf\,dV = k\int\int_W\int f\,dV, \quad \text{for all} \quad k \in \mathbb{R}$$

$$\int\int_W\int (f+g)\,dV = \int\int_W\int f\,dV + \int\int_W\int g\,dV$$

$$\int\int_W\int (f-g)\,dV = \int\int_W\int f\,dV - \int\int_W\int g\,dV$$

$$\int\int_W\int f\,dV \geq 0 \quad \text{if} \quad f \geq 0 \quad \text{on} \quad W$$

$$\int\int_W\int f\,dV \geq \int\int_W\int g\,dV \quad \text{if} \quad f \geq g \quad \text{on} \quad W$$

$$\int\int_W\int f\,dV = \int\int_{A_1}\int f\,dV + \int\int_{A_2}\int f\,dV + \cdots \int\int_{A_n}\int f\,dV$$

where W is the union of the pairwise disjoint sets $\{A_1, A_2, \ldots, A_n\}$

Theorem 7 – Fubini's theorem for triple integrals.

Suppose g is continuous on a set W in $\mathbb{R}^3$.

(i) If $W = \{(x,y,z) \in \mathbb{R}^3 : a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x),\ h_1(x,y) \leq z \leq h_2(x,y)\}$, where f_1, f_2, h_1, h_2 are continuous functions, then

$$\int\int_W\int g(x,y,z)\,dV = \int_a^b \int_{f_1(x)}^{f_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} g(x,y,z)\,dzdydx$$

(ii) If $W = \{(x,y,z) \in \mathbb{R}^3 : a \leq x \leq b,\ f_1(x) \leq z \leq f_2(x),\ h_1(x,z) \leq y \leq h_2(x,z)\}$, where f_1, f_2, h_1, h_2 are continuous functions, then

$$\int\int_W\int g(x,y,z)\,dV = \int_a^b \int_{f_1(x)}^{f_2(x)} \int_{h_1(x,z)}^{h_2(x,z)} g(x,y,z)\,dydzdx$$

(iii) If $W = \{(x,y,z) \in \mathbb{R}^3 : c \leq y \leq d,\ k_1(y) \leq x \leq k_2(y),\ h_1(x,y) \leq z \leq h_2(x,y)\}$, where k_1, k_2, h_1, h_2 are continuous functions, then

$$\int\int_W\int g(x,y,z)\,dV = \int_c^d \int_{k_1(y)}^{k_2(y)} \int_{h_1(x,y)}^{h_2(x,y)} g(x,y,z)\,dzdxdy$$

(iv) If $W = \{(x,y,z) \in \mathbb{R}^3 : c \leq y \leq d,\ k_1(y) \leq z \leq k_2(y),\ h_1(y,z) \leq x \leq h_2(y,z)\}$, where k_1, k_2, h_1, h_2 are continuous functions, then

$$\int\int_W\int g(x,y,z)\,dV = \int_c^d \int_{k_1(y)}^{k_2(y)} \int_{h_1(y,z)}^{h_2(y,z)} g(x,y,z)\,dxdzdy$$

(v) If $W = \{(x,y,z) \in \mathbb{R}^3 : a \leq z \leq b,\ k_1(z) \leq x \leq k_2(z),\ s_1(x,z) \leq y \leq s_2(x,z)\}$, where k_1, k_2, s_1, s_2 are continuous functions, then

$$\int\int_W\int g(x,y,z)\,dV = \int_a^b \int_{k_1(z)}^{k_2(z)} \int_{s_1(x,z)}^{s_2(x,z)} g(x,y,z)\,dydxdz$$

(vi) If $W = \{(x,y,z) \in \mathbb{R}^3 : a \leq z \leq b,\ k_1(z) \leq y \leq k_2(z),\ s_1(y,z) \leq x \leq s_2(y,z)\}$, where k_1, k_2, s_1, s_2 are continuous functions, then

$$\int\int_W\int g(x,y,z)\,dV = \int_a^b \int_{k_1(z)}^{k_2(z)} \int_{s_1(y,z)}^{s_2(y,z)} g(x,y,z)\,dxdydz$$

Remark 17.

In Fubini's theorem (i) and (ii) we say W is simple in the x–direction. In these cases we integrate w.r.t. x last. In Fubini's theorem (iii) and (iv) we say W is simple in the y–direction. In these cases we integrate w.r.t. y last. In Fubini's theorem (v) and (vi) we say W is simple in the z–direction. In these cases we integrate w.r.t. z last. In relation to what variables we integrate w.r.t. first and second one should look at which of the six statements (in theorem 7) apply. All the information one needs is in the six statements.

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Lecture 22

Example 19.

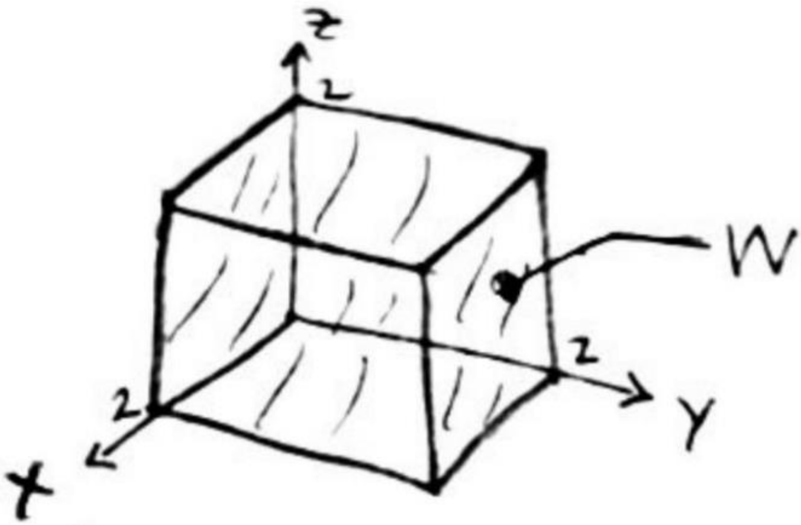
Suppose W is the cube bounded by the planes, $z = 0$, $x = 0$, $y = 0$, $x = 2$, $y = 2$, $z = 2$ in the first octant. Suppose the density at the point (x, y, z) is the square of the distance from (x, y, z) to the origin. Find the mass M of W where M is given by the triple integral.

$$\int\int_W\int k(x, y, z)\, dV$$

where $k(x, y, z)$ is the density function.

Solution.

See the picture below.



Note that

$$W = \{(x, y, z) : 0 \leq x \leq 2,\ 0 \leq y \leq 2,\ 0 \leq z \leq 2\}$$

We are given that $k(x, y, z) = x^2 + y^2 + z^2$

So

$$M = \int\int_W\int k(x, y, z)\, dV$$

$$= \int_0^2 \int_0^2 \int_0^2 (x^2 + y^2 + z^2)\, dxdydz,$$

$$= \int_0^2 \int_0^2 \left[\frac{x^3}{3} + x(y^2 + z^2)\right]_0^2 dydz$$

$$= \int_0^2 \int_0^2 \left(\frac{8}{3} + 2(y^2 + z^2)\right) dydz$$

$$= \int_0^2 \left[\frac{8y}{3} + \frac{2y^3}{3} + 2yz^2\right]_0^2 dz$$

$$= \int_0^2 \left(\frac{32}{3} + 4z^2\right) dz$$

$$= \left[\frac{32z}{3} + \frac{4z^3}{3}\right]_0^2$$

$$= \left(\frac{64}{3} + \frac{32}{3}\right)$$

$$= \frac{96}{3}$$

$$= 32$$

Chapter 5 – Mean Value Theorem.

Recall the Mean Value Theorem for functions of one variable.

Mean Value Theorem (MVT1).

Suppose $f(x)$ is a continuous function on the closed interval $[a, b]$ in $\mathbb{R}$ and suppose f is differentiable on the open interval (a, b). Then, there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Remark 1.

We will now discuss the Mean Value Theorem for functions of several variables. We first need to define what a convex set is.

Definition 1.

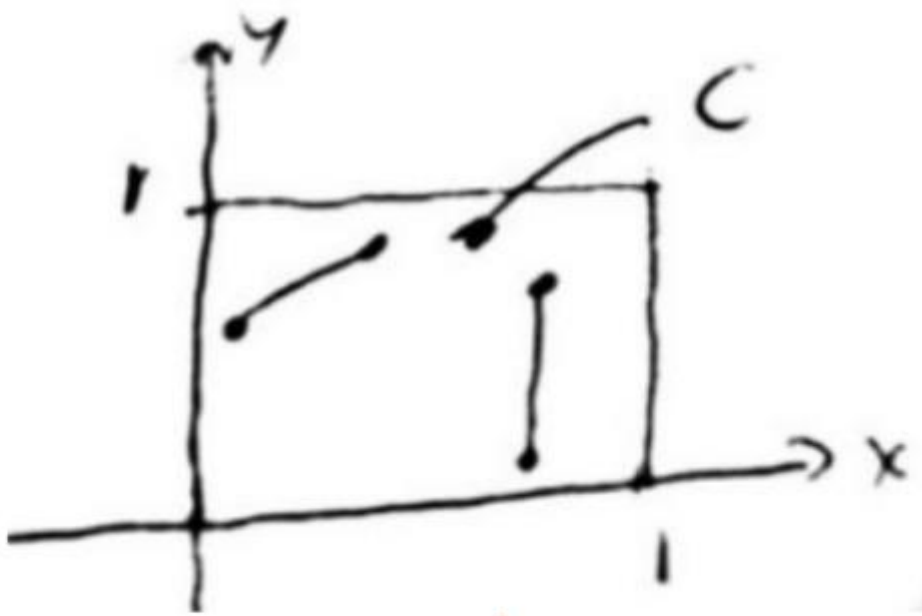
A convex set in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a set C such that

$$(1-t)z + tw \in C, \quad \text{for all} \quad z, w \in C, \ t \in [0,1]$$

This means that if $z, w \in C$, then the line segment joining z to w is a subset of C.

Example 1.

$C = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \ 0 \leq y \leq 1\}$ is a convex set. See the picture below.



Example 2.

See the picture below for a set in $\mathbb{R}^2$ that is not a convex set.

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Lecture 23

Remark 2.

We will now state the following Mean Value Theorem for several variables.

Mean Value Theorem for several variables (MVT2).

Suppose C is an open convex set in $\mathbb{R}^2$ or $\mathbb{R}^3$. Suppose $f : C \to \mathbb{R}$ is a differentiable function. Fix points $x, y \in C$. Then, there exists $m \in (0,1)$ such that

$$f(y) - f(x) = \nabla f_{((1-m)x+my)} \cdot (y - x)$$

where . means the dot product of two vectors.

Remark 3.

You may have seen how the MVT1 can be used to prove that if the real valued function $g(x)$ is differentiable on the open interval, (a, b), in $\mathbb{R}$ and $g'(x) = 0$, for all $x \in (a, b)$, then $g(x)$ is constant on (a, b). Well, the MVT2 can also be used to prove something similar in remark 4 below. We first define what connected means.

Definition 2.

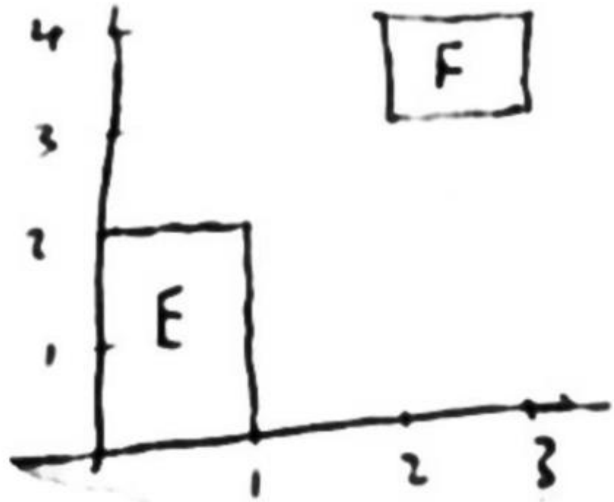
Suppose E is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. Then, E is connected if E cannot be expressed as the union of two non-empty disjoint open sets.

Remark 4.

Suppose E is an open connected set in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $f : E \to \mathbb{R}$ is a differentiable function. Suppose all the partial derivatives of f are zero. Then, f is constant on E.

Example 3.

If $E = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 2\}$ and $F = \{(x, y) \in \mathbb{R}^2 : 2 < x < 3,\ 3 < y < 4\}$, then E is connected and F is connected but $E \cup F$ is not connected because E and F are non-empty open sets that are disjoint. See the picture below.



Remark 5.

If you apply the Cauchy–Schwarz inequality to

$$f(y) - f(x) = \nabla f_{((1-m)x+my)} \cdot (y-x) \quad \text{in the MVT2 above}$$

you get

$$|f(y) - f(x)| = |\nabla f_{((1-m)x+my)} \cdot (y-x)|$$

$$\leq ||\nabla f_{((1-m)x+my)}|| \; ||(y-x)||$$

and so we have the inequality

$$|f(y) - f(x)| \leq ||\nabla f_{((1-m)x+my)}|| \; ||(y-x)||$$

Definition 3.

Suppose $g : A \to \mathbb{R}$ is a function where A is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. Then, we say g is bounded on A if there exists a real number W such that $|g(x)| \leq W$, for all $x \in A$.

Remark 6.

If all the partial derivatives of f are bounded on C in the MVT2 then from remark 5 we get that there exists a real number K such that

$$|f(y) - f(x)| \leq K||(y-x)||, \quad \text{for all} \quad x, y \in C$$

This will mean that f is (what is called) Lipschitz continuous, which is a much stronger form of continuity than the usual notion of continuity.